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Logarithmic Sobolev and Poincaré inequalities for the circular Cauchy distribution*

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Abstract

In this paper, we consider the circular Cauchy distribution μ_x on the unit circle S with index $0 \le |x| < 1$ and we study the spectral gap and the optimal logarithmic Sobolev constant for μ_x, denoted respectively by $\lambda_1(\mu_x)$ and $C_{\mathrm{LS}}(\mu_x)$. We prove that $\frac{1}{1+|x|} \le \lambda_1(\mu_x) \le 1$ while $C_{\mathrm{LS}}(\mu_x)$ behaves like $\log(1+\frac{1}{1-|x|})$ as $|x| \to 1$.

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0.1 Circular Cauchy distribution

Let S be the unit circle in $\mathbb{R}^2$ with the Riemannian structure induced by $\mathbb{R}^2$ and write ∇_S for the spherical gradient. For any $x \in \mathbb{R}^2$ with $|x| < 1$, we consider the probability measure μ_x on S which has density

$$h(x,y) = \frac{1}{2\pi}\frac{1-|x|^2}{|y-x|^2}, \quad y \in S$$

with respect to the arc length μ on the unit circle S. The form of the density h makes μ_x known as circular Cauchy distribution or wrapped Cauchy distribution (see [10, 11]).

On the one hand, it enjoys the following property: if f is an integrable function on S, then $\tilde{f}(x) = \int_S f(y)d\mu_x(y)$ solves the following Cauchy problem:

$$\begin{cases} \triangle u = 0, & \text{in } B(0,1) \\ u|_S = f, \end{cases}$$

where $B(0,1) = \{y||y| < 1\}$ is the unit ball in $\mathbb{R}^2$. For this reason, μ_x is also called the harmonic probability associated with x on S. Obviously $\mu_0 = \mu$.

On the other hand, due to the connection with Brownian motion as first identified by Kakutani [9], harmonic probabilities play an important role in probability theory. Indeed, if $\mathbb{P}^x$ denotes the probability distribution of a standard two-dimensional Brownian motion B_t starting from x, and τ the first time for B_t to hit S, μ_x is nothing but the distribution of B_τ under $\mathbb{P}^x$ (see [7]).

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Furthermore, consider the following Mӧbius Markov process (see [10]):

$$W_n = \frac{W_{n-1} + \beta}{\bar{\beta} W_{n-1} + 1} \varepsilon_n, \quad n = 1, 2, \cdots,$$

where $\beta = (x_1, x_2) \in B(0,1)$ and $\bar{\beta} = (x_2, x_1)$. Suppose that W_0 is a constant or a random variable which takes values in S and $(\varepsilon_n)_{n\geq 1}$ are independent identically distributed random variables taking values in S with common distribution μ_{x_0} for some $x_0 \in B(0,1)$ fixed. Define

$$x = \begin{cases} \dfrac{|x_0| - 1 + \sqrt{(1-|x_0|)^2 + 4|x_0||\beta|^2}}{2|\beta|^2}\beta, & \text{if } 0 < |\beta| < 1; \\ 0, & |\beta| = 0. \end{cases}$$

Kato [10] proved that μ_x is the unique invariant probability of the Möbius Markov process $(W_n)_{n\geq 1}$.

The aim of this paper is to estimate the spectral gap and logarithmic Sobolev constants of μ_x.

Let $\lambda_1(\mu_x)$ be the spectral gap of the circular Cauchy distribution μ_x associated with the Dirichlet form

$$\mathcal{E}_{\mu_x}(f,f) = \int_S |\nabla_S f|^2 d\mu_x, \quad \forall\ f : S \to \mathbb{R} \quad \text{smooth function},$$

which has a classical variational formula

$$\lambda_1(\mu_x) = \inf\{\frac{\mathcal{E}_{\mu_x}(f,f)}{\mathrm{Var}_{\mu_x}(f)} : f \text{ non constant }\}, \tag{0.1}$$

where $\mathrm{Var}_{\mu_x}(f) = \int_S f^2 d\mu_x - (\int_S f d\mu_x)^2$ is the variance of f with respect to μ_x. The constant $\lambda_1(\mu_x)$ is thus the best constant in the following Poincaré inequality

$$C\mathrm{Var}_{\mu_x}(f) \leq \mathcal{E}_{\mu_x}(f,f).$$

We say μ_x satisfies a logarithmic Sobolev inequality if there exists a non-negative constant C such that for any smooth function $f : S \to \mathbb{R}$,

$$\mathrm{Ent}_{\mu_x}(f^2) \leq 2C \int_S |\nabla_S f|^2 d\mu_x,$$

where

$$\mathrm{Ent}_{\mu_x}(f^2) := \mu_x(f^2 \log f^2) - \mu_x(f^2)\log(\mu_x(f^2))$$

is the entropy of f^2 under μ_x. We will denote by $C_{\mathrm{LS}}(\mu_x)$ the optimal logarithmic Sobolev constant of μ_x.

An effective method to prove Poincaré or logarithmic Sobolev inequalities is the Bakry-Émery curvature-dimension criterion [1]. It gives, in particular, that $\lambda_1(\mu) = C_{\mathrm{LS}}(\mu) = 1$. It is classical for the Poincaré inequality and for logarithmic Sobolev inequality as in [8]. Nevertheless, this criterion cannot be applied for all x as the generalized curvature is not bounded from below when x tends to the unit circle. Another natural approach would be to use the Brownian motion interpretation of μ_x together with stochastic calculus, as in [12], for which the stopping time τ was involved. In detail, in [12] with this method, G. Schechtman and M. Schmuckenschläger proved that harmonic measures μ_x^n on S^{n-1} with $n \geq 3$ and $|x| < 1$ had a uniform Gaussian concentration.

In [3], with F. Barthe, we used another method to work on harmonic measures μ_x^n on the unit spheres S^{n-1}. Precisely, we took advantage of the fact that the density of the harmonic measures only depends on one coordinate, based on which, we proved respectively that

$$\min\{\lambda_1(\nu_{|x|,n}), n-2\} \le \lambda_1(\mu_x^n) \le \lambda_1(\nu_{|x|,n}) \tag{0.2}$$

and

$$C_{\mathrm{LS}}(\nu_{|x|,n}) \le C_{\mathrm{LS}}(\mu_x^n) \le \max\{C_{\mathrm{LS}}(\nu_{|x|,n}), \frac{1}{n-2}\}. \tag{0.3}$$

Here $\nu_{|x|,n}$ is the image probability of μ_x^n by the map $y \to d(y, e_1)$ with e_1 the first component of the canonical basis in $\mathbb{R}^n$. From this comparison, we proved that for harmonic measures μ_x^n on S^{n-1} with $n \ge 3$, $\lambda_1(\mu_x^n)$ satisfied $\frac{n-2}{2} \le \lambda_1(\mu_x^n) \le n-1$ and the optimal logarithmic Sobolev constant $C_{\mathrm{LS}}(\mu_x^n)$ satisfied

$$\frac{1}{2(n-1)} \log(1 + \frac{2}{n(1-|x|)}) \le C_{\mathrm{LS}}(\mu_x^n) \le \frac{C}{n} \log(1 + \frac{1}{1-|x|})$$

with C a positive universal constant.

However when $n = 2$, for the circular Cauchy distribution μ_x, $n-2 = 0$, the inequalities (0.2), (0.3) do not apply. So in this paper, we follow the main idea of [3] while adjust the estimates.

Our main results are the following:

Theorem 0.1. *For any $x \in \mathbb{R}^2$ with $0 \le |x| < 1$, the following statements hold:*

(a) The spectral gap $\lambda_1(\mu_x)$ satisfies

$$\frac{1}{1+|x|} \le \lambda_1(\mu_x) \le 1 = \lambda_1(\mu).$$

(b) The optimal constant $C_{\mathrm{LS}}(\mu_x)$ satisfies

$$\max\{1, \frac{1}{2} \log(1 + \frac{1}{1-|x|})\} \le C_{\mathrm{LS}}(\mu_x) \le 8\pi \log(1 + \frac{e^2\pi}{2(1-|x|)}) + 2.$$

Remark 0.2. The estimate for $\lambda_1(\mu_x)$ is sharp since when $x = 0$, the lower and upper bounds coincide with $\lambda_1(\mu) = 1$.

Remark 0.3. Since the diameter of the unit circle S is π, the result in [15] ensures that for any $f : S \to \mathbb{R}$ with $\mu_x(f^2) = 1$, one has

$$W_d^2(f^2\mu_x, \mu_x) \le 4(8 \log 2 + \pi)\mathrm{Ent}_{\mu_x}(f^2),$$

that is to say μ_x satisfies the so called L^2-transportation inequalities W_2H introduced by Talagrand [13]. Here $W_d^2(\nu, \mu)$ is the L^2-Wasserstein distance between ν and μ, which is defined as

$$W_d^2(\nu, \mu) = \inf_{\pi} \int_{S^2} d^2(x, y) d\pi(x, y),$$

with π the coupling of ν and μ. However by Theorem 0.1, when x approaches S, the optimal logarithmic Sobolev constant explodes with speed $\log(1 + \frac{1}{1-|x|})$. That is, the circular Cauchy distribution μ_x is a natural counter-example to declare the real gap between logarithmic Sobolev and W_2H inequalities as in [3, 4, 14].

1 Prelimilaries

Given any $x \in S$, it can be written as $x = (\cos\theta, \omega\sin\theta)$, where $\theta \in [0, \pi]$ is the geodestic distance $d(x, e_1)$ between x and the first component of the canonical basis in $\mathbb{R}^2$, and $\omega \in \{-1, 1\}$. We then consider the path γ_0 defined as

$$\gamma_0(t) = (\cos(\theta + t), \omega\sin(\theta + t)),\ t \in \mathbb{R},$$

which is a path on S satisfying $\gamma_0(0) = x$ and $|\gamma_0'(0)| = 1$, then $\nabla_S f(x) = (f \circ \gamma_0)'(0)$.

For $\theta \in (0, \pi)$, define

$$S(\theta) := \{x \in S; d(x, e_1) = \theta\} = \{(\cos\theta, \omega\sin\theta),\ \omega \in \{-1, 1\}\}.$$

The conditional probability μ_θ on $S(\theta)$ is a Bernoulli distribution with parameter $1/2$.

Lemma 1.1. *Let M be a probability measure on S with*

$$M(dy) = \frac{1}{2\pi}\varphi(d(y, e_1))\mu(dy),\quad y \in S,$$

where φ is non-negative and measurable. Let ν be the image probability of M by the map $y \to d(y, e_1)$, which is a probability on the interval $[0, \pi]$.

We have respectively

(1). The corresponding spectral gaps satisfy

$$\min\{\lambda_1(\nu), \lambda^{DD}(\nu)\} \le \lambda_1(M) \le \lambda_1(\nu).$$

(2). Similarly, the optimal logarithmic Sobolev constants satisfy

$$C_{\mathrm{LS}}(\nu) \le C_{\mathrm{LS}}(M) \le C_{\mathrm{LS}}(\nu) + \frac{1}{\lambda^{DD}(\nu)}.$$

Here $\lambda_1(\nu)$ is the spectral gap of ν and $\lambda^{DD}(\nu)$ is the first eigenvalue of ν with Dirichlet boundary conditions at 0 and π, which has a classical variational formula as

$$\lambda^{DD}(\nu) := \inf\left\{\frac{\int_0^\pi (f')^2 d\nu}{\nu(f^2)} :\ f(0) = f(\pi) = 0,\ f \text{ non constant}\right\}.$$

Proof. Let F be any every smooth function $F : [0, \pi] \to \mathbb{R}$, and apply the Poincaré inequality for M to the function $f(x) = F(d(x, e_1)) = F(\arccos x)$. By definition $\mathrm{Var}_M(f) = \mathrm{Var}_\nu(F)$. If $x \neq \pm e_1$, f is differentiable $M - a.e.$, moreover,

$$|\nabla_S f|^2(x) = |(f \circ \gamma_0)'(0)|^2.$$

Clearly, $f(\gamma_0(t)) = f(\cos(\theta + t), \sin(\theta + t)\omega) = F(\theta + t)$ and $(f \circ \gamma_0)'(0) = F'(\theta)$. So,

$$|\nabla_S f|^2(x) = (F'(\theta))^2 = (F'(d(x, e_1)))^2,$$

which implies $\int_S |\nabla_S f|^2 dM = \int_0^\pi (F')^2 d\nu$. It holds by the classical variational formula (0.1) that $\lambda_1(M) \le \lambda_1(\nu)$ since the family of non constant functions $f : S \to \mathbb{R}$ is larger than that of non constant functions $F : [0, \pi] \to \mathbb{R}$.

Replacing the **Variaance** by **Entropy**, we get $C_{\mathrm{LS}}(\nu) \le C_{\mathrm{LS}}(M)$.

For the lower bound of $\lambda_1(M)$, we use the notations presented at the beginning of this section.

For any f measurable on S, we have

$$F(\theta) := \int_{S(\theta)} f(\cos\theta, \omega\sin\theta) d\mu_\theta = \frac{1}{2} f(\cos\theta, \sin\theta) + \frac{1}{2} f(\cos\theta, -\sin\theta)$$

and

$$g(\theta) := \int_{S(\theta)} f(\cos\theta, \omega\sin\theta)\omega d\mu_\theta = \frac{1}{2} f(\cos\theta, \sin\theta) - \frac{1}{2} f(\cos\theta, -\sin\theta). \tag{1.1}$$

It is clear that g satisfies $g(0) = g(\pi) = 0$. Observe that

$$\mathrm{Var}_M(f) = \mathrm{Var}_\nu(F) + \int_0^\pi \mathrm{Var}_{\mu_\theta}(f|_{S(\theta)}) d\nu(\theta) = \mathrm{Var}_\nu(F) + \nu(g^2).$$

Therefore

$$\begin{aligned}
\mathrm{Var}_M(f) &\leq \frac{1}{\lambda_1(\nu)} \int_0^\pi (F')^2 d\nu + \frac{1}{\lambda^{DD}(\nu)} \int_0^\pi g'^2 d\nu \\
&\leq \max\left\{\frac{1}{\lambda_1(\nu)}, \frac{1}{\lambda^{DD}(\nu)}, \right\} \int_0^\pi \left\{ \left(\int_{S(\theta)} (f\circ\gamma_0)'(0) d\mu_\theta \right)^2 \right. \\
&\left. + \left(\int_{S(\theta)} (f\circ\gamma_0)'(0)\omega d\mu_\theta \right)^2 \right\} d\nu \qquad . \\
&= \frac{1}{\min\{\lambda_1(\nu), \lambda^{DD}(\nu)\}} \int_0^\pi \int_{S(\theta)} \left((f\circ\gamma_0)'(0)\right)^2 d\mu_\theta d\nu(\theta) \\
&= \frac{1}{\min\{\lambda_1(\nu), \lambda^{DD}(\nu)\}} \int_S |\nabla_S f|^2 dM,
\end{aligned}$$

which immediately offers $\lambda_1(M) \geq \min\{\lambda_1(\nu), \lambda^{DD}(\nu)\}$.

Given smooth function $f : S \to \mathbb{R}$, define $G^2(\theta) := \int_{S(\theta)} f^2(\cos\theta, \omega\sin\theta) d\mu(\theta)$. Notice then that

$$\begin{aligned}
\mathrm{Ent}_M(f^2) &= \mathrm{Ent}_\nu\left(\int_{S(\theta)} f^2 d\mu_\theta\right) + \int_0^\pi \mathrm{Ent}_{\mu_\theta}(f^2|_{S(\theta)}) d\nu(\theta) \\
&\leq \mathrm{Ent}_\nu(G^2) + \frac{1}{2}\int_0^\pi (f(\cos\theta, \sin\theta) - f(\cos\theta, -\sin\theta))^2 d\nu(\theta) \qquad (1.2) \\
&\leq 2C_{\mathrm{LS}}(\nu) \int_0^\pi (G'(\theta))^2 d\nu(\theta) + \frac{2}{\lambda^{DD}(\nu)} \int_0^\pi (g'(\theta))^2 d\nu(\theta),
\end{aligned}$$

where g is given in (1.1) and the first inequality is true since the optimal logarithmic Sobolev constant for the Bernoulli distribution with parameter $1/2$ is 1.

By definition,

$$2G(\theta)G'(\theta) = 2\int_{S(\theta)} f(\cos\theta, \omega\sin\theta)(f\circ\gamma_0)'(0) d\mu_\theta,$$

which implies

$$\begin{aligned}
(G'(\theta))^2 &= \frac{\left(\int_{S(\theta)} f(\cos\theta, \omega\sin\theta)(f\circ\gamma_0)'(0) d\mu_\theta\right)^2}{G^2(\theta)} \\
&\leq \frac{\int_{S(\theta)} f^2(\cos\theta, \omega\sin\theta) d\mu_\theta}{G^2(\theta)} \int_{S(\theta)} \left((f\circ\gamma_0)'(0)\right)^2 d\mu_\theta \\
&= \int_{S(\theta)} \left((f\circ\gamma_0)'(0)\right)^2 d\mu_\theta.
\end{aligned}$$

And similarly we have

$$g'(\theta)^2 \leq \int_{S(\theta)} \left((f\circ\gamma_0)'(0)\right)^2 d\mu_\theta.$$

Thus from (1.2),

$$\mathrm{Ent}_M(f^2) \le 2(C_{\mathrm{LS}}(\nu) + \frac{1}{\lambda^{DD}(\nu)}) \int_S |\nabla_S f|^2 dM, \tag{1.3}$$

where implies immediately that

$$C_{\mathrm{LS}}(\mu_x) \le C_{\mathrm{LS}}(\nu) + \frac{1}{\lambda^{DD}(\nu)}.$$

The proof is complete now.

□

2 Proof of Theorem 0.1

By rotation invariance of the unit circle, without loss of generality, take $x = ae_1$. Let ν_a be the image probability of μ_x by the map $y \to d(y, e_1)$. Precisely,

$$d\nu_a(\theta) = \frac{1}{\pi} \frac{1-a^2}{1+a^2-2a\cos\theta} d\theta =: h_a(\theta) d\theta, \quad \theta \in [0,\pi]. \tag{2.1}$$

When $a = 0, \nu_0$ is the uniform probability on $[0,\pi]$, whose spectral gap and optimal logarithmic Sobolev constant are known to be 1.

Consider the associated Dirichlet form of ν_a

$$\mathcal{E}_a(f,f) = \int_0^\pi (f')^2 d\nu_a = \int_0^\pi f(-\mathcal{L}_a f) d\nu_a,$$

where the generator $\mathcal{L}_a$ is given as for any $f \in C^2([0,\pi])$,

$$\mathcal{L}_a f(\theta) = f''(\theta) - \frac{2a\sin\theta}{1+a^2-2a\cos\theta} f'(\theta).$$

Proof of the item (a) of Theorem 0.1. Take $f(\theta) = \cos\theta$, we have

$$\begin{aligned} \nu_a(f) &= \frac{1-a^2}{\pi} \int_0^\pi \frac{\cos\theta}{1+a^2-2a\cos\theta} d\theta = \frac{1-a^2}{2a\pi} \int_0^\pi (-1 + \frac{1+a^2}{1+a^2-2a\cos\theta}) d\theta \\ &= -\frac{1-a^2}{2a} + \frac{1+a^2}{2a} = a, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \nu_a(f^2) &= \frac{1-a^2}{\pi} \int_0^\pi \frac{\cos^2\theta}{1+a^2-2a\cos\theta} d\theta \\ &= \frac{1-a^2}{\pi} \int_0^{\pi/2} (\frac{\cos^2\theta}{1+a^2-2a\cos\theta} + \frac{\cos^2(\pi-\theta)}{1+a^2-2a\cos(\pi-\theta)}) d\theta \\ &= \frac{1-a^2}{\pi} \int_0^{\pi/2} \frac{2(1+a^2)\cos^2\theta}{(1+a^2)^2 - 4a^2\cos^2\theta} \\ &= \frac{1-a^4}{2a^2\pi} \int_0^{\pi/2} (-1 + \frac{(1+a^2)^2}{(1+a^2)^2-4a^2\cos^2\theta}) d\theta \\ &= -\frac{1-a^4}{4a^2} + \frac{(1+a^2)^2}{4a^2} = \frac{1+a^2}{2}, \end{aligned} \tag{2.3}$$

which implies

$$\mathcal{E}_a(f,f) = \int_0^\pi \sin^2\theta d\nu_a = 1 - \nu_a(f^2) = \frac{1-a^2}{2} = \nu_a(f^2) - (\nu_a(f))^2.$$

Thereby by classical variational formula (0.1),

$$\lambda_1(\nu_a) \leq \frac{\mathcal{E}_a(f,f)}{\mathrm{Var}_a(f)} = 1. \tag{2.4}$$

For the upper bound of $1/\lambda_1(\nu_a)$, we turn to Chen's original variational formula of $\lambda_1(\nu)$ (see [5]). Precisely, it is

$$\lambda_1(\nu_a)^{-1} = \inf_{\rho\in\mathcal{F}} \sup_{x\in[0,\pi]} \frac{1+a^2-2a\cos x}{\rho'(x)} \int_x^\pi \frac{\rho(y)-\nu_a(\rho)}{1+a^2-2a\cos y} dy, \tag{2.5}$$

where $\mathcal{F}$ is the set of strictly increasing functions on $[0,\pi]$.

Choose then $\rho(\theta) = -\cos\theta + a$ a strictly increasing function on $[0,\pi]$ with $\nu_a(\rho) = 0$ by (2.2). By the expression (2.5), we have

$$\begin{aligned}
\frac{1}{\lambda_1(\nu_a)} &\leq \sup_{\theta\in(0,\pi)} \frac{1+a^2-2a\cos\theta}{\sin\theta} \int_\theta^\pi \frac{(-\cos\xi+a)}{1+a^2-2a\cos\xi} d\xi \\
&= \sup_{\theta\in(0,\pi)} \frac{1+a^2-2a\cos\theta}{2a\sin\theta} \left(\pi - \theta - 2\arctan\left(\frac{1-a}{1+a}\cot(\frac{\theta}{2})\right)\right) \\
&= \sup_{\theta\in(0,\pi)} \frac{1+a^2-2a\cos\theta}{a\sin\theta} \left(\arctan\left(\cot(\frac{\theta}{2})\right) - \arctan\left(\frac{1-a}{1+a}\cot(\frac{\theta}{2})\right)\right) \\
&\leq \sup_{\theta\in(0,\pi)} \frac{1+a^2-2a\cos\theta}{a\sin\theta} \frac{(1-\frac{1-a}{1+a})\cot(\frac{\theta}{2})}{1+(\frac{1-a}{1+a}\cot(\frac{\theta}{2}))^2} \\
&= 1+a,
\end{aligned}$$

where the first equality is due to

$$\int_\theta^\pi \frac{1}{1+a^2-2a\cos\theta} = \frac{2}{1-a^2}\arctan\left(\frac{1-a}{1+a}\cot(\frac{\theta}{2})\right) \tag{2.6}$$

and the last but second inequality holds since

$$\arctan x - \arctan y \leq (x-y)(\arctan y)', \quad \forall\, 0 \leq y < x \leq \pi/2.$$

To estimate $\lambda^{DD}(\nu_a)$, we take $\rho(\theta) = \sin\theta$ on $[0,\pi]$, which satisfies

$$\rho(0) = \rho(\pi) = 0, \rho'(\theta)|_{\theta\in(0,\pi/2)} > 0 \text{ and } \rho'(\theta)|_{\theta\in(\pi/2,\pi)} < 0.$$

Therefore it follows from Theorem 1.1 in [6] that

$$\begin{aligned}
\frac{1}{\lambda^{DD}(\nu_a)} &\leq \sup_{x\in(0,\pi/2)} \frac{1}{\sin x}\int_0^x (1+a^2-2a\cos y)dy \int_x^{\pi/2} \frac{\sin u}{1+a^2-2a\cos u} du \\
&\quad \vee \sup_{x\in(\pi/2,\pi)} \frac{1}{\sin x}\int_x^\pi (1+a^2-2a\cos y)dy \int_{\pi/2}^x \frac{\sin u}{1+a^2-2a\cos u} du \\
&\leq \sup_{x\in(0,\pi/2)\cup(\pi/2,\pi)} \frac{1+a^2-2a\cos x}{\cos x} \int_x^{\pi/2} \frac{\sin u}{1+a^2-2a\cos u} du \\
&= \sup_{x\in(0,\pi/2)\cup(\pi/2,\pi)} \frac{1+a^2-2a\cos x}{2a\cos x} \log(\frac{1+a^2}{1+a^2-2a\cos x}) \\
&= \sup_{|t|<2a/(1+a^2)} (1-\frac{1}{t})\log(1-t) = \frac{(1+a)^2}{2a}\log\frac{(1+a)^2}{1+a^2},
\end{aligned} \tag{2.7}$$

where the second inequality follows from the proportional property and the last equality holds since $(1-\frac{1}{t})\log(1-t)$ is decreasing on $t\in[-1,1]$.

Finally, we have for any x with $0 \le |x| = a < 1$,

$$\frac{1}{1+a} = \min\{\frac{2a}{(1+a)^2\log\frac{(1+a)^2}{1+a^2}}, \frac{1}{1+a}\} \le \lambda_1(\mu_x) \le 1.$$

The proof of the item (**a**) of Theorem 0.1 is complete.

Proof of the item (b) of Theorem 0.1. Recall that for the function $f := \cos$, in the third section, it was proved that $\nu_a(f) = a, \nu_a(f^2) = (1+a^2)/2$ and $\mathcal{E}_a(f,f) = (1-a^2)/2$. Define $g = (1-f)/(1-a)$, then

$$\nu_a(g) = 1, \quad \nu_a(g^2) = \frac{3-a}{2(1-a)}, \quad \mathcal{E}_a(g,g) = \frac{\mathcal{E}_a(f,f)}{(1-a)^2} = \frac{1+a}{2(1-a)}.$$

Therefore with the help of an elementary inequality $\mathrm{Ent}_{\nu_a}(g^2) \ge \nu_a(g^2)\log(\nu_a(g^2))$ (see [3]), we have

$$2C_{\mathrm{LS}}(\nu_a) \ge \frac{\mathrm{Ent}_a(g^2)}{\mathcal{E}_a(g,g)} \ge \frac{3-a}{1+a}\log(1+\frac{1+a}{2(1-a)}) \ge \log(1+\frac{1}{1-a}). \tag{2.8}$$

Next we work on the upper bound. It is clear that $\theta_a := 2\arctan\frac{1-a}{1+a}$ is the median of ν_a since by (2.6),

$$\frac{1-a^2}{\pi}\int_{\theta_a}^{\pi}\frac{1}{1+a^2-2a\cos\theta} = \frac{2}{\pi}\arctan(\frac{1-a}{1+a}\cot(\frac{\theta_a}{2})) = \frac{1}{2}.$$

Define

$$B_-(a) := \sup_{\alpha\in(0,\theta_a)}\int_0^{\alpha}\frac{d\theta}{1+a^2-2a\cos\theta}\log\left(1+\frac{e^2\pi}{(1-a^2)\int_0^{\alpha}\frac{1}{1+a^2-2a\cos\theta}d\theta}\right)$$
$$\cdot\int_{\alpha}^{\theta_a}(1+a^2-2a\cos\theta)d\theta,$$

$$B_+(a) := \sup_{\alpha\in(\theta_a,\pi)}\int_{\alpha}^{\pi}\frac{d\theta}{1+a^2-2a\cos\theta}\log\left(1+\frac{e^2\pi}{(1-a^2)\int_{\alpha}^{\pi}\frac{1}{1+a^2-2a\cos\theta}d\theta}\right)$$
$$\cdot\int_{\theta_a}^{\alpha}(1+a^2-2a\cos\theta)d\theta.$$

By the equality (2.6) and $\dfrac{x}{1+x^2} \le \arctan x \le x$, we have

$$\frac{\sin\alpha}{1+a^2-2a\cos\alpha} \le \int_{\alpha}^{\pi}\frac{1}{1+a^2-2a\cos\theta}d\theta \le \frac{2}{(1+a)^2\sin\frac{\alpha}{2}} \tag{2.9}$$

and

$$\int_0^{\alpha}\frac{1}{1+a^2-2a\cos\theta}d\theta \le \frac{\pi}{1-a^2} - \frac{\sin\alpha}{1+a^2-2a\cos\alpha} \le \frac{\pi}{1-a^2}. \tag{2.10}$$

On the one hand, by the monotonicity of $x\log(1+\frac{b}{x})$ in $x>0$ for any $b>0$ and (2.9), we obtain

$$\begin{aligned}
B_+(a) \le& \sup_{\alpha\in(\theta_a,\pi)}\frac{2}{(1+a)^2\sin(\frac{\alpha}{2})}\log\left(1+\frac{e^2\pi(1+a)}{2(1-a)}\right)\left((1+a^2)\alpha - 2a\sin\alpha\right) \\
\le& \frac{4}{(1+a)^2}\log\left(1+\frac{e^2\pi(1+a)}{2(1-a)}\right)\sup_{\alpha\in(\theta_a/2,\pi/2)}\frac{(1+a^2)\alpha}{\sin\alpha} \\
=& \frac{2\pi(1+a^2)}{(1+a)^2}\log(1+\frac{e^2\pi(1+a)}{2(1-a)}) \\
\le& 2\pi\log(1+\frac{e^2\pi}{2(1-a)}).
\end{aligned}$$

On the other hand, combining the inequality (2.10), the monotonicity of $x\log(1+\frac{b}{x})$ for $b>0$ fixed and the fact that

$$\frac{2}{\pi}\theta_a \le \sin\theta_a = \frac{2\tan(\theta_a/2)}{1+\tan^2(\theta_a/2)} = \frac{1-a^2}{1+a^2},$$

we have

$$\begin{aligned} B_-(a) \le& \frac{\pi}{1-a^2}\log(1+e^2)\left((1+a^2)\theta_a - a\sin\theta_a\right) \\ \le& \pi\log(1+e^2)\frac{\theta_a}{\sin\theta_a} \le \frac{\pi^2}{2}\log(1+e^2). \end{aligned}$$

By Theorem 3 in [2],

$$C_{\mathrm{LS}}(\nu_a) \le 4\max\{B_+(a), B_-(a)\} \le 8\pi\log(1+\frac{e^2\pi}{2(1-a)}). \tag{2.11}$$

The proof is complete due to (2.8),(2.11) and the classical result

$$C_{\mathrm{LS}}(\mu_x) \ge \frac{1}{\lambda_1(\mu_x)} \ge 1.$$

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