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Concentration estimates for the isoperimetric constant of the supercritical percolation cluster

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Abstract

We consider the Cheeger constant $\phi(n)$ of the giant component of supercritical bond percolation on $\mathbb{Z}^d/n\mathbb{Z}^d$. We show that the variance of $\phi(n)$ is bounded by $\frac{\xi}{n^d}$, where ξ is a positive constant that depends only on the dimension d and the percolation parameter.

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1 Introduction

Let $\mathbb{T}^d(n)$ be the d dimensional torus with side length n, i.e. $\mathbb{Z}^d/n\mathbb{Z}^d$, and denote by $\mathbb{E}_d(n)$ the set of edges of the graph $\mathbb{T}^d(n)$. Let $p_c(\mathbb{Z}^d)$ denote the critical value for bond percolation on $\mathbb{Z}^d$, and fix some $p_c(\mathbb{Z}^d) < p \leq 1$. We apply a p-bond Bernoulli percolation process on the torus $\mathbb{T}^d(n)$ and denote by $C_d(n)$ the largest open component of the percolated graph (In case of two or more identically sized largest components, choose one by some arbitrary but fixed method). Let $\Omega = \Omega_n = \{0,1\}^{\mathbb{E}_d(n)}$ be the space of configurations for the percolation process and denote by $\mathbf{P} = \mathbf{P}_p$ the probability measure associated with the percolation process. For a subset $A \subset C_d(n)(\omega)$ we denote by $\partial_{C_d(n)}A$ the boundary of the set A in $C_d(n)$, i.e. the set of edges $(x,y) \in \mathbb{E}_d(n)$ such that $\omega((x,y)) = 1$ and with either $x \in A$ and $y \notin A$ or $x \notin A$ and $y \in A$. Throughout this paper c, C and c_i for $i \in \mathbb{N}$ denote positive constants which may depend on the dimension d and the percolation parameter p but not on n. The value of the constants may change from one line to the next.

Next we define the Cheeger constant

Definition 1.1. *For a set $\emptyset \neq A \subset C_d(n)$, we denote*

$$\psi_A = \frac{|\partial_{C_d(n)}A|}{|A|}.$$

where $|\cdot|$ denotes the cardinality of a set. The Cheeger constant of $C_d(n)$ is defined by:

$$\phi = \phi(n) := \min_{\substack{\emptyset \neq A \subset C_d(n) \\ |A| \leq |C_d(n)|/2}} \psi_A.$$

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In [5] Benjamini and Mossel studied the robustness of the mixing time and Cheeger constant of $\mathbb{Z}^d$ under a percolation perturbation. They showed that for $p_c(\mathbb{Z}^d) < p < 1$ large enough $n\phi(n)$ is bounded between two constants with high probability. In [7], Mathieu and Remy improved the result and proved the following on the Cheeger constant

Theorem 1.2. *For every $p > p_c(\mathbb{Z}^d)$, there exist constants $c_2(p), c_3(p), c(p) > 0$ such that for every $n \in \mathbb{N}$*

$$\boldsymbol{P}\left(\frac{c_2}{n} \leq \phi(n) \leq \frac{c_3}{n}\right) \geq 1 - e^{-c\log^{\frac{3}{2}} n}.$$

Recently, Marek Biskup and Gábor Pete brought to our attention that better bounds on the Cheeger constant exist in both [8] and [3]. The following theorem is stated in [8] Corollary 1.4 without asymptotic rate, however going over the proof one obtains the following statement:

Theorem 1.3. *[8] For $d \geq 2$ and $p > p_c(\mathbb{Z}^d)$ and for every $C > 0$, there are constants $\alpha(d,p) > 0$ and $\beta(d,p) > 0$ such that*

$$\boldsymbol{P}\left(\begin{array}{c} \forall S \subset \mathcal{C}_d(n) \text{ connected,} \\ \text{if } Cn \leq |S| < \frac{|\mathcal{C}_d(n)|}{2} \text{ then } \frac{|\partial_{\mathcal{C}} S|}{|S|^{(d-1)/d}} \geq \alpha \end{array}\right) \geq 1 - \exp\left(-\beta n^{(d-1)/d}\right).$$

Our result can be obtained with the use of [7] however we use Theorem 1.3 as it simplifies the proofs.

In 2011 Itai Benjamini gave the following conjecture as an extension to the known results about the Cheeger constant:

Conjecture 1.4. *The limit $\lim_{n\to\infty} n\phi(n)$ exists.*

Even though the last conjecture is still open, and the expectation of the Cheeger constant is quite evasive, we managed to give a good bound on the variance of the Cheeger constant. This is given in the main Theorem of this paper (The proof is presented in page 9):

Theorem 1.5. *There exists a constant $\xi = \xi(p,d) > 0$ such that*

$$\boldsymbol{Var}(\phi) \leq \frac{\xi}{n^d}.$$

A major ingredient of the proof is Talagrand's inequality for concentration of measure on product spaces. Talagrand's inequality requires control over the influence of a single edge on the Cheeger constant. Such a bound can be achieved using results on the isoperimetric profile of the giant component and the fact that with high probability edges outside the giant component have little effect over the Cheeger constant. This inequality is used by Benjamini, Kalai and Schramm in [4] to prove concentration of first passage percolation distance. A related study that uses another inequality by Talagrand is [1], where Alon, Krivelevich and Vu prove a concentration result for eigenvalues of random symmetric matrices.

2 The Cheeger constant

Before turning to the proof of Theorem 1.5, we give the following definitions:

Definition 2.1. *For a function $f : \Omega \to \mathbb{R}$ and an edge $e \in \mathbb{E}_d(n)$ we define $\nabla_e f : \Omega \to \mathbb{R}$ by*

$$\nabla_e f(\omega) = f(\omega) - f(\omega^e)$$

where

$$\omega^e(e') = \begin{cases} \omega(e') & e' \neq e \\ 1 - \omega(e') & e' = e \end{cases}.$$

In addition, for a configuration $\omega \in \Omega$ *and an edge* $e \in \mathbb{E}_d(n)$*, let* $\hat{\omega}^e = \min\{\omega, \omega^e\}$ *and* $\check{\omega}^e = \max\{\omega, \omega^e\}$*.*

Definition 2.2. *For* $n \in \mathbb{N}$ *we define the following events:*

$$\begin{aligned} H_n^1(c_1) &= \left\{\omega \in \Omega \ : \ |C_d(n)(\omega)| > c_1 n^d\right\} \\ H_n^2(c_2, c_3) &= \left\{\omega \in \Omega \ : \ \frac{c_2}{n} < \phi(n)(\omega) < \frac{c_3}{n}\right\} \\ H_n^3 &= \left\{\omega \in \Omega \ : \ \forall e \in \mathbb{E}_d(n) \quad |C_d(n)(\omega) \triangle C_d(n)(\omega^e)| \leq \sqrt{n}\right\} \\ H_n^4(c_4) &= \{\omega \in \Omega \ : \ \exists A : |A| > c_4 n^d, \psi_A(\omega) = \phi(n)(\omega)\} \\ H_n^5(c_5) &= \{\omega \in \Omega \ : \forall e \in \mathbb{E}_d(n) \ \exists A : |A| > c_5 n^d, \psi_A(\omega^e) = \phi(n)(\omega^e)\} \end{aligned} \quad , \tag{2.1}$$

and

$$H_n = H_n(c_1, c_2, c_3, c_4, c_5) = H_n^1(c_1) \cap H_n^2(c_2, c_3) \cap H_n^3 \cap H_n^4(c_4) \cap H_n^5(c_5). \tag{2.2}$$

We start with the following deterministic claim:

Claim 2.3. *Given* $c_1, c_2, c_3, c_4, c_5 > 0$*, there exists a constant* $C = C(c_1, c_2, c_3, c_4, c_5, d, p) > 0$ *such that if* $\omega \in H_n(c_1, c_2, c_3, c_4, c_5)$ *then for every* $e \in \mathbb{E}_d(n)$

$$|\nabla_e \phi(\omega)| \leq \frac{C}{n^d}.$$

In order to prove Claim 2.3 we will need the following two lemmas:

Lemma 2.4. *Fix a configuration* $\omega \in \Omega$ *and an edge* $e \in \mathbb{E}_d(n)$*. Let* $A \subset C_d(n)(\hat{\omega}^e)$ *be a subset such that* $|A| = \alpha n^d$*. Then*

$$|\nabla_e \psi_A| \leq \frac{1}{\alpha n^d}.$$

Proof. Since A is a subset of $C_d(n)(\hat{\omega}^e)$ it follows that A is also contained in $C_d(n)(\check{\omega}^e)$ and the size of $\partial_{C_d(n)} A$ is changed by at most 1 by adding an edge e. It therefore follows that

$$\begin{aligned} |\nabla_e \psi(A)| &= |\psi_A(\omega) - \psi_A(\omega^e)| = |\psi_A(\hat{\omega}^e) - \psi_A(\check{\omega}^e)| \\ &\leq \left| \frac{|\partial A|}{|A|} - \frac{|\partial A| + 1}{|A|} \right| = \frac{1}{|A|}. \end{aligned} \tag{2.3}$$

□

Lemma 2.5. *Let* G *be a finite graph, and let* $A, B \subset G$ *be disjoint such that there exists a unique edge* $e = (x, y)$*, such that* $x \in A$ *and* $y \in B$*, then*

$$\psi_{A \cup B} \geq \min\{\psi_A, \psi_B\} - \frac{2}{|A| + |B|}.$$

Proof. From the assumptions on A and B it follows that

$$\psi_{A \cup B} = \frac{|\partial(A \cup B)|}{|A \cup B|} = \frac{|\partial A| + |\partial B| - 2}{|A| + |B|} \geq \min\left\{\frac{|\partial A|}{|A|}, \frac{|\partial B|}{|B|}\right\} - \frac{2}{|A| + |B|}, \tag{2.4}$$

and so the lemma follows. □

Proof of Claim 2.3. We separate the proof into six different cases according to the following table:

$e=(x,y)$	$\omega(e)=0$ $(\omega=\hat{\omega}^e)$	$\omega(e)=1$ $(\omega=\check{\omega}^e)$
$x,y\notin C_d(n)$	1	2
$x,y\in C_d(n)$	3	4
$x\in C_d(n)\,,\,y\notin C_d(n)$ or $y\in C_d(n)\,,\,x\notin C_d(n)$	5	6

- **Cases 1 and 2:** In those cases the set $C_d(n)$ and the edges available from it are the same for both configurations ω and ω^e. It therefore follows that $\nabla_e\phi(\omega)=0$. See Figure 1a, and 1b.
- **Case 3:** In this case the set $C_d(n)$ is the same for both configurations ω and ω^e, however the set of edges available in $C_d(n)$ is increased by one when moving to the configuration ω^e, see figure 1c. Fix a set $A\subset C_d(n)(\omega)$ of size bigger than c_4n^d which realizes the Cheeger constant. It follows that

$$\psi_A(\omega)=\phi(\omega)\le\phi(\omega^e)\le\psi_A(\omega^e),$$

and therefore by Lemma 2.4 we have

$$|\phi(\omega^e)-\phi(\omega)|\le\psi_A(\omega^e)-\psi_A(\omega)\le\frac{1}{c_4n^d},$$

as required.
- **Case 4:** We separate this case into two subcases according to wether the set $C_d(n)(\omega)\backslash C_d(n)(\omega^e)$ is an empty set or not. If $C_d(n)(\omega)\backslash C_d(n)(\omega^e)=\emptyset$ then we are in the same situation as in **Case 3**, see Figure 1d, and so the same argument gives the desired result. So, let us assume that $C_d(n)(\omega)\backslash C_d(n)(\omega^e)\neq\emptyset$, see Figure 1e. Since $\omega\in H_n^3$, we know that

$$|C_d(n)(\omega)\backslash C_d(n)(\omega^e)|\le\sqrt{n}, \tag{2.5}$$

and since $\omega\in H_n^1$, $C_d(n)(\omega)$ and $C_d(n)(\omega^e)$ are not disjoint. Since $\omega\in H_n^4$, there exists a set $A\subset C_d(n)(\omega)$ of size bigger than c_4n^d realizing the Cheeger constant in the configuration ω. We denote $A_1=A\cap C_d(n)(\omega^e)$ and $A_2=A\cap(C_d(n)(\omega)\backslash C_d(n)(\omega^e))$. Applying Lemma 2.5 to A_1 and A_2 we see that

$$\psi_A(\omega)=\psi_{A_1\cup A_2}(\omega)\ge\min\{\psi_{A_1}(\omega),\psi_{A_2}(\omega)\}-\frac{2}{|A|}. \tag{2.6}$$

From (2.5) it follows that $|A_2|\le\sqrt{n}$ and therefore $\psi_{A_2}(\omega)\ge\frac{1}{\sqrt{n}}$ which gives us that $\min\{\psi_{A_1}(\omega),\psi_{A_2}(\omega)\}=\psi_{A_1}(\omega)$. Indeed, if the last equality doesn't hold then

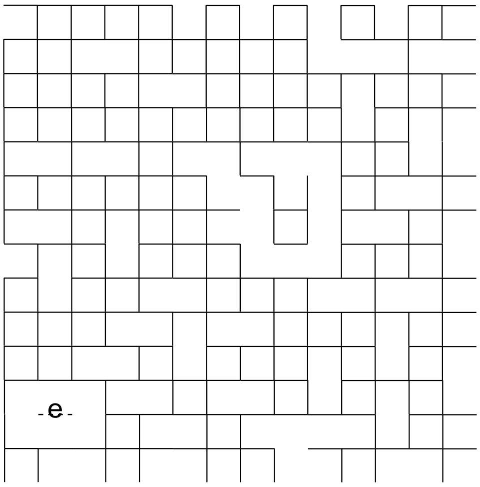
$$\frac{c_2}{n}\ge\psi_A(\omega)\ge\psi_{A_2}(\omega)-\frac{2}{|A|}\ge\frac{1}{\sqrt{n}}-\frac{2}{c_4n^d},$$

which for large enough n yields a contradiction. Consequently from (2.6) we get that

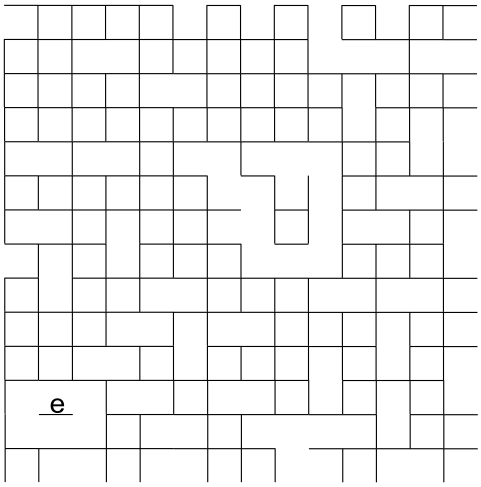
$$\psi_{A_1}(\omega)-\frac{2}{c_4n^d}\le\phi(\omega)\le\psi_{A_1}(\omega),$$

and so

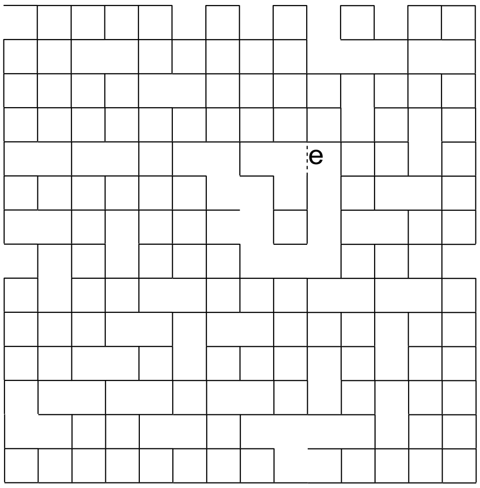
$$\phi(\omega^e)-\frac{2}{c_4n^d}\le\psi_{A_1}(\omega^e)-\frac{2}{c_4n^d}\le\psi_{A_1}(\omega)-\frac{2}{c_4n^d}\le\phi(\omega),$$



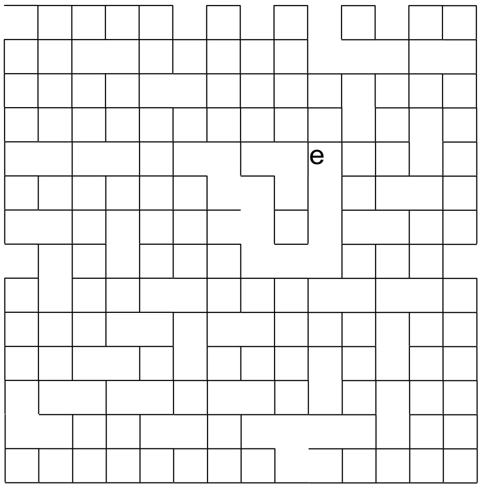
(a) Case 1



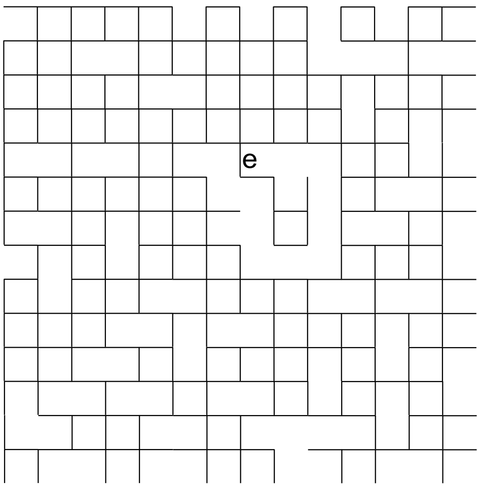
(b) Case 2



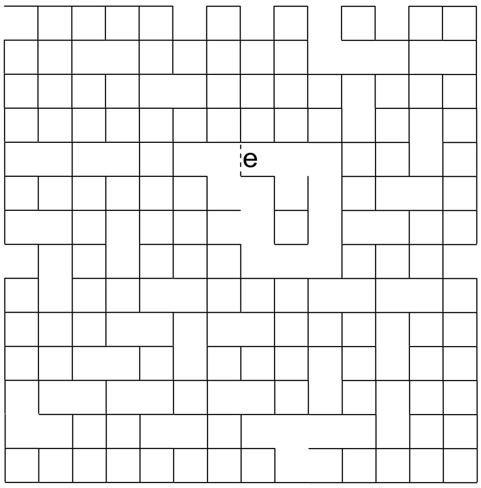
(c) Case 3



(d) Case 4a



(e) Case 4b



(f) Case 5

Figure 1: Illustrations of the different cases

i.e. $\phi(\omega^e) - \phi(\omega) \leq \frac{2}{c_4 n^d}$.

For the other direction, since $\omega \in H_n^5$, there exists a set $B \subset C_d(n)(\omega^e)$ of size bigger than $c_5 n^d$ realizing the Cheeger constant in ω^e, then

$$\phi(\omega) \leq \psi_B(\omega) \leq \psi_B(\omega^e) + \frac{1}{|B|} = \phi(\omega^e) + \frac{1}{|B|} \leq \phi(\omega^e) + \frac{1}{c_5 n^d}.$$

Consequently,

$$|\phi(\omega) - \phi(\omega^e)| \leq \max\left\{\frac{2}{c_4 n^d}, \frac{1}{c_5 n^d}\right\},$$

as required.

- **Case 5:** The proof of this case follows the proof of **case 4** above, see Figure 1f.
- **Case 6:** This case is impossible by the definition of the set $C_d(n)(\omega)$.

□

Next we turn to estimate the probability of the event H_n.

Claim 2.6. *There exist constants* $c_1, c_2, c_3, c_4, c_5 > 0$ *and a constant* $c > 0$ *such that for large enough* $n \in \mathbb{N}$ *we have*

$$\boldsymbol{P}(H_n^c) \leq e^{-c \log^{\frac{3}{2}} n}. \tag{2.7}$$

Proof. Since $\mathbf{P}(H_n^c) \leq \sum_{i=1}^5 \mathbf{P}((H_n^i)^c)$, it's enough to bound each of the last probabilities separately. The proof of the exponential decay of $\mathbf{P}((H_n^1)^c)$ for appropriate constant is presented in the Appendix.

By [7] Theorem 3.1 and section 3.4, there exists a constant $c > 0$ such that for n large enough, $\mathbf{P}((H_n^2)^c) \leq e^{-c \log^{3/2} n}$ for some constants $c_2, c_3 > 0$.

Turning to bound $\mathbf{P}((H_n^3)^c)$, we notice that the set $C_d(n)(\omega) \triangle C_d(n)(\omega^e)$ is independent of the status of the edge e and therefore

$$\begin{aligned} \mathbf{P}((H_n^3)^c) &= \frac{1}{1-p}\, \mathbf{P}\left(\left\{\omega \in \Omega \,:\, \exists e \in \mathbb{E}_d(n) \quad |C_d(n)(\omega) \triangle C_d(n)(\omega^e)| \geq \sqrt{n}\,,\ e \text{ is closed}\right\}\right) \\ &\leq \frac{1}{1-p}\, \mathbf{P}\left(\left\{\omega \in \Omega \,:\, \exists e \in \mathbb{E}_d(n) \quad |C_d(n)(\omega) \triangle C_d(n)(\omega^e)| \geq \sqrt{n}\,,\ e \text{ is closed}\right\} \cap H_n^1\right) \\ &+ \frac{1}{1-p}\, \mathbf{P}((H_n^1)^c). \end{aligned} \tag{2.8}$$

We already gave appropriate bound for the last term and therefore we are left to bound the probability of $\{\omega \in \Omega \,:\, \exists e \in E_d(n) \quad |C_d(n)(\omega) \triangle C_d(n)(\omega^e)| \geq \sqrt{n}\,,\ e \text{ is closed}\} \cap H_n^1$. Notice that the occurrence of this event implies the existence of an open cluster of size bigger than $\sqrt{n}$ which is not connected to $C_d(n)$. An appropriate bound for this event can be found in Lemma 3.2.

In order to deal with the event $(H_n^4)^c$ we denote G_n the event in Theorem 1.3,

$$G_n = \left\{\forall S \subset \mathcal{C}_d(n) \text{ connected } : Cn \leq |S| < \frac{|\mathcal{C}_d(n)|}{2}\ ,\ \frac{|\partial_{\mathcal{C}} S|}{|S|^{(d-1)/d}} \geq \alpha\right\}.$$

By [8] there exists a constant $\beta > 0$ such that $\mathbf{P}(G_n^c) < e^{-\beta n^{\left(\frac{d-1}{d}\right)}}$ for large enough $n \in \mathbb{N}$. As before we write

$$\mathbf{P}((H_n^4)^c) \leq \mathbf{P}((H_n^4)^c \cap H_n^1 \cap H_n^2 \cap G_n) + \mathbf{P}((H_n^1)^c \cup (H_n^2)^c \cup G_n^c),$$

and by the probability bound mentioned so far it's enough to bound the probability of the first event $(H_n^4)^c \cap H_n^1 \cap H_n^2 \cap G_n$. What we will actually show is that for appropriate choice of $0 < c_4 < \frac{1}{2}$ we have $(H_n^4)^c \cap H_n^1 \cap H_n^2 \cap G_n = \emptyset$. Indeed, since we assumed the event G_n occurs we have that for large enough $n \in \mathbb{N}$ and every set $A \subset C_d(n)(\omega)$ of size smaller than $c_4 n^d$,

$$|\partial_{C_d(n)} A| \geq \alpha |A|^{\frac{d-1}{d}}.$$

It follows that

$$\psi_A \geq \alpha \frac{1}{|A|^{1/d}} \geq \frac{\alpha}{c_4^{1/d} n}. \tag{2.9}$$

Choosing $c_4 > 0$ such that for large enough $n \in \mathbb{N}$ we have $\frac{\alpha}{c_4^{1/d}} > c_3$, we get a contradiction to the event H_n^2, which proves that the event $(H_n^4)^c \cap H_n^1 \cap H_n^2 \cap G_n$ is indeed empty.

Finally we turn to deal with the event $(H_n^5)^c$. As before it's enough to bound the probability of the event $(H_n^5)^c \cap H_n^1 \cap H_n^2 \cap H_n^3 \cap H_n^4 \cap G_n$. We divide the last event into two disjoint events according to the status of the edge e, namely

$$\begin{aligned} V_n^0 &:= (H_n^5)^c \cap H_n^1 \cap H_n^2 \cap H_n^3 \cap H_n^4 \cap G_n \cap \{\omega(e) = 0\} \\ V_n^1 &:= (H_n^5)^c \cap H_n^1 \cap H_n^2 \cap H_n^3 \cap H_n^4 \cap G_n \cap \{\omega(e) = 1\}, \end{aligned} \tag{2.10}$$

and will show that for right choice of $c_5 > 0$ both V_n^0 and V_n^1 are empty events.

Let us start with V_n^0. Going back to the proof of Claim 2.3 one can see that under the event $H_n^1 \cap H_n^2 \cap H_n^3 \cap H_n^4$ there exists a constant $c > 0$ such that

$$\phi(\omega^e) \leq \phi(\omega) + \frac{c}{n^d} \leq \frac{c_3}{n} + \frac{c}{n^d}, \tag{2.11}$$

and therefore $\phi(\omega^e) \leq \frac{\tilde{c}_3}{n}$ for any $\tilde{c}_3 > c_3$ and $n \in \mathbb{N}$ large enough. If $\emptyset \neq A \subset C_d(n)(\omega^e)$ is a set of size smaller than $\frac{n}{\tilde{c}_3}$ then

$$\psi_A(\omega^e) \geq \frac{1}{|A|} > \frac{\tilde{c}_3}{n}, \tag{2.12}$$

and therefore A cannot realize the Cheeger constant. On the other hand, if $A \subset C_d(n)(\omega^e)$ satisfy $\frac{n}{\tilde{c}_3} \leq |A| \leq c_5 n^d$ then

$$|\partial_{C_d(n)(\omega^e)} A| \geq |\partial_{C_d(n)(\omega^e)}(A \cap C_d(n)(\omega))| - 1 \geq |\partial_{C_d(n)(\omega)}(A \cap C_d(n)(\omega)| - 2,$$

and therefore (Since we assumed the event G_n occurs)

$$\begin{aligned} \psi_A(\omega^e) &\geq \frac{|\partial_{C_d(n)(\omega)}(A \cap C_d(n)(\omega))|}{|A|} - \frac{2}{|A|} \\ &= \frac{|\partial_{C_d(n)(\omega)}(A \cap C_d(n)(\omega))|}{|A \cap C_d(n)(\omega)|} \frac{|A \cap C_d(n)(\omega)|}{|A|} - \frac{2}{|A|} \\ &\geq \frac{\alpha}{|A \cap C_d(n)(\omega)|^{\frac{1}{d}}} \cdot \frac{|A| - \sqrt{n}}{|A|} - \frac{2\tilde{c}_3}{n} \\ &\geq \frac{\alpha}{2c_5^{\frac{1}{d}} n} - \frac{2\tilde{c}_3}{n}, \end{aligned} \tag{2.13}$$

where the last inequality holds for large enough n, since $\lim_{n\to\infty} \frac{|A|-\sqrt{n}}{|A|} = 1$. Taking $c_5 > 0$ small enough such that $\frac{\alpha}{2c_5^{\frac{1}{d}}} - 2\tilde{c}_3 > c_3$ we get a contradiction to (2.11). It follows that no set $A \subset C_d(n)(\omega^e)$ of size smaller than $c_5 n^d$ can realize the Cheeger constant which contradicts $(H_n^5)^c$, i.e, $V_n^0 = \emptyset$.

Finally, for V_n^1. The case $A \subset C_d(n)(\omega^e)$ such that $|A| < \frac{n}{\tilde{c}_3}$ is the same as for the event V_n^0. If $A \subset C_d(n)(\omega^e)$ satisfy $\frac{n}{\tilde{c}_3} \le |A| \le c_5 n^d$ then

$$|\partial_{C_d(n)(\omega^e)} A| \ge |\partial_{C_d(n)(\omega)} A| - 1.$$

and therefore as in the case of V_n^0

$$\begin{aligned}\psi_A(\omega^e) &\ge \frac{|\partial_{C_d(n)(\omega)} A| - 1}{|A|} \\ &\ge \alpha \frac{|A|^{\frac{d-1}{d}}}{|A|} - \frac{1}{|A|} \ge \frac{c_6}{2c_5^{1/d} n} - \frac{\tilde{c}_3}{n},\end{aligned} \tag{2.14}$$

where again the last inequality holds only for large enough n. Choosing c_5 small enough, we again get a contradiction to (2.11), and as before this yields that $V_n^1 = \emptyset$. □

Proof of theorem 1.5. By [10] (Theorem 1.5) the following inequality holds for some $K = K(p)$,

$$\mathbf{Var}(\phi) \le K \cdot \sum_{e \in \mathbb{E}_d(n)} \frac{\|\nabla_e \phi\|_2^2}{1 + \log\left(\|\nabla_e \phi\|_2 / \|\nabla_e \phi\|_1\right)}, \tag{2.15}$$

where $\|\nabla_e \phi\|_2^2 = \mathbf{E}[(\nabla_e \phi)^2]$ and $\|\nabla_e \phi\|_1 = \mathbf{E}[|\nabla_e \phi|]$. Observe that

$$\|\nabla_e \phi\|_1 = \|\nabla_e \phi \mathbb{1}_{\{\nabla_e \phi \neq 0\}}\|_1 \le \|\nabla_e \phi\|_2 \|\mathbb{1}_{\{\nabla_e \phi \neq 0\}}\|_2,$$

and therefore

$$\frac{\|\nabla_e \phi\|_2}{\|\nabla_e \phi\|_1} \ge \frac{1}{\sqrt{\mathbf{P}(\nabla_e \phi \neq 0)}} \ge 1.$$

Consequently, if we fix some edge $e_0 \in \mathbb{E}_d(n)$,

$$\mathbf{Var}(\phi) \le K \sum_{e \in \mathbb{E}_d(n)} \|\nabla_e \phi\|_2^2 = K|\mathbb{E}_d(n)| \cdot \|\nabla_{e_0} \phi\|_2^2 = K d n^d \cdot \|\nabla_{e_0} \phi\|_2^2, \tag{2.16}$$

where the first equality follows from the symmetry of $\mathbb{T}_d(n)$.

$$\|\nabla_{e_0} \phi\|_2^2 = \mathbf{E}[|\nabla_{e_0} \phi|^2 \mathbb{1}_{H_n}] + \mathbf{E}[|\nabla_{e_0} \phi|^2 \mathbb{1}_{H_n^c}]. \tag{2.17}$$

Notice that since $|\nabla_{e_0} \phi| \le 2d$ we have $\mathbf{E}[|\nabla_{e_0} \phi|^2 \mathbb{1}_{H_n^c}] \le 4d^2 \mathbb{P}(H_n^c)$. Thus applying Lemma 2.6,

$$\mathbf{E}[|\nabla_{e_0} \phi|^2 \mathbb{1}_{H_n^c}] \le 4d^2 e^{-c \log^{\frac{3}{2}}(n)}, \tag{2.18}$$

and by Lemma 2.3

$$\mathbf{E}[|\nabla_{e_0} \phi|^2 \mathbb{1}_{H_n}] \le \frac{C^2}{n^{2d}}. \tag{2.19}$$

Thus combing equations (2.18) and (2.19) with equation (2.16) the result follows. □

3 Appendix

In this Appendix for completeness and future reference, we sketch a proof of the exponential decay of $\mathbf{P}((H_n^1)^c)$ and the decay of probability for the size of the second largest component of percolation in a box.

The proof of the first estimate follows directly from two papers [6] by Deuschel and Pisztora and [2] by Antal Pisztora, which together gives a proof by a renormalization argument. We borrow the terminology of [2] without giving here the definitions.

Lemma 3.1. *Let $p > p_c(\mathbb{Z}^d)$. There exist $c_1, c > 0$ such that for n large enough*

$$\boldsymbol{P}_p(|C_d(n)(\omega)| < c_1 n^d) < e^{-cn}.$$

Proof. By [6] Theorem 1.2, for every $\epsilon > 0$ there exists a $p_c(\mathbb{Z}^d) < p^* < 1$ such that for every $p > p^*$ there exists a constant $c > 0$ for whom, $\mathbf{P}_p(|C_d(n)(\omega)| < (1-\epsilon)n^d) < e^{-cn}$. Since $\{|C_d(n)(\omega)| < \tilde{c}_1 n^d\}^c$ is an increasing event, by Proposition 2.1 of [2] for $N \in \mathbb{N}$ large enough, i.e., such that $\bar{p}(N) > p^*$,

$$\mathbb{P}_N(|C_d(n)(\omega)| < \tilde{c}_1 n^d) \leq \mathbb{P}^*_{\bar{p}(N)}(|C_d(n)(\omega)| < \tilde{c}_1 n^d) < e^{-cn}, \tag{3.1}$$

where $\mathbb{P}_N$ is the probability measure of the renormalized dependent percolation process and $\mathbb{P}^*_{\bar{p}(N)}$ is the probability measure of standard bond percolation with parameter $\bar{p}(N)$. From the definition of the event $R_i^{(N)}$, the crossing clusters of all the boxes B_i' that admit $R_i^{(N)}$ are connected to each other, thus

$$\mathbf{P}_p\left(|C_d(nN)(\omega)| < \tilde{c}_1(nN)^d\right) < e^{-cn}.$$

□

Next, for completeness, we turn to prove that all components outside the giant one are small.

Lemma 3.2. *Let $p > p_c(\mathbb{Z}^d)$ and denote by $\mathcal{K} \subset \mathbb{T}^d(n)\backslash C_d(n)$ the largest connected component of the graph $\mathbb{T}^d(n)\backslash C_d(n)$. Then there exist constants $c, C > 0$ such that*

$$\boldsymbol{P}_p\left(|\mathcal{K}| > C\sqrt{n}\right) \leq e^{-cn^{\frac{1}{4}}}.$$

Proof. We separate the proof into two parts: First, following ideas from Section 4 of [9], we prove the theorem for $p_c(\mathbb{Z}) < p < 1$ close enough to one. Secondly, we use a renormalization argument to show that the argument for large enough p can be used to prove the lemma for any $p_c(\mathbb{Z}^d) < p < 1$ in the cost of changing the value of the constant c.

Since there exists $\bar{c} > 0$ such that

$$\sharp\left\{*\text{-connected edge sets of size } k \text{ in } \mathbb{T}^d(n)\right\} \leq n^d \cdot \bar{c}^k,$$

we get by a union bound that[1]

$$\mathbf{P}_p\left(\exists A \subset \mathbb{E}^d(n)\ :\ A \text{ is } * \text{- connected}\ ,\ |A| > n^{\frac{1}{4}}\ ,\ \forall e \in A\ ,\ \omega(e) = 0\right)$$
$$\leq \sum_{k=\lfloor n^{\frac{1}{4}} \rfloor}^{\infty} n^d \cdot \bar{c}^k (1-p)^k.$$

If $p^* < p < 1$, where p^* solve the equation $\bar{c}(1-p) = 1$, we get that there exists some constant $c = c(p) > 0$ such that

$$\mathbf{P}_p\left(\exists A \subset \mathbb{E}^d(n)\ :\ A \text{ is } * \text{- connected}\ ,\ |A| > n^{\frac{1}{4}}\ ,\ \forall e \in A\ ,\ \omega(e) = 0\right) \leq e^{-cn^{\frac{1}{4}}}.$$

Using the proof of Lemma 3.1 for large values of p we see in the cost of increasing the value of p^* we can ensure that for every $p^* < p < 1$ there exists $\tilde{c} > 0$ such that for large enough $n \in \mathbb{N}$ we have $\mathbf{P}_p\left(|\mathcal{K}| \geq |\mathbb{T}^d(n)|/2\right) \leq e^{-\tilde{c}n}$. Thus we only need to deal with the

[1] The choice of $n^{\frac{1}{4}}$ is arbitrary and the only requirement it is $\gg \log(n)$ and smaller than $\sqrt{n}$.

case $\sqrt{n} < |\mathcal{K}| < |\mathbb{T}^d(n)|/2$. If $\sqrt{n} < |\mathcal{K}| < |\mathbb{T}^d(n)|/2$, by the isoperimetric inequality for $\mathbb{T}^d(n)$ there exists some $\delta > 0$ such that $|\partial\mathcal{K}| \geq \delta|\mathcal{K}|^{\frac{d-1}{d}} \geq \delta|\mathcal{K}|^{\frac{1}{2}} \geq \delta n^{\frac{1}{4}}$. Since $\mathcal{K}$ is a maximal connected set in $\mathbb{T}^d(n)\backslash C_d(n)$ we get that $\omega(e) = 0$ for every $e \in \partial\mathcal{K}$. Recalling that $\partial\mathcal{K}$ is $*$-connected (see [6] Lemma 2.1 or [11]) we can conclude that

$$\mathbf{P}_p\left(\sqrt{n} < |\mathcal{K}| < |\mathbb{T}^d(n)|/2\right) \leq \mathbf{P}_p\left(\begin{array}{c} |\partial\mathcal{K}| \geq \delta n^{\frac{1}{4}} \text{ , } \partial\mathcal{K} \text{ is } * \text{- connected ,} \\ \forall e \in \partial\mathcal{K} \text{ , } \omega(e) = 0 \end{array}\right) \leq e^{-cn^{\frac{1}{4}}}$$

Next we turn to the renormalization argument. Notice that the by the definition of $\mathcal{K}$ which ignores the percolation structure outside of $\mathbb{T}^d(n)\backslash C_d(n)$ we have that $\{|\mathcal{K}| > \sqrt{N}\}$ is a decreasing event. By Proposition 2.1 of [2] for $N \in \mathbb{N}$ large enough, i.e., such that $\bar{p}(N) > p^*$, we have

$$\mathbb{P}_N(|\mathcal{K}| > \sqrt{n}) \leq \mathbb{P}^*_{\bar{p}(N)}(|\mathcal{K}| > \sqrt{n}) < e^{-cn^{\frac{1}{4}}}, \tag{3.2}$$

where $\mathbb{P}_N$ is the probability measure of the renormalized dependent percolation process and $\mathbb{P}^*_{\bar{p}(N)}$ is the probability measure of standard bond percolation with parameter $\bar{p}(N)$. Assume that $\mathcal{K} \subset \mathbb{T}^d(n)\backslash C_d(n)$ is a connected component under the law of $\mathbf{P}_p$. By the definition of good boxes $\mathcal{K}_N$ contain a cluster that is contained in $C_d(n)$ under $\mathbf{P}_p$ and this cluster intersect every connected set of size $N/10$ (see [2]) thus there exists a connected component $\mathcal{K}_N \subset \mathbb{T}^d(n)\backslash C_d(n)$ under the law of $\mathbb{P}_N$ such that

$$\mathcal{K} \subset \bigcup_{x \in \mathcal{K}_N \cup \partial\mathcal{K}_N} B(x, N),$$

where $B(x, N)$ is the box of size N centered around x. Consequently we have the following estimate for the size of $\mathcal{K}$

$$|\mathcal{K}| \leq N^d\left(|\mathcal{K}_N| + |\partial\mathcal{K}_N|\right) \leq (2d+1)N^d|\mathcal{K}_N|. \tag{3.3}$$

Thus, using (3.3) and (3.2) we get that

$$\begin{aligned} \mathbf{P}_p\left(|\mathcal{K}| \geq \sqrt{n}\right) &\leq \mathbb{P}_N\left(|\mathcal{K}_N| \geq \frac{\sqrt{n/N}}{(2d+1)N^d}\right) \\ &\leq \mathbb{P}^*_{\bar{p}(N)}\left(|\mathcal{K}_N| \geq \frac{\sqrt{n}}{(2d+1)N^{(d+\frac{1}{2})}}\right) \leq e^{-\frac{c}{((2d+1)N^{(d+\frac{1}{2})})^{\frac{1}{4}}}n^{\frac{1}{4}}}, \end{aligned} \tag{3.4}$$

as required. □

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