DOI: 10.1214/ECP.v16-1685

Elect. Comm. in Probab. **16** (2011), 781–791

ELECTRONIC COMMUNICATIONS in PROBABILITY

SUPERCRITICAL BRANCHING DIFFUSIONS IN RANDOM ENVIRONMENT

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Submitted July 15, 2011, *accepted in final form* October 3, 2011

AMS 2000 Subject classification: 60J80; 60K37; 60J60.
Keywords: Branching process; random environment; BDRE; supercriticality; survival probability.

Abstract

Supercritical branching processes in constant environment conditioned on eventual extinction are known to be subcritical branching processes. The case of random environment is more subtle. A supercritical branching diffusion in random environment (BDRE) conditioned on eventual extinction of the population is not a BDRE. However the law of the population size of a supercritical BDRE (averaged over the environment) conditioned on eventual extinction is equal to the law of the population size of a subcritical BDRE (averaged over the environment). As a consequence, supercritical BDREs have a phase transition which is similar to a well-known phase transition of subcritical branching processes in random environment.

1 Introduction and main results

Branching processes in random environment (BPREs) have attracted considerable interest in recent years, see e.g. [3, 2, 7] and the references therein. On the one hand this is due to the more realistic model compared with classical branching processes. On the other hand this is due to interesting properties such as a phase transition in the subcritical regime. Let us recall this phase transition. In the strongly subcritical regime, the survival probability of a BPRE $(Z_t^{(1)})_{t\geq 0}$ scales like its expectation, that is, $\mathbb{P}\big(Z_t^{(1)} > 0\big) \sim const \cdot \mathbb{E}\big(Z_t^{(1)}\big)$ as $t \to \infty$ where $const$ is some constant in $(0, \infty)$. In the weakly subcritical regime, the survival probability decreases at a different exponential rate. The intermediate subcritical regime is in between the other two cases. Understanding the differences of these three regimes is one motivation of the literature cited above. The main observation of this article is a similar phase transition in the supercritical regime.
Let us introduce the model. We consider a diffusion approximation of BPREs as this is mathematically more convenient. The diffusion approximation of BPREs is due to Kurtz (1978) and had been conjectured (slightly inaccurately) by Keiding (1975). We follow Böinghoff and Hutzenthaler (2011) and denote this diffusion approximation as *branching diffusion in random environment* (BDRE). For every $n \in \mathbb{N} := \{1, 2, \ldots\}$, let $(Z_k^{(n)})_{k\in\mathbb{N}_0}$ be a branching process in the

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random environment $\left(Q_1^{(n)}, Q_2^{(n)}, \dots\right)$ which is a sequence of independent, identically distributed offspring distributions. If $m\left(Q_k^{(n)}\right)$ denotes the mean offspring number for $k \in \mathbb{N}$, then $S_k^{(n)} := \sqrt{n}\sum_{i=1}^{k-1} \log\left(m\left(Q_i^{(n)}\right)\right)$, $k \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$ denotes the associated random walk where $n \in \mathbb{N}$. Set $\lfloor t \rfloor := \max\{m \in \mathbb{N}_0 \colon m \le t\}$ for $t \ge 0$. Let the environment be such that $\left(S_{\lfloor tn \rfloor}^{(n)}/\sqrt{n}\right)_{t\ge 0}$ converges to a Brownian motion $(S_t)_{t\ge 0}$ with infinitesimal drift $\alpha \in \mathbb{R}$ and infinitesimal standard deviation $\sigma_e \in [0, \infty)$ as $n \to \infty$. Furthermore assume that the mean offspring variance converges to $\sigma_b^2 \in [0, \infty)$, that is,

$$\lim_{n\to\infty} \mathbb{E}\left[\sum_{k=0}^{\infty}\left(k - m\left(Q_1^{(n)}\right)\right)^2 Q_1^{(n)}(k)\right] = \sigma_b^2. \tag{1}$$

If $Z_0^{(n)}/n \to z \in [0, \infty)$ as $n \to \infty$ and if a third moment condition holds, then

$$\left(\frac{Z_{\lfloor tn \rfloor}^{(n)}}{n}, \frac{S_{\lfloor tn \rfloor}^{(n)}}{\sqrt{n}}\right)_{t\ge 0} \xrightarrow[n\to\infty]{\mathrm{w}} \left(Z_t, S_t\right)_{t\ge 0} \tag{2}$$

in the Skorohod topology (see e.g. [8]) where the limiting diffusion is the unique solution of the stochastic differential equations (SDEs)

$$\begin{aligned} dZ_t &= \frac{1}{2}\sigma_e^2 Z_t dt + Z_t dS_t + \sqrt{\sigma_b^2 Z_t} dW_t^{(b)} \\ dS_t &= \alpha dt + \sqrt{\sigma_e^2} dW_t^{(e)} \end{aligned} \tag{3}$$

for $t \ge 0$ where $Z_0 = z$ and $S_0 = 0$. The processes $(W_t^{(b)})_{t\ge 0}$ and $(W_t^{(e)})_{t\ge 0}$ are independent standard Brownian motions. Throughout the paper the notations $\mathbb{P}^z$ and $\mathbb{E}^z$ refer to $Z_0 = z$ and $S_0 = 0$ for $z \in [0, \infty)$. The diffusion approximation (2) is due to Kurtz (1978) (see also [5]). Note that the random environment affects the limiting diffusion only through the mean branching variance σ_b^2 and through the associated random walk.

We denote the process $(S_t)_{t\ge 0}$ as *associated Brownian motion*. This process plays a central role. For example it determines the conditional expectation of Z_t

$$\mathbb{E}^z\left[Z_t | (S_s)_{s\le t}\right] = z\exp\left(S_t\right) \tag{4}$$

for every $z \in [0, \infty)$ and $t \ge 0$. Moreover the infinitesimal drift α of the associated Brownian motion determines the type of criticality. The BDRE (3) is supercritical (i.e. positive survival probability) if $\alpha > 0$, critical if $\alpha = 0$ and subcritical if $\alpha < 0$, see Theorem 5 of Böinghoff and Hutzenthaler (2011). We will refer to α as criticality parameter.

Afanasyev (1979) was the first to discover different regimes for the survival probability of a BPRE in the subcritical regime (see [3, 4, 13, 2] for recent articles). The following characterisation for the BDRE (3) is due to Böinghoff and Hutzenthaler (2011). The survival probability of $(Z_t)_{t\ge 0}$ decays like the expectation, that is, $\mathbb{P}(Z_t > 0) \sim const \cdot \mathbb{E}(Z_t) = const \cdot \exp\left((\alpha + \frac{\sigma_e^2}{2})t\right)$ as $t \to \infty$, if and only if $\alpha < -\sigma_e^2$ (strongly subcritical regime). In the intermediate subcritical regime $\alpha = -\sigma_e^2$, we have that $\mathbb{P}(Z_t > 0) \sim const \cdot t^{-\frac{1}{2}} \exp\left(-\frac{\sigma_e^2}{2}t\right)$ as $t \to \infty$. Finally the survival probability decays like $\mathbb{P}(Z_t > 0) \sim const \cdot t^{-\frac{3}{2}} \exp\left(-\frac{\alpha^2}{2\sigma_e^2}t\right)$ as $t \to \infty$ in the weakly subcritical regime $\alpha \in (-\sigma_e^2, 0)$.

This article concentrates on the supercritical regime $\alpha > 0$. Our main observation is that there is a phase transition which is similar to the subcritical regime. Such a phase transition has not

been reported for BPREs yet. We condition on the event $\{Z_\infty = 0\} = \{\lim_{t\to\infty} Z_t = 0\}$ of eventual extinction and propose the following notation. If $\mathbb{P}(Z_t > 0 | Z_\infty = 0) \sim const \cdot \mathbb{E}(Z_t | Z_\infty = 0)$ as $t \to \infty$, then we say that the BDRE $(Z_t, S_t)_{t\geq 0}$ is *strongly supercritical*. If the probability of survival up to time $t \geq 0$ conditioned on eventual extinction decays at a different exponential rate as $t \to \infty$, then we refer to $(Z_t, S_t)_{t\geq 0}$ as *weakly supercritical*. The intermediate regime is referred to as *intermediate supercritical* regime. Our first theorem provides the following characterisation. The BDRE is strongly supercritical if $\alpha > \sigma_e^2$, intermediate supercritical if $\alpha = \sigma_e^2$ and weakly supercritical if $\alpha \in (0, \sigma_e^2)$.

Theorem 1. *Assume $\alpha, \sigma_e, \sigma_b \in (0,\infty)$. Let $(Z_t, S_t)_{t\geq 0}$ be the unique solution of* (3) *with $S_0 = 0$. Then*

$$\lim_{t\to\infty} \sqrt{t}^3 e^{\frac{\alpha^2}{2\sigma_e^2} t}\, \mathbb{P}^z\Big(Z_t > 0 \,\Big|\, Z_\infty = 0\Big) = \frac{8}{\sigma_e^3} \int_0^\infty f(za)\phi_\beta(a)\, da > 0 \qquad \text{if } \alpha \in (0, \sigma_e^2) \tag{5}$$

$$\lim_{t\to\infty} \sqrt{t} e^{\frac{\sigma_e^2}{2} t}\, \mathbb{P}^z\Big(Z_t > 0 \,\Big|\, Z_\infty = 0\Big) = z\, \frac{\sqrt{2}\sigma_e}{\sqrt{\pi}\sigma_b^2} > 0 \qquad \text{if } \alpha = \sigma_e^2 \tag{6}$$

$$\lim_{t\to\infty} e^{\left(\alpha - \frac{\sigma_e^2}{2}\right) t}\, \mathbb{P}^z\Big(Z_t > 0 \,\Big|\, Z_\infty = 0\Big) = z\, 2\frac{\alpha - \sigma_e^2}{\sigma_b^2} > 0 \qquad \text{if } \alpha > \sigma_e^2 \tag{7}$$

for every $z \in (0,\infty)$ where $\beta := \frac{2\alpha}{\sigma_e^2}$ and where $\phi_\beta : (0,\infty) \to (0,\infty)$ is defined as

$$\phi_\beta(a) = \int_0^\infty \int_0^\infty \frac{1}{\sqrt{2}\pi} \Gamma\Big(\frac{\beta+2}{2}\Big) e^{-a} a^{-\beta/2} u^{(\beta-1)/2} e^{-u} \frac{\sinh(\xi)\cosh(\xi)\xi}{(u + a(\cosh(\xi))^2)^{(\beta+2)/2}}\, d\xi\, du \tag{8}$$

for every $a \in (0,\infty)$.

The proof is deferred to Section 2.

Let us recall the behavior of Feller's branching diffusion, that is, (3) with $\sigma_e = 0$, which is a branching diffusion in a constant environment. The supercritical Feller diffusion conditioned on eventual extinction agrees in distribution with a subcritical Feller diffusion. This is a general property of branching processes in constant environment, see Jagers and Lagerås (2008) for the case of general branching processes (Crump-Mode-Jagers processes). Knowing this, Theorem 1 might not be surprising. However, the case of random environment is different. It turns out that the supercritical BDRE $(Z_t, S_t)_{t\geq 0}$ conditioned on $\{Z_\infty = 0\}$ is a two-dimensional diffusion which does *not* satisfy the SDE (3) and is not a branching diffusion in homogeneous random environment if $\sigma_b > 0$ and if $\sigma_e > 0$. More precisely, the associated Brownian motion $(S_t)_{t\geq 0}$ conditioned on $\{Z_\infty = 0\}$ has drift which depends on the current population size.

Theorem 2. *Let $\sigma_e \in (0,\infty)$, let $\sigma_b, z \in [0,\infty)$ and assume $\sigma_b + z > 0$. If $\left(Z_t, S_t\right)_{t\geq 0}$ is the solution of* (3) *with criticality parameter $\alpha \in (0,\infty)$, then*

$$\mathscr{L}\left(\left(Z_t, S_t\right)_{t\geq 0} \Big| Z_\infty = 0\right) = \mathscr{L}\left(\left(\check{Z}_t, \check{S}_t\right)_{t\geq 0}\right) \tag{9}$$

where $\left(\check{Z}_t, \check{S}_t\right)_{t\geq 0}$ is a two-dimensional diffusion satisfying $\check{Z}_0 = Z_0, \check{S}_0 = 0$ and

$$\begin{aligned} d\check{Z}_t &= \left(\frac{1}{2}\sigma_e^2 - 2\alpha \frac{\sigma_b^2}{\sigma_e^2 \check{Z}_t + \sigma_b^2}\right) \check{Z}_t dt + \check{Z}_t d\check{S}_t + \sqrt{\sigma_b^2 \check{Z}_t} dW_t^{(b)} \\ d\check{S}_t &= \left(\alpha - 2\alpha \frac{\sigma_e^2 \check{Z}_t}{\sigma_e^2 \check{Z}_t + \sigma_b^2}\right) dt + \sqrt{\sigma_e^2} dW_t^{(e)} \end{aligned} \tag{10}$$

for $t \geq 0$.

The proof is deferred to Section 2.
It is rather intuitive that the conditioned process is not a subcritical BDRE if $\sigma_b > 0$. The supercritical BDRE has a positive probability of extinction. Thus extinction does not require the associated Brownian motion to have negative drift. As long as the BDRE stays small, extinction is possible despite the positive drift of the associated Brownian motion. Note that if $\check{Z}_t$ is small for some $t \geq 0$, then the drift term of $\check{S}_t$ is close to α. Being doomed to extinction, the conditioned process $(\check{Z}_t)_{t\geq 0}$ is not allowed to grow to infinity. If $\check{Z}_t$ is large for some $t \geq 0$, then the drift term of $\check{S}_t$ is close to $-\alpha$ which leads a decrease of $(\check{Z}_t)_{t\geq 0}$. The situation is rather different in the case $\sigma_b = 0$. Then the extinction probability of the BDRE is zero. So the drift of $(\check{S}_t)_{t\geq 0}$ needs to be negative in order to guarantee $\check{Z}_t \to 0$ as $t \to \infty$. It turns out that if $\sigma_b = 0$, then the drift of $(\check{S}_t)_{t\geq 0}$ is $-\alpha$ and $(\check{Z}_t, \check{S}_t)_{t\geq 0}$ is a subcritical BDRE with criticality parameter $-\alpha$.
We have seen that conditioning a supercritical BDRE on extinction does – in general – not result in a subcritical BDRE. However, if we condition $(Z_t, S_t)_{t\geq 0}$ on $\{S_\infty = -\infty\}$, then the conditioned process turns out to be a subcritical BDRE with criticality parameter $-\alpha$.

Theorem 3. *Let $\sigma_e \in (0,\infty)$, let $\sigma_b, z \in [0,\infty)$ and assume $\sigma_b + z > 0$. Let $\left(Z_t^{(\alpha)}, S_t^{(\alpha)}\right)_{t\geq 0}$ be the solution of* (3) *with criticality parameter $\alpha \in \mathbb{R}$. If $\alpha > 0$, then*

$$\mathscr{L}\left(\left(Z_t^{(\alpha)}, S_t^{(\alpha)}\right)_{t\geq 0} \middle| S_\infty = -\infty\right) = \mathscr{L}\left(\left(Z_t^{(-\alpha)}, S_t^{(-\alpha)}\right)_{t\geq 0}\right) \tag{11}$$

where $Z_0^{(-\alpha)} = Z_0^{(\alpha)}$.

The proof is deferred to Section 2.
Now we come to a somewhat surprising observation. We will show that the law of $(Z_t)_{t\geq 0}$ conditioned on eventual extinction agrees in law with the law of the population size of a subcritical BDRE. More formally, inserting the second equation of (10) into the equation for $d\check{Z}_t$ we see that

$$d\check{Z}_t = \left(\frac{1}{2}\sigma_e^2 - \alpha\right)\check{Z}_t dt + \sigma_e \check{Z}_t dW_t^{(e)} + \sqrt{\sigma_b^2 \check{Z}_t} dW_t^{(b)} \tag{12}$$

for $t \geq 0$. This is the SDE for the population size of a subcritical BDRE with criticality parameter $-\alpha$. As the solution of (12) is unique, this proves the following corollary of Theorem 2.

Corollary 4. *Let $\sigma_e \in (0,\infty)$, let $\sigma_b, z \in [0,\infty)$ and assume $\sigma_b + z > 0$. Let $\left(Z_t^{(\alpha)}, S_t^{(\alpha)}\right)_{t\geq 0}$ be the solution of* (3) *with criticality parameter α for every $\alpha \in \mathbb{R}$. If $\alpha > 0$, then the law of the BDRE with criticality parameter α conditioned on extinction agrees with the law of the BDRE with criticality parameter $-\alpha$, that is,*

$$\mathscr{L}\left((Z_t^{(\alpha)})_{t\geq 0} \middle| Z_\infty = 0\right) = \mathscr{L}\left((Z_t^{(-\alpha)})_{t\geq 0}\right) \tag{13}$$

where $Z_0^{(-\alpha)} = Z_0^{(\alpha)}$.

So far we considered the event of extinction. Next we condition the BDRE on the event $\{Z_\infty > 0\} := \{\lim_{t\to\infty} Z_t = \infty\}$ of non-extinction. Define $U\colon [0,\infty) \to [0,\infty)$ by

$$U(z) := \left(\sigma_e^2 z + \sigma_b^2\right)^{-\frac{2\alpha}{\sigma_e^2}} \tag{14}$$

for $z \in [0,\infty)$. We agree on the convention that

$$\frac{c}{0} := \begin{cases} \infty & \text{if } c \in (0,\infty] \\ 0 & \text{if } c = 0 \end{cases} \qquad \frac{c}{\infty} := 0 \text{ for } c \in [0,\infty) \quad \text{and that } 0 \cdot \infty = 0. \tag{15}$$

Theorem 5. *Let $\sigma_e \in (0,\infty)$, let $\sigma_b, z \in [0,\infty)$ and assume $\sigma_b + z > 0$. Let $(Z_t, S_t)_{t\geq 0}$ be the solution of* (3) *with criticality parameter $\alpha > 0$. Then*

$$\mathscr{L}\left((Z_t,S_t)_{t\geq 0}\middle|Z_\infty > 0\right) = \mathscr{L}\left(\left(\hat{Z}_t,\hat{S}_t\right)_{t\geq 0}\right) \tag{16}$$

where $\left(\hat{Z}_t,\hat{S}_t\right)_{t\geq 0}$ is a two-dimensional diffusion satisfying $\hat{Z}_0 = Z_0, \hat{S}_0 = 0$ and

$$\begin{aligned} d\hat{Z}_t &= \left(\frac{1}{2}\sigma_e^2 + 2\alpha\frac{\sigma_b^2}{\sigma_e^2\hat{Z}_t + \sigma_b^2}\frac{U(\hat{Z}_t)}{U(0)-U(\hat{Z}_t)}\right)\hat{Z}_t\,dt + \hat{Z}_t d\hat{S}_t + \sqrt{\sigma_b^2\hat{Z}_t}dW_t^{(b)} \\ d\hat{S}_t &= \left(\alpha + 2\alpha\frac{\sigma_e^2\hat{Z}_t}{\sigma_e^2\hat{Z}_t + \sigma_b^2}\frac{U(\hat{Z}_t)}{U(0)-U(\hat{Z}_t)}\right)dt + \sqrt{\sigma_e^2}dW_t^{(e)} \end{aligned} \tag{17}$$

for $t \geq 0$. The law of $(Z_t)_{t\geq 0}$ conditioned on non-extinction satisfies that

$$\mathscr{L}\left((Z_t)_{t\geq 0}\middle|Z_\infty > 0\right) = \mathscr{L}\left((\hat{Z}_t)_{t\geq 0}\right) \tag{18}$$

where $\left(\hat{Z}_t\right)_{t\geq 0}$ is the solution of the one-dimensional SDE satisfying $\hat{Z}_0 = Z_0$ and

$$d\hat{Z}_t = \left(\frac{1}{2}\sigma_e^2 + \alpha + 2\alpha\frac{U(\hat{Z}_t)}{U(0)-U(\hat{Z}_t)}\right)\hat{Z}_t\,dt + \sigma_e\hat{Z}_t dW_t^{(e)} + \sqrt{\sigma_b^2\hat{Z}_t}dW_t^{(b)} \tag{19}$$

for $t \geq 0$.

The proof is deferred to Section 2.
On the event of non-extinction, the population size Z_t of a supercritical BDRE grows like its expectation $\mathbb{E}(Z_t|S_t)$ as $t \to \infty$.

Theorem 6. *Let $\sigma_e \in (0,\infty)$, let $\sigma_b, z \in [0,\infty)$ and assume $\sigma_b + z > 0$. Let $(Z_t,S_t)_{t\geq 0}$ be the solution of* (3) *with criticality parameter $\alpha \in \mathbb{R}$. Then $\left(Z_t/e^{S_t}\right)_{t\geq 0}$ is a nonnegative martingale. Consequently for every initial value $Z_0 = z \in [0,\infty)$ there exists a random variable $Y\colon \Omega \to [0,\infty)$ such that*

$$\frac{Z_t}{e^{S_t}} \longrightarrow Y \qquad \text{as } t\to\infty \quad \text{almost surely}. \tag{20}$$

The limiting variable is zero if and only if the BDRE goes to extinction, that is, $\mathbb{P}^z(Y=0) = \mathbb{P}^z(Z_\infty = 0)$. In the supercritical case $\alpha > 0$, the distribution of the limiting variable Y satisfies that

$$\mathbb{E}^z\left[\exp(-\lambda Y)\right] = \mathbb{E}\left[\exp\left(-\frac{z}{\frac{\sigma_b^2}{\sigma_e^2}G_{\frac{2\alpha}{\sigma_e^2}} + \frac{1}{\lambda}}\right)\right] \tag{21}$$

for all $z, \lambda \in [0,\infty)$ where G_ν is gamma-distributed with shape parameter $\nu \in (0,\infty)$ and scale parameter 1*, that is,*

$$\mathbb{P}(G_\nu \in dx) = \frac{1}{\Gamma(\nu)}x^{\nu-1}e^{-x}\,dx \tag{22}$$

for $x \in (0,\infty)$.

The proof is deferred to Section 2. In particular, Theorem 6 implies that $Z_\infty := \lim_{t\to\infty} Z_t$ exists almost surely and that $Z_\infty \in \{0,\infty\}$ almost surely.

2 Proofs

If $\sigma_b = 0$ and $Z_0 > 0$, then the process $(Z_t)_{t\geq 0}$ does not hit 0 in finite time almost surely. So the interval $(0,\infty)$ is a state space for $(Z_t)_{t\geq 0}$ if $\sigma_b = 0$. The following analysis works with the state space $[0,\infty)$ for the case $\sigma_b > 0$ and with the state space $(0,\infty)$ for the case $\sigma_b = 0$. To avoid case-by-case analysis we assume $\sigma_b > 0$ for the rest of this section. One can check that our proofs also work in the case $\sigma_b = 0$ if the state space $[0,\infty)$ is replaced by $(0,\infty)$.
Inserting the associated Brownian motion $(S_t)_{t\geq 0}$ into the diffusion equation of $(Z_t)_{t\geq 0}$, we see that $(Z_t)_{t\geq 0}$ solves the SDE

$$dZ_t = \left(\alpha + \frac{1}{2}\sigma_e^2\right) Z_t\, dt + \sqrt{\sigma_e^2 Z_t^2}\, dW_t^{(e)} + \sqrt{\sigma_b^2 Z_t}\, dW_t^{(b)} \tag{23}$$

for $t \in [0,\infty)$. One-dimensional diffusions are well-understood. In particular the scale functions are known. For the reason of completeness we derive a scale function for (23) in the following lemma. The generator of $(Z_t, S_t)_{t\geq 0}$ is the closure of the pregenerator $\mathcal{G}\colon \mathbf{C}_0^2([0,\infty)\times\mathbb{R}) \to \mathbf{C}([0,\infty)\times\mathbb{R})$ given by

$$\begin{aligned}\mathcal{G}f(z,s) := &\left(\alpha + \frac{\sigma_e^2}{2}\right) z \frac{\partial}{\partial z} f(z,s) + \alpha \frac{\partial}{\partial s} f(z,s) + \frac{1}{2}\left(\sigma_e^2 z^2 + \sigma_b^2 z\right) \frac{\partial^2}{\partial z^2} f(z,s) \\ &+ \frac{1}{2}\sigma_e^2 \frac{\partial^2}{\partial s^2} f(z,s) + \sigma_e^2 z \frac{\partial^2}{\partial z \partial s} f(z,s)\end{aligned} \tag{24}$$

for all $z \in [0,\infty)$, $s \in \mathbb{R}$ and every $f \in \mathbf{C}_0^2\big([0,\infty)\times\mathbb{R}\big)$.

Lemma 7. *Assume $\sigma_e, \sigma_b, \alpha \in (0,\infty)$. Define the functions $U\colon [0,\infty) \to (0,\infty)$ and $V\colon \mathbb{R} \to (0,\infty)$ through*

$$U(z) := \left(\sigma_e^2 z + \sigma_b^2\right)^{-\frac{2\alpha}{\sigma_e^2}} \quad \text{and} \quad V(s) := \exp\left(-\frac{2\alpha}{\sigma_e^2} s\right) \tag{25}$$

for $z \in [0,\infty)$ and $s \in \mathbb{R}$. Then U is a scale function for $(Z_t)_{t\geq 0}$ and V is a scale function for $(S_t)_{t\geq 0}$, that is, $\mathcal{G}U \equiv 0$ and $\mathcal{G}V \equiv 0$ so $\big(U(Z_t)\big)_{t\geq 0}$ and $\big(V(S_t)\big)_{t\geq 0}$ are martingales.

Proof. Note that U is twice continuously differentiable. Thus we get that

$$\begin{aligned}\mathcal{G}\left(\sigma_e^2 z + \sigma_b^2\right)^{-\frac{2\alpha}{\sigma_e^2}} &= \left(\alpha + \frac{\sigma_e^2}{2}\right) z \cdot \frac{-2\alpha}{\sigma_e^2}\left(\sigma_e^2 z + \sigma_b^2\right)^{-\frac{2\alpha}{\sigma_e^2}-1} \sigma_e^2 \\ &\quad + \frac{1}{2}\left(\sigma_e^2 z^2 + \sigma_b^2 z\right) \frac{2\alpha}{\sigma_e^2}\left(\frac{2\alpha}{\sigma_e^2} + 1\right)\left(\sigma_e^2 z + \sigma_b^2\right)^{-\frac{2\alpha}{\sigma_e^2}-2} \sigma_e^4 \\ &= \left(\sigma_e^2 z + \sigma_b^2\right)^{-\frac{2\alpha}{\sigma_e^2}-1} z\left(-2\alpha^2 - \alpha\sigma_e^2 + \frac{1}{2}4\alpha^2 + \frac{1}{2}2\alpha\sigma_e^2\right) \\ &= 0\end{aligned} \tag{26}$$

for all $z \in [0,\infty)$. Moreover V is twice continuously differentiable and we obtain that

$$\mathcal{G}\exp\left(-\frac{2\alpha}{\sigma_e^2} s\right) = \alpha \exp\left(-\frac{2\alpha}{\sigma_e^2} s\right)\frac{-2\alpha}{\sigma_e^2} + \frac{1}{2}\sigma_e^2 \exp\left(-\frac{2\alpha}{\sigma_e^2} s\right)\left(\frac{-2\alpha}{\sigma_e^2}\right)^2 = 0 \tag{27}$$

for all $s \in \mathbb{R}$. This shows $\mathcal{G}U \equiv 0 \equiv \mathcal{G}V$. Now Itô's formula implies that

$$\begin{aligned} dU(Z_t) &= \mathcal{G}U(Z_t)\,dt + U'(Z_t)\cdot\left(\sqrt{\sigma_e^2 Z_t^2}\,dW_t^{(e)} + \sqrt{\sigma_b^2 Z_t}\,dW_t^{(b)}\right) \\ dV(S_t) &= \mathcal{G}V(S_t)\,dt + V'(S_t)\sqrt{\sigma_e^2}dW_t^{(e)} \end{aligned} \tag{28}$$

for all $t \geq 0$. This proves that $\big(U(Z_t)\big)_{t\geq 0}$ and $\big(V(S_t)\big)_{t\geq 0}$ are martingales. □

Lemma 8. *Assume $\sigma_e, \sigma_b, \alpha \in (0,\infty)$. Then the semigroup of the BDRE $(Z_t, S_t)_{t\geq 0}$ conditioned on extinction satisfies that*

$$\mathbb{E}^{(z,s)}\left[f(Z_t,S_t)\middle|Z_\infty = 0\right] = \frac{\mathbb{E}^{(z,s)}\left[U(Z_t)f(Z_t,S_t)\right]}{U(z)}, \tag{29}$$

the semigroup of the BDRE $(Z_t, S_t)_{t\geq 0}$ conditioned on $\{S_\infty = -\infty\}$ satisfies that

$$\mathbb{E}^{(z,s)}\left[f(Z_t,S_t)\middle|S_\infty = -\infty\right] = \frac{\mathbb{E}^{(z,s)}\left[V(S_t)f(Z_t,S_t)\right]}{V(s)} \tag{30}$$

and the semigroup of the BDRE $(Z_t, S_t)_{t\geq 0}$ conditioned on $\{Z_\infty > 0\}$ satisfies that

$$\mathbb{E}^{(z,s)}\left[f(Z_t,S_t)\middle|Z_\infty > 0\right] = \frac{\mathbb{E}^{(z,s)}\left[\big(U(0)-U(Z_t)\big)f(Z_t,S_t)\right]}{U(0)-U(z)} \tag{31}$$

for every $z \in [0,\infty)$, $s \in \mathbb{R}$, $t \geq 0$ and every bounded measurable function $f : [0,\infty)\times\mathbb{R} \to \mathbb{R}$.

Proof. Define the first hitting time $T_x(\eta) := \inf\{t \geq 0 \colon \eta_t = x\}$ of $x \in \mathbb{R}$ for every continuous path $\eta \in \mathbf{C}\big([0,\infty),\mathbb{R})\big)$. As V is a scale function for $(S_t)_{t\geq 0}$, the optional sampling theorem implies that

$$\mathbb{P}^s\left(T_{-N}(S) < \infty\right) = \lim_{K\to\infty}\mathbb{P}^s\left(T_{-N}(S) < T_K(S)\right) = \lim_{K\to\infty}\frac{V(K)-V(s)}{V(K)-V(-N)} = \frac{V(s)}{V(-N)} \tag{32}$$

for all $s \in \mathbb{R}$ and $N \in \mathbb{N}$, see Section 6 in [10] for more details. Thus we get that

$$\begin{aligned} \mathbb{E}^{(z,s)}\left[f(Z_t,S_t)\middle|S_\infty = -\infty\right] &= \lim_{N\to\infty}\mathbb{E}^{(z,s)}\left[f(Z_t,S_t)\middle|T_{-N}(S) < \infty\right] \\ &= \lim_{N\to\infty}\frac{\mathbb{E}^{(z,s)}\left[f(Z_t,S_t)\mathbb{P}^{S_t}\left(T_{-N}(S) < \infty\right)\right]}{\mathbb{P}^{(z,s)}\left(T_{-N}(S) < \infty\right)} \\ &= \frac{\mathbb{E}^{(z,s)}\left[f(Z_t,S_t)V(S_t)\right]}{V(s)} \end{aligned} \tag{33}$$

for all $z \in [0,\infty)$, $s \in \mathbb{R}$ and $t \geq 0$. The proof of the assertions (29) and (31) is analogous. Note for the proof of (31) that

$$\mathbb{P}^z\left(Z_\infty > 0\right) = \mathbb{P}^z\left(\lim_{t\to\infty} Z_t = \infty\right) = \lim_{N\to\infty}\mathbb{P}^z\left(T_N(Z) < T_0(Z)\right) = \frac{U(0)-U(z)}{U(0)} \tag{34}$$

for every $z \in [0,\infty)$. □

Proof of Theorem 2. It suffices to identify the generator $\breve{\mathscr{G}}$ of the conditioned process. This generator is the time derivative of the semigroup of the conditioned process at $t = 0$. Let $f \in \mathbf{C}_0^2\big([0,\infty)\times\mathbb{R},\mathbb{R}\big)$ be fixed. Define $f_z(z,s) := \frac{\partial}{\partial z}f(z,s)$, $f_s(z,s) := \frac{\partial}{\partial s}f(z,s)$, $f_{zz}(z,s) := \frac{\partial^2}{\partial z^2}f(z,s)$, $f_{ss}(z,s) := \frac{\partial^2}{\partial s^2}f(z,s)$ and $f_{zs}(z,s) := \frac{\partial^2}{\partial z\partial s}f(z,s)$ for $z \in [0,\infty)$ and $s \in \mathbb{R}$. Lemma 8 implies that

$$\begin{aligned}
&\breve{\mathscr{G}}f(z,s)\\
&= \lim_{h\to 0}\frac{\mathbb{E}^{(z,s)}\left[U(Z_h)f(Z_h,S_h) - U(z)f(z,s)\right]/U(z)}{h} = \frac{\mathscr{G}(U\cdot f)(z,s)}{U(z)}\\
&= \frac{1}{U(z)}\Big[\Big(\alpha+\frac{\sigma_e^2}{2}\Big)z\left(U^{'}f + Uf_z\right)(z,s) + \alpha\left(Uf_s\right)(z,s) + \frac{\sigma_e^2}{2}\left(Uf_{ss}\right)(z,s)\\
&\quad + \frac{1}{2}\left(\sigma_e^2z^2+\sigma_b^2z\right)\left(U^{''}f + 2U^{'}f_z + Uf_{zz}\right)(z,s) + \sigma_e^2z\left(U^{'}f_s + Uf_{zs}\right)(z,s)\Big]\\
&= \frac{1}{U(z)}\left(\mathscr{G}f(z,s)\right)U(z) + \frac{1}{U(z)}\left(\mathscr{G}U(z)\right)f(z,s)\\
&\quad + \left(\sigma_e^2z^2+\sigma_b^2z\right)\left(\frac{U^{'}}{U}f_z\right)(z,s) + \sigma_e^2z\left(\frac{U^{'}}{U}f_s\right)(z,s)
\end{aligned} \tag{35}$$

for all $z \in [0,\infty)$ and $s \in \mathbb{R}$. Now we exploit that $\mathscr{G}U \equiv 0$ and that $\frac{U^{'}}{U}(z) = -2\alpha/(\sigma_e^2z+\sigma_b^2)$ for $z \in [0,\infty)$ to obtain that

$$\begin{aligned}
\breve{\mathscr{G}}f(z,s) &= \mathscr{G}f(z,s) - 2\alpha z f_z(z,s) - 2\alpha\frac{\sigma_e^2z}{\sigma_e^2z+\sigma_b^2}f_s(z,s)\\
&= \left(-\alpha+\frac{\sigma_e^2}{2}\right)zf_z(z,s) + \frac{1}{2}\left(\sigma_e^2z^2+\sigma_b^2z\right)f_{zz}(z,s)\\
&\quad + \left(\alpha - 2\alpha\frac{\sigma_e^2z}{\sigma_e^2z+\sigma_b^2}\right)f_s(z,s) + \frac{1}{2}\sigma_e^2f_{ss}(z,s) + \sigma_e^2zf_{zs}(z,s)
\end{aligned} \tag{36}$$

for all $z \in [0,\infty)$, $s \in \mathbb{R}$ and all $f \in \mathbf{C}_0^2\big([0,\infty)\times\mathbb{R},\mathbb{R}\big)$. This is the generator of the process (10). Therefore the BDRE conditioned on extinction has the same distribution as the solution of (10). □

Proof of Theorem 3. As in the proof of Theorem 2 we identify the generator $\bar{\mathscr{G}}$ of the BDRE conditioned on $\{S_\infty = -\infty\}$. Similar arguments as in (35) and $\mathscr{G}V \equiv 0$ result in

$$\begin{aligned}
&\bar{\mathscr{G}}f(z,s)\\
&= \mathscr{G}f(z,s) + \frac{\sigma_e^2}{2}2\left(\frac{V^{'}}{V}f_s\right)(z,s) + \sigma_e^2z\left(\frac{V^{'}}{V}f_z\right)(z,s)\\
&= \mathscr{G}f(z,s) - 2\alpha f_s(z,s) - 2\alpha z f_z(z,s)\\
&= \left(-\alpha+\frac{\sigma_e^2}{2}\right)zf_z(z,s) - \alpha f_s(z,s) + \frac{\sigma_e^2z^2+\sigma_b^2z}{2}f_{zz}(z,s) + \frac{\sigma_e^2}{2}f_{ss}(z,s) + \sigma_e^2zf_{zs}(z,s)
\end{aligned}$$

for all $z \in [0,\infty)$, $s \in \mathbb{R}$ and all $f \in \mathbf{C}_0^2\big([0,\infty)\times\mathbb{R},\mathbb{R}\big)$. This is the generator of the BDRE with criticality parameter $-\alpha$. □

Proof of Theorem 1. The assertion follows from Corollary 4 and from Theorem 5 of Böinghoff and Hutzenthaler (2011). □

Proof of Theorem 6. Itô's formula implies that

$$\begin{aligned} d\frac{Z_t}{e^{S_t}} &= e^{-S_t}dZ_t - e^{-S_t}Z_t dS_t + \frac{1}{2}e^{-S_t}Z_t\sigma_e^2\,dt - e^{-S_t}Z_t\sigma_e^2\,dt \\ &= e^{-S_t}\frac{\sigma_e^2}{2}Z_t\,dt + e^{-S_t}\sqrt{\sigma_b^2 Z_t}dW_t^{(b)} + \frac{1}{2}e^{-S_t}Z_t\sigma_e^2\,dt - e^{-S_t}Z_t\sigma_e^2\,dt \\ &= e^{-S_t}\sqrt{\sigma_b^2 Z_t}dW_t^{(b)} \end{aligned} \tag{37}$$

for all $t \geq 0$. Therefore $\left(Z_t/\exp(S_t)\right)_{t\geq 0}$ is a nonnegative martingale. The martingale convergence theorem implies the existence of a random variable $Y : \Omega \to [0,\infty)$ such that

$$\frac{Z_t}{e^{S_t}} \longrightarrow Y \qquad \text{as } t \to \infty \quad \text{almost surely.} \tag{38}$$

If $\alpha \leq 0$, then $Z_\infty = 0$ almost surely, which implies $Y = 0$ almost surely.
It remains to determine the distribution of Y in the supercritical regime $\alpha > 0$. Fix $z \in [0,\infty)$ and $\lambda \in [0,\infty)$. Dufresne (1990) (see also [14]) showed that

$$\int_0^\infty \exp\left(-\alpha s - \sigma_e W_s^{(e)}\right) ds \overset{\mathrm{d}}{=} \frac{2}{\sigma_e^2}G_{\frac{2\alpha}{\sigma_e^2}}. \tag{39}$$

Moreover we exploit an explicit formula for the Laplace transform of the BDRE (3) conditioned on the environment, see Corollary 3 of Böinghoff and Hutzenthaler (2011). Thus we get that

$$\begin{aligned} \mathbb{E}^z\left[\exp(-\lambda Y)\right] &= \lim_{t\to\infty}\mathbb{E}^z\left[\exp\left(-\lambda\frac{Z_t}{e^{S_t}}\right)\right] = \lim_{t\to\infty}\mathbb{E}^z\left[\mathbb{E}^z\left[\exp\left(-\lambda\frac{Z_t}{e^{S_t}}\right)\middle|\,(S_s)_{s\in[0,t]}\right]\right] \\ &= \lim_{t\to\infty}\mathbb{E}\left[\exp\left(-\frac{z}{\int_0^t\frac{\sigma_b^2}{2}\exp\left(-S_s\right)ds + \frac{\exp(S_t)}{\lambda}\exp(-S_t)}\right)\right] \\ &= \mathbb{E}\left[\exp\left(-\frac{z}{\frac{\sigma_b^2}{2}\int_0^\infty\exp\left(-\alpha s - \sigma_e W_s^{(e)}\right)ds + \frac{1}{\lambda}}\right)\right] \\ &= \mathbb{E}\left[\exp\left(-\frac{z}{\frac{\sigma_b^2}{\sigma_e^2}G_{2\alpha/\sigma_e^2} + \frac{1}{\lambda}}\right)\right]. \end{aligned} \tag{40}$$

This shows (21). Letting $\lambda \to \infty$ we conclude that

$$\mathbb{P}^z(Y=0) = \mathbb{E}\left[\exp\left(-\frac{z}{\frac{\sigma_b^2}{\sigma_e^2}G_{2\alpha/\sigma_e^2}}\right)\right] = \mathbb{P}^z\left(Z_\infty = 0\right). \tag{41}$$

The last equality follows from Theorem 5 of [5]. □

Proof of Theorem 5. Analogous to the proof of Theorem 2, we identify the generator $\hat{\mathscr{G}}$ of the BDRE conditioned on $\{Z_\infty > 0\}$. Note that

$$\frac{-U'(z)}{U(0)-U(z)} = \frac{2\alpha}{\sigma_e^2 z + \sigma_b^2}\frac{U(z)}{U(0)-U(z)} \tag{42}$$

for all $z \in [0,\infty)$. Similar arguments as in (35) and $\mathcal{G}U \equiv 0$ result in

$$\begin{aligned}
&\mathcal{G}f(z,s)\\
&= \mathcal{G}f(z,s) + \left(\sigma_e^2 z^2 + \sigma_b^2 z\right) \frac{-U'(z)}{U(0)-U(z)} f_z(z,s) + \sigma_e^2 z \frac{-U'(z)}{U(0)-U(z)} f_s(z,s)\\
&= \left(\alpha + 2\alpha \frac{U(z)}{U(0)-U(z)} + \frac{\sigma_e^2}{2}\right) z f_z(z,s) + \left(\alpha + \sigma_e^2 z \frac{2\alpha}{\sigma_e^2 z + \sigma_b^2} \frac{U(z)}{U(0)-U(z)}\right) f_s(z,s)\\
&\quad + \frac{\sigma_e^2 z^2 + \sigma_b^2 z}{2} f_{zz}(z,s) + \frac{\sigma_e^2}{2} f_{ss}(z,s) + \sigma_e^2 z f_{zs}(z,s)
\end{aligned}$$

for all $z \in [0,\infty)$, $s \in \mathbb{R}$ and all $f \in \mathbf{C}_0^2\big([0,\infty)\times\mathbb{R},\mathbb{R}\big)$. Comparing with (17), we see that $\hat{\mathcal{G}}$ is the generator of (17) which implies (16). Inserting $d\hat{S}_t$ into the equation of $d\hat{Z}_t$ for $t \in [0,\infty)$ shows that $(\hat{Z}_t)_{t\geq 0}$ solves the SDE (19). □

Acknowledgement

We thank two anonymous referees for very helpful comments and suggestions.

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