

ON THE ONE-SIDED TANAKA EQUATION WITH DRIFT

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Abstract

We study questions of existence and uniqueness of weak and strong solutions for a one-sided Tanaka equation with constant drift λ . We observe a dichotomy in terms of the values of the drift parameter: for $\lambda \leq 0$, there exists a strong solution which is pathwise unique, thus also unique in distribution; whereas for $\lambda > 0$, the equation has a unique in distribution weak solution, but no strong solution (and not even a weak solution that spends zero time at the origin). We also show that strength and pathwise uniqueness are restored to the equation via suitable “Brownian perturbations”.

1 Introduction

This paper studies the one-dimensional stochastic differential equation

$$dX(t) = \lambda dt + \mathbf{1}_{\{X(t) > 0\}} dW(t), \quad 0 \leq t < \infty, \quad (1.1)$$

where W is standard Brownian motion and λ a real constant. The diffusion function $\sigma(x) = \mathbf{1}_{(0, \infty)}(x)$ in this equation is both discontinuous and degenerate, so questions of existence and

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uniqueness of solutions are not covered by the classical theories of Itô, Stroock & Varadhan or Yamada & Watanabe (e.g., Chapter 5 of [16]).

When $\lambda = 0$, the equation (1.1) can be viewed as a one-sided version of the *Tanaka equation*

$$dX(t) = \operatorname{sgn}(X(t)) dW(t), \quad 0 \leq t < \infty, \quad (1.2)$$

where the signum function is defined as $\operatorname{sgn}(x) = 1$ for $x > 0$ and $\operatorname{sgn}(x) = -1$ for $x \leq 0$. It was shown by Zvonkin [26] (e.g., Example 3.5, Chapter 5 of [16]) that the equation (1.2), for which weak existence and weak uniqueness (i.e., uniqueness in distribution) both hold, does not admit a strong solution; and that strong (that is, pathwise) uniqueness fails for (1.2).

The equation (1.2) is a special case of the *Barlow equation*

$$dX(t) = \alpha \mathbf{1}_{\{X(t) > 0\}} dW(t) - \beta \mathbf{1}_{\{X(t) \leq 0\}} dW(t), \quad 0 \leq t < \infty \quad (1.3)$$

with real constants $\alpha > 0$, $\beta > 0$; as was shown by Barlow [2], for this equation weak existence and weak uniqueness hold but strong uniqueness fails.

At the same time, one can view a solution to the equation (1.1) as a degenerate version of the *skew-Brownian motion* studied by Walsh [23] and Harrison & Shepp [15], with the addition of a constant drift; see the recent paper [1] and the survey [17], as well as the references in these works. The skew-Brownian motion with constant drift is a solution of the equation

$$dX(t) = \lambda dt + \alpha \mathbf{1}_{\{X(t) > 0\}} dW(t) + \beta \mathbf{1}_{\{X(t) \leq 0\}} dW(t), \quad 0 \leq t < \infty \quad (1.4)$$

for some given real constants λ , $\alpha > 0$, $\beta > 0$. It follows from the results of Nakao [20] that this equation has a pathwise unique, strong solution.

Formally letting $\beta \downarrow 0$ in (1.3) and “arguing by analogy”, one might conjecture (as we did initially) that for the equation (1.1) with $\lambda = 0$ strong existence and strong uniqueness fail. Similarly, letting $\beta \downarrow 0$ in (1.4) with $\lambda > 0$, one might conjecture that the equation (1.1) with $\lambda > 0$ has a pathwise unique, strong solution. Both conjectures would be wrong, a fact that illustrates the pitfalls of this kind of spurious reasoning.

As it turns out, and as we show below, for $\lambda \leq 0$ there exists a strong solution which is pathwise unique, thus also unique in distribution (Theorems 1 and 2). With $\lambda > 0$ the equation has a weak solution which is unique in distribution, but has no strong solution (Theorem 3); whereas not even a weak solution exists under the “non-stickiness condition” $\operatorname{Leb} \{t \geq 0 : X(t) = 0\} = 0$ of [19], where *Leb* stands for the Lebesgue measure on $[0, \infty)$ (Theorem 4). The results for the equation (1.1) with $\lambda > 0$ extend also to the more general equation

$$dX(t) = \kappa \mathbf{1}_{\{X(t) > 0\}} dt + \lambda \mathbf{1}_{\{X(t) \leq 0\}} dt + \mathbf{1}_{\{X(t) > 0\}} dW(t), \quad 0 \leq t < \infty,$$

for arbitrary $\kappa \in \mathbb{R}$ and $\lambda > 0$.

When $\lambda > 0$, we show that suitable “Brownian perturbations” can restore to the equation (1.1) a pathwise unique, strong solution (Theorem 5).

2 The case $\lambda < 0$

In the case $\lambda < 0$ we show that the equation (1.1) possesses a strong solution. Moreover, we prove pathwise uniqueness and uniqueness in distribution for the stochastic equation (1.1).

Theorem 1. Let $\lambda < 0$. Then on each filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, which is rich enough to support a one-dimensional standard Brownian motion W and a real-valued random variable ζ , there exists a solution X to the equation (1.1), which satisfies $X(0) = \zeta$ and is adapted to the filtration $(\mathcal{F}_t^{(\zeta, W)})_{t \geq 0}$ generated by (ζ, W) .

Moreover, X is the unique process with these properties, and the distribution of every weak solution to (1.1) with the same initial distribution must coincide with the distribution of X .

Proof. Step (A): Considering a filtered probability space as posited in the statement of the theorem, we first define the stopping time

$$\tau := \inf\{t \geq 0 : \zeta + \lambda t + W(t) \leq 0\} \quad (2.1)$$

and claim that the process $X(t) = \zeta + \lambda t + W(t \wedge \tau)$, $t \geq 0$ is a strong solution of the equation (1.1). We use throughout the usual convention $\inf \emptyset = \infty$.

Indeed, $X(0) = \zeta$; the process X is adapted to the filtration $(\mathcal{F}_t^{(\zeta, W)})_{t \geq 0}$; and for all $0 \leq t < \infty$ we have

$$\begin{aligned} X(t) - X(0) &= \lambda t + \int_0^t \mathbf{1}_{\{\tau > s\}} dW(s) = \lambda t + \int_0^t \mathbf{1}_{\{\zeta + \lambda s + W(s \wedge \tau) > 0\}} dW(s) \\ &= \lambda t + \int_0^t \mathbf{1}_{\{X(s) > 0\}} dW(s). \end{aligned}$$

Step (B): Now, we claim that if (Y, B, ξ) is a weak solution of the stochastic integral equation

$$Y(t) = \xi + \lambda t + \int_0^t \mathbf{1}_{\{Y(s) > 0\}} dB(s), \quad 0 \leq t < \infty$$

on an appropriate filtered probability space, then $Y(t) = \xi + \lambda t + B(t \wedge \sigma)$ must hold for $0 \leq t < \infty$, where we have set

$$\sigma := \inf\{t \geq 0 : Y(t) \leq 0\}. \quad (2.2)$$

This will immediately imply weak uniqueness and pathwise uniqueness for the stochastic differential equation (1.1).

To prove this claim, we fix a weak solution (Y, B, ξ) with the described properties. Moreover, for every $\varepsilon > 0$ we introduce the stopping times

$$\sigma_{-\varepsilon} := \inf\{t \geq 0 : Y(t) \leq -\varepsilon\}, \quad (2.3)$$

$$\varrho_{-\varepsilon} := \inf\{t \geq \sigma_{-\varepsilon} : Y(t) \geq 0\}. \quad (2.4)$$

Suppose that for some $\varepsilon > 0$ we had $\varrho_{-\varepsilon} < \infty$ on a set of positive probability; then on this same set

$$\begin{aligned} 0 < \varepsilon &= Y(\varrho_{-\varepsilon}) - Y(\sigma_{-\varepsilon}) = \lambda(\varrho_{-\varepsilon} - \sigma_{-\varepsilon}) + \int_{\sigma_{-\varepsilon}}^{\varrho_{-\varepsilon}} \mathbf{1}_{\{Y(s) > 0\}} dB(s) \\ &= \lambda(\varrho_{-\varepsilon} - \sigma_{-\varepsilon}) < 0 \end{aligned}$$

would hold as well, which is clearly absurd. This shows that $\varrho_{-\varepsilon} = \infty$ is valid with probability one for all $\varepsilon > 0$. Using this fact and

$$Y(\sigma_{-\varepsilon} \vee t) - Y(\sigma_{-\varepsilon}) = \lambda((\sigma_{-\varepsilon} \vee t) - \sigma_{-\varepsilon}) + \int_{\sigma_{-\varepsilon}}^{\sigma_{-\varepsilon} \vee t} \mathbf{1}_{\{Y(s) > 0\}} dB(s),$$

we conclude that

$$Y(\sigma_{-\varepsilon} \vee t) = Y(\sigma_{-\varepsilon}) + \lambda((\sigma_{-\varepsilon} \vee t) - \sigma_{-\varepsilon}) \tag{2.5}$$

holds for every $0 \leq t < \infty$ and $\varepsilon > 0$, with probability one. On the other hand, we have

$$\begin{aligned} Y(t \wedge \sigma) &= \xi + \lambda \cdot (t \wedge \sigma) + \int_0^{t \wedge \sigma} \mathbf{1}_{\{Y(s) > 0\}} dB(s) \\ &= \xi + \lambda \cdot (t \wedge \sigma) + B(t \wedge \sigma), \quad 0 \leq t < \infty, \end{aligned} \tag{2.6}$$

therefore also

$$\sigma = \inf\{t \geq 0 : \xi + \lambda t + B(t) \leq 0\}. \tag{2.7}$$

Now, we claim, it is enough to show that the identity

$$\lim_{\varepsilon \downarrow 0} \sigma_{-\varepsilon} = \sigma \tag{2.8}$$

holds with probability one, in the notation of (2.2), (2.3); because then, using (2.8) in conjunction with the observations (2.5)–(2.7) and the continuity of $Y(\cdot)$, we will be able to conclude

$$\begin{aligned} Y(t) &= Y(t) \cdot \mathbf{1}_{\{t \leq \sigma\}} + Y(t) \cdot \mathbf{1}_{\{t > \sigma\}} \\ &= (\xi + \lambda \cdot (t \wedge \sigma) + B(t \wedge \sigma)) \mathbf{1}_{\{t \leq \sigma\}} + (Y(\sigma) + \lambda(t - \sigma)) \mathbf{1}_{\{t > \sigma\}} \\ &= (\xi + \lambda \cdot (t \wedge \sigma) + B(t \wedge \sigma)) \mathbf{1}_{\{t \leq \sigma\}} + (\xi + \lambda\sigma + B(\sigma) + \lambda(t - \sigma)) \mathbf{1}_{\{t > \sigma\}} \\ &= \xi + \lambda t + B(t \wedge \sigma) \end{aligned}$$

for all $0 \leq t < \infty$, as posited.

Step (C): We start by recalling the Dambis-Dubins-Schwarz Theorem (cf. Theorem 4.6 and Problem 4.7, in Chapter 3 of [16]), according to which there is a one-dimensional standard Brownian motion $\beta(\cdot)$ such that

$$M(t) := \int_0^t \mathbf{1}_{\{Y(s) > 0\}} dB(s) = \beta(\langle M \rangle(t)), \quad 0 \leq t < \infty \tag{2.9}$$

holds. Here

$$\langle M \rangle(t) = \int_0^t \mathbf{1}_{\{Y(s) > 0\}} ds = Leb(\{s \in [0, t] : Y(s) > 0\}), \quad 0 \leq t < \infty$$

is the quadratic variation process of the martingale M defined in (2.9), and Leb stands for the Lebesgue measure on $[0, \infty)$.

To prove (2.8), it suffices to show $\mathbb{P}(E_\delta) = 0$ for every $\delta > 0$, where we introduce the event

$$E_\delta := \{M(\sigma + t) - M(\sigma) \geq -\lambda t, \quad \forall t \in [0, \delta]\} \tag{2.10}$$

with the notation of (2.9). In order to prove this assertion, we fix a number $\delta > 0$, recall the notation (2.2), define the random set

$$A_\delta := \{s \in [\sigma, \sigma + \delta] : Y(s) > 0\}, \tag{2.11}$$

and note the identity

$$\langle M \rangle(\sigma + \delta) = \langle M \rangle(\sigma) + \text{Leb}(A_\delta). \quad (2.12)$$

On the event $\{\text{Leb}(A_\delta) = 0\}$ this identity gives $\langle M \rangle(\sigma + \delta) = \langle M \rangle(\sigma)$, therefore $M(\sigma + t) = M(\sigma)$ for all $t \in [0, \delta]$, and consequently $\mathbb{P}(E_\delta \cap \{\text{Leb}(A_\delta) = 0\}) = 0$. Thus, it suffices to show that the probability of the intersection of the event in (2.10) with the event $\{\text{Leb}(A_\delta) > 0\}$, is also equal to zero, namely:

$$\mathbb{P}(E_\delta \cap \{\text{Leb}(A_\delta) > 0\}) = 0. \quad (2.13)$$

In order to do this, we use the representation (2.9) to see that the event in (2.10) is contained in

$$\tilde{E}_\delta := \left\{ \inf_{\langle M \rangle(\sigma) \leq s \leq \langle M \rangle(\sigma) + \text{Leb}(A_\delta)} \left(\lambda(\langle M \rangle^{-1}(s) - \langle M \rangle^{-1}(\sigma)) + \beta(s) - \beta(\langle M \rangle(\sigma)) \right) \geq 0 \right\};$$

here we have recalled (2.12) and set

$$\langle M \rangle^{-1}(s) := \inf\{t \geq 0 : \langle M \rangle(t) > s\}, \quad s \geq 0.$$

On the intersection of events $\tilde{E}_\delta \cap \{\text{Leb}(A_\delta) > 0\}$, the one-dimensional, standard Brownian motion

$$\tilde{\beta}(u) := \beta(\langle M \rangle(\sigma) + u) - \beta(\langle M \rangle(\sigma)), \quad 0 \leq u < \infty$$

has to stay nonnegative throughout the interval $[0, \text{Leb}(A_\delta)]$ (recall here that $\lambda < 0$); but this implies $\mathbb{P}(\tilde{E}_\delta \cap \{\text{Leb}(A_\delta) > 0\}) = 0$, because a one-dimensional, standard Brownian motion changes sign infinitely often on every non-trivial time interval starting at the origin, with probability one (e.g., Problem 7.18 in Chapter 2 of [16]). In conjunction with the inclusion $E_\delta \subseteq \tilde{E}_\delta$, this observation leads to (2.13) and completes the proof. \square

3 The case $\lambda = 0$

In the case $\lambda = 0$ we prove the existence of a strong solution to the equation (1.1). Moreover, we show pathwise and weak uniqueness for the equation (1.1). Thus, the one-sided Tanaka equation $dX(t) = \mathbf{1}_{\{X(t) > 0\}} dW(t)$ of (1.1) has qualitative properties markedly different from those of the "real" Tanaka equation (1.2).

We remark at this point that this can be shown along the lines of the proof in the case $\lambda < 0$, but we prefer to give here a shorter proof which relies on the Engelbert-Schmidt [6], [7] criterion for weak uniqueness of one-dimensional stochastic differential equations without drift.

Theorem 2. *Let $\lambda = 0$. Then, on each filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ rich enough to support a one-dimensional standard Brownian motion W and an independent real-valued random variable ζ , there exists a solution X to the equation (1.1), which satisfies $X(0) = \zeta$ and is adapted to the filtration $(\mathcal{F}_t^{(\zeta, W)})_{t \geq 0}$ generated by (ζ, W) .*

Moreover, X is the unique process with these properties, and the law of every weak solution to (1.1) with the same initial distribution must coincide with the law of X .

Proof: We start by defining the stopping time

$$\vartheta := \inf\{t \geq 0 : \zeta + W(t) \leq 0\} \quad (3.1)$$

and setting $X(t) = \zeta + W(t \wedge \vartheta)$, $t \geq 0$. Arguing as in Step (A) in the proof of Theorem 1, we show that X is a strong solution of the stochastic differential equation (1.1) with initial value ζ .

We claim now that weak uniqueness holds for the equation (1.1) with $\lambda = 0$. To this end, we employ the Engelbert & Schmidt [6], [7] (see also [8]) theory in the form of Theorem 5.7 in Chapter 5 of [16], and need to show the following identity between sets:

$$\{x \in \mathbb{R} : \sigma(x) = 0\} = \left\{x \in \mathbb{R} : \int_{x-\varepsilon}^{x+\varepsilon} \frac{dy}{\sigma^2(y)} = \infty, \quad \forall \varepsilon > 0\right\}. \tag{3.2}$$

Indeed, one checks fairly easily that for the diffusion function $\sigma(x) = \mathbf{1}_{(0,\infty)}(x)$ of the equation (1.1), both sets in (3.2) are equal to $(-\infty, 0]$, so that weak uniqueness holds for the equation (1.1) with $\lambda = 0$.

It remains to show strong uniqueness. To this end, let Y be another solution of the equation (1.1) defined on the same probability space as X , adapted to the filtration $(\mathcal{F}_t^{(\zeta, W)})_{t \geq 0}$ and satisfying $Y(0) = \zeta = X(0)$. From the explicit formula for the process X we deduce

$$\vartheta = \inf\{t \geq 0 : X(t) \leq 0\}. \tag{3.3}$$

We now define a new stopping time ϑ' by

$$\vartheta' := \inf\{t \geq 0 : Y(t) \leq 0\}. \tag{3.4}$$

Due to weak uniqueness, we must have with probability one: $Y(t \vee \vartheta') = X(t \vee \vartheta) = X(\vartheta)$ for all $0 \leq t < \infty$. Moreover, from the stochastic differential equation (1.1) we conclude

$$Y(t \wedge \vartheta') = \zeta + W(t \wedge \vartheta'), \quad 0 \leq t < \infty.$$

Thus, with probability one we have: $X(t \wedge \vartheta \wedge \vartheta') = Y(t \wedge \vartheta \wedge \vartheta')$, $\forall 0 \leq t < \infty$. Combining the latter two observations we see that, in order to prove strong (pathwise) uniqueness, it suffices to show that $\vartheta = \vartheta'$ holds with probability one.

To this end, we note

$$\vartheta' = \inf\{t \geq 0 : Y(t \wedge \vartheta') \leq 0\} = \inf\{t \geq 0 : \zeta + W(t \wedge \vartheta') \leq 0\}.$$

This last expression is equal to $\inf\{t \geq 0 : \zeta + W(t) \leq 0\} = \vartheta$ if $\vartheta \leq \vartheta'$, and to infinity if $\vartheta > \vartheta'$. However, the second case occurs with zero probability, since weak uniqueness implies $\mathbb{P}(\vartheta' < \infty) = \mathbb{P}(\vartheta < \infty) = 1$. □

4 The case $\lambda > 0$

In the case $\lambda > 0$ we shall show first that the equation (1.1) has a unique weak solution, but not a strong solution. We shall also show that this solution is “sticky at the origin”, in the sense that the so-called *non-stickiness condition* $\int_0^\infty \mathbf{1}_{\{X(t)=0\}} dt = 0$ cannot possibly hold with probability one.

Theorem 3. *Let $\lambda > 0$. Then the equation (1.1) has a weak solution which is unique in the sense of the probability distribution, but does not admit a strong solution.*

Proof. Step (i): We start by constructing a weak solution to the equation (1.1). In order to do this, we use the results of [5] (consult also pages 193-205 in the book [12], and the more recent articles [24], [25]) to conclude that the equation

$$dX(t) = \lambda \cdot \mathbf{1}_{\{X(t) \leq 0\}} dt + \mathbf{1}_{\{X(t) > 0\}} dB(t), \quad 0 \leq t < \infty \tag{4.1}$$

has a weak solution on a suitable filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$ for all initial values $X(0) \in \mathbb{R}$, where B is a one-dimensional standard Brownian motion. Indeed, for $X(0) \geq 0$ one can define X to be the sticky Brownian motion started at $X(0)$ (see [5]); and for $X(0) < 0$, one can set $X(t) = X(0) + \lambda t$, $0 \leq t \leq |X(0)|/\lambda$, then let $X(t)$, $|X(0)|/\lambda \leq t < \infty$ be a sticky Brownian motion started at the origin.

Now we carry out a Cameron-Martin-Girsanov change of probability measure, from the underlying \mathbb{Q} to a probability measure \mathbb{P} under which the process $W(t) := B(t) - \lambda t$, $0 \leq t < \infty$ is a standard Brownian motion. (See Corollary 5.2 in Chapter 3 of [16], or pages 325-330 in [22], for the details; the two measures \mathbb{Q} and \mathbb{P} are equivalent when restricted to \mathcal{F}_T , for each $T \in (0, \infty)$.) Under this new measure \mathbb{P} , the process X will satisfy

$$dX(t) = \lambda dt + \mathbf{1}_{\{X(t) > 0\}} dW(t), \quad 0 \leq t < \infty,$$

and thus (X, W) will be a weak solution to the equation (1.1) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

Step (ii): Next, we prove *weak uniqueness*. To this end, let (X, W) be an arbitrary weak solution of the equation (1.1) on some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. If we carry out again a Cameron-Martin-Girsanov change of measure such that $B(t) = W(t) + \lambda t$, $0 \leq t < \infty$ becomes a standard Brownian motion under the new measure \mathbb{Q} , then under this new measure the pair of processes (X, B) will constitute a weak solution of the equation (4.1).

We show now that, if the initial condition $X(0)$ is nonnegative, then the state process X of such a weak solution remains nonnegative at all times. To do this, we pick a nonincreasing function $f : \mathbb{R} \rightarrow [0, 1]$ supported in $(-\infty, 0)$, which is twice continuously differentiable and has bounded first and second derivatives. Fixing $t \in [0, \infty)$ and combining Itô's formula with Fubini's Theorem, we deduce

$$\mathbb{E}^{\mathbb{Q}} [f(X(t))] - f(X(0)) = \lambda \cdot \int_0^t \mathbb{E}^{\mathbb{Q}} [f'(X(s)) \cdot \mathbf{1}_{\{X(s) < 0\}}] ds \leq 0,$$

where $\mathbb{E}^{\mathbb{Q}}$ denotes integration with respect to the auxiliary probability measure \mathbb{Q} . In particular, we see that $X(0) \geq 0$ implies $\mathbb{E}^{\mathbb{Q}} [f(X(t))] = 0$. Since the indicator function of every nonempty open interval in $(-\infty, 0)$ can be dominated by a function f as described above, and since the paths of X are continuous, we conclude that

$$X(0) \geq 0 \quad \text{implies} \quad X(t) \geq 0 \quad \text{for all} \quad 0 \leq t < \infty,$$

with \mathbb{Q} -probability one. Because the measures \mathbb{P} and \mathbb{Q} are equivalent when restricted to \mathcal{F}_T for each $T \in (0, \infty)$, we see that the above implication holds also with \mathbb{P} -probability one.

On the other hand, if $X(0) < 0$, then the equation (1.1) shows $X(t) = X(0) + \lambda t$ for all $0 \leq t \leq |X(0)|/\lambda$. The same argument as before, but now on the time interval $[|X(0)|/\lambda, \infty)$, yields $X(t) \geq 0$ for all $t \in [|X(0)|/\lambda, \infty)$, with \mathbb{Q} -probability one (thus also with \mathbb{P} -probability one). We conclude that, under the new measure \mathbb{Q} , the process X satisfies the stochastic differential equation

$$dX(t) = \lambda \cdot \mathbf{1}_{\{X(t)=0\}} dt + \mathbf{1}_{\{X(t)>0\}} dB(t) \tag{4.2}$$

driven by the \mathbb{Q} -Brownian motion B and therefore, on the strength of the results in [5], [24], has the distribution of the "sticky Brownian motion" for all $t \in [0, \infty)$ if $X(0) \geq 0$, and for all $t \in [|X(0)|/\lambda, \infty)$ if $X(0) < 0$. Moreover, the main result in [5] shows that the joint distribution of the pair (X, B) under \mathbb{Q} is uniquely determined. Thus, making a change of measure from \mathbb{Q} back to \mathbb{P} , we conclude that the distribution of X under \mathbb{P} must coincide with the distribution of the weak solution constructed above. This proves weak uniqueness.

Remark 1. At this point, and with $X(0) \geq 0$, it can be seen from Theorem 8.1.1 in [9] (see also [18]) that the process X is a Feller diffusion with state space $[0, \infty)$, with infinitesimal generator $\mathcal{A}f = (1/2) \mathbf{1}_{(0, \infty)} f'' + \lambda f'$ acting on functions $f \in C([0, \infty)) \cup C^2((0, \infty))$ that satisfy $f''(0+) = 0$, and with the origin as a regular boundary point in the Feller [10] classification. We are indebted to Professor T.G. Kurtz for this observation.

Step (iii): Finally, we show by contradiction that the equation (1.1) cannot have a strong solution. To this end, we suppose that X is a strong solution to (1.1) on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t^W)_{t \geq 0}, \mathbb{P})$; that is, X solves (1.1) and is adapted to the filtration $(\mathcal{F}_t^W)_{t \geq 0}$ generated by the Brownian motion W driving the equation (1.1). Then, the same argument as in the proof of weak uniqueness shows that there is a Cameron-Martin-Girsanov change of measure, such that: the process $B(t) := W(t) + \lambda t$, $0 \leq t < \infty$ is a standard Brownian motion under the new measure \mathbb{Q} ; whereas X solves under this new measure the equation (4.2) for $0 \leq t < \infty$ if $X(0) \geq 0$, and for $t \geq |X(0)|/\lambda$ if $X(0) < 0$. But the processes W and B generate exactly the same filtrations, so we conclude that for $X(0) \geq 0$ we have constructed a strong solution of the equation (4.2). This is in clear contradiction to the results in [5] and [24]; Theorem 1 in the paper [24] shows, in particular, that the conditional distribution of the sticky Brownian motion $X(t)$, given the entire path of the “driving” Brownian motion B in (4.2), is given by

$$\mathbb{Q}(X(t) \leq x \mid B(u), 0 \leq u < \infty) = \exp(-2\lambda(B(t) + S(t) - x)),$$

\mathbb{Q} -a.s., for all $x \in [0, B(t) + S(t)]$, where $S(t) := \max_{0 \leq u \leq t} (-B(u))$. Hence, a strong solution to the equation (1.1) cannot exist. \square

Next, we provide a direct argument showing that for $\lambda > 0$ the equation (1.1) does not admit a weak solution which spends zero time at the origin (the “non-stickiness condition” (4.4) below, in a terminology borrowed from [19]). Clearly, this can also be deduced from the weak uniqueness in Theorem 3, and from the fact that the weak solution constructed in the proof of that result spends at the origin a non-zero amount of time with positive probability. The method of proof of Theorem 4, however, seems to be novel; it might prove useful in the context of other stochastic differential equations, for which an analogue of Theorem 3 is not readily available.

Theorem 4. *Let κ be an arbitrary real constant and λ an arbitrary positive constant. Then the stochastic differential equation*

$$dX(t) = \kappa \mathbf{1}_{\{X(t) > 0\}} dt + \lambda \mathbf{1}_{\{X(t) \leq 0\}} dt + \mathbf{1}_{\{X(t) > 0\}} dW(t), \quad 0 \leq t < \infty \tag{4.3}$$

has no weak solution which satisfies the condition

$$\int_0^T \mathbf{1}_{\{X(t) = 0\}} dt = 0, \quad \text{a.s.} \tag{4.4}$$

for all $T \in [0, \infty)$.

In particular, this is the case for $\kappa = \lambda > 0$, which corresponds to the equation (1.1).

Proof. Step (1): We shall suppose that (X, W) is a weak solution of the equation (4.3) defined on a suitable filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and satisfying (4.4), and will derive a contradiction.

To this end, we first carry out a Cameron-Martin-Girsanov change of probability measure, from the underlying \mathbb{P} to a probability measure $\widehat{\mathbb{P}}$ under which the process

$$\widehat{W}(t) := W(t) + (\kappa + \lambda)t, \quad 0 \leq t < \infty$$

is a standard Brownian motion (the two probability measures $\widehat{\mathbb{P}}$ and \mathbb{P} are equivalent when restricted to \mathcal{F}_T , for each $T \in (0, \infty)$). Substituting this into (4.3) we see that, under $\widehat{\mathbb{P}}$, the process X satisfies the stochastic differential equation

$$dX(t) = -\lambda \cdot \operatorname{sgn}(X(t)) dt + \mathbf{1}_{\{X(t) > 0\}} d\widehat{W}(t), \quad (4.5)$$

with the distribution of $X(0)$ being unchanged.

Next, we let $\mu(t)$, $0 \leq t < \infty$ be the collection of one-dimensional marginal distributions of the process X under the probability measure $\widehat{\mathbb{P}}$, namely $\mu(t) = \widehat{\mathbb{P}} \circ (X(t))^{-1}$. We shall show in Step 2 that the family of measures $\frac{1}{T} \int_0^T \mu(t) dt$, $0 < T < \infty$ is uniformly tight, and in Step 3 that every limit point of this family must be the zero measure. This will establish the desired contradiction.

Step (2): The uniform tightness of the family $\mu(t)$, $t \geq 0$ (and, hence, also of the family of measures $\frac{1}{T} \int_0^T \mu(t) dt$, $0 < T < \infty$) will follow from the “mean stochastic comparison” results of Hajek [14].

We start by recalling the definition

$$L^\Theta(t) := \Theta^+(t) - \Theta^+(0) - \int_0^t \mathbf{1}_{\{\Theta(s) > 0\}} d\Theta(s) = \lim_{\varepsilon \downarrow 0} \frac{1}{4\varepsilon} \int_0^t \mathbf{1}_{\{|\Theta(s)| < \varepsilon\}} d\langle \Theta \rangle(s)$$

of the local time accumulated at the origin by a generic continuous semimartingale Θ during the time-interval $[0, t]$, where $\langle \Theta \rangle$ is the quadratic variation of the local martingale part of Θ . We recall also the fact that the local time $L^\Theta(\cdot)$ is flat off the set $\{t \in [0, \infty) : \Theta(t) = 0\}$; to wit, for every $T \in (0, \infty)$, we have

$$\int_0^T \mathbf{1}_{\{\Theta(t) \neq 0\}} dL^\Theta(t) = 0, \quad \text{a.s.} \quad (4.6)$$

(cf. Theorem 7.1, equation (7.2) on page 218, Chapter 3 of [16]).

With this terminology and notation in place, we apply first the generalized Itô rule (see Theorem 7.1, equation (7.4) on page 218, Chapter 3 of [16]) to the function $f(x) = |x|$ and the semimartingale X of (4.5), and obtain

$$d|X|(t) = -\lambda dt + \mathbf{1}_{\{X(t) > 0\}} d\widehat{W}(t) + 2dL^X(t). \quad (4.7)$$

Next, we apply the generalized Itô rule, once again to the function $f(x) = |x|$ but this time to the semimartingale $|X|$ of (4.7); in conjunction with (4.6), we obtain

$$\begin{aligned} d|X|(t) &= \operatorname{sgn}(|X(t)|) (-\lambda dt + \mathbf{1}_{\{X(t) > 0\}} d\widehat{W}(t) + 2dL^X(t)) + 2dL^{|X|}(t) \\ &= -\lambda(1 - 2 \cdot \mathbf{1}_{\{X(t) = 0\}}) dt + \mathbf{1}_{\{X(t) > 0\}} d\widehat{W}(t) - 2dL^X(t) + 2dL^{|X|}(t). \end{aligned}$$

Comparing this last expression with (4.7) and invoking the condition (4.4), we deduce that for every $T \in (0, \infty)$ the identity

$$L^{|X|}(T) = 2L^X(T)$$

holds almost surely under both $\widehat{\mathbb{P}}$ and \mathbb{P} . Thus, the equation (4.7) takes the form

$$d|X|(t) = -\lambda dt + \mathbf{1}_{\{X(t) > 0\}} d\widehat{W}(t) + dL^{|X|}(t). \quad (4.8)$$

Next, we consider the strong solution of the equation

$$dZ(t) = -\lambda dt + d\widehat{W}(t) + dL^Z(t) \quad (4.9)$$

satisfying the initial condition $Z(0) = |X(0)|$, where $L^Z(t)$ is the local time accumulated by Z at the origin during the time-interval $[0, t]$. The process Z can be constructed by applying the Skorohod map to the paths of the process $|X(0)| - \lambda t + \widehat{W}(t)$, $0 \leq t < \infty$.

Since Z is Brownian motion with negative drift and reflection at the origin, we know that it is a Markov process with (a unique) invariant distribution that has exponential density $2\lambda e^{-2\lambda\xi}$, $\xi > 0$. In particular, the family of one-dimensional marginal distributions of Z is uniformly tight, and applying Theorem 2 of [14] to the processes $|X|$ and Z we obtain that, for every given $\varepsilon > 0$, there exists a real number $K_\varepsilon > 0$ such that

$$\sup_{0 \leq t < \infty} \mu(t)(\mathbb{R} \setminus [-K_\varepsilon, K_\varepsilon]) \leq 2 \cdot \sup_{0 \leq t < \infty} \widehat{\mathbb{P}}(Z(t) \geq K_\varepsilon) < \varepsilon.$$

We conclude that the family $\mu(t)$, $0 \leq t < \infty$ is uniformly tight. In particular, we can find a sequence $0 < T_1 < T_2 < \dots$ of numbers that increase to infinity, for which the weak limit

$$\nu := \lim_{n \rightarrow \infty} \frac{1}{T_n} \int_0^{T_n} \mu(t) dt \tag{4.10}$$

is well-defined and a probability measure on $\mathcal{B}(\mathbb{R})$.

Step (3): We shall show now that the weak limit ν in (4.10) can only be the zero measure, and this will lead to the desired contradiction. We denote by $C_0^\infty(\mathbb{R})$ the space of continuous and infinitely continuously differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ which vanish at infinity together with all their derivatives. Applying Itô's formula under the measure $\widehat{\mathbb{P}}$, we see in conjunction with (4.5) that the family of probability measures $\mu(t) = \widehat{\mathbb{P}} \circ (X(t))^{-1}$, $0 \leq t < \infty$ on $\mathcal{B}(\mathbb{R})$ satisfies the Fokker-Planck equation

$$\forall f \in C_0^\infty(\mathbb{R}), T \in (0, \infty): \quad (\mu(T), f) = (\mu(0), f) + \int_0^T (\mu(t), \mathcal{L}f) dt. \tag{4.11}$$

Here we denote by (\cdot, \cdot) the pairing between finite measures and bounded measurable functions on \mathbb{R} , and have defined

$$\mathcal{L} := -\lambda \cdot \operatorname{sgn}(x) \frac{d}{dx} + \frac{1}{2} \cdot \mathbf{1}_{(0, \infty)} \frac{d^2}{dx^2}. \tag{4.12}$$

Now, we fix a constant $K > 0$, and pick a function $f \in C_0^\infty(\mathbb{R})$ and a constant $b > 2\lambda$ with the following properties: $f(x) = e^{bx}$, whenever $x \leq K$; $\mathcal{L}f \geq 1$, whenever $x \in [-K, K]$; and $\mathcal{L}f \geq -1$, whenever $x \geq K$. This can be achieved by taking b to be large enough first, and by choosing f on the interval $[K, \infty)$ appropriately thereafter. Plugging f into the Fokker-Planck equation (4.11) with $T = T_n$, dividing both sides of the equation by T_n and taking the limit as $n \rightarrow \infty$, we get

$$0 = \lim_{n \rightarrow \infty} \frac{1}{T_n} \int_0^{T_n} (\mu(t), \mathcal{L}f) dt.$$

Moreover, using the inequality

$$\mathcal{L}f \geq \mathbf{1}_{(-K, K)} - \mathbf{1}_{[K, \infty)},$$

and applying the Portmanteau Theorem, we end up with

$$0 \geq \nu((-K, K)) - \nu([K, \infty)).$$

Hence, by taking the limit as $K \rightarrow \infty$, we obtain $\nu((-\infty, \infty)) = 0$, which provides the desired contradiction. \square

Remark 2. The result of Theorem 4 for the equation (1.1), that is, when $\kappa = \lambda$ in (4.3), can be also obtained by the following shorter but less instructive argument.

We shall suppose that on a suitable filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ there is defined a weak solution (X, W) of the equation (1.1), which satisfies the non-stickiness condition (4.4); and will derive a contradiction. For simplicity, we shall assume that $X(0)$ is a nonnegative constant. As in step (ii) in the proof of Theorem 3, we conclude that, with probability one, we must have $X(t) \geq 0$ for all $0 \leq t < \infty$. Consequently,

$$\begin{aligned} 0 \leq X(t) &= X(0) + \lambda t + \int_0^t \mathbf{1}_{\{X(s) > 0\}} dW(s) \\ &= X(0) + \lambda t + \int_0^t \mathbf{1}_{\{X(s) \geq 0\}} dW(s) = X(0) + \lambda t + W(t), \quad 0 \leq t < \infty \end{aligned} \quad (4.13)$$

must also hold with probability one. The second equality in (4.13) is a consequence of $\int_0^t \mathbf{1}_{\{X(s)=0\}} dW(s) \equiv 0$, which is in turn a consequence of (4.4); whereas the inequality and the third equality are consequences of the nonnegativity of X .

The inequality between the left- and right-most members in (4.13) implies that the probability of the event

$$\{\lambda t + W(t) < -X(0), \quad \text{for some } t \in [0, \infty)\}$$

is zero. We know, however (e.g., [16], Exercise 5.9 on page 197), that the probability of this event is actually $e^{-2\lambda X(0)} > 0$, and the apparent contradiction completes the argument.

Remark 3. Theorem 4 provides a somewhat amusing counterpoint to the results in [3]. In that work the non-stickiness condition (4.4) was used to restore strength and pathwise uniqueness to the degenerate stochastic differential equation

$$dX(t) = |X(t)|^\alpha dW(t)$$

of Girsanov [13] with $\alpha \in (0, 1/2)$ which, in the absence of such a condition, admits several weak solutions. By contrast, Theorem 4 uses the condition (4.4) to leave the equation (1.1) with $\lambda > 0$ bereft of even weak solutions.

Remark 4. Consider the equation

$$dX(t) = \kappa \mathbf{1}_{\{X(t) > 0\}} dt + \lambda \mathbf{1}_{\{X(t) \leq 0\}} dt + \mathbf{1}_{\{X(t) \geq 0\}} dW(t), \quad 0 \leq t < \infty \quad (4.14)$$

with diffusion function $\sigma(x) = \mathbf{1}_{[0, \infty)}(x)$ and $X(0) \geq 0$. In the case $\kappa \in \mathbb{R}$, $\lambda > 0$ one can follow the lines of the proof of Theorem 4, and deduce that there can be no weak solution to (4.14) that satisfies the non-stickiness condition (4.4).

We claim that the equation (4.14) fails to have a weak solution also in the case $\kappa = \lambda = 0$, now even without having to impose the condition (4.4). Indeed, by plugging functions $f \in C_0^\infty(\mathbb{R})$ with compact support in $(-\infty, 0)$, into the Fokker-Planck equation corresponding to the stochastic differential equation (4.14), we see that

$$\mathbb{P}(X(t) < 0) = \mathbb{P}(X(0) < 0), \quad 0 \leq t < \infty$$

holds for every weak solution (X, W) of the equation (4.14). Thus, on the one hand, every weak solution X of the equation (4.14) with $X(0) = x \geq 0$, satisfies $X(t) \geq 0$ for Lebesgue almost every $t \in [0, \infty)$ by Fubini's Theorem. On the other hand, combining the latter conclusion and

the equation (4.14) with the P. Lévy characterization of Brownian motion (e.g., Theorem 3.16, page 157 in [16]), we conclude that X must be a standard Brownian motion; this is clearly a contradiction.

Finally, in the case $\lambda < 0$, one can proceed as in section 2 to construct the unique strong solution of the equation

$$dX(t) = \lambda dt + \mathbf{1}_{\{X(t) \geq 0\}} dW(t), \quad 0 \leq t < \infty. \tag{4.15}$$

4.1 Brownian Perturbations that Restore Strength

The addition of a suitably correlated Brownian motion with sufficiently high variance into (1.1), can restore a pathwise unique, strong solution to this equation when $\lambda > 0$. Our next result explains how, and its proof works just as well for every value $\lambda \in \mathbb{R}$.

Theorem 5. *For any real constant λ , and with W and V standard Brownian motions, the perturbed one-sided Tanaka equation*

$$dX(t) = \lambda dt + \mathbf{1}_{\{X(t) > 0\}} dW(t) + (\eta/2) dV(t), \quad 0 \leq t < \infty, \tag{4.16}$$

has a pathwise unique strong solution, provided either

- (i) $\eta \notin [-1, 1]$ and $\langle W, V \rangle(t) = -(t/\eta)$, $0 \leq t < \infty$, or
- (ii) $\eta \neq 0$ and W, V are independent.

Proof: It is fairly straightforward that solving (4.16) under the stated conditions amounts to solving the so-called ‘‘perturbed Tanaka equation’’

$$dX(t) = \lambda dt + \text{sgn}(X(t)) dM(t) + dN(t), \quad 0 \leq t < \infty, \tag{4.17}$$

where the processes $M := W/2$, $N := (W + \eta V)/2$ are continuous, orthogonal martingales with quadratic variations $\langle M \rangle(t) = t/4$ and $\langle N \rangle(t) = (\eta^2 - 1)t/4$, respectively. Thus, by the P. Lévy theorem once again, these are independent Brownian motions with respective variance parameters $1/4$ and $(\eta^2 - 1)/4$.

The recent work of Prokaj [21] shows that pathwise uniqueness holds for the equation (4.17). Thus, to complete the proof, it is enough to show that (4.17) admits a weak solution; for then the Yamada-Watanabe theory (e.g., Corollary 3.23, Chapter 5 in [16]) guarantees that this solution is actually strong, that is, for all $t \in [0, \infty)$ we have

$$\mathcal{F}_t^X \subseteq \mathcal{F}_t^{(M,N)} \equiv \mathcal{F}_t^{(W,V)}.$$

In order to prove existence of a weak solution for (4.17), it is enough to consider the case $\lambda = 0$; this is because a Cameron-Martin-Girsanov change of measure takes then care of any $\lambda \in \mathbb{R}$. Therefore, all we need to do is consider two independent Brownian motions U and N with variance parameters $1/4$ and $(\eta^2 - 1)/4$, respectively, along with a real-valued random variable ζ independent of the vector (U, N) , and define

$$X(t) := \zeta + U(t) + N(t), \quad M(t) := \int_0^t \text{sgn}(X(s)) dU(s), \quad 0 \leq t < \infty.$$

The process M is a continuous martingale that satisfies

$$\langle M, N \rangle(t) = \int_0^t \text{sgn}(X(s)) d\langle U, N \rangle(s) = 0 \quad \text{and} \quad \langle M \rangle(t) = \langle U \rangle(t) = t/4;$$

thus, by the P Lévy characterization once again, M is Brownian motion with variance parameter $1/4$, and is independent of the Brownian motion N . But then we have also $U(t) = \int_0^t \operatorname{sgn}(X(s)) dM(s)$, $t \geq 0$, therefore the representation

$$X(t) = \zeta + \int_0^t \operatorname{sgn}(X(s)) dM(s) + N(t), \quad 0 \leq t < \infty$$

as in (4.17) with $\lambda = 0$; this completes the proof under the conditions (i).

Under the conditions of (ii), the pathwise uniqueness of (4.16) is a consequence of Theorem 8.1 in Fernholz et al. [11], whereas weak existence follows from the results of Bass & Pardoux [4]. \square

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