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RANK PROBABILITIES FOR REAL RANDOM $N \times N \times 2$ TENSORS

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Abstract

We prove that the probability P_N for a real random Gaussian $N \times N \times 2$ tensor to be of real rank N is $P_N = (\Gamma((N+1)/2))^N / G(N+1)$, where $\Gamma(x)$, $G(x)$ denote the gamma and Barnes G-functions respectively. This is a rational number for N odd and a rational number multiplied by $\pi^{N/2}$ for N even. The probability to be of rank $N+1$ is $1 - P_N$. The proof makes use of recent results on the probability of having k real generalized eigenvalues for real random Gaussian $N \times N$ matrices. We also prove that $\log P_N = (N^2/4)\log(e/4) + (\log N - 1)/12 - \zeta'(-1) + \mathrm{O}(1/N)$ for large N, where ζ is the Riemann zeta function.

1 Introduction

The (real) rank of a real $m \times n \times p$ 3-tensor or 3-way array $\mathcal{T}$ is the well defined minimal possible value of r in an expansion

$$\mathcal{T} = \sum_{i=1}^{r} \mathbf{u}_i \otimes \mathbf{v}_i \otimes \mathbf{w}_i \qquad (\mathbf{u}_i \in \mathbb{R}^m, \mathbf{v}_i \in \mathbb{R}^n, \mathbf{w}_i \in \mathbb{R}^p) \tag{1}$$

where $\otimes$ denotes the tensor (or outer) product [1, 3, 4, 8].
If the elements of $\mathcal{T}$ are choosen randomly according to a continuous probability distribution, there is in general (for general m, n and p) no generic rank, i.e., a rank which occurs with probability 1. Ranks which occur with strictly positive probabilities are called typical ranks. We assume that all elements are independent and from a standard normal (Gaussian) distribution (mean 0, variance 1). Until now, the only analytically known probabilities for typical ranks were for $2\times2\times2$ and $3 \times 3 \times 2$ tensors [2, 7]. Thus in the $2 \times 2 \times 2$ case the probability that $r = 2$ is $\pi/4$ and the probability that $r = 3$ is $1 - \pi/4$, while in the $3 \times 3 \times 2$ case the probability of the rank equaling

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3 is the same as the probability of it equaling 4 which is 1/2. Before these analytic results the first numerical simulations were performed by Kruskal in 1989, for $2 \times 2 \times 2$ tensors [8], and the approximate values 0.79 and 0.21 obtained for the probability of ranks $r = 2$ and $r = 3$ respectively. For $N \times N \times 2$ tensors ten Berge and Kiers [10] have shown that the only typical ranks are N and $N + 1$. From ten Berge [9], it follows that the probability P_N for an $N \times N \times 2$ tensor to be of rank N is equal to the probability that a pair of real random Gaussian $N \times N$ matrices T_1 and T_2 (the two slices of $\mathscr{T}$) has N real generalized eigenvalues, i.e., the probability that $\det(T_1 - \lambda T_2) = 0$ has only real solutions λ [2, 9]. Knowledge about the expected number of real solutions to $\det(T_1 - \lambda T_2) = 0$ obtained by Edelman et al. [5] led to the analytical results for $N = 2$ and $N = 3$ in [2]. Forrester and Mays [7] have recently determined the probabilities $p_{N,k}$ that $\det(T_1 - \lambda T_2) = 0$ has k real solutions, and we here apply the results to $P_N = p_{N,N}$ to obtain explicit expressions for the probabilities for all typical ranks of $N \times N \times 2$ tensors for arbitrary N, hence settling this open problem for tensor decompositions. We also determine the precise asymptotic decay of P_N for large N and give some recursion formulas for P_N.

2 Probabilities for typical ranks of $N \times N \times 2$ tensors

As above, assume that T_1 and T_2 are real random Gaussian $N \times N$ matrices and let $p_{N,k}$ be the probability that $\det(T_1 - \lambda T_2) = 0$ has k real solutions. Then Forrester and Mays [7] prove:

Theorem 1. *Introduce the generating function*

$$Z_N(\xi) = \sum_{k=0}^{N}{}^{*} \xi^k p_{N,k} \tag{2}$$

where the asterisk indicates that the sum is over k values of the same parity as N. For N even we have

$$Z_N(\xi) = \frac{(-1)^{N(N-2)/8}\Gamma(\frac{N+1}{2})^{N/2}\Gamma(\frac{N+2}{2})^{N/2}}{2^{N(N-1)/2}\prod_{j=1}^{N}\Gamma(\frac{j}{2})^2} \prod_{l=0}^{\frac{N-2}{2}} (\xi^2\alpha_l + \beta_l), \tag{3}$$

while for N odd

$$Z_N(\xi) = \frac{(-1)^{(N-1)(N-3)/8}\Gamma(\frac{N+1}{2})^{(N+1)/2}\Gamma(\frac{N+2}{2})^{(N-1)/2}}{2^{N(N-1)/2}\prod_{j=1}^{N}\Gamma(\frac{j}{2})^2} \pi\xi$$
$$\times \prod_{l=0}^{\lceil\frac{N-1}{4}\rceil-1} (\xi^2\alpha_l + \beta_l) \prod_{\lceil\frac{N-1}{4}\rceil}^{\frac{N-3}{2}} (\xi^2\alpha_{l+1/2} + \beta_{l+1/2}) \tag{4}$$

Here

$$\alpha_l = \frac{2\pi}{N-1-4l} \frac{\Gamma(\frac{N+1}{2})}{\Gamma(\frac{N+2}{2})} \tag{5}$$

and

$$\alpha_{l+1/2} = \frac{2\pi}{N-3-4l} \frac{\Gamma(\frac{N+1}{2})}{\Gamma(\frac{N+2}{2})} \tag{6}$$

The expressions for β_l and $\beta_{l+1/2}$ are given in [7], but are not needed here, and $\lceil\cdot\rceil$ denotes the ceiling function.

The method used in [7] relies on first obtaining the explicit form of the element probability density function for

$$G = T_1^{-1} T_2. \tag{7}$$

A real Schur decomposition is used to introduce k real and $(N-k)/2$ complex eigenvalues, with the imaginary part of the latter required to be positive (the remaining $(N-k)/2$ eigenvalues are the complex conjugate of these), for $k = 0, 2, \ldots, N$ (N even) and $k = 1, 3, \ldots, N$ (N odd). The variables not depending on the eigenvalues can be integrated out to give the eigenvalue probability density function, in the event that there are k real eigenvalues. And integrating this over all allowed values of the real and positive imaginary part complex eigenvalues gives $P_{N,k}$. From Theorem 1 we derive our main result:

Theorem 2. *Let P_N denote the probability that a real $N \times N \times 2$ tensor whose elements are independent and normally distributed with mean 0 and variance 1 has rank N. We have*

$$P_N = \frac{(\Gamma((N+1)/2))^N}{G(N+1)}, \tag{8}$$

where

$$G(N+1) := (N-1)!(N-2)! \ldots 1! \qquad (N \in \mathbb{Z}^+) \tag{9}$$

is the Barnes G-function and $\Gamma(x)$ denotes the gamma function. More explicitly $P_2 = \pi/4$, and for $N \geq 4$ even

$$P_N = \frac{\pi^{N/2}(N-1)^{N-1}(N-3)^{N-3} \cdot \ldots \cdot 3^3}{2^{N^2/2}(N-2)^2(N-4)^4 \cdot \ldots \cdot 2^{N-2}}\ , \tag{10}$$

while for N odd

$$P_N = \frac{(N-1)^{N-1}(N-3)^{N-3} \cdot \ldots \cdot 2^2}{2^{N(N-1)/2}(N-2)^2(N-4)^4 \cdot \ldots \cdot 3^{N-3}}\ . \tag{11}$$

Hence P_N for N odd is a rational number but for N even it is a rational number multiplied by $\pi^{N/2}$. The probability for rank $N+1$ is $1 - P_N$.

Proof. From [2] we know that $P_N = p_{N,N}$. Hence, by Theorem 1

$$P_N = p_{N,N} = \frac{1}{N!} \frac{d^N}{d\xi^N} Z_N(\xi) \tag{12}$$

Since

$$\frac{1}{N!} \frac{d^N}{d\xi^N} \prod_{l=0}^{\frac{N-2}{2}} (\xi^2 \alpha_l + \beta_l) = \prod_{l=0}^{\frac{N-2}{2}} \alpha_l \tag{13}$$

and

$$\frac{1}{N!} \frac{d^N}{d\xi^N} \xi \prod_{l=0}^{\lceil \frac{N-1}{4} \rceil - 1} (\xi^2 \alpha_l + \beta_l) \prod_{\lceil \frac{N-1}{4} \rceil}^{\frac{N-3}{2}} (\xi^2 \alpha_{l+1/2} + \beta_{l+1/2}) = \prod_{l=0}^{\lceil \frac{N-1}{4} \rceil - 1} \alpha_l \prod_{\lceil \frac{N-1}{4} \rceil}^{\frac{N-3}{2}} \alpha_{l+1/2} \tag{14}$$

the values of β_l and $\beta_{l+1/2}$ are not needed for the determination of P_N. By (3) we immediately find

$$P_N = \frac{(-1)^{N(N-2)/8} \Gamma(\frac{N+1}{2})^{N/2} \Gamma(\frac{N+2}{2})^{N/2}}{2^{N(N-1)/2} \prod_{j=1}^{N} \Gamma(\frac{j}{2})^2} \prod_{l=0}^{\frac{N-2}{2}} \alpha_l \tag{15}$$

if N is even. For N odd we use (4) to get

$$P_N = \frac{(-1)^{(N-1)(N-3)/8}\Gamma(\frac{N+1}{2})^{(N+1)/2}\Gamma(\frac{N+2}{2})^{(N-1)/2}}{2^{N(N-1)/2}\prod_{j=1}^{N}\Gamma(\frac{j}{2})^2}\,\pi\prod_{l=0}^{\lceil\frac{N-1}{4}\rceil-1}\alpha_l\prod_{\lceil\frac{N-1}{4}\rceil}^{\frac{N-3}{2}}\alpha_{l+1/2} \tag{16}$$

Substituting the expressions for α_l and $\alpha_{l+1/2}$ into these formulas we obtain, after simplifying, for N even

$$P_N = \frac{(-1)^{N(N-2)/8}(2\pi)^{N/2}\Gamma(\frac{N+1}{2})^N}{2^{N(N-1)/2}\prod_{j=1}^{N}\Gamma(\frac{j}{2})^2}\prod_{l=0}^{\frac{N-2}{2}}\frac{1}{N-1-4l}\,, \tag{17}$$

and for N odd

$$P_N = \frac{(-1)^{(N-1)(N-3)/8}(2\pi)^{(N+1)/2}\Gamma(\frac{N+1}{2})^N}{2^{N(N-1)/2+1}\prod_{j=1}^{N}\Gamma(\frac{j}{2})^2}\prod_{l=0}^{\lceil\frac{N-1}{4}\rceil-1}\frac{1}{N-1-4l}\prod_{\lceil\frac{N-1}{4}\rceil}^{\frac{N-3}{2}}\frac{1}{N-3-4l}\,. \tag{18}$$

Now

$$\begin{aligned}\prod_{j=1}^{N}\Gamma(j/2)^2 &= \frac{\Gamma(1/2)}{\Gamma((N+1)/2)}\prod_{j=1}^{N}\Gamma(j/2)\Gamma((j+1)/2)\\ &= \frac{\Gamma(1/2)}{\Gamma((N+1)/2)}\prod_{j=1}^{N}2^{1-j}\sqrt{\pi}\,\Gamma(j)\\ &= \frac{\Gamma(1/2)}{\Gamma((N+1)/2)}2^{-N(N-1)/2}\pi^{N/2}G(N+1),\end{aligned} \tag{19}$$

where to obtain the second equality use has been made of the duplication formula for the gamma function, and to obtain the third equality the expression (9) for the Barnes G-function has been used. Furthermore, for each N even

$$\begin{aligned}(-1)^{N(N-2)/8}\prod_{l=0}^{(N-2)/2}\frac{1}{N-1-4l} &= \frac{(-1)^{N(N-2)/8}}{(N-1)(N-5)\dots(N-1-(2N-4))}\\ &= \frac{1}{(N-1)(N-3)\dots 3\cdot 1}\\ &= \frac{\Gamma(1/2)}{2^{N/2}\Gamma((N+1)/2)},\end{aligned} \tag{20}$$

where to obtain the final equation use is made of the fundamental gamma function recurrence

$$\Gamma(x+1) = x\Gamma(x), \tag{21}$$

and for N odd

$$
\begin{aligned}
&(-1)^{(N-1)(N-3)/8}\prod_{l=0}^{\lceil\frac{N-1}{4}\rceil}\frac{1}{N-1-4l}\prod_{\lceil\frac{N-1}{4}\rceil}^{\frac{N-3}{2}}\frac{1}{N-3-4l}\\
&\quad=(-1)^{(N-1)(N-3)/8}\begin{cases}\dfrac{1}{(N-1)(N-5)\dots 2}\dfrac{1}{(-4)(-8)\dots(-N+3)}, & N=3,7,11,\dots\\ \dfrac{1}{(N-1)(N-5)\dots 4}\dfrac{1}{(-2)(-6)\dots(-N+3)}, & N=5,9,13,\dots\end{cases}\\
&\quad=\frac{1}{(N-1)(N-3)\dots 4\cdot 2}\\
&\quad=\frac{1}{2^{(N-1)/2}\Gamma((N+1)/2)} \qquad (22)
\end{aligned}
$$

Substituting (19) and (20) in (17) establishes (8) for N even, while the N odd case of (8) follows by substituting (19) and (22) in (18), and the fact that

$$\Gamma(1/2)=\sqrt{\pi}. \qquad (23)$$

The forms (10) and (11) follow from (8) upon use of (9), the recurrence (21) and (for N even) (23).

□

3 Recursion formulas and asymptotic decay

By Theorem 2 it is straightforward to calculate P_{N+1}/P_N from either (8) or (10) and (11), and P_{N+2}/P_N from either (8) or (10) and (11).

Corollary 3. *For general N*

$$P_{N+1}=P_N\cdot\frac{\Gamma(N/2+1)^{N+1}}{\Gamma((N+1)/2)^N}\frac{1}{\Gamma(N+1)},\qquad P_{N+2}=P_N\cdot\frac{((N+1)/2)^{N+2}\Gamma((N+1)/2)^2}{\Gamma(N+2)\Gamma(N+1)} \qquad (24)$$

More explicitly, making use of the double factorial

$$N!!=\begin{cases}N(N-2)\dots 4\cdot 2, & N\text{ even}\\ N(N-2)\dots 3\cdot 1, & N\text{ odd,}\end{cases}$$

for N even we have the recursion formulas

$$P_{N+1}=P_N\cdot\frac{(N!!)^N}{(2\pi)^{N/2}((N-1)!!)^{N+1}}\ ,\quad P_{N+2}=P_N\cdot\frac{\pi}{2}\cdot\frac{(N+1)^{N+1}}{2^{2N+1}(N!!)^2} \qquad (25)$$

and for N odd we have

$$P_{N+1}=P_N\cdot\frac{\pi^{(N+1)/2}(N!!)^N}{2^{(3N+1)/2}((N-1)!!)^{N+1}}\ ,\quad P_{N+2}=P_N\cdot\frac{(N+1)^{N+1}}{2^{2N+1}(N!!)^2}. \qquad (26)$$

We can illustrate the pattern for P_N using Theorem 2 or Corollary 3. One finds

$$
\begin{aligned}
P_2 &= \frac{1}{2^2}\cdot\pi\,, \quad P_3 = \frac{1}{2} \\
P_4 &= \frac{3^3}{2^{10}}\cdot\pi^2\,, \quad P_5 = \frac{1}{3^2} \\
P_6 &= \frac{5^5\cdot 3^3}{2^{26}}\cdot\pi^3\,, \quad P_7 = \frac{3^2}{5^2\cdot 2^5} \\
P_8 &= \frac{7^7\cdot 5^5\cdot 3}{2^{48}}\cdot\pi^4\,, \quad P_9 = \frac{2^4}{7^2\cdot 5^4} \\
P_{10} &= \frac{7^7\cdot 5^5\cdot 3^{17}}{2^{80}}\cdot\pi^5\,, \quad P_{11} = \frac{5^4}{7^4\cdot 3^6\cdot 2^5} \\
P_{12} &= \frac{11^{11}\cdot 7^7\cdot 5^5\cdot 3^{15}}{2^{118}}\cdot\pi^6\,, \quad P_{13} = \frac{5^2}{11^2\cdot 7^6\cdot 2^4} \quad \cdots
\end{aligned} \tag{27}
$$

Numerically, it is clear that $P_N \to 0$ as $N \to \infty$. Some qualitative insight into the rate of decay can be obtained by recalling $P_N = p_{N,N}$ and considering the behaviour of $p_{N,k}$ as a function of k. Thus we know from [5] that for large N, the mean number of real eigenvalues $E_N := \langle k\rangle_{p_{N,k}}$ is to leading order equal to $\sqrt{\pi N/2}$, and from [7] that the corresponding variance $\sigma_N^2 := \langle k^2\rangle_{p_{N,k}} - E_N^2$ is to leading order equal to $(2-\sqrt{2})E_N$. The latter reference also shows that $\lim_{N\to\infty}\sigma_N p_{N,[\sigma_N x+E_N]} = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, and is thus $p_{N,k}$ is a standard Gaussian distribution after centering and scaling in k by appropriate multiples of $\sqrt{N}$. It follows that $p_{N,N}$ is, for large N, in the large deviation regime of $p_{N,k}$. We remark that this is similarly true of $p_{N,N}$ in the case of eigenvalues of $N\times N$ real random Gaussian matrices (i.e. the individual matrices T_1, T_2 of (7)), for which it is known $p_{N,N} = 2^{-N(N-1)/4}$ [5], [6, Section 15.10].
In fact from the exact expression (8) the explicit asymptotic large N form of P_N can readily be calculated. For this, let

$$A = e^{-\zeta'(-1)+1/12} = 1.28242712... \tag{28}$$

denote the Glaisher-Kinkelin constant, where ζ is the Riemann zeta function [11].

Theorem 4. *For large N,*

$$P_N = N^{1/12}(\frac{e}{4})^{N^2/4}\cdot Ae^{-1/6}(1+\mathrm{O}(N^{-1})) \tag{29}$$

or equivalently

$$\log P_N = (N^2/4)\log(e/4) + (\log N - 1)/12 - \zeta'(-1) + \mathrm{O}(1/N). \tag{30}$$

Proof. We require the $x\to\infty$ asymptotic expansions of the Barnes G-function [12] and the gamma function

$$\log G(x+1) = \frac{x^2}{2}\log x - \frac{3}{4}x^2 + \frac{x}{2}\log 2\pi - \frac{1}{12}\log x + \zeta'(-1) + \mathrm{O}\Big(\frac{1}{x}\Big), \tag{31}$$

$$\Gamma(x+1) = \sqrt{2\pi x}\Big(\frac{x}{e}\Big)^x\Big(1+\frac{1}{12x}+\mathrm{O}\Big(\frac{1}{x^2}\Big)\Big) \tag{32}$$

For future purposes, we note that a corollary of (32), and the elementary large x expansion

$$\Big(1+\frac{c}{x}\Big)^x = e^c\Big(1-\frac{c^2}{2x}+\mathrm{O}\Big(\frac{1}{x^2}\Big)\Big) \tag{33}$$

is the asymptotic formula

$$\frac{\Gamma(x+1/2)}{\Gamma(x)} = \sqrt{x}\Big(1-\frac{1}{8x}+\mathrm{O}\Big(\frac{1}{x^2}\Big)\Big). \tag{34}$$

To make use of these expansions, we rewrite (8) as

$$P_N = \frac{(\Gamma(N/2+1))^N}{G(N+1)}\Big(\frac{\Gamma((N+1)/2)}{\Gamma(N/2+1)}\Big)^N. \tag{35}$$

Now, (34) and (33) show that with

$$y := N/2 \tag{36}$$

and y large we have

$$\Big(\frac{\Gamma(y+1/2)}{\Gamma(y+1)}\Big)^N = e^{-y\log y}e^{-1/4}\Big(1+\mathrm{O}\Big(\frac{1}{y}\Big)\Big). \tag{37}$$

Furthermore, in the notation (36) it follows from (31) and (32) and further use of (33) (only the explicit form of the leading term is now required) that

$$\frac{\Gamma(N/2+1)^N}{G(N+1)} = e^{-y^2\log(4/e)}e^{y\log y+\frac{1}{12}\log 2y}e^{1/6-\zeta'(-1)}\Big(1+\mathrm{O}\Big(\frac{1}{y}\Big)\Big). \tag{38}$$

Multiplying together (37) and (38) as required by (35) and recalling (36) gives (29).
Recalling (28), the second stated result (30) is then immediate. □

Corollary 5. *For large N,*

$$\frac{P_{N+1}}{P_N} = \Big(\frac{e}{4}\Big)^{(2N+1)/4}(1+\mathrm{O}(N^{-1})) \tag{39}$$

This corollary follows trivially from Theorem 4. It can however also be derived directly from the recursion formulas in Corollary 3, without use of Theorem 4.

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