KANTER RANDOM VARIABLE AND POSITIVE FREE STABLE DISTRIBUTIONS

NIZAR DEMNI[1]

IRMAR, Rennes 1 University, France.
email: E-mail: nizar.demni@univ-rennes1.fr

Abstract

We derive the representative Bernstein measure of the density of $(X_\alpha)^{-\alpha/(1-\alpha)}, 0 < \alpha < 1$, where X_α is a positive stable random variable, as a Fox-H function. Up to a factor, this measure describes the law of some function a_α of a uniform random variable U on $(0, \pi)$. The distribution function of $a_{1-\alpha}(U)$ is then expressed through a H-function and is used to describe more explicitly the density of the analogue of X_α in the setting of free probability theory. Moreover, this density is shown to converge to a function with infinite mass when $\alpha \to 0^+$, in contrast to the classical setting where X_α is known to converge weakly to the inverse of an exponential random variable. Analytic evidences of the occurence of a_α in both the classical and the free settings conclude the exposition.

1 Motivation: Kanter's random variable

The study of stable random variables has known a considerable growth during approximately the last five decades, where they were shown to be an efficient model for various phenomena occuring in biology, quantum physics, market finance ([19], Ch.I). With certain exceptions, they do not have explicit expressions for their densities or distribution functions. Among these exceptions figure the one parameter class of positive stable variables, say $X_\alpha, 0 < \alpha < 1$, called in this fashion since supported on $(0, \infty)$ ([10], p. 50). The latter random variables may be then characterized by their Laplace transforms given by

$$\mathbb{E}\left(e^{-tX_\alpha}\right) = e^{-t^\alpha}$$

and they correspond to the set of parameter values $\{0 < \alpha < 1, \beta = c = 1, \gamma = 0\}$ in the four-parameters class of stable variables ([6]). Recall from [20] that for any $\alpha \in (0, 1)$, the density of X_α admits the following expansion:

$$-\frac{1}{\pi x} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \sin(k\pi\alpha)\Gamma(1 + k\alpha)\frac{1}{x^{k\alpha}}, \quad x > 0.$$

The non trivial positivity of this series follows for instance from the integral representation displayed p. 74 in [19], as most of analytic properties of stable distributions do from their numerous path-integral representations. Amazing and non trivial as well is the complete monotonicity of the density of $(X_\alpha)^{-r}, r \geq \alpha/(1-\alpha)$ (in particular its infinite divisibility, [15]). Indeed, there exist a standard exponential random variable L and an infinitely divisible random variable $V_{\alpha,r}$, both variables being independent, such that ([13]):

$$(X_\alpha)^r \stackrel{d}{=} \frac{e^{V_{\alpha,r}}}{L}.$$

For the special value $r = \alpha/(1-\alpha)$, the following very precise description of $V_{\alpha,r}$ traces back implicitly [5] (see also Ch.II in [10]) and is due to M. Kanter ([11] p. 703)

$$(X_\alpha)^{\alpha/(1-\alpha)} \stackrel{d}{=} \frac{a_\alpha(U)}{L},$$

where U is a uniform random variable on $(0, \pi)$ and

$$a_\alpha(u) = [b_\alpha(u)]^{1/(1-\alpha)}, \quad b_\alpha(u) = [c_\alpha(u)]^\alpha [c_{1-\alpha}(u)]^{(1-\alpha)}, \quad c_\alpha(u) := \frac{\sin(\alpha u)}{\sin(u)}.$$

Hence, the injectivity of the Laplace transform entails

$$V_{\alpha,\alpha/(1-\alpha)} \stackrel{d}{=} \log[a_\alpha(U)]$$

and we refer below to $a_\alpha(U)$ as the Kanter random variable. Since a_α is strictly increasing ([11] p. 704), let

$$a_\alpha^{-1}(u) = \pi \mathbb{P}(a_\alpha(U) \leq u), \quad u \geq a_\alpha(0) = (1-\alpha)\alpha^{\alpha/(1-\alpha)},$$

be its inverse function. Mysteriously, $a_{1-\alpha}^{-1}$ appears in the expression of the density of the analogue of $X_\alpha, 0 < \alpha < 1$ in the setting of Voiculescu's free probability theory. The analogy comes from Voiculescu's free additive convolution giving rise to the so-called free stable distributions ([1], [2], [3]). The latter distributions form a two-parameters family (up to a drift and a scaling parameters). Moreover, keeping the notations from [2], the set $\{0 < \alpha < 1, \rho = 1\}$ corresponds exactly to distributions supported in $(0, \infty)$, that we shall call in short positive free distributions. More precisely, their densities may be written in the following form ([2] p. 1053)

$$\frac{1}{\pi x} \frac{\sin(a_{1-\alpha}^{-1}(x)) \sin(\alpha a_{1-\alpha}^{-1}(x))}{\sin((1-\alpha)a_{1-\alpha}^{-1}(x))}, \tag{1}$$

for $x \geq a_{1-\alpha}(0)$. For this reason, we think that it is reasonable to have an insight on the mysterious occurence of the Kanter random variable in both probability theories. It is also of some benefit to seek an expression of the density of $a_\alpha(U)$ as a special function, and more generally of $e^{V_{\alpha,r}}$, since then (1) would be more explicit. The easiest way to address the second problem is to invert the Mellin transform of the density of $e^{-V_{\alpha,r}}$ given by ([13] p. 292)

$$\mathbb{E}(e^{-sV_{\alpha,r}}) = \frac{\Gamma((rs/\alpha) + 1)}{\Gamma(s+1)\Gamma(sr+1)}, \quad \Re(s) > -\alpha/r \geq -(1-\alpha).$$

However, Mellin's inversion formula

$$\frac{1}{2i\pi y} \int_{-i\infty}^{+i\infty} \frac{\Gamma((rs/\alpha) + 1)}{\Gamma(s+1)\Gamma(sr+1)} y^{-s} ds \tag{2}$$

applies for any $r > \alpha/(1-\alpha)$ (take $\Re(s) = 0$ as a path of integration) but does not when $r = \alpha/(1-\alpha)$. The latter fact may be seen from the following estimate of $|\Gamma(a+ib)|, a, b \in \mathbb{R}, |b| \to \infty$ displayed in formula 1.24. p. 4 from [9]:

$$|\Gamma(a + ib)| \sim \sqrt{2\pi}|b|^{a-1/2}e^{-a-\pi|b|/2}, \quad |b| \to \infty, \tag{3}$$

so that the absolute value of the integrand in (2) is equivalent to $|s|^{-1/2}$ (up to a constant). Note that when $r > \alpha/(1-\alpha)$, the integral displayed in (2) is nothing else but a so-called Fox H-function ([9] p. 3):

$$\frac{1}{2i\pi y}\int_{-i\infty}^{+i\infty} \frac{\Gamma((rs/\alpha)+1)}{\Gamma(s+1)\Gamma(sr+1)}y^{-s}ds = \frac{1}{y}H_{2,1}^{1,0}\left[y|_{(1,r/\alpha)}^{(1,1),(1,r)}\right]$$

so that the density of $e^{V_{\alpha,r}}$ reads after the change of variable $y \mapsto 1/y$,

$$\frac{1}{y}H_{2,1}^{1,0}\left[\frac{1}{y}|_{(1,r/\alpha)}^{(1,1),(1,r)}\right], \quad y > 0, r > \frac{\alpha}{1-\alpha}.$$

The first result of this paper states then that in the pathological case $r = \alpha/(1-\alpha)$, the density of $a_\alpha(U)$ is still a H-function yet represented through a different path integral than the one used in (2) (Hankel path). Using Cauchy's residue Theorem, our second result expresses $a_{1-\alpha}^{-1}$ through another H-function thereby describes more explicitly the density (1). Next, we investigate the limiting behavior of the density appearing in (1) aiming to prove a free analogue of Cressie's result on the weak convergence of stable laws when $\alpha \to 0^+$ ([6]). In the last and likely the most important part of this paper, we supply analytic arguments trying to describe the occurence of a_α in both probabilistic settings (classical and free). Loosely speaking, the inverse formula for the Fourier transform of the distribution of X_α together with admissible deformations of the paths of integration lead to the so-called Voiculescu's transform of the distribution of its free analogue. Nevertheless, we think that a group theoretical argument should exist, hidden at this point. For the sake of completeness, some basic facts on the H-function are collected in the next section. A good reference on this topic is the monograph [9].

2   On the Fox H-function

The Fox H-function is defined as a Mellin-Barnes integral

$$H_{p,q}^{m,n}\left[z|_{(b_i,B_i)_{1\leq i\leq q}}^{(a_i,A_i)_{1\leq i\leq p}}\right] = \int_L \Theta(s)z^{-s}ds$$

where $1 \leq m \leq q, 0 \leq n \leq p, a_i, b_i \in \mathbb{R}, A_i, B_i > 0$, and

$$\Theta(s) = \frac{\prod_{i=1}^m \Gamma(b_i + B_i s)\prod_{i=1}^n \Gamma(1 - a_i - A_i s)}{\prod_{i=m+1}^q \Gamma(1 - b_i - B_i s)\prod_{i=n+1}^p \Gamma(a_i + A_i s)}.$$

Here, we take the principal determination of the power function (although this is not strictly necessary) and furthermore agree that an empty product is equal to one. The integral is taken over a suitable contour L separating the poles of both products of the meromorphic (Gamma) functions in the numerator of Θ ([9] p. 3,4). The choice of L and of the domain of convergence of the Mellin-Barnes integrals defining the H-function for each contour depend on the parameters

a_i, b_i, A_i, B_i. Moreover, these integrals are evaluated by means of Cauchy's residue Theorem applied to truncated contours together with asymptotic estimates at infinity of the integrand. As a matter of fact, a contour may be deformed into another one provided that the deformation is admissible ([7], [17], see also CH.V. in [8] for a similar discussion on Meijer's G-function which coincides with the H-function when $A_i = B_j = 1, 1 \le i \le p, 1 \le j \le q$).

Eight cases are discussed in [9] p. 4. We only make use of five of them in the sequel to which we refer whenever needed. Set

$$\mu := \sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i.$$

i) If $\mu > 0, q \ge 1$, then the H-function exists in the punctured complex plane.

ii) If $\mu = 0, q \ge 1$, then the H-function exists in the open disc $|z| < \Delta$ and $L = L_{-\infty}$ is described in [9] p. 3.

iii) If $\mu = 0, p \ge 1$, then the H-function exists in the domain $|z| > \Delta$ where

$$\Delta := \prod_{i=1}^{p} A_i^{-A_i} \prod_{i=1}^{p} B_i^{B_i}$$

and $L = L_{+\infty}$ described in [9] p. 3.

iv) If $\mu = 0$ and

$$\delta := \sum_{i=1}^{q} b_i - \sum_{i=1}^{p} a_i + \frac{p-q}{2} < -1,$$

then the H-function is well-defined for complex numbers lying on the circle $|z| = \Delta$.

v) If

$$\Omega := \sum_{i=1}^{n} A_i + \sum_{i=1}^{m} B_i - \sum_{i=n+1}^{p} A_i - \sum_{i=m+1}^{q} B_i > 0,$$

then the H-function exists for all $z \neq 0$ such that $|\arg(z)| < \pi\Omega/2$ and L is the infinite semi-circle $\gamma - i\infty, \gamma + i\infty$ for a suitable real γ.

Remark 1. *In the sequel, the complex variable z will take only strictly positive values, thereby $s \mapsto z^{-s}$ is defined through the real-valued logarithm function.*

3   The density of $a_\alpha(U)$

Let h_α denote the density of $a_\alpha(U)$ supported by $(a_\alpha(0), \infty)$, then the density of $X_\alpha^{-\alpha/(1-\alpha)}$ is related to h_α by

$$-\frac{1-\alpha}{\alpha\pi x} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \sin(k\pi\alpha)\Gamma(1+k\alpha)x^{k(1-\alpha)} = \int_0^\infty e^{-xy} y h_\alpha(y) dy.$$

Hence, one needs to derive a Laplace-type integral representation of the LHS of this equality. The issue of this derivation is the following statement.

Proposition 1. *The density h_α of the Kanter variable is expressed as*

$$h_\alpha(y) = \frac{1}{y} H_{2,1}^{1,0}\left[\frac{1}{y}\Big|_{(1,1/(1-\alpha))}^{(1,\alpha/(1-\alpha)),(1,1)}\right] = \frac{2(1-\alpha)}{y} H_{2,1}^{1,0}\left[\frac{1}{y^{2(1-\alpha)}}\Big|_{(1,2)}^{(1,2\alpha),(1,2(1-\alpha))}\right]$$

for $y \in [a_\alpha(0), \infty[$.

Proof. Using the mirror formula satisfied by the Euler Gamma function ([8], p. 3),

$$\Gamma(1 + k\alpha)\Gamma(-k\alpha) = -\frac{\pi}{\sin(k\alpha\pi)},$$

the density of $X_\alpha^{-\alpha/(1-\alpha)}$ is transformed into

$$\frac{1-\alpha}{\alpha x}\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\frac{x^{k(1-\alpha)}}{\Gamma(-k\alpha)}.$$

According to A.6 p. 218 in [9], the latter density is expressed through a H-function[2]:

$$\frac{2(1-\alpha)}{\alpha x}H_{1,1}^{1,0}\left[x^{2(1-\alpha)}\Big|_{(0,2)}^{(0,2\alpha)}\right]$$

for all $x > 0$, which is well defined for all $z \neq 0$ in the complex plane by the virtue of (i) since $q \geq 1, \mu = 2 - 2\alpha > 0$. According to (v) and using the Gamma integral, one has for $\gamma > 0$

$$
\begin{aligned}
\frac{2(1-\alpha)}{\alpha x^2}H_{1,1}^{1,0}\left[x^{2(1-\alpha)}\Big|_{(0,2)}^{(0,2\alpha)}\right] &= \frac{2(1-\alpha)}{2\alpha i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(2s)}{\Gamma(2\alpha s)}\frac{ds}{x^{2(1-\alpha)s+2}} \\
&= \frac{2(1-\alpha)}{2\alpha i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(2s)}{\Gamma(2\alpha s)\Gamma(2+2(1-\alpha)s)} \\
&\quad \int_0^{\infty}e^{-xy}y^{2(1-\alpha)s+1}dyds \\
&= \frac{2(1-\alpha)}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(2s+1)}{\Gamma(2\alpha s+1)\Gamma(2+2(1-\alpha)s)} \\
&\quad \int_0^{\infty}e^{-xy}y^{2(1-\alpha)s+1}dyds.
\end{aligned}
$$

In order to change the order of integration, one needs to check that

$$\int_{-\infty}^{+\infty}\frac{|\Gamma(2is+2\gamma+1)|}{|\Gamma(2i\alpha s+2\alpha\gamma+1)\Gamma(2(1-\alpha)s+2(1-\alpha)\gamma+2)|}\int_0^{\infty}e^{-xy}y^{2(1-\alpha)\gamma}dyds < \infty.$$

Performing an integration with respect to y, it amounts to check

$$\int_{-\infty}^{+\infty}\frac{|\Gamma(2is+2\gamma+1)|}{|\Gamma(2i\alpha s+2\alpha\gamma+1)\Gamma(2i(1-\alpha)s+2(1-\alpha)\gamma+2)|}ds < \infty$$

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[2]It is a special kind of what V. M. Zolotarev called incomplete hypergeometric function in [20].

which follows again from the estimate of $|\Gamma(a+ib)|, |b| \to \infty$ (see (3)). Thus, Fubini's Theorem yields

$$\int_0^\infty e^{-xy} y h_\alpha(y) dy = 2(1-\alpha) \int_0^\infty xy e^{-xy} H_{2,1}^{1,0} \left[\frac{1}{y^{2(1-\alpha)}} \Big|_{(1,2)}^{(1,2\alpha),(2,2(1-\alpha))} \right] dy. \qquad (4)$$

Note that the H-function displayed in (4) may be represented for $y > (1-\alpha)\alpha^{\alpha/(1-\alpha)}$ through the Hankel path $L_{-\infty}$ according to (ii). It vanishes for $0 < y < (1-\alpha)\alpha^{\alpha/(1-\alpha)}$ since according to (iii) the path of integration is $L = L_{+\infty}$ and contains no pole of the meromorphic function $s \mapsto \Gamma(1+2s)$. Finally, it takes a finite value at $z = (1-\alpha)\alpha^{\alpha/(1-\alpha)}$ regarding (iv) since $\mu = 0, \delta = -3/2 < -1$ (in particular, it vanishes there by continuity). Hence, an integration by parts shows that the density h_α of the Kanter random variable $a_\alpha(U)$ is given by

$$\begin{aligned} h_\alpha(y) &= \frac{2(1-\alpha)}{y} \frac{d}{dy} \left\{ y H_{2,1}^{1,0} \left[\frac{1}{y^{2(1-\alpha)}} \Big|_{(1,2)}^{(1,2\alpha),(2,2(1-\alpha))} \right] \right\} \\ &= \frac{2(1-\alpha)}{y} \frac{d}{dy} \int_{L_{-\infty}} \frac{\Gamma(2s+1)}{\Gamma(2\alpha s+1)\Gamma(2+2(1-\alpha)s)} y^{2(1-\alpha)s+1} ds \\ &= \frac{2(1-\alpha)}{y} \int_{L_{-\infty}} \frac{\Gamma(2s+1)}{\Gamma(2\alpha s+1)\Gamma(2(1-\alpha)s+1)} y^{2(1-\alpha)s} ds \\ &= \frac{2(1-\alpha)}{y} H_{2,1}^{1,0} \left[\frac{1}{y^{2(1-\alpha)}} \Big|_{(1,2)}^{(1,2\alpha),(1,2(1-\alpha))} \right] \\ &= \frac{1}{y} \int_{L'_{-\infty}} \frac{\Gamma(s/(1-\alpha)+1)}{\Gamma(\alpha s/(1-\alpha)+1)\Gamma(s+1)} y^s ds = \frac{1}{y} H_{2,1}^{1,0} \left[\frac{1}{y} \Big|_{(1,1/(1-\alpha))}^{(1,\alpha/(1-\alpha)),(1,1)} \right], \end{aligned}$$

where $L'_{-\infty}$ is the image of $L_{-\infty}$ under the map $s \mapsto 2(1-\alpha)s$. The latter H-function is clearly well defined for $y > (1-\alpha)\alpha^{\alpha/(1-\alpha)} = a_\alpha(0)$ by (ii) and vanishes for $0 < y < (1-\alpha)\alpha^{\alpha/(1-\alpha)}$ according to (iii). The proposition is established. ∎

Remark 2. *For special indices of the form* $\alpha = 1 - (1/j), j \geq 2$, *we can prove that* $a_{1-(1/j)}(U)$ *is distributed as a product of independent Beta variables since the H-function involved in* $h_{1-(1/j)}$ *reduces to a Meijer function ([9] Ch.IV). This description is not surprising since* $X_{1/j}$ *is distributed as a product of* $(j-1)$ *independent Gamma variables ([18], see also [16] for a more larger set of special indices).*

4   Positive free stable distributions

Once h_α is written down, the distribution function of $a_\alpha(U)$ is expressed below through a H-function too. The density (1) is then described more explicitly and we shall investigate the limiting behavior, when $\alpha \to 0^+$, of its image under the map $x \mapsto x^\alpha$.

4.1   Distribution function of $a_{1-\alpha}(U)$

Before stating our second result, let us consider the case $\alpha = 1/2$. Then

$$a_{1/2}(\theta) = \frac{1}{4}[1 + \tan^2(\theta/2)]$$

and elementary computations show that

$$a_{1/2}^{-1}(x) = 2\arctan\left[2\sqrt{x - \frac{1}{4}}\right]$$

for $x > 1/4$. Using the trigonometric identity

$$\sin(\theta) = 2\frac{\tan(\theta/2)}{1 + \tan^2(\theta/2)}$$

one easily recovers the density displayed in [2], p. 1054:

$$\frac{2}{\pi x}\sin[a_{1/2}^{-1}(x)] = \frac{2}{\pi x}\frac{\tan(a_{1/2}^{-1}(x)/2)}{1 + \tan^2(a_{1/2}^{-1}(x)/2)} = \frac{2}{\pi x}\frac{\sqrt{4x - 1}}{1 + 4(x - 1/4)} = \frac{\sqrt{4x - 1}}{2\pi x^2}.$$

For general $\alpha \in (0, 1)$, one derives the following conclusion.

Proposition 2. *The inverse $a_{1-\alpha}^{-1}$ of $a_{1-\alpha}$ fits the following H-function:*

$$\frac{1}{\pi}a_{1-\alpha}^{-1}(x) = 2H_{2,1}^{1,0}\left[\frac{1}{x^{2\alpha}}\Big|_{(0,2)}^{(1,2\alpha),(1,2(1-\alpha))}\right].$$

Proof. Assuming for a while that Fubini's Theorem applies, the integral representation

$$H_{2,1}^{1,0}\left[\frac{1}{y^{2\alpha}}\Big|_{(1,2)}^{(1,2\alpha),(1,2(1-\alpha))}\right] = \frac{1}{2i\pi}\int_{L_{-\infty}}\frac{\Gamma(2s + 1)}{\Gamma(2\alpha s + 1)\Gamma(1 + 2(1 - \alpha)s)}y^{2\alpha s}ds$$

leads to

$$\frac{1}{\pi}a_{1-\alpha}^{-1}(x) = \frac{2\alpha}{2i\pi}\int_{L_{-\infty}}\frac{\Gamma(2s + 1)}{\Gamma(2\alpha s + 1)\Gamma(1 + 2(1 - \alpha)s)}\int_{\alpha(1-\alpha)^{(1-\alpha)/\alpha}}^{x}y^{2\alpha s - 1}dy\,ds$$

$$= \frac{2}{2i\pi}\int_{L_{-\infty}}\frac{\Gamma(2s)}{\Gamma(2\alpha s + 1)\Gamma(1 + 2(1 - \alpha)s)}\left[x^{2\alpha s} - (\alpha^{2\alpha}(1 - \alpha)^{2(1-\alpha)})^s\right]ds$$

for any $x > \alpha(1 - \alpha)^{(1-\alpha)/\alpha}$. Since the poles of $s \mapsto \Gamma(2s)$ are $s = -k/2, k \in \mathbb{N}$ and are simple, and since the pole $s = 0$ lies outside $L_{-\infty}$, then Cauchy's residue Theorem yields

$$\frac{1}{2i\pi}\int_{L_{-\infty}}\frac{\Gamma(2s)}{\Gamma(2\alpha s + 1)\Gamma(1 + 2(1 - \alpha)s)}x^{2\alpha s}ds = H_{2,1}^{1,0}\left[\frac{1}{x^{2\alpha}}\Big|_{(0,2)}^{(1,2\alpha),(1,2(1-\alpha))}\right] - \text{Res}(x, 0)$$

where $\text{Res}(x, 0)$ is the residue of the meromorphic function

$$s \mapsto \frac{\Gamma(2s)}{\Gamma(2\alpha s + 1)\Gamma(1 + 2(1 - \alpha)s)}x^{2\alpha s}$$

at $s = 0$ for fixed $x > \alpha(1 - \alpha)^{(1-\alpha)/\alpha}$. This residue is easily computed as

$$\lim_{s \to 0}\frac{s\Gamma(2s)}{\Gamma(2\alpha s + 1)\Gamma(1 + 2(1 - \alpha)s)}x^{2\alpha s} = \frac{1}{2}, \quad x > \alpha^{2\alpha}(1 - \alpha)^{2(1-\alpha)}.$$

Since

$$x \mapsto H_{2,1}^{1,0} \left[x \vert_{(0,2)}^{(1,2\alpha),(1,2(1-\alpha))} \right]$$

is defined on the whole real line by the virtue of (ii),(iii) and (iv), it follows that

$$\frac{1}{\pi} a_{1-\alpha}^{-1}(x) = 2 H_{2,1}^{1,0} \left[\frac{1}{x^{2\alpha}} \vert_{(0,2)}^{(1,2\alpha),(1,2(1-\alpha))} \right] - 2 H_{2,1}^{1,0} \left[\frac{1}{\alpha^{2\alpha}(1-\alpha)^{2(1-\alpha)}} \vert_{(0,2)}^{(1,2\alpha),(1,2(1-\alpha))} \right].$$

Finally, the continuity of the H-function entails

$$H_{2,1}^{1,0} \left[\frac{1}{\alpha^{2\alpha}(1-\alpha)^{2(1-\alpha)}} \vert_{(0,2)}^{(1,2\alpha),(1,2(1-\alpha))} \right] = 0$$

since

$$H_{2,1}^{1,0} \left[\frac{1}{x^{2\alpha}} \vert_{(0,2)}^{(1,2\alpha),(1,2(1-\alpha))} \right] = 0$$

for any $x < \alpha(1-\alpha)^{(1-\alpha)/\alpha}$ by the virtue of (iii).

Coming back to the validity of Fubini's Theorem assumed at the beginning of the proof, one needs to ensure, after integrating with respect to the variable y, the convergence of the integral

$$\int_{L_{-\infty}} \frac{|\Gamma(2s+1)|}{|\Gamma(2\alpha s+1)\Gamma(1+2(1-\alpha)s)|} \left[x^{2\alpha \Re(s)} - (\alpha^{2\alpha}(1-\alpha)^{2(1-\alpha)})^{\Re(s)} \right] \frac{|ds|}{|\Re(s)|}.$$

But, this holds true since the estimate of $\Gamma(a + ib)$ for large a (formula 1.23 p. 4. in [9]):

$$|\Gamma(a+ib)| \sim \sqrt{2\pi} |a|^{a-1/2} e^{-a-a(1-\mathrm{sgn}(b))/2}, \quad |a| \to \infty,$$

shows that the integrand is equivalent to

$$\frac{C}{|\Re(s)|^{3/2}} e^{-(\alpha \log(\alpha) + (1-\alpha)\log(1-\alpha))\Re(s)}, \quad |\Re(s)| \to \infty,$$

for some constant $C = C(\Im(s))$ depending only on $\Im(s)$. The proof of the proposition is therefore complete. ∎

Remarks 1. *1/Elementary variables changes give*

$$\frac{1}{\pi} a_{1-\alpha}^{-1} = \frac{1}{\alpha} H_{2,1}^{1,0} \left[\frac{1}{x} \vert_{(0,1/\alpha)}^{(1,1),(1,(1-\alpha)/\alpha)} \right] = \frac{1}{1-\alpha} H_{2,1}^{1,0} \left[\frac{1}{x^{\alpha/(1-\alpha)}} \vert_{(0,1/(1-\alpha))}^{(1,\alpha/(1-\alpha)),(1,1)} \right].$$

2/The duality relation given in [2] p. 1055 allows to derive a more explicit description of the density of a free stable distribution of index $1 < \alpha < 2$ and asymmetry coefficient $\rho = 0$.

4.2   Behavior for small indices

Motivated by a limiting result for general stable variables as $\alpha \to 0^+$ due to N. Cressie ([6], see also [4]), we now investigate the limiting behavior of $a_\alpha(U)$ and of the image of the density displayed in (1) under the map $x \mapsto x^\alpha$ as $\alpha \to 0^+$. To proceed, recall that if $Y(\alpha, \beta, \gamma, c), 0 < \alpha \le 2, |\beta| \le 1, \gamma \in \mathbb{R}, c > 0$ is a stable random variable, then

$$|Y(\alpha, \beta, \gamma, c) - \gamma|^\alpha \to \frac{1}{L}, \quad \alpha \to 0^+,$$

in distribution, where L is an exponential random variable with mean $1/c$ (when $c = 0$, $Y(\alpha, \beta, \gamma, 0)$ degenerates). Without loss of generality, we may choose $c = 1$. Noting that this limiting result does not depend on the asymmetry parameter β and that the special set of parameters $\{0 < \alpha < 1, \beta = 1, \gamma = 0, c = 1\}$ corresponds to positive stable random variables, Cressie's result reduces to

$$(X_\alpha)^\alpha \to \frac{1}{L}, \quad \alpha \to 0^+$$

in distribution. When combined with Kanter's representation

$$(X_\alpha)^\alpha \overset{d}{=} \left[\frac{a_\alpha(U)}{L} \right]^{1-\alpha},$$

the latter limiting result shows that $a_\alpha(U)$ has to be degenerate when α tends to 0^+. In the free setting, the picture is quite different.

Proposition 3. *For any $x > 1$,*

$$\lim_{\alpha \to 0^+} \frac{a_{1-\alpha}^{-1}(x^{1/\alpha})}{\pi x} \frac{\sin(\alpha a_{1-\alpha}^{-1}(x^{1/\alpha}))}{\alpha a_{1-\alpha}^{-1}(x^{1/\alpha})} \frac{\sin(a_{1-\alpha}^{-1}(x^{1/\alpha}))}{\sin((1-\alpha)a_{1-\alpha}^{-1}(x^{1/\alpha}))} = \frac{1}{x}.$$

Proof. Note first that the LHS is the image of the density displayed in (1) under the map $x \mapsto x^\alpha$ and that it is supported in $\{x > \alpha^\alpha(1-\alpha)^{1-\alpha}\}$. Now, according to Appendix 6 of [9] p. 218, one expands

$$\frac{a_{1-\alpha}^{-1}(x^{1/\alpha})}{\pi} = 2H_{2,1}^{1,0}\left[\frac{1}{x^2} \Big|_{(0,2)}^{(1,2\alpha),(1,2(1-\alpha))} \right] = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1-k\alpha)\Gamma(1-k(1-\alpha))k!} \frac{1}{x^k}.$$

Next, the mirror formula

$$\Gamma(1 - k(1-\alpha))\Gamma((1-\alpha)k) = \frac{\pi}{\sin(k(1-\alpha)\pi)}, \quad k \geq 1,$$

together with Lebesgue's convergence Theorem yield the limit

$$\lim_{\alpha \to 0^+} \frac{1}{\pi} a_{1-\alpha}^{-1}(x^{1/\alpha}) = 1 + \frac{1}{\pi} \lim_{\alpha \to 0^+} \sum_{k=1}^{\infty} \frac{\sin(k(1-\alpha)\pi)\Gamma(k(1-\alpha))}{\Gamma(1-k\alpha)} \frac{(-1)^k}{k!} \frac{1}{x^k} = 1$$

for any fixed $x > 1$. The conclusion follows.   ∎

Since the measure

$$\frac{dx}{x} \mathbf{1}_{\{x>1\}}$$

has infinite mass, then Scheffé's Lemma does not apply as it did in the classical setting. Note however that the map $x \mapsto 1/x$ transforms this measure into the Haar measure on $[0, 1]$:

$$\frac{dx}{x} \mathbf{1}_{\{0<x<1\}}.$$

5   On the function a_α

This section is likely the main contribution of the paper since it provides an explanation of the mysterious occurence of a_α in both the classical and the free settings. Our starting point is the Fourier-type integral representation of the density of X_α ([10] p.49, [19] p. 66):

$$\frac{1}{\pi}\Re\int_0^\infty e^{-itx}e^{-t^\alpha e^{-i\alpha\pi/2}}dt = \frac{1}{\pi}\Re\int_0^\infty e^{itx}e^{-(it)^\alpha}dt.$$

Then we need to deform twice the path of integration: the first deformation leads to a Laplace-type integral representation and follows from an integration over a quarter of a circle in the positive quadrant ([19] p. 70)

$$\frac{1}{\pi}\Re\int_0^\infty e^{itx}e^{-(it)^\alpha}dt = \frac{1}{\pi}\Im\int_0^\infty e^{-tx}e^{-e^{-i\alpha\pi}t^\alpha}dt$$
$$= \frac{x^{-1/(1-\alpha)}}{\pi}\Im\int_0^\infty e^{-x^{-\alpha/(1-\alpha)}(u+\phi_{1-\alpha}(u))}du$$

where $\phi_{1-\alpha}(u) := e^{-i\pi\alpha}u^\alpha$. The second deformation was used in [5] (see also Ch.II [10]) forces the exponential factor in the latter integrand to be a positive real number. Thus the new path of integration is the curve[3]

$$\mathscr{C}_\alpha := \{u = re^{i\theta}, \theta \in [-\pi, 0], \Im(u + \phi_{1-\alpha}(u)) = 0\}$$
$$= \left\{u = re^{i\theta}, \theta \in [-\pi, 0], r = \left[\frac{\sin(\alpha(\pi-\theta))}{\sin\theta}\right]^{1/(1-\alpha)}\right\}$$
$$= \left\{u = -re^{i\theta}, \theta \in [0, \pi], r = \left[\frac{\sin(\alpha\theta)}{\sin\theta}\right]^{1/(1-\alpha)}\right\}.$$

Moreover, any $u = -re^{i\theta} \in \mathscr{C}_\alpha$ satisfies

$$\Re(u + \phi_{1-\alpha}(u)) = -r\cos\theta + r^\alpha\cos(\alpha\theta)$$
$$= r^\alpha[-r^{1-\alpha}\cos\theta + \cos(\alpha\theta)]$$
$$= \left[\frac{\sin(\alpha\theta)}{\sin\theta}\right]^{\alpha/(1-\alpha)}\frac{\sin((1-\alpha)\theta)}{\sin(\theta)} = a_\alpha(\theta).$$

Now, we shall see that a conformal mapping of $\mathscr{C}_{1-\alpha}$ is closely related to the support of the density displayed in (1), yielding $a_{1-\alpha}$ after exactly the same computations performed above. To this end, let G_α be the Stieltjes transform of the positive free stable distribution. Then G_α is a one-to-one correspondence from the open upper half-plane $\mathbb{C}^+$ onto

$$\Omega_\alpha = \{z \in \mathbb{C}^-, \Im[\phi_\alpha(1/z) + (1/z)] \leq 0\}$$

where $\mathbb{C}^-$ is the open lower half-plane ([2] p. 1051). Moreover, the function ϕ_α is the substitute of the cumulants generating function of X_α for positive free stable distributions. The domain Ω_α

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[3]A continuous determination of the power function is considered.

is a Jordan domain whose boundary $\partial\Omega_\alpha$ is easily seen to be

$$\partial\Omega_\alpha = \left\{ re^{i\theta}, \theta \in [-\pi, 0], r = \left[\frac{\sin(\pi + \theta)}{\sin((1-\alpha)(\pi+\theta))} \right]^{1/\alpha} \right\}$$

$$= \left\{ -re^{i\theta}, \theta \in [0, \pi], r = \left[\frac{\sin\theta}{\sin((1-\alpha)\theta))} \right]^{1/\alpha} \right\}.$$

A comparison with the parametric equation defining $\mathscr{C}_\alpha$ shows that $\partial\Omega_\alpha$ is the image of $\mathscr{C}_{1-\alpha}$ under the map $z \mapsto 1/\bar{z}$. In order to derive $a_{1-\alpha}$, two ingredients are needed. The first one is the Stieltjes inversion formula expressing the density of the free positive stable variable as

$$-\frac{1}{\pi} \lim_{y \to 0^+} \mathfrak{J}[G_\alpha(x+iy)].$$

The second one is Caratheodory's extension Theorem ([14]) Ch. XIV p. 281) which shows that G_α extends to a homeomorphic map from the real line onto $\partial\Omega_\alpha$, thereby

$$-\frac{1}{\pi} \lim_{y \to 0^+} \mathfrak{J}[G_\alpha(x+iy)] = -\frac{1}{\pi}\mathfrak{J}[G_\alpha(x)].$$

As a matter of fact, the support of the density displayed in (1) is obviously

$$\begin{aligned}
\{x \in \mathbb{R}, \mathfrak{J}(G_\alpha(x)) < 0\} &= \{\mathfrak{R}(G_\alpha^{-1}(z)), z \in \partial\Omega_\alpha \, \mathfrak{J}(z) < 0\} \\
&= \{\mathfrak{R}(F_\alpha^{-1}(1/z)), z \in \partial\Omega_\alpha \, \mathfrak{J}(z) < 0\} \\
&= \{\mathfrak{R}(F_\alpha^{-1}(\bar{z})), z \in \mathscr{C}_{1-\alpha} \, \mathfrak{J}(z) < 0\} \\
&= \{\mathfrak{R}(\bar{z} + \phi_\alpha(\bar{z})), z \in \mathscr{C}_{1-\alpha} \, \mathfrak{J}(z) < 0\} \\
&= \{a_{1-\alpha}(\theta), z = -re^{i\theta} \in \mathscr{C}_{1-\alpha}, \theta \in]0, \pi[\}.
\end{aligned}$$

Remark 3. *When integrating along the curve $\mathscr{C}_\alpha$, the density of X_α admits the following integral representation:*

$$\frac{x^{-1/(1-\alpha)}}{\pi} \mathfrak{J} \int_{\mathscr{C}_\alpha} e^{-x^{-\alpha/(1-\alpha)}\mathfrak{R}(z+\phi_{1-\alpha}(z))} dz.$$

It would be interesting to perform a 'holomorphic' change of variable $z \mapsto 1/\bar{z}$ which maps C_α onto $\partial\Omega_{1-\alpha}$, since it may lead to a relation between the densities of positive stable variables in both probabilistic settings. Note that if

$$\psi(u) = \frac{u-i}{u+i}$$

is the Cayley map satisfying $\psi(\bar{u}) = 1/(\overline{\psi(u)})$ then $\psi^{-1}(\mathscr{C}_\alpha)$ and $\psi^{-1}(\partial\Omega_{1-\alpha})$ are complex conjugate.

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