

## MOMENT ESTIMATES FOR SOLUTIONS OF LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY ANALYTIC FRACTIONAL BROWNIAN MOTION

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### *Abstract*

As a general rule, differential equations driven by a multi-dimensional irregular path  $\Gamma$  are solved by constructing a rough path over  $\Gamma$ . The domain of definition – and also estimates – of the solutions depend on upper bounds for the rough path; these general, deterministic estimates are too crude to apply e.g. to the solutions of stochastic differential equations with linear coefficients driven by a Gaussian process with Hölder regularity  $\alpha < 1/2$ .

We prove here (by showing convergence of Chen's series) that linear stochastic differential equations driven by analytic fractional Brownian motion [6, 7] with arbitrary Hurst index  $\alpha \in (0, 1)$  may be solved on the closed upper half-plane, and that the solutions have finite variance.

## 1 Introduction

Assume  $\Gamma_t = (\Gamma_t(1), \dots, \Gamma_t(d))$  is a smooth  $d$ -dimensional path, and  $V_1, \dots, V_d : \mathbb{R}^r \rightarrow \mathbb{R}^r$  be smooth vector fields. Then (by the classical Cauchy-Lipschitz theorem for instance) the differential equation driven by  $\Gamma$

$$dy(t) = \sum_{i=1}^d V_i(y(t)) d\Gamma_t(i) \quad (1.1)$$

admits a unique solution with initial condition  $y(0) = y_0$ . The usual way to prove this is by showing (by a functional fixed-point theorem) that iterated integrals

$$y_n(t) \rightarrow y_{n+1}(t) := y_0 + \int_0^t \sum_i V_i(y_n(s)) d\Gamma_s(i) \quad (1.2)$$

converge when  $n \rightarrow \infty$ .

Expanding this expression to all orders yields formally for an arbitrary analytic function  $f$

$$f(y_t) = f(y_s) + \sum_{n=1}^{\infty} \sum_{1 \leq i_1, \dots, i_n \leq d} [V_{i_1} \dots V_{i_n} f](y_s) \Gamma_{ts}^n(i_1, \dots, i_n), \tag{1.3}$$

where

$$\Gamma_{ts}^n(i_1, \dots, i_n) := \int_s^t d\Gamma_{t_1}(i_1) \int_s^{t_1} d\Gamma_{t_2}(i_2) \dots \int_s^{t_{n-1}} d\Gamma_{t_n}(i_n), \tag{1.4}$$

provided, of course, the series converges. By specializing to the identity function  $f = \text{Id} : \mathbb{R}^r \rightarrow \mathbb{R}^r$ ,  $x \rightarrow x$ , one gets a series expansion for the solution  $(y_t)$ .

Let

$$\mathcal{E}_V^{N,t,s}(y_s) = y_s + \sum_{n=1}^N \sum_{1 \leq i_1, \dots, i_n \leq d} [V_{i_1} \dots V_{i_n} \text{Id}](y_s) \Gamma_{ts}^n(i_1, \dots, i_n) \tag{1.5}$$

be the  $N$ -th order truncation of (1.3). It may be interpreted as *one* iteration of the numerical Euler scheme of order  $N$ , which is defined by

$$y_{t_k}^{Euler;D} := \mathcal{E}_V^{N,t_k,t_{k-1}} \circ \dots \circ \mathcal{E}_V^{N,t_1,t_0}(y_0) \tag{1.6}$$

for an arbitrary partition  $D = \{0 = t_0 < \dots < t_n = T\}$  of the interval  $[0, T]$ . When  $\Gamma$  is only  $\alpha$ -Hölder with  $\frac{1}{N+1} < \alpha \leq \frac{1}{N}$ , the iterated integrals  $\Gamma^n(i_1, \dots, i_n)$ ,  $n = 2, \dots, N$  do not make sense a priori and must be substituted with a *geometric rough path* over  $\Gamma$ . A *geometric rough path* over  $\Gamma$  is a family

$$\left( (\Gamma_{ts}^1(i_1))_{1 \leq i_1 \leq d}, (\Gamma_{ts}^2(i_1, i_2))_{1 \leq i_1, i_2 \leq d}, \dots, (\Gamma_{ts}^N(i_1, \dots, i_N))_{1 \leq i_1, \dots, i_N \leq d} \right) \tag{1.7}$$

of functions of two variables such that:  $\Gamma_{ts}^1 = \Gamma_t^1 - \Gamma_s^1$  and satisfying a natural Hölder regularity condition, namely,  $\sup_{s,t \in \mathbb{R}} \left( \frac{|\Gamma_{ts}^k(i_1, \dots, i_k)|}{|t-s|^{k\alpha}} \right) < \infty$ ,  $k = 1, \dots, N$ , along with two algebraic compatibility properties (Chen/multiplicativity and shuffle/geometricity properties) for which we refer e.g. to [2]. To such data one may associate a theory of integration along  $\Gamma$ , so that (1.1), rewritten in its integral form, makes sense, see e.g. [2] or [3] for *local* solutions of differential equations in this setting.

In this article, we prove convergence of the series (1.3) when the vector fields  $V_i$  are linear and  $\Gamma$  is *analytic fBm* (afBm for short). This process – first defined in [7] –, depending on an index  $\alpha \in (0, 1)$ , is a complex-valued process, a.s.  $\kappa$ -Hölder for every  $\kappa < \alpha$ , which has an analytic continuation to the upper half-plane  $\Pi^+ := \{z = x + iy \mid x \in \mathbb{R}, y > 0\}$ . Its real part  $(2\text{Re } \Gamma_t, t \in \mathbb{R})$  has the same law as fBm with Hurst index  $\alpha$ . Trajectories of  $\Gamma$  on the closed upper half-plane  $\bar{\Pi}^+ = \Pi^+ \cup \mathbb{R}$  have the same regularity as those of fBm, namely, they are  $\kappa$ -Hölder for every  $\kappa < \alpha$ . As shown in [6], the regularized rough path – constructed by moving inside the upper half-plane through an imaginary translation  $t \rightarrow t + i\varepsilon$  – converges in the limit  $\varepsilon \rightarrow 0$  to a geometric rough path over  $\Gamma$  for *any*  $\alpha \in (0, 1)$ , which makes it possible to produce strong, local pathwise solutions of stochastic differential equations driven by  $\Gamma$  with analytic coefficients.

We do not enquire about the convergence of the series (1.3) in the general case (as mentioned before, it diverges e.g. when  $V$  is quadratic), but only in the *linear* case. One obtains, see section 3:

**Main Theorem.**

Let  $V_1, \dots, V_d$  be linear vector fields on  $\mathbb{C}^r$ . Then the series (1.3), associated to afBm  $\Gamma$  with Hurst index  $\alpha \in (0, 1)$ , converges in  $L^2(\Omega)$  on the closed upper half-plane  $\bar{\Pi}^+ = \Pi^+ \cup \mathbb{R}$ . Furthermore, there exists a constant  $C$  such that the solution  $(y_t)_{t \in \bar{\Pi}^+}$ , defined as the limit of the series, satisfies

$$\mathbb{E}|y_t - y_s|^2 \leq C|t - s|^{2\alpha}, \quad s, t \in \bar{\Pi}^+. \quad (1.8)$$

The Main Theorem depends essentially on an explicit estimate of the variance of iterated integrals of  $\Gamma$  proved in Lemma 2.2 below, which states the following:

**Lemma 2.2.**

There exists a constant  $C'$  such that, for every  $s, t \in \bar{\Pi}^+ = \Pi^+ \cup \mathbb{R}$ ,

$$\text{Var}\Gamma_{ts}^n(i_1, \dots, i_n) \leq \frac{(C'|t - s|)^{2n\alpha}}{n!}. \quad (1.9)$$

**Notation.** Constants (possibly depending on  $\alpha$ ) are generally denoted by  $C, C', C_1, c_\alpha$  and so on.

## 2 Definition of afBm and first estimates

We briefly recall to begin with the definition of the analytic fractional Brownian motion  $\Gamma$ , which is a complex-valued process defined on the closed upper half-plane  $\bar{\Pi}^+$  [6]. Its introduction was initially motivated by the possibility to construct quite easily iterated integrals of  $\Gamma$  by a contour deformation. Alternatively, its Fourier transform is supported on  $\mathbb{R}_+$ , which makes the regularization procedure in [8, 9] void.

**Proposition 2.1.** *There exists a unique analytic Gaussian process  $(\Gamma'_z, z \in \Pi^+)$  with the following properties (see [6] or [7] for its definition):*

(1)  $\Gamma'$  is a well-defined analytic process on  $\Pi^+$ , with Hermitian covariance kernel

$$\mathbb{E}\Gamma'_z \Gamma'_w = 0, \quad \mathbb{E}\Gamma'_z \bar{\Gamma}'_w = \frac{\alpha(1 - 2\alpha)}{2 \cos \pi\alpha} (-i(z - \bar{w}))^{2\alpha - 2}. \quad (2.1)$$

(2) Let  $\gamma : (0, 1) \rightarrow \Pi^+$  be any continuous path with endpoints  $\gamma(0) = 0$  and  $\gamma(1) = z$ , and set  $\Gamma_z = \int_\gamma \Gamma'_u du$ . Then  $\Gamma$  is an analytic process on  $\Pi^+$ . Furthermore, as  $z$  runs along any path in  $\Pi^+$  going to  $t \in \mathbb{R}$ , the random variables  $\Gamma_z$  converge almost surely to a random variable called again  $\Gamma_t$ .

(3) The family  $\{\Gamma_t; t \in \mathbb{R}\}$  defines a Gaussian centered complex-valued process, whose covariance function is given by:

$$\mathbb{E}[\Gamma_s \Gamma_t] = 0, \quad \mathbb{E}[\Gamma_s \bar{\Gamma}_t] = \frac{e^{-i\pi\alpha \text{sgn}(s)} |s|^{2\alpha} + e^{i\pi\alpha \text{sgn}(t)} |t|^{2\alpha} - e^{i\pi\alpha \text{sgn}(t-s)} |s - t|^{2\alpha}}{4 \cos(\pi\alpha)}.$$

The paths of this process are almost surely  $\kappa$ -Hölder for any  $\kappa < \alpha$ .

(4) Both real and imaginary parts of  $\{\Gamma_t; t \in \mathbb{R}\}$  are (non independent) fractional Brownian motions indexed by  $\mathbb{R}$ , with covariance given by

$$\mathbb{E}[\text{Re } \Gamma_s \text{Im } \Gamma_t] = -\frac{\tan \pi\alpha}{8} [-\text{sgn}(s)|s|^{2\alpha} + \text{sgn}(t)|t|^{2\alpha} - \text{sgn}(t-s)|t-s|^{2\alpha}]. \quad (2.2)$$

**Definition 2.2.** Let  $Y_t := \text{Re } \Gamma_{it}$ ,  $t \in \mathbb{R}_+$ . More generally,  $Y_t = (Y_t(1), \dots, Y_t(d))$  is a vector-valued process with  $d$  independent, identically distributed components.

The above results imply that  $Y_t$  is real-analytic on  $\mathbb{R}_+^*$ .

**Lemma 2.3.** *The infinitesimal covariance function of  $Y_t$  is:*

$$\mathbb{E}Y'_s Y'_t = \frac{\alpha(1-2\alpha)}{4 \cos \pi\alpha} (s+t)^{2\alpha-2}. \quad (2.3)$$

**Proof.** Let  $X_t := \text{Im } \Gamma_{it}$ . Since  $\mathbb{E}\Gamma_s \Gamma_t = 0$ ,  $(Y_s, s \geq 0)$  and  $(X_s, s \geq 0)$  have same law, with covariance kernel  $\mathbb{E}Y_s Y_t = \mathbb{E}X_s X_t = \frac{1}{2} \text{Re } \Gamma_{is} \bar{\Gamma}_{it}$ . Hence

$$\mathbb{E}[Y'_s Y'_t] = \frac{1}{2} \text{Re } \mathbb{E}\Gamma'_{is} \bar{\Gamma}'_{it} = \frac{\alpha(1-2\alpha)}{4 \cos \pi\alpha} (s+t)^{2\alpha-2}. \quad (2.4)$$

□

Note that  $\mathbb{E}Y'_s Y'_t > 0$ . From this simple remark follows (see proof of a similar statement in [5] concerning usual fractional Brownian motion with Hurst index  $\alpha > 1/2$ ):

**Lemma 2.4.** *Let  $\mathbf{Y}_{ts}^n(i_1, \dots, i_n)$ ,  $n \geq 2$  be the iterated integrals of  $Y$ . Then there exists a constant  $C > 0$  such that*

$$\text{Var}\mathbf{Y}_{ts}^n(i_1, \dots, i_n) \leq C \frac{(C|t-s|)^{2n\alpha}}{n!}. \quad (2.5)$$

**Proof.** Let  $\Pi$  be the set of all pairings  $\pi$  of the set  $\{1, \dots, 2n\}$  such that  $((k_1, k_2) \in \pi) \Rightarrow (i_{k'_1} = i_{k'_2})$ , where  $k'_1 = k_1$  if  $k_1 \leq n$ ,  $k_1 - n$  otherwise, and similarly for  $k'_2$ . By Wick's formula,

$$\begin{aligned} & \text{Var}\mathbf{Y}_{ts}^n(i_1, \dots, i_n) \\ &= \sum_{\pi \in \Pi} \left( \int_s^t dx_1 \dots \int_s^{x_{n-1}} dx_n \right) \left( \int_s^t dx_{n+1} \dots \int_s^{x_{2n-1}} dx_{2n} \right) \\ & \quad \prod_{(k_1, k_2) \in \pi} \mathbb{E}[Y'_{x_{k_1}} Y'_{x_{k_2}}]. \end{aligned} \quad (2.6)$$

Since the process  $Y'$  is positively correlated, and  $\Pi$  is largest when all indices  $i_1, \dots, i_n$  are equal, one gets  $\text{Var}\mathbf{Y}_{ts}^n(i_1, \dots, i_n) \leq \text{Var}\mathbf{Y}_{ts}^n(1, \dots, 1)$ . On the other hand,  $\mathbf{Y}_{ts}^n(1, \dots, 1) = \frac{1}{n!} (Y_t - Y_s)^n$ , hence

$$\text{Var}\mathbf{Y}_{ts}^n(1, \dots, 1) = \frac{[\text{Var}(Y_t - Y_s)]^n}{(n!)^2} \cdot \frac{(2n)!}{2^n \cdot n!} \leq \frac{[2\text{Var}(Y_t - Y_s)]^n}{n!}. \quad (2.7)$$

Now (assuming for instance  $0 < s < t$ )

$$\text{Var}(Y_t - Y_s) = c_\alpha \int_s^t \int_s^t (u+v)^{2\alpha-2} dudv \leq c_\alpha s^{2\alpha-2} (t-s)^2 \leq c_\alpha (t-s)^{2\alpha} \quad (2.8)$$

if  $\frac{t}{2} \leq s \leq t$ , and

$$\text{Var}(Y_t - Y_s) = \frac{c_\alpha}{2\alpha(2\alpha-1)} \left[ (2t)^{2\alpha} + (2s)^{2\alpha} - 2(t+s)^{2\alpha} \right] \leq C t^{2\alpha} \leq C'(t-s)^{2\alpha} \quad (2.9)$$

if  $s < t/2$ . Hence the result. □

### 3 Estimates for iterated integrals of $\Gamma$

The main tool for the study of  $\Gamma$  is the use of contour deformation. Iterated integrals of  $\Gamma$  are particular cases of analytic iterated integrals, see [7] or [6]. In particular, the following holds:

**Lemma 3.1.** *Let  $\gamma : (0, 1) \rightarrow \Pi^+$  be the piecewise linear contour with affine parametrization defined by :*

- (i)  $\gamma([0, 1/3]) = [s, s + i|\operatorname{Re}(t - s)|]$ ;
- (ii)  $\gamma([1/3, 2/3]) = [s + i|\operatorname{Re}(t - s)|, t + i|\operatorname{Re}(t - s)|]$ ;
- (iii)  $\gamma([2/3, 1]) = [t + i|\operatorname{Re}(t - s)|, t]$ .

If  $z = \gamma(x) \in \gamma([0, 1])$ , we let  $\gamma_z$  be the same path stopped at  $z$ , i.e.  $\gamma_z = \gamma([0, x])$ , with the same parametrization. Then (letting  $c_\alpha = \frac{\alpha(1-2\alpha)}{2\cos\pi\alpha}$ )

$$\begin{aligned} \operatorname{Var}\Gamma_{ts}^n(i_1, \dots, i_n) = & c_\alpha^n \sum_{\sigma \in \Sigma_I} \int_{\gamma} dz_1 \int_{\tilde{\gamma}} d\bar{w}_1 (-i(z_1 - \bar{w}_{\sigma(1)}))^{2\alpha-2} \cdot \int_{\gamma_{z_1}} dz_2 \int_{\tilde{\gamma}_{\bar{w}_1}} d\bar{w}_2 (-i(z_2 - \bar{w}_{\sigma(2)}))^{2\alpha-2} \dots \\ & \int_{\gamma_{z_{n-1}}} dz_n \int_{\tilde{\gamma}_{\bar{w}_{n-1}}} d\bar{w}_n (-i(z_n - \bar{w}_{\sigma(n)}))^{2\alpha-2} \end{aligned} \quad (3.1)$$

where  $\Sigma_I$  is the subset of permutations of  $\{1, \dots, n\}$  such that  $(i_j = i_k) \Rightarrow (\sigma(j) = \sigma(k))$ .

**Proof.** Note first that, similarly to eq. (2.6),

$$\begin{aligned} \operatorname{Var}\Gamma_{ts}^n(i_1, \dots, i_n) = & \sum_{\sigma \in \Sigma_I} \left( \int_1^t dz_1 \dots \int_s^{z_{n-1}} dz_n \right) \left( \int_{\bar{s}}^{\bar{t}} d\bar{w}_1 \dots \int_{\bar{s}}^{\bar{w}_{n-1}} d\bar{w}_n \right) \\ & \prod_{j=1}^n \mathbb{E} \left[ \Gamma'_{z_j} \bar{\Gamma}'_{\bar{w}_{\sigma(j)}} \right] \end{aligned} \quad (3.2)$$

(the difference with respect to eq. (2.6) comes from the fact that contractions only operate between  $\Gamma$ 's and  $\bar{\Gamma}$ 's, since  $\mathbb{E}[\Gamma_{z_j} \Gamma_{z_k}] = \mathbb{E}[\bar{\Gamma}_{\bar{w}_j} \bar{\Gamma}_{\bar{w}_k}] = 0$  by Proposition 2.1). Now the result comes from a deformation of contour, see [7].  $\square$

**Lemma 3.2.** *There exists a constant  $C'$  such that, for every  $s, t \in \bar{\Pi}^+ = \Pi^+ \cup \mathbb{R}$ ,*

$$\operatorname{Var}\Gamma_{ts}^n(i_1, \dots, i_n) \leq \frac{(C'|t-s|)^{2n\alpha}}{n!}. \quad (3.3)$$

**Proof.** We assume (without loss of generality) that  $\operatorname{Im} s \leq \operatorname{Im} t$ . If  $|\operatorname{Im}(t-s)| \geq c\operatorname{Re}|t-s|$  for some positive constant  $c$  (or equivalently  $|\operatorname{Re}(t-s)| \leq c'|\operatorname{Im}(t-s)|$  for some  $0 \leq c' < 1$ ) then it is preferable to integrate along the straight line  $[s, t] = \{z \in \mathbb{C} \mid z = (1-u)s + ut, 0 \leq u \leq 1\}$  instead of  $\gamma$ , and use the parametrization  $y = \operatorname{Im} z$ . If  $z_1, z_2 \in [s, t]$ ,  $y_1 = \operatorname{Im} z_1, y_2 = \operatorname{Im} z_2$ , then  $|(-i(z_1 - \bar{z}_2))^{2\alpha-2}| \leq C(y_1 + y_2)^{2\alpha-2}$ , hence  $\operatorname{Var}\Gamma_{ts}^n(i_1, \dots, i_n) \leq C'^n \operatorname{Var}\mathbf{Y}_{y_2, y_1}^n(i_1, \dots, i_n)$ , which yields the result by Lemma 2.4. So we shall assume that  $|\operatorname{Re}(t-s)| > c|\operatorname{Im}(t-s)|$  for some constant  $c > 0$ .

Let us use as new variable the parametrization coordinate  $x$  along  $\gamma$ . Then formula (3.1) reads

$$\begin{aligned} \text{Var}\Gamma_{ts}^n(i_1, \dots, i_n) &= c_\alpha^n \sum_{\sigma \in \Sigma_l} \int_0^1 dx_1 \int_0^1 dy_1 K'(x_1, y_{\sigma(1)}) \cdot \int_0^{x_1} dx_2 \int_0^{y_1} dy_2 K'(x_2, y_{\sigma(2)}) \dots \\ &\quad \int_0^{x_{n-1}} dx_n \int_0^{x_n} dy_n K'(x_n, y_{\sigma(n)}), \end{aligned} \tag{3.4}$$

where  $K'(x, y) = (3|\text{Re}(t-s)|)^2(3(x+y)|\text{Re}(t-s)| + 2\text{Im } s)^{2\alpha-2}$  if  $0 < x, y < 1/3$ ,  $(3|\text{Re}(t-s)|)^2(3((1-x) + (1-y))|\text{Re}(t-s)| + 2\text{Im } t)^{2\alpha-2}$  if  $2/3 < x, y < 1$ , and is bounded by a constant times  $|t-s|^{2\alpha}$  otherwise thanks to the condition  $|\text{Re}(t-s)| > c|t-s|$ . Note that  $(x+y)^{2\alpha-2} > 2^{2\alpha-2}$  if  $0 < x, y < 1$ . Hence (if  $0 < x, y < 1$ )  $|K'(x, y)| \leq (C_1|t-s|)^{2\alpha} [(x+y)^{2\alpha-2} + ((1-x) + (1-y))^{2\alpha-2}]$ ,

which is (up to a coefficient) the infinitesimal covariance of  $|t-s|^\alpha(Y_x + \check{Y}_{1-x}, 0 < x < 1)$  if  $\check{Y} \stackrel{(law)}{=} Y$  is independent of  $Y$ . A slight modification of the argument of Lemma 2.4 yields

$$\begin{aligned} \text{Var}\Gamma_{ts}^n(i_1, \dots, i_n) &\leq (C_1|t-s|)^{2n\alpha} \frac{[\text{Var}(Y_1 - Y_0) + \text{Var}(\check{Y}_1 - \check{Y}_0)]^n}{(n!)^2} \cdot \frac{(2n)!}{2^n \cdot n!} \\ &\leq C_1^{2n\alpha} \cdot \frac{(2C|t-s|)^{2n\alpha}}{n!}. \end{aligned} \tag{3.5}$$

□

### 4 Proof of main theorem

We now prove the theorem stated in the introduction, which is really a simple corollary of Lemma 3.2.

Let  $C$  be the maximum of the matrix norms  $\|V_i\| = \sup_{\|x\|_\infty=1} \|V_i x\|_\infty$  for the supremum norm  $\|x\|_\infty = \max(|x_1|, \dots, |x_r|)$ . Rewrite eq. (1.5) as

$$\mathcal{E}_V^{N,t,s}(y_s) = y_s + \sum_{n=1}^N \sum_{1 \leq i_1, \dots, i_n \leq d} a_{i_1, \dots, i_n} \Gamma_{ts}^n(i_1, \dots, i_n). \tag{4.1}$$

Then  $\|a_{i_1, \dots, i_n}\|_\infty \leq C^n$  and  $\mathbb{E}|\Gamma_{ts}^n(i_1, \dots, i_n)|^2 \leq \frac{(C|t-s|)^{2n\alpha}}{n!}$ . Hence (by the Cauchy-Schwarz inequality)

$$\begin{aligned} \mathbb{E} \left( \mathcal{E}_V^{N,t,s}(y_s) - y_s \right)^2 &\leq \sum_{m,n=1}^N \frac{(C''|t-s|)^{(m+n)\alpha}}{\sqrt{m!n!}} \\ &= \left( \sum_{m=1}^N \frac{(C''|t-s|)^{m\alpha}}{\sqrt{m!}} \right)^2 \leq C''' \cdot |t-s|^{2\alpha} \end{aligned} \tag{4.2}$$

independently of  $N$ . The series obviously converges and yields eq. (1.8) for  $p = 1$ . □

It should be easy to prove along the same lines that the series defining  $\mathbb{E}|y_t - y_s|^{2p}$  converges for every  $p \geq 1$ , and that there exists a constant  $C_p$  such that  $\mathbb{E}|y_t - y_s|^{2p} \leq C_p |t-s|^{2\alpha p}$  for every  $s, t \in \bar{\Pi}^+$ . The most obvious consequence – using Kolmogorov’s lemma – would be that  $y_t$  has Hölder regularity of any order less than  $\alpha$ . But this follows from standard rough path theory, so we skip the proof.

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