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RANDOM STRICT PARTITIONS AND DETERMINANTAL POINT PROCESSES

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Abstract
We present new examples of determinantal point processes with infinitely many particles. The particles live on the half-lattice $\{1,2,\dots\}$ or on the open half-line $(0,+\infty)$. The main result is the computation of the correlation kernels. They have integrable form and are expressed through the Euler gamma function (the lattice case) and the classical Whittaker functions (the continuous case). Our processes are obtained via a limit transition from a model of random strict partitions introduced by Borodin (1997) in connection with the problem of harmonic analysis for projective characters of the infinite symmetric group.

1 Introduction

In this paper we present new examples of determinantal point processes and compute their correlation kernels. About determinantal processes, e.g., see [34, 15] and the recent survey [4].

1.1 A model of random strict partitions

We begin with describing a family of probability measures on the set of all strict partitions. These measures depend on two real parameters $\alpha\in(0,+\infty)$ and $\xi\in(0,1)$. By a *strict partition* we mean a partition without equal parts, that is, a sequence of any length of the form $\lambda=(\lambda_1>\dots>\lambda_{\ell(\lambda)})$, where $\lambda_i\in\mathbb{Z}_{>0}:=\{1,2,\dots\}$. Set $|\lambda|:=\lambda_1+\dots+\lambda_{\ell(\lambda)}$, this is the *weight* of the partition (we agree that the empty partition $\lambda=\emptyset$ has zero weight). Let Pl_n (where $n=0,1,2,\dots$) denote the *Plancherel measure* on the set of strict partitions of weight n:

$$\mathrm{Pl}_n(\lambda):=\frac{2^{n-\ell(\lambda)}\cdot n!}{(\lambda_1!\dots\lambda_{\ell(\lambda)}!)^2}\prod_{1\le i<j\le\ell(\lambda)}\left(\frac{\lambda_i-\lambda_j}{\lambda_i+\lambda_j}\right)^2,\qquad |\lambda|=n. \tag{1}$$

This is a probability measure on $\{\lambda\colon|\lambda|=n\}$ which is an analogue (in the theory of projective representations of symmetric groups) of the well-known Plancherel measure on ordinary partitions.

The Plancherel measure on strict partitions was studied in, e.g., [2, 17, 18, 31].
A certain mixing procedure (called *poissonization*) for the Plancherel measures on ordinary partitions was considered in [1, 5]. This procedure leads to determinantal point processes. In our situation we define the following *poissonized Plancherel measure* on strict partitions:

$$\mathrm{Pl}_\theta := \sum_{n=0}^{\infty} \frac{(\theta/2)^n e^{-\theta/2}}{n!} \mathrm{Pl}_n, \qquad \theta > 0, \tag{2}$$

that is, we mix the measures Pl_n on $\{\lambda\colon |\lambda| = n\}$ using the Poisson distribution on the set $\{0, 1, \dots\}$ of indices n. As a result we obtain a probability measure on all strict partitions. In [24] it was proved that the poissonized Plancherel measure on strict partitions gives rise to a Pfaffian point process. We improve this result and show that this point process is determinantal (§2.5).
In [2] Borodin has introduced a deformation $\mathsf{M}_n^{(\alpha)}$ of the Plancherel measure Pl_n depending on a parameter $\alpha > 0$ (in [2] this parameter is denoted by x):

$$\mathsf{M}_n^{(\alpha)}(\lambda) = const_{\alpha,n} \cdot \mathrm{Pl}_n(\lambda) \cdot \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} \big(j(j-1) + \alpha\big), \qquad |\lambda| = n.$$

Here $const_{\alpha,n}$ is the normalizing constant. As explained in [2], the deformed measure $\mathsf{M}_n^{(\alpha)}$ preserves certain important properties of the Plancherel measure Pl_n. For $n = 0, 1, \dots$, the measure Pl_n is the limit of $\mathsf{M}_n^{(\alpha)}$ as $\alpha \to +\infty$.
Similarly to the mixing of the Plancherel measures Pl_n (2), we consider a mixing of the deformed measures $\mathsf{M}_n^{(\alpha)}$. But now as the mixing distribution we take the negative binomial distribution $\{(1-\xi)^{\alpha/2} \frac{(\alpha/2)_n}{n!} \xi^n\}$ on nonnegative integers n with parameter $\xi \in (0,1)$ (here $(a)_k := a(a+1)\dots(a+k-1)$ is the Pochhammer symbol). As a result we again obtain a probability measure on the set of all strict partitions. This measure also gives rise to a determinantal point process.
It is convenient to switch from the parameter $\alpha > 0$ to a new parameter $\nu := \frac{1}{2}\sqrt{1-4\alpha}$. The parameter ν can be either a real number $0 \le \nu < \frac{1}{2}$ (if $0 < \alpha \le \frac{1}{4}$), or a pure imaginary number (if $\alpha > \frac{1}{4}$). All our formulas below are symmetric with respect to the replacement of ν by $(-\nu)$.
We denote the above mixing of the measures $\mathsf{M}_n^{(\alpha)}$ by $\mathsf{M}_{\nu,\xi}$. The poissonized Plancherel measure Pl_θ is the limit of $\mathsf{M}_{\nu,\xi}$ as $\xi \searrow 0$, $\alpha = \frac{1}{4} - \nu^2 \to +\infty$ such that $\alpha\xi \to \theta$. In the sequel we call this limit transition the *Plancherel degeneration*.

1.2 Point processes

We identify every strict partition λ with the point configuration $\big\{\lambda_1, \dots, \lambda_{\ell(\lambda)}\big\}$ on the lattice $\mathbb{Z}_{>0}$. In this way, our two-parameter measure $\mathsf{M}_{\nu,\xi}$ on all strict partitions gives rise to a point process $\mathbf{P}_{\nu,\xi}$ on $\mathbb{Z}_{>0}$. The poissonized Plancherel measure Pl_θ also defines a point process on $\mathbb{Z}_{>0}$, denote this process by $\mathbf{P}_\theta$. By the very definition, $\mathbf{P}_{\nu,\xi}$ and $\mathbf{P}_\theta$ are supported by finite configurations. They have a general structure described in the following definition.

Definition 1.1. Let $\mathbf{P}^{(\psi)}$ be the point process on $\mathbb{Z}_{>0}$ that lives on finite configurations and assigns the following probability to every configuration $X = \{x_1, \dots, x_N\}$:

$$\mathbf{P}^{(\psi)}(X) := const \cdot (U(X))^2 \cdot \prod_{i=1}^{N} \psi(x_i). \tag{3}$$

Here ψ is a nonnegative function such that $\sum_{x=1}^{\infty}\psi(x)<\infty$, *const* is the normalizing constant and $U(X):=\prod_{1\le i<j\le N}\frac{x_i-x_j}{x_i+x_j}$.

The process $\mathbf{P}_{\nu,\xi}$ has the form (3) if as the function $\psi(x)$ we take

$$\psi_{\nu,\xi}(x):=\frac{\xi^x\cos(\pi\nu)}{2\pi}\frac{\Gamma(\frac{1}{2}-\nu+x)\Gamma(\frac{1}{2}+\nu+x)}{(x!)^2}. \tag{4}$$

The process $\mathbf{P}_\theta$ also has the form (3) if as $\psi(x)$ we take $\psi_\theta(x):=\frac{\theta^x}{2(x!)^2}$ which is the Plancherel degeneration of $\psi_{\nu,\xi}(x)$.

1.3 Correlation kernels

1.3.1 The pre-limit kernels

We observe (§2.1) that any point process of the form $\mathbf{P}^{(\psi)}$ (3) is determinantal and explicitly compute correlation kernels in the special cases $\mathbf{P}_{\nu,\xi}$ (§2.2–2.3) and $\mathbf{P}_\theta$ (§2.5). The kernel $\mathsf{K}_{\nu,\xi}$ of the process $\mathbf{P}_{\nu,\xi}$ has integrable form and is expressed through the Gauss hypergeometric function. We call $\mathsf{K}_{\nu,\xi}$ the *hypergeometric–type kernel*. In §2.4 we present alternative double contour integral representations for $\mathsf{K}_{\nu,\xi}$.

For any function ψ, the correlation kernel K of $\mathbf{P}^{(\psi)}$ is symmetric. However, viewed as an operator in the Hilbert space $\ell^2(\mathbb{Z}_{>0})$, K is *not* a projection operator as it happens in many other (in particular, random matrix) models with symmetric correlation kernels.

1.3.2 Limit transitions

Recall that the process $\mathbf{P}_{\nu,\xi}$ lives on finite configurations on $\mathbb{Z}_{>0}$. We consider two limit regimes as $\xi\nearrow 1$. In §3.1 we examine a limit of $\mathbf{P}_{\nu,\xi}$ on the lattice $\mathbb{Z}_{>0}$. This limit regime corresponds to studying the asymptotics of smallest parts of the random strict partition distributed according to the measure $\mathsf{M}_{\nu,\xi}$. In §3.2 we consider a scaling limit of $\mathbf{P}_{\nu,\xi}$. We embed the lattice $\mathbb{Z}_{>0}$ into the half-line $\mathbb{R}_{>0}$, $x\mapsto(1-\xi)x$, where $x\in\mathbb{Z}_{>0}$, and then pass to the limit as $\xi\nearrow 1$. This limit regime corresponds to studying the asymptotics of scaled largest parts of the random strict partition distributed according to the measure $\mathsf{M}_{\nu,\xi}$.

The resulting limit point processes live on infinite configurations (on $\mathbb{Z}_{>0}$ and $\mathbb{R}_{>0}$, respectively). One cannot describe the processes in terms of probabilities of individual configurations. We use the description in terms of correlation functions. We show that both limit processes are determinantal and explicitly compute their correlation kernels. The first kernel $\mathsf{K}_\nu^{\mathrm{gamma}}$ is expressed in terms of the Euler gamma function, and the second kernel $\mathscr{K}_\nu$ is expressed in terms of the Macdonald functions (they are certain versions of the Bessel functions). The kernel $\mathscr{K}_\nu$ is called the *Macdonald kernel*. In §3.2 we also give an alternative description of $\mathscr{K}_\nu$ in terms of a certain Sturm–Liouville operator. The Macdonald kernel has already appeared in the recent paper [22, §10.2] and also in [29, §5] in a different context.

1.4 Comparison with other models

1.4.1 z-measures and log-gas systems

Our determinantal processes arise from the measures $\mathsf{M}_n^{(\alpha)}$ on strict partitions introduced in [2] which are related to the problem of harmonic analysis for projective characters of the infinite symmetric group. About projective representations of symmetric groups, e.g., see [33, 14, 25, 17]. Harmonic analysis for ordinary characters of the infinite symmetric group leads to the z–measures on ordinary partitions [20, 21]. Determinantal processes corresponding to the z–measures are widely studied, e.g., see [6, 1, 7, 5, 19, 3, 26, 8, 9]. Recently Strahov studied another example of point processes of representation–theoretic origin arising from the z–measures with the deformation (Jack) parameter 2 [35, 36]. In that case the point processes are Pfaffian. The conventional z–measures correspond to the Jack parameter 1. In Remark 6 we compare some of the operators considered in the present paper and the corresponding objects for the z-measures.

On the other hand, note a similarity of our model (3) to lattice log-gas systems [13]. The major difference however is that in our model the pair interaction is directed by the factor $(U(X))^2$ instead of the conventional $(V(X))^\beta$, where $V(X) := \prod_{1\le i<j\le N}(x_i - x_j)$. Lattice log–gas systems have representation–theoretic interpretation for $\beta = 2$ (the z–measures with Jack parameter 1) and for $\beta = 1$ or 4 (the deformed z–measures studied in [35, 36]). Our factor $(U(X))^2$ comes from the structure of the Plancherel measures Pl_n (1) on strict partitions and is specific to the study of projective representations of symmetric groups.

Note also that the determinantal and Pfaffian processes coming from the z–measures on ordinary partitions are closely related to orthogonal polynomial ensembles. Our model seems to lack this property.

1.4.2 Shifted Schur measure

The z–measures with Jack parameter 1 are a special case of the Schur measure on ordinary partitions introduced in [27]. On strict partitions there exists an analogue of the Schur measure, namely, the shifted Schur measure introduced in [37]. In [24, §4] it was pointed out that the poissonized Plancherel measure Pl_θ can be interpreted as a special case of the shifted Schur measure. However, it seems that the measures $\mathsf{M}_{\nu,\xi}$ have no such interpretation. The correlation functions of the shifted Schur measure were computed in [24], they are expressed in terms of certain Pfaffians. For the poissonized Plancherel measure these Pfaffians turn into determinants, see §2.5 below.

There exists another family of (complex-valued) probability measures on strict partitions which under certain specializations becomes $\mathsf{M}_{\nu,\xi}$ or Pl_θ. These measures were introduced by Rains [32, §7]. We discuss them below in §2.6.

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2 Hypergeometric–type kernel

2.1 The process $\mathbf{P}^{(\psi)}$ as an L–ensemble

Let $\mathbf{P}^{(\psi)}$ be the point process defined by (3) with arbitrary nonnegative function $\psi(x)$ such that $\sum_{x=1}^{\infty}\psi(x) < \infty$. Let L be the following $\mathbb{Z}_{>0}\times\mathbb{Z}_{>0}$ matrix:

$$\mathsf{L}(x,y) := \frac{2\sqrt{xy\psi(x)\psi(y)}}{x+y}, \qquad x,y\in\mathbb{Z}_{>0}. \tag{5}$$

The condition $\sum_{x=1}^{\infty}\psi(x)<\infty$ ensures that the operator in $\ell^2(\mathbb{Z}_{>0})$ corresponding to L is of trace class. Therefore, the Fredholm determinant $\det(1+\mathsf{L})$ is well defined.

Proposition 1. *Let $X=\{x_1,\dots,x_n\}\subset\mathbb{Z}_{>0}$ be any finite subset. We have $\mathbf{P}^{(\psi)}(X)=\frac{\det \mathsf{L}_X}{\det(1+\mathsf{L})}$, where by L_X we denote the submatrix $\left[\mathsf{L}(x_i,x_j)\right]_{i,j=1}^{n}$.*

This follows from the Cauchy determinant identity [23, Ch. I, §4, Ex. 6].
Proposition 1 implies that the random point process $\mathbf{P}^{(\psi)}$ is an L–ensemble corresponding to the matrix L (e.g., see [4, §5]). It follows from general properties of determinantal point processes (for example, see [7, Prop. 2.1]) that the L–ensemble $\mathbf{P}^{(\psi)}$ is determinantal, and its correlation kernel has the form $\mathsf{K}=\mathsf{L}(1+\mathsf{L})^{-1}$. Since L is symmetric, the kernel K is also symmetric. However, the operator of the form $\mathsf{L}(1+\mathsf{L})^{-1}$ in $\ell^2(\mathbb{Z}_{>0})$ cannot be a projection operator.

2.2 Correlation kernel of the process $\mathbf{P}_{\nu,\xi}$

Here we present explicit expressions for the correlation kernel $\mathsf{K}_{\nu,\xi}$ of the point process $\mathbf{P}_{\nu,\xi}$ defined by (3)–(4). To shorten the notation, set

$$\begin{aligned}\phi_i(x) &:= {}_2F_1\left(-\tfrac12-\nu+i,-\tfrac12+\nu+i;x+i;\tfrac{\xi}{\xi-1}\right), \qquad i=0,1,2,\dots;\\ \widetilde{\phi}(x) &:= {}_2F_1\left(\tfrac32+\nu,-\tfrac12-\nu;x;\tfrac{\xi}{\xi-1}\right).\end{aligned}$$

where ${}_2F_1$ is the Gauss hypergeometric function. Since $x\in\mathbb{Z}_{>0}$ and $i\in\mathbb{Z}_{\ge0}$, the third parameter of the above hypergeometric functions is a positive integer, therefore, $\phi_i(x)$ and $\widetilde{\phi}(x)$ are well defined. Also set

$$\Xi(x,y):=\left\{\Gamma\left(\tfrac12-\nu+x\right)\Gamma\left(\tfrac12+\nu+x\right)\Gamma\left(\tfrac12-\nu+y\right)\Gamma\left(\tfrac12+\nu+y\right)\right\}^{\frac12}, \tag{6}$$

where $x,y\in\mathbb{Z}_{>0}$. Note that due to our assumptions on the parameter ν (§1.1), the above expression in the curved brackets is strictly positive, so we choose the square root to be real positive.

Theorem 2.1. *We have*

$$\mathsf{K}_{\nu,\xi}(x,y)=\frac{2\,\Xi\left(x,y\right)\sqrt{xy}}{x+y}\sum_{j=0}^{\infty}\frac{\xi^{j+(x+y)/2}(1-\xi)^{-2j}\phi_{j+1}(x)\phi_{j+1}(y)}{2^{\delta(j)}(x+j)!(y+j)!\Gamma(\frac12-\nu-j)\Gamma(\frac12+\nu-j)}, \tag{7}$$

where $\delta(x):=\delta_{x0}$ is the Kronecker delta.

Theorem 2.2. *We have*

$$\mathsf{K}_{\nu,\xi}(x,y)=\frac{\cos(\pi\nu)}{\pi}\frac{\xi^{\frac{x+y}{2}}\,\Xi(x,y)}{\sqrt{x!y!(x-1)!(y-1)!}}\cdot\frac{\mathsf{A}(x)\mathsf{B}(y)-\mathsf{B}(x)\mathsf{A}(y)}{x^2-y^2}, \tag{8}$$

where $\mathsf{B}(x)=\phi_1(x)$ and $\mathsf{A}(x)$ can be written in one of the two following forms:
(1) $\mathsf{A}^{(1)}(x):=x\left(2\phi_0(x)-\phi_1(x)\right)$;
(2) $\mathsf{A}^{(2)}(x):=\frac{x}{1+\xi}[2\widetilde{\phi}(x)-(1-\xi)\phi_1(x)]$.

If $x=y$, formula (8) is also true when understood according to the L'Hospital's rule. This agreement is also applicable to similar formulas below.

Remark 1. Note that the kernel $\mathsf{K}_{\nu,\xi}$ given by (8) can be viewed as a discrete analogue of an integrable operator if as variables we take x^2 and y^2. About integrable operators, e.g., see [16, 11]. Discrete integrable operators are discussed in [3] and [7, §6].

Remark 2. There is an identity

$$\phi_0(x) = \frac{\widetilde{\phi}(x)}{1+\xi} - \frac{\xi(1+2\nu - x(1-\xi))}{x(1-\xi^2)}\phi_1(x), \tag{9}$$

which is a combination of 2.8(38), 2.8(39) and 2.9(2) in [12]. Therefore, $\mathsf{A}^{(2)}(x) = \mathsf{A}^{(1)}(x) + c\,\mathsf{B}(x)$, where c does not depend on x. Thus, the kernel (8) with $\mathsf{A}^{(1)}$ is identical to the one with $\mathsf{A}^{(2)}$.

Furthermore, all our formulas must be symmetric with respect to the replacement of ν by $(-\nu)$. Clearly, $\Xi(x,y)$ and all the functions $\phi_i(x)$, $i \in \mathbb{Z}_{\ge 0}$, possess this property, so the kernel (7) and the kernel (8) with $\mathsf{A}^{(1)}$ do not change under the substitution $\nu \to (-\nu)$. The same holds for the kernel (8) with $\mathsf{A}^{(2)}$, because from (9) we have $\widetilde{\phi}(x)|_{\nu\to(-\nu)} = \widetilde{\phi}(x) + \frac{\widetilde{c}}{x}\phi_1(x)$, where $\widetilde{c}$ does not depend on x.

2.3 Scheme of proof of Theorems 2.1 and 2.2

We begin with the argument similar to [26], but instead of the infinite wedge space we take the Fock space with the orthonormal basis $v_\lambda = e_{\lambda_1} \wedge e_{\lambda_2} \wedge \cdots \wedge e_{\lambda_{\ell(\lambda)}}$ indexed by all strict partitions (in particular, $v_\emptyset = 1$). A similar space is used in [24, §3] and [38, §5.2]. By calculations in this Fock space we first obtain a *Pfaffian formula* for the correlation functions $\rho_{\nu,\xi}$ of the point process $\mathbf{P}_{\nu,\xi}$ (and not a determinantal formula as it was in [26]).

Proposition 2. *There exists a function $\Phi_{\nu,\xi}\colon (\mathbb{Z}\setminus\{0\})^2 \to \mathbb{C}$ such that for every finite subset $X = \{x_1,\dots,x_n\} \subset \mathbb{Z}_{>0}$ we have*

$$\rho_{\nu,\xi}(X) = (-1)^{\sum_{i=1}^n x_i}\cdot \operatorname{Pf}\big(\Phi(X)\big),$$

where Pf *means Pfaffian. Here $\Phi(X)$ is the $2n\times 2n$ skew-symmetric matrix with rows and columns indexed by $x_1,\dots,x_n,-x_n,\dots,-x_1$ such that the ij-th element of the matrix $\Phi(X)$ above the main diagonal is $\Phi_{\nu,\xi}(i,j)$, where i and j take values $x_1,\dots,x_n,-x_n,\dots,-x_1$.*

Now we explain how one can convert the above Pfaffian formula for the correlation functions of $\mathbf{P}_{\nu,\xi}$ to a determinantal one. It turns out that $\Phi_{\nu,\xi}$ satisfies the following identities (here $x,y \in \mathbb{Z}\setminus\{0\}$):

- If $x \ne y$, then $\Phi_{\nu,\xi}(x,-y) = (-1)^y \frac{x+y}{x-y}\Phi_{\nu,\xi}(x,y)$.
- If $x \ne -y$, then $\Phi_{\nu,\xi}(y,x) = -\Phi_{\nu,\xi}(x,y)$ and, moreover, $\Phi_{\nu,\xi}(-x,-y) = (-1)^{x+y+1}\Phi_{\nu,\xi}(x,y)$.
- If $x \ne 0$, then $\Phi_{\nu,\xi}(x,-x) + \Phi_{\nu,\xi}(-x,x) = (-1)^x$.

Fix a finite subset $X = \{x_1,\dots,x_n\} \subset \mathbb{Z}_{>0}$. Define $C_{kl} := \delta_{kl} + (-1)^{x_{k\wedge l}} \frac{x_{k\wedge l}-x_n}{x_{k\wedge l}+x_n} I_{\{k+l=2n+1\}}$ $(k,l = 1,\dots,2n)$, where $k\wedge l$ means the minimum of k and l, and I means the indicator. Clearly, the $2n\times 2n$ matrix $C = [C_{kl}]$ is invertible. Using the above identities for $\Phi_{\nu,\xi}$, we obtain

$$C\Phi(X)C' = \begin{bmatrix} 0 & M \\ -M' & 0 \end{bmatrix},$$

where $(..)'$ means the matrix transpose and M has format $n\times n$. It follows from properties of Pfaffians that $\operatorname{Pf}\big(\Phi(X)\big) = (-1)^{n(n-1)/2}(\det C)^{-1}\det M$. There exist two diagonal $n\times n$ matrices D_1

and D_2 such that $\det(D_1D_2) = (-1)^{\sum_{i=1}^n x_i}(\det C)^{-1}$ and $D_1 M^{\circlearrowleft} D_2 = \mathsf{K}_{\nu,\xi}(X) = [\mathsf{K}_{\nu,\xi}(x_i,x_j)]_{i,j=1}^n$ for some $\mathbb{Z}_{>0}\times\mathbb{Z}_{>0}$ matrix $\mathsf{K}_{\nu,\xi}$. Here $M^{\circlearrowleft}$ is the matrix that is obtained from M by rotation by 90 degrees counter–clockwise. Note that $\det\left(M^{\circlearrowleft}\right) = (-1)^{n(n-1)/2}\det M$.

Thus, $\rho_{\nu,\xi}(X) = (-1)^{\sum_{i=1}^n x_i}\operatorname{Pf}\left(\Phi(X)\right) = \det \mathsf{K}_{\nu,\xi}(X)$, which means that $\mathsf{K}_{\nu,\xi}$ is the desired correlation kernel. The kernel $\mathsf{K}_{\nu,\xi}$ is related to $\Phi_{\nu,\xi}$ as follows:

$$\mathsf{K}_{\nu,\xi}(x,y) = \frac{2(-1)^y\sqrt{xy}}{x+y}\Phi_{\nu,\xi}(x,-y), \qquad x,y\in\mathbb{Z}_{>0}. \tag{10}$$

We obtain explicit expressions for $\Phi_{\nu,\xi}$ in terms of the Gauss hypergeometric function. Their form is similar to formulas (3.16) and (3.17) in [26]. These expressions for $\Phi_{\nu,\xi}$ together with relation (10) imply Theorems 2.1 and 2.2, respectively.

Remark 3. Let $\mathsf{L}_{\nu,\xi}$ be the operator defined by (5) with $\psi = \psi_{\nu,\xi}$ given by (4). Once formula (8) for $\mathsf{K}_{\nu,\xi}$ is obtained, one can directly check that $\mathsf{K}_{\nu,\xi} = \mathsf{L}_{\nu,\xi}(1+\mathsf{L}_{\nu,\xi})^{-1}$. Indeed, this relation is equivalent to $\mathsf{K}_{\nu,\xi} + \mathsf{K}_{\nu,\xi}\mathsf{L}_{\nu,\xi} - \mathsf{L}_{\nu,\xi} = 0$, and the computation of the matrix product $\mathsf{K}_{\nu,\xi}\mathsf{L}_{\nu,\xi}$ mainly reduces to the computation of sums of the form $\sum_{k=1}^{\infty}\frac{\xi^k\Gamma(\frac12+\nu+k)\Gamma(\frac12-\nu+k)}{k!(k-1)!}\frac{f(k)}{k+a}$, where $a\neq -1,-2,\dots$ is some constant and $f(k)$ is one of the functions $k\phi_0(k)$, $k\phi_1(k)$, or $\phi_1(k)$. These sums can be computed using Lemma 3.4 in Appendix in [7].

2.4 Double contour integral representations

Here we present two double contour integral expressions for the hypergeometric–type kernel $\mathsf{K}_{\nu,\xi}$. Formulas of this type are useful in certain limit transitions, e.g., see [28, 9, 30]. To obtain double contour integral formulas for the correlation kernel $\mathsf{K}_{\nu,\xi}$, we write $\mathsf{K}_{\nu,\xi}$ as the sum (7) and use the contour integral representation for the hypergeometric function [9, Lemma 2.2] combined with the identity [12, 2.9(2)]. To shorten the notation, set $\mathrm{g}(x) := \frac{\sqrt{\Gamma(\frac12+\nu+x)\Gamma(\frac12-\nu+x)}}{\Gamma(\frac12+\nu+x)}$. Note that for our values of ν (see the end of §1.1) the expression under the square root is positive for all $x\in\mathbb{Z}$.

Proposition 3. *For all $x,y\in\mathbb{Z}_{>0}$ we have*

$$\begin{aligned}\frac{\mathrm{g}(x)}{\mathrm{g}(y)}\mathsf{K}_{\nu,\xi}(x,y) = {}& \frac{2\sqrt{xy}}{x+y}\frac{1}{(2\pi i)^2}\oint_{\{w_1\}}\oint_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac12+\nu}\left(1-\frac{\sqrt{\xi}}{w_1}\right)^{\frac12+\nu}\times\\ &\times\left(1-w_2\sqrt{\xi}\right)^{-\frac12-\nu}\left(1-\frac{\sqrt{\xi}}{w_2}\right)^{\frac12-\nu}\frac{w_1^{-x}w_2^{-y}}{w_1w_2-1}dw_1dw_2\\ &-\frac{\sqrt{xy}}{x+y}\frac{1-\xi}{(2\pi i)^2}\oint_{\{w_1\}}\oint_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac12+\nu}\left(1-\frac{\sqrt{\xi}}{w_1}\right)^{-\frac12+\nu}\times\\ &\times\left(1-w_2\sqrt{\xi}\right)^{-\frac12-\nu}\left(1-\frac{\sqrt{\xi}}{w_2}\right)^{-\frac12-\nu}\frac{dw_1dw_2}{w_1^{x+1}w_2^{y+1}}.\end{aligned}$$

The contours $\{w_1\}$ and $\{w_2\}$ go around 0 and $\sqrt{\xi}$ in positive direction leaving $1/\sqrt{\xi}$ outside. Moreover, in the first integral we have to impose an extra condition: the contour $\{w_1^{-1}\}$ lies in the interior of the contour $\{w_2\}$.

Proposition 4. *Let $x, y \in \mathbb{Z}_{>0}$. Then*

$$\frac{\mathsf{g}(-y)}{\mathsf{g}(-x)}\mathsf{K}_{\nu,\xi}(x,y) = \frac{\sqrt{xy}}{x+y}\frac{1-\xi}{(2\pi i)^2}\oint_{\{w_1\}}\oint_{\{w_2\}}\left(1-w_1\sqrt{\xi}\right)^{-\frac{1}{2}+\nu+x}\left(1-\frac{\sqrt{\xi}}{w_1}\right)^{-\frac{1}{2}+\nu-x}\times$$

$$\times\left(1-w_2\sqrt{\xi}\right)^{-\frac{1}{2}-\nu+y}\left(1-\frac{\sqrt{\xi}}{w_2}\right)^{-\frac{1}{2}-\nu-y} w_1^{-x}w_2^{-y}\frac{w_1w_2+1}{w_1w_2-1}\cdot\frac{dw_1dw_2}{w_1w_2}.$$

Here the contours $\{w_1\}$ and $\{w_2\}$ are as in the first integral in Proposition 3.

2.5 Poissonized Plancherel measure

The poissonized Plancherel measure Pl_θ defined by (2) gives rise to the point process $\mathbf{P}_\theta$ on $\mathbb{Z}_{>0}$, see §1.2. Denote the L–operator corresponding to $\mathbf{P}_\theta$ by L_θ (see §2.1). The operator L_θ is given by (5) with $\psi(x)$ replaced by $\psi_\theta(x) = \frac{\theta^x}{2(x!)^2}$.

Theorem 2.3. *The point process $\mathbf{P}_\theta$ is determinantal with the correlation kernel*

$$\mathsf{K}_\theta(x,y) = \frac{\sqrt{xy}}{x^2-y^2}\left(2\sqrt{\theta}J_{x-1}J_y - 2\sqrt{\theta}J_{y-1}J_x - (x-y)J_xJ_y\right), \qquad x,y\in\mathbb{Z}_{>0}.$$

Here $J_k = J_k(2\sqrt{\theta})$ is the Bessel function of the first kind.

The correlation kernel K_θ is similar to the *discrete Bessel kernel* from [19] and [5] (but note the appearance of additional summands in K_θ). The kernel K_θ is obtained from the hypergeometric–type kernel $\mathsf{K}_{\nu,\xi}$ via the Plancherel degeneration (§1.1). Moreover, one can check that $\mathsf{K}_\theta = \mathsf{L}_\theta(1+\mathsf{L}_\theta)^{-1}$ using the identities for the Bessel functions $J_k(2\sqrt{\theta})$ from §2 in [5].
The poissonized Plancherel measure is a special case of the shifted Schur measure introduced and studied in [37, 24]. In [24, §3] a Pfaffian formula for the correlation functions of the shifted Schur measure was obtained. This Pfaffian formula essentially coincides with the Plancherel degeneration of the formula from Proposition 2. Therefore as in §2.3 the Pfaffian formula from [24] turns into a determinantal formula from Theorem 2.3 above.

2.6 Schur–type measure

The measures $\mathsf{M}_{\nu,\xi}$ and Pl_θ defined in §1.1 can be included in a wider family of (complex-valued) measures on strict partitions (and, equivalently, on finite point configurations on $\mathbb{Z}_{>0}$). The latter measures were introduced in [32, §7]. They are similar to the Schur measure on ordinary partitions introduced in [27] and are defined as

$$\mathfrak{M}(\lambda) := \frac{1}{Z}\pi(s_{(\lambda|\lambda-1)}),$$

where λ is an arbitrary strict partition, $s_{(\lambda|\lambda-1)}$ is the Schur function indexed by the Young diagram written in Frobenius notation as $(\lambda_1,\dots,\lambda_{\ell(\lambda)} \mid \lambda_1-1,\dots,\lambda_{\ell(\lambda)}-1)$ (see [23, Ch. I, §1]), and Z is the normalizing constant. Here π is a *specialization* of the algebra of symmetric functions Λ (that is, a multiplicative homomorphism $\pi\colon\Lambda\to\mathbb{C}$) such that the series $Z=\sum_\lambda \pi(s_{(\lambda|\lambda-1)})$ converges.

The difference between $\mathfrak{M}$ and the Schur measure is that in $\mathfrak{M}$ we have only one Schur function instead of two functions for the Schur measure.
The probability measure $\mathsf{M}_{\nu,\xi}$ is obtained from $\mathfrak{M}$ if we take the specialization defined on the Newton power sums as

$$\pi(p_k) = (\nu + \tfrac{1}{2}) i^k \xi^{k/2}, \qquad k = 1, 2, \ldots. \tag{11}$$

Here $i = \sqrt{-1}$. Recall that the Newton power sums are algebraically independent generators of Λ. Though the specialization (11) is complex-valued, the values $\mathsf{M}_{\nu,\xi}(\lambda)$ are real positive for all strict partitions λ. The measure Pl_θ is obtained in the same way if we take the Plancherel degeneration (§1.1) of the specialization (11):

$$\pi(p_1) = \sqrt{\theta}, \qquad \pi(p_k) = 0, \quad k = 2, 3, \ldots.$$

For a wide class of "admissible" specializations Theorem 7.1 in Rains' paper [32] gives a determinantal formula for the measure $\mathfrak{M}$ viewed as a complex-valued measure on point configurations on $\mathbb{Z}_{>0}$. Denote by $\mathsf{K}_R(x,y)$ the correlation kernel [32, (7.6)] under the specialization (11). In contrast to our kernel $\mathsf{K}_{\nu,\xi}$ (§2.2), K_R is not symmetric. Numerical computations suggest the following relation between K_R and $\mathsf{K}_{\nu,\xi}$. Fix any $a \in \mathbb{Z}_{>0}$. Set $F(x) := (-1)^x \frac{\sqrt{\mathsf{K}_R(a,x)\mathsf{K}_R(x,a)}}{\mathsf{K}_R(x,a)}$ (the expression under the square root is real positive). Then $\mathsf{K}_{\nu,\xi}(x,y) = \frac{F(x)}{F(y)} \mathsf{K}_R(x,y)$ for all $x, y \in \mathbb{Z}_{>0}$. This relation between K_R and $\mathsf{K}_{\nu,\xi}$ is an instance of a so-called "gauge transformation" which does not change the correlation functions and hence the point process. However, we do not dispose of a rigorous proof of the above relation.

3 Limit transitions

Recall that the measures $\mathbf{P}_{\nu,\xi}$ defined in Introduction live on finite configurations on $\mathbb{Z}_{>0}$. As $\xi \nearrow 1$, the probability $\mathbf{P}_{\nu,\xi}(X)$ of every configuration $X \subset \mathbb{Z}_{>0}$ (given by (3)–(4)) tends to zero. However, it is possible to study limits of $\mathbf{P}_{\nu,\xi}$ as $\xi \nearrow 1$ in spaces larger than the space of finite configurations in $\mathbb{Z}_{>0}$. Here we consider two limit regimes described in §1.3.2.

3.1 Limit on the lattice

The space of all (possibly infinite) configurations on $\mathbb{Z}_{>0}$ can be identified with $\{0,1\}^{\mathbb{Z}_{>0}}$. This is a compact space and the point process $\mathbf{P}_{\nu,\xi}$ can be viewed as a probability measure on it.

Theorem 3.1. *As $\xi \nearrow 1$, there exists a weak limit of the measures $\mathbf{P}_{\nu,\xi}$ on the space $\{0,1\}^{\mathbb{Z}_{>0}}$. The limit point process on $\mathbb{Z}_{>0}$ is supported by infinite configurations and is determinantal with the kernel*

$$\mathsf{K}_\nu^{\mathrm{gamma}}(x,y) = \frac{\sqrt{xy} \cdot \mathrm{ctg}(\pi\nu)}{\pi\, \Xi(x,y)} \frac{\Gamma(\frac{1}{2}+\nu+x)\Gamma(\frac{1}{2}-\nu+y) - \Gamma(\frac{1}{2}+\nu+y)\Gamma(\frac{1}{2}-\nu+x)}{x^2 - y^2}.$$

Here $\Xi(x,y)$ is given by (6).

The proof of this theorem uses certain asymptotic relations for the hypergeometric function, cf. [8, §2]. Similar correlation kernels expressed in terms of the Euler gamma function have been studied in [8, 30]. However, it seems that there is no direct link between our point process (corresponding to $\mathsf{K}_\nu^{\mathrm{gamma}}$) and processes from [8].

3.2 Scaling limit and the Macdonald kernel

Consider embeddings of $\mathbb{Z}_{>0}$ into $\mathbb{R}_{>0}$, $x \mapsto u := x(1-\xi) \in \mathbb{R}_{>0}$, where $x \in \mathbb{Z}_{>0}$.

Theorem 3.2. *Under these embeddings, as $\xi \nearrow 1$, the point processes $\mathbf{P}_{\nu,\xi}$ converge to a determinantal point process $\mathscr{P}_\nu$ in $\mathbb{R}_{>0}$. The correlation kernel $\mathscr{K}_\nu$ of $\mathscr{P}_\nu$ can be expressed in terms of the Whittaker functions* (*see* [12, §6.9] *for definition*):

$$\mathscr{K}_\nu(u,v) = \frac{\cos(\pi\nu)}{\pi}\frac{2W_{1,\nu}(u)W_{0,\nu}(v) - 2W_{1,\nu}(v)W_{0,\nu}(u) - (u-v)W_{0,\nu}(u)W_{0,\nu}(v)}{u^2-v^2}, \tag{12}$$

and also in terms of the Macdonald functions (*see* [12, §7.2.2] *for definition*):

$$\mathscr{K}_\nu(u,v) = \frac{\sqrt{uv}\cos(\pi\nu)}{\pi^2}\frac{uK_{\nu+1}(\frac{u}{2})K_\nu(\frac{v}{2}) - vK_{\nu+1}(\frac{v}{2})K_\nu(\frac{u}{2})}{u^2-v^2}. \tag{13}$$

For generalities on point processes and correlation functions on continuous spaces, e.g., see the survey [34].

Formulas (12) and (13) are proved using the asymptotics [12, 6.8(1)] for the hypergeometric function: one should write the kernel $\mathsf{K}_{\nu,\xi}$ using formula (8) with $\mathsf{A}^{(1)}$ and $\mathsf{A}^{(2)}$, respectively. See also Theorem 5.4 in [7].

The kernel $\mathscr{K}_\nu$ is called the *Macdonald kernel*. Note that $\mathscr{K}_\nu$ is an integrable operator in the variables u^2 and v^2 (see also Remark 1). The Macdonald kernel has already appeared in [29, §5] and [22, §10.2]. Observe that the kernel [29, (5.3)] takes the form (13) if we choose the parameters $z_0 = \frac{1}{4} - \frac{\nu}{2}$, $z_0' = \frac{1}{4} + \frac{\nu}{2}$ and change the coordinates as $\xi = \frac{u^2}{16}$, $\eta = \frac{v^2}{16}$. Note that for our values of ν (§1.1) the parameters z_0 and z_0' are of principal or complementary series (e.g., see [10, §3.7] for definition).

Remark 4. There exists a simple connection between large n limit of the measures $\mathsf{M}_n^{(\alpha)}$ (see §1.1) and $\xi \nearrow 1$ limit of the point processes $\mathbf{P}_{\nu,\xi}$. These limits are related via the *lifting* construction described in [7, §5].

Remark 5. One can directly check that the kernels (12) and (13) are the same. To do this, one should express $W_{\kappa,\mu}(u)$ and $K_\nu(u)$ through the confluent hypergeometric function $\Psi(a,c;u)$ and use the identity for Ψ which follows from [12, 6.6(4)–(6)]:

$$\begin{aligned}\Psi(-\tfrac{1}{2}-\nu, 1-2\nu; u) - (\tfrac{1}{4}-\nu^2)\Psi(\tfrac{3}{2}-\nu, 1-2\nu; u)-&\\ -\Psi(-\tfrac{1}{2}-\nu, -1-2\nu; u) + (1+2\nu)\Psi(\tfrac{1}{2}-\nu, 1-2\nu; u) &= 0.\end{aligned}$$

Consider an integral operator in $L^2(\mathbb{R}_{>0})$ with the following kernel:

$$\mathscr{L}_\nu(u,v) := \frac{\cos(\pi\nu)}{\pi}\frac{e^{-\frac{u+v}{2}}}{u+v}, \qquad u,v \in \mathbb{R}_{>0}. \tag{14}$$

The operators $\mathscr{K}_\nu$ and $\mathscr{L}_\nu$ satisfy the operator relation $\mathscr{K}_\nu = \mathscr{L}_\nu(1+\mathscr{L}_\nu)^{-1}$ which is the same as the relation between the pre-limit operators $\mathsf{K}_{\nu,\xi}$ and $\mathsf{L}_{\nu,\xi}$ on the lattice (§2). However, the limit process $\mathscr{P}_\nu$ cannot be interpreted as an L–ensemble because it has infinite configurations almost surely.

Now let us present another description of the Macdonald kernel $\mathscr{K}_\nu$ (12)–(13). Namely, we interpret the operator $\mathscr{K}_\nu$ as a function (in operator calculus sense) of a Sturm–Liouville differential operator. Set $f_m(u) := \frac{1}{u}W_{0,im}(u)$, where $m \in [0,+\infty)$ is a parameter.

Proposition 5. (1) *The operator $\mathscr{K}_\nu$ commutes with the second order differential operator*

$$\mathfrak{D} = -\frac{d}{du}u^2\frac{d}{du} + \frac{1}{4}u^2.$$

That is, the Macdonald kernel $\mathscr{K}_\nu(u,v)$ satisfies $\mathfrak{D}_u\mathscr{K}_\nu(u,v) = \mathfrak{D}_v\mathscr{K}_\nu(u,v)$, where the subscript u or v indicates the variable on which the differential operator acts.
(2) *For every $m \geq 0$ we have*

$$\mathfrak{D}f_m = \left(m^2 + \frac{1}{4}\right)f_m, \qquad \mathscr{K}_\nu f_m = \frac{\cos(\pi\nu)}{\cos(\pi\nu) + \cosh(\pi m)}f_m.$$

Note that $\mathfrak{D}$ and $\mathscr{K}_\nu$ are self-adjoint in $L^2(\mathbb{R}_{>0})$. One can say that $\mathscr{K}_\nu = h_\nu(\mathfrak{D})$, where

$$h_\nu(r) := \frac{\cos(\pi\nu)}{\cos(\pi\nu) + \cosh\left(\pi\sqrt{r - \frac{1}{4}}\right)}, \qquad r \geq \frac{1}{4}.$$

Proposition 5 is suggested by results in [29]. Indeed, observe that the above operator $\mathscr{L}_\nu$ coincides with the operator A [29, (2.29)] if we set $a = 0$ and $\sigma = \cos(\pi\nu)$. Thus, from [29, §3] it follows that $\mathscr{L}_\nu$ commutes with $\mathfrak{D}$ and that $\mathscr{L}_\nu f_m = \frac{\cos(\pi\nu)}{\cosh(\pi m)}f_m$. This implies Proposition 5 because $\mathscr{K}_\nu = \mathscr{L}_\nu(1 + \mathscr{L}_\nu)^{-1}$.
The functions $\{f_m\}_{m\geq 0}$ form a continual basis in $L^2(\mathbb{R}_{>0})$, and an explicit Plancherel formula [29, (3.5)–(3.6)] (where one must set $a = 0$) holds. The operators $\mathscr{K}_\nu$ for our values of ν (see §1.1) form a commutative family. As $\nu \to i\infty$, the spectrum of $\mathscr{K}_\nu$ becomes closer to 1, and the norm of $\mathscr{L}_\nu$ tends to infinity.

Remark 6. There are certain formal relations between some of the operators considered in the present paper and the corresponding objects for the z-measures. More precisely, the pairs of corresponding objects are: the $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ matrix $\mathsf{L}_{\nu,\xi}$ from Remark 3 and the matrix [7, (3.3)]; the operator $\mathscr{L}_\nu$ in $L^2(\mathbb{R}_{>0})$ given by (14) and the operator [29, (2.28)–(2.30)]; the Macdonald kernel $\mathscr{K}_\nu$ from Theorem 3.2 and the matrix Whittaker kernel from [7, §5]. The relations between these objects are realized by taking *non-admissible* values of the parameters (z, z') of the z-measures, namely, $z = -z' = \nu - \frac{1}{2}$. Clearly, for our values of ν the parameters (z, z') are not of principal or complementary series. Therefore, it seems that there is no direct connection between our model and the z-measures at the level of random point processes.
Let us describe how the operator [29, (2.28)–(2.30)] is related to $\mathscr{L}_\nu$. If we set $z = -z' = \nu - \frac{1}{2}$, the parameter a in [29, (2.29)–(2.30)] vanishes, and the parameter $\sigma = \sqrt{\sin(\pi z)\sin(\pi z')}$ should be understood as $\sigma = i\cos(\pi\nu)$. We see that the operator [29, (2.28)] takes the form

$$\begin{bmatrix} 0 & i\mathscr{L}_\nu \\ -i\mathscr{L}_\nu & 0 \end{bmatrix}.$$

This fact implies that under the above choice of non-admissible values of (z, z') we have[1]

$$\mathscr{K}_\nu = \mathscr{K}_{++} - i\mathscr{K}_{+-}, \tag{15}$$

where $\mathscr{K}_{++}$ and $\mathscr{K}_{+-}$ are the blocks of the matrix Whittaker kernel, see [7, §5].
The pre-limit relation between $\mathsf{L}_{\nu,\xi}$ and the matrix [7, (3.3)] has a more complicated structure and involves the same non-admissible (z, z'). However, it seems that (15) does not have a pre-limit analogue, that is, there is no tractable relation between the pre-limit correlation kernel $\mathsf{K}_{\nu,\xi}$ and the matrix hypergeometric kernel from [7, §3].

[1] This formula was suggested to the author by A. Borodin in a private communication.

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