DOI: 10.1214/ECP.v15-1533

Elect. Comm. in Probab. **15** (2010), 106–118

ELECTRONIC COMMUNICATIONS in PROBABILITY

CONSISTENT MINIMAL DISPLACEMENT OF BRANCHING RANDOM WALKS

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Submitted December 7, 2009, *accepted in final form* March 12, 2010

AMS 2000 Subject classification: 60G50; 60J80
Keywords: Branching Random Walk; Consistent Minimal Displacement

Abstract
Let $\mathbb{T}$ denote a rooted b-ary tree and let $\{S_v\}_{v\in\mathbb{T}}$ denote a branching random walk indexed by the vertices of the tree, where the increments are i.i.d. and possess a logarithmic moment generating function $\Lambda(\cdot)$. Let m_n denote the minimum of the variables S_v over all vertices at the nth generation, denoted by $\mathbb{D}_n$. Under mild conditions, m_n/n converges almost surely to a constant, which for convenience may be taken to be 0. With $\bar{S}_v = \max\{S_w : w \text{ is on the geodesic connecting the root to } v\}$, define $L_n = \min_{v\in\mathbb{D}_n} \bar{S}_v$. We prove that $L_n/n^{1/3}$ converges almost surely to an explicit constant l_0. This answers a question of Hu and Shi.

1 Introduction

A branching random walk, as its name suggests, is a process describing a particle performing random walk while branching. In this paper, we consider the 1-dimensional case as follows. At time 0, there is one particle at location 0. At time 1, the particle splits into b particles ($b \in \mathbb{Z}_+$ deterministic and $b \geq 2$ to avoid trivial cases), each of which moves independently to a new position according to some distribution function $F(x)$. Then at time 2, each of the b particles splits again into b particles, which again move independently according to the distribution function $F(x)$. The splitting and moving continue at each integer time and are independent of each other. This procedure produces a 1-dimensional branching random walk.
To describe the relation between particles, we associate to each particle a vertex in a b-ary rooted tree $\mathbb{T} = \{V, E\}$ with root o, where each vertex has b children; V is the set of vertices in $\mathbb{T}$ and

[1]RESEARCH PARTIALLY SUPPORTED BY NSF GRANT DMS-0804133
[2]RESEARCH PARTIALLY SUPPORTED BY NSF GRANT DMS-0804133 AND BY THE ROBERT AND GIAMPIERO ALHADEFF RESEARCH AWARD AT THE WEIZMANN INSTITUTE.

E is the set of edges in $\mathbb{T}$. The root o is associated with the original particle. The b children of a vertex $v \in V$ correspond to the b particles from the splitting of the particle corresponding to v. In particular, the vertices whose distance from o is n, denoted by $\mathbb{D}_n$, correspond to particles at time n. To describe the displacement between particles, we assign i.i.d. random variables X_e with common distribution $F(x)$ to each edge $e \in E$. (Throughout, we let $e = uv$ denote the edge e connecting two vertices $u, v \in V$.) For each vertex $v \in V$, we use $|v|$ to denote its distance from o and use v^k to denote the ancestor of v in $\mathbb{D}_k$ for any $0 \le k \le |v|$. Then the positions of particles at time n can be described by $\{S_v | v \in \mathbb{D}_n\}$, where for $v \in \mathbb{D}_n$, $S_v = \sum_{i=0}^{n-1} X_{v^i v^{i+1}}$.
The limiting behavior of the maximal displacement $M_n = \max_{v\in\mathbb{D}_n} S_v$ or the minimal displacement $m_n = \min_{v\in\mathbb{D}_n} S_v$ as $n \to \infty$ has been extensively studied in the literature (See in particular Bramson [2],[3], Addario-Berry and Reed [1], and references therein.) Throughout this paper, we assume that

$$Ee^{\lambda X_e} < \infty \quad \text{for some } \lambda < 0 \text{ and some } \lambda > 0. \tag{1}$$

Then the Fenchel-Legendre transform of the log-moment generating function $\Lambda(\lambda) = \log Ee^{\lambda X_e}$,

$$\Lambda^*(x) = \sup_{\lambda\in\mathbb{R}}(\lambda x - \Lambda(\lambda)), \tag{2}$$

is the large deviation rate function (see [4, Ch. 1,2]) of a random walk with step distribution $F(x)$. In addition to (1), we also assume that, for some $\lambda_- < 0$ and $\lambda_+ > 0$ in the interior of $\{\lambda : \Lambda(\lambda) < \infty\}$,

$$\lambda_\pm \Lambda'(\lambda_\pm) - \Lambda(\lambda_\pm) = \log b, \tag{3}$$

which implies that $\Lambda^*(\Lambda'(\lambda_\pm)) = \log b$. These assumptions imply that

$$M := \lim_{n\to\infty} \frac{M_n}{n} = \Lambda'(\lambda_+) \quad \text{and} \quad m := \lim_{n\to\infty} \frac{m_n}{n} = \Lambda'(\lambda_-) \quad a.s.. \tag{4}$$

See [1] for more details on (4).
The *offset* of the branching random walk is defined as the minimal deviation of the path up to time n from the line leading to mn (roughly, the minimal position at time n). Explicitly, set

$$L_n = \min_{v\in\mathbb{D}_n} \max_{k=0}^{n} (S_{v^k} - mk). \tag{5}$$

(see Figure 1 for a pictorial description of L_3.) Without loss of generality, subtracting the deterministic constant $\Lambda'(\lambda_-)$ from each increment $\{X_e\}$, we can and will assume that

$$m = \Lambda'(\lambda_-) = 0. \tag{6}$$

Under this assumption, (3) and (5) simplify to

$$-\Lambda(\lambda_-) = \log b, \tag{3'}$$

$$L_n = \min_{v\in\mathbb{D}_n} \max_{k=0}^{n} S_{v^k}. \tag{5'}$$

In the process of studying random walks in random environments on trees, Hu and Shi [5] (2007) discovered that the offset has order $n^{1/3}$ in the following sense: there exist constants $c_1, c_2 > 0$ such that

$$c_1 \le \liminf_{n\to\infty} \frac{L_n}{n^{1/3}} \le \limsup_{n\to\infty} \frac{L_n}{n^{1/3}} \le c_2. \tag{7}$$

They raised and advertised the question as to whether the limit of $L_n/n^{1/3}$ exists. In this note, we answer this affirmatively and prove the following.

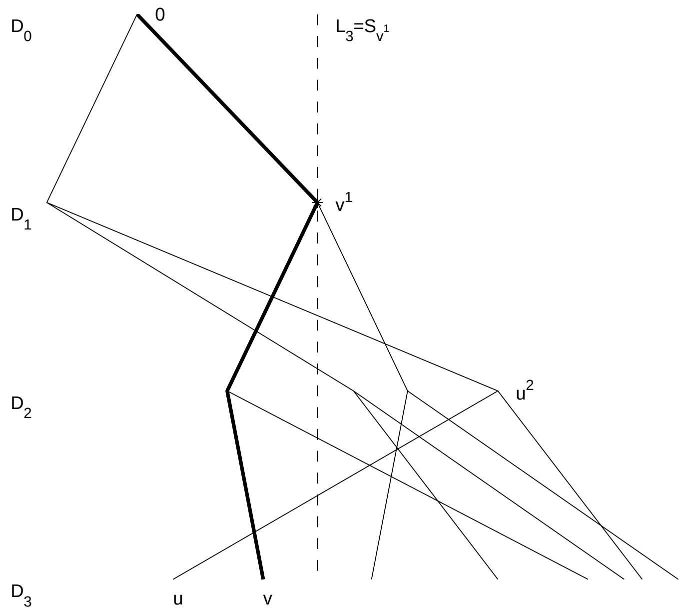

Figure 1: Figure for L_3 when m=0 and b=2

Theorem 1. *Under assumption (1) and (3) and with* $l_0 = \sqrt[3]{\frac{3\pi^2\sigma_Q^2}{-2\lambda_-}}$*, it holds that*

$$\lim_{n\to\infty} \frac{L_n}{n^{1/3}} = l_0 \quad a.s.. \tag{8}$$

In the expression for l_0, $\lambda_- < 0$ by the definition (3) and σ_Q^2 is a certain variance defined in (10). The proof of the theorem is divided into two parts - the lower bound (21) and the upper bound (33). In Section 2, we review a result from Mogul'skii [7], which will be the key estimate in our proof. In Section 3, we apply a first moment argument (with a twist) in order to study the minimal positions for intermediate levels with the restriction that the walks do not exceed $ln^{1/3}$ for some $l > 0$ at all time. This yields the lower bound for L_n. In section 4, we apply a second moment argument to lower bound $P(L_n \le ln^{1/3})$ for certain values of l. Compared with standard applications of the second moment method in related problems, the analysis here requires the control of second order terms in the large deviation estimates. Truncation of the tree is then used to get independence and complete the proof of the upper bound.

The offset is determined by a competition between two terms: a displacement term (whose cost is exponential in the displacement) and an entropy term (reflecting the difficulty in keeping the walk confined in a narrow tube, and with cost proportional to the exponent of the time divided by width squared; this is made precise in Theorem 2). Roughly speaking, in a time interval of length Δt, and displacement width Δl, the cost is of the form $e^{c_1\Delta l - c_2\Delta t/(\Delta l)^2}$. One then sees that the optimum is achieved at Δl proportional to $(\Delta t)^{1/3}$. This gives the scaling on $n^{1/3}$ to the displacement. In the actual proof, when optimizing the cost, a certain curve $s(t)$, see (17), emerges. The curve $s(t)$ reflects the location of the minimal position of intermediate levels, and plays an important role also in the second moment computation, see a discussion in Section 5.1.

Acknowledgement

While this work was being completed, we learnt that as part of their study of RWRE on trees, G. Faraud, Y. Hu and Z. Shi had independently obtained Theorem 1, using a related but slightly different method [6]. In particular, their work handles also the case of Galton–Watson trees. We thank Y. Hu for discussing this problem with one of us (O.Z.) and for providing us with the reference [7], which allowed us to skip tedious details in our original proof.

2 An Auxiliary Estimate: the absorption problem for random walk

We derive in this section some estimates for random walk with i.i.d. increments $\{X_i\}_{i\geq 1}$ distributed according to a law P with $P((-\infty,x])=F(x)$ satisfying (1), (3) and (6). Define

$$S_n(t)=\frac{X_0+X_1+\cdots+X_k}{n^{1/3}} \quad \text{for } \frac{k}{n}\leq t<\frac{k+1}{n},\ k=0,1,\ldots,n-1,$$

where $X_0=0$. Note that due to (6), $EX_i>0$. Introduce the auxiliary law

$$\frac{dQ}{dP}=e^{\lambda_- X_1-\Lambda(\lambda_-)}. \tag{9}$$

Under Q, $E_Q X_1=0$. The variance of X_1 under Q is denoted by

$$\sigma_Q^2=E_Q X_1^2. \tag{10}$$

In the following estimates, $f_1(t)$ and $f_2(t)$, which may take the value $\pm\infty$, are right-continuous and piecewise constant functions on $[0,1]$. $G=\cup_{0\leq t\leq 1}\{(f_1(t),f_2(t))\times t\}$ is a region bounded by $f_1(t)$ and $f_2(t)$. Assume also that G contains the graph of a continuous function.

Theorem 2. *(Mogul'skii [7, Theorem 3]) Under the above assumptions,*

$$Q(S_n(t)\in G, t\in[0,1])=e^{-\frac{\pi^2\sigma_Q^2}{2}H_2(G)n^{1/3}+o(n^{1/3})}, \tag{11}$$

where

$$H_2(G)=\int_0^1\frac{1}{(f_1(t)-f_2(t))^2}dt. \tag{12}$$

In the following, we will need to control the dependence of the estimate (11) on the starting point.

Corollary 1. *With notation and assumptions as in Theorem 2, for any $\epsilon>0$, there is a $\delta>0$ such that, for any interval $I\subset(f_1(0),f_2(0))$ with length $|I|\leq\delta$, we have*

$$\sup_{x\in I}Q(x+S_n(\cdot)\in G)\leq e^{-(\frac{\pi^2\sigma_Q^2}{2}H_2(G)-\epsilon)n^{1/3}+o(n^{1/3})}. \tag{13}$$

Proof Let $I=(a,b)$ and $G_x:=\cup_{0\leq t\leq 1}\{(f_1(t)-x,f_2(t)-x)\times t\}$ be the shift of G by x. Set $G'=G_a\cup G_b$. We have

$$\sup_{x\in I}Q(x+S_n(\cdot)\in G)=\sup_{x\in I}Q(S_n(\cdot)\in G_x)\leq Q(S_n(\cdot)\in G')=e^{-\frac{\pi^2\sigma_Q^2}{2}H_2(G')n^{1/3}+o(n^{1/3})}.$$

Since $H_2(G')=\int_0^1\frac{1}{(f_2(t)-f_1(t)+(b-a))^2}dt\uparrow H_2(G)$ as $|I|=(b-a)\to 0$ uniformly in the position of I, the lemma is proved. □

3 Lower Bound

Consider the branching random walk up to level n. In this and the next section, we estimate the number of particles that stay constantly below $ln^{1/3}$, i.e.,

$$N_n^l = \sum_{v \in \mathbb{D}_n} 1_{\{S_{v_k} \le ln^{1/3} \text{ for } k=0,1,\ldots,n\}}. \tag{14}$$

In order to get a lower bound on the offset, we apply a first moment method with a small twist: while it is natural to just calculate the first moment of N_n^l, such a computation ignores the constraint on the number of particles at level k imposed by the tree structure. In particular, EN_n^l for branching random walks is the same as the one for b^n independent random walks. An easy first and second moment argument shows that the limit in (8) is 0 for b^n independent random walks, and thus no useful upper bound can be derived in this way.
To address this issue, we use a more delicate first moment argument. Namely, we look at the vertices not only at level n but also at some intermediate levels. Divide the interval $[0, n]$ into $1/\epsilon$ equidistant levels, with $1/\epsilon$ an integer. Define recursively, for any $\delta > 0$,

$$\begin{cases} s_0 = 0, w_0 = l + \delta; \\ s_k = s_{k-1} - \frac{\pi^2 \sigma_Q^2}{2\lambda_- w_{k-1}^2}\epsilon, \ w_k = l + \delta - s_k \text{ for } k = 1, \ldots, \frac{1}{\epsilon}. \end{cases} \tag{15}$$

For particles staying below $ln^{1/3}$, s_k will be interpreted as values such that the walks between times $k\epsilon n$ and $(k+1)\epsilon n$ never go below $(s_k - \delta)n^{1/3}$, and $w_k n^{1/3}$ will correspond to the width of the window $W_k = ((s_k - \delta)n^{1/3}, ln^{1/3})$ that we allow between level $k\epsilon n$ and $(k+1)\epsilon n$, when considering those walks that do not go below $(s_k - \delta)n^{1/3}$ or go above $ln^{1/3}$.
Before calculating the first moment, consider the recursion (15) for s_k. Rewrite it as

$$s_k = s_{k-1} - \frac{\pi^2 \sigma_Q^2}{2\lambda_-(l + \delta - s_{k-1})^2}\epsilon. \tag{16}$$

This is an Euler's approximation sequence for the solution of the following differential equation

$$s'(t) = -\frac{\pi^2 \sigma_Q^2}{2\lambda_-(\alpha - s(t))^2}, \ s(0) = 0, \tag{17}$$

where $\alpha = l + \delta$. The above initial value problem has the solution $s^\alpha(t) = \alpha + \sqrt[3]{-\frac{3\pi^2\sigma_Q^2}{2\lambda_-}t - \alpha^3}$.
Here we find

$$l_0 = \sqrt[3]{\frac{3\pi^2 \sigma_Q^2}{-2\lambda_-}} \tag{18}$$

such that $s^{l_0}(1) = l_0$. For any $ł_1 < l_0$, we can choose $\delta > 0$ and $l_1 + \delta < l_0$. In this case, $s^{l_1+\delta}(1) > l_1 + \delta > l_1$. If we choose such l_1 and δ in (15), it is easy to check that the sequence $\{s_k\}_{k=0}^{\frac{1}{\epsilon}}$ will be greater than l_1 somewhere in the sequence. Define

$$K = \min\{k : s_k \ge l_1\}. \tag{19}$$

For fixed $\gamma > 0$ small enough, we can choose ϵ small such that

$$K\epsilon < 1 - \gamma. \tag{20}$$

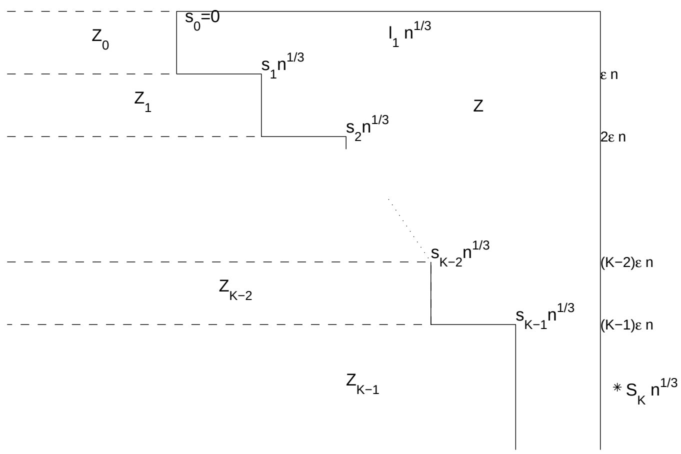

Figure 2: The relation between Z_k's and s_k's.

For $k < K-1$, let Z_k denote the number of vertices v between level $k\epsilon n$ and $(k+1)\epsilon n$ with $S_v < (s_k - \delta)n^{1/3}$. Denote by Z_{K-1} the number of vertices w between level $(K-1)\epsilon n$ and n with $S_w < (s_{K-1} - \delta)n^{1/3}$. Denote by Z the number vertices $v \in \mathbb{D}_n$ whose associated walks stay in W_k between level $k\epsilon n$ and $(k+1)\epsilon n$ for $k < K$ and then stay in W_{K-1} up to level n. Explicitly,

$$Z_0 = \sum_{i=1}^{\lfloor \epsilon n \rfloor} \sum_{v \in \mathbb{D}_i} 1_{\{S_v < -\delta n^{1/3}\}},$$

$$Z_k = \sum_{i=\lfloor k\epsilon n \rfloor+1}^{\lfloor (k+1)\epsilon n \rfloor} \sum_{v \in \mathbb{D}_i} 1_{\{S_v < (s_k - \delta)n^{1/3},\ S_{v_d} \in W_j \text{ for } j\epsilon n \le d \le (j+1)\epsilon n \text{ and } j<k\}},\ 0 < k < K-1,$$

$$Z_{K-1} = \sum_{i=\lfloor K\epsilon n \rfloor+1}^{n} \sum_{v \in \mathbb{D}_i} 1_{\{S_v < (s_{K-1} - \delta)n^{1/3},\ S_{v_d} \in W_j \text{ for } j\epsilon n \le d \le (j+1)\epsilon n \text{ and } j<K-1\}},$$

$$Z = \sum_{v \in \mathbb{D}_n} 1_{\{S_{v_d} \in W_j \text{ for } j\epsilon n \le d \le (j+1)\epsilon n \text{ and } j<K,\ S_{v_d} \in W_{K-1} \text{ for } K\epsilon n \le d \le n\}}.$$

Observe that $N_n^{l_1} \le \sum_{k=0}^{K-1} Z_k + Z$. Using Theorem 2, we provide upper bounds for the first moment of the Z_ks and Z. Starting with Z_0, we have

$$\begin{aligned} EZ_0 &= \sum_{i=1}^{\lfloor \epsilon n \rfloor} b^i E 1_{\{S_i < -\delta n^{1/3}\}} = \sum_{i=1}^{\lfloor \epsilon n \rfloor} b^i E_Q e^{-\lambda_- S_i + i\Lambda(\lambda_-)} 1_{\{S_i < -\delta n^{1/3}\}} \\ &\le \sum_{i=1}^{\lfloor \epsilon n \rfloor} e^{\lambda_- \delta n^{1/3}} E_Q 1_{\{S_i < -\delta n^{1/3}\}} \le \sum_{i=1}^{\lfloor \epsilon n \rfloor} e^{\lambda_- \delta n^{1/3}} \le e^{\lambda_- \delta n^{1/3} + o(n^{1/3})}, \end{aligned}$$

where we used the change of measure (9) in the second equality, and (3′) and the fact that $\lambda_- < 0$

in the first inequality. For $0 < k < K-1$, using again the change of measure (9), we get

$$
\begin{aligned}
EZ_k &= \sum_{i=\lfloor k\epsilon n\rfloor+1}^{\lfloor (k+1)\epsilon n\rfloor} b^i E 1_{\{S_i<(s_k-\delta)n^{1/3},\ S_d\in W_j \text{ for } j\epsilon n\le d\le (j+1)\epsilon n \text{ and } j<k\}} \\
&= \sum_{i=\lfloor k\epsilon n\rfloor+1}^{\lfloor (k+1)\epsilon n\rfloor} E_Q e^{-\lambda_- S_i} 1_{\{S_i<(s_k-\delta)n^{1/3},\ S_d\in W_j \text{ for } j\epsilon n\le d\le (j+1)\epsilon n \text{ and } j<k\}} \\
&\le e^{-\lambda_-(s_k-\delta)n^{1/3}} \sum_{i=\lfloor k\epsilon n\rfloor+1}^{\lfloor (k+1)\epsilon n\rfloor} E_Q 1_{\{S_i<(s_k-\delta)n^{1/3},\ S_d\in W_j \text{ for } j\epsilon n\le d\le (j+1)\epsilon n \text{ and } j<k\}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
EZ_k &\le e^{-\lambda_-(s_k-\delta)n^{1/3}} \sum_{i=\lfloor k\epsilon n\rfloor+1}^{\lfloor (k+1)\epsilon n i\rfloor} Q(S_d\in W_j \text{ for } j\epsilon n\le d\le (j+1)\epsilon n \text{ and } j<k) \\
&= e^{-\lambda_-(s_k-\delta)n^{1/3}} \sum_{i=\lfloor k\epsilon n\rfloor+1}^{\lfloor (k+1)\epsilon n\rfloor} e^{-\sum_{j=0}^{k-1} \frac{\pi^2\sigma_Q^2}{2w_j^2}\epsilon n^{1/3}+o(n^{1/3})} \\
&\le e^{-\lambda_-(s_k-\delta)n^{1/3}-\sum_{j=0}^{k-1} \frac{\pi^2\sigma_Q^2}{2w_j^2}\epsilon n^{1/3}+o(n^{1/3})} = e^{\lambda_-\delta n^{1/3}+o(n^{1/3})},
\end{aligned}
$$

where (11) with the choice of $G = \{\cup_{j=0}^{k-1} W_j/n^{1/3}\times[j\epsilon,(j+1)\epsilon)\}\cup\{(-\infty,\infty)\times[k\epsilon,1]\}$ is applied in the first equality, and (15) in the second. The calculation of EZ_{K-1} is almost the same as EZ_k except that we replace the summation limits above by $(K-1)\epsilon n+1$ and n and that we replace the k in the summand by $K-1$. Thus, we get the same upper bound for EZ_{K-1},

$$
EZ_{K-1} \le e^{\lambda_-\delta n^{1/3}+o(n^{1/3})}.
$$

We estimate EZ similarly as follows. First, use the change of measure (9) to get

$$
\begin{aligned}
EZ &= b^n E 1_{\{S_d\in W_j \text{ for } j\epsilon n\le d\le (j+1)\epsilon n \text{ and } j<K,\ S_d\in W_{K-1}, \text{ for } K\epsilon n\le d\le n\}} \\
&= E_Q e^{-\lambda_- S_n} 1_{\{S_d\in W_j \text{ for } j\epsilon n\le d\le (j+1)\epsilon n \text{ and } j<K,\ S_d\in W_{K-1}, \text{ for } K\epsilon n\le d\le n\}} \\
&\le e^{-\lambda_- l_1 n^{1/3}} E_Q 1_{\{S_d\in W_j \text{ for } j\epsilon n\le d\le (j+1)\epsilon n \text{ and } j<K,\ S_d\in W_{K-1}, \text{ for } K\epsilon n\le d\le n\}}.
\end{aligned}
$$

Then, applying (11) with $G = \{\cup_{j=0}^{K-1} W_j/n^{1/3}\times[j\epsilon,(j+1)\epsilon)\}\cup\{W_{K-1}/n^{1/3}\times[K\epsilon,1]\}$ in the first equality, we get

$$
\begin{aligned}
EZ &\le e^{-\lambda_- l_1 n^{1/3}} E_Q 1_{\{S_d\in W_j \text{ for } j\epsilon n\le d\le (j+1)\epsilon n \text{ and } j<K,\ S_d\in W_{K-1}, \text{ for } K\epsilon n\le d\le n\}} \\
&= e^{-\lambda_- l_1 n^{1/3}-\sum_{i=0}^{K-1}\frac{\pi^2\sigma_Q^2}{2w_i^2}\epsilon n^{1/3}-\frac{\pi^2\sigma_Q^2}{2w_{K-1}^2}(1-K\epsilon)n^{1/3}+o(n^{1/3})} \\
&\le e^{-\gamma\frac{\pi^2\sigma_Q^2}{2l_1^2}n^{1/3}+o(n^{1/3})},
\end{aligned}
$$

where the last inequality is obtained by noting that $l_1 \le S_K = -\sum_{i=0}^{K-1}\frac{\pi^2\sigma_Q^2}{2\lambda_- w_i^2}\epsilon$ by (19) and (15), and then recalling (20) and $w_{K-1} < l_1$.

In conclusion, we proved that $E(\sum_{k=0}^{K-1} Z_k+Z) \le e^{-c_3 n^{1/3}+o(n^{1/3})}$ for some $0 < c_3 < \min\{-\lambda_-\delta, \gamma\frac{\pi^2\sigma_Q^2}{2l_1^2}\}$. Since $\sum_{k=0}^{K-1} Z_k + Z$ is an integer valued random variable, we have

$$P(\sum_{k=0}^{K-1} Z_k + Z > 0) = P(\sum_{k=0}^{K-1} Z_k + Z \ge 1) \le E(\sum_{k=0}^{K-1} Z_k + Z) \le e^{-c_3 n^{1/3}+o(n^{1/3})}.$$

By the Borel-Cantelli lemma, we have $\sum_{k=0}^{K-1} Z_k + Z = 0$ a.s. for all large n. So is $N_n^{l_1} = 0$, which means that $L_n > l_1 n^{1/3}$ a.s. for all large n. Since $l_1 < l_0$ is arbitrary, we conclude that

$$\liminf_{n\to\infty} \frac{L_n}{n^{1/3}} \ge l_0 \quad a.s.. \tag{21}$$

This completes the proof of the lower bound in Theorem 1.

4 Upper Bound

4.1 A Second Moment Method Estimate

In this section, we consider any fixed $l_2 > l_0$. A second moment argument will provide a lower bound for the probability that we can find at least one walk which stays in the interval W_k between level $k\epsilon n$ and $(k+1)\epsilon n$ for all k. A truncation (of the tree) argument will complete the proof of the upper bound.

As a first step, consider the sequence $\{s_k\}$ in (15) with $l_2 > l_0$. Then for any $\delta > 0$, it is easy to see that $s^{l_2+\delta}(t)$ is increasing and convex for $0 \le t \le 1$. Thus in Euler's approximation,

$$s_{\frac{1}{\epsilon}} < s^{l_2+\delta}(1) < s^{l_2}(1) < l_2. \tag{22}$$

It follows from (15) that

$$w_k \ge \delta \quad \text{for all } 0 \le k \le \frac{1}{\epsilon} - 1. \tag{23}$$

Define $\tilde{N}_n^{l_2}$ as follows.

$$\tilde{N}_n^{l_2} = \sum_{v\in\mathbb{D}_n} 1_{\{S_{v^j}\in W_k,\ \text{for } k\epsilon n\le j\le(k+1)\epsilon n,\ k=0,\ldots,\frac{1}{\epsilon}-1\}}.$$

We will apply second moment method to $\tilde{N}_n^{l_2}$. $E\tilde{N}_n^{l_2}$ is calculated the same way as EZ in the previous section. But this time we consider $G = \{\cup_{j=0}^{\frac{1}{\epsilon}-1} W_j/n^{1/3} \times [j\epsilon,(j+1)\epsilon)\} \cup \{(l_2-\Delta l_2, l_2)\times\{1\}\}$ in (11) with $\Delta l_2 \to 0$, so

$$\begin{aligned} E\tilde{N}_n^{l_2} &= b^n E 1_{\{S_j\in W_k,\ \text{for } k\epsilon n\le j\le(k+1)\epsilon n,\ k=0,\ldots,\frac{1}{\epsilon}-1\}} \\ &= E_Q e^{-\lambda_- S_n} 1_{\{S_j\in W_k,\ \text{for } k\epsilon n\le j\le(k+1)\epsilon n,\ k=0,\ldots,\frac{1}{\epsilon}-1\}} \\ &= e^{(-\lambda_- l_2 - \sum_{k=0}^{\frac{1}{\epsilon}-1}\frac{\pi^2\sigma_Q^2}{2w_k^2}\epsilon)n^{1/3}+o(n^{1/3})}. \end{aligned} \tag{24}$$

From (22) and the definition (15) of s_k, $-\lambda_- l_2 - \sum_{k=0}^{\frac{1}{\epsilon}-1}\frac{\pi^2\sigma_Q^2}{2w_k^2}\epsilon > 0$ and thus $E\tilde{N}_n^{l_2} \to \infty$. Therefore, we will be ready to apply the second moment method after the following calculations.

$$\begin{aligned} E(\tilde{N}_n^{l_2})^2 &= E\sum_{u,v\in\mathbb{D}_n} 1_{\{S_{u^j},S_{v^j}\in W_k,\ \text{for } k\epsilon n\le j\le (k+1)\epsilon n,\ k=0,\dots,\frac{1}{\epsilon}-1\}} \\ &= \sum_{h=0}^{n-1} E\sum_{\substack{u,v\in\mathbb{D}_n\\ u\wedge v\in\mathbb{D}_h}} 1_{\{S_{u^j},S_{v^j}\in W_k,\ \text{for } k\epsilon n\le j\le (k+1)\epsilon n,\ k=0,\dots,\frac{1}{\epsilon}-1\}} + E\tilde{N}_n^{l_2}. \end{aligned} \tag{25}$$

In the last expression above, $u\wedge v$ is the largest common ancestor of u and v. Write $h=q\epsilon n+r$ for $0\le q\le \frac{1}{\epsilon}-1$ and $0\le r<\epsilon n$. There are $b^{2n-h-1}(b-1)$ indices in the second sum in the right side of (25). We estimate the probability for one such pair to stay in W_k's. In order to simplify the notation, define

$$p_1(0,h,x)=P(S_h\in dx,\ S_j\in W_k,\ \text{for } k\epsilon n\le j\le (k+1)\epsilon n\wedge h,\ k=0,\dots,q),$$

$$p_2(h,x,n,y)=P(S_n\in dy,\ S_j\in W_k,\ \text{for } h\vee k\epsilon n\le j\le (k+1)\epsilon n,\ k=q,\dots,n|S_h=x).$$

Similarly, define $q_1(0,h,x)$ and $q_2(h,x,n,y)$ to be the probability of the same events under Q. Then we have

$$\begin{aligned} E(\tilde{N}_n^{l_2})^2 &= E\tilde{N}_n^{l_2}+\sum_{h=0}^{n-1} b^{2n-h-1}(b-1)\int_{W_q}(\int_{W_n} p_2(h,x,n,y)dy)^2 p_1(0,h,x)dx \\ &= E\tilde{N}_n^{l_2}+\sum_{h=0}^{n-1} b^{2n-h-1}(b-1)\int_{W_q}(\int_{W_n} e^{-\lambda_-(y-x)+(n-h)\Lambda(\lambda_-)}q_2(h,x,n,y)dy)^2 \\ &\qquad \cdot e^{-\lambda_- x+h\Lambda(\lambda_-)}q_1(0,h,x)dx \\ &\le E\tilde{N}_n^{l_2}+\sum_{h=0}^{n-1}\frac{b-1}{b}e^{(-2\lambda_- l_2+\lambda_-(s_q-\delta))n^{1/3}} \\ &\qquad \cdot\int_{W_q}(\int_{W_n} q_2(h,x,n,y)dy)^2 q_1(0,h,x)dx. \end{aligned} \tag{26}$$

We now provide an upper bound for the integral term in the right side of (26). We have

$$\begin{aligned}
&\int_{W_q}\Big(\int_{W_n} q_2(h,x,n,y)dy\Big)^2 q_1(0,h,x)dx \\
\leq\ & \Big(\sup_{x\in W_q}\int_{W_n} q_2(h,x,n,y)dy\Big)^2\int_{W_q} q_1(0,h,x)dx \\
\leq\ & \Big(\sup_{x\in W_q}\int_{W_n}\int_{W_{q+1}} q_2(h,x,(q+1)\epsilon n,z)q_2((q+1)\epsilon n,z,n,y)dzdy\Big)^2\int_{W_q} q_1(0,q\epsilon n,x)dx \\
\leq\ & \Big(\sup_{x\in W_q}\int_{W_{q+1}} q_2(h,x,(q+1)\epsilon n,z)q_2((q+1)\epsilon n,z,n,W_n)dz\Big)^2 e^{-\sum_{k=0}^{q-1}\frac{\pi^2\sigma_Q^2}{2w_k^2}n^{1/3}+o(n^{1/3})} \\
\leq\ & \Big(\sup_{x\in W_q}\sum_i\int_{I_i} q_2(h,x,(q+1)\epsilon n,z)q_2((q+1)\epsilon n,z,n,W_n)dz\Big)^2 e^{-\sum_{k=0}^{q-1}\frac{\pi^2\sigma_Q^2}{2w_k^2}n^{1/3}+o(n^{1/3})} \\
\leq\ & \Big(\sum_i\sup_{z\in I_i} q_2((q+1)\epsilon n,z,n,W_n)\Big)^2 e^{-\sum_{k=0}^{q-1}\frac{\pi^2\sigma_Q^2}{2w_k^2}\epsilon n^{1/3}+o(n^{1/3})}. \qquad (27)
\end{aligned}$$

In the above, $\cup_i I_i = W_{q+1}$. Due to (13), for any small $\epsilon_1 > 0$, we can choose a finite number of I_is and $|I_i| \leq \delta_1 n^{1/3}$ such that for each i,

$$\sup_{z\in I_i}(q_2((q+1)\epsilon n,z,n,W_n) \leq e^{-(\sum_{k=q+1}^{\frac{1}{\epsilon}-1}\frac{\pi^2\sigma_Q^2}{2w_k^2}\epsilon-\epsilon_1)n^{1/3}+o(n^{1/3})}.$$

After splitting $\sum_{h=0}^{n-1}$ to $\sum_{q=0}^{\frac{1}{\epsilon}-1}\sum_{r=0}^{\epsilon n-1}$ in (26), we obtain the upper bound for $E(\tilde{N}_n^{l_2})^2$ as follows,

$$\begin{aligned}
E(\tilde{N}_n^{l_2})^2 \ &\leq\ E\tilde{N}_n^{l_2} + \sum_{q=0}^{1/\epsilon-1} e^{(-2\lambda_-l_2+\lambda_-(s_q-\delta))n^{1/3}-\sum_{k=0}^{q-1}\frac{\pi^2\sigma_Q^2}{2w_k^2}\epsilon n^{1/3}-2(\sum_{k=q+1}^{\frac{1}{\epsilon}-1}\frac{\pi^2\sigma_Q^2}{2w_k^2}\epsilon-\epsilon_1)n^{1/3}+o(n^{1/3})} \\
&\leq\ \sum_{q=0}^{1/\epsilon-1} e^{(-2\lambda_-l_2+\lambda_-(s_q-\delta))n^{1/3}-\sum_{k=0}^{q-1}\frac{\pi^2\sigma_Q^2}{2w_k^2}\epsilon n^{1/3}-2(\sum_{k=q+1}^{\frac{1}{\epsilon}-1}\frac{\pi^2\sigma_Q^2}{2w_k^2}\epsilon-\epsilon_1)n^{1/3}+o(n^{1/3})}. \qquad (28)
\end{aligned}$$

With the bounds for $E\tilde{N}_n^{l_2}$ (24) and $E(\tilde{N}_n^{l_2})^2$ (28), we have

$$\begin{aligned}
P(\tilde{N}_n^{l_2} > 0) \ &\geq\ \frac{(E\tilde{N}_n^{l_2})^2}{E(\tilde{N}_n^{l_2})^2} \geq \frac{1}{\sum_{q=0}^{1/\epsilon-1} e^{(\lambda_-(s_q-\delta)+\sum_{k=0}^{q-1}\frac{\pi^2\sigma_Q^2}{2w_k^2}\epsilon+2\frac{\pi^2\sigma_Q^2}{2w_q^2}\epsilon+2\epsilon_1)n^{1/3}+o(n^{1/3})}} \\
&=\ \frac{1}{\sum_{q=0}^{1/\epsilon-1} e^{(-\lambda_-\delta+\frac{\pi^2\sigma_Q^2}{w_q^2}\epsilon+2\epsilon_1)n^{1/3}+o(n^{1/3})}} \geq e^{(\lambda_-\delta-\frac{\pi^2\sigma_Q^2}{\delta^2}\epsilon-2\epsilon_1)n^{1/3}+o(n^{1/3})} \\
&=\ e^{-\epsilon_2 n^{1/3}+o(n^{1/3})}, \qquad (29)
\end{aligned}$$

where $\epsilon_2 := -\lambda_-\delta + \frac{\pi^2\sigma_Q^2}{\delta^2}\epsilon + 2\epsilon_1$, and we use (15) in the first equality and $w_q \geq \delta$ (see (23)) in the last inequality. We can make ϵ_2 arbitrarily small by first choosing δ small then choosing ϵ and ϵ_1 small. Therefore, we get

$$P(L_n \leq l_2 n^{1/3}) \geq P(\tilde{N}_n^{l_2} > 0) \geq e^{-\epsilon_2 n^{1/3}+o(n^{1/3})}. \qquad (30)$$

4.2 A Truncation Argument

In view of the lower bound (30), we truncate the tree at level $\lfloor \epsilon_3 n^{1/3} \rfloor = \lfloor 2\epsilon_2 n^{1/3}/\log b \rfloor$ to get $b^{\lfloor \epsilon_3 n^{1/3} \rfloor} \geq e^{2\epsilon_2 n^{1/3}}/b$ independent branching random walks. We take care of the path before and after level $\lfloor \epsilon_3 n^{1/3} \rfloor$ separately.

Define L_n^v similarly as L_n for each branching random walk starting from $v \in \mathbb{D}_{\lfloor \epsilon_3 n^{1/3} \rfloor}$, i.e., letting $z = \lfloor \epsilon_3 n^{1/3} \rfloor$,

$$L_n^v = \min_{u \in \mathbb{D}_{z+n}, u^z = v} \max_{k=z}^{z+n} (S_{u^k} - S_v).$$

Then

$$\begin{aligned} P(L_n^v > l_2 n^{1/3} \text{ for every } v) &= (1 - P(L_n \leq l_2 n^{1/3}))^{b^{\lfloor \epsilon_3 n^{1/3} \rfloor}} \\ &\leq (1 - e^{-\epsilon_2 n^{1/3} + o(n^{1/3})})^{e^{2\epsilon_2 n^{1/3}}/b} \leq e^{-e^{\epsilon_2 n^{1/3} + o(n^{1/3})}}, \end{aligned} \tag{31}$$

when n is large. By the Borel-Cantelli lemma, the above double exponential guarantees that almost surely for all large n, there exists a $v \in \mathbb{D}_{\lfloor \epsilon_3 n^{1/3} \rfloor}$ such that

$$L_n^v \leq l_2 n^{1/3}. \tag{32}$$

This is an upper bound for the deviation of paths after level $\lfloor \epsilon_3 n^{1/3} \rfloor$. We also need to control the paths before that level, which is a standard large deviation computation. Indeed, for q integer (later, we take $q = \lfloor \epsilon_3 n^{1/3} \rfloor$), set

$$\tilde{Z}_q = \sum_{k=1}^{q} \sum_{v \in \mathbb{D}_k} 1_{\{S_v \geq 2Mq\}}.$$

Recall the definition for M in (4). Let Q' be defined by $\frac{dQ'}{dP} = e^{\lambda_+ X_e - \Lambda(\lambda_+)}$. We have

$$\begin{aligned} E\tilde{Z}_q &= \sum_{k=1}^{q} b^k E 1_{\{S_k \geq 2Mq\}} = \sum_{k=1}^{q} b^k E_{Q'} e^{-\lambda_+ S_k + k\Lambda(\lambda_+)} 1_{\{S_k \geq 2Mq\}} \\ &\leq \sum_{k=1}^{q} b^k e^{-2\lambda_+ Mq + k\Lambda(\lambda_+)} E_{Q'} 1_{\{S_k \geq 2Mq\}} \\ &\leq \sum_{k=1}^{q} b^k e^{-\lambda_+ Mk + k\Lambda(\lambda_+)} e^{-\lambda_+ Mq} = e^{-\lambda_+ Mq + o(q)}, \end{aligned}$$

where, in the last equality, we use the definitions of M and λ_+ (see (3) and (4)). It follows that

$$P(\tilde{Z}_q \geq 1) \leq E\tilde{Z}_q \leq e^{-\lambda_+ Mq + o(q)}.$$

Again by the Borel-Cantelli lemma, $\tilde{Z}_q = 0$ for all large q almost surely. Taking $q = \lfloor \epsilon_3 n^{1/3} \rfloor$ and combining with (32), we obtain that

$$L_n \leq L_{n + \lfloor \epsilon_3 n^{1/3} \rfloor} \leq (l_2 + 2M\epsilon_3) n^{1/3}$$

is true for all large n almost surely. That is,

$$\limsup_{n \to \infty} \frac{L_n}{n^{1/3}} \leq l_2 + 2M\epsilon_3 \qquad a.s..$$

Since $\epsilon_3 > 0$ and $l_2 > l_0$ are arbitrary, we conclude that

$$\limsup_{n\to\infty} \frac{L_n}{n^{1/3}} \le l_0 \qquad a.s.. \tag{33}$$

Together with (21), this completes the proof of Theorem 1. □

5 Concluding Remarks

5.1 The Curve s(t) of (17)

We comment in this subsection on the appearance of the curve $s(t)$ of (17) as a solution to an appropriate variational principle. By the computation in Section 2, $s(t)n^{1/3}$ denotes the minimal possible position for vertices at level tn. However, in Section 3, it is not apriori clear that $s(t)$ will be our best choice. To see why this must indeed be the best choice for the upper bound argument, let us consider a general curve $\phi(t) \le l_2$ as the lower bound for the region. Examining the second moment computation, we need

$$\max_t \{-\phi(t) + \int_0^t \frac{c}{(l_2 - \phi(u))^2} du\} \le 0$$

to make the argument work, where c is some constant. Define $w(t) = l_2 - \phi(t) \ge 0$. The above condition is equivalent to

$$l_2 \ge \max_t \{w(t) + \int_0^t \frac{c}{w(u)^2} du\}.$$

Therefore, the best (smallest) upper bound that we can hope is the result of the following optimization problem

$$\min_{w:(0,1)\to\mathbb{R}_+} \max_t \{w(t) + \int_0^t \frac{c}{w(u)^2} du\}. \tag{34}$$

The solution to this variational problem, denoted by $w^*(\cdot)$, satisfies $s(t) = l_2 - w^*(t)$.

5.2 Generalizations

The approach in this note seems to apply, under natural assumptions, to the situation where the b-ary tree is replaced by a Galton-Watson tree whose offspring distribution possesses high enough exponential moments. We do not pursue such an extension here.

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