

## AN OPTIMAL ITÔ FORMULA FOR LÉVY PROCESSES

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### Abstract

Several Itô formulas have been already established for Lévy processes. We explain according to which criteria they are not *optimal* and establish an extended Itô formula that satisfies that criteria. The interest, in particular, of this formula, is to obtain the explicit decomposition of  $F(X_t, t)$ , for  $X$  Lévy process and  $F$  deterministic function with locally bounded first order Radon-Nikodym derivatives, as the sum of a Dirichlet process and a bounded variation process.

## 1 Introduction and main results

Let  $X$  be a general real-valued Lévy process with characteristic triplet  $(a, \sigma, \nu)$ , i.e. its characteristic exponent is equal to

$$\psi(u) = iua - \sigma^2 \frac{u^2}{2} + \int_{\mathbb{R}} (e^{iuy} - 1 - iuy1_{\{|y| \leq 1\}}) \nu(dy)$$

where  $a$  and  $\sigma$  are real numbers and  $\nu$  is a Lévy measure. We will denote by  $(\sigma B_t, t \geq 0)$  the Brownian component of  $X$ . Let  $F$  be a  $C^{2,1}$  function from  $\mathbb{R} \times \mathbb{R}^+$  to  $\mathbb{R}$ . The classical Itô formula gives

$$\begin{aligned} F(X_t, t) &= F(X_0, 0) + \int_0^t \frac{\partial F}{\partial t}(X_{s-}, s) ds \\ &+ \int_0^t \frac{\partial F}{\partial x}(X_{s-}, s) dX_s + \frac{\sigma^2}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(X_s, s) ds \\ &+ \sum_{0 < s \leq t} \{F(X_s, s) - F(X_{s-}, s) - \frac{\partial F}{\partial x}(X_{s-}, s) \Delta X_s\} \end{aligned} \quad (1.1)$$

This formula can be rewritten under the following form (see [8]):  $(F(X_t, t), t \geq 0)$  is a semi-martingale admitting the decomposition

$$F(X_t, t) = F(X_0, 0) + M_t + V_t \tag{1.2}$$

where the local martingale  $M$  and the adapted with bounded variation process  $V$  are given by

$$M_t = \sigma \int_0^t \frac{\partial F}{\partial x}(X_{s-}, s) dB_s + \int_0^t \int_{\{|y| < 1\}} \{F(X_{s-} + y, s) - F(X_{s-}, s)\} \tilde{\mu}_X(dy, ds) \tag{1.3}$$

$$V_t = \sum_{0 < s \leq t} \{F(X_s, s) - F(X_{s-}, s)\} 1_{\{|\Delta X_s| \geq 1\}} + \int_0^t \mathcal{A}F(X_s, s) ds \tag{1.4}$$

where  $\tilde{\mu}_X(dy, ds)$  denotes the compensated Poisson measure associated to the jumps of  $X$ , and  $\mathcal{A}$  is the operator associated to  $X$  defined by

$$\begin{aligned} \mathcal{A}G(x, t) &= \frac{\partial G}{\partial t}(x, t) + a \frac{\partial G}{\partial x}(x, s) + \frac{1}{2} \sigma^2 \frac{\partial^2 G}{\partial x^2}(x, t) \\ &+ \int_{\mathbb{R}} \{G(x + y, t) - G(x, t) - y \frac{\partial G}{\partial x}(x, t)\} 1_{\{|y| < 1\}} \nu(dy) \end{aligned}$$

for any function  $G$  defined on  $\mathbb{R} \times \mathbb{R}^+$ , such that  $\frac{\partial G}{\partial x}$ ,  $\frac{\partial G}{\partial t}$  and  $\frac{\partial^2 G}{\partial x^2}$  exist as Radon-Nikodym derivatives with respect to the Lebesgue measure and the integral is well defined. The later condition is satisfied when  $\frac{\partial^2 G}{\partial x^2}$  is locally bounded.

Note that the existence of locally bounded first order Radon-Nikodym derivatives alone guarantees the existence of

$$F(X_t, t) - F(X_0, 0) - \int_0^t \frac{\partial F}{\partial t}(X_{s-}, s) ds - \int_0^t \frac{\partial F}{\partial x}(X_{s-}, s) dX_s \tag{1.5}$$

but then to say that this expression coincides with

$$\frac{\sigma^2}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(X_s, s) ds + \sum_{0 < s \leq t} \{F(X_s, s) - F(X_{s-}, s) - \frac{\partial F}{\partial x}(X_{s-}, s) \Delta X_s\}$$

we need to assume much more on  $F$ .

In that sense one might say that the classical Itô formula is not *optimal*. The interest of an optimal formula is two-fold. It allows to expand  $F(X_t, t)$  under minimal conditions on  $F$  but also to know explicitly the structure of the process  $F(X_t, t)$ . Such an optimal formula has been established in the particular case when  $X$  is a Brownian motion [4]. Indeed in that case, under the minimal assumption on  $F$  for the existence of (1.5), namely that  $F$  admits locally bounded first order Radon-Nikodym derivatives, we know that this expression coincides with

$$-\frac{1}{2} \int_0^t \int_{\mathbb{R}} \frac{\partial F}{\partial x}(x, s) dL_s^x$$

where  $(L_s^x, x \in \mathbb{R}, s \geq 0)$  is the local time process of  $X$ . Moreover the process

$(\int_0^t \int_{\mathbb{R}} \frac{\partial F}{\partial x}(x, s) dL_s^x, t \geq 0)$  has a 0-quadratic energy.

In the general case, various extensions of (1.1) have been established. We will quote here only the extensions exploiting the notion of local times, we send to [4] for a more exhaustive bibliography. Meyer [9] has been the first to relax the assumption on  $F$  by introducing an integral with respect to local time, followed then by Bouleau and Yor [3], Azéma et al [1], Eisenbaum [4], [5], Ghomrasni and Peskir [7], Eisenbaum and Kyprianou [6]. In the discontinuous case, none of the obtained Itô formulas is optimal because of the presence of the expression  $\sum_{0 < s \leq t} \{F(X_s, s) - F(X_{s-}, s) - \frac{\partial F}{\partial x}(X_{s-}, s) \Delta X_s\}$ .

The Itô formula for Lévy processes presented below in Theorem 1.1, is available for  $X$  admitting a Brownian component. It lightens the condition on the jumps of  $X$  required by [5], and it also lightens the condition on the first order derivatives of  $F$  required by [6]. Besides it is optimal. To introduce it we need the operator  $I$  defined on the set of locally bounded measurable functions  $G$  on  $\mathbb{R} \times \mathbb{R}^+$  by

$$IG(x, t) = \int_0^x G(y, t) dy.$$

We will denote the Markov local time process of  $X$  by  $(L_t^x, x \in \mathbb{R}, t \geq 0)$ .

**Theorem 1.1.** : Assume that  $\sigma \neq 0$ . Let  $F$  be a function from  $\mathbb{R} \times \mathbb{R}^+$  to  $\mathbb{R}$  such that  $\frac{\partial F}{\partial x}$  and  $\frac{\partial F}{\partial t}$  exist as Radon-Nikodym derivatives with respect to the Lebesgue measure and are locally bounded. Then the process  $(F(X_t, t), t \geq 0)$  admits the following decomposition

$$F(X_t, t) = F(X_0, 0) + M_t + V_t + Q_t$$

with  $M$  the local martingale given by (1.3),  $V$  is the bounded variation process

$$V_t = \sum_{0 \leq s \leq t} \{F(X_s, s) - F(X_{s-}, s)\} 1_{\{|\Delta X_s| \geq 1\}}$$

and  $Q$  the following adapted process with 0-quadratic variation

$$Q_t = - \int_0^t \int_{\mathbb{R}} \mathcal{A}IF(x, s) dL_s^x.$$

As a simple application of Theorem 1.1 consider the example of the function  $F(x, s) = |x|$  in the case  $\int_0^1 y \nu(dy) = +\infty$ . This function does not satisfy the assumption of Theorem 3 of [6] nor  $X$  does satisfy the assumption of Theorem 2.2 in [5]. But, thanks to Theorem 1.1, we immediately obtain Tanaka's formula.

The proofs are presented in Section 2.

## 2 Proofs

We first remind the meaning of integration with respect to the semimartingale local time process of  $X$  denoted  $(\ell_s^x, x \in \mathbb{R}, s \geq 0)$ . Theorem 1.1 is expressed in terms of the Markov local time process  $(L_s^x, x \in \mathbb{R}, s \geq 0)$ . The two processes are connected by:

$$(L_s^x, x \in \mathbb{R}, s \geq 0) = \left(\frac{1}{\sigma^2} \ell_s^x, x \in \mathbb{R}, s \geq 0\right).$$

Let  $\sigma B$  be the Brownian component of  $X$ . Defined the norm  $\|\cdot\|$  of a measurable function  $f$  from  $\mathbb{R} \times \mathbb{R}_+$  to  $\mathbb{R}$  by

$$\|f\| = 2\mathbb{E}\left(\int_0^1 f^2(X_s, s) ds\right)^{1/2} + \mathbb{E}\left(\int_0^1 |f(X_s, s)| \frac{|B_s|}{s} ds\right).$$

In [6], integration with respect to  $\ell$  of locally bounded measurable function  $f$  has been defined by

$$\int_0^t \int_{\mathbb{R}} f(x, s) d\ell_s^x = \sigma \int_0^t f(X_{s-}, s) dB_s + \sigma \int_{1-t}^1 f(\tilde{X}_{s-}, 1-s) d\tilde{B}_s, \quad 0 \leq t \leq 1 \quad (2.1)$$

where  $\tilde{B}$  and  $\tilde{X}$  are the time reversal at 1 of  $B$  and  $X$ .

We have the following properties:

- (i)  $\mathbb{E}(|\int_0^t \int_{\mathbb{R}} f(x, s) d\ell_s^x|) \leq |\sigma| \|f\|$ .
- (ii) If  $f$  admits a locally bounded Radon-Nikodym derivative with respect to  $x$ , then:  $\int_0^t \int_{\mathbb{R}} f(x, s) d\ell_s^x = -\sigma^2 \int_0^t \frac{\partial f}{\partial x}(X_s, s) ds$ .
- (iii) The process  $(\int_0^t \int_{\mathbb{R}} f(x, s) d\ell_s^x, 0 \leq t \leq 1)$  has 0-quadratic variation.

**Proof of Theorem 1.1 :** We start by assuming that  $F$  and  $\frac{\partial F}{\partial x}$  are bounded. We set

$$F_n(x, t) = \int \int_{\mathbb{R}^2} F(x - y/n, t - s/n) f(y) h(s) dy ds$$

where  $f$  and  $h$  are nonnegative  $C^\infty$  functions with compact supports such that  $\int_{\mathbb{R}} f(x) dx = \int_{\mathbb{R}} h(x) dx = 1$ . Thanks to the usual Itô formula we have:

$$\begin{aligned} F_n(X_t, t) = & F_n(0, 0) + \sigma \int_0^t \frac{\partial F_n}{\partial x}(X_{s-}, s) dB_s + \int_0^t \frac{\partial F_n}{\partial t}(X_s, s) ds \\ & + a \int_0^t \frac{\partial F_n}{\partial x}(X_s, s) ds + \sum_{0 \leq s \leq t} \{F_n(X_s, s) - F_n(X_{s-}, s)\} 1_{\{|\Delta X_s| \geq 1\}} \\ & + \int_0^t \int_{\mathbb{R}} \{F_n(X_{s-} + y, s) - F_n(X_{s-}, s)\} 1_{\{|y| < 1\}} \tilde{\mu}(ds, dy) \\ & + \frac{\sigma^2}{2} \int_0^t \frac{\partial^2 F_n}{\partial x^2}(X_s, s) ds \\ & + \int_0^t \int_{-1}^1 \{F_n(X_s + y, s) - F_n(X_s, s) - \frac{\partial F_n}{\partial x}(X_s, s) y\} \nu(dy) ds \end{aligned} \quad (2.2)$$

With the same arguments as in the proof of Theorem 2.2 of [5], we see that as  $n$  tends to  $\infty$ ,  $F_n(X_t, t)$  and each of the first five terms of the RHS of (2.2) converges at least in probability to the corresponding expression with  $F$  replacing  $F_n$ . Besides we note that

$$\begin{aligned} \int_0^t \frac{\partial F}{\partial t}(X_s, s) ds &= -\frac{1}{\sigma^2} \int_0^t \int_{\mathbb{R}} \left( \int_0^x \frac{\partial F}{\partial t}(y, s) dy \right) d\ell_s^x \\ &= -\int_0^t \int_{\mathbb{R}} \left( \frac{\partial}{\partial t} \int_0^x F(y, s) dy \right) dL_s^x \end{aligned}$$

since  $\frac{\partial F}{\partial t}$  is locally bounded. Hence we have:

$$\int_0^t \frac{\partial F}{\partial t}(X_s, s) ds = -\int_0^t \int_{\mathbb{R}} \frac{\partial(IF)}{\partial t}(x, s) dL_s^x. \quad (2.3)$$

Since :  $F(x, s) = \frac{\partial(IF)}{\partial x}(x, s)$ , we immediately obtain:

$$a \int_0^t \frac{\partial F}{\partial x}(X_s, s) ds = - \int_0^t \int_{\mathbb{R}} a \frac{\partial(IF)}{\partial x}(x, s) dL_s^x. \tag{2.4}$$

The convergence in  $L^2$  of the sixth term of the RHS is obtained with the same proof as in [6]. The limit is equal to

$$\int_0^t \int_{\mathbb{R}} \{F(X_{s-} + y, s) - F(X_{s-}, s)\} 1_{\{|y| < 1\}} \tilde{\mu}(ds, dy) \tag{2.5}$$

For the seventh term of the RHS of (2.2), we note that :

$\frac{\sigma^2}{2} \int_0^t \frac{\partial^2 F_n}{\partial x^2}(X_s, s) ds = -\frac{1}{2} \int_0^t \int_{\mathbb{R}} \frac{\partial F_n}{\partial x}(x, s) d\ell_s^x$ . Thanks to the properties (i) and (ii) of the integration with respect to the local times, this expression converges in  $L^1$  to  $-\frac{1}{2} \int_0^t \int_{\mathbb{R}} \frac{\partial F}{\partial x}(x, s) d\ell_s^x$ . We can obviously write:

$$-\frac{1}{2} \int_0^t \int_{\mathbb{R}} \frac{\partial F}{\partial x}(x, s) d\ell_s^x = -\frac{\sigma^2}{2} \int_0^t \int_{\mathbb{R}} \frac{\partial^2(IF)}{\partial x^2}(x, s) dL_s^x. \tag{2.6}$$

We now study the convergence of the last term of the RHS of (2.2). We have:

$$\begin{aligned} & \int_0^t \int_{-1}^1 \{F_n(X_s + y, s) - F_n(X_s, s) - \frac{\partial F_n}{\partial x}(X_s, s)y\} \nu(dy) ds \\ &= - \int_0^t \int_{\mathbb{R}} H_n(x, s) dL_s^x \end{aligned} \tag{2.7}$$

where:  $H_n(x, s) = \int_0^x \int_{-1}^1 \{F_n(z + y, s) - F_n(z, s) - \frac{\partial F_n}{\partial x}(z, s)y\} \nu(dy) dz$ . We have:

$$\begin{aligned} & |F_n(z + y, s) - F_n(z, s) - \frac{\partial F_n}{\partial x}(z, s)y| 1_{\{|y| < 1\}} \\ &= \left| \int_z^{z+y} \frac{\partial F_n}{\partial x}(v, t) - \frac{\partial F_n}{\partial x}(z, t) dv \right| 1_{\{|y| < 1\}} \\ &\leq y^2 \sup \left| \frac{\partial^2 F_n}{\partial x^2} \right| 1_{\{|y| < 1\}}. \end{aligned}$$

Noting that:  $\frac{\partial^2 F_n}{\partial x^2}(x, t) = n^2 \int_{\mathbb{R}^2} F(x - y/n, t - s/n) f''(y) h(s) dy ds$ , we obtain  $|F_n(z + y, s) - F_n(z, s) - \frac{\partial F_n}{\partial x}(z, s)y| 1_{\{|y| < 1\}} \leq cste n^2 y^2 1_{\{|y| < 1\}} \sup |F|$

Consequently :

$$\begin{aligned} H_n(x, s) &= \int_{-1}^1 \int_0^x \{F_n(z + y, s) - F_n(z, s) - \frac{\partial F_n}{\partial x}(z, s)y\} dz \nu(dy) \\ &= \int_{-1}^1 \left\{ \int_0^{x+y} F_n(z, s) dz - \int_0^x F_n(z, s) dz - yF_n(x, s) + yF_n(0, s) - \int_0^y F_n(z, s) dz \right\} \nu(dy) \\ &= G_n(x, s) + \int_{-1}^1 (yF_n(0, s) - \int_0^y F_n(z, s) dz) \nu(dy) \end{aligned}$$

where  $G_n(x, s) = \int_{-1}^1 (IF_n(x + y, s) - IF_n(x, s) - yF_n(x, s))\nu(dy)$ . Thanks to Corollary 8 of [6], we know that

$$\int_0^t \int_{\mathbb{R}} H_n(x, s) dL_s^x = \int_0^t \int_{\mathbb{R}} G_n(x, s) dL_s^x. \quad (2.8)$$

By dominated convergence, we have as  $n$  tends to  $\infty$  for every  $(x, s)$

$$IF_n(x + y, s) - IF_n(x, s) - yF_n(x, s) \rightarrow IF(x + y, s) - IF(x, s) - yF(x, s).$$

Besides, for every  $n : |IF_n(x + y, s) - IF_n(x, s) - yF_n(x, s)| \leq y^2 \mathbf{1}_{\{|y| < 1\}} \sup |\frac{\partial F}{\partial x}|$ , hence for every  $(x, s) : G_n(x, s)$  tends to  $G(x, s)$ , where

$$G(x, s) = \int_{\mathbb{R}} (IF(x + y, s) - IF(x, s) - yF(x, s)) \mathbf{1}_{\{|y| < 1\}} \nu(dy).$$

By dominated convergence,  $(G_n)_{n > 0}$  converges for the norm  $\|\cdot\|$  to  $G$ . Consequently the limit of the last term of the RHS of (2.2) is equal by (2.7) and (2.8) to

$$- \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} (IF(x + y, s) - IF(x, s) - yF(x, s)) \mathbf{1}_{\{|y| < 1\}} \nu(dy) dL_s^x. \quad (2.9)$$

Summing all the limits (2.3), (2.4), (2.5), (2.6) and (2.9), we finally obtain

$$\begin{aligned} F(X_t, t) &= F(X_0, 0) + \sigma \int_0^t \frac{\partial F}{\partial x}(X_{s-}, s) dB_s \\ &+ \int_0^t \int_{\mathbb{R}} \{F(X_{s-} + y, s) - F(X_{s-}, s)\} \mathbf{1}_{\{|y| < 1\}} \tilde{\mu}(ds, dy) \\ &+ \sum_{0 < s \leq t} \{F(X_s, s) - F(X_{s-}, s)\} \mathbf{1}_{\{|\Delta X_s| \geq 1\}} \\ &- \int_0^t \int_{\mathbb{R}} \left\{ \frac{\partial(IF)}{\partial t}(x, s) + a \frac{\partial(IF)}{\partial x}(x, s) + \frac{\sigma^2}{2} \frac{\partial^2(IF)}{\partial x^2}(x, s) \right\} dL_s^x \\ &- \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \{IF(x + y, s) - IF(x, s) - yF(x, s)\} \mathbf{1}_{\{|y| < 1\}} \nu(dy) dL_s^x. \end{aligned} \quad (2.10)$$

which summarizes in

$$\begin{aligned} F(X_t, t) &= F(X_0, 0) + \sigma \int_0^t \frac{\partial F}{\partial x}(X_{s-}, s) dB_s \\ &+ \int_0^t \int_{\mathbb{R}} \{F(X_{s-} + y, s) - F(X_{s-}, s)\} \mathbf{1}_{\{|y| < 1\}} \tilde{\mu}(ds, dy) \\ &+ \sum_{0 < s \leq t} \{F(X_s, s) - F(X_{s-}, s)\} \mathbf{1}_{\{|\Delta X_s| \geq 1\}} - \int_0^t \int_{\mathbb{R}} \mathcal{A}IF(x, s) dL_s^x. \end{aligned}$$

In the general case, we set:

$$\tilde{F}_n(x, s) = F(x, s) \mathbf{1}_{[a_n, b_n]}(x) + F(a_n, s) \mathbf{1}_{(-\infty, a_n)}(x) + F(b_n, s) \mathbf{1}_{(b_n, \infty)}(x)$$

where  $(-a_n)_{n \in \mathbb{N}}$  and  $(b_n)_{n \in \mathbb{N}}$  are two positive real sequences increasing to  $\infty$ .

We write (2.10) for  $\tilde{F}_n$  and stop the process  $(\tilde{F}_n(X_s, s), 0 \leq s \leq 1)$  at  $T_m = 1 \wedge \inf\{s \geq 0 : |X_s| > m\}$ . We let  $n$  tend to  $\infty$  and then  $m$  tend to  $\infty$ . The behavior of two terms deserves specific explanations, the other terms converging respectively to the expected expressions.

The first one is :  $\int_0^{t \wedge T_m} \int_{\mathbb{R}} \{I\tilde{F}_n(x+y, s) - I\tilde{F}_n(x, s) - y\tilde{F}_n(x, s)\} 1_{\{|y| < 1\}} \nu(dy) dL_s^x$ . Thanks to the definition of the integral with respect to local time (2.1), it is equal to

$$\frac{1}{\sigma} \int_0^{t \wedge T_m} \tilde{H}_n(X_{s-}, s) dB_s + \frac{1}{\sigma} \int_{1-(t \wedge T_m)}^1 \tilde{H}_n(\tilde{X}_{s-}, s) d\tilde{B}_s \quad (2.11)$$

where  $\tilde{H}_n(x, s) = \int \{I\tilde{F}_n(x+y, s) - I\tilde{F}_n(x, s) - y\tilde{F}_n(x, s)\} 1_{\{|y| < 1\}} \nu(dy)$ .

We set  $H(x, s) = \int \{IF(x+y, s) - IF(x, s) - yF(x, s)\} 1_{\{|y| < 1\}} \nu(dy)$ .

We can choose  $n$  big enough to have  $|a_n|$  and  $b_n$  bigger than  $m+1$ . Hence (2.11) is equal to

$$\frac{1}{\sigma} \int_0^{t \wedge T_m} H(X_{s-}, s) dB_s + \frac{1}{\sigma} \int_{1-(t \wedge T_m)}^1 H(\tilde{X}_{s-}, s) d\tilde{B}_s.$$

For every  $\epsilon > 0$

$$\begin{aligned} \mathbb{P}(\sup_{0 \leq t \leq 1} \left| \int_{1-(t \wedge T_m)}^1 H(\tilde{X}_{s-}, s) d\tilde{B}_s - \int_{1-t}^1 H(\tilde{X}_{s-}, s) d\tilde{B}_s \right| \geq \epsilon) \\ \leq \mathbb{P}(T_m < 1) \\ = \mathbb{P}(\sup_{0 \leq t \leq 1} |X_t| > m) \end{aligned}$$

which shows that as  $m$  tends to  $\infty$ ,  $\int_{1-(t \wedge T_m)}^1 H(\tilde{X}_{s-}, s) d\tilde{B}_s$  converges in probability uniformly on  $[0, 1]$  to  $\int_{1-t}^1 H(\tilde{X}_{s-}, s) d\tilde{B}_s$ . Similarly  $\int_0^{t \wedge T_m} H(X_{s-}, s) dB_s$  converges in probability to  $\int_0^t H(X_{s-}, s) dB_s$ . Consequently as  $m$  tends to  $\infty$ , (2.11) converges to

$$\int_0^t \int_{\mathbb{R}} \{IF(x+y, s) - IF(x, s) - yF(x, s)\} 1_{\{|y| < 1\}} \nu(dy) dL_s^x.$$

The second term is :  $\int_0^t \int_{\mathbb{R}} \{\tilde{F}_n(X_{s-} + y, s) - \tilde{F}_n(X_{s-}, s)\} 1_{\{s < T_m\}} 1_{\{|y| < 1\}} \tilde{\mu}(ds, dy)$ . For  $n$  big enough such that  $|a_n|, b_n > m$ , this term is equal to

$\int_0^t \int_{\mathbb{R}} \{F(X_{s-} + y, s) - F(X_{s-}, s)\} 1_{\{s < T_m\}} 1_{\{|y| < 1\}} \tilde{\mu}(ds, dy)$ . As Ikeda and Watanabe [8], we then denote by  $(\int_0^t \int_{\mathbb{R}} \{F(X_{s-} + y, s) - F(X_{s-}, s)\} 1_{\{|y| < 1\}} \tilde{\mu}(ds, dy), 0 \leq t \leq 1)$  the local martingale  $(Y_t, 0 \leq t \leq 1)$  defined by :

$$Y_{t \wedge T_m} = \int_0^t \int_{\mathbb{R}} \{\tilde{F}_n(X_{s-} + y, s) - \tilde{F}_n(X_{s-}, s)\} 1_{\{s < T_m\}} 1_{\{|y| < 1\}} \tilde{\mu}(ds, dy). \square$$

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