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A NOTE ON *R*-BALAYAGES OF MATRIX-EXPONENTIAL LÉVY PROCESSES

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Abstract

In this note we give semi-explicit solutions for r-balayages of matrix-exponential Lévy processes. To this end, we turn to an identity for the joint Laplace transform of the first entry time and the undershoot and a semi-explicit solution of the negative Wiener-Hopf factor. Our result is closely related to the works by Mordecki in [11], Asmussen, Avram and Pistorius in [3], Chen, Lee and Sheu in [7], and many others.

1 Introduction

Given a Lévy process $X = (X_t, t \geq 0)$ on $\mathbb{R}$. For $x \in \mathbb{R}$, denote by $\mathbb{P}_x$ the law of X under which $X_0 = x$ and write simply $\mathbb{P}_0 = \mathbb{P}$. Let ψ be the Laplace exponent of X, that is, $\mathbb{E}[e^{i\xi X_1}] = \exp\{\psi(i\xi)\}$ for $\xi \in \mathbb{R}$. For every killing rate $r \geq 0$, consider the r-balayage operator P^r defined by

$$P^r g(x) \equiv \mathbb{E}_x\left[e^{-r\tau} g(X_\tau)\right], \quad g \in \mathscr{B}_b(\mathbb{R}_-), \tag{1.1}$$

with the convention that $r \cdot \infty = \infty$. Here, by writing $\mathbb{R}_- = (-\infty, 0]$, we take $\tau = \inf\{t \geq 0; X_t \in \mathbb{R}_-\}$ and $\mathscr{B}_b(\mathbb{R}_-)$ as the set of all bounded Borel measurable functions defined on $\mathbb{R}_-$. Solutions of r-balayages have been a major concern not only in probability theory itself but also in applied sciences such as mathematical finance and insurance mathematics (see [8] and [9]). It is, however, an undeniable fact that only few Lévy processes allow solutions for their r-balayages. Besides the classical case of Lévy processes with no negative jumps, various authors have found that r-balayages have semi-explicit solutions when the triplet (r, g, X) is well chosen. See the works by Mordecki in [11], Avram and Usabel in [4], Asmussen, Avram and Pistorius in [3], Alili and Kyprianou in [1], Boyarchenko and Levendorskiǐ in [6], Chen, Lee and Sheu in [7], and

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many others. In particular, the salient feature of their choices of the underlying process X is that X is taken from the class of matrix-exponential Lévy processes. A natural question then arises. We ask whether the class of matrix-exponential Lévy processes allow semi-explicit solutions for r-balayages for all bounded measurable functions g and all $r \geq 0$.

We say that a Lévy process is a **matrix-exponential Lévy process** if its downward Lévy measure is a finite measure and has a rational Laplace transform. In other words, the upward Lévy measure of $-X$ is proportional to a matrix-exponential distribution dF given by

$$dF(y) = \sum_{j=1}^{n} \sum_{k=1}^{n_j} A_{j,k} y^{k-1} e^{-\lambda_j y} dy, \quad y > 0. \tag{1.2}$$

Note that the parameters $A_{j,k}$ and λ_j are not necessarily real; see [10]. The two dense classes of distributions on $(0,\infty)$ – phase-type distributions and distributions which are *linear* combinations of exponential distributions – are both subclasses of matrix-exponential distributions. See [2] Theorem 4.2 and [5]. In particular, by Proposition 1 in [3], matrix-exponential Lévy processes are dense in the class of Lévy processes.

The aim of this paper is to show that r-balayages admit semi-explicit solutions whenever X is a matrix-exponential Lévy process, g is a bounded Borel measurable function on $\mathbb{R}_-$ and $r > 0$. The case that $r = 0$ will also be considered under some restrictions on the moment of X. In this paper, we obtain the desired semi-explicit solutions of r-balayages with the help of a recent result on the negative Wiener-Hopf factor by Lewis and Mordecki in [10] and an identity for the joint Laplace transform of the first exit time and the undershoot by Alili and Kyprianou in [1]. The results are stated in Theorem 2.2. In this way, we overcome the difficulties of either giving a probabilistic interpretation for the jump distribution or transforming the generator of X into an differential operator. The former approach was treated by Asmussen, Avram and Pistorius in [3] for the case that (1.2) is phase-type, while the latter approach was expatiated by Chen, Lee and Sheu in [7] for the case that the Lévy measure is a two-sided phase-type distributions. It is worth noting that in [3], the authors showed that a semi-explicit solution of an r-balayage can be determined implicitly by a system of linear equations. At the end of the paper, by Lagrange's interpolation, we demonstrate the consistency of our result with an existing formula. In addition, an application of the formula of the r-balayage to another passage functional of X is briefly discussed.

2 r-Balayages and Negative Wiener-Hopf Factor

Let $X^{(+)}$ be a Lévy process with no negative jumps and Z be a compound Poisson process with jump rate $\lambda > 0$ whose jump distribution is a matrix-exponential distribution given by (1.2). Assume that $X^{(+)}$ and Z are independent. Consider a **matrix-exponential Lévy process** X given by

$$X_t = X_t^{(+)} - Z_t, \quad t \geq 0. \tag{2.1}$$

It follows from (1.2) and the Lévy-Khintchine formula that the Laplace exponent ψ takes the following form:

$$\psi(z) = \frac{\sigma^2}{2} z^2 + \mu z + \int_{\mathbb{R}_+} \left(e^{zy} - 1 - zy \mathbb{1}_{y \leq 1} \right) \Pi(dy) + \lambda(R(z) - 1). \tag{2.2}$$

Here, $\sigma \geq 0$, $\mu \in \mathbb{R}$, Π is a Radon measure on $\mathbb{R}_+$ satisfying $\Pi(\{0\}) = 0$ and $\int_{\mathbb{R}_+} 1 \wedge y^2 \Pi(dy) < \infty$, $\lambda > 0$, and R is a rational function such that $R(z) \equiv \int_0^\infty e^{-zy} dF(y)$ on $i\mathbb{R}$. The function on the right-hand side of (2.2) admits an analytic continuation into $\{z; \Re z < 0\}$ except at the poles of R and is continuous up $i\mathbb{R}$. As a result, we can extend ψ by (2.2) and the resulting extension is analytic on $\{z; \Re z < 0\}$ except at the poles of R.

For $r > 0$, let $\mathbf{e}_r$ be an independent exponential random variable with mean $1/r$. Recall that the negative Wiener-Hopf factor ϕ_r^- is defined by

$$\phi_r^-(z) = \mathbb{E}\left[\exp\left(z \inf_{0\le s\le \mathbf{e}_r} X_s\right)\right], \quad \Re z \ge 0.$$

If $\mathbb{E}[(X_1)^-] < \infty$ and $\mathbb{E}[X_1] \in (0,\infty]$, then $\inf_{0\le s<\infty} X_s > -\infty$ almost surely by the strong law of large numbers for Lévy processes. (See [12] Theorem 36.5 and Theorem 36.6.) It follows that $\lim_{r\downarrow 0} \phi_r^-(z)$ exists for each $z \in \mathbb{C}$ with $\Re z \ge 0$, and hence we can define

$$\phi_0^-(z) \triangleq \mathbb{E}\left[\exp\left(z \inf_{0\le s<\infty} X_s\right)\right] = \lim_{r\downarrow 0} \phi_r^-(z). \tag{2.3}$$

For convenience, we will slightly abuse the notation to write $\inf_{0\le s\le \mathbf{e}_0} X_s$ for $\inf_{0\le s<\infty} X_s$. *From now on, we always assume that $r \ge 0$ and that $\mathbb{E}[(X_1)^-] < \infty$ and $\mathbb{E}[X_1] \in (0,\infty]$ for the case in which $r = 0$.* For matrix-sexponential Lévy processes, the negative Wiener-Hopf factor has a semi-explicit solution. We quote the following results by Lewis and Mordecki in [10]. See also [3].

Theorem 2.1. *(1) The equation $\psi - r = 0$ has solutions in $\{z; \Re z < 0\}$. Moreover, if $(\rho_j; 1 \le j \le m)$ are the distinct zeros of $\psi - r = 0$ in $\{z; \Re z < 0\}$ and m_j is the multiplicity of ρ_j, then $\sum_{j=1}^m m_j$ is equal to $\overline{n}$ when $X^{(+)}$ is a subordinator and $\overline{n}+1$ when $X^{(+)}$ is not a subordinator. Here, $\overline{n} = \sum_{j=1}^n n_j$.*
(2) ϕ_r^- is a rational function given by

$$\phi_r^-(z) = \frac{\prod_{j=1}^m (-\rho_j)^{m_j}}{\prod_{j=1}^n \lambda_j^{n_j}} \frac{P(z)}{Q(z)}. \tag{2.4}$$

Here, P and Q are monic polynomials given respectively by

$$P(z) = \prod_{j=1}^n (z + \lambda_j)^{n_j}, \tag{2.5}$$

$$Q(z) = \prod_{j=1}^m (z - \rho_j)^{m_j}. \tag{2.6}$$

(3) The distribution of $\inf_{0\le s\le \mathbf{e}_r} X_s$ is given by

$$\mathbb{P}\left[\inf_{0\le s\le \mathbf{e}_r} X_s \in dy\right] = \frac{\prod_{j=1}^m (-\rho_j)^{m_j}}{\prod_{j=1}^n \lambda_j^{n_j}} \left[\boldsymbol{U}_0 \delta_0(dy) + \left(\sum_{j=1}^m \sum_{k=1}^{m_j} \boldsymbol{U}_{j,m_j-k} \frac{(-y)^{k-1}}{(k-1)!} e^{-\rho_j y}\right) \mathbb{1}_{(-\infty,0)}(y) dy\right].$$

Here, the constants $\boldsymbol{U}_0$ and $\boldsymbol{U}_{j,k}$ are defined by the partial fraction decomposition of ϕ_r^-:

$$\phi_r^-(z) \equiv \frac{\prod_{j=1}^m (-\rho_j)^{m_j}}{\prod_{j=1}^n \lambda_j^{n_j}} \left(\boldsymbol{U}_0 + \sum_{j=1}^m \sum_{k=1}^{m_j} \frac{\boldsymbol{U}_{j,m_j-k}}{(z-\rho_j)^k}\right)$$

We now sketch our plan to find semi-explicit solutions for r-balayages. Recall that a function g defined on $\mathbb{R}$ is a Schwartz function, denoted by $g \in \mathscr{S}$, if $\sup_x |x^j g^{(\ell)}(x)| < \infty$ for all $j, \ell \geq 0$. For properties of Schwartz functions, see [13]. Since the class $\mathscr{S}$ is sufficiently large, we may obtain, in principle, solutions of general r-balayages by suitable approximation once we have found solutions of r-balayages of Schwartz functions. Set

$$a_r(x,z) \equiv \mathbb{E}_x \left[e^{-r\tau + zX_\tau} \right], \quad \Re z \geq 0.$$

For every $g \in \mathscr{S}$, it follows from the Fourier inversion formula that

$$P^r g(x) = \int_{-\infty}^{\infty} a_r(x, 2\pi i \zeta) \mathscr{F} g(\zeta) d\zeta, \tag{2.7}$$

with

$$\mathscr{F} g(\zeta) = \int_{-\infty}^{\infty} e^{-2\pi i y \zeta} g(y) dy$$

being the Fourier transform of g. Define a Fourier multiplier operator $T_x^r : \mathscr{S} \longrightarrow L_2$ by

$$\mathscr{F} T_x^r g(\zeta) \equiv a_r(x, 2\pi i \zeta) \mathscr{F} g(\zeta). \tag{2.8}$$

By (2.7), it follows that $P^r g(x) = \int_{-\infty}^{\infty} \mathscr{F} T_x^r g(\zeta) d\zeta$. If the Fourier inversion formula is applicable in a suitable sense, we should have

$$P^r g(x) = T_x^r g(0).$$

Hence, if a_r has a form which allows us to identify T_x^r semi-explicitly by (2.8), the desired solutions of r-balayages of Schwartz functions will fall right into our lap.

To carry out the plan, we first recall an identity for a_r which states that for all z with $\Re z \geq 0$

$$a_r(x,z) = \frac{e^{zx} \mathbb{E} \left[\exp\left(z \inf_{0 \leq s \leq \mathbf{e}_r} X_s \right) \mathbb{1}_{[\inf_{0 \leq s \leq \mathbf{e}_r} X_s < -x]} \right]}{\mathbb{E} \left[\exp\left(z \inf_{0 \leq s \leq \mathbf{e}_r} X_s \right) \right]}. \tag{2.9}$$

(See [1] Corollary 2.) We can use Theorem 2.1 to calculate a_r.

Lemma 2.1. *For any $x > 0$ and any $z \in \mathbb{C}$ such that $\Re z \geq 0$,*

$$a_r(x,z) = \sum_{j=1}^{m} \sum_{k=0}^{m_j - 1} \frac{x^{m_j - 1 - k} e^{\rho_j x}}{(m_j - 1 - k)!} \sum_{\ell=0}^{k} \boldsymbol{U}_{j,k-\ell} \frac{Q_{j,\ell}(z)}{P(z)}.$$

Here, $Q_{j,\ell}(z) = (z - \rho_j)^{-(\ell+1)} Q(z)$ for $0 \leq \ell \leq m_j - 1$.

Proof. Using Theorem 2.1, we get that

$$\begin{aligned}\mathbb{E}_x\left[e^{-r\tau+zX_\tau}\right] &= \frac{Q(z)}{P(z)}\sum_{j=1}^{m}\sum_{k=1}^{m_j}\boldsymbol{U}_{j,m_j-k}\sum_{\ell=0}^{k-1}\frac{x^{k-\ell-1}e^{\rho_j x}}{(k-\ell-1)!(z-\rho_j)^{\ell+1}}\\ &= \frac{Q(z)}{P(z)}\sum_{j=1}^{m}\sum_{k=0}^{m_j-1}\boldsymbol{U}_{j,m_j-k-1}\sum_{\ell=0}^{k}\frac{x^{k-\ell}e^{\rho_j x}}{(k-\ell)!(z-\rho_j)^{\ell+1}}\\ &= \frac{Q(z)}{P(z)}\sum_{j=1}^{m}\sum_{k=0}^{m_j-1}\boldsymbol{U}_{j,m_j-k-1}\sum_{\ell=0}^{k}\frac{x^{\ell}e^{\rho_j x}}{\ell!(z-\rho_j)^{k-\ell+1}}\\ &= \frac{Q(z)}{P(z)}\sum_{j=1}^{m}\sum_{\ell=0}^{m_j-1}\frac{x^\ell}{\ell!}\sum_{k=\ell}^{m_j-1}\frac{\boldsymbol{U}_{j,m_j-k-1}e^{\rho_j x}}{(z-\rho_j)^{k-\ell+1}}\\ &= \frac{Q(z)}{P(z)}\sum_{j=1}^{m}\sum_{\ell=0}^{m_j-1}\frac{x^{m_j-1-\ell}}{(m_j-1-\ell)!}\sum_{k=m_j-1-\ell}^{m_j-1}\frac{\boldsymbol{U}_{j,m_j-k-1}e^{\rho_j x}}{(z-\rho_j)^{k-(m_j-1-\ell)+1}}\\ &= \frac{Q(z)}{P(z)}\sum_{j=1}^{m}\sum_{\ell=0}^{m_j-1}\frac{x^{m_j-1-\ell}}{(m_j-1-\ell)!}\sum_{k=0}^{\ell}\frac{\boldsymbol{U}_{j,\ell-k}e^{\rho_j x}}{(z-\rho_j)^{k+1}}.\end{aligned}$$

The desired equation then follows from the definition of $Q_{j,\ell}$. □

The following proposition is the key step to identify T_x^r.

Proposition 2.1. *For any $\gamma\in\mathbb{C}$ with $\Re\gamma>0$, the exponential integral operator $\mathscr{E}_\gamma$ defined by*

$$\mathscr{E}_\gamma g(x)=\int_0^\infty e^{-\gamma t}g(x-t)dt=\int_{-\infty}^{x}e^{-\gamma(x-t)}g(t)dt \tag{2.10}$$

is a bijection on the space $\mathscr{S}$. Moreover, for any $q\in\mathbb{N}$,

$$\frac{\mathscr{F}g(\zeta)}{(\gamma+2\pi i\zeta)^q}=\mathscr{F}(\mathscr{E}_\gamma)^q g(\zeta),\quad \zeta\in\mathbb{R}. \tag{2.11}$$

Proof. The case $q=1$ for (2.11) follows from a straightforward application of Fourier inversion formula. Recall that $\mathscr{F}$ is a bijection from $\mathscr{S}$ onto $\mathscr{S}$. Using (2.11) for $q=1$ and the fact that $\mathscr{F}g\in\mathscr{S}$, we see that $\mathscr{F}\mathscr{E}_\gamma g\in\mathscr{S}$ and hence $\mathscr{E}_\gamma g\in\mathscr{S}$. This shows that $\mathscr{E}_\gamma$ is a bijection from $\mathscr{S}$ to $\mathscr{S}$. Finally, by the fact that $\mathscr{E}_\gamma$ is bijective and iteration, we deduce that (2.11) holds for all $q\geq 1$. □

Remark 1. (1) It is plain that the operator $\mathscr{E}_\gamma$ is well-defined on the set of all bounded Borel measurable functions.
(2) For all $q\geq 1$, we have

$$\left(\mathscr{E}_\gamma\right)^q g(x)=\int_{-\infty}^{x}(x-t)^{q-1}e^{-\gamma(x-t)}g(t)dt. \tag{2.12}$$

In particular, for $x\leq 0$, the integral depends only on the restriction of h to $\mathbb{R}_-$, for any $h\in\mathscr{B}_{b+}(\mathbb{R})$. □

We now state the main result of this paper.

Theorem 2.2. **(Main Result)** *Suppose that g is a bounded measurable function defined on $\mathbb{R}_-$. Then the r-balayage $P^r g$ is given by*

$$P^r g(x) = \sum_{j=1}^{m} \sum_{k=0}^{m_j-1} \left[\frac{\sum_{\ell=0}^{k} \boldsymbol{U}_{j,k-\ell} \boldsymbol{V}_{j,\ell}(g)}{(m_j-1-k)!} \right] x^{m_j-1-k} e^{\rho_j x}. \tag{2.13}$$

Here, we firstly define the constants $\boldsymbol{W}^{(j,\ell)}_{p,q}$ by the partial fraction decomposition of $Q_{j,\ell}/P$:

$$\frac{Q_{j,\ell}(2\pi i\zeta)}{P(2\pi i\zeta)} = \begin{cases} \delta_{0,\ell} + \sum_{p=1}^{n} \sum_{q=1}^{n_p} \frac{\boldsymbol{W}^{(j,\ell)}_{p,q}}{(2\pi i\zeta+\lambda_p)^q}, & \text{if } X^{(+)} \text{ is not a subordinator,} \\ \sum_{p=1}^{n} \sum_{q=1}^{n_p} \frac{\boldsymbol{W}^{(j,\ell)}_{p,q}}{(2\pi i\zeta+\lambda_p)^q}, & \text{if } X^{(+)} \text{ is a subordinator,} \end{cases} \tag{2.14}$$

where $\delta_{0,\ell}$ is Kronecker's delta, and then set the linear functionals $\boldsymbol{V}_{j,\ell}$ by

$$\boldsymbol{V}_{j,\ell}(h) = \begin{cases} h(0)\delta_{0,\ell} + \sum_{p=1}^{n} \sum_{q=1}^{n_p} \boldsymbol{W}^{(j,\ell)}_{p,q} \left(\mathscr{E}_{\lambda_p}\right)^q h(0), & \text{if } X^{(+)} \text{ is not a subordinator,} \\ \sum_{p=1}^{n} \sum_{q=1}^{n_p} \boldsymbol{W}^{(j,\ell)}_{p,q} \left(\mathscr{E}_{\lambda_p}\right)^q h(0), & \text{if } X^{(+)} \text{ is a subordinator.} \end{cases} \tag{2.15}$$

The formula of each $(\mathscr{E}_{\lambda_p})^q$ is given by (2.12).

Proof. We prove the desired conclusion holds whenever $g \in \mathscr{S}$. By (2.7) and Lemma 2.1, we obtain

$$\mathbb{E}_x\left[e^{-r\tau} g(X_\tau)\right] = \sum_{j=1}^{m} \sum_{k=0}^{m_j-1} \frac{x^{m_j-1-k} e^{\rho_j x}}{(m_j-1-k)!} \sum_{\ell=0}^{k} \boldsymbol{U}_{j,k-\ell} \int_{-\infty}^{\infty} \frac{Q_{j,\ell}(2\pi i\zeta)}{P(2\pi i\zeta)} \mathscr{F} g(\zeta) d\zeta. \tag{2.16}$$

By Proposition 2.1 and the Fourier inversion formula, we deduce that the formula (2.13) holds for every $g \in \mathscr{S}$.
Next, we show that (2.13) holds for all bounded Borel measurable functions g. Given a bounded Borel measurable function g, choose a uniformly bounded sequence (g_q) of Schwartz functions such that g_q converges to g almost everywhere on $(-\infty, 0)$ and $g_q(0) = g(0)$ for all q. (See [7] Lemma A.1 and [14] Chapter 2 Theorem 2.4.) Recall the law of X_τ restricted on $(-\infty, 0)$ is absolutely continuous under $\mathbb{P}_x$ for $x > 0$ (see [7] Lemma A.2). Hence, by passing limit, we see that $P^r g_q(x) \longrightarrow P^r g(x)$. On the other hand, note that for each $q \geq 1$, the formula (2.13) holds with g replaced by g_q. If we denote by $\widetilde{P}^r g$ the function on the right hand side of (2.13), then by Remark 1 and dominated convergence $P^r g_q(x) \longrightarrow \widetilde{P}^r g(x)$. This gives $P^r g(x) = \widetilde{P}^r g(x)$, and the proof is complete. □

Remark 2. (1) To avoid using (2.9), one can first consider the Laplace transform of r-balayages for $g \in \mathscr{S}$. Recall that the Pecherskii-Rogzin identity expresses the Laplace transform of $a_r(x,z)$ in x in terms of the negative Wiener-Hopf factor. This implies that one may identify the Fourier symbol of the operator $\mathscr{L}P^r$ in terms of the negative Wiener-Hopf factor, where the operator $\mathscr{L}$ is the Laplace transform operator. Using a version of Laplace inversion formula and residue calculus, one obtains the same result as in Theorem 2.2.

(2) Take $g \equiv 1$, $g = \mathbb{1}_{\{0\}}$ and $g = \mathbb{1}_{(-\infty,0)}$. If X stands for the surplus process of an insurance company, then the resulting balayages represent respectively the discounted ruin probability, the discounted measure of ruin caused by diffusion, and the discounted measure of ruin caused by jump. □

3 Consistency with an Existing Formula

As in [7], we consider the case that the downward jump distribution F is a convex combination of exponential distributions:

$$dF(y) = \sum_{j=1}^{n} p_j \lambda_j e^{-\lambda_j y} 1_{y>0} dy,$$

where $p_j > 0$ and $\lambda_j > 0$ for all $1 \le j \le n$ and λ_j are distinct. We assume further that $X^{(+)}$ is not a subordinator.
In this case, we have the followings.

(1) $n_j = 1$ for all $1 \le j \le n$ and $m_j = 1$ for all $1 \le j \le m = n+1$. See (1.2) and Theorem 2.1.

(2) $\phi_r^-(z) = \frac{\prod_{j=1}^{m}(-\rho_j)}{\prod_{j=1}^{n}\lambda_j} \frac{P(z)}{Q(z)}$, where $P(z) = \prod_{j=1}^{n}(z+\lambda_j)$ and $Q(z) = \prod_{j=1}^{n+1}(z-\rho_j)$. See Theorem 2.1.

(3) $\phi_r^-(z) = \frac{\prod_{j=1}^{m}(-\rho_j)}{\prod_{j=1}^{n}\lambda_j} \sum_{j=1}^{n+1} \frac{\boldsymbol{U}_{j,0}}{z-\rho_j}$, where $\boldsymbol{U}_{j,0}$ are given by

$$\boldsymbol{U}_{j,0} = \frac{\prod_{k=1}^{n}(\rho_j+\lambda_k)}{\prod_{k\neq j, 1\le k\le n+1}(\rho_j-\rho_k)}, \quad \forall\ 1 \le j \le n+1. \tag{3.1}$$

See Theorem 2.1.

(4) $Q_{j,0}(z) = \prod_{\ell=1,\ell\neq j}^{n+1}(z-\rho_\ell)$ and the constants $\left(\boldsymbol{W}_{k,1}^{(j,0)}\right)$ are defined by the equation:

$$\frac{Q_{j,0}(z)}{P(z)} = 1 + \sum_{k=1}^{n} \frac{\boldsymbol{W}_{k,1}^{(j,0)}}{z+\lambda_k}.$$

Hence,

$$\boldsymbol{W}_{k,1}^{(j,0)} = \frac{\prod_{\ell=1,\ell\neq j}^{n+1}(-\lambda_k-\rho_\ell)}{\prod_{\ell=1,\ell\neq k}^{n}(-\lambda_k+\lambda_\ell)} = \frac{-\prod_{\ell=1,\ell\neq j}^{n+1}(\lambda_k+\rho_\ell)}{\prod_{\ell=1,\ell\neq k}^{n}(\lambda_k-\lambda_\ell)}. \tag{3.2}$$

See Theorem 2.2.

(5) The linear functionals $(\boldsymbol{V}_{j,0}; 1 \le j \le n+1)$ are given by

$$\boldsymbol{V}_{j,0}(g) = g(0) + \sum_{k=1}^{n} \boldsymbol{W}_{k,1}^{(j,0)} \mathscr{E}_{\lambda_k} g(0). \tag{3.3}$$

See Theorem 2.2.

(6) If g is a bounded measurable function defined on $\mathbb{R}_-$, then we have

$$P^r g(x) = \sum_{j=1}^{m} e^{\rho_j x} \boldsymbol{U}_{j,0} \boldsymbol{V}_{j,0}(g), \quad x > 0. \tag{3.4}$$

where $\boldsymbol{U}_{j,0}$ and $\boldsymbol{V}_{j,0}(g)$ are given in (3.1) and (3.3) respectively. See Theorem 2.2.

In the following, we will show that our formula (3.4) coincides with the one in [7] Theorem 4.3. We write $P^r g(x) = \sum_{j=1}^{m} e^{\rho_j x} \boldsymbol{T}_{j,0}(g)$, where $\boldsymbol{T}_{j,0}(g)$ is given in (4.6) of Chen et al.(2007). We verify that $\boldsymbol{T}_{j,0}(g) = \boldsymbol{U}_{j,0} \boldsymbol{V}_{j,0}(g)$. By linearity of the operator P^r, it suffices to consider the two cases: (I) $g \equiv \mathbb{1}_{\{0\}}$, and (II) $g \in \mathscr{B}_b(\mathbb{R})$ such that $g(0) = 0$.
In case (I), the formula in [7] Example 4.5 states that, by setting $\lambda_{n+1} = 0$,

$$\begin{aligned}
\boldsymbol{T}_{j,0}(g) =& \frac{1}{\rho_j \prod_{k=1,k\neq j}^{n+1}(-\rho_j + \rho_k)} \sum_{k=1}^{n+1} \left(\prod_{\ell=1}^{n+1} (\lambda_k + \rho_\ell) \prod_{i=1,i\neq k}^{n+1} \frac{-\rho_j - \lambda_i}{\lambda_k - \lambda_i} \right) \\
=& \frac{1}{\rho_j \prod_{k=1,k\neq j}^{n+1}(\rho_j - \rho_k)} \sum_{k=1}^{n+1} \left(\prod_{\ell=1}^{n+1} (\lambda_k + \rho_\ell) \prod_{i=1,i\neq k}^{n+1} \frac{\rho_j + \lambda_i}{\lambda_k - \lambda_i} \right) \\
=& \frac{1}{\rho_j \prod_{k=1,k\neq j}^{n+1}(\rho_j - \rho_k)} \sum_{k=1}^{n+1} \left(\prod_{\ell=1,\ell\neq j}^{n+1} (\lambda_k + \rho_\ell) \prod_{i=1,i\neq k}^{n+1} \frac{1}{\lambda_k - \lambda_i} \prod_{i=1,i\neq k}^{n+1} (\rho_j + \lambda_i)(\rho_j + \lambda_k) \right) \\
=& \frac{\prod_{i=1}^{n+1}(\rho_j + \lambda_i)}{\prod_{k=1,k\neq j}^{n+1}(\rho_j - \rho_k)} \frac{1}{\rho_j} \sum_{k=1}^{n+1} \left(\prod_{\ell=1,\ell\neq j}^{n+1} (\lambda_k + \rho_\ell) \prod_{i=1,i\neq k}^{n+1} \frac{1}{\lambda_k - \lambda_i} \right) \\
=& \frac{\prod_{i=1}^{n}(\rho_j + \lambda_i)}{\prod_{k=1,k\neq j}^{n+1}(\rho_j - \rho_k)} \sum_{k=1}^{n+1} \left(\prod_{\ell=1,\ell\neq j}^{n+1} (\lambda_k + \rho_\ell) \prod_{i=1,i\neq k}^{n+1} \frac{1}{\lambda_k - \lambda_i} \right). \\
=& \boldsymbol{U}_{j,0} \sum_{k=1}^{n+1} \left(\prod_{\ell=1,\ell\neq j}^{n+1} (\lambda_k + \rho_\ell) \prod_{i=1,i\neq k}^{n+1} \frac{1}{\lambda_k - \lambda_i} \right),
\end{aligned} \tag{3.5}$$

where the last equality follows from (3.1). Since $\boldsymbol{V}_{j,0}(g) = 1$, it suffices to show that

$$\sum_{k=1}^{n+1} \left(\prod_{\ell=1,\ell\neq j}^{n+1} (\lambda_k + \rho_\ell) \prod_{i=1,i\neq k}^{n+1} \frac{1}{\lambda_k - \lambda_i} \right) = 1. \tag{3.6}$$

Note that, since $\lambda_{n+1} = 0$, we have

$$\begin{aligned}
&\sum_{k=1}^{n+1} \left(\prod_{\ell=1,\ell\neq j}^{n+1} (\lambda_k + \rho_\ell) \prod_{i=1,i\neq k}^{n+1} \frac{1}{\lambda_k - \lambda_i} \right) \\
=& \sum_{k=1}^{n} \left(\prod_{\ell=1,\ell\neq j}^{n+1} (\lambda_k + \rho_\ell) \prod_{i=1,i\neq k}^{n} \frac{1}{\lambda_k - \lambda_i} \right) \frac{1}{\lambda_k} + \prod_{\ell=1,\ell\neq j}^{n+1} \rho_\ell \prod_{i=1}^{n} \frac{1}{-\lambda_i}.
\end{aligned} \tag{3.7}$$

Next, fixed $1 \leq j \leq n+1$ and set $\lambda_{n+2} = -\rho_j$. Then $Q(-\lambda_{n+2}) = 0$. Since Q is a polynomial of order $n+1$, we obtain by Lagrange's interpolation that

$$
\begin{aligned}
Q(-x) =& \sum_{k=1}^{n+1} Q(-\lambda_k) \prod_{i=1, i\neq k}^{n+2} \frac{x-\lambda_i}{\lambda_k-\lambda_i} \\
=& (x+\rho_j) \sum_{k=1}^{n+1} \frac{Q(-\lambda_k)}{\lambda_k+\rho_j} \prod_{i=1, i\neq k}^{n+1} \frac{x-\lambda_i}{\lambda_k-\lambda_i} \\
=& (x+\rho_j) P(-x) x \sum_{k=1}^{n+1} \frac{\prod_{\ell=1}^{n+1}(\lambda_k+\rho_\ell)}{(-x+\lambda_k)(\lambda_k+\rho_j)} \prod_{i=1, i\neq k}^{n+1} \frac{1}{\lambda_k-\lambda_i}
\end{aligned}
$$

Recall $Q_{j,0}(x) = Q(x)/(x-\rho_j)$. Then

$$
Q_{j,0}(-x) = P(-x)(-x) \sum_{k=1}^{n+1} \frac{\prod_{\ell=1,\ell\neq j}^{n+1}(\lambda_k+\rho_\ell)}{-x+\lambda_k} \prod_{i=1, i\neq k}^{n+1} \frac{1}{\lambda_k-\lambda_i}
$$

This implies that, using the fact that $\lambda_{n+1} = 0$ again,

$$
\begin{aligned}
\frac{Q_{j,0}(-x)}{P(-x)} =& -x \sum_{k=1}^{n} \frac{\prod_{\ell=1,\ell\neq j}^{n+1}(\lambda_k+\rho_\ell)}{(-x+\lambda_k)} \prod_{i=1, i\neq k}^{n+1} \frac{1}{\lambda_k-\lambda_i} - x \frac{\prod_{\ell=1,\ell\neq j}^{n+1}\rho_\ell}{-x} \prod_{i=1}^{n} \frac{1}{-\lambda_i} \\
=& -x \sum_{k=1}^{n} \frac{\prod_{\ell=1,\ell\neq j}^{n+1}(\lambda_k+\rho_\ell)}{(-x+\lambda_k)} \prod_{i=1, i\neq k}^{n} \frac{1}{\lambda_k-\lambda_i} \frac{1}{\lambda_k} + \prod_{\ell=1,\ell\neq j}^{n+1} \rho_\ell \prod_{i=1}^{n} \frac{1}{-\lambda_i}.
\end{aligned}
$$

Since $Q_{j,0}$ and P are both monic polynomials of order n, we see that, by letting $x \longrightarrow \infty$,

$$
1 = \sum_{k=1}^{n} \prod_{\ell=1,\ell\neq j}^{n+1} (\lambda_k+\rho_\ell) \prod_{i=1, i\neq k}^{n} \frac{1}{\lambda_k-\lambda_i} \frac{1}{\lambda_k} + \prod_{\ell=1,\ell\neq j}^{n+1} \rho_\ell \prod_{i=1}^{n} \frac{1}{(-\lambda_i)}
$$

Using this equation, we deduce from (3.7) that (3.6) holds. We have proved that our result coincides with the one in [7] Example 4.5 for $g \equiv \mathbb{1}_{\{0\}}$.
Next, consider case (II). Then the formula (4.6) in [7] Theorem 4.3 states that

$$
\begin{aligned}
\boldsymbol{T}_{j,0}(g) =& \frac{-1}{\rho_j \prod_{k=1,k\neq j}^{n+1}(-\rho_j+\rho_k)} \sum_{k=1}^{n} \left(\prod_{\ell=1}^{n+1} (\lambda_k+\rho_\ell) \prod_{i=1, i\neq k}^{n+1} \frac{-\rho_j-\lambda_i}{\lambda_k-\lambda_i} \int_{-\infty}^{0} g(y) \lambda_k e^{\lambda_k y} dy \right) \\
=& -\frac{\prod_{i=1}^{n}(\rho_j+\lambda_i)}{\prod_{k=1,k\neq j}^{n+1}(\rho_j-\rho_k)} \sum_{k=1}^{n} \left(\prod_{\ell=1,\ell\neq j}^{n+1} (\lambda_k+\rho_\ell) \prod_{i=1, i\neq k}^{n+1} \frac{1}{\lambda_k-\lambda_i} \int_{-\infty}^{0} g(y) \lambda_k e^{\lambda_k y} dy \right) \\
=& -\boldsymbol{U}_{j,0} \sum_{k=1}^{n} \left(\prod_{\ell=1,\ell\neq j}^{n+1} (\lambda_k+\rho_\ell) \prod_{i=1, i\neq k}^{n} \frac{1}{\lambda_k-\lambda_i} \mathscr{E}_{\lambda_k} g(0) \right) \\
=& \boldsymbol{U}_{j,0} \sum_{k=1}^{n} \boldsymbol{W}_{k,1}^{(j,0)} \mathscr{E}_{\lambda_k} g(0),
\end{aligned}
$$

where the last equality follows from (3.2). By (3.3) and the fact that $g(0) = 0$, the last equality gives $\boldsymbol{T}_{j,0}(g) = \boldsymbol{U}_{j,0} \boldsymbol{V}_{j,0}(g)$. The conclusion is that our result coincides with the one in [7] Theorem 4.3 in case (II) as well.

4 Another First-Passage Functional

Let $f \in \mathscr{B}_b(\mathbb{R})$ and $r > 0$. Consider the first passage functional

$$I^r f(x) \triangleq \mathbb{E}_x \left[\int_0^\tau e^{-rs} f(X_s) ds \right].$$

By strong Markov property, we have

$$I^r f(x) = \mathbb{E}_x \left[\int_0^\infty e^{-rs} f(X_s) ds \right] - \mathbb{E}_x \left[e^{-r\tau} \mathbb{E}_{X_\tau} \left[\int_0^\infty e^{-rs} f(X_s) ds \right] \right] = U^r f - P^r U^r f(x), \quad (4.1)$$

where U^r is the r-resolvent kernel of X. Hence, to derive a semi-explicit solution of $I^r f$, it suffices to find a semi-explicit solution of $U^r f$. Suppose that the upper Lévy measure of X is also matrix-exponential. In this case, X is merely a perturbed compound Poisson process both sides of whose jump distributions are matrix-exponential. We can use Theorem 2.1 to calculate $U^r f$. Recall that $\mathbf{e}_r$ is an exponential random variable independent of X and has mean $1/r$. By Wiener-Hopf factorization of Lévy processes,

$$U^r f(x) = \mathbb{E}\left[f(x + X_{\mathbf{e}_r}) \right] = \int_{\mathbb{R}_+} \mathbb{P}\left[\sup_{0 \le s \le \mathbf{e}_r} X_s \in dy \right] \int_{\mathbb{R}_-} \mathbb{P}\left[\inf_{0 \le s \le \mathbf{e}_r} X_s \in dz \right] f(x + y + z). \quad (4.2)$$

By Theorem 2.1, one can write down semi-explicit formulas for the distributions of $\sup_{0 \le s \le \mathbf{e}_r} X_s$ and $\inf_{0 \le s \le \mathbf{e}_r} X_s$. A semi-explicit solution of $U^r f$ will then follow from (4.2). By (4.1), this further implies a semi-explicit solution of $I^r f$. We leave the details to the readers.

References

[1] Alili, L., Kyprianou, A.E. *Some remarks on first passage of Lévy processes, the American put and pasting principles*, Annals of Applied Probability Vol.**15** No.3 (2005), 2062-2080. MR2152253

[2] Asmussen, S. *Applied Probability and Queues*, Vol.**51** of *Applications of Mathematics* (New York). 2nd edition, Springer-Verlag, Berlin, (2003). MR1978607

[3] Asmussen, S., Avram, F., Pistorius, M.R. *Russian and American put options under exponential phase-type Lévy modesl*, Stochastic Processes and Their Applications Vol.**109** (2004), 79-111. MR2024845

[4] Avram, F., Usabel, M. *Finite time ruin probabilities with one Laplace transform*, Insurance, Mathematics and Economics Vol.**32** (2003), 371-377. MR1994498

[5] Botta, R.F., Harris, C.M. *Approximation with generalized hyperexponential distributions: weak convergence results*, Queueing Systems Vol.**2** (1986), 169-190. MR0896234

[6] Boyarchenko, S., Levendorskiĭ, S. *American options: the EPV pricing model*, Annals of Finance Vol.**1** (2005), 267-292.

[7] Chen, Y.T., Lee, C.F., Sheu, Y.C. *An ODE approach for the expected discounted penalty at ruin in a jump-diffusion model*, Finance and Stochastics Vol. **11** (2007), 323-355. MR2322916

[8] Dixit, A.K., Pyndick, R.S. *Investment under Uncertainty*. Princeton University Press, New Jersey, (1994).

[9] Gerber, H.U., Shiu, E.S.W. *On the time value of ruin*, North American Actuarial Journal Vol.**2** (1998), 48-78. MR1988433

[10] Lewis, A.L., Mordecki, E. *Wiener-Hopf factorization for Lévy processes having positive jumps with rational transforms*, Journal of Applied Probability Vol.**45** (2008), 118-134. MR2409315

[11] Mordecki, E. *Optimal stopping and perpetual options for Lévy processes*, Finance and Stochastics Vol.**6** (2002), 473-493. MR1932381

[12] Sato, K. *Lévy processes and infinitely divisible distributions*, Cambridge University Press, Cambridge, (1999).

[13] Stein, E.M., Shakarchi, R. *Fourier Analysis, Princeton Lectures in Analysis I*, Princeton University Press, New Jersey, (2003). MR1970295

[14] Stein, E.M., Shakarchi, R. *Real Analysis, Princeton Lectures in Analysis III*, Princeton University Press, New Jersey,(2005). MR2129625