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THE SCALING LIMIT OF SENILE REINFORCED RANDOM WALK.

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Abstract
We prove that the scaling limit of nearest-neighbour senile reinforced random walk is Brownian Motion when the time T spent on the first edge has finite mean. We show that under suitable conditions, when T has heavy tails the scaling limit is the so-called *fractional kinetics process*, a random time-change of Brownian motion. The proof uses the standard tools of time-change and invariance principles for additive functionals of Markov chains.

1 Introduction

The senile reinforced random walk is a toy model for a much more mathematically difficult model known as edge-reinforced random walk (for which many basic questions remain open [e.g. see [15]]). It is characterized by a reinforcement function $f : \mathbb{N} \to [-1,\infty)$, and has the property that only the most recently traversed edge is reinforced. As soon as a new edge is traversed, reinforcement begins on that new edge and the reinforcement of the previous edge is forgotten. Such walks may get stuck on a single (random) edge if the reinforcement is strong enough, otherwise (except for one degenerate case) they are recurrent/transient precisely when the corresponding simple random walk is [9].
Formally, a nearest-neighbour *senile reinforced random walk* is a sequence $\{S_n\}_{n\geq 0}$ of $\mathbb{Z}^d$-valued random variables on a probability space $(\Omega,\mathscr{F},\mathbb{P}_f)$, with corresponding filtration $\{\mathscr{F}_n=\sigma(S_0,\dots,S_n)\}_{n\geq 0}$, defined by:

- $S_0=o$, $\mathbb{P}_f$-almost surely, and $\mathbb{P}_f(S_1=x)=(2d)^{-1}I_{\{|x|=1\}}$.
- For $n\in\mathbb{N}$, $e_n=(S_{n-1},S_n)$ is an $\mathscr{F}_n$-measurable *undirected* edge and

$$m_n=\max\{k\geq 1 : e_{n-l+1}=e_n \text{ for all } 1\leq l\leq k\} \quad (1.1)$$

  is an $\mathscr{F}_n$-measurable, $\mathbb{N}$-valued random variable.

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- For $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$ such that $|x| = 1$,

$$\mathbb{P}_f(S_{n+1} = S_n + x|\mathscr{F}_n) = \frac{1 + f(m_n)I_{\{(S_n,S_n+x)=e_n\}}}{2d + f(m_n)}. \tag{1.2}$$

Note that the triple (S_n, e_n, m_n) (equivalently (S_n, S_{n-1}, m_n)) is a Markov chain. Hereafter we suppress the f dependence of the probability $\mathbb{P}_f$ in the notation.

The diffusion constant is defined as $\lim_{n\to\infty} n^{-1}\mathbb{E}[|S_n|^2]$ (=1 for simple random walk) whenever this limit exists. Let T denote the random number of consecutive traversals of the first edge traversed, and $p = \mathbb{P}(T \text{ is odd})$. Then when $\mathbb{E}[T] < \infty$, the diffusion constant is given by ([9] and [11])

$$v = \frac{dp}{(d-p)\mathbb{E}[T]}, \tag{1.3}$$

which is not monotone in the reinforcement. Indeed one can prove that (1.3) holds for all f (in the case $d = 1$ and $f(1) = -1$ this must be interpreted as " $1/0 = \infty$"). The reinforcement regime of most interest is that of linear reinforcement $f(n) = Cn$ for some C. In this case, by the second order mean-value theorem applied to $\log(1-x)$, $x < 1$ we have

$$\begin{aligned}
\mathbb{P}(T \geq n) := \prod_{j=1}^{n-1} \frac{1+f(j)}{2d+f(j)} &= \exp\left\{\sum_{j=1}^{n-1} \log\left(1 - \frac{2d-1}{2d+Cj}\right)\right\} \\
&= \exp\left\{-\sum_{j=1}^{n-1} \frac{2d-1}{2d+Cj} - \sum_{j=1}^{n-1} \frac{(2d-1)^2}{2(2d+Cj)^2(1-u_j)^2}\right\} \\
&= \exp\left\{-\sum_{j=1}^{n-1} \frac{2d-1}{2d+Cj} - \sum_{j=1}^{\infty} \frac{(2d-1)^2}{2(2d+Cj)^2(1-u_j)^2} + o(1)\right\} \\
&= \exp\left\{-\frac{2d-1}{C}\log(2d + C(n-1)) + \gamma + o(1)\right\} \sim \frac{\kappa}{n^{\frac{2d-1}{C}}},
\end{aligned} \tag{1.4}$$

where $u_i \in (0, \frac{2d-1}{2d+Cj})$, and γ is a constant arising from the summable infinite series and the approximation of the finite sum by a log. An immediate consequence of (1.4) is that for $f(n) = Cn$, $\mathbb{E}[T]$ is finite if and only if $C < 2d - 1$.

A different but related model, in which the current direction (rather than the current edge) is reinforced according to the function f was studied in [12, 10]. For such a model, T is the number of consecutive steps in the same direction before turning. In [10], the authors show that in all dimensions the scaling limit is Brownian motion when σ^2 =Var$(T) < \infty$ and $\sigma^2 + 1 - 1/d > 0$. In the language of this paper, the last condition corresponds to the removal of the special case $d = 1$ and $f(1) = -1$. Moreover when $d = 1$ and T has heavy tails (in the sense of (2.1) below) they show that the scaling limit is an α-stable process when $1 < \alpha < 2$ and a random time change of an α-stable process when $0 < \alpha < 1$. See [10] for more details.

Davis [4] showed that the scaling limit of once-reinforced random walk in one dimension is not Brownian motion (see [15] for further discussion).

In Section 2 we state and discuss the main result of this paper, which describes the scaling limit of S_n when either $\mathbb{E}[T] < \infty$ or $\mathbb{P}(T \geq n) \sim n^{-\alpha}L(n)$ for some $\alpha > 0$ and L slowly varying at infinity. When $\mathbb{P}(T < \infty) < 1$ the walk has finite range since it traverses a random (geometric) number of edges before getting stuck on a random edge. To prove the main result, in Section 3 we observe the walk at the times that it has just traversed a new edge and describe this as an additive functional of a particular Markov chain. In Section 4 we prove the main result assuming the joint convergence of this time-changed walk and the associated time-change process. In Section 5 we prove the convergence of this joint process.

2 Main result

The assumptions that will be necessary to state the main theorem of this paper are as follows:

(A1) $\mathbb{P}(T < \infty) = 1$, and either $d > 1$ or $\mathbb{P}(T = 1) < 1$.

(A2a) Either $\mathbb{E}[T] < \infty$, or for some $\alpha \in (0,1]$ and L slowly varying at infinity,

$$\mathbb{P}(T \geq n) \sim L(n)n^{-\alpha}. \tag{2.1}$$

(A2b) If (2.1) holds but $\mathbb{E}[T] = \infty$, then we also assume that

$$\begin{aligned} &\text{when } \alpha = 1,\ \exists\ \ell(n) \uparrow \infty \text{ such that } (\ell(n))^{-1}L(n\ell(n)) \to 0, \text{ and } (\ell(n))^{-1}\sum_{j=1}^{\lfloor n\ell(n)\rfloor} j^{-1}L(j) \to 1, \\ &\text{when } \alpha < 1,\ \mathbb{P}(T \geq n, T \text{ odd}) \sim L_o(n)n^{-\alpha_o}, \text{ and } \mathbb{P}(T \geq n, T \text{ even}) \sim L_e(n)n^{-\alpha_e}, \end{aligned} \tag{2.2}$$

where ℓ, L_o and L_e are slowly varying at ∞ and L_o and L_e are such that if $\alpha_o = \alpha_e$ then $L_o(n)/L_e(n) \to \beta \in [0,\infty]$ as $n \to \infty$.

Note that both (2.1) and $\mathbb{E}[T] < \infty$ may hold when $\alpha = 1$ (e.g. take $L(n) = (\log n)^{-2}$).
By [6, Theorem XIII.6.2], when $\alpha < 1$ there exists $\ell(\cdot) > 0$ slowly varying such that

$$(\ell(n))^{-\alpha} L\left(n^{\frac{1}{\alpha}}\ell(n)\right) \to (\Gamma(1-\alpha))^{-1}. \tag{2.3}$$

For $\alpha > 0$ let

$$g_\alpha(n) = \begin{cases} \mathbb{E}[T]n & \text{, if } \mathbb{E}[T] < \infty \\ n^{\frac{1}{\alpha}}\ell(n) & \text{, otherwise} \end{cases} \tag{2.4}$$

By [3, Theorem 1.5.12], there exists an asymptotic inverse function $g_\alpha^{-1}(\cdot)$ (unique up to asymptotic equivalence) satisfying $g_\alpha(g_\alpha^{-1}(n)) \sim g_\alpha^{-1}(g_\alpha(n)) \sim n$, and by [3, Theorem 1.5.6] we may assume that g_α and g_α^{-1} are monotone nondecreasing.
A *subordinator* is a real-valued process starting at 0, with stationary, independent increments, such that almost every path is nondecreasing and right continuous. Let $B_d(t)$ be a standard d-dimensional Brownian motion. For $\alpha \geq 1$, let $V_\alpha(t) = t$ and for $\alpha \in (0,1)$, let V_α be a standard α-stable subordinator (independent of $B_d(t)$). This is a strictly increasing pure-jump Levy process whose law is specified by the Laplace transform of its one-dimensional distributions (see e.g. [2, Sections 1.1 and 1.2])

$$\mathbb{E}[e^{-\lambda V_\alpha(t)}] := e^{-t\lambda^\alpha}. \tag{2.5}$$

Define the right-continuous inverse of $V_\alpha(t)$ and (when $\alpha < 1$ the fractional-kinetics process) $Z_\alpha(s)$ by

$$V_\alpha^{-1}(s) := \inf\{t : V_\alpha(t) > s\}, \quad Z_\alpha(s) = B_d(V_\alpha^{-1}(s)). \tag{2.6}$$

Since V_α is strictly increasing, both V_α^{-1} and Z_α are continuous (almost-surely). The main result of this paper is the following theorem, in which $D(E, \mathbb{R}^d)$ is the set of *cadlag* paths from E to $\mathbb{R}^d$. Throughout this paper $\Longrightarrow$ denotes weak convergence.

Theorem 2.1. *Suppose that f is such that (2.1) holds for some $\alpha > 0$, then as $n \to \infty$,*

$$\frac{S_{\lfloor nt \rfloor}}{\sqrt{\frac{p}{d-p} g_\alpha^{-1}(n)}} \Longrightarrow Z_\alpha(t), \tag{2.7}$$

where the convergence is in $D([0,1], \mathbb{R}^d)$ equipped with the uniform topology.

2.1 Discussion

The limiting object in Theorem 2.1 is the scaling limit of a simple random walk jumping at random times τ_i with i.i.d. increments $T_i = \tau_i - \tau_{i-1}$ (e.g. see [13]) that are independent of the position and history of the walk. In [1] the same scaling limit is obtained for a class of (continuous time) trap models with $d \geq 2$, where a random jump rate or waiting time is chosen initially at each site and remains fixed thereafter. In that work, when $d = 1$, the mutual dependence of the time spent at a particular site on successive returns remains in the scaling limit, where the time change/clock process depends on the (local time of the) Brownian motion itself. The independence on returning to an edge is the feature which makes our model considerably easier to handle. For the senile reinforced random walk, the direction of the steps of the walk is dependent on the clock and we need to prove that the dependence is sufficiently weak so that it disappears in the limit.

While the slowly varying functions in g_α and g_α^{-1} are not given explicitly, in many cases of interest one can use [6, Theorem XIII.6.2] and [3, Section 1.5.7] to explicitly construct them. For example, let $L(n) = \kappa(\log n)^\beta$ for some $\beta \geq -1$. For $\alpha = 1$ we can take

$$\ell(n) = \begin{cases} \kappa \log n, & \text{if } \beta = 0 \\ \kappa(\log\log n), & \text{if } \beta = -1 \\ |\beta^{-1}|\kappa(\log n)^{\beta+1}, & \text{otherwise,} \end{cases} \quad \text{and} \quad g_\alpha^{-1}(n) = \begin{cases} n(\kappa \log n)^{-1}, & \text{if } \beta = 0 \\ n(\kappa \log\log n)^{-1}, & \text{if } \beta = -1 \\ n|\beta|(\kappa \log n)^{-(\beta+1)}, & \text{otherwise.} \end{cases} \tag{2.8}$$

If $\alpha < 1$ we can take

$$\ell(n) = \left(\kappa\Gamma(1-\alpha)\left(\frac{\log n}{\alpha}\right)^\beta\right)^{\frac{1}{\alpha}}, \qquad \text{and} \qquad g_\alpha^{-1}(n) = n^\alpha \left(\kappa(\alpha \log n)^\beta\right)^{-\alpha}. \tag{2.9}$$

Assumption (A1) is simply to avoid the trivial cases where the walk gets stuck on a single edge (i.e. when $(1 + f(n))^{-1}$ is summable [9]) or is a self-avoiding walk in one dimension. For linear reinforcement $f(n) = Cn$, (1.4) shows that assumption (A2) holds with $\alpha = (2d-1)/C$. It may be of interest to consider the scaling limit when $f(n)$ grows like $n\ell(n)$, where $\liminf_{n\to\infty} \ell(n) = \infty$ but such that $(1 + f(n))^{-1}$ is not summable. An example is $f(n) = n \log n$, for which $\mathbb{P}(T \geq n) \sim (C \log n)^{-1}$ satisfies (2.1) with $\alpha = 0$.

The condition (2.2) when $\alpha = 1$ is so that one can apply a weak law of large numbers. The condition holds for example when $L(n) = (\log n)^k$ for any $k \geq -1$. For the $\alpha < 1$ case, the condition (2.2) holds (with $\alpha_o = \alpha_e$ and $L_o = L_e$) whenever there exists n_0 such that for all $n \geq n_0$, $f(n) \geq f(n-1) - (2d-1)$ (so in particular when f is non-decreasing). To see this, observe that for all $n \geq n_0$

$$\begin{aligned}\mathbb{P}(T \geq n, T \text{ even}) &= \sum_{m=\lfloor \frac{n+1}{2} \rfloor}^{\infty} \mathbb{P}(T = 2m) = \sum_{m=\lfloor \frac{n+1}{2} \rfloor}^{\infty} \mathbb{P}(T = 2m+1) \frac{2d + f(2m+1)}{1 + f(2m)} \\ &\geq \sum_{m=\lfloor \frac{n+1}{2} \rfloor}^{\infty} \mathbb{P}(T = 2m+1) = \mathbb{P}(T \geq n+1, T \text{ odd}).\end{aligned} \tag{2.10}$$

Similarly, $\mathbb{P}(T \geq n, T \text{ odd}) \geq \mathbb{P}(T \geq n+1, T \text{ even})$ for all $n \geq n_0$. If $\alpha_o \neq \alpha_e$ in (2.2), then (2.1) implies that $\alpha = \alpha_o \wedge \alpha_e$ and L is the slowly varying function corresponding to $\alpha \in \{\alpha_o, \alpha_e\}$ in (2.2). If $\alpha_o = \alpha_e$ then trivially $L \sim L_o + L_e$ ($\sim L_o$ if $L_o(n)/L_e(n) \to \infty$). One can construct examples of reinforcement functions giving rise to different asymptotics for the even and odd cases in (2.2), for example by taking $f(2m) = m^2$ and $f(2m+1) = Cm$ for some well chosen constant $C > 0$ depending on the dimension.

3 Invariance principle for the time-changed walk

In this section we prove an invariance principle for any senile reinforced random walk (satisfying (A1)) observed at stopping times τ_n defined by

$$\tau_0 = 0, \quad \tau_k = \inf\{n > (\tau_{k-1} \vee 1) : S_n \neq S_{n-2}\}. \tag{3.1}$$

It is easy to see that $\tau_n = 1 + \sum_{i=1}^{n} T_i$ for each $n \geq 1$, where the T_i, $i \geq 1$ are independent and identically distributed random variables (with the same distribution as T), corresponding to the number of consecutive traversals of successive edges traversed by the walk.

Proposition 3.1. *If (A1) is satisfied, then* $\left(\frac{p}{d-p} n\right)^{-\frac{1}{2}} S_{\tau_{\lfloor nt \rfloor}} \Longrightarrow B_d(t)$ *as* $n \to \infty$*, where the convergence is in* $D([0,1], \mathbb{R}^d)$ *with the uniform topology.*

The process S_{τ_n} is a simpler one than S_n and one may use many different methods to prove Proposition 3.1 (see for example the martingale approach of [11]). We give a proof based on describing S_{τ_n} as an additive functional of a Markov chain. This is not necessarily the simplest representation, but it is the most natural to the author.

Let $\mathscr{X}$ denote the collection of pairs (u, v) such that v is one of the unit vectors $u_i \in \mathbb{Z}^d$, for $i \in \{\pm 1, \pm 2, \cdots \pm d\}$ (labelled so that $u_{-i} = -u_i$) and u is either $0 \in \mathbb{Z}^d$ or one of the unit vectors $u_i \neq -v$. The cardinality of $\mathscr{X}$ is then $|\mathscr{X}| = 2d + 2d(2d-1) = (2d)^2$.

Given a senile reinforced random walk S_n with parameter $p = \mathbb{P}(T \text{ odd}) \in (0, 1]$, we define an irreducible, aperiodic Markov chain $X_n = (X_n^{[1]}, X_n^{[2]})$ with natural filtration $\mathscr{G}_n = \sigma(X_1, \ldots, X_n)$, and finite state-space $\mathscr{X}$, as follows.

For $n \geq 1$, let $X_n = (S_{\tau_n - 1} - S_{\tau_{(n-1)}}, S_{\tau_n} - S_{\tau_n - 1})$, and $Y_n = X_n^{[1]} + X_n^{[2]}$. It follows immediately that $S_{\tau_n} = \sum_{m=1}^{n} Y_m$ and

$$\mathbb{P}(X_1 = (0, u_i)) = \frac{1-p}{2d}, \quad \text{and} \quad \mathbb{P}(X_1 = (u_i, u_j)) = \frac{p}{2d(2d-1)}, \quad \text{for each } i, j, \quad (j \neq -i). \tag{3.2}$$

Now T_n is independent of $X_1,\dots,X_{n-1}$, and conditionally on T_n being odd (resp. even), $S_{\tau_n}-S_{\tau_{(n-1)}}$ (resp. $S_{\tau_n}-S_{\tau_n-1}$) is uniformly distributed over the $2d-1$ unit vectors in $\mathbb{Z}^d$ other than $-X^{[2]}_{n-1}$ (resp. other than $X^{[2]}_{n-1}$). It is then an easy exercise to verify that $\{X_n\}_{n\geq 1}$ is a finite, irreducible and aperiodic Markov chain with initial distribution (3.2) and transition probabilities given by

$$\mathbb{P}\left(X_n=(u,v)|X_{n-1}=(u',v')\right)=\frac{1}{2d-1}\times\begin{cases}p, & \text{if } u=0 \text{ and } v\neq -v',\\ 1-p, & \text{if } u=-v' \text{ and } v\neq v',\\ 0, & \text{otherwise.}\end{cases} \tag{3.3}$$

By symmetry, the first $2d$ entries of the unique stationary distribution $\vec{\pi}\in M_1(\mathscr{X})$ are all equal (say π_a) and the remaining $2d(2d-1)$ entries are all equal (say π_b), and it is easy to check that

$$\pi_a=\frac{p}{2d},\qquad \pi_b=\frac{1-p}{2d(2d-1)}. \tag{3.4}$$

As an irreducible, aperiodic, finite-state Markov chain, $\{X_n\}_{n\geq 1}$ has *exponentially fast, strong mixing*, i.e. there exists a constant c and $t<1$ such that for every $k\geq 1$,

$$\alpha(k):=\sup_n\Big\{|\mathbb{P}(F\cap G)-\mathbb{P}(F)\mathbb{P}(G)|:F\in\sigma(X_j,j\leq n),G\in\sigma(X_j,j\geq n+k)\Big\}\leq ct^k. \tag{3.5}$$

Since Y_n is measurable with respect to X_n, the sequence Y_n also has exponentially fast, strong mixing. To verify Proposition 3.1, we use the following multidimensional result that follows easily from [8, Corollary 1] using the Cramér-Wold device.

Corollary 3.2. *Suppose that $W_n=(W_n^{(1)},\dots,W_n^{(d)})$, $n\geq 0$ is a sequence of $\mathbb{R}^d$-valued random variables such that $\mathbb{E}[W_n]=0$, $\mathbb{E}[|W_n|^2]<\infty$ and $\mathbb{E}[n^{-1}\sum_{i=1}^n\sum_{i'=1}^n W_i^{(j)}W_{i'}^{(l)}]\to\sigma^2 I_{j=l}$, as $n\to\infty$. Further suppose that W_n is α-strongly mixing and that there exists $\beta\in(2,\infty]$ such that*

$$\sum_{k=1}^{\infty}\alpha(k)^{1-2/\beta}<\infty,\quad \textit{and}\quad \limsup_{n\to\infty}\|W_n\|_\beta<\infty, \tag{3.6}$$

then $\mathscr{W}_n(t):=(\sigma^2 n)^{-\frac{1}{2}}\sum_{i=1}^{\lfloor nt\rfloor}W_i\implies B_d(t)$ as $n\to\infty$, where the convergence is in $D([0,1],\mathbb{R}^d)$ equipped with the uniform topology.

3.1 Proof of Proposition 3.1

Since $S_{\tau_n}=\sum_{m=1}^n Y_m$ where $|Y_m|\leq 2$, and the sequence $\{Y_n\}_{n\geq 0}$ has exponentially fast strong mixing, Proposition 3.1 will follow from Corollary 3.2 provided we show that

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n\sum_{i'=1}^n Y_i^{(j)}Y_{i'}^{(l)}\right]\to\frac{p}{d-p}I_{j=l}, \tag{3.7}$$

where the superscript (j) denotes the jth component of the vector, e.g. $Y_m=(Y_m^{(1)},\dots,Y_m^{(d)})$. By symmetry, $\mathbb{E}[Y_i^{(j)}Y_{i'}^{(l)}]=0$ for all i,i' and $j\neq l$, and it suffices to prove (3.7) with $j=l=1$.
For $n\geq 2$, $\mathbb{E}[X_n^{[2],(1)}|X_{n-1}]=\frac{2p-1}{2d-1}X_{n-1}^{[2],(1)}$, so by induction and the Markov property,

$$\mathbb{E}[X_n^{[2],(1)}|X_m]=\left(\frac{2p-1}{2d-1}\right)^{n-m}X_m^{[2],(1)},\quad \text{for every } n\geq m\geq 1. \tag{3.8}$$

For $n \geq 2$, $\mathbb{E}[Y_n^{(1)}|X_{n-1}] = \frac{p-2d(1-p)}{2d-1} X_{n-1}^{[2],(1)}$, and the Markov property for X_n implies that

$$\mathbb{E}[Y_n^{(1)}|X_m] = \frac{p-2d(1-p)}{2d-1}\left(\frac{2p-1}{2d-1}\right)^{n-1-m} X_m^{[2],(1)}, \qquad \text{for } n > m \geq 1. \tag{3.9}$$

For $n > m \geq 1$, and letting $r = \frac{2p-1}{2d-1}$ we have

$$\begin{aligned}\mathbb{E}[Y_n^{(1)}Y_m^{(1)}] =& \mathbb{E}[Y_m^{(1)}\mathbb{E}[Y_n^{(1)}|X_m]] = \frac{p-2d(1-p)}{2d-1} r^{n-1-m}\mathbb{E}[Y_m^{(1)}X_m^{[2],(1)}] \\ =& \frac{p-2d(1-p)}{2d-1} r^{n-1-m}\left(\mathbb{E}[X_m^{[1],(1)}X_m^{[2],(1)}] + \mathbb{E}[(X_m^{[2],(1)})^2]\right) \\ =& \frac{p-2d(1-p)}{2d-1} r^{n-1-m} \times \begin{cases} \frac{1-p}{d(2d-1)} + \frac{1}{d}, & m \geq 2 \\ \frac{p}{d(2d-1)} + \frac{1}{d}, & m = 1. \end{cases}\end{aligned} \tag{3.10}$$

Lastly $\mathbb{E}[|Y_1|^2] = (1-p) + \frac{4dp}{2d-1}$ and $\mathbb{E}[|Y_m|^2] = p + \frac{4d(1-p)}{2d-1}$, for $m \geq 2$.
Combining these results, we get that

$$\begin{aligned}\mathbb{E}\left[\sum_{l=1}^{n}\sum_{m=1}^{n} Y_l^{(1)}Y_m^{(1)}\right] =& 2\sum_{l=2}^{n}\sum_{m=2}^{l-1}\mathbb{E}[Y_l^{(1)}Y_m^{(1)}] + 2\sum_{l=2}^{n}\mathbb{E}[Y_l^{(1)}Y_1^{(1)}] + \sum_{l=1}^{n}\mathbb{E}[|Y_l^{(1)}|^2] \\ =& \frac{2}{d}\frac{p-2d(1-p)}{2d-1}\left[\left(\frac{2d-p}{2d-1}\right)\sum_{l=2}^{n}\sum_{k=0}^{l-2} r^k + \sum_{l=2}^{n}\frac{2d-1+p}{2d-1} r^{l-2}\right] \\ &+ \frac{1-p}{d} + \frac{4p}{2d-1} + (n-1)\left(\frac{p}{d} + \frac{4(1-p)}{2d-1}\right).\end{aligned} \tag{3.11}$$

Since $r < 1$, the second sum over l is bounded by a constant, uniformly in n. Thus, this is equal to

$$\begin{aligned}&\frac{2}{d}\frac{p-2d(1-p)}{2d-1}\left(\frac{2d-p}{2d-1}\right)\sum_{l=2}^{n}\frac{1-r^{l-2}}{1-r} + n\left(\frac{p}{d} + \frac{4(1-p)}{2d-1}\right) + \mathcal{O}(1) \\ =& n\left[\frac{2}{d(1-r)}\frac{p-2d(1-p)}{2d-1}\left(\frac{2d-p}{2d-1}\right) + \frac{p}{d} + \frac{4(1-p)}{2d-1}\right] + \mathcal{O}(1) \\ =& n\left[\frac{(p-2d(1-p))(2d-p)}{d(d-p)(2d-1)} + \frac{p}{d} + \frac{4(1-p)}{2d-1}\right] + \mathcal{O}(1) = n\frac{p}{d-p} + \mathcal{O}(1).\end{aligned} \tag{3.12}$$

Dividing by n and taking the limit as $n \to \infty$ verifies (3.7) and thus completes the proof of Proposition 3.1. □

4 Proof of Theorem 2.1

Theorem 2.1 is a consequence of convergence of the joint distribution of the rescaled stopping time process and the random walk at those stopping times as in the following proposition.

Proposition 4.1. *Suppose that assumptions (A1) and (A2) hold for some $\alpha > 0$, then as $n \to \infty$,*

$$\left(\frac{S_{\tau_{\lfloor nt\rfloor}}}{\sqrt{\frac{p}{d-p}n}}, \frac{\tau_{\lfloor nt\rfloor}}{g_\alpha(n)}\right) \Longrightarrow \left(B_d(t), V_\alpha(t)\right), \tag{4.1}$$

where the convergence is in $\left(D([0,1],\mathbb{R}^d),\mathscr{U}\right) \times \left(D([0,1],\mathbb{R}),J_1\right)$*, and where* $\mathscr{U}$ *and* J_1 *denote the uniform and Skorokhod* J_1 *topologies respectively.*

Proof of Theorem 2.1 assuming Proposition 4.1. Since $\lfloor g_\alpha^{-1}(n)\rfloor$ is a sequence of positive integers such that $\lfloor g_\alpha^{-1}(n)\rfloor \to \infty$ and $n/g_\alpha(\lfloor g_\alpha^{-1}(n)\rfloor) \to 1$ as $n\to\infty$, it follows from (4.1) that as $n\to\infty$,

$$\left(\frac{S_{\tau_{\lfloor\lfloor g_\alpha^{-1}(n)\rfloor t\rfloor}}}{\sqrt{\frac{p}{d-p}\lfloor g_\alpha^{-1}(n)\rfloor}}, \frac{\tau_{\lfloor\lfloor g_\alpha^{-1}(n)\rfloor t\rfloor}}{n}\right) \Longrightarrow \left(B_d(t),V_\alpha(t)\right), \tag{4.2}$$

in $\left(D([0,1],\mathbb{R}^d),\mathscr{U}\right) \times \left(D([0,1],\mathbb{R}),J_1\right)$.
Let

$$Y_n(t) = \frac{S_{\tau_{\lfloor\lfloor g_\alpha^{-1}(n)\rfloor t\rfloor}}}{\sqrt{\frac{p}{d-p}\lfloor g_\alpha^{-1}(n)\rfloor}}, \quad \textit{and} \quad \mathscr{T}_n(t) = \frac{\tau_{\lfloor\lfloor g_\alpha^{-1}(n)\rfloor t\rfloor}}{n}, \tag{4.3}$$

and let $\mathscr{T}_n^{-1}(t) := \inf\{s\ge 0 : \mathscr{T}_n(s) > t\} = \inf\{s \ge 0 : \tau_{\lfloor\lfloor g_\alpha^{-1}(n)\rfloor s\rfloor} > nt\}$. It follows (e.g. see the proof of Theorem 1.3 in [1]) that $Y_n(\mathscr{T}_n^{-1}(t)) \Longrightarrow B_d(V_\alpha^{-1}(t))$ in $\left(D([0,1],\mathbb{R}^d),\mathscr{U}\right)$. Thus,

$$\frac{S_{\tau_{\lfloor\lfloor g_\alpha^{-1}(n)\rfloor \mathscr{T}_n^{-1}(t)\rfloor}}}{\sqrt{\frac{p}{d-p} g_\alpha^{-1}(n)}} \Longrightarrow B_d(V_\alpha^{-1}(t)). \tag{4.4}$$

By definition of $\mathscr{T}_n^{-1}$, we have $\tau_{\lfloor\lfloor g_\alpha^{-1}(n)\rfloor \mathscr{T}_n^{-1}(t)\rfloor-1} \le nt \le \tau_{\lfloor\lfloor g_\alpha^{-1}(n)\rfloor \mathscr{T}_n^{-1}(t)\rfloor}$ and hence $|S_{\lfloor nt\rfloor} - S_{\tau_{\lfloor\lfloor g_\alpha^{-1}(n)\rfloor \mathscr{T}_n^{-1}(t)\rfloor}}| \le 3$. Together with (4.4) and the fact that $g_\alpha^{-1}(n)/\lfloor g_\alpha^{-1}(n)\rfloor \to 1$, this proves Theorem 2.1. □

5 Proof of Proposition 4.1

The proof of Proposition 4.1 is broken into two parts. The first part is the observation that the marginal processes converge, i.e. that the time-changed walk and the time-change converge to $B_d(t)$ and $V_\alpha(t)$ respectively, while the second is to show that these two processes are asymptotically independent.

5.1 Convergence of the time-changed walk and the time-change.

Lemma 5.1. *Suppose that assumptions (A1) and (A2) hold for some* $\alpha > 0$*, then as* $n\to\infty$*,*

$$\frac{S_{\tau_{\lfloor nt\rfloor}}}{\sqrt{\frac{p}{d-p}n}} \Longrightarrow B_d(t) \quad \textit{in } (D([0,1],\mathbb{R}^d),\mathscr{U}), \quad \textit{and} \quad \frac{\tau_{\lfloor nt\rfloor}}{g_\alpha(n)} \Longrightarrow V_\alpha(t) \quad \textit{in } (D([0,1],\mathbb{R}),J_1). \tag{5.1}$$

Proof. The first claim is the conclusion of Proposition 3.1, so we need only prove the second claim. Recall that $\tau_n = 1 + \sum_{i=1}^n T_i$ where the T_i are i.i.d. with distribution T. Since $g_\alpha(n)\to\infty$, it is enough to show convergence of $\tau^*_{\lfloor nt\rfloor} = (\tau_{\lfloor nt\rfloor} - 1)/g_\alpha(n)$.
For processes with independent and identically distributed increments, a standard result of Skorokhod essentially extends the convergence of the one-dimensional distributions to a functional central limit theorem. When $\mathbb{E}[T]$ exists, convergence of the one-dimensional marginals $\tau^*_{\lfloor nt\rfloor}/n\mathbb{E}[T] \Longrightarrow t$ is immediate from the law of large numbers. The case $\alpha < 1$ is well known, see for example [6, Section XIII.6] and [16, Section 4.5.3]. The case where $\alpha = 1$ but (2.1) is not summable is perhaps less well known. Here the result is immediate from the following lemma.

Lemma 5.2. *Let $T_k \geq 0$ be independent and identically distributed random variables satisfying (2.1) and (2.2) with $\alpha = 1$. Then for each $t \geq 0$,*

$$\frac{\tau^*_{\lfloor nt \rfloor}}{n\ell(n)} \xrightarrow{P} t. \tag{5.2}$$

Lemma 5.2 is a corollary of the following weak law of large numbers due to Gut [7].

Theorem 5.3 ([7], Theorem 1.3)**.** *Let X_k be i.i.d. random variables and $S_n = \sum_{k=1}^n X_k$. Let $g_n = n^{1/\alpha}\ell(n)$ for $n \geq 1$, where $\alpha \in (0,1]$ and $\ell(n)$ is slowly varying at infinity. Then*

$$\frac{S_n - n\mathbb{E}\left[XI_{\{|X| \leq g_n\}}\right]}{g_n} \xrightarrow{P} 0, \quad \text{as } n \to \infty, \tag{5.3}$$

if and only if $n\mathbb{P}(|X| > g_n) \to 0$.

Proof of Lemma 5.2. Note that

$$\mathbb{E}\left[TI_{\{T \leq n\ell(n)\}}\right] = \sum_{j=1}^{\lfloor n\ell(n) \rfloor} \mathbb{P}(n\ell(n) \geq T \geq j) = \sum_{j=1}^{\lfloor n\ell(n) \rfloor} \mathbb{P}(T \geq j) - \lfloor n\ell(n) \rfloor \mathbb{P}(T \geq n\ell(n)). \tag{5.4}$$

Now by assumption (A2b),

$$\begin{aligned} \frac{n}{n\ell(n)}\mathbb{E}\left[TI_{\{|T| \leq n\ell(n)\}}\right] &= \frac{\sum_{j=1}^{\lfloor n\ell(n) \rfloor} \mathbb{P}(T \geq j)}{\ell(n)} - \frac{\lfloor n\ell(n) \rfloor}{\ell(n)} \mathbb{P}(T \geq n\ell(n)) \\ &\sim \frac{\sum_{j=1}^{\lfloor n\ell(n) \rfloor} j^{-1}L(j)}{\ell(n)} - \frac{\lfloor n\ell(n) \rfloor}{\ell(n)} (n\ell(n))^{-1} L(n\ell(n)) \to 1. \end{aligned} \tag{5.5}$$

Theorem 5.3 then implies that $(n\ell(n))^{-1}\tau_n \xrightarrow{P} 1$, from which it follows immediately that

$$(n\ell(n))^{-1}\tau_{\lfloor nt \rfloor} = (n\ell(n))^{-1}\lfloor nt \rfloor \ell(\lfloor nt \rfloor)(\lfloor nt \rfloor \ell(\lfloor nt \rfloor))^{-1}\tau_{\lfloor nt \rfloor} \xrightarrow{P} t. \tag{5.6}$$

This completes the proof of Lemma 5.2, and hence Lemma 5.1. □

5.2 Asymptotic Independence

Tightness of the joint process in Proposition 4.1 is an easy consequence of the tightness of the marginal processes (Lemma 5.1), so we need only prove convergence of the finite-dimensional distributions (f.d.d.s). For $\alpha \geq 1$ this is simple and is left as an exercise. To complete the proof of Proposition 4.1, it remains to prove convergence of the f.d.d.s when $\alpha < 1$ (hence $p < 1$).

Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be convergence determining classes of bounded, $\mathbb{C}$-valued functions on $\mathbb{R}^d$ and $\mathbb{R}_+$ respectively, each closed under conjugation and containing a non-zero constant function, then $\{g(x_1, x_2) := g_1(x_1)g_2(x_2) : g_i \in \mathscr{G}_i\}$ is a convergence determining class for $\mathbb{R}^d \times \mathbb{R}_+$. This follows as in [5, Proposition 3.4.6] where the closure under conjugation allows us to extend the proof to complex-valued functions. Therefore, to prove convergence of the finite-dimensional distributions in (4.1) it is enough to show that for every $0 \leq t_1 < \cdots < t_r \leq 1$, $k_1, \ldots, k_r \in \mathbb{R}^d$ and $\eta_1, \ldots, \eta_r \geq 0$,

$$\mathbb{E}\left[\exp\left\{i\sum_{j=1}^r \frac{k_j \cdot S_{\tau_{\lfloor nt_j \rfloor}}}{\sqrt{\frac{p}{d-p}}n} - \eta_j \frac{\tau_{\lfloor nt_j \rfloor}}{g_\alpha(n)}\right\}\right] \to \mathbb{E}\left[\exp\left\{i\sum_{j=1}^r k_j \cdot B_d(t_j)\right\}\right] \mathbb{E}\left[\exp\left\{-\sum_{j=1}^r \eta_j V_\alpha(t_j)\right\}\right]. \tag{5.7}$$

From (2.6) and the fact that V_α has independent increments, the rightmost expectation can be written as $\exp\{-\sum_{l=1}^r(\eta_l^*)^\alpha(t_l-t_{l-1})\}$, where $\eta_l^*=\sum_{j=l}^r\eta_j$.
Let $\mathscr{A}_n=\{i\in\{1,\dots,n\}:T_i \text{ is odd}\}$, $\mathscr{A}_{\lfloor n\vec{t}\rfloor}=(\mathscr{A}_{\lfloor nt_1\rfloor}\setminus\mathscr{A}_{\lfloor nt_0\rfloor},\dots,\mathscr{A}_{\lfloor nt_r\rfloor}\setminus\mathscr{A}_{\lfloor nt_{r-1}\rfloor})$ and $t_0=0$. For fixed n and $\vec{t}$, we write $A=(A^{(1)},\dots,A^{(r)})$ to denote an element of the range of the random variable $\mathscr{A}_{\lfloor n\vec{t}\rfloor}$, where $A^{(i)}\subseteq\{\lfloor nt_{i-1}\rfloor+1,\dots,\lfloor nt_i\rfloor\}$ for each $i\in 1,\dots,r$. Observe that $|\mathscr{A}^{(l)}_{\lfloor n\vec{t}\rfloor}|$ has a binomial distribution with parameters $\lfloor nt_l\rfloor-\lfloor nt_{l-1}\rfloor$ and p. Then for $\epsilon\in(0,\frac{1}{2})$ and $B_n(\vec{t}):=\{A:||A^{(l)}|-(\lfloor nt_lp\rfloor-\lfloor nt_{l-1}p\rfloor)|\le n^{1-\epsilon}$ for each $l\}$, we have that $\mathbb{P}(B_n(\vec{t})^c)\to 0$ as $n\to\infty$. Defining $Q^n_{\vec{k}}(\vec{t})=\exp\left\{i\sum_{j=1}^r\frac{k_j\cdot S_{\tau_{\lfloor nt_j\rfloor}}}{\sqrt{\frac{p}{d-p}n}}\right\}$, and conditioning on $\mathscr{A}_{\lfloor n\vec{t}\rfloor}$, the left hand side of (5.7) is equal to

$$
\begin{aligned}
&e^{-\frac{1}{g_\alpha(n)}\sum_{j=1}^r\eta_j}\sum_A\mathbb{E}\left[Q^n_{\vec{k}}(\vec{t})\exp\left\{-\sum_{j=1}^r\eta_j\frac{\tau^*_{\lfloor nt_j\rfloor}}{g_\alpha(n)}\right\}\middle|\{\mathscr{A}_{\lfloor n\vec{t}\rfloor}=A\}\right]\mathbb{P}(\mathscr{A}_{\lfloor n\vec{t}\rfloor}=A)\\
&=\sum_{A\in B_n(\vec{t})}\mathbb{E}\left[Q^n_{\vec{k}}(\vec{t})\exp\left\{-\sum_{j=1}^r\eta_j\frac{\tau^*_{\lfloor nt_j\rfloor}}{g_\alpha(n)}\right\}\middle|\{\mathscr{A}_{\lfloor n\vec{t}\rfloor}=A\}\right]\mathbb{P}(\mathscr{A}_{\lfloor n\vec{t}\rfloor}=A)+o(1)\\
&=\sum_{A\in B_n(\vec{t})}\mathbb{E}\left[Q^n_{\vec{k}}(\vec{t})\middle|\{\mathscr{A}_{\lfloor n\vec{t}\rfloor}=A\}\right]\mathbb{E}\left[\exp\left\{-\sum_{j=1}^r\eta_j\frac{\sum_{i=1}^{\lfloor nt_j\rfloor}T_i}{g_\alpha(n)}\right\}\middle|\{\mathscr{A}_{\lfloor n\vec{t}\rfloor}=A\}\right]\mathbb{P}(\mathscr{A}_{\lfloor n\vec{t}\rfloor}=A)+o(1),
\end{aligned}
\tag{5.8}
$$

where we have used the fact that S_{τ_n} is conditionally independent of the collection $\{T_i\}_{i\ge1}$ given $I_{\{T_i \text{ even}\}}$, $i=1,\dots,n$, to obtain the last equality.
Writing $\sum_{i=1}^{\lfloor nt_j\rfloor}T_i=\sum_{l=1}^j\sum_{i=\lfloor nt_{l-1}\rfloor+1}^{\lfloor nt_l\rfloor}T_i$ and using the mutual independence of T_i, $i\ge1$, the last line of (5.8) is equal to a term $o(1)$ plus

$$
\sum_{A\in B_n(\vec{t})}\mathbb{E}\left[Q^n_{\vec{k}}(\vec{t})\middle|\{\mathscr{A}_{\lfloor n\vec{t}\rfloor}=A\}\right]\mathbb{P}(\mathscr{A}_{\lfloor n\vec{t}\rfloor}=A)\prod_{l=1}^r\mathbb{E}\left[\exp\left\{-\eta_l^*\frac{\sum_{i=\lfloor nt_{l-1}\rfloor+1}^{\lfloor nt_l\rfloor}T_i}{g_\alpha(n)}\right\}\middle|\{\mathscr{A}^{(l)}_{\lfloor n\vec{t}\rfloor}=A^{(l)}\}\right].
\tag{5.9}
$$

Let $\{T_i^o\}_{i\in\mathbb{N}}$ be i.i.d. random variables satisfying $\mathbb{P}(T_i^o=k)=\mathbb{P}(T=k|T \text{ odd})$, and similarly define T_i^e to be i.i.d. with $\mathbb{P}(T_i^e=k)=\mathbb{P}(T=k|T \text{ even})$. The l^{th} term in the product in (5.9) is

$$
\mathbb{E}\left[\exp\left\{-\eta_l^*\frac{\sum_{i=1}^{|A^{(l)}|}T_i^o}{g_\alpha(n)}\right\}\right]\mathbb{E}\left[\exp\left\{-\eta_l^*\frac{\sum_{i=1}^{\lfloor nt_l\rfloor-\lfloor nt_{l-1}\rfloor-|A^{(l)}|}T_i^e\}}{g_\alpha(n)}\right\}\right].
\tag{5.10}
$$

For T^o we have $\mathbb{P}(T^o\ge n)=p^{-1}\mathbb{P}(T\ge n, T \text{ odd})\sim p^{-1}n^{-\alpha_o}L_o(n)$ and there exists ℓ_o such that $(\ell_o(n))^{-\alpha_o}p^{-1}L_o(n^{\frac{1}{\alpha_o}}\ell_o(n))\to(\Gamma(1-\alpha))^{-1}$. Define $g^o_{\alpha_o}(n)=n^{\frac{1}{\alpha_o}}\ell_o(n)$. Similarly define $g^e_{\alpha_e}(n)=n^{\frac{1}{\alpha_e}}\ell_e(n)$.
Observe that

$$
\frac{\sum_{i=1}^{|A^{(l)}|}T_i^o}{g_\alpha(n)}=\frac{\sum_{i=1}^{n_l}T_i^o}{g^o_{\alpha_o}(n_l)}\frac{g^o_{\alpha_o}(n_l)}{g_\alpha(n_l)}\frac{g_\alpha(n_l)}{g_\alpha(n)}+\mathscr{O}\left(\frac{\sum_{i=1}^{|A^{(l)}|}T_i^o-\sum_{i=1}^{n_l}T_i^o}{g^o_{\alpha_o}(n_l^*)}\frac{g^o_{\alpha_o}(n_l^*)}{g_\alpha(n_l^*)}\frac{g_\alpha(n_l^*)}{g_\alpha(n)}\right),
\tag{5.11}
$$

where $n_l:=\lfloor nt_lp\rfloor-\lfloor nt_{l-1}p\rfloor$ and $n_l^*:=||A^{(l)}|-n_l|\le n^{1-\epsilon}$ since $A\in B_n(\vec{t})$. By definition of g_α and standard results on regular variation we have that $g_\alpha(n_l)/g_\alpha(n)\to(p(t_l-t_{l-1}))^{\frac{1}{\alpha}}$ and

$g_\alpha(n_l^*)/g_\alpha(n) \to 0$. Since $\alpha = \alpha_o \wedge \alpha_e \le \alpha_o$, the $\mathcal{O}$ term on the right of (5.11) converges in probability to 0. Thus, as in the second claim of Lemma 5.1, we get that

$$\frac{\sum_{i=1}^{|A^{(l)}|} T_i^o}{g_\alpha(n)} \Longrightarrow V_\alpha(1)(p(t_l - t_{l-1}))^{\frac{1}{\alpha}} \lim_{n\to\infty} \frac{g_{\alpha_o}^o(n_l)}{g_\alpha(n_l)}, \tag{5.12}$$

where for $\alpha < 1$ the limit $\rho_o := \lim_{n\to\infty} \frac{g_{\alpha_o}^o(n_l)}{g_\alpha(n_l)}$ exists in $[0,\infty]$ since $\alpha \le \alpha_o$ and in the case of equality, the limit L_o/L_e exists in $[0,\infty]$. Note that we were able to replace α_o with α in various places in (5.12) due to the presence of the factor $\frac{g_{\alpha_o}^o(n_l)}{g_\alpha(n_l)}$ which is zero when $\alpha_o > \alpha$. Therefore

$$\mathbb{E}\left[\exp\left\{-\eta_l^* \frac{\sum_{i=1}^{|A^{(l)}|} T_i^o}{g_\alpha(n)}\right\}\right] \to \mathbb{E}\left[\exp\{-\eta_l^* V_\alpha(1)(p(t_l - t_{l-1}))^{\frac{1}{\alpha}} \rho_o\}\right], \text{ and similarly,}$$

$$\mathbb{E}\left[\exp\left\{-\eta_l^* \frac{\sum_{i=1}^{\lfloor nt_l\rfloor - \lfloor nt_{l-1}\rfloor - |A^{(l)}|} T_i^e}{g_\alpha(n)}\right\}\right] \to \mathbb{E}\left[\exp\{-\eta_l^* V_\alpha(1)((1-p)(t_l - t_{l-1}))^{\frac{1}{\alpha}} \rho_e\}\right]. \tag{5.13}$$

Since $\mathbb{E}[e^{-\eta V_\alpha(1)}] = \exp\{-\eta^\alpha\}$, it remains to show that

$$\left((p(t_l - t_{l-1}))^{\frac{1}{\alpha}} \rho_o\right)^\alpha + \left(((1-p)(t_l - t_{l-1}))^{\frac{1}{\alpha}} \rho_e\right)^\alpha = t_l - t_{l-1}, \quad \text{i.e.} \quad p\rho_o^\alpha + (1-p)\rho_e^\alpha = 1. \tag{5.14}$$

If $\alpha_o < \alpha_e$ (or $\alpha_o = \alpha_e$ and $L_o/L_e \to \infty$), then $\alpha = \alpha_o$, and $L \sim L_o$. It is then an easy exercise in manipulating slowly varying functions to show that $\ell_o \sim p^{-1/\alpha}\ell$ and therefore $\rho_o = p^{-1/\alpha}$ and $\rho_e = 0$, giving the desired result. Similarly if $\alpha_o > \alpha_e$ (or $\alpha_o = \alpha_e$ and $L_o/L_e \to 0$) we get the desired result. When $\alpha_o = \alpha_e < 1$ and $L_o/L_e \to \beta \in (0,\infty)$ we have that $L \sim L_o + L_e \sim (1+\beta)L_e \sim (1+\beta^{-1})L_o$. It follows that $\ell_e \sim ((1-p)(1+\beta))^{-1/\alpha}\ell$. Similarly $\ell_o \sim (p(1+\beta^{-1}))^{-1/\alpha}\ell$, and therefore $\rho_o = (p(1+\beta^{-1}))^{-1/\alpha}$ and $\rho_e = ((1-p)(1+\beta))^{-1/\alpha}$. The result follows since $(1+\beta)^{-1} + (1+\beta^{-1})^{-1} = 1$. □

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