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# SHARP INEQUALITY FOR BOUNDED SUBMARTINGALES AND THEIR DIFFERENTIAL SUBORDINATES

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*Abstract*

Let $\alpha$ be a fixed number from the interval $[0,1]$. We obtain the sharp probability bounds for the maximal function of the process which is $\alpha$-differentially subordinate to a bounded submartingale. This generalizes the previous results of Burkholder and Hammack.

## 1 Introduction

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, equipped with a discrete filtration $(\mathscr{F}_n)$. Let $f = (f_n)_{n=0}^\infty$, $g = (g_n)_{n=0}^\infty$ be adapted integrable processes taking values in a certain separable Hilbert space $\mathscr{H}$. The difference sequences $df = (df_n)$, $dg = (dg_n)$ of these processes are given by

$$df_0 = f_0, \quad df_n = f_n - f_{n-1}, \quad dg_0 = g_0, \quad dg_n = g_n - g_{n-1}, \quad n = 1,\ 2,\ \ldots.$$

Let $g^*$ stand for the maximal function of $g$, that is, $g^* = \max_n |g_n|$.
The following notion of differential subordination is due to Burkholder. The process $g$ is differentially subordinate to $f$ (or, in short, subordinate to $f$) if for any nonnegative integer $n$ we have, almost surely,

$$|dg_n| \leq |df_n|.$$

We will slightly change this definition and say that $g$ is differentially subordinate to $f$, if the above inequality for the differences holds for any *positive* integer $n$.
Let $\alpha$ be a fixed nonnegative number. Then $g$ is $\alpha$-differentially subordinate to $f$ (or, in short, $\alpha$-subordinate to $f$), if it is subordinate to $f$ and for any positive integer $n$ we have

$$|\mathbb{E}(dg_n|\mathscr{F}_{n-1})| \leq \alpha |\mathbb{E}(df_n|\mathscr{F}_{n-1})|.$$

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This concept was introduced by Burkholder in [2] in the special case $\alpha = 1$. In general form, it first appeared in the paper by Choi [3].

In the sequel it will sometimes be convenient to work with simple processes. A process $f$ is called simple, if for any $n$ the variable $f_n$ is simple and there exists $N$ such that $f_N = f_{N+1} = f_{N+2} = \dots$. Given such a process, we will identify it with the finite sequence $(f_n)_{n=0}^N$.

Assume that the processes $f$ and $g$ are real-valued and fix $\alpha \in [0,1]$. The objective of this paper is to establish a sharp exponential inequality for the distribution function of $g^*$ under the assumption that $f$ is a submartingale satisfying $||f||_\infty \leq 1$ and $g$ is $\alpha$-subordinate to $f$. To be more precise, for any $\lambda > 0$ define the function $V_{\alpha,\lambda} : [-1,1] \times \mathbb{R} \to \mathbb{R}$ by the formula

$$V_{\alpha,\lambda}(x_0, y_0) = \sup \mathbb{P}(g^* \geq \lambda). \tag{1}$$

Here the supremum is taken over all pairs $(f, g)$ of integrable adapted processes, such that $(f_0, g_0) \equiv (x_0, y_0)$ almost surely, $f$ is a submartingale satisfying $||f||_\infty \leq 1$ and $g$ is $\alpha$-subordinate to $f$. The filtration must also vary, as well as the probability space, unless it is nonatomic. Our main result is an explicit formula for the functions $V_{\alpha,\lambda}$, $\lambda > 0$. Usually we will omit the index $\alpha$ and write $V_\lambda$ instead of $V_{\alpha,\lambda}$.

Let us discuss some related results which appeared in the literature. In [1] Burkholder studied the analogous question in the case of $f$, $g$ being Hilbert space-valued martingales. The paper [1] contains also a related one-sided sharp exponential inequality for real martingales. This work was later extended by Hammack [4], who established a similar (two-sided) inequality under the assumption that $f$ is a submartingale bounded by 1 and $g$ is $R^\nu$-valued, $\nu \geq 1$, and strongly 1-subordinate to $f$. Both papers present applications to stochastic integrals.

The paper is organized as follows. In the next section we introduce a family of special functions $U_\lambda$, $\lambda > 0$ and study their properties. This enables us to establish the inequality $V_\lambda \leq U_\lambda$ in Section 3. Then we prove the reverse inequality in the last section.

Throughout the paper, $\alpha$ is a fixed number from the interval $[0,1]$. All the considered processes are assumed to be real valued.

## 2 The explicit formulas

Let $S$ be the strip $[-1,1] \times \mathbb{R}$. Consider the following subsets of $S$: for $0 < \lambda \leq 2$,

$$\begin{aligned}
A_\lambda &= \{(x,y) \in S : |y| \geq x + \lambda - 1\}, \\
B_\lambda &= \{(x,y) \in S : 1 - x \leq |y| < x + \lambda - 1\}, \\
C_\lambda &= \{(x,y) \in S : |y| < 1 - x \text{ and } |y| < x + \lambda - 1\}.
\end{aligned}$$

For $\lambda \in (2,4)$, define

$$\begin{aligned}
A_\lambda &= \{(x,y) \in S : |y| \geq \alpha x + \lambda - \alpha\}, \\
B_\lambda &= \{(x,y) \in S : \alpha x + \lambda - \alpha > |y| \geq x - 1 + \lambda\}, \\
C_\lambda &= \{(x,y) \in S : x - 1 + \lambda > |y| \geq 1 - x\}, \\
D_\lambda &= \{(x,y) \in S : 1 - x > |y| \geq -x - 3 + \lambda \text{ and } |y| < x - 1 + \lambda\}, \\
E_\lambda &= \{(x,y) \in S : -x - 3 + \lambda > |y|\}.
\end{aligned}$$

Finally, for $\lambda \geq 4$, let

$$\begin{array}{rcl} A_\lambda & = & \{(x,y)\in S : |y| \geq \alpha x + \lambda - \alpha\}, \\ B_\lambda & = & \{(x,y)\in S : \alpha x + \lambda - \alpha > |y| \geq x - 1 + \lambda\}, \\ C_\lambda & = & \{(x,y)\in S : x - 1 + \lambda > |y| \geq -x - 3 + \lambda\}, \\ D_\lambda & = & \{(x,y)\in S : -x - 3 + \lambda > |y| \geq 1 - x\}, \\ E_\lambda & = & \{(x,y)\in S : 1 - x > |y|\}. \end{array}$$

Let $H : S \times (-1,\infty) \to \mathbb{R}$ be a function given by

$$H(x,y,z) = \frac{1}{\alpha+2}\Big[1 + \frac{(x+1+|y|)^{1/(\alpha+1)}((\alpha+1)(x+1)-|y|)}{(1+z)^{(\alpha+2)/(\alpha+1)}}\Big]. \tag{2}$$

Now we will define the special functions $U_\lambda : S \to \mathbb{R}$. For $0 < \lambda \leq 2$, let

$$U_\lambda(x,y) = \begin{cases} 1 & \text{if } (x,y)\in A_\lambda, \\ \frac{2-2x}{1+\lambda-x-|y|} & \text{if } (x,y)\in B_\lambda, \\ 1 - \frac{(\lambda-1+x-|y|)(\lambda-1+x+|y|)}{\lambda^2} & \text{if } (x,y)\in C_\lambda. \end{cases} \tag{3}$$

For $2 < \lambda < 4$, set

$$U_\lambda(x,y) = \begin{cases} 1 & \text{if } (x,y)\in A_\lambda, \\ 1 - (\alpha(x-1) - |y| + \lambda)\cdot \frac{2\lambda-4}{\lambda^2} & \text{if } (x,y)\in B_\lambda, \\ \frac{2-2x}{1+\lambda-x-|y|} - \frac{2(1-x)(1-\alpha)(\lambda-2)}{\lambda^2} & \text{if } (x,y)\in C_\lambda, \\ \frac{2(1-x)}{\lambda}\Big[1 - \frac{(1-\alpha)(\lambda-2)}{\lambda}\Big] - \frac{(1-x)^2-|y|^2}{\lambda^2} & \text{if } (x,y)\in D_\lambda, \\ a_\lambda H(x,y,\lambda-3) + b_\lambda & \text{if } (x,y)\in E_\lambda, \end{cases} \tag{4}$$

where

$$a_\lambda = -\frac{2(1+\alpha)(\lambda-2)^2}{\lambda^2}, \quad b_\lambda = 1 - \frac{4(\lambda-2)(1-\alpha)}{\lambda^2}. \tag{5}$$

For $\lambda \geq 4$, set

$$U_\lambda(x,y) = \begin{cases} 1 & \text{if } (x,y)\in A_\lambda, \\ 1 - \frac{\alpha(x-1)-|y|+\lambda}{4} & \text{if } (x,y)\in B_\lambda, \\ \frac{2-2x}{1+\lambda-x-|y|} - \frac{(1-x)(1-\alpha)}{4} & \text{if } (x,y)\in C_\lambda, \\ \frac{(1-x)(1+\alpha)}{4}\exp\Big(\frac{3+x+|y|-\lambda}{2(\alpha+1)}\Big) & \text{if } (x,y)\in D_\lambda, \\ a_\lambda H(x,y,1) + b_\lambda & \text{if } (x,y)\in E_\lambda, \end{cases} \tag{6}$$

where

$$a_\lambda = -b_\lambda = -\frac{(1+\alpha)}{2}\exp\Big(\frac{4-\lambda}{2\alpha+2}\Big). \tag{7}$$

For $\alpha = 1$, the formulas (3), (4), (6) give the special functions constructed by Hammack [4]. The key properties of $U_\lambda$ are described in the two lemmas below.

**Lemma 1.** *For $\lambda > 2$, let $\phi_\lambda$, $\psi_\lambda$ denote the partial derivatives of $U_\lambda$ with respect to $x$, $y$ on the interiors of $A_\lambda$, $B_\lambda$, $C_\lambda$, $D_\lambda$, $E_\lambda$, extended continuously to the whole of these sets. The following statements hold.*

*(i) The functions $U_\lambda$, $\lambda > 2$, are continuous on $S \setminus \{(1, \pm\lambda)\}$ .*
*(ii) Let*

$$S_\lambda = \{(x, y) \in [-1, 1] \times \mathbb{R} : |y| \neq \alpha x + \lambda - \alpha \text{ and } |y| \neq x + \lambda - 1\}.$$

*Then*

$$\phi_\lambda,\ \psi_\lambda,\ \lambda > 2, \text{ are continuous on } S_\lambda. \tag{8}$$

*(iii) For any $(x, y) \in S$, the function $\lambda \mapsto U_\lambda(x, y)$, $\lambda > 0$, is left-continuous.*
*(iv) For any $\lambda > 2$ we have the inequality*

$$\phi_\lambda \leq -\alpha|\psi_\lambda|. \tag{9}$$

*(v) For $\lambda > 2$ and any $(x, y) \in S$ we have $\chi_{\{|y| \geq \lambda\}} \leq U_\lambda(x, y) \leq 1$.*

*Proof.* We start with computing the derivatives. Let $y' = y/|y|$ stand for the sign of $y$, with $y' = 0$ if $y = 0$. For $\lambda \in (2, 4)$ we have

$$\phi_\lambda(x,y) = \begin{cases} 0 & \text{if } (x,y) \in A_\lambda, \\ -\frac{(2\lambda-4)\alpha}{\lambda^2} & \text{if } (x,y) \in B_\lambda, \\ -\frac{2\lambda-2|y|}{(1+\lambda-x-|y|)^2} + \frac{(2\lambda-4)(1-\alpha)}{\lambda^2} & \text{if } (x,y) \in C_\lambda, \\ -\frac{2}{\lambda}\left[1 - \frac{(1-\alpha)(\lambda-2)}{\lambda}\right] + \frac{2(1-x)}{\lambda^2} & \text{if } (x,y) \in D_\lambda, \\ -c_\lambda(x+|y|+1)^{-\alpha/(\alpha+1)}(x+1+\frac{\alpha}{\alpha+1}|y|) & \text{if } (x,y) \in E_\lambda, \end{cases}$$

$$\psi_\lambda(x,y) = \begin{cases} 0 & \text{if } (x,y) \in A_\lambda, \\ \frac{2\lambda-4}{\lambda^2}y' & \text{if } (x,y) \in B_\lambda, \\ \frac{2-2x}{(1+\lambda-x-|y|)^2}y' & \text{if } (x,y) \in C_\lambda, \\ \frac{2y}{\lambda^2} & \text{if } (x,y) \in D_\lambda, \\ c_\lambda(x+|y|+1)^{-\alpha/(\alpha+1)}\frac{y}{1+\alpha} & \text{if } (x,y) \in E_\lambda, \end{cases}$$

where

$$c_\lambda = 2(1+\alpha)(\lambda-2)^{\alpha/(\alpha+1)}\lambda^{-2}.$$

Finally, for $\lambda \geq 4$, set

$$\phi_\lambda(x,y) = \begin{cases} 0 & \text{if } (x,y) \in A_\lambda, \\ -\frac{\alpha}{4} & \text{if } (x,y) \in B_\lambda, \\ -\frac{2\lambda-2|y|}{(1+\lambda-x-|y|)^2} + \frac{1-\alpha}{4} & \text{if } (x,y) \in C_\lambda, \\ -\frac{x+1+2\alpha}{8}\exp\left(\frac{x+|y|+3-\lambda}{2(\alpha+1)}\right) & \text{if } (x,y) \in D_\lambda, \\ -c_\lambda(x+|y|+1)^{-\alpha/(\alpha+1)}(x+1+\frac{\alpha}{\alpha+1}|y|) & \text{if } (x,y) \in E_\lambda, \end{cases}$$

$$\psi_\lambda(x,y) = \begin{cases} 0 & \text{if } (x,y) \in A_\lambda, \\ \frac{1}{4}y' & \text{if } (x,y) \in B_\lambda, \\ \frac{2-2x}{(1+\lambda-x-|y|)^2}y' & \text{if } (x,y) \in C_\lambda, \\ \frac{(1-x)}{8}\exp\left(\frac{x+|y|+3-\lambda}{2(\alpha+1)}\right)y' & \text{if } (x,y) \in D_\lambda, \\ c_\lambda(x+|y|+1)^{-\alpha/(\alpha+1)}\frac{y}{1+\alpha} & \text{if } (x,y) \in E_\lambda, \end{cases}$$

where

$$c_\lambda = (1+\alpha)2^{-(2\alpha+3)/(\alpha+1)}\exp\Big(\frac{4-\lambda}{2(\alpha+1)}\Big).$$

Now the properties (i), (ii), (iii) follow by straightforward computation. To prove (iv), note first that for any $\lambda > 2$ the condition (9) is clearly satisfied on the sets $A_\lambda$ and $B_\lambda$. Suppose $(x,y)\in C_\lambda$. Then $\lambda-|y|\in[0,4]$, $1-x\leq\min\{\lambda-|y|,4-\lambda+|y|\}$ and (9) takes form

$$-2(\lambda-|y|)+\frac{2\lambda-4}{\lambda^2}(1-\alpha)(1-x+\lambda-|y|)^2+2\alpha(1-x)\leq 0,$$

or

$$-2(\lambda-|y|)+\frac{1-\alpha}{4}\cdot(1-x+\lambda-|y|)^2+2\alpha(1-x)\leq 0, \tag{10}$$

depending on whether $\lambda<4$ or $\lambda\geq 4$. As $(2\lambda-4)/\lambda^2\leq\frac{1}{4}$, it suffices to show (10). If $\lambda-|y|\leq 2$, then, as $1-x\leq\lambda-|y|$, the left-hand side does not exceed

$$\begin{aligned}-2(\lambda-|y|)+(1-\alpha)(\lambda-|y|)^2+2\alpha(\lambda-|y|) &= (\lambda-|y|)(-2+(1-\alpha)(\lambda-|y|)+2\alpha)\\ &\leq(\lambda-|y|)(-2+2(1-\alpha)+2\alpha)=0.\end{aligned}$$

Similarly, if $\lambda-|y|\in(2,4]$, then we use the bound $1-x\leq 4-\lambda+|y|$ and conclude that the left-hand side of (10) is not greater than

$$-2(\lambda-|y|)+4(1-\alpha)+2\alpha(4-\lambda+|y|)=-2(\lambda-|y|-2)(1+\alpha)\leq 0$$

and we are done with the case $(x,y)\in C_\lambda$.
Assume that $(x,y)\in D_\lambda$. For $\lambda\in(2,4)$, the inequality (9) is equivalent to

$$-\frac{2}{\lambda}\Big[1-\frac{(1-\alpha)(\lambda-2)}{\lambda}\Big]+\frac{2-2x}{\lambda^2}\leq-\frac{2\alpha|y|}{\lambda^2},$$

or, after some simplifications, $\alpha|y|+1-x\leq 2+\alpha\lambda-2\alpha$. It is easy to check that $\alpha|y|+1-x$ attains its maximum for $x=-1$ and $|y|=\lambda-2$ and then we have the equality. If $(x,y)\in D_\lambda$ and $\lambda\geq 4$, then (9) takes form $-(2\alpha+1+x)\leq-\alpha(1-x)$, or $(x+1)(\alpha+1)\geq 0$. Finally, on the set $E_\lambda$, the inequality (9) is obvious.
(v) By (9), we have $\phi_\lambda\leq 0$, so $U_\lambda(x,y)\geq U_\lambda(1,y)=\chi_{\{|y|\geq\lambda\}}$. Furthermore, as $U_\lambda(x,y)=1$ for $|y|\geq\lambda$ and $\psi_\lambda(x,y)y'\geq 0$ on $S_\lambda$, the second estimate follows. □

**Lemma 2.** *Let $x$, $h$, $y$, $k$ be fixed real numbers, satisfying $x$, $x+h\in[-1,1]$ and $|k|\leq|h|$. Then for any $\lambda>2$ and $\alpha\in[0,1)$,*

$$U_\lambda(x+h,y+k)\leq U_\lambda(x,y)+\phi_\lambda(x,y)h+\psi_\lambda(x,y)k. \tag{11}$$

We will need the following fact, proved by Burkholder; see page 17 of [1].

**Lemma 3.** *Let $x$, $h$, $y$, $k$, $z$ be real numbers satisfying $|k|\leq|h|$ and $z>-1$. Then the function*

$$F(t)=H(x+th,y+tk,z),$$

*defined on $\{t:|x+th|\leq 1\}$, is convex.*

*Proof of the Lemma 2.* Consider the function

$$G(t) = G_{x,y,h,k}(t) = U_\lambda(x + th, y + tk),$$

defined on the set $\{t : |x + th| \leq 1\}$. It is easy to check that $G$ is continuous. As explained in [1], the inequality (11) follows once the concavity of $G$ is established. This will be done by proving the inequality $G'' \leq 0$ at the points, where $G$ is twice differentiable and checking the inequality $G'_+(t) \leq G'_-(t)$ for those $t$, for which $G$ is not differentiable (even once). Note that we may assume $t = 0$, by a translation argument $G''_{x,y,h,k}(t) = G''_{x+th,y+tk,h,k}(0)$, with analogous equalities for one-sided derivatives. Clearly, we may assume that $h \geq 0$, changing the signs of both $h$, $k$, if necessary. Due to the symmetry of $U_\lambda$, we are allowed to consider $y \geq 0$ only.
We start from the observation that $G''(0) = 0$ on the interior of $A_\lambda$ and $G'_+(0) \leq G'_-(0)$ for $(x, y) \in A_\lambda \cap \overline{B}_\lambda$. The latter inequality holds since $U_\lambda \equiv 1$ on $A_\lambda$ and $U_\lambda \leq 1$ on $B_\lambda$. For the remaining inequalities, we consider the cases $\lambda \in (2, 4)$, $\lambda \geq 4$ separately.
*The case* $\lambda \in (2, 4)$. The inequality $G''(0) \leq 0$ is clear for $(x, y)$ lying in the interior of $B_\lambda$. On $C_\lambda$, we have

$$G''(0) = -\frac{4(h+k)(h(\lambda - y) - k(1 - x))}{(1 - x - y + \lambda)^3} \leq 0, \tag{12}$$

which follows from $|k| \leq h$ and the fact that $\lambda - y \geq 1 - x$. For $(x, y)$ in the interior of $D_\lambda$,

$$G''(0) = \frac{-h^2 + k^2}{\lambda^2} \leq 0,$$

as $|k| \leq h$. Finally, on $E_\lambda$, the concavity follows by Lemma 3.
It remains to check the inequalities for one-sided derivatives. By Lemma 1 (ii), the points $(x, y)$, for which $G$ is not differentiable at 0, do not belong to $S_\lambda$. Since we excluded the set $A_\lambda \cap \overline{B}_\lambda$, they lie on the line $y = x - 1 + \lambda$. For such points $(x, y)$, the left derivative equals

$$G'_-(0) = -\frac{2\lambda - 4}{\lambda^2}(\alpha h - k),$$

while the right one is given by

$$G'_+(0) = \frac{-h + k}{2(\lambda - y)} + \frac{(2\lambda - 4)(1 - \alpha)h}{\lambda^2},$$

or

$$G'_+(0) = -\frac{2h}{\lambda}\Big[1 - \frac{(1 - \alpha)(\lambda - 2)}{\lambda}\Big] + \frac{2(1 - x)h + 2yk}{\lambda^2},$$

depending on whether $y \geq 1 - x$ or $y < 1 - x$. In the first case, the inequality $G'_+(0) \leq G'_-(0)$ reduces to

$$(h - k)\Big(\frac{1}{2(\lambda - y)} - \frac{2(\lambda - 2)}{\lambda^2}\Big) \geq 0,$$

while in the remaining one,

$$\frac{2}{\lambda^2}(h - k)(y - (\lambda - 2)) \geq 0.$$

Both inequalities follow from the estimate $\lambda - y \leq 2$ and the condition $|k| \leq h$.

*The case* $\lambda \geq 4$. On the set $B_\lambda$ the concavity is clear. For $C_\lambda$, we have that the formula (12) holds. If $(x,y)$ lies in the interior of $D_\lambda$, then

$$G''(0) = \frac{1}{8}\exp\Big(\frac{3+x+y-\lambda}{2(\alpha+1)}\Big)\Big[\frac{1-x}{2(\alpha+1)}\cdot(-h^2+k^2) - \Big(2-\frac{1-x}{\alpha+1}\Big)(h^2+hk)\Big] \leq 0,$$

since $|k| \leq h$ and $(1-x)/(\alpha+1) \leq 2$. The concavity on $E_\lambda$ is a consequence of Lemma 3. It remains to check the inequality for one-sided derivatives. By Lemma 1 (ii), we may assume $y = x+\lambda-1$, and the inequality $G'_+(0) \leq G'_-(0)$ reads

$$\frac{1}{2}(h-k)\Big(\frac{1}{\lambda-y} - \frac{1}{2}\Big) \geq 0,$$

an obvious one, as $\lambda - y \leq 2$. □

# 3 The main theorem

Now we may state and prove the main result of the paper.

**Theorem 1.** *Suppose $f$ is a submartingale satisfying $||f||_\infty \leq 1$ and $g$ is an adapted process which is $\alpha$-subordinate to $f$. Then for all $\lambda > 0$ we have*

$$\mathbb{P}(g^* \geq \lambda) \leq \mathbb{E}U_\lambda(f_0, g_0). \tag{13}$$

*Proof.* If $\lambda \leq 2$, then this follows immediately from the result of Hammack [4]; indeed, note that $U_\lambda$ coincides with Hammack's special function and, furthermore, since $g$ is $\alpha$-subordinate to $f$, it is also 1-subordinate to $f$.
Fix $\lambda > 2$. We may assume $\alpha < 1$. It suffices to show that for any nonnegative integer $n$,

$$\mathbb{P}(|g_n| \geq \lambda) \leq \mathbb{E}U_\lambda(f_0, g_0). \tag{14}$$

To see that this implies (13), fix $\varepsilon > 0$ and consider a stopping time $\tau = \inf\{k : |g_k| \geq \lambda - \varepsilon\}$. The process $f^\tau = (f_{\tau\wedge n})$, by Doob's optional sampling theorem, is a submartingale. Furthermore, we obviously have that $||f^\tau||_\infty \leq 1$ and the process $g^\tau = (g_{\tau\wedge n})$ is $\alpha$-subordinate to $f^\tau$. Therefore, by (14),

$$\mathbb{P}(|g_n^\tau| \geq \lambda - \varepsilon) \leq \mathbb{E}U_{\lambda-\varepsilon}(f_0^\tau, g_0^\tau) = \mathbb{E}U_{\lambda-\varepsilon}(f_0, g_0).$$

Now if we let $n \to \infty$, we obtain $\mathbb{P}(g^* \geq \lambda) \leq \mathbb{E}U_{\lambda-\varepsilon}(f_0, g_0)$ and by left-continuity of $U_\lambda$ as a function of $\lambda$, (13) follows.
Thus it remains to establish (14). By Lemma 1 (v), $\mathbb{P}(|g_n| \geq \lambda) \leq \mathbb{E}U_\lambda(f_n, g_n)$ and it suffices to show that for all $1 \leq j \leq n$ we have

$$\mathbb{E}U_\lambda(f_j, g_j) \leq \mathbb{E}U_\lambda(f_{j-1}, g_{j-1}). \tag{15}$$

To do this, note that, since $|dg_j| \leq |df_j|$ almost surely, the inequality (11) yields

$$U_\lambda(f_j, g_j) \leq U_\lambda(f_{j-1}, g_{j-1}) + \phi_\lambda(f_{j-1}, g_{j-1})df_j + \psi_\lambda(f_{j-1}, g_{j-1})dg_j \tag{16}$$

with probability 1. Assume for now that $\phi_\lambda(f_{j-1}, g_{j-1})df_j$, $\psi_\lambda(f_{j-1}, g_{j-1})dg_j$ are integrable. By $\alpha$-subordination, the condition (9) and the submartingale property $\mathbb{E}(d_j|\mathscr{F}_{j-1}) \geq 0$, we have

$$\mathbb{E}\big[\phi_\lambda(f_{j-1}, g_{j-1})df_j + \psi_\lambda(f_{j-1}, g_{j-1})dg_j|\mathscr{F}_{j-1}\big]$$

$$
\begin{aligned}
&\leq \phi_\lambda(f_{j-1},g_{j-1})\mathbb{E}(df_j|\mathscr{F}_{j-1}) + \left|\psi_\lambda(f_{j-1},g_{j-1})\right| \cdot \left|\mathbb{E}(dg_j|\mathscr{F}_{j-1})\right| \\
&\leq \left[\phi_\lambda(f_{j-1},g_{j-1}) + \alpha|\psi_\lambda(f_{j-1},g_{j-1})|\right]\mathbb{E}(df_j|\mathscr{F}_{j-1}) \leq 0.
\end{aligned}
$$

Therefore, it suffices to take the expectation of both sides of (16) to obtain (15).
Thus we will be done if we show the integrability of $\phi_\lambda(f_{j-1},g_{j-1})df_j$ and $\psi_\lambda(f_{j-1},g_{j-1})dg_j$. In both the cases $\lambda \in (2,4)$, $\lambda \geq 4$, all we need is that the variables

$$
\frac{2\lambda - 2|g_{j-1}|}{(1-f_{j-1}-|g_{j-1}|+\lambda)^2}df_j \quad \text{and} \quad \frac{2-2f_{j-1}}{(1-f_{j-1}-|g_{j-1}|+\lambda)^2}dg_j \tag{17}
$$

are integrable on the set $K = \{|g_{j-1}| < f_{j-1}+\lambda-1,\ |g_{j-1}| \geq \lambda-1\}$, since outside it the derivatives $\phi_\lambda$, $\psi_\lambda$ are bounded by a constant depending only on $\alpha$, $\lambda$ and $|df_j|$, $|dg_j|$ do not exceed 2. The integrability is proved exactly in the same manner as in [4]. We omit the details. □

We will now establish the following sharp exponential inequality.

**Theorem 2.** *Suppose $f$ is a submartingale satisfying $||f||_\infty \leq 1$ and $g$ is an adapted process which is $\alpha$-subordinate to $f$. In addition, assume that $|g_0| \leq |f_0|$ with probability 1. Then for $\lambda \geq 4$ we have*

$$
\mathbb{P}(g^* \geq \lambda) \leq \gamma e^{-\lambda/(2\alpha+2)}, \tag{18}
$$

*where*

$$
\gamma = \frac{1+\alpha}{2\alpha+4}\left(\alpha+1+2^{-\frac{\alpha+2}{\alpha+1}}\right)\exp\left(\frac{2}{\alpha+1}\right).
$$

*The inequality is sharp.*

This should be compared to Burkholder's estimate (Theorem 8.1 in [1])

$$
\mathbb{P}(g^* \geq \lambda) \leq \frac{e^2}{4} \cdot e^{-\lambda}, \quad \lambda \geq 2,
$$

in the case when $f$, $g$ are Hilbert space-valued martingales and $g$ is subordinate to $f$. For $\alpha = 1$, we obtain the inequality of Hammack [4],

$$
\mathbb{P}(g^* \geq \lambda) \leq \frac{(8+\sqrt{2})e}{12} \cdot e^{-\lambda/4}, \quad \lambda \geq 4.
$$

*Proof of the inequality (18).* We will prove that the maximum of $U_\lambda$ on the set $K = \{(x,y) \in S : |y| \leq |x|\}$ is given by the right hand side of (18). This, together with the inequality (13) and the assumption $\mathbb{P}((f_0,g_0) \in K) = 1$, will imply the desired estimate. Clearly, by symmetry, we may restrict ourselves to the set $K^+ = K \cap \{y \geq 0\}$. If $(x,y) \in K^+$ and $x \geq 0$, then it is easy to check that

$$
U_\lambda(x,y) \leq U_\lambda((x+y)/2,(x+y)/2).
$$

Furthermore, a straightforward computation shows that the function $F : [0,1] \to \mathbb{R}$ given by $F(s) = U_\lambda(s,s)$ is nonincreasing. Thus we have $U_\lambda(x,y) \leq U_\lambda(0,0)$. On the other hand, if $(x,y) \in K^+$ and $x \leq 0$, then it is easy to prove that $U_\lambda(x,y) \leq U_\lambda(-1,x+y+1)$ and the function $G : [0,1] \to \mathbb{R}$ given by $G(s) = U_\lambda(-1,s)$ is nondecreasing. Combining all these facts we have that for any $(x,y) \in K^+$,

$$
U_\lambda(x,y) \leq U_\lambda(-1,1) = \gamma e^{-\lambda/(2\alpha+2)}. \tag{19}
$$

Thus (18) holds. The sharpness will be shown in the next section. □

## 4 Sharpness

Recall the function $V_\lambda = V_{\alpha,\lambda}$ defined by (1) in the introduction. The main result in this section is Theorem 3 below, which, combined with Theorem 1, implies that the functions $U_\lambda$ and $V_\lambda$ coincide. If we apply this at the point $(-1,1)$ and use the equality appearing in (19), we obtain that the inequality (18) is sharp.

**Theorem 3.** *For any $\lambda > 0$ we have*

$$U_\lambda \leq V_\lambda. \tag{20}$$

The main tool in the proof is the following "splicing" argument. Assume that the underlying probability space is the interval $[0,1]$ with the Lebesgue measure.

**Lemma 4.** *Fix $(x_0, y_0) \in [-1,1] \times \mathbb{R}$. Suppose there exists a filtration and a pair $(f,g)$ of simple adapted processes, starting from $(x_0, y_0)$, such that $f$ is a submartingale satisfying $||f||_\infty \leq 1$ and $g$ is $\alpha$-subordinate to $f$. Then $V_\lambda(x_0,y_0) \geq \mathbb{E}V_\lambda(f_\infty, g_\infty)$ for $\lambda > 0$.*

*Proof.* Let $N$ be such that $(f_N, g_N) = (f_\infty, g_\infty)$ and fix $\varepsilon > 0$. With no loss of generality, we may assume that $\sigma$-field generated by $f$, $g$ is generated by the family of intervals $\{[a_i, a_{i+1}) : i = 1, 2, \ldots, M-1\}$, $0 = a_1 < a_2 < \ldots < a_M = 1$. For any $i \in \{1, 2, \ldots, M-1\}$, denote $x_0^i = f_N(a_i)$, $y_0^i = g_N(a_i)$. There exists a filtration and a pair $(f^i, g^i)$ of adapted processes, with $f$ being a submartingale bounded in absolute value by 1 and $g$ being $\alpha$-subordinate to $f$, which satisfy $f_0^i = x_0^i \chi_{[0,1)}$, $g_0^i = y_0^i \chi_{[0,1)}$ and $\mathbb{P}((g^i)^* \geq \lambda) > \mathbb{E}V_\lambda(f_0^i, g_0^i) - \varepsilon$. Define the processes $F$, $G$ by $F_k = f_k$, $G_k = g_k$ if $k \leq N$ and

$$F_k(\omega) = \sum_{i=1}^{M-1} f_{k-N}^i((\omega - a_i)/(a_{i+1} - a_i))\chi_{[a_i, a_{i+1})}(\omega),$$
$$G_k(\omega) = \sum_{i=1}^{M-1} g_{k-N}^i((\omega - a_i)/(a_{i+1} - a_i))\chi_{[a_i, a_{i+1})}(\omega)$$

for $k > N$. It is easy to check that there exists a filtration, relative to which the process $F$ is a submartingale satisfying $||F||_\infty \leq 1$ and $G$ is an adapted process which is $\alpha$-subordinate to $F$. Furthermore, we have

$$\begin{aligned}\mathbb{P}(G^* \geq \lambda) &\geq \sum_{i=1}^{M-1}(a_{i+1} - a_i)\mathbb{P}((g^i)^* \geq \lambda) \\ &> \sum_{i=1}^{M-1}(a_{i+1} - a_i)\big(\mathbb{E}V_\lambda(f_0^i, g_0^i) - \varepsilon\big) = \mathbb{E}V_\lambda(f_\infty, g_\infty) - \varepsilon.\end{aligned}$$

Since $\varepsilon$ was arbitrary, the result follows. □

*Proof of Theorem 3.* First note the following obvious properties of the functions $V_\lambda$, $\lambda > 0$: we have $V_\lambda \in [0,1]$ and $V_\lambda(x,y) = V_\lambda(x,-y)$. The second equality is an immediate consequence of the fact that if $g$ is $\alpha$-subordinate to $f$, then so is $-g$.

In the proof of Theorem 3 we repeat several times the following procedure. Having fixed a point $(x_0, y_0)$ from the strip $S$, we construct certain simple finite processes $f$, $g$ starting from $(x_0, y_0)$, take their natural filtration $(\mathscr{F}_n)$, apply Lemma 4 and thus obtain a bound for $V_\lambda(x_0, y_0)$. All the constructed processes appearing in the proof below are easily checked to satisfy the conditions

of this lemma: the condition $||f||_\infty \le 1$ is straightforward, while the $\alpha$-subordination and the fact that $f$ is a submartingale are implied by the following. For any $n \ge 1$, either $df_n$ satisfies $\mathbb{E}(df_n|\mathscr{F}_{n-1}) = 0$ and $dg_n = \pm df_n$, or $df_n \ge 0$ and $dg_n = \pm\alpha df_n$.
We will consider the cases $\lambda \le 2$, $2 < \lambda < 4$, $\lambda \ge 4$ separately. Note that by symmetry, it suffices to establish (20) on $S \cap \{y \ge 0\}$.
*The case* $\lambda \le 2$. Assume $(x_0, y_0) \in A_\lambda$. If $y_0 \ge \lambda$, then $g^* \ge \lambda$ almost surely, so $V_\lambda(x_0, y_0) \ge 1 = U_\lambda(x_0, y_0)$. If $\lambda > y_0 \ge \alpha x_0 - \alpha + \lambda$, then let $(f_0, g_0) \equiv (x_0, y_0)$,

$$df_1 = (1 - x_0)\chi_{[0,1]} \quad \text{and} \quad dg_1 = \alpha df_1. \tag{21}$$

Then we have $g_1 = y_0 + \alpha - \alpha x_0 \ge \lambda$, which implies $g^* \ge \lambda$ almost surely and (20) follows. Now suppose $(x_0, y_0) \in A_\lambda$ and $y_0 < \alpha x_0 - \alpha + \lambda$. Let $(f, g) \equiv (x_0, y_0)$,

$$df_1 = \frac{y_0 - x_0 + 1 - \lambda}{1 - \alpha}\chi_{[0,1]},\ dg_1 = \alpha df_1 \tag{22}$$

and

$$df_2 = dg_2 = \beta\chi_{[0,1-\beta/2)} + (\beta - 2)\chi_{[1-\beta/2,1]}, \tag{23}$$

where

$$\beta = \frac{\alpha x_0 - y_0 - \alpha + \lambda}{1 - \alpha} \in [0, 2]. \tag{24}$$

Then $(f_2, g_2)$ takes values $(-1, \lambda - 2)$, $(1, \lambda)$ with probabilities $\beta/2$, $1 - \beta/2$, respectively, so, by Lemma 4,

$$V_\lambda(x_0, y_0) \ge \frac{\beta}{2}V_\lambda(-1, \lambda - 2) + \left(1 - \frac{\beta}{2}\right)V_\lambda(1, \lambda) = \frac{\beta}{2}V_\lambda(-1, 2 - \lambda) + 1 - \frac{\beta}{2}. \tag{25}$$

Note that $(-1, 2 - \lambda) \in A_\lambda$. If $2 - \lambda \ge \alpha \cdot (-1) - \alpha + \lambda$, then, as already proved, $V_\lambda(-1, 2 - \lambda) = 1$ and $V_\lambda(x_0, y_0) \ge 1 = U_\lambda(x_0, y_0)$. If the converse inequality holds, i.e., $2 - \lambda < -2\alpha + \lambda$, then we may apply (25) to $x_0 = -1$, $y_0 = 2 - \lambda$ to get

$$V_\lambda(-1, 2 - \lambda) \ge \frac{\beta}{2}V_\lambda(-1, 2 - \lambda) + 1 - \frac{\beta}{2},$$

or $V_\lambda(-1, 2 - \lambda) \ge 1$. Thus we established $V_\lambda(x_0, y_0) = 1$ for any $(x_0, y_0) \in A_\lambda$.
Suppose then, that $(x_0, y_0) \in B_\lambda$. Let

$$\beta = \frac{2(1 - x_0)}{1 - x_0 - y_0 + \lambda} \in [0, 1] \tag{26}$$

and consider a pair $(f, g)$ starting from $(x_0, y_0)$ and satisfying

$$df_1 = -dg_1 = -\frac{x_0 - y_0 - 1 + \lambda}{2}\chi_{[0,\beta)} + (1 - x_0)\chi_{[\beta,1]}. \tag{27}$$

On $[0, \beta)$, the pair $(f_1, g_1)$ lies in $A_\lambda$; Lemma 4 implies $V_\lambda(x_0, y_0) \ge \beta = U_\lambda(x_0, y_0)$.
Finally, for $(x_0, y_0) \in C_\lambda$, let $(f, g)$ start from $(x_0, y_0)$ and

$$df_1 = -dg_1 = \frac{-x_0 - \lambda + 1 + y_0}{2}\chi_{[0,\gamma)} + \frac{y_0 - x_0 + 1}{2}\chi_{[\gamma,1]},$$

where

$$\gamma = \frac{y_0 - x_0 + 1}{\lambda} \in [0,1].$$

On $[0,\gamma)$, the pair $(f_1, g_1)$ lies in $A_\lambda$, while on $[\gamma, 1]$ we have $(f_1, g_1) = ((x_0 + y_0 + 1)/2, (x_0 + y_0 - 1)/2) \in B_\lambda$. Hence

$$V_\lambda(x_0, y_0) \geq \gamma \cdot 1 + (1-\gamma) \cdot \frac{1 - x_0 - y_0}{\lambda} = U_\lambda(x_0, y_0).$$

*The case* $2 < \lambda < 4$. For $(x_0, y_0) \in A_\lambda$ we prove (20) using the same processes as in the previous case, i.e. the constant ones if $y_0 \geq \lambda$ and the ones given by (21) otherwise. The next step is to establish the inequality

$$V_\lambda(-1, \lambda-2) \geq U_\lambda(-1, \lambda - 2) = \frac{1+\alpha}{2} + \frac{1-\alpha}{2} \cdot \Big(\frac{4-\lambda}{\lambda}\Big)^2. \tag{28}$$

To do this, fix $\delta \in (0,1]$ and set

$$\beta = \frac{\delta(1-\alpha)}{\lambda},\ \kappa = \frac{4 - \lambda - \delta(1+\alpha)}{\lambda} \cdot \beta,\ \gamma = \beta + (1-\beta) \cdot \frac{\delta(1+\alpha)}{4},\ \nu = \kappa \cdot \frac{\lambda}{4}.$$

We have $0 \leq \nu \leq \kappa \leq \beta \leq \gamma \leq 1$. Consider processes $f$, $g$ given by $(f_0, g_0) \equiv (-1, \lambda - 2)$, $(df_1, dg_1) \equiv (\delta, \alpha\delta)$,

$$df_2 = -dg_2 = \frac{\lambda - \delta(1-\alpha)}{2} \chi_{[0,\beta)} - \frac{\delta(1-\alpha)}{2} \chi_{[\beta,1]},$$

$$\begin{aligned} df_3 = dg_3 = -\big(\lambda - 2 + \frac{\delta(1+\alpha)}{2}\big)\chi_{[0,\kappa)} + \big(2 - \frac{\lambda + \delta(1+\alpha)}{2}\big)\big)\chi_{[\kappa,\beta)} \\ + \big(2 - \frac{\delta(1+\alpha)}{2}\big)\chi_{[\beta,\gamma)} - \frac{\delta(1+\alpha)}{2}\chi_{[\gamma,1)}, \end{aligned}$$

$$df_4 = -dg_4 = \big(-2 + \frac{\lambda}{2}\big)\chi_{[0,\nu)} + \frac{\lambda}{2}\chi_{[\nu,\kappa)}.$$

As $(f_4, |g_4|)$ takes values $(1,\lambda)$, $(1,0)$ and $(-1, \lambda-2)$ with probabilities $(\gamma - \beta) + (\kappa - \nu)$, $\beta - \kappa$ and $1 - \gamma + \nu$, respectively, we have

$$V_\lambda(-1, \lambda-2) \geq \gamma - \beta + \kappa - \nu + (1 - \gamma + \nu) V_\lambda(-1, \lambda - 2),$$

or

$$V_\lambda(-1, \lambda - 2) \geq \frac{\gamma - \beta + \kappa - \nu}{\gamma - \nu} = \frac{1+\alpha}{2} + \frac{1-\alpha}{2} \cdot \Big(\frac{4-\lambda}{\lambda}\Big)^2 - \frac{\delta(1-\alpha^2)}{\lambda^2}.$$

As $\delta$ is arbitrary, we obtain (28). Now suppose $(x_0, y_0) \in B_\lambda$ and recall the pair $(f, g)$ starting from $(x_0, y_0)$ given by (22) and (23) (with $\beta$ defined in (24)). As previously, it leads to (25), which takes form

$$\begin{aligned} V_\lambda(x_0, y_0) &\geq \frac{\beta}{2}\Big[\frac{1+\alpha}{2} + \frac{1-\alpha}{2} \cdot \Big(\frac{4-\lambda}{\lambda}\Big)^2\Big] + 1 - \frac{\beta}{2} \\ &= \frac{\beta(1-\alpha)}{4}\Big[\Big(\frac{4-\lambda}{\lambda}\Big)^2 - 1\Big] + 1 = \frac{(\alpha x_0 - \alpha - y_0 + \lambda)(4 - 2\lambda)}{\lambda^2} + 1 = U_\lambda(x_0, y_0). \end{aligned}$$

For $(x_0, y_0) \in C_\lambda$, consider a pair $(f,g)$, starting from $(x_0, y_0)$ defined by (27) (with $\beta$ given by (26)). On $[0,\beta)$ we have $(f_1, g_1) = ((x_0+y_0+1-\lambda)/2, (x_0+y_0-1+\lambda)/2) \in B_\lambda$, so Lemma 4 yields

$$\begin{aligned} V_\lambda(x_0,y_0) &\geq \beta V_\lambda\Big(\frac{x_0+y_0+1-\lambda}{2}, \frac{x_0+y_0-1+\lambda}{2}\Big) \\ &= \frac{2(1-x_0)}{1+\lambda-x_0-y_0}\cdot\Big\{1-\Big[\alpha\Big(\frac{x_0+y_0-1-\lambda}{2}\Big)-\frac{x_0+y_0-1-\lambda}{2}\Big]\cdot\frac{2\lambda-4}{\lambda^2}\Big\} \\ &= U_\lambda(x_0,y_0). \end{aligned}$$

For $(x_0, y_0) \in D_\lambda$, set $\beta = (y_0 - x_0 + 1)/\lambda \in [0,1]$ and let a pair $(f,g)$ be given by $(f_0, g_0) \equiv (x_0, y_0)$ and

$$df_1 = -dg_1 = \frac{-x_0+y_0+1-\lambda}{2}\chi_{[0,\beta)} + \frac{-x_0+y_0+1}{2}\chi_{[\beta,1]}.$$

As $(f_1, g_1)$ takes values

$$\Big(\frac{x_0+y_0+1-\lambda}{2}, \frac{x_0+y_0-1+\lambda}{2}\Big) \in B_\lambda \text{ and } \Big(\frac{x_0+y_0+1}{2}, \frac{x_0+y_0-1}{2}\Big) \in C_\lambda$$

with probabilites $\beta$ and $1-\beta$, respectively, we obtain $V_\lambda(x_0, y_0)$ is not smaller than

$$\begin{aligned} &\beta V_\lambda\Big(\frac{x_0+y_0+1-\lambda}{2}, \frac{x_0+y_0-1+\lambda}{2}\Big) + (1-\beta)V_\lambda\Big(\frac{x_0+y_0+1}{2}, \frac{x_0+y_0-1}{2}\Big) \\ &= \frac{y_0-x_0+1}{\lambda}\cdot\Big\{1-\Big[\alpha\Big(\frac{x_0+y_0-1-\lambda}{2}\Big)-\frac{x_0+y_0-1-\lambda}{2}\Big]\cdot\frac{2\lambda-4}{\lambda^2}\Big\} \\ &\qquad + \frac{\lambda-y_0+x_0-1}{\lambda}\Big[\frac{1-x_0-y_0}{\lambda} - \frac{(1-x_0-y_0)(1-\alpha)(\lambda-2)}{\lambda^2}\Big] \\ &= I + II + III + IV, \end{aligned}$$

where

$$I + III = \frac{y_0-x_0+1}{\lambda} + \frac{(\lambda-y_0+x_0-1)(1-x_0-y_0)}{\lambda^2} = \frac{2(1-x_0)}{\lambda} - \frac{(1-x_0)^2-y_0^2}{\lambda^2}$$

and

$$\begin{aligned} II + IV &= \frac{(1-\alpha)(\lambda-2)}{\lambda^3}\big[(y_0-x_0+1)(y_0+x_0-1-\lambda) - (1-x_0-y_0)(\lambda-y_0+x_0-1)\big] \\ &= -\frac{(1-\alpha)(\lambda-2)}{\lambda^3}\cdot\lambda(2-2x_0). \end{aligned}$$

Combining these facts, we obtain $V_\lambda(x_0, y_0) \geq U_\lambda(x_0, y_0)$.

For $(x_0, y_0) \in E_\lambda$ with $(x_0, y_0) \neq (-1, 0)$, the following contruction will turn to be useful. Denote $w = \lambda - 3$, so, as $(x_0, y_0) \in E_\lambda$, we have $x_0 + y_0 < w$. Fix positive integer $N$ and set $\delta = \delta_N = (w - x_0 - y_0)/[N(\alpha+1)]$. Consider sequences $(x_j^N)_{j=1}^{N+1}$, $(p_j)_{j=1}^{N+1}$, defined by

$$x_j^N = x_0 + y_0 + (j-1)\delta(\alpha+1),\ j = 1,\ 2,\ \ldots, N+1,$$

and $p_1^N = (1+x_0)/(1+x_0+y_0)$,

$$p_{j+1}^N = \frac{(1+x_j^N)\big(1+x_j^N+\frac{\delta(\alpha-1)}{2}\big)p_j^N}{(1+x_{j+1}^N)\big(1+x_j^N+\frac{\delta(\alpha+1)}{2}\big)} + \frac{\delta}{1+x_{j+1}^N},\ j=1,\ 2,\ \ldots,\ N. \tag{29}$$

We construct a process $(f,g)$ starting from $(x_0,y_0)$ such that for $j=1,2,\ldots,\ N+1$,

$$\begin{gathered}\text{the variable } (f_{3j},|g_{3j}|) \text{ takes values } (x_j^N,0) \text{ and } (-1,1+x_j^N)\\ \text{with probabilities } p_j^N \text{ and } 1-p_j^N, \text{ respectively.}\end{gathered} \tag{30}$$

We do this by induction. Let

$$df_1 = -dg_1 = y_0\chi_{[0,p_1^N)} + (-1-x_0)\chi_{[p_1^N,1]},\ df_2 = dg_2 = df_3 = dg_3 = 0.$$

Note that (30) is satisfied for $j=1$. Now suppose we have a pair $(f,g)$, which satisfies (30) for $j=1,\ 2,\ \ldots,\ n,\ n\le N$. Let us describe $f_k$ and $g_k$ for $k=3n+1,\ 3n+2,\ 3n+3$. The difference $df_{3n+1}$ is determined by the following three conditions: it is a martingale difference, i.e., satisfies $\mathbb{E}(df_{3n+1}|\mathscr{F}_{3n})=0$; conditionally on $\{f_{3n}=x_n^N\}$, it takes values in $\{-1-x_n^N,\ \delta(\alpha+1)/2\}$; and vanishes on $\{f_{3n}\neq x_n^N\}$. Furthermore, set $dg_{3n+1}=df_{3n+1}$. Moreover,

$$df_{3n+2} = \delta\chi_{\{f_{3n+1}=-1\}},\quad dg_{3n+2} = \frac{g_{3n+1}}{|g_{3n+1}|}\alpha\cdot df_{3n+2}.$$

Finally, the variable $df_{3n+3}$ satisfies $\mathbb{E}(df_{3n+3}|\mathscr{F}_{3n+2})=0$, and, in addition, the variable $f_{3n+3}$ takes values in $\{-1,x_n^N+\delta(\alpha+1)\}=\{-1,x_n^{N+1}\}$. The description is completed by

$$dg_{3n+3} = -\frac{g_{3n+2}}{|g_{3n+2}|}df_{3n+3}.$$

One easily checks that $(f_{3n+3},|g_{3n+3}|)$ takes values in $\{(x_{n+1}^N,0),(-1,1+x_{n+1}^N)\}$; moreover, since

$$\begin{aligned}\mathbb{E}f_{3n+3} &= \mathbb{E}f_{3n}+\mathbb{E}df_{3n+2} = x_n^Np_n^N-(1-p_n^N)+\delta\mathbb{P}(f_{3n+1}=-1)\\ &= x_n^Np_n^N-(1-p_n^N)+\delta\left(1-p_n^N+p_n^N\frac{\delta(\alpha+1)}{2(1+x_n^N)+\delta(\alpha+1)}\right)\\ &= p_n^N\cdot\frac{(x_n^N+1)(1+x_n^N+\delta(\alpha-1)/2)}{1+x_n^N+\delta(\alpha+1)/2}+\delta-1,\end{aligned}$$

we see that $\mathbb{P}(f_{3n+3}=x_{n+1}^N)=p_{n+1}^N$ and the pair $(f,g)$ satisfies (29) for $j=n+1$. Thus there exists $(f,g)$ satisfying (29) for $j=1,\ 2,\ \ldots,N+1$. In particular, $(f_{3(N+1)},|g_{3(N+1)}|)$ takes values $(w,0)$, $(-1,w+1)\in D_\lambda$ with probabilities $p_{N+1}^N$, $1-p_{N+1}^N$. By Lemma 4,

$$V_\lambda(x_0,y_0)\ge p_{N+1}^NV_\lambda(w,0)+(1-p_{N+1}^N)V_\lambda(-1,w+1). \tag{31}$$

Recall the function $H$ defined by (2). The function $h:[x_0+y_0,w]\to\mathbb{R}$ given by $h(t)=H(x_0,y_0,t)$, satisfies the differential equation

$$h'(t)+\frac{\alpha+2}{\alpha+1}\cdot\frac{h(t)}{1+t}=\frac{1}{(\alpha+1)(1+t)}.$$

As we assumed $x_0+y_0>-1$, the expression $(h(x+\delta)-h(x))/\delta$ converges uniformly to $h'(x)$ on $[x_0+y_0,\lambda-3]$. Therefore there exist constants $\varepsilon_N$, which depend only on $N$ and $x_0+y_0$ satisfying $\lim_{N\to\infty}\varepsilon_N=0$ and for $1\le j\le N$,

$$\left|\frac{h(x_{j+1}^N)-h(x_j^N)}{(\alpha+1)\delta_N}+\frac{\left[\frac{\alpha+2}{\alpha+1}(1+x_j^N)-\frac{\delta_N(\alpha+1)}{2}\right]h(x_j^N)}{(1+x_{j+1}^N)\left(1+x_j^N+\frac{\delta_N(\alpha+1)}{2}\right)}-\frac{1}{(\alpha+1)(1+x_{j+1}^N)}\right|\le\varepsilon_N,$$

or, equivalently,

$$\left|h(x_{j+1}^N)-\frac{(1+x_j^N)\left(1+x_j^N+\frac{\delta_N(\alpha-1)}{2}\right)h(x_j^N)}{(1+x_{j+1}^N)\left(1+x_j^N+\frac{\delta_N(\alpha+1)}{2}\right)}-\frac{\delta_N}{1+x_{j+1}^N}\right|\le(\alpha+1)\delta_N\varepsilon_N.$$

Together with (29), this leads to

$$|h(x_{j+1}^N)-p_{j+1}^N|\le\frac{(1+x_j^N)\left(1+x_j^N+\frac{\delta_N(\alpha-1)}{2}\right)}{(1+x_{j+1}^N)\left(1+x_j^N+\frac{\delta_N(\alpha+1)}{2}\right)}|h(x_j^N)-p_j^N|+(\alpha+1)\delta_N\varepsilon_N.$$

Since $p_1^N=h(x_1^N)$, we have

$$|h(w)-p_{N+1}^N|\le(\alpha+1)N\delta_N\varepsilon_N=(\lambda-3-x_0-y_0)\varepsilon_N$$

and hence $\lim_{N\to\infty}p_{N+1}^N=h(w)$. Combining this with (31), we obtain

$$V_\lambda(x_0,y_0)\ge h(w)(V_\lambda(w,0)-V_\lambda(-1,w+1))+V_\lambda(-1,w+1).$$

As $w=\lambda-3$, it suffices to check that we have

$$a_\lambda=V_\lambda(\lambda-3,0)-V_\lambda(-1,\lambda-2))\text{ and }b_\lambda=V_\lambda(-1,\lambda-2),$$

where $a_\lambda$, $b_\lambda$ were defined in (5). Finally, if $(x_0,y_0)=(-1,0)$, then considering a pair $(f,g)$ starting from $(x_0,y_0)$ and satisfying $df_1\equiv\delta$, $dg_1\equiv\alpha\delta$, we get

$$V(-1,0)\ge V(-1+\delta,\alpha\delta). \tag{32}$$

Now let $\delta\to0$ to obtain $V(-1,0)\ge U(-1,0)$.
*The case* $\lambda\ge4$. We proceed as in previous case. We deal with $(x_0,y_0)\in A_\lambda$ exactly in the same manner. Then we establish the analogue of (28), which is

$$V(-1,\lambda-2)\ge U_\lambda(-1,\lambda-2)=\frac{1+\alpha}{2}. \tag{33}$$

To do this, fix $\delta\in(0,1)$ and set

$$\beta=\frac{4-2\delta}{4-\delta(1+\alpha)},\quad\gamma=\beta\cdot\Big(1-\frac{\delta(\alpha+1)}{4}\Big).$$

Now let a pair $(f,g)$ be defined by $(f_0,g_0)\equiv(-1,\lambda-2)$, $(df_1,dg_1)\equiv(\delta,\ \alpha\delta)$,

$$df_2=-dg_2=-\frac{\delta(1-\alpha)}{2}\chi_{[0,\beta)}+(2-\delta)\chi_{[\beta,1]},$$

$$df_3 = dg_3 = -\frac{\delta(1+\alpha)}{2}\chi_{[0,\gamma)} + \Big(2 - \frac{\delta(1+\alpha)}{2}\Big)\chi_{[\gamma,\beta)}.$$

Then $(f_3, g_3)$ takes values $(-1, \lambda-2)$, $(1, \lambda)$ and $(1, \lambda - 4 + \delta(\alpha+1))$ with probabilities $\gamma$, $\beta - \gamma$ and $1-\beta$, respectively, and Lemma 4 yields

$$V(-1, \lambda-2) \geq \gamma V(-1, \lambda-2) + (\beta-\gamma)V(1,\lambda),$$

or

$$V(-1,\lambda-2) \geq \frac{\beta-\gamma}{1-\gamma} = \frac{(\alpha+1)(2-\delta)}{4-\delta(\alpha+1)}.$$

It suffices to let $\delta \to 0$ to obtain (33). The cases $(x_0, y_0) \in B_\lambda$, $C_\lambda$ are dealt with using the same processes as in the case $\lambda \in (2,4)$. If $(x_0, y_0) \in D_\lambda$, then Lemma 4, applied to the pair $(f, g)$ given by $(f_0, g_0) \equiv (x_0, y_0)$, $df_1 = -dg_1 = -(1+x_0)\chi_{[0,(1-x_0)/2)} + (1-x_0)\chi_{[(1-x_0)/2,1]}$, yields

$$V(x_0, y_0) \geq \frac{1-x_0}{2} V(-1, x_0 + y_0 + 1). \tag{34}$$

Furthermore, for any number $y$ and any $\delta \in (0,1)$, we have

$$V(-1, y) \geq V(-1+\delta, y + \alpha\delta), \tag{35}$$

which is proved in the same manner as (32). Hence, for large $N$, if we set $\delta = (\lambda - 3 - x_0 - y_0)/(N(\alpha+1))$, the inequalities (34) and (35) give

$$\begin{aligned} V(x_0,y_0) \geq \frac{1-x_0}{2}V(-1,x_0+y_0+1) &\geq \frac{1-x_0}{2}V(-1+\delta, x_0+y_0+1+\alpha\delta) \\ &\geq \frac{1-x_0}{2}\Big(1-\frac{\delta}{2}\Big)V(-1,x_0+y_0+1+(\alpha+1)\delta) \\ &\geq \frac{1-x_0}{2}\Big(1-\frac{\delta}{2}\Big)^N V(-1,x_0+y_0+1+N(\alpha+1)\delta) \\ &= \frac{1-x_0}{2}\Big(1-\frac{\lambda-3-x_0-y_0}{2N(\alpha+1)}\Big)^N V(-1,\lambda-2) \\ &= \frac{(1-x_0)(1+\alpha)}{4}\Big(1-\frac{\lambda-3-x_0-y_0}{2N(\alpha+1)}\Big)^N. \end{aligned}$$

Now take $N \to \infty$ to obtain $V_\lambda(x_0, y_0) \geq U_\lambda(x_0, y_0)$.
Finally, if $(x_0, y_0) \in E_\lambda$ we use the pair $(f, g)$ used in the proof of the case $(x_0, y_0) \in E_\lambda$, $\lambda \in (2,4)$, with $\omega = 1$. Then the process $(f, |g|)$ ends at the points $(1,0)$ and $(-1,2)$ with probabilities, which can be made arbitrarily close to $H(x_0, y_0, 1)$ and $1 - H(x_0, y_0, 1)$, respectively. It suffices to apply Lemma 4 and check that it gives $V_\lambda(x_0, y_0) \geq U_\lambda(x_0, y_0)$. □

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