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EXACT CONVERGENCE RATE FOR THE MAXIMUM OF STANDARDIZED GAUSSIAN INCREMENTS

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Abstract

We prove an almost sure limit theorem on the exact convergence rate of the maximum of standardized gaussian random walk increments. This gives a more precise version of Shao's theorem (*Shao, Q.-M., 1995. On a conjecture of Révész. Proc. Amer. Math. Soc. 123, 575-582*) in the gaussian case.

1 Introduction

Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. standard gaussian variables. Let $S_n = \sum_{i=1}^n \xi_i$, $S_0 = 0$ be the gaussian random walk and define the maximum of standardized gaussian random walk increments by

$$L_n = \max_{0\le i<j\le n} \frac{S_j - S_i}{\sqrt{j-i}}. \tag{1}$$

Recently, the precise behaviour of L_n and related quantities has become important in statistics [15], [16], [5], [8], [2]. For example, in the context of nonparametric function estimation, the statistical multiscale paradigm suggests to select a function from a set of candidate functions such that the resulting residuals immitate L_n, i.e. they behave as white noise simultaneously on all scales, see [5], [8], [2].

It follows from a more general result of Shao [14], see Theorem 2.1 below, that

$$\lim_{n\to\infty} L_n/\sqrt{2\log n} = 1 \text{ a.s.}$$

Our goal is to determine the exact convergence rate in Shao's theorem.

Theorem 1.1. *We have*

$$\limsup_{n\to\infty} \sqrt{2\log n}\,(L_n - \sqrt{2\log n})/\log\log n = 3/2 \ \textit{a.s.}$$

and

$$\liminf_{n\to\infty} \sqrt{2\log n}\,(L_n - \sqrt{2\log n})/\log\log n = 1/2 \ \textit{a.s.}$$

For comparison, we cite a result of [12], see also [19], [11], [18], [1] for extensions and improvements. Let $\{\xi_i\}_{i=1}^{\infty}$ be a stationary centered gaussian sequence with covariance function $r_n = \mathbb{E}(\xi_1\xi_n)$, $r_1 = 1$. Suppose that $r_n = O(n^{-\varepsilon})$, $n \to \infty$, for some $\varepsilon > 0$. Then, with $M_n = \max_{i=1,\dots,n} \xi_i$,

$$\limsup_{n\to\infty} \sqrt{2\log n}\,(M_n - \sqrt{2\log n})/\log\log n = 1/2 \text{ a.s.}$$

and

$$\liminf_{n\to\infty} \sqrt{2\log n}\,(M_n - \sqrt{2\log n})/\log\log n = -1/2 \text{ a.s.}$$

There is a continuous-time version of Theorem 1.1. To state it, let $\{B(x), x \geq 0\}$ be the standard Brownian motion. For $n > 1$ define

$$L_n^{\mathrm{Br},1} = \sup_{\substack{x_1,x_2\in[0,1]\\ x_2-x_1\geq 1/n}} \frac{B(x_2)-B(x_1)}{\sqrt{x_2-x_1}} \quad \text{and} \quad L_n^{\mathrm{Br},2} = \sup_{\substack{x_1,x_2\in[0,n]\\ x_2-x_1\geq 1}} \frac{B(x_2)-B(x_1)}{\sqrt{x_2-x_1}}.$$

Although for each fixed n the distributions of $L_n^{\mathrm{Br},1}$ and $L_n^{\mathrm{Br},2}$ coincide, the distributions of stochastic processes $\{L_n^{\mathrm{Br},1}, n > 1\}$ and $\{L_n^{\mathrm{Br},2}, n > 1\}$ are clearly different.

Theorem 1.2. *For $i = 1, 2$ we have*

$$\limsup_{n\to\infty} \sqrt{2\log n}\,(L_n^{Br,i} - \sqrt{2\log n})/\log\log n = 5/2 \ \textit{a.s.}$$

and

$$\liminf_{n\to\infty} \sqrt{2\log n}\,(L_n^{Br,i} - \sqrt{2\log n})/\log\log n = 3/2 \ \textit{a.s.}$$

The $\limsup$ part of Theorem 1.1 and Theorem 1.2 may be generalized to the following integral test.

Theorem 1.3. *Let $\{f_n\}_{n=1}^{\infty}$ be a positive non-decreasing sequence. Then*

$$\mathbb{P}[L_n > f_n \textit{ for infinitely many values of } n] = 1 \textit{ iff } \sum_{n=1}^{\infty} f_n e^{-f_n^2/2} = \infty$$

and, for i=1,2,

$$\mathbb{P}[L_n^{Br,i} > f_n \textit{ for infinitely many values of } n] = 1 \textit{ iff } \sum_{n=1}^{\infty} f_n^3 e^{-f_n^2/2} = \infty.$$

In the case $i = 1$, the second statement is a consequence of the Chung-Erdös-Sirao integral test [4]. Since $L_n^{\mathrm{Br},2}$ may be treated by essentially the same method as L_n, we omit the proof of the second half of the above theorem.
The corresponding result for M_n instead of L_n and $L_n^{\mathrm{Br},i}$ reads as follows, see [11, Th.B]:

$$\mathbb{P}[M_n > f_n \text{ for infinitely many values of } n] = 1 \text{ iff } \sum_{n=1}^{\infty} f_n^{-1} e^{-f_n^2/2} = \infty.$$

The limiting behavior as $n \to \infty$ of the *distribution* of L_n was studied in [15] (alternatively, see Th. 1.3 of [9]) and that of $L_n^{\mathrm{Br},i}$, $i = 1, 2$, in Th 1.6 of [9]. It was shown there that the appropriately normalized distributions of L_n and $L_n^{\mathrm{Br},i}$ converge to the Gumbel (double-exponential) distribution. The next theorem is a simple consequence of these results.

Theorem 1.4. *We have*

$$\lim_{n\to\infty} \sqrt{2\log n}\,(L_n - \sqrt{2\log n})/\log\log n = 1/2$$

and, for $i = 1, 2$,

$$\lim_{n\to\infty} \sqrt{2\log n}\,(L_n^{Br,i} - \sqrt{2\log n})/\log\log n = 3/2,$$

where the convergence is in probability.

Above, we have considered increments of *all* possible lengths of the gaussian random walk and Brownian motion. The increments of *fixed* length were extensively studied in the past, the most well-known result being the Erdös-Rényi-Shepp law of large numbers. Results similar to ours in the case of increments of fixed length were obtained in [7], [6], [13].

2 The non-gaussian case

It seems difficult to obtain the exact convergence rate for the maximum of standardized increments in the case of non-gaussian summands. Let us recall the following result of [14], see also [17].

Theorem 2.1. *Let* $\{\xi_i\}_{i=1}^{\infty}$ *be i.i.d. random variables. Suppose that* $\mathbb{E}\xi_1 = 0$, $\mathbb{E}\xi_1^2 = 1$ *and that* $\varphi(t) = \log \mathbb{E}e^{t\xi_1}$ *exists finitely in some interval containing* 0. *Let* $S_n = \sum_{i=1}^n \xi_i$, $S_0 = 0$ *and define* L_n *as in* (1). *Then*

$$\lim_{n\to\infty} L_n/\sqrt{2\log n} = \alpha^* \qquad a.s.,$$

where $\alpha^* \in [1, \infty]$ *is a constant defined as follows. Let* $I(t) = \sup_{x\in\mathbb{R}}(xt - \varphi(x))$, $\alpha(c) = \sup\{t \geq 0 : I(t) \leq 1/c\}$. *Then*

$$\alpha^* = \sup_{c>0}(\alpha(c)\sqrt{c/2}). \tag{2}$$

In particular, if ξ_i are standard normal, then $\alpha^* = 1$. If the supremum in (2) is attained at some number $c^* \in (0, \infty)$ then Shao's proof shows that the almost sure limiting behavior of L_n coincides with the behavior of the Erdös-Rényi statistic

$$L_n^{c^*} = \max_{\substack{0\leq i<j\leq n \\ j-i=[c^*\log n]}} \frac{S_j - S_i}{\sqrt{j-i}}.$$

If the supremum is attained at, say, $c^* = \infty$, then dominating in L_n are terms of the form $(S_j - S_i)/\sqrt{j-i}$ with $j - i \gg \log n$. Now, in the case of standard normal variables we have $\alpha(c) = \sqrt{2/c}$ for every $c \in (0, \infty)$. Thus, in the gaussian case, dominating in L_n are terms of the form $(S_j - S_i)/\sqrt{j-i}$ with $j - i \approx c \log n$, where c varies in $(0, \infty)$. Our proofs use this fact extensively. As the above discussion suggests, the exact convergence rate in the case of non-gaussian summands may depend strongly on the way in which the supremum in (2) is attained. Note also that for statistical purposes the square-root normalization in (1) seems to be natural only in the gaussian case. See [16] for a different normalization having a natural statistical interpretation.

In the rest of the paper we prove Theorem 1.3 and Theorem 1.1.

3 Standardized Brownian motion increments

We denote by $\mathbb{H} = \{t = (x, y) \in \mathbb{R}^2 | \ y > 0\}$ the upper half-plane. Let $\{B(x), x \in \mathbb{R}\}$ be the standard Brownian motion. Then the gaussian field $\{X(t), t = (x, y) \in \mathbb{H}\}$ of *standardized Brownian motion increments* is defined by

$$X(t) = \frac{B(x+y) - B(x)}{\sqrt{y}}. \tag{3}$$

We shall need the following theorem, see Theorem 2.1 and Example 2.10 in [3], which describes the precise asymptotics of the high excursion probability of the field X.

Theorem 3.1. *Let $K \subset \mathbb{H}$ be a compact set with positive Jordan measure. Then, for some constant $C_K > 0$,*

$$\mathbb{P}\left[\sup_{t \in K} X(t) > u\right] \sim C_K u^3 e^{-u^2/2} \text{ as } u \to +\infty.$$

The next theorem, although not stated explicitly in [3], may be proved by the methods of [3], cf. also [10, Lemma 12.2.4].

Theorem 3.2. *Let $K \subset \mathbb{H}$ be a compact set with positive Jordan measure. Let $u \to +\infty$ and $q \to +0$ in such a way that $qu^2 \to a$ for some constant $a > 0$. Then, for some constant $C_{K,a} > 0$,*

$$\mathbb{P}\left[\sup_{t \in K \cap q\mathbb{Z}^2} X(t) > u\right] \sim C_{K,a} u^3 e^{-u^2/2} \text{ as } u \to +\infty.$$

4 Proof of Theorem 1.3.

We prove only the first statement. In the sequel, C, C', C'', etc. are constants whose values are irrelevant and may change from line to line.

Suppose that $\sum_{n=1}^{\infty} f_n e^{-f_n^2/2} < \infty$. We are going to prove that a.s. only finitely many events $L_n > f_n$ occur. A simple argument, see e.g. [4], shows that we may suppose that $\frac{1}{2}\sqrt{2 \log n} < f_n < 2\sqrt{2 \log n}$.

Let $\{t_n\}_{n=0}^{\infty}$ be an increasing integer sequence such that $t_0 = 0$ and $t_n = [n \log n]$ for sufficiently large n. Note that

$$C'(\log n)^{1/2} < f_{t_n} < C''(\log n)^{1/2}. \tag{4}$$

It is easy to see that

$$\sum_{n=1}^{\infty} f_{t_n}^3 \exp(-f_{t_n}^2/2) < \infty. \tag{5}$$

For $m, k \in \mathbb{Z}_{\geq 0}$ define $l_{m,k} = t_{2^k(m+1)} - t_{2^k m}$. We need the following technical lemma.

Lemma 4.1. *There is a constant c_1 such that for all $m, k \in \mathbb{Z}_{\geq 0}$ we have $l_{m+1,k} < c_1 l_{m,k}$.*

Proof. Let N be a sufficiently large integer. Suppose first that $m = 0$ and $k > N$. Then

$$l_{m+1,k} = [2^{k+1}\log 2^{k+1}] - [2^k \log 2^k] < 5[2^k \log 2^k] = 5(t_{2^k} - t_0) = 5l_{m,k}.$$

Suppose now that $2^k m > N$. Then

$$\begin{aligned} l_{m+1,k} &\leq 2^k(m+2)\log(2^k(m+2)) - 2^k(m+1)\log(2^k(m+1)) + 1 \\ &= \int_{2^k(m+1)}^{2^k(m+2)} (1+\log u)du + 1 = \int_{2^k m}^{2^k(m+1)} (1 + \log(u + 2^k))du + 1 \\ &< \int_{2^k m}^{2^k(m+1)} 2(1+\log u)du - 2 < 2l_{m,k}. \end{aligned}$$

Note that for all but finitely many pairs $(m, k) \in \mathbb{Z}^2_{\geq 0}$ we have either $m = 0$ and $k > N$ or $2^k m > N$. This finishes the proof of the lemma. $\square$

Put $c_2 = c_1 + 1$. For $m, k \in \mathbb{Z}_{\geq 0}$ let the set $\mathcal{A}_{m,k}$ be defined by

$$\mathcal{A}_{m,k} = \{(i,j) \in \mathbb{Z}^2 : t_{2^k m} \leq j \leq t_{2^k(m+1)}, l_{m,k} \leq j - i \leq c_2 l_{m,k}\}.$$

For $m \in \mathbb{Z}_{\geq 0}$ define

$$\mathcal{B}_m = \{(i,j) \in \mathbb{Z}^2 : t_m < j \leq t_{m+1}, 1 \leq j - i < l_{m,0}\}.$$

We are going to show that each pair $(i,j) \in \mathbb{Z}^2$ such that $0 \leq i < j$ is contained in some $\mathcal{A}_{m,k}$, $k \in \mathbb{Z}_{\geq 0}$, $m \in \mathbb{Z}_{>0}$ or in some $\mathcal{B}_m$, $m \in \mathbb{Z}_{\geq 0}$. Take some $0 \leq i < j < \infty$. We may suppose that $j \geq t_1$, since otherwise $(i,j) \in \mathcal{B}_0$. Let $\tilde{k} = \max\{k : t_{2^k} \leq j\}$. For each $k = 0, \ldots, \tilde{k}$ there is $m(k) \in \mathbb{Z}_{>0}$ such that the interval $[t_{2^k m(k)}, t_{2^k(m(k)+1)}]$ contains j. Note that $m(\tilde{k}) = 1$. For $k = 0, \ldots, \tilde{k}$ set $d_k = l_{m(k),k}$ and let $d_{\tilde{k}+1} = t_{2^{\tilde{k}+1}}$. Then it follows from Lemma 4.1 that $d_{k+1} < c_2 d_k$ for each $k = 0, \ldots, \tilde{k}$. If $j - i < d_0$, then $(i,j) \in \mathcal{B}_{m(0)}$. Otherwise, we can find a number $k = k(i,j) \in \{0, \ldots, \tilde{k}\}$ such that $j - i \in [d_k, d_{k+1}]$ (note that $j - i \leq j \leq t_{2^{\tilde{k}+1}} = d_{\tilde{k}+1}$). Thus, we have $j \in [t_{2^k m(k)}, t_{2^k(m(k)+1)}]$ and $j - i \in [d_k, d_{k+1}] \subset [d_k, c_2 d_k]$. It follows that $(i,j) \in \mathcal{A}_{m(k),k}$.

We are going to show that a.s. only finitely many events

$$A_{m,k} = \text{"} \max_{(i,j)\in\mathcal{A}_{m,k}} \frac{S_j - S_i}{\sqrt{j-i}} > f_{t_{2^k m}}\text{"}, k \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}_{>0}$$

and

$$B_m = \text{"} \max_{(i,j)\in\mathcal{B}_m} \frac{S_j - S_i}{\sqrt{j-i}} > f_{t_m}\text{"}, m \in \mathbb{Z}_{\geq 0}$$

occur. Recall that $\{X(t), t = (x,y) \in \mathbb{H}\}$ is the field of standardized Brownian motion increments defined in (3). Let K be the set $\{(x,y) \in \mathbb{H} : y \in [1, c_2], x + y \in [0,1]\}$. Note that the sets $\mathcal{A}_{m,k}$ and $K \cap l_{m,k}^{-1}\mathbb{Z}^2$ may be identified by taking $i = t_{2^k m} + l_{m,k}x$, $j = i + l_{m,k}y$. Using this and the scaling property of Brownian motion it is easy to see that the gaussian vector

$$\left\{ \frac{S_j - S_i}{\sqrt{j-i}}; (i,j) \in \mathcal{A}_{m,k} \right\}$$

has the same distribution as

$$\{X(t); t \in K \cap l_{m,k}^{-1}\mathbb{Z}^2\}.$$

It follows that

$$\mathbb{P}[A_{m,k}] \le \mathbb{P}[\max_{t\in K} X(t) > f_{t_{2^k m}}].$$

Thus, by Theorem 3.1, we obtain

$$\mathbb{P}[A_{m,k}] \le C f_{t_{2^k m}}^3 \exp(-f_{t_{2^k m}}^2/2).$$

In order to be able to apply the Borel-Cantelli lemma we have to show that

$$\sum_{k=0}^{\infty}\sum_{m=1}^{\infty} f_{t_{2^k m}}^3 \exp(-f_{t_{2^k m}}^2/2) < \infty.$$

It is easy to verify that $\log t_{2^k m} < C \log t_i$ for all $i \in [2^k(m-1)+1, 2^k m]$, $k \in \mathbb{Z}_{\ge 0}$, $m \in \mathbb{Z}_{>0}$. It follows that

$$(\log t_{2^k m})^{3/2} \exp(-f_{t_{2^k m}}^2/2) < C 2^{-k} \sum_{i=2^k(m-1)+1}^{2^k m} (\log t_i)^{3/2} \exp(-f_{t_i}^2/2).$$

and thus, by summing over m,

$$\sum_{m=1}^{\infty} (\log t_{2^k m})^{3/2} \exp(-f_{t_{2^k m}}^2/2) < C 2^{-k} \sum_{i=1}^{\infty} (\log t_i)^{3/2} \exp(-f_{t_i}^2/2).$$

It follows, using (4), the previous inequality and (5),

$$\begin{aligned}
\sum_{k=0}^{\infty}\sum_{m=1}^{\infty} f_{t_{2^k m}}^3 \exp(-f_{t_{2^k m}}^2/2) &< C_1 \sum_{k=0}^{\infty}\sum_{m=1}^{\infty} (\log t_{2^k m})^{3/2} \exp(-f_{t_{2^k m}}^2/2) \\
&< C_2 \sum_{k=0}^{\infty} 2^{-k} \sum_{i=1}^{\infty} (\log t_i)^{3/2} \exp(-f_{t_i}^2/2) \\
&= 2C_2 \sum_{i=1}^{\infty} (\log t_i)^{3/2} \exp(-f_{t_i}^2/2) \\
&< C_3 \sum_{i=1}^{\infty} f_{t_i}^3 \exp(-f_{t_i}^2/2) < \infty.
\end{aligned}$$

By the Borel-Cantelli lemma, a.s. only finitely many events $A_{m,k}$, $k \in \mathbb{Z}_{\ge 0}$, $m \in \mathbb{Z}_{>0}$ occur. Now we are going to show that $\sum_{m=0}^{\infty} \mathbb{P}[B_m] \le \infty$. Let $\mathcal{N}$ be standard gaussian random variable. Since $\#\mathcal{B}_m \le (t_{m+1}-t_m)^2$, we have

$$\begin{aligned}\sum_{m=1}^{\infty}\mathbb{P}[B_m] &\le \sum_{m=1}^{\infty}(t_{m+1}-t_m)^2\mathbb{P}[\mathcal{N} > f_{t_m}]\\ &\le C_1\sum_{m=1}^{\infty}(\log m)^2 f_{t_m}^{-1}\exp(-f_{t_m}^2/2)\\ &\le C_2\sum_{m=1}^{\infty} f_{t_m}^3\exp(-f_{t_m}^2/2) < \infty.\end{aligned}$$

Thus, by the Borel-Cantelli lemma, a.s. only finitely many events B_m, $m \in \mathbb{Z}_{\ge 0}$ occur. Since any pair (i,j) such that $0 \le i < j < \infty$ is contained in some $\mathcal{A}_{m,k}$, $k \in \mathbb{Z}_{\ge 0}$, $m \in \mathbb{Z}_{>0}$ (resp. some $\mathcal{B}_m$, $m \in \mathbb{Z}_{\ge 0}$), with probability 1 for all but finitely many (i,j) we have $(S_j - S_i)/\sqrt{j-i} \le f_{t_{2^k m}} \le f_j$ (resp. $(S_j - S_i)/\sqrt{j-i} \le f_{t_m} \le f_j$). It follows that a.s. only finitely many events $L_n > f_n$ occur.
Now suppose that $\sum_{n=1}^{\infty} f_n e^{-f_n^2/2} = \infty$. Recall that $t_n = [n\log n]$ for sufficiently large n and let $l_n = [\log n]$. Again, it is not difficult to see that we may suppose that $\frac{1}{2}\sqrt{2\log n} < f_n < 2\sqrt{2\log n}$ and even $f_n/\sqrt{2\log n} \to 1$ as $n \to \infty$. It follows easily that $\sum_{n=1}^{\infty} f_{t_n}^3 e^{-f_{t_n}^2/2} = \infty$. We have to show that a.s. $L_n > f_n$ for infinitely many values of n. For n sufficiently large let

$$\mathcal{C}_n = \{(i,j) \in \mathbb{Z}^2 : i \in [t_{n-1}, t_{n-1} + 1/4 l_n], j - i \in [1/4 l_n, 1/2 l_n]\}.$$

Then the events

$$C_n = \text{"} \max_{(i,j)\in\mathcal{C}_n} \frac{S_j - S_i}{\sqrt{j-i}} > f_{t_n}\text{"}$$

are independent. Let $K = [0,1/4] \times [1/4,1/2]$. The sets $\mathcal{C}_n$ and $K \cap l_n^{-1}\mathbb{Z}^2$ may be identified by taking $i = t_{n-1} + l_n x$, $j = i + l_n y$. Thus, the distribution of the gaussian vector

$$\left\{\frac{S_j - S_i}{\sqrt{j-i}}; (i,j) \in \mathcal{C}_n\right\}$$

coincides with the distribution of

$$\{X(t); t \in K \cap l_n^{-1}\mathbb{Z}^2\},$$

and we have, by Theorem 3.2,

$$\sum_{n=1}^{\infty}\mathbb{P}[C_n] \ge C\sum_{n=1}^{\infty} f_{t_n}^3 \exp(-f_{t_n}^2/2) = \infty.$$

By the Borel-Cantelli lemma infinitely many events C_n occur a.s. and thus, for infinitely many pairs (i,j) we have $(S_j - S_i)/\sqrt{j-i} > f_{t_n} > f_j$. This finishes the proof.

5 Proof of Theorem 1.1.

The lim sup part of the theorem follows from Theorem 1.3 by setting $f_n(\varepsilon) = \sqrt{2\log n} + (3/2 + \varepsilon)\log\log n/\sqrt{2\log n}$ and noting that $\sum_{n=1}^{\infty} f_n(\varepsilon)e^{-f_n(\varepsilon)^2/2} < \infty$ iff $\varepsilon > 0$.

We prove the lim inf part. Let $f_n = \sqrt{2\log n} + (1/2+\varepsilon)\log\log n/\sqrt{2\log n}$ for some $\varepsilon > 0$. Then $\lim_{n\to\infty} \mathbb{P}[L_n < f_n] = 1$ by [9, Th. 1.3]. Thus, $L_n < f_n$ for infinitely many values of n a.s.
Let $f_n(\varepsilon) = \sqrt{2\log n} + (1/2-\varepsilon)\log\log n/\sqrt{2\log n}$ for some $\varepsilon > 0$. It remains to show that $L_n < f_n(\varepsilon)$ for at most finitely many values of n a.s. Let $l_n = [\log n]$. For $k = 1,\ldots,[n/(2l_n)]$ define

$$\mathcal{A}_k^{(n)} = \{(i,j) \in \mathbb{Z}^2 : (2k-1)l_n \le i \le 2kl_n, l_n/4 \le j-i \le l_n/2\}$$

and let $A_k^{(n)}$ be the event

$$A_k^{(n)} = \text{"} \max_{(i,j)\in\mathcal{A}_k^{(n)}} \frac{S_j - S_i}{\sqrt{j-i}} < f_n(\varepsilon)\text{"}.$$

Then

$$\mathbb{P}[L_n < f_n(\varepsilon)] < \mathbb{P}[\cap_{k=1}^{[n/(2l_n)]} A_k^{(n)}] = \mathbb{P}[A_1^{(n)}]^{[n/(2l_n)]}. \tag{6}$$

Let $K = [0,1]\times[1/4,1/2]$. Again, we may identify $\mathcal{A}_k^{(n)}$ and $K \cap l_n^{-1}\mathbb{Z}^2$ by taking $i = (2k-1)l_n + l_n x$, $j = i + l_n y$. Thus, the gaussian vector

$$\left\{ \frac{S_j - S_i}{\sqrt{j-i}}; (i,j) \in \mathcal{A}_k^{(n)} \right\}$$

has the same distribution as

$$\{X(t); t \in K \cap l_n^{-1}\mathbb{Z}^2\},$$

and we have, by Theorem 3.2,

$$\mathbb{P}[A_1^{(n)}] = 1 - \mathbb{P}[A_1^{(n)c}] < 1 - cf_n(\varepsilon)^3 \exp(-f_n(\varepsilon)^2/2). \tag{7}$$

A simple calculation using (6) and (7) shows that

$$\mathbb{P}[L_n < f_n(\varepsilon)] < \exp(-c(\log n)^\varepsilon).$$

It follows from the Borel-Cantelli lemma and the above inequality with ε replaced by $\varepsilon/2$ that a.s. only finitely many events $L_{n_i} < f_{n_i}(\varepsilon/2)$ take place, where $n_i = 2^i$. To finish the proof note that $f_{n_{i+1}}(\varepsilon) < f_{n_i}(\varepsilon/2)$, $n > N$. Since each n is contained in an interval of the form $[n_i, n_{i+1}]$, we have $L_n \ge L_{n_i} \ge f_{n_i}(\varepsilon/2) > f_{n_{i+1}}(\varepsilon) \ge f_n(\varepsilon)$ for all but finitely many values of n.
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