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Scaling limit of an equilibrium surface under the Random Average Process*

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Abstract

We consider the equilibrium surface of the Random Average Process started from an inclined plane, as seen from the height of the origin, obtained in [8], where its fluctuations were shown to be of order of the square root of the distance to the origin in one dimension, and the square root of the log of that distance in two dimensions (and constant in higher dimensions). Remarkably, even if not pointed out explicitly in [8], the covariance structure of those fluctuations is given in terms of the Green's function of a certain random walk, and thus corresponds to those of Discrete Gaussian Free Fields. In the present paper we obtain the scaling limit of those fluctuations in one and two dimensions, in terms of Gaussian processes, in the sense of finite dimensional distributions. In one dimension, the limit is given by Brownian Motion; in two dimensions, we get a process with a discontinuous covariance function.

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1 Introduction

This paper may be seen as a followup to [8], even if after a long span. In the latter paper, the Random Average Process (RAP) was introduced as a dynamical random surface/field, whose heights, indexed by d-dimensional (discrete) space, evolve in discrete time by taking averages of neighboring heights.[1] The average weights are random, hence the terminology. The initial condition is important for the behavior of the dynamics, and in [8] the case of an inclined hyperplane was considered, and, among other results on the time asymptotics of the RAP, a CLT for the height at the origin, as well as the existence

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[1] [8] also considers a continuous time version of the RAP.

of a limiting (in time) surface as seen from the height of the origin, were established. We stress that this involves a time limit only; space is kept fixed (and discrete). This limiting surface may be seen also as an invariant surface under the RAP dynamics.

Since that initial paper, considerable attention has been devoted to that model. We mention [15, 17, 19, 9, 20, 1, 4, 3, 5, 11].

In the present paper, we consider the above mentioned limiting/invariant surface obtained in [8], and obtain the (full) spatial scaling limit of its fluctuations in dimensions one and two (where they are unbounded; they are bounded in higher dimensions). In [8], the order of magnitude of those fluctuations were shown to be the square root of the distance to the origin in one dimension, and the square root of the log of that distance in dimension two. The (limiting) shape of the fluctuations was not addressed, and we seek to complete the picture now.

A remarkable feature of the fluctuations of the above mentioned invariant surface obtained in [8] is that its covariance structure is given by the Green's function of a certain random walk (whose jump distribution depends on the weights of the random averages of the RAP). In this sense, there is a relation with Discrete Gaussian Free Fields (by which we mean Gaussian fields indexed by $\mathbb{Z}^d$ with the same covariance structure).[2]

As an aside, it is natural to wonder whether the invariant surface is itself Gaussian or not. A positive answer would reduce the efforts in this paper to a straightforward computation of scaled covariances. Analysis of a simpler case (in one dimension, and where the above mentioned random walk is simple) suggests that this is not the case in general, and probably never for weights of bounded range (which is the case we address in this paper).

Be that as it may, upon taking spatial scaling limits of the invariant surfaces (in one and two dimensions, as aforementioned), we get Gaussian fields (in continuous space). In one dimension, it is Brownian motion, and in two dimensions it has a discontinuous covariance function.

Our results are in the sense of finite dimensional distributions only, in both cases. In the one dimensional case, it is conceivable that this may be strengthened to convergence in the usual space of continuous trajectories, without much departure from methods of the present paper and of [8] (although preliminary computations indicate quite a laborious effort on an approach at verifying classic tightness criteria for that).

The two dimensional scaling limit cannot be continuous, so we would seem to be limited in options for going beyond our present results, on the one hand. On the other hand, the same issue of course comes up in the two dimensional Discrete Gaussian Free Field with the same covariance, and an investigation of a possible connection with the continuous two dimensional Gaussian Free Field suggests itself, even if that sounds to the authors like an uncertain project at the moment (given their scant understanding of the latter object currently; one mismatch can however be spotted at this point, to the effect that our scaling of the two dimensional invariant surface leading to our scaling limit result differs from the one of the DGFF leading to the continous GFF — this is a point left to be understood).

Before closing this introductory discussion, and moving to the details of the RAP and our results, it is perhaps worth mentioning that scaling limit results were obtained in [1], for the RAP in one dimension, along a characteristic direction, with a possibly random initial surface. We failed to find a connection to our results, even if merely in a broad or conceptual sense, but that may be due to a lack of depth in our search.

A final point concerns previous attempts at proving our results. One reason for the long delay since [8] may be traced to an unsuccessful previous approach, aiming at verifying conditions in the literature for the CLT for processes with stationary increments,

[2] But this was not pointed out explicitly in [8].

such as the object of this paper in the one dimensional case. Common such conditions involve obtaining good estimates on decay of correlations, but these seem inadequate to deal with our case, which is more amenable to second moment estimation. The only set of conditions we found previously involving moments came from [12], a reference used in [8] for a Martingale CLT. It so happens that these conditions all require control over second moments of quantities involving certain conditional expectations, with some liberty over which σ-algebra to condition over. With such a choice made, one has to compute the limits of two different quantities, which have to come out equal. On and off over the years, time was spent making such (quite laborious and intricate) computations, with different choices of σ-algebra, and more recently we became convinced that this approach would not work (whatever the choice of σ-algebra), so, after some further investigation, we came up with the present, more direct approach (involving nonetheless verifying the conditions of a CLT from [12], but a Martingale CLT, as in [8]). Curiously, in the present approach we also need, at a point, to compute two limits which have to come out equal, and, in this case, they do. It is also perhaps remarkable that the present approach goes through a CLT in time for the process in order to obtain a CLT for the spatial fluctuations of (one of) its invariant measure(s).

1.1 Notation

Let us denote by $\{u_n(i,i+\cdot),\, n\geq 1,\, i\in\mathbb{Z}^d\}$ a collection of i.i.d. random probability vectors distributed in $[0,1]^{\mathbb{Z}^d}$ with finite range, and by $\mathcal{F}_n$ the σ-algebra generated by $\{u_i,\, 1\leq i\leq n\}$. As in [8], we also assume that

$$\mathbb{E}[u_1(0,j)]>0,\ \text{ for } |j|\leq 1.$$

By $\{X_n(i), i\in\mathbb{Z}^d, n\geq 0\}$, we refer to the discrete-time version of the Random Average Process (RAP) defined in (2.3) at [8] as follows

$$X_n(i)=\sum_{j\in\mathbb{Z}^d} u_n(i,j)X_{n-1}(j),\ \text{for } n\geq 1 \text{ and } i\in\mathbb{Z}^d.$$

We denote the inner product between two vectors $x=(x_1,\dots,x_d)\in\mathbb{Z}^d$ and $\lambda=(\lambda_1,\dots,\lambda_d)\in\mathbb{R}^d$ by $x\lambda^*$; i.e. $x\lambda^*=\sum_{i=1}^d x_i\lambda_i$.

Given $\lambda\in\mathbb{R}^d$, let us denote by $\widehat{X}_\infty$ the weak limit of the RAP seen from the height at the origin with the initial configuration being a hyper-plane, that is

$$(\widehat{X}_\infty(x))_{x\in\mathbb{Z}^d}\stackrel{d}{=}\lim_{n\to\infty}\left(X_n(x)-X_n(0)\right)_{x\in\mathbb{Z}^d},\ \text{where } X_0(x)=x\lambda^*. \tag{1.1}$$

The existence of $\widehat{X}_\infty$ is proved in Corollary 5.2 in [8]. As pointed out in [8], we may consider the random walk in a random environment $\tilde{Y}_k^x$, with $\tilde{Y}_0^x=x$ and conditional probability transitions

$$\mathbb{P}\left(\tilde{Y}_k^x=j\middle|\tilde{Y}_{k-1}^x=i,\mathcal{F}_n\right)=u_k(i,j),\ \text{ for } 1\leq k\leq n,$$

such that for every $n\geq 1$

$$\big(X_n(x)\big)_{x\in\mathbb{Z}^d}\stackrel{d}{=}\left(\mathbb{E}\big[X_0(\tilde{Y}_n^x)\big|\mathcal{F}_n\big]\right)_{x\in\mathbb{Z}^d}. \tag{1.2}$$

Through (1.2) we can get the following representation (see (2.21) and (2.22) in [8]), that will be crucial in our way of dealing with $\widehat{X}_\infty$:

$$\widehat{X}_\infty(x)-x\lambda^*\stackrel{d}{=}\sum_{i=1}^{\infty}\left(W_i^x-W_i^0\right)\lambda^*,\ \text{for } x\in\mathbb{Z}^d, \tag{1.3}$$

where

$$W_i^x = \mathbb{E}[\theta_i(\tilde{Y}_{i-1}^x)|\mathcal{F}_i], \text{ for } i \geq 1, x \in \mathbb{Z}^d,$$

and

$$\theta_n(k) = \sum_{j \in \mathbb{Z}^d} (j-k) u_n(k,j), \text{ for } n \geq 1, k \in \mathbb{Z}^d.$$

We assume that

$$\sigma^2 := \mathbb{V}(\theta_1(0)\lambda^*) \in (0,\infty),$$

where $\mathbb{V}$ stands for variance.

1.2 Main results

We state now the main results of this work. In Theorem 1.1 below, we enunciate a Central Limit Theorem for the process $\{\widehat{X}_\infty(x),\, x \in \mathbb{Z}^d\}$ in dimensions $d = 1, 2$, and in Theorem 1.4, its finite dimension convergence. Both theorems are obtained (directly or through corollaries) from Proposition 1.2 and Proposition 1.3, also stated in this section.

Theorem 1.1. *Let $\widehat{X}_\infty$ be the weak limit of the RAP seen from the height at the origin and with the initial configuration being a hyper-plane as defined in* (1.1)*. There exists a positive constant $c = c(d)$ such that*

$$\frac{\widehat{X}_\infty(x) - x\lambda^*}{\sqrt{\mathcal{P}_x}} \xrightarrow[|x|\to\infty]{d} \mathcal{N}(0,c),$$

where $\mathcal{P}_x = |x|$ for $d = 1$, $\mathcal{P}_x = \log|x|$ for $d = 2$ and $\mathcal{N}(0,c)$ is a mean zero Gaussian r.v. with variance c.

Propositions 1.2 and 1.3, stated next, allow us to split the infinite series in (1.3) in two sums, such that, after normalization, one sum converges to a Gaussian law, and the second moment of the other is close to zero in a certain way to be made precise, leading to Theorem 1.1.

Proposition 1.2. *Let us consider $\mathcal{P}_x$ as defined in Theorem 1.1. The following limits exist*

$$h(A) := \lim_{|x|\to\infty} \frac{1}{\mathcal{P}_x} \mathbb{E}\Big(\sum_{i=1}^{A|x|^2} (W_i^x - W_i^0)\lambda^* \Big)^2, \text{ for any } A \geq 1,$$
$$c(d) := \lim_{A\to\infty} h(A).$$

Proposition 1.3. *Let us consider $\mathcal{P}_x$ as in Theorem 1.1 and h as in Proposition 1.2. Then*

$$\sum_{i=1}^{A|x|^2} \frac{\left(W_i^x - W_i^0\right)\lambda^*}{\sqrt{\mathcal{P}_x}} \xrightarrow[|x|\to\infty]{d} \mathcal{N}(0, h(A)), \text{ for any } A \geq 1.$$

In Section 3 and Section 5, we obtain corollaries from Proposition 1.2 and Proposition 1.3, respectively. These results lead us to Theorem 1.4 as Proposition 1.2 and Proposition 1.3 do for Theorem 1.1. For the sake of simplicity, we decide not to include the statement of these corollaries in this section.

Theorem 1.4. *Let us consider the following rescaled process.*

(i) *For $d = 1$ define*

$$X_n(t) := \frac{\widehat{X}_\infty(\lfloor nt \rfloor) - \lfloor nt \rfloor \lambda}{\sqrt{c\mathcal{P}_n}}, \text{ for } t \geq 0 \text{ and } n \geq 1,$$

where $\mathcal{P}_n$ and c are taken as in Theorem 1.1. Let $\{B(t), t \geq 0\}$ be a Standard Brownian motion. Then for $0 < t_1 < \cdots < t_k$ we have

$$(X_n(t_1), \ldots, X_n(t_k)) \xrightarrow[n\to\infty]{d} (B(t_1), \ldots, B(t_k)). \tag{1.4}$$

(ii) *In case $d = 2$, given $z \in \mathbb{Z}^2 \setminus (0, 0)$, let us define*

$$\tilde{x}_n(z) := (\lfloor n^{|z(1)|} \rfloor, \lfloor n^{|z(2)|} \rfloor) \text{ for } n \geq 1, z \in \mathbb{Z}^2, \tag{1.5}$$

and

$$X_n(z) := \frac{\widehat{X}_\infty(\tilde{x}_n(z)) - \tilde{x}_n(z)\lambda^*}{\sqrt{c\mathcal{P}_n}}, \text{ for } z \in \mathbb{Z}^2 \text{ and } n \geq 1.$$

Then for $z_1, \ldots, z_k$ in $\mathbb{Z}^2$ we have

$$(X_n(z_1), X_n(z_2), \ldots, X_n(z_k)) \xrightarrow[n\to\infty]{d} (Z_1, \ldots, Z_k), \tag{1.6}$$

where $(Z_1, \ldots, Z_k)$ is a Gaussian vector with covariance matrix $(C_{j,l})_{1\leq j,l\leq k}$, defined as follows

$$C_{j,l} := \begin{cases} \max\{|z_j(1)|, |z_j(2)|\}, & \text{for } j = l, \\ \frac{1}{2}\min\{\max\{|z_l(1)|, |z_l(2)|\}, \max\{|z_j(1)|, |z_j(2)|\}\}, & \text{for } j \neq l. \end{cases}$$

To conclude this introduction, we establish that the structure of the article. In Section 2, we prove Proposition 1.2. In Section 3, we state and prove a corollary of Proposition 1.2: Corollary 3.1. In Section 4, we prove Proposition 1.3. In Section 5, we enunciate and prove a corollary of Proposition 1.3: Corollary 5.1. From Proposition 1.2 and Proposition 1.3, in Section 6, we obtain Theorem 1.1. From Corollary 3.1 and Corollary 5.1, in Section 7, we obtain Theorem 1.4. Finally, in Appendix A we present some technical calculations needed for the proof of Proposition 1.3. These calculations are inspired by and are very similar to ones that appear in the proof of Theorem 4.1 in [8]. We include them for the sake of completeness.

2 Proof of Proposition 1.2

We split the proof of Proposition 1.2 into two cases, $d = 1$ and $d = 2$. Each case will be dealt with in separate subsections, 2.1 and 2.2, respectively. Later in Section 3, we prove a corollary of Proposition 1.2 (Corollary 3.1) that is used to obtain Theorem 1.4 in the same way that Proposition 1.2 is used to get Theorem 1.1.

As in [8] let us denote by $D = \{D_n, n \geq 0\}$ and $H = \{H_n, n \geq 0\}$ two Markov chains in $\mathbb{Z}^d$ with the following transition probabilities,

$$\mathbb{P}(D_{n+1} = k | D_n = l) = \sum_{j\in\mathbb{Z}^d} \mathbb{E}[u_1(0, j)u_1(l, j + k)],$$

and

$$\mathbb{P}(H_{n+1} = k | H_n = l) = \sum_{j\in\mathbb{Z}^d} \mathbb{E}[u_1(0, j)]\, \mathbb{E}[u_1(l, j + k)].$$

Also let us consider the following stopping time

$$\tau = \inf\{k \geq 0 : D_k = 0\}. \tag{2.1}$$

It follows from a standard argument using the Markov property (see (5.9) in [8]) that for $x \in \mathbb{Z}^d$ we get

$$\sum_{k=0}^{n} \{\mathbb{P}_0(D_k = 0) - \mathbb{P}_x(D_k = 0)\} = \sum_{k=0}^{n} \mathbb{P}_0(D_{n-k} = 0)\mathbb{P}_x(\tau > k). \tag{2.2}$$

The following results are also used to prove Proposition 1.2.

Lemma 2.1. *Let us consider*

$$a(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \{\mathbb{P}_0\left(H_k = 0\right) - \mathbb{P}_x(H_k = 0)\} \text{ for } x \in \mathbb{Z}^d,$$

the potential kernel of the Markov chain H, and set $\mathfrak{A} = \mathbb{E}_0\big[a(D_1)\big]$. Consider also the following quadratic form:

$$Q(\theta) = \mathbb{E}_0\left[(H_1 \cdot \theta)^2\right] = \theta \cdot Q \cdot \theta^t,\, \theta \in \mathbb{Z}^d, \tag{2.3}$$

when $d = 1$, $Q = \mathbb{E}_0\left[H_1^2\right] =: \sigma_H^2$, and when $d = 2$, Q is the convariance matrix of H_1 with the chain starting from the origin. Then, in the case $d = 1$ we have that

$$\mathbb{P}_0(D_n = 0) \sim \frac{1}{\mathfrak{A}} \frac{1}{\sqrt{2\pi}\,\sigma_H} \frac{1}{\sqrt{n}} \text{ as } n \to \infty, \tag{2.4}$$

and when $d = 2$

$$\sum_{k=0}^{n} \mathbb{P}_0(D_k = 0) \sim \frac{1}{\mathfrak{A}} \frac{1}{2\pi\sqrt{det(Q)}} \ln(n) \text{ as } n \to \infty. \tag{2.5}$$

Proof. By P7.9 (pag. 75) in [18] we have that

$$\lim_{n\to\infty} (2\pi n)^{d/2}\mathbb{P}_0(H_n = 0) = \frac{1}{\sqrt{det(Q)}}. \tag{2.6}$$

This implies that

$$\sum_{k=0}^{n} \mathbb{P}_0(H_k = 0) \overset{3}{\sim} \begin{cases} \frac{2}{\sqrt{2\pi}\,\sigma_H}\sqrt{n}, & \text{when } d = 1, \\ \frac{1}{2\pi\sqrt{det(Q)}} \ln n, & \text{when } d = 2. \end{cases} \tag{2.7}$$

Let

$$f(s) = \sum_{n\geq 0} \mathbb{P}\left(D_n = 0|D_0 = 0\right) s^n \text{ and } g(s) = \sum_{n\geq 0} \mathbb{P}\left(H_n = 0|H_0 = 0\right) s^n$$

be the power series of $\mathbb{P}(D_n = 0|D_0 = 0)$ and $\mathbb{P}(H_n = 0|H_0 = 0)$, respectively. By Theorem 5 in [7] (page 447) and (2.7), we obtain

$$g(s) \overset{4}{\sim} \begin{cases} \frac{1}{\sqrt{2}\,\sigma_H} \frac{1}{\sqrt{1-s}}, & \text{when } d = 1, \\ \frac{1}{2\pi\sqrt{det(Q)}} \ln\left(\frac{1}{1-s}\right), & \text{when } d = 2 \end{cases} \quad \text{as } s \to 1^-. \tag{2.8}$$

In Lemma 3.2 of [8] it is proved that

$$\lim_{s\to 1} \frac{f(s)}{g(s)} = \frac{1}{\mathfrak{A}}.^5 \tag{2.9}$$

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[3] Let r_n and s_n be two sequences of real numbers, we write $r_n \sim s_n$ when $\lim_{n\to\infty} \frac{r_n}{s_n} = 1$.

[4] Let $h_1(s)$ and $h_2(s)$ be two real functions, we write $h_1 \sim h_2$ as $s \to s_0$ when $\lim_{s\to s_0} \frac{h_1(s)}{h_2(s)} = 1$.

[5] (2.9) corresponds to Equation 3.14 in [8], except that there are (minor) mistakes in the argument in [8] leading to the latter equation, causing the appearance of the extraneous factor of $1 - \gamma'$ in (3.14) of [8], which should not be there, and also causing the expression $(1 - \gamma) \sum_x a(x)p_x$, which we presently write as $\mathfrak{A} = \mathbb{E}_0(a(D_1))$, to appear in the numerator (so to say), rather than in the denominator.

Therefore,

$$f(s) \sim \begin{cases} \frac{1}{\mathfrak{A}} \frac{1}{\sqrt{2}\,\sigma_H} \frac{1}{\sqrt{1-s}}, & \text{when } d=1, \\ \frac{1}{\mathfrak{A}} \frac{1}{2\pi\sqrt{det(Q)}} \ln\left(\frac{1}{1-s}\right), & \text{when } d=2 \end{cases} \quad \text{as } s \to 1^-. \tag{2.10}$$

Again by Theorem 5 in [7] we get (2.5). In Lemma 3.1 of [8] is proved that $\mathbb{P}_0(D_n = 0)$ is non-increasing in n. Therefore, we can use the second part of Theorem 5 in [7] for the case $d = 1$, and we obtain (2.4). □

One last comment before the proof Proposition 1.2 is that by equation (5.8) in [8] we have that

$$\mathbb{E}\Big(\sum_{i=1}^{A|x|^2} \left(W_i^x - W_i^0\right) \lambda^* \Big)^2 = 2\sigma^2 \sum_{i=1}^{A|x|^2} \left\{ \mathbb{P}(D_i = 0 | D_0 = 0) - \mathbb{P}(D_i = 0 | D_0 = x) \right\}. \tag{2.11}$$

Hence, to get Proposition 1.2, it is enough to show that

(i) when $d = 1$,

$$\frac{2\sigma^2}{|x|} \sum_{k=1}^{Ax^2} \left\{ \mathbb{P}(D_k = 0 | D_0 = 0) - \mathbb{P}(D_k = 0 | D_0 = x) \right\} \underset{|n| \to \infty}{\longrightarrow} h(A) \underset{A \to \infty}{\longrightarrow} c, \tag{2.12}$$

(ii) when $d = 2$,

$$\frac{2\sigma^2}{\log |x|} \sum_{k=1}^{A|x|^2} \left\{ \mathbb{P}(D_k = 0 | D_0 = 0) - \mathbb{P}(D_k = 0 | D_0 = x) \right\} \underset{|x| \to \infty}{\longrightarrow} c, \text{ for all } A \geq 1. \tag{2.13}$$

2.1 Proof of Proposition 1.2 when $\mathrm{d} = 1$

Since the chains D and H are symmetric (see Lemma 2.5 in [8]), we may consider (2.12) with $x = n > 0$. By (2.2) and Lemma 2.1, the left hand side of (2.12) becomes

$$\frac{c'}{n} \sum_{k=1}^{An^2} \frac{1}{\sqrt{An^2 - k}} \mathbb{P}_n(\tau > k), \text{ where } c' = \frac{2\sigma^2}{\mathfrak{A}\sqrt{2\pi}\sigma_H}. \tag{2.14}$$

The left hand side of (2.14) then equals

$$c' \sum_{k=1}^{An^2} \frac{1}{\sqrt{A - \frac{k}{n^2}}} \mathbb{P}_0\left(\frac{\tau_n}{n^2} > \frac{k}{n^2} \right) \frac{1}{n^2}, \tag{2.15}$$

where τ_n is the hitting time of n by H.

It follows from standard properties of Brownian motion that τ_n / n^2 converges in distribution to T_{1/σ_H}, where for each fixed $a \in \mathbb{R}$, T_a is the passage time of a by a standard one-dimensional Brownian motion $(B_s)_{s \geq 0}$. We spell this out at the end of this proof. Since the latter random variable has a continuous distribution, we have that

$$\mathbb{P}_0\left(\frac{\tau_n}{n^2} > t \right) \underset{n \to \infty}{\longrightarrow} \mathbb{P}(T_{1/\sigma_H} > t), \tag{2.16}$$

uniformly (the latter claim is a standard exercise; see, e.g., Exercise 3.2.9 in [6]). It follows that (2.15) is asymptotically equivalent to

$$c' \sum_{k=1}^{An^2} \frac{1}{\sqrt{A - \frac{k}{n^2}}} \mathbb{P}\left(T_{1/\sigma_H} > \frac{k}{n^2} \right) \frac{1}{n^2}. \tag{2.17}$$

We now observe that

$$\lim_{n\to\infty} c' \sum_{k=1}^{An^2} \frac{1}{\sqrt{A-\frac{k}{n^2}}} \mathbb{P}\left(T_{1/\sigma_H} > \frac{k}{n^2}\right)\frac{1}{n^2} = c'\int_0^A \frac{1}{\sqrt{A-y}}\mathbb{P}(T_{1/\sigma_H} > y)dy$$
$$= c'\int_0^1 \frac{\sqrt{A}\mathbb{P}(T_{1/\sigma_H} > Ay)}{\sqrt{1-y}}dy, \tag{2.18}$$

which proves the first part of Proposition 1.2, namely

$$\lim_{|x|\to\infty} \frac{1}{\mathcal{P}_x}\mathbb{E}\left(\sum_{i=1}^{A|x|^2}(W_i^x - W_i^0)\lambda^*\right)^2 = c'\int_0^1 \frac{\sqrt{A}\mathbb{P}(T_{1/\sigma_H} > Ay)}{\sqrt{1-y}}dy =: h(A). \tag{2.19}$$

Hence, to conclude the proof, we verify the following claim.

Claim 2.2.

$$\lim_{A\to\infty}\int_0^1 \frac{\sqrt{A}\,\mathbb{P}(T_{1/\sigma_H} > Ax)}{\sqrt{1-x}}dx = \frac{\sqrt{2\pi}}{\sigma_H}. \tag{2.20}$$

Proof of Claim 2.2. From a well-known formula,[6] we have

$$\mathbb{P}(T_{1/\sigma_H} \le t) = \frac{2}{\sqrt{2\pi}}\int_{\frac{1}{\sigma_H\sqrt{t}}}^{\infty} e^{-\frac{x^2}{2}}dx. \tag{2.21}$$

Applying L'Hôpital's rule, we find that

$$\lim_{t\to\infty}\frac{1-\frac{2}{\sqrt{2\pi}}\int_{\frac{1}{\sigma_H\sqrt{t}}}^{\infty} e^{-\frac{x^2}{2}}dx}{1/\sqrt{t}} = \lim_{t\to\infty}\frac{2}{\sqrt{2\pi}\sigma_H}e^{-\frac{1}{2\sigma_H^2 t}} = \frac{2}{\sqrt{2\pi}\sigma_H};$$

it then follows that, for every $x\in(0,1)$

$$\lim_{A\to\infty}\sqrt{A}\mathbb{P}(T_1 \ge Ax) = \frac{2}{\sqrt{2\pi}\sigma_H\sqrt{x}}. \tag{2.22}$$

It also follows from (2.21) that

$$\mathbb{P}(T_{1/\sigma_H} > t) \le \frac{1}{\sigma_H\sqrt{t}}, \tag{2.23}$$

for all $t>0$. We prove this by computing the derivative of

$$\frac{1}{\sigma_H\sqrt{t}} - 1 + \frac{2}{\sqrt{2\pi}}\int_{\frac{1}{\sigma_H\sqrt{t}}}^{\infty} e^{-\frac{x^2}{2}}dx, \tag{2.24}$$

and checking that it is negative for all $t>0$, which implies that the expression in (2.24) is non-increasing in $(0,\infty)$. Since $\int_0^\infty e^{-\frac{x^2}{2}}dx = \sqrt{2\pi}/2$, (2.24) vanishes as $t\to\infty$. Hence, the equation (2.24) is greater or equal to zero for all $t>0$, and (2.23) follows. Now (2.22) and (2.23) allow us to use the Dominated Convergence Theorem to get

$$\int_0^1 \frac{\sqrt{A}\mathbb{P}(T_{1/\sigma_H} > Ax)}{\sqrt{1-x}}dx \underset{A\to\infty}{\longrightarrow} \frac{2}{\sqrt{2\pi}\sigma_H}\int_0^1\frac{1}{\sqrt{x(1-x)}}dx = \frac{2}{\sqrt{2\pi}\sigma_H}\pi = \frac{\sqrt{2\pi}}{\sigma_H}.$$

Therefore, by (2.19) and Claim 2.2 we obtain

$$\lim_{A\to\infty}\lim_{|x|\to\infty}\frac{1}{\mathcal{P}_x}\mathbb{E}\left(\sum_{i=1}^{A|x|^2}[W_i^x - W_i^0]\lambda^*\right)^2 = c'\frac{\sqrt{2\pi}}{\sigma_H} = \frac{2\sigma^2}{\mathfrak{A}\,\sigma_H^2} =: c(1).$$

and the proof of Proposition 1.2 is concluded for $d=1$. □

[6] See, e.g., Remark 2.8.3 in page 92 of [13].

Argument for (2.16) **for a fixed** $t > 0$ Let us write τ_n as $\tau_n^+ + \tau_n'$, where τ_n^+ is the hitting time of $\{n, n+1, \ldots\}$ by H, namely,

$$\tau_n^+ = \inf\{k \geq 0 : \, H_k \geq n\},$$

and τ_n' is the time elapsed since then to reach n, namely,

$$\tau_n' = \inf\{k \geq \tau_n^+ : \, H_k = n\} - \tau_n^+.$$

From our assumptions on the jumps of H, we have that τ_n' is a tight sequence, and thus it is enough to show (2.16) with τ_n replaced by τ_n^+, namely

$$\mathbb{P}_0\left(\frac{\tau_n^+}{n^2} > t\right) \underset{n\to\infty}{\longrightarrow} \mathbb{P}(T_{1/\sigma_H} > t). \tag{2.25}$$

In order to do that, we first note that the event on the left hand side of (2.25) equals $\{M_\ell < n\}$, where $\ell = \lfloor n^2 t \rfloor$ and $M_s = \max_{0\leq i\leq s} H_i$. Using this, and rewriting the event, we find that the left hand side of (2.25) equals $P_0(M^{(\ell)} < u_n/\sqrt{t})$, where $M^{(s)} = M_s/\sqrt{s}$ and $u_n = n\sqrt{t}/\sqrt{\lfloor n^2 t\rfloor}$. Donsker's Theorem and a straightforward continuity argument now tells us that the limit as $n \to \infty$ of the latter probability equals $P_0(\sigma_H \mathcal{M}_1 < 1/\sqrt{t}) = P_0(\mathcal{M}_t < \sigma_H^{-1})$, where $\mathcal{M}_s = \max_{0\leq r\leq s} B_r$; Brownian scaling justifies the equality of the two latter probabilities, the latter of which is readily seen to equal the right hand side of (2.25) — again, the events are the same.

2.2 Proof of Proposition 1.2 when $\mathrm{d} = 2$

Rewriting the sum in the left hand side of (2.13) we obtain

$$\sum_{k=1}^{A|x|^2} \{\mathbb{P}(D_k = 0|D_0 = 0) - \mathbb{P}(D_k = 0|D_0 = x)\} = \mathbb{E}_0 \sum_{k=1}^{A|x|^2} \mathbb{1}_{\{D_k=0\}} - \mathbb{E}_x \sum_{k=1}^{A|x|^2} \mathbb{1}_{\{D_k=0\}}. \tag{2.26}$$

Let us work first with the second expected value in the right hand side of (2.26). Consider τ as defined in (2.1). Then by the Markov property we get that

$$\mathbb{E}_x\Big(\sum_{k=1}^{A|x|^2} \mathbb{1}_{\{D_k=0\}}\Big) = \mathbb{E}_x\Big(\sum_{k=\tau}^{A|x|^2} \mathbb{1}_{\{D_k=0\}}; \tau < A|x|^2\Big) = \sum_{j=1}^{A|x|^2} \mathbb{E}_0\Big(\sum_{k=0}^{A|x|^2-j} \mathbb{1}_{\{D_k=0\}}\Big)\mathbb{P}_x(\tau = j). \tag{2.27}$$

Substituting (2.27) into (2.26), we find that

$$\begin{aligned}
&\sum_{k=1}^{A|x|^2} \{\mathbb{P}(D_k = 0|D_0 = 0) - \mathbb{P}(D_k = 0|D_0 = x)\} \\
&= \mathbb{E}_0\Big(\sum_{k=1}^{A|x|^2} \mathbb{1}_{\{D_k=0\}}\Big) - \sum_{j=1}^{A|x|^2} \mathbb{E}_0\Big(\sum_{k=0}^{A|x|^2-j} \mathbb{1}_{\{D_k=0\}}\Big)\mathbb{P}_x(\tau = j) \\
&= \mathbb{P}_x\left(\tau > A|x|^2\right) \mathbb{E}_0\Big(\sum_{k=1}^{A|x|^2} \mathbb{1}_{\{D_k=0\}}\Big) \\
&\quad + \sum_{j=1}^{A|x|^2} \Big\{\mathbb{E}_0\Big(\sum_{k=1}^{A|x|^2} \mathbb{1}_{\{D_k=0\}}\Big) - \mathbb{E}_0\Big(\sum_{k=0}^{A|x|^2-j} \mathbb{1}_{\{D_k=0\}}\Big)\Big\}\mathbb{P}_x(\tau = j) \\
&= \mathbb{P}_x\left(\tau > A|x|^2\right) \mathbb{E}_0\Big(\sum_{k=1}^{A|x|^2} \mathbb{1}_{\{D_k=0\}}\Big) + \sum_{j=1}^{A|x|^2} \mathbb{E}_0\Big(\sum_{k=A|x|^2-j+1}^{A|x|^2} \mathbb{1}_{\{D_k=0\}}\Big)\mathbb{P}_x(\tau = j).
\end{aligned} \tag{2.28}$$

Then, by equation (2.11) and (2.28), we have that

$$\begin{aligned}&\frac{1}{\log|x|}\,\mathbb{E}\Big(\sum_{i=1}^{A|x|^2}\left[W_i^x-W_i^0\right]\lambda^*\Big)^2\\&=\frac{2\sigma^2}{\log|x|}\Big[\mathbb{P}_x(\tau>A|x|^2)\,\mathbb{E}_0\Big(\sum_{k=1}^{A|x|^2}\mathbb{1}_{\{D_k=0\}}\Big)+\sum_{j=1}^{A|x|^2}\mathbb{E}_0\Big(\sum_{k=A|x|^2-j+1}^{A|x|^2}\mathbb{1}_{\{D_k=0\}}\Big)\mathbb{P}_x(\tau=j)\Big]\\&\geq\frac{2\sigma^2}{\log|x|}\,\mathbb{P}_x(\tau>A|x|^2)\sum_{k=1}^{A|x|^2}\mathbb{P}_0(D_k=0).\end{aligned}\tag{2.29}$$

By (2.25) in [8], we have that

$$\mathbb{E}\left(W_i^x\lambda^*|\mathcal{F}_{i-1}\right)=\mathbb{E}[\theta_1(0)\lambda^*]=:\mu,\text{ for } i\geq 1 \text{ and } x\in\mathbb{Z}^d.\tag{2.30}$$

From this and time independence, it follows that

$$\mathbb{E}\left[\left(W_i^y\lambda^*-W_i^0\lambda^*\right)\left(W_j^z\lambda^*-W_j^0\lambda^*\right)\right]=0\ \text{ for } i\neq j \text{ and } y,z\in\mathbb{Z}^d.\tag{2.31}$$

By (1.3) and (2.31) we have

$$\begin{aligned}\frac{\mathbb{E}\left(\widehat{X}_\infty(x)-x\lambda^*\right)^2}{\mathcal{P}_x}&=\frac{1}{\mathcal{P}_x}\,\mathbb{E}\Big(\sum_{i=1}^{A|x|^2}(W_i^x-W_i^0)\lambda^*\Big)^2+\frac{1}{\mathcal{P}_x}\,\mathbb{E}\Big(\sum_{i=A|x|^2+1}^{\infty}(W_i^x-W_i^0)\lambda^*\Big)^2\\&\geq\frac{1}{\log|x|}\,\mathbb{E}\Big(\sum_{i=1}^{A|x|^2}(W_i^x-W_i^0)\lambda^*\Big)^2.\end{aligned}\tag{2.32}$$

Then, by (2.29) and (2.32), we obtain

$$\frac{2\sigma^2\,\mathbb{P}_x(\tau>A|x|^2)\sum_{k=1}^{A|x|^2}\mathbb{P}_0(D_k=0)}{\log|x|}\leq\frac{\mathbb{E}\Big(\sum_{i=1}^{A|x|^2}(W_i^x-W_i^0)\lambda^*\Big)^2}{\log|x|}\leq\frac{\mathbb{E}\Big(\widehat{X}_\infty(x)-x\lambda^*\Big)^2}{\log|x|}.\tag{2.33}$$

By (5.14) in [8] and Theorem 1 in [10], when $d=2$ we have

$$\lim_{|x|\to\infty}\frac{\mathbb{E}\left(\widehat{X}_\infty(x)-x\lambda^*\right)^2}{\log|x|}=\frac{2\sigma^2}{\mathfrak{A}\pi\sqrt{det(Q)}},\tag{2.34}$$

with Q as defined in the paragraph of (2.3).

We now want to prove that the limit as $|x|\to\infty$ of the left hand side in (2.33) is also equal to the right hand side of (2.34). In order to do that, we use Lemma 2.1, that implies that

$$\sum_{k=1}^{A|x|^2}\mathbb{P}_0(D_k=0)\sim\frac{1}{\mathfrak{A}2\pi\sqrt{det(Q)}}\log(A|x|^2)\text{ as }|x|\to\infty,$$

and by the corollary of Theorem 1 in [16],

$$\mathbb{P}_x(\tau>A|x|^2)=\frac{[1+o(1)]2\log(|x|)}{\log(A|x|^2)}\text{ for all } x\neq 0 \text{ and } A\geq 1.$$

Therefore,

$$\lim_{|x|\to\infty}\frac{2\sigma^2}{\log|x|}\,\mathbb{P}_x(\tau>A|x|^2)\sum_{k=1}^{A|x|^2}\mathbb{P}_0(D_k=0)=\frac{2\sigma^2}{\mathfrak{A}\pi\sqrt{det(Q)}}.\tag{2.35}$$

Then by (2.33), (2.34) and (2.35) we conclude that

$$\lim_{|x|\to\infty}\frac{1}{\log|x|}\mathbb{E}\left(\sum_{i=1}^{A|x|^2}(W_i^x - W_i^0)\lambda^*\right)^2 = \frac{2\sigma^2}{\mathfrak{A}\pi\sqrt{det(Q)}} := c(2), \text{ for all } A \geq 1, \tag{2.36}$$

thus completing the proof of Proposition 1.2 for $d = 2$.

Remark 2.3. It will be useful later to mention that for $d = 2$ the equality in (2.32), (2.34) and (2.36) imply that the rest of the infinite sum

$$\frac{\sum_{i=|x|^2+1}^{\infty}(W_i^x - W_i^0)\lambda^*}{\sqrt{\mathcal{P}_x}},$$

converges to zero in L^2 as $|x|$ goes to infinity.

3 Corollary of Proposition 1.2

By the Cramér-Wold theorem, convergence in distribution of a sequence of random vectors is equivalent to that of arbitrary linear combinations of its coordinates. So, in order to obtain Theorem 1.4, it suffices that we state and prove in this section the following result.

Corollary 3.1. *Given $k \geq 1$, let $\bar{\alpha} = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$. Let us also consider c and $\mathcal{P}_n$ as in Propostion 1.2.*

(i) In case $d = 1$, for $\bar{t} = (t_1, \dots, t_k) \in \mathbb{R}^k$ with $0 < t_1, \dots < t_k$, the following limit exits

$$g(A, \bar{t}, \bar{\alpha}) := \lim_{n\to\infty}\frac{1}{c\mathcal{P}_n}\mathbb{E}\left[\Big(\sum_{j=1}^{k}\alpha_j\sum_{i=1}^{An^2}\Big[W_i^{\lfloor nt_j\rfloor} - W_i^0\Big]\lambda\Big)^2\right], \text{ for } A \geq 1,$$

and

$$\lim_{A\to\infty} g(A, \bar{t}, \bar{\alpha}) = \sum_{j=1}^{k}\alpha_j^2 t_j + 2\sum_{1\leq j<l\leq k}\alpha_j\alpha_l t_j.$$

(ii) In case $d = 2$, for $\bar{z} = (z_1, \dots, z_k) \in (\mathbb{Z}^2)^k$ and $\tilde{x}_n$ as defined in (1.5) we have

$$\begin{aligned} g(\bar{z}, \bar{\alpha}) &:= \lim_{n\to\infty}\frac{1}{c\mathcal{P}_n}\mathbb{E}\left[\Big(\sum_{j=1}^{k}\alpha_j\sum_{i=1}^{M_{k,n}}\Big[W_i^{\tilde{x}_n(z_j)} - W_i^0\Big]\lambda^*\Big)^2\right] \\ &= \Big(\sum_{j=1}^{k}\alpha_j^2\max\{|z_j(1)|, |z_j(2)|\} \\ &\quad + \sum_{1\leq j<l\leq k}\alpha_j\alpha_l\min\{\max\{|z_l(1)|, |z_l(2)|\}, \max\{|z_j(1)|, |z_j(2)|\}\}\Big), \end{aligned}$$

where $M_{k,n} = \max_{1\leq j\leq k}|\tilde{x}_n(z_j)|^2$.

The proof of Corollary 3.1 contains two parts, one for each dimension, but before starting, let us point out something useful for both dimensions. For $x, y \in \mathbb{Z}^d$ and μ as in (2.30) we have

$$\begin{aligned} \left(W_i^x\lambda^* - W_i^0\lambda^*\right)\left(W_i^y\lambda^* - W_i^0\lambda^*\right) &= \left(W_i^x\lambda^* - \mu\right)\left(W_i^y\lambda^* - \mu\right) - \left(W_i^x\lambda^* - \mu\right)\left(W_i^0\lambda^* - \mu\right) \\ &\quad - \left(W_i^0\lambda^* - \mu\right)\left(W_i^y\lambda^* - \mu\right) + \left(W_i^0\lambda^* - \mu\right)^2. \end{aligned} \tag{3.1}$$

By the translation invariance of the model, and reasoning as in [8] to get (5.7), we obtain

$$\mathbb{E}\left[\left(W_i^x\lambda^*-\mu\right)\left(W_i^y\lambda^*-\mu\right)\right]=\mathbb{E}\left[\left(W_i^{x-y}\lambda^*-\mu\right)\left(W_i^0-\mu\right)\right]=\sigma^2\mathbb{P}\left(D_{i-1}=0|D_0=x-y\right). \tag{3.2}$$

Using now (3.1) and (3.2) we get

$$\begin{aligned}\mathbb{E}\left[\left(W_i^x\lambda^*-W_i^0\lambda^*\right)\left(W_i^y\lambda^*-W_i^0\lambda^*\right)\right]=&\sigma^2\left\{\mathbb{P}\left(D_{i-1}=0|D_0=x-y\right)-\mathbb{P}\left(D_{i-1}=0|D_0=x\right)\right\}\\&-\sigma^2\left\{\mathbb{P}\left(D_{i-1}=0|D_0=y\right)-\mathbb{P}\left(D_{i-1}=0|D_0=0\right)\right\}.\end{aligned} \tag{3.3}$$

3.1 Proof of Corollary 3.1 in $\mathrm{d}=1$.

Notice that

$$\begin{aligned}&\mathbb{E}\Big[\Big(\sum_{j=1}^{k}\alpha_j\sum_{i=1}^{An^2}\Big[W_i^{\lfloor nt_j\rfloor}-W_i^0\Big]\lambda\Big)^2\Big]\\&=\sum_{j=1}^{k}\alpha_j^2\mathbb{E}\Big[\Big(\sum_{i=1}^{An^2}\Big[W_i^{\lfloor nt_j\rfloor}-W_i^0\Big]\lambda\Big)^2\Big]\\&\quad+2\sum_{1\le j<l\le k}\alpha_j\alpha_l\sum_{i=1}^{An^2}\sum_{m=1}^{An^2}\mathbb{E}\Big[\Big(\Big[W_i^{\lfloor nt_j\rfloor}-W_i^0\Big]\lambda\Big)\Big(\Big[W_m^{\lfloor nt_l\rfloor}-W_m^0\Big]\lambda\Big]\Big).\end{aligned} \tag{3.4}$$

By (2.31) we have that

$$\mathbb{E}\Big[\Big(\sum_{i=1}^{An^2}\Big[W_i^{\lfloor nt_j\rfloor}-W_i^0\Big]\lambda\Big)^2\Big]=\sum_{i=1}^{An^2}\mathbb{E}\Big[\Big(\Big[W_i^{\lfloor nt_j\rfloor}-W_i^0\Big]\lambda\Big)^2\Big] \tag{3.5}$$

and

$$\begin{aligned}&\sum_{i=1}^{An^2}\sum_{m=1}^{An^2}\mathbb{E}\Big[\Big(\Big[W_i^{\lfloor nt_j\rfloor}-W_i^0\Big]\lambda\Big)\Big(\Big[W_m^{\lfloor nt_l\rfloor}-W_m^0\Big]\lambda\Big]\Big)\\&=\sum_{i=1}^{An^2}\mathbb{E}\Big[\Big(\Big[W_i^{\lfloor nt_j\rfloor}-W_i^0\Big]\lambda\Big)\Big(\Big[W_i^{\lfloor nt_l\rfloor}-W_i^0\Big]\lambda\Big]\Big).\end{aligned} \tag{3.6}$$

By Proposition 1.2 and (3.5), we have that

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{c\mathcal{P}_n}\sum_{j=1}^{k}\alpha_j^2\mathbb{E}\Big[\Big(\sum_{i=1}^{An^2}\Big[W_i^{\lfloor nt_j\rfloor}-W_i^0\Big]\lambda\Big)^2\Big]&=\lim_{n\to\infty}\frac{1}{c\mathcal{P}_n}\sum_{j=1}^{k}\alpha_j^2\sum_{i=1}^{An^2}\mathbb{E}\Big[\Big(\Big[W_i^{\lfloor nt_j\rfloor}-W_i^0\Big]\lambda\Big)^2\Big]\\&=\frac{1}{c}\sum_{j=1}^{k}\alpha_j^2t_jh(A/t_j^2)\\&\to\sum_{j=1}^{k}\alpha_j^2t_j,\ \text{as } A\to\infty.\end{aligned} \tag{3.7}$$

Hence, by (3.4), (3.6) and (3.7), to finish the proof of the corollary for dimension 1, it is enough to compute

$$\lim_{n\to\infty}\frac{1}{\mathcal{P}_n}\sum_{i=1}^{An^2}\mathbb{E}\left[\left(\left[W_i^{\lfloor nt_j\rfloor}-W_i^0\right]\lambda\right)\left(\left[W_i^{\lfloor nt_l\rfloor}-W_i^0\right]\lambda\right]\right)\ \text{for } 1\le j<l\le k. \tag{3.8}$$

and its limit as $A \to \infty$. By (3.3) we obtain

$$\begin{aligned}
&\sum_{i=1}^{An^2} \mathbb{E}\left[\left(W_i^{\lfloor nt_j \rfloor}\lambda - W_i^0\lambda\right)\left(W_i^{\lfloor nt_l \rfloor}\lambda - W_i^0\lambda\right)\right] \\
&= \sigma^2 \sum_{i=1}^{An^2} \left\{\mathbb{P}\left(D_{i-1} = 0 | D_0 = \lfloor nt_l \rfloor - \lfloor nt_j \rfloor\right) - \mathbb{P}\left(D_{i-1} = 0 | D_0 = \lfloor nt_j \rfloor\right)\right\} \\
&+ \sigma^2 \sum_{i=1}^{An^2} \left\{-\mathbb{P}\left(D_{i-1} = 0 | D_0 = \lfloor nt_l \rfloor\right) + \mathbb{P}\left(D_{i-1} = 0 | D_0 = 0\right)\right\} \\
&= -\sigma^2 \sum_{i=1}^{An^2} \left\{\mathbb{P}\left(D_{i-1} = 0 | D_0 = 0\right) - \mathbb{P}\left(D_{i-1} = 0 | D_0 = \lfloor nt_l \rfloor - \lfloor nt_j \rfloor\right)\right\} \\
&+ \sigma^2 \sum_{i=1}^{An^2} \left\{\mathbb{P}\left(D_{i-1} = 0 | D_0 = 0\right) - \mathbb{P}\left(D_{i-1} = 0 | D_0 = \lfloor nt_j \rfloor\right)\right\} \\
&+ \sigma^2 \sum_{i=1}^{An^2} \left\{\mathbb{P}\left(D_{i-1} = 0 | D_0 = 0\right) - \mathbb{P}\left(D_{i-1} = 0 | D_0 = \lfloor nt_l \rfloor \rfloor\right)\right\}, \qquad (3.9)
\end{aligned}$$

and, from (2.11), (2.12) and (3.9), we find that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{c\mathcal{P}_N} \sum_{i=1}^{An^2} \mathbb{E}\left[\left(W_i^{\lfloor nt_j \rfloor}\lambda - W_i^0\lambda\right)\left(W_i^{\lfloor nt_l \rfloor}\lambda - W_i^0\lambda\right)\right] \\
&= -(t_l - t_j)\frac{h(A/(t_l - t_j)^2)}{2c} + t_j \frac{h(A/(t_l - t_j)^2)}{2c} + t_l \frac{h(A/(t_l - t_j)^2)}{2c} \\
&\to t_j \text{ as } A \to \infty,
\end{aligned}$$

thus concluding the proof of Corollary 3.1 in $d = 1$.

3.2 Proof of Corollary 3.1 in $\mathrm{d} = 2$

By (3.4), (3.5) and (3.6), we get that

$$\begin{aligned}
&\frac{1}{c\mathcal{P}_n}\mathbb{E}\Big[\Big(\sum_{j=1}^{k} \alpha_j \sum_{i=1}^{M_{k,n}} \Big[W_i^{\tilde{x}_n(z_j)} - W_i^0\Big]\lambda^*\Big)^2\Big] \\
&= \frac{1}{c\mathcal{P}_n} \sum_{j=1}^{k} \alpha_j^2 \sum_{i=1}^{M_{k,n}} \mathbb{E}\left([W_i^{\tilde{x}_n(z_j)} - W_i^0]\lambda^*\right)^2 \\
&+ \frac{2}{c\mathcal{P}_n} \sum_{1\le j<l\le k} \alpha_j\alpha_l \sum_{i=1}^{M_{k,n}} \mathbb{E}\Big(W_i^{\tilde{x}_n(z_j)}\lambda^* - W_i^0\lambda^*\Big)\Big(W_i^{\tilde{x}_n(z_l)}\lambda^* - W_i^0\lambda^*\Big). \qquad (3.10)
\end{aligned}$$

Let us start with the first term in the right hand side of (3.10). For each fixed j, we have that

$$\begin{aligned}
\sum_{i=1}^{|\tilde{x}_n(z_j)|^2} \mathbb{E}\left(W_i^{\tilde{x}_n(z_j)}\lambda^* - W_i^0\lambda^*\right)^2 &\le \sum_{i=1}^{M_{k,n}} \mathbb{E}\left(W_i^{\tilde{x}_n(z_j)}\lambda^* - W_i^0\lambda^*\right)^2 \\
&\le \sum_{i=1}^{\infty} \mathbb{E}\left(W_i^{\tilde{x}_n(z_j)}\lambda^* - W_i^0\lambda^*\right)^2. \qquad (3.11)
\end{aligned}$$

Using equations (2.34), (2.36) and (3.11), we have that

$$\lim_{n\to\infty} \frac{1}{c\log(|\tilde{x}_n(z_j)|)} \sum_{i=1}^{M_{k,n}} \mathbb{E}\left(W_i^{\tilde{x}_n(z_j)}\lambda^* - W_i^0\lambda^*\right)^2 = 1.$$

Hence, for every fixed j, we have that

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{c\mathcal{P}_n} \sum_{i=1}^{M_{k,n}} \mathbb{E}\left(W_i^{\tilde{x}_n(z_j)}\lambda^* - W_i^0\lambda^*\right)^2 &= \lim_{n\to\infty} \frac{\log(|\tilde{x}_n(z_j)|)}{\mathcal{P}_n} \\ &= \lim_{n\to\infty} \frac{\log(n^{\max\{|z_j(1)|,|z_j(2)|\}}(1+o(1)))}{\mathcal{P}_n} \\ &= \max\{|z_j(1)|,|z_j(2)|\}. \qquad (3.12)\end{aligned}$$

Therefore

$$\lim_{n\to\infty} \frac{1}{c\mathcal{P}_n} \sum_{j=1}^{k} \alpha_j^2 \sum_{i=1}^{M_{k,n}} \mathbb{E}\left([W_i^{\tilde{x}_n(z_j)} - W_i^0]\lambda^*\right)^2 = \sum_{j=1}^{k} \alpha_j^2 \max\{|z_j(1)|,|z_j(2)|\}.$$

To deal with the second term on the right member of (3.10) observe that, for $1 \le j < l \le k$, by (3.3) we have

$$\begin{aligned}&\sum_{i=1}^{M_{k,n}} \mathbb{E}\left[(W_i^{\tilde{x}_n(z_j)}\lambda^* - W_i^0\lambda^*)(W_i^{\tilde{x}_n(z_l)}\lambda^* - W_i^0\lambda^*)\right] \\ &= \sum_{i=1}^{M_{k,n}} \sigma^2\mathbb{P}(D_{i-1}=0|D_0=\tilde{x}_n(z_l)-\tilde{x}_n(z_j)) - \sum_{i=1}^{M_{k,n}} \sigma^2\mathbb{P}(D_{i-1}=0|D_0=\tilde{x}_n(z_j))) \\ &\quad - \sum_{i=1}^{M_{k,n}} \sigma^2\mathbb{P}(D_{i-1}=0|D_0=\tilde{x}_n(z_l)) + \sum_{i=1}^{M_{k,n}} \sigma^2\mathbb{P}(D_{i-1}=0|D_0=0)). \qquad (3.13)\end{aligned}$$

Notice that, for any $M(x) \ge |x|^2$, we may obtain

$$\lim_{|x|\to\infty} \frac{1}{\log|x|} \sum_{i=1}^{M(x)} \sigma^2 \left\{\mathbb{P}(D_{i-1}=0|D_0=0) - \mathbb{P}(D_{i-1}=0|D_0=x)\right\} = c$$

in the same way as in the proof of (2.13). Hence

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{c\mathcal{P}_n} \sum_{i=1}^{M_{k,n}} \sigma^2 \left\{\mathbb{P}(D_{i-1}=0|D_0=0) - \mathbb{P}(D_{i-1}=0|D_0=\tilde{x}_n(z_l)-\tilde{x}_n(z_j))\right\} \\ &= \lim_{n\to\infty} \frac{\log(|\tilde{x}_n(z_l)-\tilde{x}_n(z_j)|)}{2\log n} \\ &= \lim_{n\to\infty} \frac{\log[n^{\max\{|z_j(1)|,|z_j(2)|,|z_l(1)|,|z_l(2)|\}}(1+o(1))]}{2\log n} \\ &= \frac{\max\{|z_j(1)|,|z_j(2)|,|z_l(1)|,|z_l(2)|\}}{2}. \qquad (3.14)\end{aligned}$$

Then, subtracting $\sum_{i=1}^{M_{k,n}} \sigma^2 \mathbb{P}(D_{i-1}=0|D_0=0)$ from the right hand side of (3.13), and using (3.14), we find that

$$\lim_{n\to\infty} \frac{1}{c\mathcal{P}_n} \sum_{i=1}^{M_{k,n}} \mathbb{E}\left[(W_i^{\tilde{x}_n(t_j)}\lambda^* - W_i^0\lambda^*)(W_i^{\tilde{x}_n(t_l)}\lambda^* - W_i^0\lambda^*)\right]$$
$$= -\frac{\max\{|z_j(1)|,|z_j(2)|,|z_l(1)|,|z_l(2)|\}}{2} + \frac{\max\{|z_l(1)|,|z_l(2)|\}}{2} + \frac{\max\{|z_j(1)|,|z_j(2)|\}}{2}$$
$$= \frac{\min\{\max\{|z_l(1)|,|z_l(2)|\},\max\{|z_j(1)|,|z_j(2)|\}\}}{2}.$$

Therefore,

$$\lim_{n\to\infty} \frac{2}{c\mathcal{P}_n} \sum_{1\le j<l\le k} \alpha_j\alpha_l \sum_{i=1}^{M_{k,n}} \mathbb{E}\left(W_i^{\tilde{x}_n(t_j)}\lambda^* - W_i^0\lambda^*\right)\left(W_i^{\tilde{x}_n(t_l)}\lambda^* - W_i^0\lambda^*\right)$$
$$= \sum_{1\le j<l\le k}^{k} \alpha_l\alpha_j \min\{\max\{|z_l(1)|,|z_l(2)|\},\max\{|z_j(1)|,|z_j(2)|\}\}, \tag{3.15}$$

and the proof of Corollary 3.1 for $d=2$ is finished.

4 Proof of Proposition 1.3

In this section, we prove Proposition 1.3. We follow the strategy adopted in the proof of Theorem 4.1 in [8]. The main difference in our case is in the following Lemma 4.1, which is analogous to Lemma 4.3 in [8], on the one hand, but the proof of the latter result does not apply in our more general, not necessarily nearest neighbor case.

Lemma 4.1. *Let us consider $d=1,2$. Given any positive integer K, we have that*

$$\sum_{i=j}^{n} \left[\mathbb{P}(D_i=0|D_j=l) - \mathbb{P}(D_i=0|D_j=l')\right],$$

is uniformly bounded in n, j, l and l' such that $|l-l'|\le K$.

Proof. To avoid the trivial case, let us assume $l\neq l'$. Also, let us consider τ as defined in (2.1). By the Strong Markov property, we have that

$$\mathbb{P}\left(D_i=0|D_0=x\right) = \sum_{k=0}^{i} \mathbb{P}_0\left(D_{i-k}=0\right)\mathbb{P}_x(\tau=k),$$

Hence,

$$\sum_{i=0}^{n} \mathbb{P}(D_i=0|D_0=x) = \sum_{i=0}^{n}\sum_{k=0}^{i} \mathbb{P}_0\left(D_{i-k}=0\right)\mathbb{P}_x(\tau=k)$$
$$= \sum_{k=0}^{n}\Big(\sum_{i=0}^{n-k}\mathbb{P}_0(D_i=0)\Big)\mathbb{P}_x(\tau=k),$$

for all $x\in\mathbb{Z}^d$. Then

$$\sum_{i=0}^{n} \mathbb{P}(D_i=0|D_0=l) - \sum_{i=0}^{n} \mathbb{P}(D_i=0|D_0=l')$$
$$= \sum_{i=0}^{n}\big(\sum_{k=0}^{n-i}\mathbb{P}_0(D_k=0)\big)\left[\mathbb{P}_l(\tau=i) - \mathbb{P}_{l'}(\tau=i)\right]. \tag{4.1}$$

Using (2.4), (2.5), and (2.7) in (4.1), and the fact that τ has the same distribution for the chain D and H, we obtain that

$$\begin{aligned}&\left|\sum_{i=0}^{n}[\mathbb{P}(D_i=0|D_0=l+(1,0))-\mathbb{P}(D_i=0|D_0=l)]\right|\\&\le C\left|\sum_{i=0}^{n}[\mathbb{P}(H_i=0|H_0=l+(1,0))-\mathbb{P}(H_i=0|H_0=l)]\right|\end{aligned}$$

for some positive constant C. Therefore, we will prove the lemma for the homogeneous chain H. Let us take $L=l-l'$. Then, by the spatial homogeneity of the Markov chain H, we have that

$$\begin{aligned}\sum_{k=0}^{n}\mathbb{P}_0(H_k=l') &= \mathbb{E}_0\left(\sum_{k=0}^{n}\mathbf{1}_{\{H_k=l'\}}\right)\\&=\mathbb{E}_L\left(\sum_{k=0}^{n}\mathbf{1}_{\{H_k=l\}}\right)\\&=\mathbb{E}_L\left(\sum_{k=0}^{n\wedge\tau-1}\mathbf{1}_{\{H_k=l\}}\right)+\mathbb{E}_L\left(\sum_{k=n\wedge\tau}^{n}\mathbf{1}_{\{H_k=l\}}\right).\end{aligned}\tag{4.2}$$

Observe that

$$\mathbb{E}_L\Big(\sum_{k=n\wedge\tau}^{n}\mathbf{1}_{\{H_k=l\}}\Big)=\sum_{i\ge0}\sum_{k=n\wedge i}^{n}\mathbb{P}_0(H_{k-i}=l)\mathbb{P}_L(\tau=i)=\sum_{i\ge0}\mathbb{P}_L(\tau=i)\sum_{k=0}^{n-n\wedge i}\mathbb{P}_0(H_k=l).\tag{4.3}$$

Using (4.3) in (4.2), we obtain

$$\sum_{k=0}^{n}\{\mathbb{P}_0(H_k=l)-\mathbb{P}_0(H_k=l')\}=\sum_{i\ge1}\mathbb{P}_L(\tau=i)\sum_{k=n-n\wedge i+1}^{n}\mathbb{P}_0(H_k=l)-\mathbb{E}_L\sum_{k=0}^{n\wedge\tau-1}\mathbf{1}_{\{H_k=l\}}.\tag{4.4}$$

For the first term in the right hand side of (4.4), observe that

$$\begin{aligned}&\sum_{i\ge1}\mathbb{P}_L(\tau=i)\sum_{k=n-n\wedge i+1}^{n}\mathbb{P}_0(H_k=l)\\&=\sum_{i=1}^{n}\mathbb{P}_L(\tau=i)\sum_{k=n-i+1}^{n}\mathbb{P}_0(H_k=l)+\mathbb{P}_L(\tau>n)\sum_{k=1}^{n}\mathbb{P}_0(H_k=l)\\&=\sum_{k=1}^{n}\mathbb{P}_0(H_k=l)\sum_{i=n-k+1}^{n}\mathbb{P}_L(\tau=i)+\mathbb{P}_L(\tau>n)\sum_{k=1}^{n}\mathbb{P}_0(H_k=l)\\&=\sum_{k=1}^{n}\mathbb{P}_0(H_k=l)\mathbb{P}_L(n-k+1\le\tau\le n)+\mathbb{P}_L(\tau>n)\sum_{k=1}^{n}\mathbb{P}_0(H_k=l)\\&=\sum_{k=1}^{n}\mathbb{P}_0(H_k=l)\mathbb{P}_L(\tau\ge n-k+1)\\&=\sum_{k=0}^{n-1}\mathbb{P}_0(H_{n-k}=l)\mathbb{P}_L(\tau\ge k+1).\end{aligned}\tag{4.5}$$

Using (4.5) in (4.4), we find that

$$\sum_{k=0}^{n}\{\mathbb{P}_0(H_k=l)-\mathbb{P}_0(H_k=l')\}=\sum_{k=0}^{n-1}\mathbb{P}_0(H_{n-k}=l)\mathbb{P}_L(\tau\ge k+1)-\mathbb{E}_L\Big(\sum_{k=0}^{n\wedge\tau-1}\mathbf{1}_{\{H_k=l\}}\Big).\tag{4.6}$$

In the rest of the proof, we deal separately with the two terms in the right hand side of (4.6). For the first term, observe that by (2.6), P4 in [18] (page 382) and the corollary of Theorem 1 in [16] imply that

$$\sum_{k=0}^{n-1} \mathbb{P}_0(H_{n-k}=l)\mathbb{P}_L(\tau \geq k+1) \leq \begin{cases} C_1 \sum_{k=1}^{n-1} \frac{1}{\sqrt{n-k}} \frac{a(L)}{\sqrt{k}}, & \text{in } d=1, \\ C_2 \sum_{k=1}^{n-1} \frac{1}{n-k} \frac{\log |L|}{\log k}, & \text{in } d=2. \end{cases} \tag{4.7}$$

Notice that

$$\sum_{i=1}^{n-1} \frac{1}{\sqrt{n-i}} \frac{1}{\sqrt{i}} \underset{n\to\infty}{\longrightarrow} \int_0^1 \frac{1}{\sqrt{(1-y)y}} dy.$$

Since this integral is finite and $|L| \leq K$, there exists a uniform upper bound for the left hand side expression in (4.7) when $d=1$.

For the bound in $d=2$, notice that

$$\begin{aligned} \sum_{i=1}^{n-1} \frac{1}{n-i} \frac{1}{\log i} &\leq \sum_{i=1}^{(1-\epsilon)n} \frac{1}{n-i} \frac{1}{\log i} + \sum_{i=(1-\epsilon)n}^{n-1} \frac{1}{n-i} \frac{1}{\log i} \\ &\leq \frac{(1-\epsilon)n}{n} + \frac{1}{\log((1-\epsilon)n)} \sum_{i=(1-\epsilon)n}^{n-1} \frac{1}{n-i} \\ &\leq \frac{(1-\epsilon)n}{n} + \frac{1}{\log((1-\epsilon)n)} \int_{(1-\epsilon)n}^{n-1} \frac{1}{n-x} dx. \\ &= (1-\epsilon) + \frac{\log(\epsilon n)}{\log((1-\epsilon)n)} \\ &\leq C_2, \end{aligned}$$

for some positive constant C_2 and for all $n \geq 1$, and we have that the left hand side of (4.7) is bounded for $d=2$ also.

Back to the second term in (4.6), we observe that it is bounded above by

$$\tilde{g}_{\{0\}}(L,l) := \mathbb{E}_L \left(\sum_{k=0}^{\tau} \mathbf{1}_{\{H_k=l\}} \right). \tag{4.8}$$

It is proved in [14] that

$$\lim_{l\to\infty} \tilde{g}_{\{0\}}(L,l) < \infty, \text{ when } d=2 \tag{4.9}$$

and

$$\lim_{l\to\infty} \frac{1}{2} (\tilde{g}_{\{0\}}(L,l) + \tilde{g}_{\{0\}}(L,-l)) < \infty, \text{ when } d=1. \tag{4.10}$$

(4.9) and (4.10) correspond to (1.16) and (1.17) in [14], respectively. By (4.9) and (4.10) we have that the second term in the right hand side of (4.8) is uniformly bounded for $|L| \leq K$. □

Proof of Proposition 1.3: To prove Proposition 1.3 we will apply Corollary 3.1 in [12] for the variables

$$X_{x,i} = \frac{W_i^x \lambda^* - W_i^0 \lambda^*}{\sqrt{\mathcal{P}_x}}, \, 1 \leq i \leq A|x|^2.$$

In Lemma 2.2 in [8] is proved that

$$\Big\{ \sum_{i=1}^{n} W_i^x \lambda^* - n\mu\lambda^* \Big\}_{n\geq 1},$$

is a martingale with respect to the filtration $\mathcal{F}_n$ defined in Section 1.1. Observe that a linear combination of martingales is also a martingale. Therefore, $X_{x,i}$ is the increment of a nested mean zero and square-integrable martingale with respect to the nested filtration $\mathcal{F}_{x,i} = \mathcal{F}_i$. Hence, by Corollary 3.1 in [12], if we check

$$\max_{1\le i\le A|x|^2} |X_{x,i}| \xrightarrow{p} 0, \tag{4.11}$$

$$\mathbb{E}\left(\max_{1\le i\le A|x|^2} X_{x,i}\right) \text{ is bounded in } x, \tag{4.12}$$

and

$$\sum_{i=1}^{A|x|^2} \mathbb{E}\left(X_{x,i}^2|\mathcal{F}_{x,i-1}\right) \xrightarrow{p} h(A) \text{ as } |x|\to\infty, \tag{4.13}$$

we obtain

$$\sum_{i=1}^{A|x|^2} X_{x,i} \xrightarrow{d} Z \text{ as } |x|\to\infty, \tag{4.14}$$

where Z is a mean zero Gaussian r.v. with variance $h(A)$.

Notice that conditions (4.11) and (4.12) follow straightforwardly from the fact that $u_n(i,i+\cdot)$ have bounded support. It remains to prove (4.13). To do this, observe that by the definition of $h(A)$ and (2.31), we have that

$$h(A) = \lim_{|x|\to\infty} \frac{1}{\mathcal{P}_x}\mathbb{E}\sum_{i=1}^{A|x|^2} \left([W_i^x - W_i^0]\lambda^*\right)^2, \text{ for all } A>0. \tag{4.15}$$

Given (4.15), to obtain (4.13), it is enough to prove that

$$\frac{1}{\mathcal{P}_x}\Bigg\{\sum_{i=1}^{A|x|^2} \mathbb{E}\left([W_i^x\lambda^* - W_i^0\lambda^*]^2|\mathcal{F}_{i-1}\right) - \mathbb{E}\left(W_i^x\lambda^* - W_i^0\lambda^*\right)^2\Bigg\} \xrightarrow{p} 0, \text{ as } |x|\to\infty. \tag{4.16}$$

Let us denote the range of $u_n(i,i+\cdot)$ by R.

We consider $\tilde{D}_n^x = \tilde{Y}_n^x - \hat{Y}_n^0$, with $\hat{Y}_n^0$ a copy of $\tilde{Y}_n^0$ independent of $\tilde{Y}_n^x$ given $\mathcal{F}_n$. With the same ideas used to get (2.33) in [8], we obtain

$$\begin{aligned}
&\mathbb{E}\left((W_i^x\lambda^* - \mu\lambda^*)(W_i^0\lambda^* - \mu)\Big|\mathcal{F}_{i-1}\right)\\
&= \sum_{k,l}\mathbb{E}\left((\theta_i(k)\lambda^* - \mu)(\theta_i(l)\lambda^* - \mu)\right)\mathbb{P}\left(\tilde{Y}_{i-1}^x = k|\mathcal{F}_{i-1}\right)\mathbb{P}\left(\tilde{Y}_{i-1}^0 = l|\mathcal{F}_{i-1}\right)\\
&= \sigma^2\sum_k \mathbb{P}\left(\tilde{Y}_{i-1}^x = k|\mathcal{F}_{i-1}\right)\mathbb{P}\left(\tilde{Y}_{i-1}^0 = k|\mathcal{F}_{i-1}\right)\\
&= \sigma^2\mathbb{P}\left(\tilde{D}_{i-1}^x = 0\big|\mathcal{F}_{i-1}\right),
\end{aligned} \tag{4.17}$$

The first equality in (4.17) is a consequence of the independence of $\theta_i's$ with $\mathcal{F}_{i-1}$. The second equality use the fact that $\theta_i(k)$ and $\theta_i(l)$ are independent when $k\neq l$. Hence, equation (4.17) implies

$$\begin{aligned}
&\mathbb{E}[(W_i^x - W_i^0)^2|\mathcal{F}_{i-1}] - \mathbb{E}(W_i^x - W_i^0)^2\\
&= 2\sigma^2\left[\mathbb{P}(\tilde{D}_{i-1}^0 = 0|\mathcal{F}_{i-1}) - \mathbb{P}(\tilde{D}_{i-1}^0 = 0)\right] - 2\sigma^2\left[\mathbb{P}(\tilde{D}_{i-1}^x = 0|\mathcal{F}_{i-1}) - \mathbb{P}(\tilde{D}_{i-1}^x = 0)\right].
\end{aligned}$$

Therefore, it is sufficient to prove

$$\frac{1}{\mathcal{P}_x}\sum_{i=1}^{A|x|^2}\Big(\mathbb{P}(\tilde{D}^x_{i-1}=0|\mathcal{F}_{i-1})-\mathbb{P}(\tilde{D}^x_{i-1}=0)\Big)\xrightarrow{p}0,\ \text{as } |x|\to\infty, \tag{4.18}$$

and

$$\frac{1}{\mathcal{P}_x}\sum_{i=1}^{A|x|^2}\Big(\mathbb{P}(\tilde{D}^0_{i-1}=0|\mathcal{F}_{i-1})-\mathbb{P}(\tilde{D}^0_{i-1}=0)\Big)\xrightarrow{p}0,\ \text{as } |x|\to\infty. \tag{4.19}$$

The computations to obtain (4.18) and (4.19) are very similar to the ones in the proof of Theorem 4.1 in [8]. To avoid repetition and for the sake of completeness, we give them in Appendix A. To conclude for now, let us point out that Lemma 4.1 is used in the proof of (4.18) and (4.19). □

5 Corollary of Proposition 1.3

As mentioned in Section 3, Cramér-Wold theorem allows us to deduce the convergence of a sequence of vectors through the study of linear combinations of its components. Hence, to obtain Theorem 1.4, we state and prove the following result in this section.

Corollary 5.1. *Given $k\geq 1$, let $\bar{\alpha}=(\alpha_1,\dots,\alpha_k)\in\mathbb{R}^k$. Let us also consider c and $\mathcal{P}_n$ as in Proposition 1.2.*

(i) In case $d=1$, let $\bar{t}=(t_1,\dots,t_k)\in\mathbb{R}^k$ with $0=t_0<t_1,\dots<t_k$. Then we have

$$\frac{1}{\sqrt{c\mathcal{P}_n}}\sum_{j=1}^{k}\sum_{i=1}^{An^2}\alpha_j\left(W_i^{\lfloor nt_j\rfloor}-W_i^0\right)\lambda\xrightarrow[n\to\infty]{d}\mathcal{N}(0,g(A,\bar{t},\bar{\alpha})),$$

where the function g is as in Corollary 3.1 for dimension one.

(ii) In case $d=2$, let $\bar{z}=(z_1,\dots,z_k)\in\mathbb{Z}^{2k}$. Then we have

$$\frac{1}{\sqrt{c\mathcal{P}_n}}\sum_{j=1}^{k}\sum_{i=1}^{M_{k,n}}\alpha_j\left(W_i^{\bar{x}_n(z_j)}-W_i^0\right)\lambda^*\xrightarrow[n\to\infty]{d}\mathcal{N}(0,g(\bar{z},\bar{\alpha})),$$

where $M_{k,n}$ and g are as in Corollary 3.1 for dimension two.

Proof. To avoid being repetitive, we only write the proof for dimension one, but the reader can check that the same ideas apply to dimension 2.

As in the proof of Proposition 1.3 we will apply Corollary 3.1 in [12] but now for the variables

$$X^k_{n,i}=\frac{1}{\sqrt{c\mathcal{P}_n}}\sum_{j=1}^{k}\alpha_j\left(W_i^{\lfloor nt_j\rfloor}-W_i^0\right)\lambda,\ 1\leq i\leq An^2.$$

Again conditions (4.11) and (4.12) follow readily from the fact that we are assuming finite support for $u_n(i,i+\cdot)$.

It remains to argue (4.13). In order to that, let us first notice that equation (2.31) implies (after straightforward computations) that

$$\sum_{i=1}^{An^2}\mathbb{E}\Big[\Big(\sum_{j=1}^{k}\alpha_j\big[W_i^{\lfloor nt_j\rfloor}-W_i^0\big]\lambda\Big)^2\Big]=\mathbb{E}\Big[\Big(\sum_{j=1}^{k}\alpha_j\sum_{i=1}^{An^2}\big[W_i^{\lfloor nt_j\rfloor}-W_i^0\big]\lambda\Big)^2\Big]. \tag{5.1}$$

Hence, by Corollary 3.1 we have

$$\lim_{n\to\infty}\sum_{i=1}^{An^2}\mathbb{E}\left[\left(X^k_{n,i}\right)^2\right]=g(A,\bar{t},\bar{\alpha}). \tag{5.2}$$

Since

$$(X^k_{n,i})^2 = \frac{1}{c\mathcal{P}_n}\sum_{j=1}^{k}\left(W_i^{\lfloor nt_j\rfloor}\lambda - W_i^0\lambda\right)^2 + 2\sum_{1\le j<l\le k}\alpha_j\alpha_l\left(W_i^{\lfloor nt_j\rfloor}\lambda - W_i^0\lambda\right)\left(W_i^{\lfloor nt_l\rfloor}\lambda - W_i^0\lambda\right),$$

and, from (4.18),

$$\sum_{j=1}^{k}\frac{1}{c\mathcal{P}_n}\sum_{i=1}^{An^2}\left\{\mathbb{E}\Big(\big[W_i^{\lfloor nt_j\rfloor}\lambda - W_i^0\lambda\big]^2\Big|\mathcal{F}_{i-1}\Big) - \mathbb{E}\Big(\big[W_i^{\lfloor nt_j\rfloor}\lambda n - W_i^0\lambda\big]^2\Big)\right\}\xrightarrow{p} 0 \text{ as } n\to\infty,$$

in order to obtain Condition (4.13) (and conclude the proof), it is enough to show that

$$\frac{1}{c\mathcal{P}_n}\sum_{i=1}^{An^2}\left\{\mathbb{E}\Big((W_i^{\lfloor nt_j\rfloor}\lambda - W_i^0\lambda)(W_i^{\lfloor nt_l\rfloor}\lambda - W_i^0\lambda)\big|\mathcal{F}_{i-1}\Big) - \mathbb{E}(W_i^{\lfloor nt_j\rfloor}\lambda - W_i^0\lambda)(W_i^{\lfloor nt_l\rfloor}\lambda - W_i^0\lambda)\right\} \tag{5.3}$$

converges in probability to zero as $n\to\infty$, for any $1\le j<l\le k$.
Set

$$S_n^{y,z} := \frac{\sigma^2}{c\mathcal{P}_n}\sum_{i=1}^{An^2}\Big\{\mathbb{E}\left((W_i^z\lambda - \mu\lambda)(W_i^y\lambda - \mu\lambda)\Big|\mathcal{F}_{i-1}\right) - \mathbb{E}\left((W_i^z\lambda - \mu\lambda)(W_i^y\lambda - \mu\lambda)\right)\Big\}.$$

Then we can write (5.3) as

$$S_n^{\lfloor nt_j\rfloor,\lfloor nt_l\rfloor} - S_n^{\lfloor nt_j\rfloor,0\rfloor} - S_n^{\lfloor nt_l\rfloor,0} + S_n^{0,0}. \tag{5.4}$$

Using the invariance translation property of the model and (4.17) we obtain

$$S_n^{y,z} \overset{d}{=} \frac{\sigma^2}{c\mathcal{P}_n}\sum_{i=1}^{An^2}\left\{\mathbb{P}\left(\tilde{D}_{i-1}^{z-y}=0\big|\mathcal{F}_{i-1}\right) - \mathbb{P}\left(\tilde{D}_{i-1}^{z-y}=0\right)\right\}. \tag{5.5}$$

Now, by (5.5), (4.18) and (4.19), we have that each term in equation (5.4) goes to zero in probability as n goes to infinity, and this establishes (4.13). □

6 Proof of Theorem 1.1

Now we have all the ingredients to prove Theorem 1.1.

Proof of Theorem 1.1. By (1.3) we have that

$$\frac{\widehat{X}_x - x\lambda^*}{\sqrt{\mathcal{P}_x}} \overset{d}{=} \frac{1}{\sqrt{\mathcal{P}_x}}\sum_{i=0}^{A|x|^2}\left(W_i^x - W_i^0\right)\lambda^* + \frac{1}{\sqrt{\mathcal{P}_x}}\sum_{i=A|x|^2+1}^{\infty}\left(W_i^x - W_i^0\right)\lambda^*, \text{ for any } A\ge 1. \tag{6.1}$$

By equation (5.14) in [8] and **P28.4**[7] (page 345) in [18], for $d=1$ and equation (2.34) for $d=2$, we have that

$$\lim_{|x|\to\infty}\frac{\mathbb{E}\left(\widehat{X}_\infty(x) - x\lambda^*\right)^2}{\mathcal{P}_x} = c. \tag{6.2}$$

[7] To apply **28.4** is necessary to note that the function a is even because the chain H is symmetric.

By (1.3) and (2.31) we obtain

$$\frac{\mathbb{E}\Big(\widehat{X}_\infty(x) - x\lambda^*\Big)^2}{\mathcal{P}_x} = \frac{1}{\mathcal{P}_x}\mathbb{E}\Big(\sum_{i=1}^{A|x|^2}(W_i^x - W_i^0)\lambda^*\Big)^2 + \frac{1}{\mathcal{P}_x}\mathbb{E}\Big(\sum_{i=A|x|^2+1}^{\infty}(W_i^x - W_i^0)\lambda^*\Big)^2. \quad (6.3)$$

Hence by (6.2), (6.3), and Proposition 1.2 we obtain

$$\lim_{A\to\infty}\lim_{|x|\to\infty}\mathbb{E}\Big[\Big(\frac{1}{\sqrt{\mathcal{P}_x}}\sum_{i=A|x|^2+1}^{\infty}\big(W_i^x - W_i^0\big)\,\lambda^*\Big)^2\Big] = 0. \quad (6.4)$$

Only in the following calculations and to simplify notations we denote the left member in (6.1) by X_x, and the first and second term by Y_x^A and Z_x^A, respectively.

Take f a uniformly continuous and bounded function and we denote by M_f the bound of f. Also, for a fixed $\epsilon > 0$ we take δ such that $|f(z) - f(z+y)| \le \epsilon/6$ for all z and all y such that $|y| \le \delta$. In addition, we take A large enough such that

$$\Big|\int f(Y^A)d\mathbb{P} - \int f(Y)d\mathbb{P}\Big| \le \frac{\epsilon}{3}, \quad (6.5)$$

where Y^A and Y are distributed $\mathcal{N}(0, h(A))$ and $\mathcal{N}(0, c)$, respectively. Now, for this A we take $|x|$ large enough such that

$$\mathbb{P}(|Z_x^A| > \delta) \le \frac{\epsilon}{12M_f} \text{ and } \Big|\int f(Y_x^A)d\mathbb{P} - \int f(Y^A)d\mathbb{P}\Big| \le \frac{\epsilon}{3}. \quad (6.6)$$

The first part of (6.6) is a consequence of (6.4) and the second is the result in Proposition 1.3. Observe that

$$\Big|\int f(X_x)d\mathbb{P} - \int f(Y)d\mathbb{P}\Big| \le \Big|\int f(X_x)d\mathbb{P} - \int f(Y_x^A)d\mathbb{P}\Big| + \Big|\int f(Y_x^A)d\mathbb{P} - \int f(Y^A)d\mathbb{P}\Big| + \Big|\int f(Y^A)d\mathbb{P} - \int f(Y)d\mathbb{P}\Big|. \quad (6.7)$$

By (6.6) and (6.5) the second and third term in the right member of (6.7) are both less than $\epsilon/3$. For the first term, using (6.6) and our choice of δ we have that

$$\Big|\int f(X_x)d\mathbb{P} - \int f(Y_x^A)d\mathbb{P}\Big| \le \frac{2\epsilon}{12M_f} + \Big|\int_{|Z_x^A|<\delta}(f(X_x) - f(Y_x^A))d\mathbb{P}\Big| \le \frac{\epsilon}{3}.$$

Therefore, for every $\epsilon > 0$ and $|x|$ large enough, we have concluded that

$$\Big|\int f(X_x)d\mathbb{P} - \int f(Y)d\mathbb{P}\Big| \le \epsilon,$$

obtaining the desired converge in distribution. □

7 Proof of Theorem 1.4

Although the proof of Theorem 1.4 is very similar for dimension one and two, it has some specific calculations that are different for each case. Therefore, we start with the proof in dimension one, and after that, we prove the core different part for dimension two.

Proof for $d=1$. By Cramér-Wold theorem, the convergence in (1.4) is equivalent to

$$\sum_{j=1}^{k}\alpha_j X_n(t_j) \xrightarrow[n\to\infty]{d} \sum_{j=1}^{k}\alpha_j B(t_j), \text{ for any } (\alpha_1,\dots,\alpha_k)\in\mathbb{R}^k. \tag{7.1}$$

Hence, we will prove (7.1). To do this, first notice that $\sum_{j=1}^{k}\alpha_j B(t_j)$ is a Gaussian random variable with mean zero and variance $\sum_{j=1}^{k}\alpha_j^2 t_j + 2\sum_{1\le j<l\le k}\alpha_j\alpha_l t_j$. As in the proof of Theorem 1.1, we use that

$$\begin{aligned}\sum_{j=1}^{k}\alpha_j X_n(t_j) &\overset{d}{=} \frac{1}{\sqrt{c\mathcal{P}_n}}\sum_{j=1}^{k}\sum_{i=1}^{\infty}\alpha_j\left(W_i^{\lfloor nt_j\rfloor}-W_i^0\right)\lambda\\ &= \frac{1}{\sqrt{c\mathcal{P}_n}}\sum_{j=1}^{k}\sum_{i=1}^{An^2}\alpha_j\left(W_i^{\lfloor nt_j\rfloor}-W_i^0\right)\lambda + \frac{1}{\sqrt{c\mathcal{P}_n}}\sum_{j=1}^{k}\sum_{i>An^2}\alpha_j\left(W_i^{\lfloor nt_j\rfloor}-W_i^0\right)\lambda.\end{aligned} \tag{7.2}$$

To deal with the first sum in the right hand side of (7.2), we use Corollary 3.1 and Corollary 5.1. It will remain to compute the limit as $n\to\infty$ of the second moment of the left hand side of (7.2). If this limit is equal to the one in Corollary 3.1, then the second sum in right hand side of (7.2) will be small in probability when A is large, and this allows us to follow the proof of Theorem 1.1 in a straightforward fashion to obtain (7.1). Summing up, it is enough to show that

$$\lim_{n\to\infty}\frac{1}{c\mathcal{P}_n}\mathbb{E}\Big[\Big(\sum_{j=1}^{k}\alpha_j\sum_{i=1}^{\infty}\Big[W_i^{\lfloor nt_j\rfloor}-W_i^0\Big]\lambda\Big)^2\Big] = \sum_{j=1}^{k}\alpha_j^2 t_j + 2\sum_{1\le j<l<\le k}\alpha_j\alpha_l t_j. \tag{7.3}$$

Expanding the square inside the mean in the left member of (7.3) we have that

$$\begin{aligned}\frac{1}{c\mathcal{P}_n}\mathbb{E}\Big[\Big(\sum_{j=1}^{k}\alpha_j\sum_{i=1}^{\infty}\Big[W_i^{\lfloor nt_j\rfloor}-W_i^0\Big]\lambda\Big)^2\Big] &= \sum_{j=1}^{k}\alpha_j^2\frac{1}{c\mathcal{P}_n}\mathbb{E}\Big[\Big(\sum_{i=1}^{\infty}\Big[W_i^{\lfloor nt_j\rfloor}-W_i^0\Big]\lambda\Big)^2\Big]\\ &+2\sum_{1\le j<l\le k}\frac{1}{c\mathcal{P}_n}\alpha_j\alpha_l\mathbb{E}\Big[\Big(\sum_{i=1}^{\infty}\Big[W_i^{\lfloor nt_j\rfloor}-W_i^0\Big]\lambda\Big)\Big(\sum_{m=1}^{\infty}\Big[W_m^{\lfloor nt_l\rfloor}-W_m^0\Big]\lambda\Big)\Big].\end{aligned} \tag{7.4}$$

Now, we deal with the first sum in the right hand side of (7.4). Expanding the square inside the mean, and using (2.31), we find that

$$\sum_{j=1}^{k}\alpha_j^2\frac{1}{c\mathcal{P}_n}\mathbb{E}\Big[\Big(\sum_{i=1}^{\infty}\Big[W_i^{\lfloor nt_j\rfloor}-W_i^0\Big]\lambda\Big)^2\Big] = \sum_{j=1}^{k}\alpha_j^2\frac{1}{c\mathcal{P}_n}\sum_{i=1}^{\infty}\mathbb{E}\Big[\Big(W_i^{\lfloor nt_j\rfloor}-W_i^0\Big)\lambda\Big]^2. \tag{7.5}$$

Equation 5.14 in [8] implies that the limit of the sum in the right hand side of (7.5) equals $\sum_{j=1}^{k}\alpha_j^2 t_j$. Therefore, to get the convergence in (7.3), it is enough to prove that

$$\lim_{n\to\infty}\frac{1}{c\mathcal{P}_n}\mathbb{E}\Big[\Big(\sum_{i=1}^{\infty}\Big[W_i^{\lfloor nt_j\rfloor}-W_i^0\Big]\lambda\Big)\Big(\sum_{m=1}^{\infty}\Big[W_m^{\lfloor nt_l\rfloor}-W_m^0\Big]\lambda\Big)\Big] = t_j, \text{ for any } 1\le j<l\le k. \tag{7.6}$$

Again, multiplying the sums inside the mean in (7.6) and using (2.31) we find that

$$\begin{aligned}&\mathbb{E}\Big[\Big(\sum_{i=1}^{\infty}\Big[W_i^{\lfloor nt_j\rfloor}-W_i^0\Big]\lambda\Big)\Big(\sum_{m=1}^{\infty}\Big[W_m^{\lfloor nt_l\rfloor}-W_m^0\Big]\lambda\Big)\Big]\\ &= \sum_{i=1}^{\infty}\mathbb{E}\Big[\Big(W_i^{\lfloor nt_j\rfloor}\lambda-W_i^0\lambda\Big)\Big(W_i^{\lfloor nt_l\rfloor}\lambda-W_i^0\lambda\Big)\Big].\end{aligned}$$

Hence, what we need to prove is

$$\lim_{n\to\infty}\frac{1}{c\mathcal{P}_n}\sum_{i=1}^{\infty}\mathbb{E}\left[\left(W_i^{\lfloor nt_j\rfloor}\lambda-W_i^0\lambda\right)\left(W_i^{\lfloor nt_l\rfloor}\lambda-W_i^0\lambda\right)\right]=t_j,\ \text{ for any } 1\le j<l\le k. \tag{7.7}$$

By (3.3) we have

$$\begin{aligned}&\mathbb{E}\left[\left(W_i^{\lfloor nt_j\rfloor}\lambda-W_i^0\lambda\right)\left(W_i^{\lfloor nt_l\rfloor}\lambda-W_i^0\lambda\right)\right]\\&=-\sigma^2\left\{\mathbb{P}\left(D_{i-1}=0|D_0=0\right)-\mathbb{P}\left(D_{i-1}=0|D_0=\lfloor nt_l\rfloor-\lfloor nt_j\rfloor\right)\right\}\\&+\sigma^2\left\{\mathbb{P}\left(D_{i-1}=0|D_0=0\right)-\mathbb{P}\left(D_{i-1}=0|D_0=\lfloor nt_j\rfloor\right)\right\}\\&+\sigma^2\left\{\mathbb{P}\left(D_{i-1}=0|D_0=0\right)-\mathbb{P}\left(D_{i-1}=0|D_0=\lfloor nt_l\rfloor\right)\right\}.\end{aligned}$$

By (5.14) in [8] the following sum

$$\sum_{i=1}^{\infty}\left\{\mathbb{P}\left(D_{i-1}=0|D_0=0\right)-\mathbb{P}\left(D_{i-1}=0|D_0=k\right)\right\}$$

is finite for any $k\in\mathbb{Z}$. Then

$$\begin{aligned}&\frac{1}{c\mathcal{P}_n}\sum_{i=1}^{\infty}\mathbb{E}\left[\left(W_i^{\lfloor nt_j\rfloor}\lambda-W_i^0\lambda\right)\left(W_i^{\lfloor nt_l\rfloor}\lambda-W_i^0\lambda\right)\right]\\&=-\frac{\sigma^2}{c\mathcal{P}_n}\sum_{i=1}^{\infty}\left\{\mathbb{P}\left(D_{i-1}=0|D_0=0\right)-\mathbb{P}\left(D_{i-1}=0|D_0=\lfloor nt_l\rfloor-\lfloor nt_j\rfloor\right)\right\}\\&+\frac{\sigma^2}{c\mathcal{P}_n}\sum_{i=1}^{\infty}\left\{\mathbb{P}\left(D_{i-1}=0|D_0=0\right)-\mathbb{P}\left(D_{i-1}=0|D_0=\lfloor nt_j\rfloor\right)\right\}\\&+\frac{\sigma^2}{c\mathcal{P}_n}\sum_{i=1}^{\infty}\left\{\mathbb{P}\left(D_{i-1}=0|D_0=0\right)-\mathbb{P}\left(D_{i-1}=0|D_0=\lfloor nt_l\rfloor\right)\right\}.\end{aligned} \tag{7.8}$$

By (5.14) in [8] and **P28.4** (page 345) in [18], for $s<t$ we have that

$$\lim_{n\to\infty}\frac{2\sigma^2}{\mathcal{P}_n}\sum_{i=1}^{\infty}\left\{\mathbb{P}\left(D_{i-1}=0|D_0=0\right)-\mathbb{P}\left(D_{i-1}=0|D_0=\lfloor nt\rfloor-\lfloor ns\rfloor\right)\right\}=(t-s)c. \tag{7.9}$$

Then

$$\lim_{n\to\infty}\frac{1}{c\mathcal{P}_n}\sum_{i=1}^{\infty}\mathbb{E}\left[\left(W_i^{\lfloor nt_j\rfloor}\lambda-W_i^0\lambda\right)\left(W_i^{\lfloor nt_l\rfloor}\lambda-W_i^0\lambda\right)\right]=-\frac{1}{2}(t_l-t_j)+\frac{1}{2}t_j+\frac{1}{2}t_l=t_j,$$

and the proofs of (7.7) and of Theorem 1.4 for $d=1$ is concluded. □

Proof for $d=2$. As in dimension one, by Cramér-Wold theorem, the convergence in (1.6) is equivalent to

$$\sum_{j=1}^{k}\alpha_jX_n(z_j)\xrightarrow[n\to\infty]{d}\sum_{j=1}^{k}\alpha_jZ_j,\ \text{ for any } (\alpha_1,\dots,\alpha_k)\in\mathbb{R}^k. \tag{7.10}$$

Again, we split the infinite sum as we did for dimension one, but this time we do not split at n^2, but at $M_{k,n}$. More precisely,

$$\sum_{j=1}^{k}\alpha_j X_n(z_j) \stackrel{d}{=} \frac{1}{\sqrt{c\mathcal{P}_n}}\sum_{j=1}^{k}\sum_{i=1}^{M_{k,n}}\alpha_j\left(W_i^{\tilde{x}_n(z_j)}-W_i^0\right)\lambda^* + \frac{1}{\sqrt{c\mathcal{P}_n}}\sum_{j=1}^{k}\sum_{i>M_{k,n}}\alpha_j\left(W_i^{\tilde{x}_n(z_j)}-W_i^0\right)\lambda^*. \tag{7.11}$$

By Remark 2.3 the second term in the right member of (7.11) converges to zero in L^2. Also, in Corollary 5.1 we have the limit in distribution of the first term in the right member of (7.11). Therefore, the limit in distribution of the left member in (7.11) is the same of the finite sum in the right member. □

A Finishing the proof of Proposition 1.3

Since the arguments for (4.18) and (4.19) are entirely similar, we give only the proof of (4.18).

Proof of (4.18). Recycling the arguments given in the equations (4.3), (4.4) and (4.5) in [8], we could write the variance of the sum at (4.18) as

$$\sum_{j=1}^{A|x|^2}\mathbb{E}\Big(\sum_{i=j}^{A|x|^2-1}\Big\{\mathbb{P}(\tilde{D}_i^x=0|\mathcal{F}_j)-\mathbb{P}(\tilde{D}_i^x=0|\mathcal{F}_{j-1})\Big\}\Big)^2. \tag{A.1}$$

Conditioning in $\tilde{D}_j^x$ we have that

$$\mathbb{P}(\tilde{D}_i^x{=}0|\mathcal{F}_j)-\mathbb{P}(\tilde{D}_i^x=0|\mathcal{F}_{j-1})=\sum_k\mathbb{P}(\tilde{D}_i^x=0|\tilde{D}_j^x=k)\big[\mathbb{P}(\tilde{D}_j^x=k|\mathcal{F}_j)-\mathbb{P}(\tilde{D}_j^x=k|\mathcal{F}_{j-1})\big]. \tag{A.2}$$

Now, conditioning on $\tilde{Y}_{j-1}^x$ and $\hat{Y}_{j-1}^0$ we have that $\mathbb{P}(\tilde{D}_j^x=k|\mathcal{F}_j)-\mathbb{P}(\tilde{D}_j^x=k|\mathcal{F}_{j-1})$ is equal to

$$\begin{aligned}\sum_{l,l'}&\Big[\mathbb{P}(\tilde{D}_j^x=k|\tilde{Y}_{j-1}^x=l'+l,\hat{Y}_{j-1}^0=l',\mathcal{F}_j)-\mathbb{P}(\tilde{D}_j^x=k|\tilde{Y}_{j-1}^x=l'+l,\hat{Y}_{j-1}^0=l')\Big]\\ &\times\mathbb{P}(\tilde{Y}_{j-1}^x=l'+l,\hat{Y}_{j-1}^0=l'|\mathcal{F}_{j-1}).\end{aligned} \tag{A.3}$$

In the rest of the proof, we consider $d=2$ and the proof for $d=1$ is similar. Back to equation (A.3), observe that we have the following relation between k and l

$$k=l+(\tilde{Y}_j^x-\tilde{Y}_{j-1}^x)-(\hat{Y}_j^0-\hat{Y}_{j-1}^0).$$

Hence, we could write $k=l+b_1-b_2$ for b_1, $b_2\in\mathbb{Z}^2$ with $\max\{|b_1|,|b_2|\}\le K$, where K is the range of the measure u_0. The finite support assumption is only required when we use Lemma 4.1. We denote by V a K-neighborhood of zero. Also, we use the following notation

$$F(k,l',l):=\mathbb{P}(\tilde{D}_j^x=k|\tilde{Y}_{j-1}^x=l'+l,\hat{Y}_{j-1}^0=l',\mathcal{F}_j)-\mathbb{P}(\tilde{D}_j^x=k|\tilde{Y}_{j-1}^x=l'+l,\hat{Y}_{j-1}^0=l')$$
$$u_{j,l,l',b}=u_j(l'+l,l'+l+b) \text{ and } u_{j,l',b}=u_j(l',l'+b)$$

where b is in V. With these new notations and (A.3), we rearrange the sum in (A.2) as follows

$$\begin{aligned}\sum_{l,l'}\sum_{k}&\mathbb{P}(\tilde{D}_i^x=0|\tilde{D}_j^x=k)F(k,l',l)\mathbb{P}(\tilde{Y}_{j-1}^x=l'+l,\hat{Y}_{j-1}^0=l'|\mathcal{F}_{j-1})\\ &=\sum_{l,l'}\mathbb{P}(\tilde{Y}_{j-1}^x=l'+l,\hat{Y}_{j-1}^0=l'|\mathcal{F}_{j-1})\\ &\qquad\times\Big[\sum_{k\neq l}F(k,l',l)\mathbb{P}(\tilde{D}_i^x=0|\tilde{D}_j^x=k)+F(l,l',l)\mathbb{P}(\tilde{D}_i^x=0|\tilde{D}_j^x=l)\Big].\end{aligned}\tag{A.4}$$

Notice that for $k\neq l$, we have that

$$\begin{aligned}F(k,l',l)&=\sum_{b_1,b_2\in V;b_1\neq b_2;b_1-b_2=k-l}\Big\{u_j(l'+l,l'+l+b_1)u_j(l',l'+b_2)\\ &\qquad\qquad-\mathbb{E}\left[u_j(l'+l,l'+l+b_1)u_j(l',l'+b_2)\right]\Big\}\\ &=\sum_{b_1,b_2\in V;b_1\neq b_2;b_1-b_2=k-l}u_{j,l',l,b_1}u_{j,l',b_2}-\mathbb{E}\left[u_{j,l',l,b_1}u_{j,l',b_2}\right].\end{aligned}\tag{A.5}$$

Also, observe that

$$\begin{aligned}F(l,l',l)&=\sum_{b_1\in V}\left\{u_j(l'+l,l'+l+b_1)u_j(l',l'+b_1)-\mathbb{E}\left[u_j(l'+l,l'+l+b_1)u_j(l',l'+b_1)\right]\right\}\\ &=\sum_{b_1\in V}\left\{u_{j,l',l,b_1}u_{j,l',b_1}-\mathbb{E}\left[u_{j,l',l,b_1}u_{j,l',b_1}\right]\right\},\end{aligned}\tag{A.6}$$

substituting $u_j(l',l'+b_1)$ by $1-\sum_{b_2\in V;b_2\neq b_1}u_j(l',l'+b_2)$ into (A.6) we obtain

$$\begin{aligned}&F(l,l',l)\\ &=-\sum_{b_1,b_2\in V;b_1\neq b_2}\Big(u_j(l'+l,l'+l+b_1)u_j(l',l'+b_2)-\mathbb{E}\left[u_j(l'+l,l'+l+b_1)u_j(l',l'+b_2)\right]\Big)\\ &=-\sum_{b_1,b_2\in V;b_1\neq b_2}\Big(u_{j,l',l,b_1}u_{j,l',b_2}-\mathbb{E}\left[u_{j,l',l,b_1}u_{j,l',b_2}\right]\Big).\end{aligned}\tag{A.7}$$

Substituting (A.7) and (A.5) into (A.4) we obtain that the sum in (A.2) is equal to

$$\begin{aligned}&\sum_{l,l'}\mathbb{P}(\tilde{Y}_{j-1}^x=l'+l,\hat{Y}_{j-1}^0=l'|\mathcal{F}_{j-1})\times\\ &\Big(\sum_{b_1,b_2\in V;b_1\neq b_2}\{\mathbb{P}(\tilde{D}_i^x=0|\tilde{D}_j^x=l+b_1-b_2)-\mathbb{P}(\tilde{D}_i^x=0|\tilde{D}_j^x=l)\}\\ &\qquad\qquad\times\{u_{j,l',l,b_1}u_{j,l',b_2}-\mathbb{E}\left[u_{j,l',l,b_1}u_{j,l',b_2}\right]\}\Big)\\ &=\sum_{b_1,b_2\in V;b_1\neq b_2}\sum_{l',l}\{\mathbb{P}(\tilde{D}_i^x=0|\tilde{D}_j^x=l+b_1-b_2)-\mathbb{P}(\tilde{D}_i^x=0|\tilde{D}_j^x=l)\}\\ &\qquad\times\{u_{j,l',l,b_1}u_{j,l',b_2}-\mathbb{E}\left[u_{j,l',l,b_1}u_{j,l',b_2}\right]\}\times\mathbb{P}(\tilde{Y}_{j-1}^x{=}l'+l,\hat{Y}_{j-1}^0{=}l'|\mathcal{F}_{j-1}).\end{aligned}$$

Hence, there is some positive constant $C_2=C_2(K)$, such that

$$\sum_{j=1}^{A|x|^2}\mathbb{E}\Big(\sum_{i=j}^{A|x|^2-1}\Big\{\mathbb{P}(\tilde{D}_i^x=0|\mathcal{F}_j)-\mathbb{P}(\tilde{D}_i^x=0|\mathcal{F}_{j-1})\Big\}\Big)^2\leq C_2\sum_{b_1,b_2\in V;b_1\neq b_2}G(b_1,b_2),$$

where

$$G(b_1,b_2) := \sum_{j=1}^{A|x|^2} \mathbb{E}\Big(\sum_{i=j}^{A|x|^2-1} \sum_{l',l} \{\mathbb{P}(\tilde{D}_i^x = 0 | \tilde{D}_j^x = l + b_1 - b_2) - \mathbb{P}(\tilde{D}_i^x = 0 | \tilde{D}_j^x = l)\}$$
$$\times \{u_j(l'+l, l'+l+b_1) u_j(l', l'+b_2)$$
$$- \mathbb{E}\left[u_j(l'+l, l'+l+b_1) u_j(l', l'+b_2)\right]\}$$
$$\times \mathbb{P}(\tilde{Y}_{j-1}^x = l'+l, \hat{Y}_{j-1}^0 = l' | \mathcal{F}_{j-1})\Big)^2.$$

We only analyze $G((1,0),(0,0))$, and the same arguments used for this term work for the other terms $G(b_1,b_2)$ with $b_1, b_2 \in V, b_1 \neq b_2$. Also, to make the notation more compact, we define

$$A_{j,l}^x = \sum_{i=j}^{A|x|^2-1} \left[\mathbb{P}(\tilde{D}_i^x = 0 | \tilde{D}_j^x = l + (1,0)) - \mathbb{P}(\tilde{D}_i^x = 0 | \tilde{D}_j^x = l)\right]$$
$$\overline{u_{j,l,l'}} = u_j(l'+l, l'+l+(1,0)) u_j(l', l'+(0,0)) - \mathbb{E}\left(u_j(l'+l, l'+l+(1,0)) u_j(l', l'+(0,0))\right)$$
$$p_j^x(l) = \mathbb{P}(\tilde{Y}_{j-1}^x = l | \mathcal{F}_{j-1}),\ p_j^0(l) = \mathbb{P}(\hat{Y}_{j-1}^0 = l | \mathcal{F}_{j-1}).$$

Then

$$G((1,0),(0,0)) = \sum_{j=1}^{A|x|^2-1} \mathbb{E}\left(\sum_{l',l} A_{j,l}^x \overline{u_{j,l,l'}} p_j^x(l'+l) p_j^0(l') \right)^2. \tag{A.8}$$

In Lemma 4.1, we proved that the terms $|A_{j,l}^x|$ are bounded uniformly in l, j, and x, and therefore (A.8) is bounded by

$$C^2 \sum_{j=1}^{A|x|^2-1} \mathbb{E} \sum_{l',l} \sum_{k,k'} \mathbb{E}(\overline{u_{j,l,l'}}\, \overline{u_{j,k,k'}}) p_j^x(l'+l) p_j^0(l') p_j^x(k'+k) p_j^0(k'). \tag{A.9}$$

In (A.9), we have used the independence of the $u_j's$ with $\mathcal{F}_{j-1}$. Now, if $\{l+l', l'\} \cap \{k+k', k'\} = \emptyset$, then $\overline{u_{j,l,l'}}$ and $\overline{u_{j,k,k'}}$ are independent and the expectation of both terms is zero. When $l+l' = k+k'$ and $k' = l'$, (A.9) becomes

$$\begin{aligned}
\sum_{j=1}^{A|x|^2-1} \mathbb{E} \sum_{l',l} \mathbb{E}(\overline{u_{j,l,l'}}^2)(p_j^x(l'+l))^2 (p_j^0(l'))^2 &\leq \sum_{j=1}^{A|x|^2-1} \mathbb{E} \sum_{l',l} p_j^x(l'+l)(p_j^0(l'))^2 \\
&= \sum_{j=1}^{A|x|^2-1} \mathbb{E} \sum_{l'} (p_j^0(l'))^2 \\
&= \sum_{j=1}^{A|x|^2-1} \mathbb{P}(\tilde{D}_j^0 = 0).
\end{aligned} \tag{A.10}$$

When $l' = k+k'$ and $l+l' = k'$, we have that

$$\sum_{j=1}^{A|x|^2-1} \mathbb{E} \sum_{l',l} \mathbb{E}(\overline{u_{j,l,l'}}\, \overline{u_{j,-l,l+l'}}) p_j^x(l'+l) p_j^0(l') p_j^x(l') p_j^0(l+l')$$
$$\leq \sum_{j=1}^{A|x|^2-1} \mathbb{E} \sum_{l',l} p_j^0(l') p_j^x(l') p_j^0(l+l')$$

$$= \sum_{j=1}^{A|x|^2-1} \mathbb{E}\sum_{l'} p_j^0(l')p_j^x(l')$$
$$= \sum_{j=1}^{A|x|^2-1} \mathbb{P}(\tilde{D}_j^x = 0). \tag{A.11}$$

For the case $l'+l = k'+k$ and $k' \neq l'$ we obtain

$$\sum_{j=1}^{A|x|^2-1} \mathbb{E}\sum_{l',l}\sum_{k'} \mathbb{E}(\overline{u_{j,l,l'}}\,\overline{u_{j,l'+l-k',k'}})p_j^x(l'+l)^2 p_j^0(l')p_j^0(k') \leq \sum_{j=1}^{A|x|^2-1} \mathbb{E}\sum_{l',l} p_j^x(l')^2 p_j^0(l'+l)$$
$$= \sum_{j=1}^{A|x|^2-1} \mathbb{E}\sum_{l'} p_j^x(l')^2$$
$$= \sum_{j=1}^{A|x|^2-1} \mathbb{P}(\tilde{D}_j^0 = 0). \tag{A.12}$$

Finally, when $l'+l = k'$ and $k'+k \neq l'$

$$\sum_{j=1}^{A|x|^2-1} \mathbb{E}\sum_{l',l}\sum_{k} \mathbb{E}(\overline{u_{j,l,l'}}\,\overline{u_{j,k,l+l'}})p_j^x(l+l')p_j^0(l+l')p_j^0(l')p_j^x(l+l'+k)$$
$$\leq \sum_{j=1}^{A|x|^2-1} \mathbb{E}\sum_{l,l'} p_j^x(l+l')p_j^0(l+l')p_j^0(l')$$
$$= \sum_{j=1}^{A|x|^2-1} \mathbb{E}\sum_{l,l'} p_j^x(l)p_j^0(l)p_j^0(l-l')$$
$$= \sum_{j=1}^{A|x|^2-1} \mathbb{E}\sum_{l} p_j^x(l)p_j^0(l)$$
$$= \sum_{j=1}^{A|x|^2-1} \mathbb{P}(\tilde{D}_j^x = 0). \tag{A.13}$$

Reasoning as in the proof of Proposition 2.3 in [8] we have that

$$\mathbb{P}\left(D_n = 0 | D_0 = x\right) = \mathbb{P}(\tilde{D}_n = x), \text{ for } n \geq 0 \text{ and } x \in \mathbb{Z}^d.$$

Hence, equation (2.5) implies that (A.10) and (A.12) are both $O(\ln(A|x|^2))$. Using (2.13), we conclude that (A.11) and (A.13) are also $O(\ln(A|x|^2))$. Summing up, the variance of the sum at the left hand side of (4.18) is bounded by an $O(\ln(A|x|^2))$. Therefore, the variance of the whole term in (4.18) is bounded by an $o(1)$, and we have proved the result. □

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