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Generalized transport inequalities and concentration bounds for Riesz-type gases*

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Abstract

This article explores the connection between a generalized Riesz electric energy and norms on the set of probability measures defined in terms of duality. We derive functional inequalities linking these two notions, recovering and generalizing existing Coulomb transport inequalities. We then use them to prove concentration of measure around the equilibrium and thermal equilibrium measures. Finally, we leverage these concentration inequalities to obtain Moser-Trudinger-type inequalities, which may also be interpreted as bounds on the Laplace transform of fluctuations.

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1 Introduction and motivation

We will study a many-particle system, in which the particles interact via a repulsive kernel, and are confined by an external potential. This is modelled by the energy

$$\mathcal{H}_N(X_N) = \sum_{i\neq j} g(x_i - x_j) + N\sum_{i=1}^{N} V(x_i), \tag{1.1}$$

where $g : \mathbf{R}^d \to \mathbf{R}$ is the repulsive interaction potential, $V : \mathbf{R}^d \to \mathbf{R}$ is the confining potential, N is the number of particles, and $X_N = (x_1, \dots, x_N) \in (\mathbf{R}^d)^N$. We are interested in the behaviour for large but finite N.

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At zero temperature, the particles will arrange themselves into the configuration that minimizes the energy $\mathcal{H}_N$. However, we may also study many-particle systems at positive temperature. In this case, the position of the particles is a random variable in $(\mathbf{R}^d)^N$ whose law is given by

$$\mathrm{d}\mathbf{P}_{N,\beta}(X_N) = \frac{1}{Z_{N,\beta}} \exp\left(-\beta \mathcal{H}_N(X_N)\right) \mathrm{d}X_N, \tag{1.2}$$

where

$$Z_{N,\beta} = \int_{\mathbf{R}^{d\times N}} \exp\left(-\beta \mathcal{H}_N(X_N)\right) \mathrm{d}X_N \tag{1.3}$$

is the partition function, and $\beta > 0$ is the inverse temperature which may depend on N. An element of $(\mathbf{R}^d)^N$ following the law in (1.2) will be called an *interacting gas* with total potential energy $\mathcal{H}_N$ and inverse temperature β.

A very frequent form of the repulsive interaction g is given by

$$g(x) = \begin{cases} \frac{c_d}{|x|^{d-2}} & \text{if } d \geq 3, \\ -c_2 \log(|x|) & \text{if } d = 2, \end{cases} \tag{1.4}$$

where c_d is such that

$$-\Delta g = \delta_0. \tag{1.5}$$

We will refer to this setting as the *Coulomb case.* Another frequent form of g is given by

$$g(x) = \frac{c_{d,s}}{|x|^{d-2s}}, \tag{1.6}$$

with $0 < s < \min\{\frac{d}{2}, 1\}$, $d \geq 1$, and $c_{d,s}$ such that

$$(-\Delta)^s g = \delta_0. \tag{1.7}$$

We will refer to this setting as the *Riesz case*. Here, we will deal with a generalization of Riesz interactions, which are very similar to ones first introduced in [31, 35].

Apart from the Riesz case, it is also possible to study a many-particle system with interactions given by a hypersingular (non-integrable) interaction, i.e., g given by Equation (1.6) with $s < 0$. See, for example, [21, 20, 19].

Coulomb and Riesz gases are classical subjects with applications to spherical packing [43, 13, 14, 45, 34], statistical mechanics [1, 23, 39, 18, 42], random matrix theory [24, 8, 9, 10, 25, 22, 27], and other topics in mathematical physics [6, 37].

The study of a general interacting gas has recently begun to attract attention as can be seen from [17, 26, 11, 31, 35].

2 Main definitions

In this section, we introduce the objects and notation needed to state our main results.

2.1 Interacting gases

We start with notions related to interacting gases. The most fundamental observable is the empirical measure. The empirical measure emp_N is defined as

$$\mathrm{emp}_N(X_N) = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}. \tag{2.1}$$

In order to ease notation, we will often write emp_N instead of $\mathrm{emp}_N(X_N)$. The main functionals of interest in our setting are related to the macroscopic interaction energy and the entropy. First, the *macroscopic interaction energy* of a measure μ is given by

$$\mathcal{E}(\mu) = \int_{\mathbf{R}^d \times \mathbf{R}^d} g(x-y)\mathrm{d}(\mu \otimes \mu)(x,y) \tag{2.2}$$

whenever this integral makes sense and infinite if not. Moreover, the mean field limit of $\mathcal{H}_N$ is given by $\mathcal{E}_V$ defined as

$$\mathcal{E}_V(\mu) = \mathcal{E}(\mu) + \int_{\mathbf{R}^d} V \mathrm{d}\mu = \int_{\mathbf{R}^d \times \mathbf{R}^d} \left(g(x-y) + \frac{V(x)}{2} + \frac{V(y)}{2} \right) \mathrm{d}(\mu \otimes \mu)(x,y) \tag{2.3}$$

whenever the integral in the right-hand side makes sense and we define it as infinite if not. By adding a multiple of the entropy, we get the free energy $\mathcal{E}_V^\theta$, given by

$$\mathcal{E}_V^\theta(\mu) = \mathcal{E}_V(\mu) + \frac{1}{\theta}\mathrm{ent}[\mu], \tag{2.4}$$

with the *entropy*

$$\mathrm{ent}[\mu] = \begin{cases} \int_{\mathbf{R}^d} \log(\mathrm{d}\mu/\mathrm{d}\mathcal{L})\,\mathrm{d}\mu & \text{if } \mu \ll \mathcal{L},\ \log(\mathrm{d}\mu/\mathrm{d}\mathcal{L}) \in L^1(\mu) \\ \infty & \text{otherwise,} \end{cases} \tag{2.5}$$

where $\mathcal{L}$ denotes the Lebesgue measure on $\mathbf{R}^d$. We define $\mathcal{E}^{\neq}$ for a measure μ as

$$\mathcal{E}^{\neq}(\mu) = \iint_{\mathbf{R}^d \times \mathbf{R}^d \setminus \Delta} g(x-y)\mathrm{d}(\mu \otimes \mu)(x,y), \tag{2.6}$$

where Δ is the diagonal on $\mathbf{R}^d \times \mathbf{R}^d$. For $X_N = (x_1, \ldots, x_N)$ that satisfy $x_i \neq x_j$ for $i \neq j$, we define the energy of $\mu - \mathrm{emp}_N$ by

$$\mathrm{F}_N(X_N, \mu) = \mathcal{E}^{\neq}(\mu - \mathrm{emp}_N(X_N)). \tag{2.7}$$

We have excluded the diagonal in the integral so that there are no self-interactions and the quantity in (2.7) can be finite. Lastly, given a measure μ on $\mathbf{R}^d$, we define the potential h^μ of μ as

$$h^\mu = \mu * g \tag{2.8}$$

whenever it makes sense. We proceed to define the equilibrium and thermal equilibrium measures, which will play a central role in the rest of the article. We denote by μ_∞ the minimizer of $\mathcal{E}_V$ over the set of probability measures $\mathcal{P}(\mathbf{R}^d)$

$$\mu_\infty := \underset{\mu \in \mathcal{P}(\mathbf{R}^d)}{\mathrm{argmin}}\ \mathcal{E}_V(\mu). \tag{2.9}$$

We will refer to μ_∞ as the *equilibrium measure* and denote by Σ its support. We denote by μ_θ the minimizer of $\mathcal{E}_V^\theta$ over $\mathcal{P}(\mathbf{R}^d)$

$$\mu_\theta := \underset{\mu \in \mathcal{P}(\mathbf{R}^d)}{\mathrm{argmin}}\ \mathcal{E}_V^\theta(\mu). \tag{2.10}$$

We will refer to μ_θ as the *thermal equilibrium measure*. We refer to Section 6 for existence, uniqueness and basic properties of μ_∞ and μ_θ.

Remark 2.1. As long as the temperature is not too big ($\frac{1}{N} \ll \beta$), the equilibrium measure provides a good approximation to the empirical measure. A better approximation, however, is provided by $\mu_{N\beta}$, the thermal equilibrium measure with parameter $\theta = N\beta$. This difference becomes bigger as the temperature becomes bigger. In the extreme case $N\beta$ when converges to θ, the empirical measure converges to μ_θ so that μ_∞ is no longer a good approximation while $\mu_{N\beta}$ is always a good approximation, see Remark 3.5.

Remark 2.2. For the rest of this article, we commit the abuse of notation of not distinguishing between a measure and its density.

As mentioned before, this work deals with general interactions that qualitatively behave like Riesz interactions. We now specify this class of interactions.

Notation 2.3. *We denote by* $\mathfrak{F}(f)$*, or alternatively by* $\widehat{f}$ *the Fourier transform of* f*.*

Definition 2.4. *Let* $s \in (0, \min\{1, \frac{d}{2}\})$*. A continuous function* $g : \mathbf{R}^d \setminus \{0\} \to \mathbf{R}$ *that defines a tempered distribution on* $\mathbf{R}^d$ *is called a Riesz-type kernel of order* s *if there exists an integer* $m \geq 0$*, a continuous function* $G : \mathbf{R}^d \times \mathbf{R}^m \setminus \{0\} \to \mathbf{R}$ *that defines a tempered distribution on* $\mathbf{R}^d \times \mathbf{R}^m$ *and constants* $C_1, C_2, C \geq 0$ *depending only on* d, s, m*, and* G *such that*

1. $G(x, 0) = g(x)$ *for every* $x \in \mathbf{R}^d \setminus \{0\}$*.*
2. $G(X) = G(-X)$ *for every* $X \in \mathbf{R}^d \times \mathbf{R}^m \setminus \{0\}$*.*
3. $\lim_{X \to 0} G(X) = \infty$*.*
4. *There exists* $r_0 > 0$ *such that* $\Delta G \leq 0$ *in* $\{X \in \mathbf{R}^d \times \mathbf{R}^m : \|X\| < r_0\}$ *as a distribution.*
5. $|G(X)| \leq C\left(\frac{1}{|X|^{d-2s}}\right)$ *for every* $X \in \mathbf{R}^d \times \mathbf{R}^m \setminus \{0\}$*.*
6. *The distribution* ∇G *is a locally integrable function and* $|\nabla G(X)| \leq C\left(\frac{1}{|X|^{d-2s+1}}\right)$ *for every* $X \in \mathbf{R}^d \times \mathbf{R}^m \setminus \{0\}$*.*
7. *The distribution* $\hat{G}$ *is a locally integrable function and* $\frac{C_1}{|\Xi|^{m+2s}} \leq \widehat{G}(\Xi) \leq \frac{C_2}{|\Xi|^{m+2s}}$*.*
8. *The distribution* $\hat{g}$ *is a locally integrable function and* $\frac{C_1}{|\xi|^{2s}} \leq \widehat{g}(\xi) \leq \frac{C_2}{|\xi|^{2s}}$*.*
9. *There exist* $c_s < 1$ *and* $r_0 > 0$ *such that, for every pair* $X, Y \in B(0, r_0) \setminus \{0\}$ *with* $|Y| \geq 2|X|$*, the inequality* $G(Y) < c_s G(X)$ *is satisfied.*
10. *The distribution* $h = \mathfrak{F}(1/\widehat{g})$ *is a locally integrable function and the inequalities* $\frac{C_1}{|x|^{d+2s}} \leq h(x) \leq \frac{C_2}{|x|^{d+2s}}$ *hold.*

Remark 2.5. Items 1–9 are basically found in [31, 35], but unlike [31, 35], we only impose a growth condition on G and its first derivative. Item 10 is not found in [31, 35], but it is necessary to derive a monotonicity property, see Section 7.

Remark 2.6. The most important example of a Riesz-type kernel of order s is, of course, the Riesz kernel (1.6) of order s. In this case, $G(X) = \frac{c_{d,s}}{|X|^{d-2s}}$, which satisfies Definition 2.4 for m large enough.

Apart from a general interaction, we will deal with a potential which is general except for growth and regularity conditions. We now specify the exact class of potentials that we will deal with.

Definition 2.7. *Let* $s \in (0, \frac{d}{2}) \cap (0, 1]$ *and either let* g *be a Riesz-type kernel of order* s *if* $s < 1$ *or let* g *be the Coulomb kernel if* $s = 1$*. We call* $V : \mathbf{R}^d \to \mathbf{R}$ admissible *if*

1. $V \in C^2$*.*
2. $\lim_{|x| \to \infty} V(x) = \infty$*.*
3. *For all* $\beta > 0$*,* $\int_{\mathbf{R}^d} \exp(-\beta V(x))\,\mathrm{d}x < \infty$*.*
4. $\mu_\infty \in L^\infty$*.*

We say that the pair (g, V) *is an* admissible *pair of order* s*.*

2.2 Functional analysis

In this subsection, we define some norms and spaces used in the main results.

2.2.1 Hölder norms

We start by defining the Hölder seminorm and the full Hölder norm as well as their duals.

Definition 2.8. *For a continuous function $f : \Omega \to \mathbf{R}$, we define the Hölder α seminorm as*

$$|f|_{\dot{C}^{0,\alpha}} = \sup_{x,y\in\Omega, x\neq y} \frac{f(x)-f(y)}{|x-y|^\alpha}.$$

whose kernel is the space of constant functions. We denote by $\dot{C}^{0,\alpha}(\Omega)$ the space

$$\dot{C}^{0,\alpha}(\Omega) = \{f \in C(\Omega) : |f|_{\dot{C}^{0,\alpha}} < \infty\}. \tag{2.11}$$

For a continuous function $f : \Omega \to \mathbf{R}$, we define the full Hölder α norm as

$$\|f\|_{C^{0,\alpha}} = |f|_{\dot{C}^{0,\alpha}} + \|f\|_\infty, \tag{2.12}$$

where $\|f\|_\infty$ is defined for a continuous function f as the supremum of $|f|$. We now define the corresponding dual norms. For a measure μ on Ω, we define

$$\|\mu\|_{\dot{C}^{0,\alpha}_*} = \sup_{f\in\dot{C}^{0,\alpha}(\Omega)} \frac{\int_\Omega f\,\mathrm{d}\mu}{|f|_{\dot{C}^{0,\alpha}}}, \tag{2.13}$$

where the supremum is taken over non-constant functions. For a measure μ on Ω, we define

$$\|\mu\|_{C^{0,\alpha}_*} = \sup_{f\in C^{0,\alpha}(\Omega)} \frac{\int_\Omega f\,\mathrm{d}\mu}{\|f\|_{C^{0,\alpha}}}. \tag{2.14}$$

We also define the $\dot{C}^{1,\alpha}$ seminorm of $f : \Omega \to \mathbf{R}$ as

$$|f|_{\dot{C}^{1,\alpha}} = |f|_{\dot{C}^{0,\alpha}} + \max_i |\partial_i f|_{\dot{C}^{0,\alpha}}. \tag{2.15}$$

Remark 2.9. Notice that for μ and ν probability measures, $\|\mu-\nu\|_{\dot{C}^{0,\alpha}_*}$ is the Wasserstein $1-$ distance between μ and ν, and that $\|\cdot\|_{C^{0,\alpha}_*}$ is the bounded-Lipschitz norm.

Remark 2.10. Note that if $\mu(\Omega) \neq 0$, then $\|\mu\|_{\dot{C}^{0,\alpha}_*} = \infty$.

Remark 2.11. Whenever Ω is bounded, the norms $\|\cdot\|_{C^{0,\alpha}_*}$ and $\|\cdot\|_{\dot{C}^{0,\alpha}_*}$ are equivalent on the set of measures of total mass zero.

2.2.2 On H^s norms

We deal with the L^2-based norm $\dot{H}^s$ for $s \in (0,1)$. Consider a tempered distribution f on $\mathbf{R}^d$ such that its Fourier transform is locally integrable. We define is (Fourier) $\dot{H}^s$ norm

$$|f|^2_{\dot{H}^s_F} = \int_{\mathbf{R}^d} |\widehat{f}(\xi)|^2 |\xi|^{2s} \mathrm{d}\xi. \tag{2.16}$$

For a measurable function f on $\mathbf{R}^d$, we can define the (difference quotient) $\dot{H}^s$ seminorm

$$|f|^2_{\dot{H}^s_{dq}} = \iint_{\mathbf{R}^d\times\mathbf{R}^d} \frac{|f(x)-f(y)|^2}{|x-y|^{d+2s}}\,\mathrm{d}x\mathrm{d}y.$$

Both are equivalent norms on the space

$$\dot{H}^s(\mathbf{R}^d) := \{f \text{ tempered distribution on } \mathbf{R}^d : \hat{f} \text{ is locally integrable and } |f|^2_{\dot{H}^s_F} < \infty\}.$$

Lemma 2.12. *$\dot{H}^s(\mathbf{R}^d) \subset L^2_{\mathrm{loc}}(\mathbf{R}^d)$ and the norms $|\cdot|_{\dot{H}^s_{dq}}$ and $|\cdot|_{\dot{H}^s_F}$ are equivalent on $\dot{H}^s(\mathbf{R}^d)$. In fact, one is a multiple of the other.*

Proof. See [4, Proposition 1.37]. □

We will write $|\cdot|_{\dot{H}^s}$ to denote either of the two norms unless a distinction between them is necessary.

3 Main results

Here we state the main results of the paper, using the notation introduced above. We start with a proposition about elementary properties of the thermal equilibrium measure.

Proposition 3.1. *Let (g, V) be an admissible pair of order s and let $\theta_0 > 0$. Then, there exists a constant $C_{\theta_0} > 0$ such that, for every $\theta > \theta_0$ and every $x \in \mathbf{R}^d$,*

$$\mu_\theta(x) \leq C_{\theta_0}. \tag{3.1}$$

Furthermore, there exists a constant $C > 0$ that does not depend on θ_0 such that

$$\mu_\theta(x) \leq \exp\left(C_{\theta_0} - \theta(V(x) - C)\right). \tag{3.2}$$

The proof is found in Section 7. We proceed to state generalized transport inequalities, which are extensions of Theorems 1.1 and 1.2 of [12].

Theorem 3.2. *Let (g, V) be an admissible pair of order s and let $\alpha > s$.*

1. *Let $\Omega \subset \mathbf{R}^d$ be a compact set, then there exists a constant $C_{\alpha,\Omega} > 0$ which depends only on g, α and Ω such that, for any $\mu, \nu \in \mathcal{P}(\Omega)$,*

$$\|\mu - \nu\|^2_{\dot{C}^{0,\alpha}_*} \leq C_{\alpha,\Omega}\mathcal{E}(\mu - \nu). \tag{3.3}$$

2. *There exists a constant $C_\alpha > 0$ which depends only on V, g and α such that, for every $\mu \in \mathcal{P}(\mathbf{R}^d)$,*

$$\|\mu - \mu_\infty\|^2_{C^{0,\alpha}_*} \leq C_\alpha\left(\mathcal{E}_V(\mu) - \mathcal{E}_V(\mu_\infty)\right). \tag{3.4}$$

3. *Assume that*

$$\liminf_{x\to\infty} \frac{V(x)}{|x|^{2\alpha}} > 0. \tag{3.5}$$

Then there exists a constant $C_\alpha > 0$ which depends only on V, g and α such that, for every $\mu \in \mathcal{P}(\mathbf{R}^d)$, we have

$$\|\mu - \mu_\infty\|^2_{\dot{C}^{0,\alpha}_*} \leq C_\alpha\left(\mathcal{E}_V(\mu) - \mathcal{E}_V(\mu_\infty)\right). \tag{3.6}$$

The proof is found in Section 8. We then apply these results to obtain a quantitative understanding of the convergence of the thermal equilibrium measure to the equilibrium measure.

Corollary 3.3. *Let (g, V) be an admissible pair of order s Then, the function $\theta \mapsto \mathrm{ent}[\mu_\theta]$ is non-decreasing. Moreover, if $\alpha > s$, there exists a constant $C_\alpha > 0$, which depends only on V, g, and α such that*

$$\|\mu_\infty - \mu_\theta\|^2_{C^{0,\alpha}_*} \leq \frac{C_\alpha}{\theta}\left|\mathrm{ent}[\mu_\infty] - \mathrm{ent}[\mu_\theta]\right|. \tag{3.7}$$

Additionally, if

$$\liminf_{x\to\infty} \frac{V(x)}{|x|^{2\alpha}} > 0 \tag{3.8}$$

then there exits a constant C_α, which depends only on V, g, and α such that

$$\|\mu_\infty - \mu_\theta\|^2_{\dot{C}^{0,\alpha}_*} \leq \frac{C_\alpha}{\theta}\left|\mathrm{ent}[\mu_\infty] - \mathrm{ent}[\mu_\theta]\right|. \tag{3.9}$$

The proof is found in Section 8. We then use the previous results to prove concentration of measure around the equilibrium measure and thermal equilibrium measure. The next theorem is a generalization of [12, Theorem 1.5] and [32, Theorem 2.2] to general interactions.

Theorem 3.4. *Let (g, V) be an admissible pair of order s and let $\alpha > s$. Assume that $N\beta \to \infty$ and let $r > 0$. Then*

1. *There exists a constant C, which depends only on V, g and α, such that*

$$\begin{aligned}&\mathbf{P}_{N,\beta}\left(\|\mathrm{emp}_N - \mu_\infty\|_{C^{0,\alpha}_*} > r\right) \le \\ &\exp\Big\{ - N^2\beta\big(C[r - N^{-\frac{\alpha}{d}}]_+^2 - C\|\mu_\infty\|_{L^\infty} N^{-\frac{2s}{d}}\big) + \\ &\quad N\big(\log|\Sigma| + \mathrm{ent}[\mu_\infty] + \beta\mathcal{E}(\mu_\infty)\big) + o(N)\Big\},\end{aligned} \tag{3.10}$$

   where $(\cdot)_+$ denotes the positive-part function.

2. *Set $\theta = N\beta$. Then there exists a constant C depending only on V, g and α such that*

$$\mathbf{P}_{N,\beta}\left(\|\mathrm{emp}_N - \mu_\theta\|_{C^{0,\alpha}_*} > r\right) \le \exp\left(-N^2\beta\left(C\left[r - CN^{-\frac{\alpha}{d}}\right]_+^2 - CN^{-\frac{2s}{d}}\right)\right). \tag{3.11}$$

The proof is found in Section 9. The previous theorem implies, with high probability, an upper bound for the distances between the empirical measure and equilibrium and thermal equilibrium measures:

Remark 3.5. Theorem 3.4 implies that, with high probability, the $C^{0,\alpha}_*$ distance between emp_N and μ_θ is of order $N^{\frac{-\alpha}{d}}$ for any value of β. On the other hand, from Theorem 3.4, we may only infer that the $C*^{0,\alpha}$ distance between emp_N and μ_∞ is of order $N^{\frac{-\alpha}{d}}$ with high probability if β is not too small. Nevertheless, Theorem 3.4 still implies that the $C^{0,\alpha}_*$ distance between emp_N and μ_∞ tends to 0 with high probability if $\frac{1}{N} \ll \beta$.

We complement this with a lower bound for such distance.

Proposition 3.6. *Let μ be a probability measure on $\mathbf{R}^d$ such that $\|\mu\|_{L^\infty} < L$ for some finite $L > 0$. Let ν_N be a probability measure on $\mathbf{R}^d$ such that*

$$\nu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}. \tag{3.12}$$

Then

$$\|\mu - \nu_N\|_{C^{0,\alpha}_*} \ge CN^{-\frac{\alpha}{d}}, \tag{3.13}$$

where C only depends on d and L.

The proof is found in Section 10. Our final result is a series of Moser-Trudinger like inequalities, which may also be interpreted as bounds on the Laplace transform of fluctuations. This result is new also in the Coulomb case. Notice that we implicitly assume that $f : \mathbf{R}^d \to \mathbf{R}$ has enough regularity so that the involved norms make sense.

Theorem 3.7. *Let (g, V) be an admissible pair of order s, and assume $N\beta \to \infty$. Consider a continuous function $f : \mathbf{R}^d \to \mathbf{R}$ and define the random variable $\mathrm{Fluct}[f]$ by*

$$\mathrm{Fluct}[f] = \int_{\mathbf{R}^d} f \mathrm{d}\left(\mathrm{emp}_N - \mu_\theta\right), \tag{3.14}$$

where $\theta = N\beta$.

1. *If g is the Coulomb kernel, there exists $C > 0$ depending only on V such that*

$$\log\left(\mathbf{E}_{\mathbf{P}_{N,\beta}} \exp\left(N^2\beta\,|t\mathrm{Fluct}[f]|\right)\right) \le N^2\beta\left(Ct^2\|f\|^2_{W^{1,\infty}} + N^{-\frac{2}{d}}C\right), \qquad (3.15)$$

where $\|f\|_{W^{1,\infty}} := \|f\|_{L^\infty} + \|\nabla f\|_{L^\infty}$.

2. *If g is a Riesz-type kernel of order $s < 1$ then for any $\alpha > s$ there exists $C > 0$ depending only on V, g, and α such that*

$$\log\left(\mathbf{E}_{\mathbf{P}_{N,\beta}} \exp\left(N^2\beta\,|t\mathrm{Fluct}[f]|\right)\right) \le N^2\beta\left(Ct^2\|f\|^2_{C^{0,\alpha}} + N^{-\frac{2\alpha}{d}}C\right). \qquad (3.16)$$

3. *If g is the Coulomb kernel, $i \in \{0,1\}$ and $\alpha \in (0,1)$ then there exists $C > 0$ depending only on V such that*

$$\log\left(\mathbf{E}_{\mathbf{P}_{N,\beta}} \exp\left(N^2\beta\,|t\mathrm{Fluct}[f]|\right)\right) \le$$
$$\beta N^2\left(\frac{1}{4}t^2|f|^2_{\dot H^1} + N^{-\frac{i+\alpha}{d}}t|f|_{\dot C^{i,\alpha}} + CN^{-\frac{2}{d}}\right) + \frac{d}{2}\left(\log(N^2\beta) + \log(1+t|f|_{\dot H^1})\right). \qquad (3.17)$$

4. *If g is a Riesz-type kernel of order $s < 1$, $i \in \{0,1\}$ and $\alpha \in (0,1)$, then there exists $C > 0$ depending only on V and g such that*

$$\log\left(\mathbf{E}_{\mathbf{P}_{N,\beta}} \exp\left(N^2\beta\,|t\mathrm{Fluct}[f]|\right)\right) \le \beta N^2\left(Ct^2|f|^2_{\dot H^s} + CN^{-\frac{i+\alpha}{d}}t|f|_{\dot C^{i,\alpha}} + CN^{-\frac{2s}{d}}\right). \qquad (3.18)$$

The proof is found in Section 11.

4 Further work

1. We expect that an analogue of the main results in this paper holds for compact manifolds. In this case, the Riesz kernel would be defined as the fundamental solution of the fractional Laplacian, and may be written in terms of the eigenfunctions of the Laplace-Beltrami operator.
2. Theorems 3.4 and 3.2 hold for $\alpha > s$. The reason is in Remark 6.14. Do Theorems 3.4 and 3.2 hold also in the endpoint case $\alpha = s$? We emphasise that this s true in the case $s = 1$.
3. Do transport inequalities hold for the thermal equilibrium measure as well? That is, is it true (in the Coulomb case) that

$$\begin{aligned} \|\mu - \mu_\beta\|^2_{\mathrm{BL}} &\le C\left(\mathcal{E}^V_\beta(\mu) - \mathcal{E}^V_\beta(\mu_\beta)\right), \\ W_1(\mu,\mu_\beta)^2 &\le C\left(\mathcal{E}^V_\beta(\mu) - \mathcal{E}^V_\beta(\mu_\beta)\right), \end{aligned} \qquad (4.1)$$

   for some constant C, where $\|\cdot\|_{\mathrm{BL}}$ and W_1 denote the bounded-Lipschitz norm and Wasserstein $1-$ distance, respectively? These inequalities would prove a more intimate link between optimal transport and Coulomb gases than the ones available in the literature.
4. Theorem 3.4 implies, with high probability, an upper bound for the bounded-Lipschitz distance between the empirical measure and thermal equilibrium measure. However, unlike the Coulomb case, this inequality is not geometrically optimal, i.e. it is not of order $N^{-\frac{1}{d}}$. Is it true that

$$\mathbf{P}_{N,\beta}(\limsup N^{\frac{1}{d}}\|\mathrm{emp} - \mu_\beta\|_{\mathrm{BL}} = \infty) = 0? \qquad (4.2)$$

   In the case of Coulomb interactions, it is shown in [32] that (4.2) holds, and that it cannot hold for any exponent higher than $\frac{1}{d}$.

5. We expect that generalized transport inequalities and concentration inequalities are also valid in the high temperature regime ($\beta = \frac{1}{N}$); and very high temperature regime ($\beta \ll \frac{1}{N}$). In the case of the very high temperature regime, it would be necessary to consider either a compact manifold, or a Hamiltonian in which the confining potential term has weight big enough to be comparable to the effect of the entropy.
6. We expect that generalized transport inequalities (Theorem 3.2) and concentration inequalities (Theorem 3.4) also hold for subCoulomb interactions, i.e. interactions of the form $g(x) = |x|^{d-2s}$ for $d \geq 3$ and $s \in \left(1, \frac{d}{2}\right)$. In this case, the inequalities would be for the $C_*^{k,\alpha}$ or $\dot{C}_*^{k,\alpha}$ norms, for $k \in \left(0, \frac{d}{2}\right) \cap \mathbf{N}$.

5 Literature comparison

The thermal equilibrium measure was introduced in this context in [3] and [2], and further explored in [41]. [3] analyzes fundamental qualitative and quantitative properties, while [2] introduces the splitting formula, and exploits it to obtain local laws for Coulomb gases at arbitrary temperature. [41] Continues the ideas of [2] by deriving a precise expansion of the partition function (1.3), and applying these estimates to obtain a CLT for fluctuations around the thermal equilibrium measure. This paper begins the investigation of the thermal equilibrium measure for Riesz and more general interactions.

The type of interactions that we treat in this paper are, roughly speaking, interactions that qualitatively behave like Riesz potentials in both real space and Fourier space. Very similar kernels were introduced in [31] and [35]. In the last two references, the goal was to understand the passage to the mean-field limit of a system of ODE's modelling the dynamic behaviour of a many-particle system. More specifically, the authors show that if $X_N \in \mathbf{R}^{d\times N}$ satisfies the ODE

$$\begin{aligned} \dot{x}_i &= \frac{1}{N}\sum_{i\neq j} \nabla g(x_i - x_j) \\ x_i(0) &= x_i^0, \end{aligned} \tag{5.1}$$

then in the limit as N tends to infinity, $\mathrm{emp}_N(X_N(0))$ converges to a measure μ_0, and under additional conditions, then $\mathrm{emp}_N(X_N(t))$ converges to a solution of the PDE

$$\begin{aligned} \partial_t \mu &= -\mathrm{div}((\nabla g * \mu)\mu) \\ \mu(0) &= \mu_0. \end{aligned} \tag{5.2}$$

The exact result is more general, since it covers possible stochastic noise, among other things, see [31, 35] for further details.

The Moser-Trudinger-type inequalities of Theorem 3.7 (which may also be interpreted as bounds on the Laplace transform of the fluctuations) may be compared to [41, Theorem 1], and [7, Theorems 1.2, 1.5]. [7, Theorems 1.2, 1.5] apply only to two-dimensional Coulomb gases, but they are sharp in that case. On the other hand the inequalities of Theorem 3.7 hold in arbitrary dimension and temperature regime, but we do not believe them to be sharp. [41, Theorem 1] also holds in arbitrary dimension and temperature regime. It contains a bound in terms of the maximum absolute value of f and its higher order derivatives (or ξ in the notation of [41]). This bound requires higher regularity (namely, C^3 regularity), and also that $Ct\max\{|f|_{C^1}, |f|_{C^2}\} < 1$ for a given constant C, but is more accurate than Theorem 3.7 for small values of t. Unlike the bounds in both [41] and [7], Theorem 3.7 holds for Riesz kernels and more general interactions.

[38, 36] obtain concentration inequalities for Coulomb gases as a consequence of an analysis of the renormalized energy. [29] obtains bounds on the Laplace transform of

fluctuations for Coulomb gases in 2d, which imply concentration inequalities. [5] obtains bounds on the Laplace transform of fluctuations and also concentration bounds for log gases in 1d. [28] derives and LDP for Coulomb and Riesz gases; in particular, the upper bound of the LDP may be interpreted as a concentration inequality in which the error depends on the set in question. We expect that concentration inequalities for Riesz gases may also be derived using the ideas of [33].

This paper is inspired by [12, 30]. [30] is an early reference in the connection between optimal transport and Coulomb gases. This work was later expanded in [12]. The main results in [12] are analogues of Theorems 3.2 and 3.4 item 1 for the Coulomb kernel, with the $C_*^{0,\alpha}$ norm replaced by the bounded-Lipschitz norm, and the $\dot{C}_*^{0,\alpha}$ replaced by the Wasserstein 1-norm. This concentration inequality was later extended to compact manifolds in [16], and to the thermal equilibrium measure in [32]. In this paper, we present an alternative approach to the one in [12]. This approach, based on splitting formulas, a dual representation of the electric energy, and a localization inequality for the H^{-s} norm (see Proposition 8.1) allows us to recover the results of [12] and extend them to more general interactions.

To the best of our knowledge, this is the first concentration inequality for Riesz gases (at least the first one in which the error is independent of the set in question).

6 Preliminaries

This section will lay the groundwork needed to prove the results introduced earlier.

6.1 Splitting formulas and partition functions

We start by recalling some well-known facts about the equilibrium and thermal equilibrium measures. First, we recall existence and uniqueness.

Lemma 6.1. *Assume (g, V) is an admissible pair of order s. Then the functional $\mathcal{E}_V$ has a unique minimizer μ_∞ in the set of probability measures. This minimizer has compact support and satisfies the First Order Condition*

$$\begin{aligned} &2h^{\mu_\infty} + V - c_\infty \geq 0 \text{ on } \mathbf{R}^d \\ &2h^{\mu_\infty} + V - c_\infty \leq 0 \text{ on } \operatorname{supp}(\mu_\infty), \end{aligned} \tag{6.1}$$

for $c_\infty = 2\mathcal{E}_V(\mu_\infty)$.

Proof. The theorem in the Coulomb case is classical ([15], see also [40, Theorem 1]). The proof extends to the Riesz-type case without difficulty. □

Lemma 6.2. *The functional $\mathcal{E}_V^\theta$ has a unique minimizer μ_θ in the set of probability measures. This minimizer has a density (that we still call μ_θ) which is everywhere positive, bounded from above (with a bound that may depend on θ), satisfies the First Order Condition*

$$2h^{\mu_\theta} + V + \frac{1}{\theta}\log(\mu_\theta) = c_\theta, \tag{6.2}$$

for some constant c_θ, and also that

$$\lim_{|x|\to\infty} h^{\mu_\theta}(x) = 0. \tag{6.3}$$

Proof. See [3, Lemma 2.1] for the Coulomb case, which extends to the Riesz-type case. □

Next, we derive expansions of the Hamiltonian around the equilibrium and thermal equilibrium measures. These expansions are splitting formulas.

Proposition 6.3 (Splitting formula)**.** *For every $X_N = (x_1, \ldots, x_N) \in (\mathbf{R}^d)^N$ such that $x_i \neq x_j$ whenever $i \neq j$, we have that the Hamiltonian $\mathcal{H}_N$ can be split into*

$$\mathcal{H}_N(X_N) = N^2 \left(\mathcal{E}_V(\mu_\infty) + \mathrm{F}_N(X_N, \mu_\infty) + \int_{\mathbf{R}^d} \zeta_\infty \, \mathrm{d}\,\mathrm{emp}_N \right), \tag{6.4}$$

where

$$\zeta_\infty = V + 2h^{\mu_\infty} - c_\infty. \tag{6.5}$$

Proof. See [40, Lemma 3.1] for the Coulomb case, which extends to the Riesz-type case. □

Proposition 6.4 (Thermal splitting formula)**.** *We introduce the notation*

$$\zeta_\theta = -\frac{1}{\theta} \log(\mu_\theta). \tag{6.6}$$

For every $X_N = (x_1, \ldots, x_N) \in (\mathbf{R}^d)^N$ such that $x_i \neq x_j$ whenever $i \neq j$, we have that the Hamiltonian $\mathcal{H}_N$ can be split into

$$\mathcal{H}_N(X_N) = N^2 \left(\mathcal{E}_V^\theta(\mu_\theta) + \mathrm{F}_N(X_N, \mu_\theta) + \int_{\mathbf{R}^d} \zeta_\theta \, \mathrm{d}\,\mathrm{emp}_N \right). \tag{6.7}$$

Proof. See [2, Lemma2.1] for the Coulomb case, which extends to the Riesz-type case. □

Motivated by the splitting formulas, we can identify the leading order term of the partition function, and also prove elementary results about the next order components.

Lemma 6.5. *Assume that (g, V) is an admissible pair of order s and that $\mathrm{ent}[\mu_\infty] < \infty$. Then, for any $N \geq 2$,*

$$Z_\beta^N \geq \exp\left(-N^2 \beta \mathcal{E}_V(\mu_\infty) + N \left(\beta \mathcal{E}(\mu_\infty) - \mathrm{ent}[\mu_\infty] \right) \right). \tag{6.8}$$

Proof. See [12, Lemma 4.1] for the Coulomb case, which extends to the Riesz-type case. □

Definition 6.6. *Set $\theta = N\beta$. We define the next order partition function $K_{N,\beta}$ as*

$$K_{N,\beta} = \frac{Z_{N,\beta}}{\exp\left(N^2 \mathcal{E}_V^\theta(\mu_\theta) \right)}. \tag{6.9}$$

Lemma 6.7. *Let (g, V) be an admissible pair of order s. Then the next order partition function is greater that 1, i.e.,*

$$\log(K_{N,\beta}) > 0. \tag{6.10}$$

Proof. See [32, Proposition 5.10] for the Coulomb case, which extends to the Riesz-type case. □

6.2 More on H^s norms

In this subsection, we introduce additional properties of the H^s norm. We start by turning the $\dot{H}^s$ semi-norm into a full norm.

Definition 6.8. *We define the H^s norm of a function $f : \mathbf{R}^d \to \mathbf{R}$ by*

$$\|f\|_{H^s}^2 = \|f\|_{L^2}^2 + |f|_{\dot{H}^s}^2 \tag{6.11}$$

We now extend the definition of H^s norms to negative indices.

Definition 6.9. *Let $s > 0$. Given a tempered distribution μ on $\mathbf{R}^d$ such that $\hat{\mu}$ is a locally integrable function, we define the $\dot{H}^{-s}$ norm of μ by*

$$\|\mu\|_{\dot{H}^{-s}}^2 = \int_{\mathbf{R}^d} |\widehat{\mu}(\xi)|^2 |\xi|^{-2s} d\xi. \tag{6.12}$$

Now we introduce an analogue of the H^s norms for functions that are not defined on the whole space.

Definition 6.10. *Given a domain Ω, and a measurable function $f : \Omega \to \mathbf{R}$ we define the $\dot{H}^s$ semi-norm restricted to Ω by*

$$|f|_{\dot{H}^s(\Omega)}^2 = \iint_{\Omega\times\Omega} \frac{|f(x) - f(y)|^2}{|x - y|^{d+2s}} \,\mathrm{d}x\mathrm{d}y. \tag{6.13}$$

and the $H^s(\Omega)$ norm by

$$\|f\|_{H^s(\Omega)}^2 = \|f\|_{L^2}^2 + |f|_{\dot{H}^s(\Omega)}^2. \tag{6.14}$$

An alternative way to define an H^s space on a bounded set is to take restrictions of H^s functions on the whole space. These two approaches are equivalent, and this is the focus of the next lemma.

Lemma 6.11. *Given a domain Ω with a smooth boundary and $f : \Omega \to \mathbf{R}$, the following are equivalent.*

1. $|f|_{\dot{H}^s(\Omega)} < \infty$
2. *There exists $\overline{f} : \mathbf{R}^d \to \mathbf{R}$ such that $\overline{f}|_\Omega = f$ and $|\overline{f}|_{\dot{H}^s} < \infty$.*

Proof. See [44, Theorem 1.7]. ☐

Next, we introduce the extension operator for H^s functions.

Lemma 6.12 (Extensions)**.** *Let $\Omega \subset \mathbf{R}^d$ be a compact domain with a smooth boundary. There exists a linear operator from $H^s(\Omega)$ to $\dot{H}^s(\mathbf{R}^d)$, which takes a function $f \in H^s(\Omega)$ to a function $\overline{f} \in \dot{H}^s(\mathbf{R}^d)$ that satisfies*

1. $\overline{f}|_\Omega = f$ *and*
2. $\|\overline{f}\|_{H^s} \le C\|f\|_{H^s(\Omega)}$,

where C depends on Ω but does not depend on f.

Proof. See [44, Theorem 1.105]. ☐

We also introduce the $H^{-s}(\Omega)$ norm, analogous to the $H^{-1}(\Omega)$ norm introduced in [32].

Definition 6.13. *Given a bounded open set Ω, a Radon measure μ on Ω, we define the $H^{-s}(\Omega)$ norm as*

$$\|\mu\|_{H^{-s}(\Omega)} = \sup \frac{\int_\Omega \phi d\mu}{\|\phi\|_{H^s(\Omega)}}. \tag{6.15}$$

A key point in this paper is that the H^s and $C^{0,\alpha}$ norms can be easily compared in bounded sets.

Remark 6.14. Note that, for any $s < \alpha$ and any bounded set $\Omega \subset \mathbf{R}^d$, there exists $C > 0$ such that for any $f : \Omega \to \mathbf{R}^d$

$$|f|_{\dot{H}^s} \le C|f|_{\dot{C}^{0,\alpha}}, \tag{6.16}$$

which implies that for any Radon measure μ on Ω,

$$\|\mu\|_{C_*^{0,\alpha}} \le C\|\mu\|_{H^{-s}(\Omega)}. \tag{6.17}$$

We end this section with a lemma that relates the H^{-s} norm of a measure to its electric energy.

Lemma 6.15. *Let g be the Riesz kernel of order s (Equation (1.6)) or the Coulomb kernel (in which case we take $s = 1$), and let μ be a Radon measure on $\mathbf{R}^d$. Then*

$$\mathcal{E}(\mu) = \|\mu\|^2_{\dot{H}^{-s}} = \left(\sup_{\phi \in C_0^\infty} \frac{\int_{\mathbf{R}^d} \phi \, d\mu}{|\phi|_{\dot{H}^s_F}} \right)^2. \tag{6.18}$$

Proof. It is clear that

$$\mathcal{E}(\mu) = \|\mu\|^2_{\dot{H}^{-s}}. \tag{6.19}$$

By taking as a test functions a sequence of smooth functions with compact support that converge to h^μ, we also have that

$$\sup_{\phi \in C_0^\infty} \frac{\int_{\mathbf{R}^d} \phi \, d\mu}{|\phi|_{\dot{H}^s_F}} \geq \frac{\int_{\mathbf{R}^d} h^\mu \, d\mu}{|h^\mu|_{\dot{H}^s_F}} = \|\mu\|_{\dot{H}^{-s}}. \tag{6.20}$$

Hence, we will prove that

$$\sup_{\phi \in C_0^\infty} \frac{\int_{\mathbf{R}^d} \phi \, d\mu}{|\phi|_{\dot{H}^s_F}} \leq \|\mu\|_{\dot{H}^{-s}}. \tag{6.21}$$

This is because, for any $\phi \in C_0^\infty$, we have that

$$\begin{aligned} \int_{\mathbf{R}^d} \phi \, d\mu &= \int_{\mathbf{R}^d} \widehat{\phi} \widehat{\mu} \, d\xi \\ &= \int_{\mathbf{R}^d} |\xi|^s \widehat{\phi} \frac{1}{|\xi|^s} \widehat{\mu} \, d\xi \\ &\leq \||\xi|^s \widehat{\phi}\|_{L^2} \||\xi|^{-s} \widehat{\mu}\|_{L^2} \\ &= |\phi|_{\dot{H}^s_F} \|\mu\|_{\dot{H}^{-s}}. \end{aligned} \tag{6.22}$$

From this, we can conclude. □

7 Proof of statements about the thermal equilibrium measure

This section is devoted to the proof of Proposition 3.1 as a consequence of a series of lemmas.

Lemma 7.1. *Let g be a Riesz-type kernel of order s. Then, the assignation $\mu \mapsto \mu * g$ can be extended to an invertible operator $U : L^2(\mathbf{R}^d) \to \dot{H}^{2s}(\mathbf{R}^d)$. Moreover, if $\mathfrak{D} = U^{-1}$,*

$$\mathfrak{D}(f)(x) = \frac{1}{2} \int_{\mathbf{R}^d} \left(-2f(x) + f(x+y) + f(x-y) \right) h(y) \, \mathrm{d}y, \tag{7.1}$$

where h satisfies $\widehat{h} = 1/\widehat{g}$.

Remark 7.2. Note that the integral in (7.1) is well defined for any $f \in H^{2s}$ because of item 10 in the definition of Riesz-type kernel, Definition 2.4.

Proof. First notice that the inverse is a well defined map on from $\dot{H}^{2s}$ to L^2, since $\|h^\mu\|_{L^2}$ is equivalent to $\|\mu\|_{H^{-2s}}$ by item 8 of Definition 2.4. The we take the Fourier transform

$$\widehat{\mathfrak{D}(f)}(\xi) = \frac{1}{2} \int_{\mathbf{R}^d} \left(-2 + e^{i\xi \cdot y} + e^{-i\xi \cdot y} \right) h(y) \widehat{f}(\xi) \, \mathrm{d}y. \tag{7.2}$$

We now need to show that

$$F(\xi) := \frac{1}{2}\int_{\mathbf{R}^d}\left(-2+e^{i\xi\cdot y}+e^{-i\xi\cdot y}\right)h(y)\,\mathrm{d}y \\ = \frac{1}{\widehat{g}(\xi)}. \tag{7.3}$$

To prove this, we apply Plancherel's Theorem,

$$F(\xi) = \frac{1}{2}\langle -2\delta_0+\delta_\xi+\delta_{-\xi},\widehat{h}\rangle. \tag{7.4}$$

Since $\widehat{h}$ has a well-defined, finite value at 0, we have that

$$\langle \delta_0,\widehat{h}\rangle = 0. \tag{7.5}$$

On the other hand, since g is even (item 2 of definition 2.4), $\widehat{h}$ is even, and therefore

$$\frac{1}{2}\langle \delta_\xi+\delta_{-\xi},\widehat{h}\rangle = \widehat{h}(\xi). \tag{7.6}$$

From this we can conclude. □

Corollary 7.3. *As a consequence of the previous lemma, the operator $\mathfrak{D}$ satisfies a monotonicity property, i.e. if x is a maximum of f, then $\mathfrak{D}f(x)\le 0$. Furthermore, if the inequality is not strict, then f is constant almosy everywhere.*

After having access to a monotonicity property, we can now follow the same strategy as in [3] to prove Proposition 3.1. We begin with an analogue of [3, Lemma 3.2].

Lemma 7.4. *Let*

$$m_\theta = \sup_{\mathbf{R}^d}\mu_\theta. \tag{7.7}$$

Then

$$-\frac{\log m_\theta}{\theta} \le h^{\mu_\theta} - c_\theta - (h^{\mu_\infty}-c_\infty). \tag{7.8}$$

Proof. We first show that we have

$$\liminf_{|x|\to\infty}\left(h^{\mu_\theta}+c_\infty-c_\theta+\frac{\log m_\theta}{\theta}\right)\ge 0, \tag{7.9}$$

which, since h^{μ_θ} is zero at infinity as seen in (6.3), is equivalent to showing that

$$c_\infty - c_\theta + \frac{\log m_\theta}{\theta} \ge 0. \tag{7.10}$$

To prove this claim, we proceed by contradiction and assume that

$$c_\infty - c_\theta + \frac{\log m_\theta}{\theta} < 0. \tag{7.11}$$

We recall that Σ is the support of μ_∞ and define ψ as the unique function satisfying that

$$\psi|_\Sigma \equiv 0$$

$$(\mathfrak{D}\psi)|_{\mathbf{R}^d\setminus\Sigma} \equiv 0$$

$$\lim_{x\to\infty}\psi(x) = c_\infty - c_\theta + \frac{\log m_\theta}{\theta}.$$

The existence of such a ψ follows from standard arguments: We let $\lambda = -c_\infty + c_\theta - \frac{\log m_\theta}{\theta}$ and define C_λ as the affine space of compactly supported smooth functions on $\mathbf{R}^d$ that are λ on Σ. We define H_λ as the closure of C_λ in the H^s norm, and the set $H_0 = H_\lambda - \lambda$. The function ψ is then a minimizer of $\mu \mapsto \int_{\mathbf{R}^d} \widehat{g}^{-1}|\widehat{\mu}|^2$ on the space H_0.

Since $c_\infty - c_\theta + \frac{\log m_\theta}{\theta} < 0$ by hypothesis, we have that $\psi - (c_\infty - c_\theta + \frac{\log m_\theta}{\theta})$ decays at infinity like the Green's function associated to the operator $\mathfrak{D}$, i.e., for x large enough, $\psi(x) - (c_\infty - c_\theta + \frac{\log m_\theta}{\theta})$ is bounded above and below by a constant multiple of $|x|^{2s-d}$ by item 5 of Definition 2.4.

On the other hand, we define

$$\varphi := h^{\mu_\theta} - h^{\mu_\infty} + c_\infty - c_\theta + \frac{\log m_\theta}{\theta}. \tag{7.12}$$

Note that μ_θ satisfies

$$h^{\mu_\theta} + V - c_\theta + \frac{\log m_\theta}{\theta} \geq 0, \tag{7.13}$$

and because of the First Order Condition for μ_∞ from Equation (6.1), we have that φ satisfies

$$\begin{aligned} \varphi \geq 0 \quad \text{in} \quad \Sigma \\ \mathfrak{D}(\varphi) \geq 0 \quad \text{in} \quad \mathbf{R}^d \setminus \Sigma. \end{aligned} \tag{7.14}$$

We then have that $\mathfrak{D}(\varphi - \psi) \geq 0$ on $\mathbf{R}^d \setminus \Sigma$, $\lim_{x\to\infty}(\varphi(x) - \psi(x)) = 0$, and $\varphi \geq \psi$ in Σ. Then by monotonicity $\varphi - \psi \geq 0$ in $\mathbf{R}^d \setminus \Sigma$. Furthermore, since

$$\int_{\mathbf{R}^d} \mathfrak{D}(\varphi) = \int_{\mathbf{R}^d} (\mu_\theta - \mu_\infty) = 0, \tag{7.15}$$

we also have that $\varphi - (c_\infty - c_\theta + \frac{\log m_\theta}{\theta})$ satisfies that, for x large enough, it is bounded above and below by a constant multiple of $|x|^{2s-d-1}$ by item 6 of Definition 2.4. This last statement, the fact that $\varphi - (c_\infty - c_\theta + \frac{\log m_\theta}{\theta})$ decays at infinity like $|x|^{2s-d}$ and the fact that $\varphi \geq \psi$ imply a contradiction. Therefore

$$\liminf_{x\to\infty} \varphi(x) \geq 0. \tag{7.16}$$

Hence, by Equation (7.14) and monotonicity, we deduce that $\varphi \geq 0$ in all of $\mathbf{R}^d$. This implies the desired result. □

We can now prove Proposition 3.1, restated here for convenience. The proof is analogous to the proof of Lemma 3.3 in [3].

Proposition 7.5. *Let (g, V) be an admissible pair of order s and let $\theta_0 > 0$. Then, there exists a constant $C_{\theta_0} > 0$ such that, for every $\theta > \theta_0$ and every $x \in \mathbf{R}^d$,*

$$\mu_\theta(x) \leq C_{\theta_0}. \tag{7.17}$$

Furthermore, there exists a constant $C > 0$ that does not depend on θ_0 such that

$$\mu_\theta(x) \leq \exp\left(C_{\theta_0} - \theta(V(x) - C)\right). \tag{7.18}$$

Proof. We start with Lemma 7.4, which along with the hypothesis that μ_∞ is bounded gives

$$h^{\mu_\theta} - c_\theta \geq -C - \frac{\log m_\theta}{\theta}, \tag{7.19}$$

for some $C > 0$ such that $h^{\mu_\infty} \geq -C$ (which depends only on g and V). Inserting into the First Order Condition for μ_θ (Equation (6.2)), we get that

$$\begin{aligned}\log \mu_\theta &= \theta(c_\theta - h^{\mu_\theta}) - \theta V \\ &\leq \theta C + \log m_\theta - \theta V.\end{aligned} \tag{7.20}$$

Since $V(x)$ tends to ∞ as x tends to ∞, we have that for every θ_0 there exists R_0 such that for every $\theta \geq \theta_0$ and $x \notin B(0, R_0)$ we have

$$\log \mu_\theta(x) \leq \log m_\theta - 1. \tag{7.21}$$

Therefore the supremum m_θ of μ_θ is achieved in $B(0, R_0)$ for every $\theta \geq \theta_0$.

Let $x_\theta \in B(0, R_0)$ be the maximizer of μ_θ. By monotonicity we have that $\mathfrak{D}(\log(\mu_\theta)) \geq 0$, and therefore by the First Order Condition for μ_θ (Equation (6.2)),

$$\mu_\theta(x_\theta) + \mathfrak{D}V(x_\theta) \leq 0. \tag{7.22}$$

Therefore

$$m_\theta \leq C_{\theta_0} := \sup_{x\in B(0,R_0)} |\mathfrak{D}V(x)|, \tag{7.23}$$

which implies that μ_θ is bounded independently of θ, and therefore Equation (7.17). Note that the hypothesis that $V \in C^2$, along with item 10 of a Riesz-type kernel (Definition 2.4) imply that $\sup_{x\in B(0,R)} |\mathfrak{D}V(x)| < \infty$. This uniform bound, together with (7.20) implies that

$$\log \mu_\theta \leq \theta C + \log C_{\theta_0} - \theta V, \tag{7.24}$$

which implies the exponential decay (7.18), which implies the uniform bound (7.17). □

8 Proof of transport inequalities

This section is devoted to proving Theorem 3.2. The proof will rely on the following localization inequality, which is an extension of an analogous proposition in [32] to the H^{-s} norm.

Proposition 8.1. *Let $\nu \in H^{-s}(\mathbf{R}^d)$ and assume that there exists a compact set Ω such that ν is nonpositive or nonnegative outside of Ω. Then there exists a compact set Ω_1 which contains Ω, and a constant C such that*

$$\| \nu \|_{H^{-s}} \geq C \| \nu|_{\Omega_1} \|_{H^{-s}(\Omega_1)} . \tag{8.1}$$

Furthermore, C and Ω_1 depend only on Ω.

The proof is found in Section 12. We now prove the first item of Theorem 3.2, restated here for convenience.

Theorem 8.2. *Let (g, V) be an admissible pair of order s and let $\alpha > s$.*

1. *Let $\Omega \subset \mathbf{R}^d$ be a compact set, then there exists a constant $C_{\alpha,\Omega} > 0$ which depends only on g, α and Ω such that, for any $\mu, \nu \in \mathcal{P}(\Omega)$.*

Proof. First, note that the $\dot{C}^{0,\alpha}_*$ norm can be rewritten as

$$\|\mu - \nu\|_{\dot{C}^{0,\alpha}_*} = \sup_{|\phi|_{\dot{C}^{0,\alpha}} \leq 1,\ \phi(0)=0} \int_{\mathbf{R}^d} \phi \, \mathrm{d}(\mu - \nu). \tag{8.2}$$

Since Ω is compact, there exists $R > 0$ such that

$$\Omega \subset B(0, R). \tag{8.3}$$

Hence,

$$\|\mu-\nu\|_{\dot{C}^{0,\alpha}_*} \le \sup_{|\phi|_{\dot{C}^{0,\alpha}}\le R,\ \|\phi\|_{L^\infty}\le R^\alpha} \int_{\mathbf{R}^d} \phi \, \mathrm{d}(\mu-\nu). \tag{8.4}$$

Note that if

$$|\phi|_{\dot{C}^{0,\alpha}} \le R,\ \|\phi\|_{L^\infty} \le R^\alpha \tag{8.5}$$

then

$$\|\phi\|_{H^s} \le C, \tag{8.6}$$

where C depends on Ω. Hence

$$\|\mu-\nu\|_{\dot{C}^{0,\alpha}_*} \le C\|\mu-\nu\|_{H^{-s}(\Omega)}, \tag{8.7}$$

where C depends on s, α, and Ω. Lastly, note that Lemma 6.12 implies that

$$\|\mu-\nu\|_{H^{-s}(\Omega)} \le C\|\mu-\nu\|_{H^{-s}}, \tag{8.8}$$

where C depends on s and Ω. Putting everything together, and noting that $\mathcal{E}(\cdot)$ is equivalent to $\|\cdot\|^2_{\dot{H}^{-s}}$ by property 8 of Riesz-type gases (Definition 2.4) we have that

$$\|\mu-\nu\|^2_{\dot{C}^{0,\alpha}_*} \le C_{\alpha,\Omega}\mathcal{E}(\mu-\nu), \tag{8.9}$$

where C depends on s, α, and Ω. □

We proceed to prove the second item of Theorem 3.2, restated here for convenience.

Theorem 8.3. *Let (g, V) be an admissible pair of order s and let $\alpha > s$.*

2. *There exists a constant $C_\alpha > 0$ which depends only on V, g and α such that, for every $\mu \in \mathcal{P}(\mathbf{R}^d)$,*

$$\|\mu-\mu_\infty\|^2_{C^{0,\alpha}_*} \le C_\alpha \left(\mathcal{E}_V(\mu) - \mathcal{E}_V(\mu_\infty)\right). \tag{8.10}$$

Proof. Since $\mu - \mu_\infty$ is nonnegative outside of Σ, by Proposition 8.1, there exists Ω containing Σ such that

$$\| \mu-\mu_\infty \|_{H^{-s}} \ge C \parallel (\mu-\mu_\infty)\mathbf{1}_\Omega \parallel_{H^{-s}(\Omega)}, \tag{8.11}$$

where C and Ω depends only on V and s. We may assume without loss of generality that, in addition,

$$V - 2h^{\mu_\infty} \ge 1 \tag{8.12}$$

outside of Ω by item 2 of Definition 2.7. We then have that

$$\begin{aligned} \|\mu-\mu_\infty\|^2_{C^{0,\alpha}_*} &\le 2\left(\|\mu\mathbf{1}_\Omega - \mu_\infty\|^2_{C^{0,\alpha}_*} + \|\mu\mathbf{1}_{\mathbf{R}^d\setminus\Omega}\|^2_{C^{0,\alpha}_*}\right) \\ &\le C\left(\|\mu\mathbf{1}_\Omega - \mu_\infty\|^2_{H^{-s}(\Omega)} + \mu\left(\mathbf{R}^d\setminus\Omega\right)\right), \end{aligned} \tag{8.13}$$

where C depends on s, α, and Ω. By (8.11) and Property 8 of Riesz-type gases (Definition 2.4), we have that

$$\|\mu\mathbf{1}_\Omega - \mu_\infty\|^2_{H^{-s}(\Omega)} \le C\mathcal{E}\left(\mu-\mu_\infty\right), \tag{8.14}$$

where C depends on s, Ω, and g. On the other hand, by (8.12) we have that

$$\mu\left(\mathbf{R}^d\setminus\Omega\right) \le \int_{\mathbf{R}^d} \zeta_\infty d\mu, \tag{8.15}$$

where

$$\zeta_\infty = V - 2h^{\mu_\infty}. \tag{8.16}$$

Putting everything together, and using the splitting formula (6.4), we have that

$$\begin{aligned}\|\mu-\mu_\infty\|^2_{C^{0,\alpha}_*} &\le C_\alpha\left(\mathcal{E}\left(\mu-\mu_\infty\right)+\int_{\mathbf{R}^d}\zeta_\infty d\mu\right)\\ &= C_\alpha\left(\mathcal{E}_V\left(\mu\right)-\mathcal{E}_V\left(\mu_\infty\right)\right),\end{aligned} \tag{8.17}$$

where C_α depends only on V, s, α, and g. □

We now prove the last item of Theorem 3.2, restated here for convenience.

Theorem 8.4. *Let (g, V) be an admissible pair of order s and let $\alpha > s$.*

3. Assume that

$$\liminf_{x\to\infty}\frac{V(x)}{|x|^{2\alpha}}>0. \tag{8.18}$$

Then there exists a constant $C_\alpha > 0$ which depends only on V, g and α such that, for every $\mu \in \mathcal{P}(\mathbf{R}^d)$, we have

$$\|\mu-\mu_\infty\|^2_{\dot{C}^{0,\alpha}_*}\le C_\alpha\left(\mathcal{E}_V(\mu)-\mathcal{E}_V(\mu_\infty)\right). \tag{8.19}$$

Proof. Since $\mu - \mu_\infty$ is nonnegative outside of Σ, by Proposition 8.1, there exists Ω containing Σ such that

$$\|\ \mu-\mu_\infty\ \|_{H^{-s}}\ge C\ \|\ (\mu-\mu_\infty)\mathbf{1}_\Omega\ \|_{H^{-s}(\Omega)}, \tag{8.20}$$

where C and Ω depends on s and V. Let

$$\gamma:=\liminf_{x\to\infty}\frac{V(x)}{|x|^{2\alpha}}. \tag{8.21}$$

We may also chose Ω such that

$$\zeta_\infty(x)\ge\gamma\left(|x|^\alpha-1\right)^2 \tag{8.22}$$

for x outside of Ω. We then have that

$$\begin{aligned}\|\mu-\mu_\infty\|^2_{\dot{C}^{0,\alpha}_*} &= \left(\sup_{|\phi|_{\dot{C}^{0,\alpha}}\le 1}\int_{\mathbf{R}^d}\phi\, d(\mu-\mu_\infty)\right)^2\\ &= \left(\sup_{|\phi|_{\dot{C}^{0,\alpha}}\le 1,\ \phi(0)=-1}\int_{\mathbf{R}^d}\phi\, d(\mu-\mu_\infty)\right)^2\\ &\le 2\left(\sup_{|\phi|_{\dot{C}^{0,\alpha}}\le 1,\ \phi(0)=-1}\int_{\Omega}\phi\, d(\mu-\mu_\infty)\right)^2+2\left(\sup_{|\phi|_{\dot{C}^{0,\alpha}}\le 1,\ \phi(0)=-1}\int_{\mathbf{R}^d\setminus\Omega}\phi\, d\mu\right)^2.\end{aligned} \tag{8.23}$$

We will deal with the first term now. Since Ω is compact, there exists $R > 0$ such that

$$\Omega\subset B(0,R). \tag{8.24}$$

Then by Proposition 8.1, we have that

$$\begin{aligned}\sup_{|\phi|_{\dot{C}^{0,\alpha}}\le 1,\ \phi(0)=-1}\int_{\Omega}\phi\, d(\mu-\mu_\infty) &\le \sup_{|\phi|_{\dot{C}^{0,\alpha}}\le 1,\ \|\phi\|_{L^\infty}\le 1+R}\int_{\Omega}\phi\, d(\mu-\mu_\infty)\\ &\le C\sup_{|\phi|_{\dot{H}^s}\le 1,\ \|\phi\|_{L^2}\le 1}\int_{\Omega}\phi\, d(\mu-\mu_\infty)\\ &\le C\|(\mu-\mu_\infty)\mathbf{1}_\Omega\|_{H^{-s}(\Omega)}\\ &\le C\sqrt{\mathcal{E}\left(\mu-\mu_\infty\right)},\end{aligned} \tag{8.25}$$

where C depends only on Ω, α, s, and g. We now deal with the second term in the last line of Equation (8.23). It is easy to see that the supremum is achieved at $\phi = \phi^*$, where

$$\phi^* := |x|^\alpha - 1. \tag{8.26}$$

Using Jensen's inequality, and the fact that $\mu\left(\mathbf{R}^d \setminus \Omega\right) \leq 1$, we have that

$$\begin{aligned}\left(\sup_{|\phi|_{\dot{C}^{0,\alpha}} \leq 1,\ \phi(0)=-1} \int_{\mathbf{R}^d\setminus\Omega} \phi\, d\mu\right)^2 &= \left(\int_{\mathbf{R}^d\setminus\Omega} \phi^*\, d\mu\right)^2 \\ &\leq \int_{\mathbf{R}^d\setminus\Omega} \left(\phi^*\right)^2\, d\mu.\end{aligned} \tag{8.27}$$

Using (8.22), we have that

$$\begin{aligned}\left(\sup_{|\phi|_{\dot{C}^{0,\alpha}} \leq 1,\ \phi(0)=-1} \int_{\mathbf{R}^d\setminus\Omega} \phi\, d\mu\right)^2 &\leq \int_{\mathbf{R}^d\setminus\Omega} \left(\phi^*\right)^2\, d\mu \\ &\leq \frac{1}{\gamma}\int_{\mathbf{R}^d} \zeta_\infty\, d\mu.\end{aligned} \tag{8.28}$$

Putting everything together, and using the splitting formula (6.4), we have that

$$\begin{aligned}\|\mu - \mu_\infty\|^2_{\dot{C}^{0,\alpha}_*} &\leq C_\alpha\left(\mathcal{E}\left(\mu - \mu_\infty\right) + \int_{\mathbf{R}^d} \zeta_\infty d\mu\right) \\ &= C_\alpha\left(\mathcal{E}_V\left(\mu\right) - \mathcal{E}_V\left(\mu_\infty\right)\right),\end{aligned} \tag{8.29}$$

where C_α depends on V, α, s, and g. □

Remark 8.5. The same proof could be applied to the Coulomb kernel instead of a Riesz-type kernel, thus recovering Theorems 1.1 and 1.2 of [12].

We now apply Theorem 3.2 to prove Corollary 3.3, restated here.

Corollary 8.6. *Let (g, V) be an admissible pair of order s. Then, the function $\theta \mapsto \mathrm{ent}[\mu_\theta]$ is non-decreasing. Moreover, if $\alpha > s$, there exists a constant $C_\alpha > 0$, which depends only on V, g, and α such that*

$$\|\mu_\infty - \mu_\theta\|^2_{C^{0,\alpha}_*} \leq \frac{C_\alpha}{\theta}\left|\mathrm{ent}[\mu_\infty] - \mathrm{ent}[\mu_\theta]\right|. \tag{8.30}$$

Additionally, if

$$\liminf_{x\to\infty} \frac{V(x)}{|x|^{2\alpha}} > 0 \tag{8.31}$$

then there exits a constant C_α, which depends only on V, g, and α such that

$$\|\mu_\infty - \mu_\theta\|^2_{\dot{C}^{0,\alpha}_*} \leq \frac{C_\alpha}{\theta}\left|\mathrm{ent}[\mu_\infty] - \mathrm{ent}[\mu_\theta]\right|. \tag{8.32}$$

Proof. Let $\theta_1 < \theta_2$ in $(0, \infty]$. We would like to show that $\mathrm{ent}[\mu_{\theta_1}] \leq \mathrm{ent}[\mu_{\theta_2}]$. Since μ_{θ_1} is the minimizer of $\mathcal{E}_{\theta_1}$ and μ_{θ_2} is the minimizer of $\mathcal{E}_{\theta_2}$ we have the two inequalities $\mathcal{E}_{\theta_1}(\mu_{\theta_1}) \leq \mathcal{E}_{\theta_1}(\mu_{\theta_2})$ and $\mathcal{E}_{\theta_2}(\mu_{\theta_2}) \leq \mathcal{E}_{\theta_2}(\mu_{\theta_1})$ that written more explicitly give

$$\begin{aligned}\mathcal{E}_V(\mu_{\theta_1}) + \frac{1}{\theta_1}\mathrm{ent}[\mu_{\theta_1}] &\leq \mathcal{E}_V(\mu_{\theta_2}) + \frac{1}{\theta_1}\mathrm{ent}[\mu_{\theta_2}], \\ \mathcal{E}_V(\mu_{\theta_2}) + \frac{1}{\theta_2}\mathrm{ent}[\mu_{\theta_2}] &\leq \mathcal{E}_V(\mu_{\theta_1}) + \frac{1}{\theta_2}\mathrm{ent}[\mu_{\theta_1}].\end{aligned}$$

Here we use the convention $\frac{1}{\infty}\mathrm{ent} = 0$. We may add both inequalities, cancel out the terms $\mathcal{E}_V(\mu_{\theta_1})$ and $\mathcal{E}_V(\mu_{\theta_2})$, and rearrange the entropies to obtain

$$\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)\mathrm{ent}[\mu_{\theta_1}] \leq \left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)\mathrm{ent}[\mu_{\theta_2}].$$

Since $1/\theta_2 < 1/\theta_1$ we may cancel the positive coefficient $1/\theta_1 - 1/\theta_2$ and obtain what we wanted. By rearranging terms, $\mathcal{E}_\theta(\mu_\theta) \leq \mathcal{E}_\theta(\mu_\infty)$ gives us that

$$\mathcal{E}_V(\mu_\theta) \leq \mathcal{E}_V(\mu_\infty) + \frac{1}{\theta}(\mathrm{ent}[\mu_\infty] - \mathrm{ent}[\mu_\theta])$$

and the conclusion follows from items 2 and 3 of Theorem 3.2. □

9 Proof of concentration inequalities

In this section, we will prove Theorem 3.4. The proof will be a consequence of Proposition 8.1, as well as some preliminary results which we now state.

Proposition 9.1. *Let g be a Riesz-type kernel of order s, let μ have compact support and let $\alpha > s$. Then there exists a constant $C > 0$, which depends on $V, g, \operatorname{supp}\mu$, and α such that*

$$\|\mathrm{emp}_N - \mu\|_{C_*^{0,\alpha}} \leq N^{-\frac{\alpha}{d}} + C\left(\mathrm{F}_N(X_N, \mu) + C\|\mu\|_{L^\infty} N^{-\frac{2s}{d}}\right)^{\frac{1}{2}}. \tag{9.1}$$

The proof is found in Section 13. We need one more lemma.

Lemma 9.2. *Let $\mu, \nu \in \mathcal{P}(\mathbf{R}^d)$ and let $\Omega \subset \mathbf{R}^d$ be such that supp$(\mu) \subset \Omega$. Then for any $\alpha \in (0, 1]$,*

$$\|(\mu - \nu)\mathbf{1}_\Omega\|_{C_*^{0,\alpha}} \geq \frac{1}{2}\|\mu - \nu\|_{C_*^{0,\alpha}}. \tag{9.2}$$

Proof. Proceed by contradiction and assume that

$$\|(\mu - \nu)\mathbf{1}_\Omega\|_{C_*^{0,\alpha}} < \frac{1}{2}\|\mu - \nu\|_{C_*^{0,\alpha}}. \tag{9.3}$$

Then by triangle inequality, we have that

$$\|(\mu - \nu)\mathbf{1}_{\mathbf{R}^d \setminus \Omega}\|_{C_*^{0,\alpha}} \geq \frac{1}{2}\|\mu - \nu\|_{C_*^{0,\alpha}}. \tag{9.4}$$

However, it may be seen that

$$\|(\mu - \nu)\mathbf{1}_{\mathbf{R}^d \setminus \Omega}\|_{C_*^{0,\alpha}} = \nu\left(\mathbf{R}^d \setminus \Omega\right). \tag{9.5}$$

This implies that

$$\nu(\Omega) \leq 1 - \frac{1}{2}\|\mu - \nu\|_{C_*^{0,\alpha}}, \tag{9.6}$$

and therefore

$$\begin{aligned} \|(\mu - \nu)\mathbf{1}_\Omega\|_{C_*^{0,\alpha}} &\geq |\mu(\Omega) - \nu(\Omega)| \\ &\geq \frac{1}{2}\|\mu - \nu\|_{C_*^{0,\alpha}}. \end{aligned} \tag{9.7}$$

This is a contradiction and, therefore,

$$\|(\mu - \nu)\mathbf{1}_\Omega\|_{C_*^{0,\alpha}} \geq \frac{1}{2}\|\mu - \nu\|_{C_*^{0,\alpha}}. \tag{9.8}$$

□

We now prove the first item of Theorem 3.4, restated here.

Theorem 9.3. *Assume (g, V) is an admissible pair of order s and let $\alpha > s$. Assume that $N\beta \to \infty$ and let $r > 0$. Then*

1. *There exists a constant C, which depends only on V, g, and α such that*

$$\begin{aligned}&\mathbf{P}_{N,\beta}\left(\|\mathrm{emp}_N - \mu_\infty\|_{C_*^{0,\alpha}} > r\right) \leq \\ &\exp\Big\{ - N^2\beta\big(C[r - N^{-\frac{\alpha}{d}}]_+^2 - C\|\mu_\infty\|_{L^\infty} N^{-\frac{2s}{d}}\big) + \\ &\quad N\big(\log|\Sigma| + \mathrm{ent}[\mu_\infty] + \beta\mathcal{E}(\mu_\infty)\big) + o(N)\Big\},\end{aligned} \tag{9.9}$$

where $(\cdot)_+$ denotes the positive-part function.

Proof. With the help of Lemmas 6.5 and 6.3 we can write

$$\begin{aligned}&\mathbf{P}_{N,\beta}\left(\|\mathrm{emp}_N - \mu_\infty\|_{C_*^{0,\alpha}} > r\right) = \\ &\frac{1}{Z_{N,\beta}} \int_{\|\mathrm{emp}_N - \mu_\infty\|_{C_*^{0,\alpha}} > r} \exp\left(-\beta\mathcal{H}_N(X_N)\right) \mathrm{d}X_N = \\ &\frac{1}{Z_{N,\beta}} \int_{\|\mathrm{emp}_N - \mu_\infty\|_{C_*^{0,\alpha}} > r} \exp\left(-N^2\beta\left(\mathcal{E}_V(\mu_\infty) + \mathrm{F}_N(X_N, \mu_\infty) + \frac{1}{N}\sum_{i=1}^N \zeta_\infty(x_i)\right)\right) \mathrm{d}X_N \leq \\ &\left(\int_{\mathbf{R}^d} \exp\left(-N\beta\zeta_\infty(x)\right) \mathrm{d}x\right)^N \times \\ &\quad \max_{\|\mathrm{emp}_N - \mu_\infty\|_{C_*^{0,\alpha}} > r} \exp\left(-N^2\beta\mathrm{F}_N(X_N, \mu_\infty) - N\beta\mathcal{E}(\mu_\infty) + N\mathrm{ent}[\mu_\infty]\right).\end{aligned} \tag{9.10}$$

The hypothesis $N\beta \to \infty$, implies that

$$\left(\int_{\mathbf{R}^d} \exp\left(-N\beta\zeta_\infty(x_i)\right) \mathrm{d}X_N\right)^N = \exp\left(N\log|\Sigma| + o(N)\right), \tag{9.11}$$

since $N\beta\zeta_\infty(x_i) = 0$ for $x_i \in \Sigma$ and $N\beta\zeta_\infty(x_i) \to \infty$ for $x_i \notin \Sigma$. On the other hand, by Proposition 9.1, there exists a constant C depending only on V such that

$$\begin{aligned}&\max_{\|\mathrm{emp}_N - \mu_\infty\|_{C_*^{0,\alpha}} > r} \exp\left(-N^2\beta\mathrm{F}_N(X_N, \mu_\infty) - N\beta\mathcal{E}(\mu_\infty) + N\mathrm{ent}[\mu_\infty]\right) = \\ &\exp\left(-N^2\beta \min_{\|\mathrm{emp}_N - \mu_\infty\|_{C_*^{0,\alpha}} > r} \{\mathrm{F}_N(X_N, \mu_\infty)\} - N\beta\mathcal{E}(\mu_\infty) + N\mathrm{ent}[\mu_\infty]\right) \leq \\ &\exp\left(-N^2\beta\left(C[r - N^{-\frac{\alpha}{d}}]_+^2 - C\|\mu_\infty\|_{L^\infty} N^{-\frac{2s}{d}}\right) - N\beta\mathcal{E}(\mu_\infty) + N\mathrm{ent}[\mu_\infty]\right).\end{aligned} \tag{9.12}$$

Putting everything together, we have that

$$\begin{aligned}&\mathbf{P}_{N,\beta}\left(\|\mathrm{emp}_N - \mu_\infty\|_{C_*^{0,\alpha}} > r\right) \leq \\ &\exp\left(-N^2\beta\left(C[r - N^{-\frac{\alpha}{d}}]_+^2 - C\|\mu_\infty\|_{L^\infty} N^{-\frac{2s}{d}}\right) + N\left(\log|\Sigma| + \mathrm{ent}[\mu_\infty] + \beta\mathcal{E}(\mu_\infty)\right) + o(N)\right).\end{aligned} \tag{9.13}$$

□

We proceed to the proof of the last item of Theorem 3.4, which we restate here.

Theorem 9.4. *Assume (g, V) is an admissible pair of order s and let $\alpha > s$. Assume that $N\beta \to \infty$ and let $r > 0$.*

2. *Set $\theta = N\beta$, then there exists a constant C, which depends only on V, g, and α such that*

$$\mathbf{P}_{N,\beta}\left(\|\mathrm{emp}_N - \mu_\theta\|_{C_*^{0,\alpha}} > r\right) \leq \exp\left(-N^2\beta\left(C\left[r - CN^{-\frac{\alpha}{d}}\right]_+^2 - CN^{-\frac{2s}{d}}\right)\right). \quad (9.14)$$

Proof. We begin by using the splitting formula (6.7) and Lemma 6.7 to write

$$\begin{aligned}\mathbf{P}_{N,\beta}\left(\|\mathrm{emp}_N - \mu_\theta\|_{C_*^{0,\alpha}} > r\right) &= \frac{1}{K_{N,\beta}}\int_{\|\mathrm{emp}_N - \mu_\theta\|_{C_*^{0,\alpha}} > r} \exp\left(-N^2\beta \mathrm{F}_N(X_N, \mu_\theta)\right)\, d\mu_\theta(x_i) \\ &\leq \exp\left(-N^2\beta \min_{\|\mathrm{emp}_N - \mu_\theta\|_{C_*^{0,\alpha}} > r} \mathrm{F}_N(X_N, \mu_\theta)\right).\end{aligned} \quad (9.15)$$

Let Ω be a compact set such that for $x \notin \Omega$,

$$\mu_\theta(x) \leq C\exp\left(-N\beta V(x)\right), \quad (9.16)$$

for C depending only on V and g. Let

$$\overline{\mu}_\theta = \frac{\mu_\theta \mathbf{1}_\Omega}{\int_\Omega \mu_\theta\,\mathrm{d}x}. \quad (9.17)$$

By Proposition 9.1, since $\overline{\mu}_\theta$ is a probability measure with compact support

$$\|\mathrm{emp}_N - \overline{\mu}_\theta\|_{C_*^{0,\alpha}} \leq N^{-\frac{\alpha}{d}} + C\left(\mathrm{F}_N(X_N, \overline{\mu}_\theta) + CN^{-\frac{2s}{d}}\right)^{\frac{1}{2}}, \quad (9.18)$$

for some C depending only V and g. Using (9.16), we have

$$\|\mathrm{emp}_N - \mu_\theta\|_{C_*^{0,\alpha}} - \mathrm{err}_N \leq N^{-\frac{\alpha}{d}} + C\left(\mathrm{F}_N(X_N, \mu_\theta) + \mathrm{err}_N + CN^{-\frac{2s}{d}}\right)^{\frac{1}{2}}, \quad (9.19)$$

where

$$\mathrm{err}_N = o(N^p) \quad (9.20)$$

for every $p \in \mathbf{R}$. This implies that

$$\min_{\|\mathrm{emp}_N - \mu_\theta\|_{C_*^{0,\alpha}} > r} \mathrm{F}_N(X_N, \mu_\theta) \geq C\left[r + \mathrm{err}_N - CN^{-\frac{\alpha}{d}}\right]_+^2 + \mathrm{err}_N - CN^{-\frac{2s}{d}}, \quad (9.21)$$

and, therefore,

$$\mathbf{P}_{N,\beta}\left(\|\mathrm{emp}_N - \mu_\theta\|_{C_*^{0,\alpha}} > r\right) \leq \exp\left(-N^2\beta\left(C\left[r + \mathrm{err}_N - CN^{-\frac{\alpha}{d}}\right]_+^2 + \mathrm{err}_N - CN^{-\frac{2s}{d}}\right)\right), \quad (9.22)$$

which is equivalent to the desired result by absorbing the err_N term into the constants. □

10 Lower bound for the distance

In this section, we prove Proposition 3.6, which we restate here.

Proposition 10.1. *Let μ be a probability measure on $\mathbf{R}^d$ such that $\|\mu\|_{L^\infty} < L$ for some finite $L > 0$. Let ν_N be a probability measure on $\mathbf{R}^d$ such that*

$$\nu_N = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}.$$

Then

$$\|\mu - \nu_N\|_{C_*^{0,\alpha}} \geq CN^{-\frac{\alpha}{d}}, \quad (10.1)$$

where $C > 0$ only depends on α, d and L.

Proof. Let $\overline{X}_N = \bigcup_{i=1}^N \{x_i\}$. For $\lambda > 0$, to be determined later, define $\varphi_\lambda : \mathbf{R}^d \to \mathbf{R}$ as

$$\varphi_\lambda(x) = \left(\lambda^\alpha N^{-\frac{\alpha}{d}} - \left[\operatorname{dist}(x, \overline{X}_N)\right]^\alpha\right)_+ , \tag{10.2}$$

where $(\cdot)_+$ denotes the positive part of $(\cdot)$. Note that for every $\lambda > 0$, the function φ_λ satisfies

$$|\varphi_\lambda|_{\dot{C}^{0,\alpha}} = 1 \quad \text{and} \quad \operatorname{supp}(\varphi_\lambda) = \bigcup_{i=1}^N B\left(x_i, \lambda N^{-\frac{1}{d}}\right).$$

Our aim is now to prove that

$$\int_{\mathbf{R}^d} \varphi_\lambda \mathrm{d}\left(\nu_N - \mu\right) \geq k N^{-\frac{\alpha}{d}}, \tag{10.3}$$

for some $k \in \mathbf{R}^+$. In order to do this, we introduce the function

$$\widetilde{\mu}_N = L \sum_{i=1}^N \mathbf{1}_{B(x_i, \lambda N^{-\frac{1}{d}})}. \tag{10.4}$$

Note that

$$\int_{\mathbf{R}^d} \varphi_\lambda \mathrm{d}\left(\nu_N - \mu\right) \geq \int_{\mathbf{R}^d} \varphi_\lambda \mathrm{d}\left(\nu_N - \widetilde{\mu}_N\right), \tag{10.5}$$

since

$$\begin{aligned}\int_{\mathbf{R}^d} \varphi_\lambda \,\mathrm{d}\mu &\leq L \sum_{i=1}^N \int_{\mathbf{R}^d} \mathbf{1}_{B(x_i, \lambda N^{-\frac{1}{d}})} \varphi_\lambda \,\mathrm{d}x \\ &= \int_{\mathbf{R}^d} \varphi_\lambda \,\mathrm{d}\widetilde{\mu}_N.\end{aligned} \tag{10.6}$$

We now compute

$$\begin{aligned}\int_{\mathbf{R}^d} \varphi_\lambda \mathrm{d}\left(\nu_N - \widetilde{\mu}_N\right) &= \lambda^\alpha N^{-\frac{\alpha}{d}} - \int_{\mathbf{R}^d} \varphi_\lambda \,\mathrm{d}\widetilde{\mu}_N \\ &\geq \lambda^\alpha N^{-\frac{\alpha}{d}} - L\left(\sum_{i=1}^N \int_{B(x_i, \lambda N^{-\frac{1}{d}})} \lambda^\alpha N^{-\frac{\alpha}{d}} \,\mathrm{d}x\right) \\ &= \lambda^\alpha N^{-\frac{\alpha}{d}} - \lambda^\alpha L N^{-\frac{\alpha}{d}}\left(N k_d (\lambda N^{-\frac{1}{d}})^d\right) \\ &= \lambda^\alpha N^{-\frac{\alpha}{d}}\left(1 - L k_d \lambda^d\right),\end{aligned} \tag{10.7}$$

where k_d is the volume of the unit ball of $\mathbb{R}^d$. Taking

$$\lambda = \frac{1}{(2Lk_d)^{\frac{1}{d}}}, \tag{10.8}$$

we have that

$$\int_{\mathbf{R}^d} \varphi_\lambda \mathrm{d}\left(\nu_N - \widetilde{\mu}_N\right) \geq N^{-\frac{\alpha}{d}}\left(\frac{1}{2^{\frac{d+\alpha}{d}} (Lk_d)^{\frac{\alpha}{d}}}\right), \tag{10.9}$$

which implies (10.1). □

11 Bound on the Laplace transform

This section is devoted to proving Theorem 3.7. We shall denote $\theta = N\beta$. The proof will rely on the following concentration result from [32].

Theorem 11.1. *Let $d \geq 2$, assume that $\frac{1}{N} \ll \beta$ and take an admissible pair (g, V) of order s. Then there exists a constant C depending only on (g, V) such that for any $r > 0$, we have*

$$\mathbf{P}_{N,\beta}\left(\| \operatorname{emp}_N - \mu_\theta \|_{BL} \leq \frac{r}{N^{\frac{1}{d}}}\right) \geq 1 - \exp\left(-N^{2-\frac{2}{d}}\beta\left(C(r-C)_+^2 - C\right)\right), \tag{11.1}$$

We will also use a simple concentration inequality, which is an immediate consequence of the splitting formula (Proposition 6.3) and the fact that the log of the next-order partition function is positive (Lemma 6.7). The proof is omitted.

Lemma 11.2. *For any $r > 0$,*

$$\mathbf{P}_{N,\beta}(\mathrm{F}_N(X_N, \mu_\theta) > r) \leq \exp\left(-N^2\beta r\right). \tag{11.2}$$

We now prove Theorem 3.7, restated here,

Theorem 11.3. *Let (g, V) be an admissible pair of order s and assume $N\beta \to \infty$. Consider a continuous function $f : \mathbf{R}^d \to \mathbf{R}$ and define the random variable $\mathrm{Fluct}[f]$ by*

$$\mathrm{Fluct}[f] = \int_{\mathbf{R}^d} f \mathrm{d}\left(\operatorname{emp}_N - \mu_\theta\right). \tag{11.3}$$

1. *If g is the Coulomb kernel, there exists $C > 0$ depending only on V such that*
$$\log\left(\mathbf{E}_{\mathbf{P}_{N,\beta}} \exp\left(N^2\beta\left|t\mathrm{Fluct}[f]\right|\right)\right) \leq N^2\beta\left(Ct^2\|f\|_{W^{1,\infty}}^2 + N^{-\frac{2}{d}}C\right). \tag{11.4}$$
2. *If g is a Riesz-type kernel of order s then for any $\alpha > s$ there exists a constant C, which depends only on V, g, and α such that*
$$\log\left(\mathbf{E}_{\mathbf{P}_{N,\beta}} \exp\left(N^2\beta\left|t\mathrm{Fluct}[f]\right|\right)\right) \leq N^2\beta\left(Ct^2\|f\|_{C^{0,\alpha}}^2 + N^{-\frac{2\alpha}{d}}C\right). \tag{11.5}$$
3. *If g is the Coulomb kernel, $i \in \{0, 1\}$ and $\alpha \in (0, 1)$ then there exists a constant C, which depends only on V such that*
$$\log\left(\mathbf{E}_{\mathbf{P}_{N,\beta}} \exp\left(N^2\beta\left|t\mathrm{Fluct}[f]\right|\right)\right) \leq$$
$$\beta N^2\left(\frac{1}{4}t^2|f|_{\dot{H}^1}^2 + N^{-\frac{i+\alpha}{d}}t|f|_{\dot{C}^{i,\alpha}} + CN^{-\frac{2}{d}}\right) + \frac{d}{2}\left(\log(N^2\beta) + \log(1 + t|f|_{\dot{H}^1})\right). \tag{11.6}$$
4. *If g is a Riesz-type kernel of order s, $i \in \{0, 1\}$ and $\alpha \in (0, 1)$, then there exists a constant C depending only on V and g such that*
$$\log\left(\mathbf{E}_{\mathbf{P}_{N,\beta}} \exp\left(N^2\beta\left|t\mathrm{Fluct}[f]\right|\right)\right) \leq \beta N^2\left(Ct^2|f|_{\dot{H}^s}^2 + CN^{-\frac{i+\alpha}{d}}t|f|_{\dot{C}^{i,\alpha}} + CN^{-\frac{2s}{d}}\right). \tag{11.7}$$

Proof. **Step 1:** Proof of item 1.

We start with the case of g given by the Coulomb kernel. Since $\mathbf{E}_{\mathbf{P}_{N,\beta}} \exp\left(N^2\beta\left|t\mathrm{Fluct}[f]\right|\right)$ is a non-negative random variable, we can rewrite the expectation as

$$\begin{aligned}
\mathbf{E}_{\mathbf{P}_{N,\beta}} \exp\left(N^2\beta\left|t\mathrm{Fluct}[f]\right|\right) &= \int_0^\infty \mathbf{P}_{N,\beta}\left(\exp\left(N^2\beta\left|t\mathrm{Fluct}[f]\right|\right) > x\right) \mathrm{d}x \\
&= \int_0^\infty \mathbf{P}_{N,\beta}\left(\left|t\mathrm{Fluct}[f]\right| > \frac{\log x}{N^2\beta}\right) \mathrm{d}x \\
&\leq \int_0^\infty \mathbf{P}_{N,\beta}\left(t\|f\|_{W^{1,\infty}}\|\operatorname{emp}_N - \mu_\theta\|_{\mathrm{BL}} > \frac{\log x}{N^2\beta}\right) \mathrm{d}x.
\end{aligned} \tag{11.8}$$

Note that if $x \le 1$, then

$$\mathbf{P}_{N,\beta}\left(t\|f\|_{W^{1,\infty}}\|\mathrm{emp}_N-\mu_\theta\|_{\mathrm{BL}}>\frac{\log x}{N^2\beta}\right)=1. \tag{11.9}$$

On the other hand, if $x>1$ then by Theorem 11.1

$$\mathbf{P}_{N,\beta}\left(t\|f\|_{W^{1,\infty}}\|\mathrm{emp}_N-\mu_\theta\|_{\mathrm{BL}}>\frac{\log x}{N^2\beta}\right)\le \exp\left(-N^2\beta\left[C\left(\frac{\log x}{N^2\beta t\|f\|_{W^{1,\infty}}}\right)^2-CN^{-\frac{2}{d}}\right]\right). \tag{11.10}$$

Therefore, we infer that

$$\begin{aligned}&\mathbf{E}_{\mathbf{P}_{N,\beta}}\exp\left(N^2\beta\,|t\mathrm{Fluct}[f]|\right)\\ \le& 1+\exp\left(CN^{2-\frac{2}{d}}\beta\right)\int_0^\infty \exp\left(-\left[C\frac{[\log x]^2}{N^2\beta t^2\|f\|^2_{W^{1,\infty}}}\right]\right)\,\mathrm{d}x.\end{aligned} \tag{11.11}$$

Performing the change of variables $y=\log x$ we can transform the integral into

$$\begin{aligned}&\mathbf{E}_{\mathbf{P}_{N,\beta}}\exp\left(N^2\beta\,|t\mathrm{Fluct}[f]|\right)\\ \le& 1+\exp\left(CN^{2-\frac{2}{d}}\beta\right)\int_{-\infty}^\infty \exp\left(-\left[C\frac{y^2}{N^2\beta t^2\|f\|^2_{W^{1,\infty}}}-y\right]\right)\,\mathrm{d}y.\end{aligned} \tag{11.12}$$

The last integral is a Gaussian density that can be computed exactly by completing squares,

$$\begin{aligned}&\int_{-\infty}^\infty \exp\left(-\left[C\frac{y^2}{N^2\beta t^2\|f\|^2_{W^{1,\infty}}}-y\right]\right)\,\mathrm{d}y\\ =&\int_{-\infty}^\infty \exp\left(-\frac{C}{N^2\beta t^2\|f\|^2_{W^{1,\infty}}}\left(y-\frac{N^2\beta t^2\|f\|^2_{W^{1,\infty}}}{2C}\right)^2+\frac{N^2\beta t^2\|f\|^2_{W^{1,\infty}}}{4C}\right)\,\mathrm{d}y\\ =&\exp\left(\frac{N^2\beta t^2\|f\|^2_{W^{1,\infty}}}{4C}\right)\left(\frac{N^2\beta t^2\|f\|^2_{W^{1,\infty}}}{2C}\right)^{\frac{d}{2}}\end{aligned} \tag{11.13}$$

We conclude that

$$\begin{aligned}&\log\left(\mathbf{E}_{\mathbf{P}_{N,\beta}}\exp\left(N^2\beta\,|t\mathrm{Fluct}[f]|\right)\right)\\ \le&\log\left(1+\exp\left(CN^{2-\frac{2}{d}}\beta\right)\exp\left(\frac{N^2\beta t^2\|f\|^2_{W^{1,\infty}}}{4C}\right)\left(\frac{N^2\beta t^2\|f\|^2_{W^{1,\infty}}}{2C}\right)^{\frac{d}{2}}\right)\\ \le& N^2\beta\left(Ct^2\|f\|^2_{W^{1,\infty}}+N^{-\frac{2}{d}}C\right).\end{aligned} \tag{11.14}$$

Step 2: Proof of item 2.
The proof of the statement in the case of Riesz-type kernels is the same, after noting that if g is a Riesz-type kernel of order s, then for any $\alpha>s$ there exist a constant C depending on V,d,g and α such that

$$\mathbf{P}_{N,\beta}\left(\|\,\mathrm{emp}_N-\mu_\theta\,\|_{C_*^{0,\alpha}}\ge r\right)\le \exp\left(-N^2\beta\left(Cr^2-CN^{-\frac{2\alpha}{d}}\right)\right). \tag{11.15}$$

Step 3: Proof of item 3.
We now turn to proving the third item. We introduce the notation

$$\mathrm{emp}_N^*=\mathrm{emp}_N*\delta_{N^{-\frac{1}{d}}}, \tag{11.16}$$

where $\delta_{N^{-\frac{1}{d}}}$ denotes the uniform probability measure on $B(0,N^{-\frac{1}{d}})$, and

$$\mathrm{Fluct}^*[f]=\int_{\mathbf{R}^d} f\,d(\mathrm{emp}_N^*-\mu_\theta). \tag{11.17}$$

Note that if $i = 0$, then

$$\left|\int_{\mathbf{R}^d} f\, d(\mathrm{emp}_N^* - \mathrm{emp}_N)\right| \le \frac{1}{N}\sum_{j=1}^N \left| f(x_j) - \fint_{B(0,N^{-\frac{1}{d}})} f(x_j) + (f(y) - f(x_j))\,\mathrm{d}y\right| \le N^{-\frac{\alpha}{d}} |f|_{\dot{C}^{0,\alpha}}. \tag{11.18}$$

On the other hand, for $i = 1$, we have

$$\left|\int_{\mathbf{R}^d} f\, d(\mathrm{emp}_N^* - \mathrm{emp}_N)\right| \le \frac{1}{N}\sum_{j=1}^N \left| f(x_j) - \fint_{B(0,N^{-\frac{1}{d}})} f(x_j) + \nabla f_{x_j}\cdot(y - x_j) - (f(y) - f(x_j) + \nabla f_{x_j}\cdot(y-x_j))\,\mathrm{d}y\right| \le N^{-\frac{1+\alpha}{d}} |f|_{\dot{C}^{1,\alpha}}. \tag{11.19}$$

In either case, we get that

$$|\mathrm{Fluct}^*[f] - \mathrm{Fluct}[f]| \le N^{-\frac{i+\alpha}{d}} |f|_{\dot{C}^{i,\alpha}}. \tag{11.20}$$

We now proceed as in steps 1 and 2, and write

$$\begin{aligned}
&\mathbf{E}_{\mathbf{P}_{N,\beta}} \exp\left(N^2\beta\, |t\mathrm{Fluct}[f]|\right)\\
&= \int_0^\infty \mathbf{P}_{N,\beta}\left(\exp\left(N^2\beta\,|t\mathrm{Fluct}[f]|\right) > x\right)\,\mathrm{d}x\\
&= \int_0^\infty \mathbf{P}_{N,\beta}\left(t\mathrm{Fluct}[f] > \frac{\log x}{N^2\beta}\right)\,\mathrm{d}x\\
&\le \int_0^\infty \mathbf{P}_{N,\beta}\left(t\mathrm{Fluct}^*[f] > \frac{\log x}{N^2\beta} - N^{-\frac{i+\alpha}{d}}|f|_{\dot{C}^{i,\alpha}}\right)\,\mathrm{d}x\\
&\le \int_0^\infty \mathbf{P}_{N,\beta}\left(t|f|_{\dot{H}^1}\|\mathrm{emp}_N^* - \mu_\theta\|_{H^{-1}} > \frac{\log x}{N^2\beta} - N^{-\frac{i+\alpha}{d}}t|f|_{\dot{C}^{i,\alpha}}\right)\mathrm{d}x\\
&= \int_0^\infty \mathbf{P}_{N,\beta}\left(\|\mathrm{emp}_N^* - \mu_\theta\|_{H^{-1}} > \frac{\frac{\log x}{N^2\beta} - N^{-\frac{i+\alpha}{d}}t|f|_{\dot{C}^{i,\alpha}}}{t|f|_{\dot{H}^1}}\right)\,\mathrm{d}x.
\end{aligned} \tag{11.21}$$

Note that if $\frac{\log x}{N^2\beta} - N^{-\frac{i+\alpha}{d}}t|f|_{\dot{C}^{i,\alpha}} < 0$, i.e. if $x < \exp\left(N^{2-\frac{i+\alpha}{d}}\beta t|f|_{\dot{C}^{i,\alpha}}\right)$, then

$$\mathbf{P}_{N,\beta}\left(\|\mathrm{emp}_N^* - \mu_\theta\|_{H^{-1}} > \frac{\frac{\log x}{N^2\beta} - N^{-\frac{i+\alpha}{d}}t|f|_{\dot{C}^{i,\alpha}}}{t|f|_{\dot{H}^1}}\right) = 1. \tag{11.22}$$

Otherwise, if $x \ge \exp\left(N^2\beta t|f|_{\dot{C}^{i,\alpha}}\right)$, then by Lemma 11.2,

$$\begin{aligned}
&\mathbf{P}_{N,\beta}\left(\|\mathrm{emp}_N^* - \mu_\theta\|_{H^{-1}} > \frac{\frac{\log x}{N^2\beta} - N^{-\frac{i+\alpha}{d}}t|f|_{\dot{C}^{i,\alpha}}}{t|f|_{\dot{H}^1}}\right)\\
=&\mathbf{P}_{N,\beta}\left(\|\mathrm{emp}_N^* - \mu_\theta\|_{H^{-1}}^2 > \left[\frac{\frac{\log x}{N^2\beta} - N^{-\frac{i+\alpha}{d}}t|f|_{\dot{C}^{i,\alpha}}}{t|f|_{\dot{H}^1}}\right]^2\right)\\
\le&\mathbf{P}_{N,\beta}\left(\mathrm{F}_N(X_N,\mu_\theta) > \left[\frac{\frac{\log x}{N^2\beta} - N^{-\frac{i+\alpha}{d}}t|f|_{\dot{C}^{i,\alpha}}}{t|f|_{\dot{H}^1}}\right]^2 - CN^{-\frac{2}{d}}\right)\\
\le&\exp\left(CN^{2-\frac{2}{d}}\beta\right)\exp\left(\left[\frac{\frac{\log x}{N^2\beta} - N^{-\frac{i+\alpha}{d}}t|f|_{\dot{C}^{i,\alpha}}}{t|f|_{\dot{H}^1}}\right]^2\right),
\end{aligned} \tag{11.23}$$

for some C depending only on V. Therefore,

$$\begin{aligned}&\mathbf{E}_{\mathbf{P}_{N,\beta}} \exp\left(N^2\beta\,|t\mathrm{Fluct}[f]|\right)\\ &\leq \exp\left(N^{2-\frac{i+\alpha}{d}}\beta t|f|_{\dot{C}^{i,\alpha}}\right) + \exp\left(CN^{2-\frac{2}{d}}\beta\right)\int_0^\infty \exp\left(\left[\frac{\frac{\log x}{N^2\beta} - N^{-\frac{i+\alpha}{d}}t|f|_{\dot{C}^{i,\alpha}}}{t|f|_{\dot{H}^1}}\right]^2\right)\,\mathrm{d}x.\end{aligned} \tag{11.24}$$

Proceeding as in the previous step, we can perform the change of variables $y = \log x$, then complete squares, and use the formula for the integral of a Gaussian density, and get

$$\begin{aligned}&\int_0^\infty \exp\left(\left[\frac{\frac{\log x}{N^2\beta} - N^{-\frac{i+\alpha}{d}}t|f|_{\dot{C}^{i,\alpha}}}{t|f|_{\dot{H}^1}}\right]^2\right)\,\mathrm{d}x = \\ &\left(\frac{N^2\beta t|f|_{\dot{H}^1}}{2}\right)^{\frac{d}{2}} \exp\left(N^2\beta\left[\frac{t^2|f|_{\dot{H}^1}^2}{4} + N^{-\frac{i+\alpha}{d}}t|f|_{\dot{C}^{i,\alpha}}\right]\right).\end{aligned} \tag{11.25}$$

Note that

$$\begin{aligned}&\max\Bigg\{N^{2-\frac{i+\alpha}{d}}\beta t|f|_{\dot{C}^{i,\alpha}},\\ &\log\left(\exp\left(CN^{2-\frac{2}{d}}\beta\right)\left(\frac{N^2\beta t|f|_{\dot{H}^1}}{2}\right)^{\frac{d}{2}}\exp\left(N^2\beta\left[\frac{t^2|f|_{\dot{H}^1}^2}{4} + N^{-\frac{i+\alpha}{d}}t|f|_{\dot{C}^{i,\alpha}}\right]\right)\right)\Bigg\} \leq \\ &CN^{2-\frac{2}{d}}\beta + \frac{d}{2}\left(\log(N^2\beta) + \log(1+t|f|_{\dot{H}^1})\right) + N^2\beta\left[\frac{t^2|f|_{\dot{H}^1}^2}{4} + N^{-\frac{i+\alpha}{d}}t|f|_{\dot{C}^{i,\alpha}}\right].\end{aligned} \tag{11.26}$$

From this we can conclude.

Step 4: Proof of item 4.

We now turn to proving the last item of the Theorem. The procedure is similar to the proof of the last item, the difference is that the smearing procedure goes through higher dimensions.

Let $\overline{f} : \mathbf{R}^{d+m} \to \mathbf{R}$ be such that $\overline{f}(x,0) = f(x)$ for every x and

$$\begin{aligned}|\overline{f}|_{\dot{H}^{s+\frac{m}{2}}} &\leq C|f|_{\dot{H}^s}\\ |\overline{f}|_{\dot{C}^{0,\alpha}} &\leq C|f|_{\dot{C}^{0,\alpha}},\end{aligned} \tag{11.27}$$

for some constant depending only on d and m. For example, take $\overline{f}(x,z) = \left(e^{|z|\Delta_{\mathbf{R}^d}}f\right)(x)$.

We introduce the notation $\delta_x^{(\eta)}$ for the uniform probability measure on $\partial B(x,\eta) \subset \mathbf{R}^{d+m}$, and define

$$\mathrm{emp}_N^* = \mathrm{emp}_N * \delta_x^{(N^{-\frac{1}{d}})}, \tag{11.28}$$

and

$$\mathrm{Fluct}^*[f] = \int_{\mathbf{R}^{d+m}} \overline{f}\, d(\mu_\theta - \mathrm{emp}_N^*). \tag{11.29}$$

Proceeding as in the proof of the last item, we have

$$\begin{aligned}|\mathrm{Fluct}^*[f] - \mathrm{Fluct}[f]| &\leq N^{-\frac{i+\alpha}{d}}|\overline{f}|_{\dot{C}^{i,\alpha}}\\ &\leq CN^{-\frac{i+\alpha}{d}}|f|_{\dot{C}^{i,\alpha}}.\end{aligned} \tag{11.30}$$

Proceeding as in step 3, we have that

$$\begin{aligned}
&\mathbf{E}_{\mathbf{P}_{N,\beta}} \exp\left(N^2\beta\left|t\mathrm{Fluct}[f]\right|\right) = \\
&\int_0^\infty \mathbf{P}_{N,\beta}\left(\exp\left(N^2\beta\left|t\mathrm{Fluct}[f]\right|\right) > x\right) \mathrm{d}x = \\
&\int_0^\infty \mathbf{P}_{N,\beta}\left(t\mathrm{Fluct}[f] > \frac{\log x}{N^2\beta}\right) \mathrm{d}x \leq \\
&\int_0^\infty \mathbf{P}_{N,\beta}\left(t\mathrm{Fluct}^*[f] > \frac{\log x}{N^2\beta} - CN^{-\frac{i+\alpha}{d}}|f|_{\dot{C}^{i,\alpha}}\right) \mathrm{d}x \leq \\
&\int_0^\infty \mathbf{P}_{N,\beta}\left(Ct|\overline{f}|_{\dot{H}^{s+\frac{m}{2}}}\left\|\mathrm{emp}_N^* - \mu_\theta\right\|_{H^{-s-\frac{m}{2}}} > \frac{\log x}{N^2\beta} - CN^{-\frac{i+\alpha}{d}}t|f|_{\dot{C}^{i,\alpha}}\right) \mathrm{d}x = \\
&\int_0^\infty \mathbf{P}_{N,\beta}\left(\left\|\mathrm{emp}_N^* - \mu_\theta\right\|_{H^{-s-\frac{m}{2}}} > \frac{\frac{\log x}{N^2\beta} - CN^{-\frac{i+\alpha}{d}}t|f|_{\dot{C}^{i,\alpha}}}{Ct|f|_{\dot{H}^s}}\right) \mathrm{d}x.
\end{aligned} \tag{11.31}$$

Proceeding as in the previous step, we can split split the integral into the domains $\frac{\log x}{N^2\beta} - N^{-\frac{i+\alpha}{d}}t|f|_{\dot{C}^{i,\alpha}} > 0$ and $\frac{\log x}{N^2\beta} - N^{-\frac{i+\alpha}{d}}t|f|_{\dot{C}^{i,\alpha}} < 0$, then apply Lemma 13.1 with $\eta_i = N^{-\frac{1}{d}}$ for each i, and Lemma 11.2, then bound/compute the integrals, and get

$$\begin{aligned}
&\mathbf{E}_{\mathbf{P}_{N,\beta}} \exp\left(N^2\beta\left|t\mathrm{Fluct}[f]\right|\right) \leq \\
&\exp\left(N^{2-\frac{i+\alpha}{d}}\beta t|f|_{\dot{C}^{i,\alpha}}\right) + \\
&\quad \exp\left(CN^{2-\frac{2s}{d}}\beta\right)\left(N^2\beta t|f|_{\dot{H}^s}\right)^{\frac{d}{2}} \exp\left(CN^2\beta\left[t^2|f|_{\dot{H}^s}^2 + N^{-\frac{i+\alpha}{d}}t|f|_{\dot{C}^{i,\alpha}}\right]\right).
\end{aligned} \tag{11.32}$$

From this we can conclude. □

12 Appendix A: the $H^{-s}(\Omega)$ norm

This section is devoted to proving Proposition 8.1. The strategy is an extension of the proof [32] for an analogous result. The proof will rely on a few lemmas.

Lemma 12.1. *Let $\Omega \subset \mathbf{R}^d$ be an open bounded set with a C^2 boundary, and let $f \in H^s(\Omega)$. Let*

$$\epsilon_* = \sup\{\epsilon > 0 | x \mapsto x + \epsilon\widehat{n}(x) \text{ is a diffeomorphism for all } |\delta| < \epsilon\}, \tag{12.1}$$

where $\widehat{n}(x)$ is the unit normal to $\partial\Omega$ at x. Then for every $0 < \epsilon < \epsilon_*$ there exists $f_\epsilon \in \dot{H}^s(\mathbf{R}^d)$ such that

1. $f_\epsilon|_\Omega = f$,
2. $\mathrm{supp}(f_\epsilon) \subset \overline{\Omega}^\epsilon$*, where* $\Omega^\epsilon = \{x \in \mathbf{R}^d | d(x,\Omega) < \epsilon\}$ *and*
3. *we have the inequalities*

$$\| f_\epsilon \|_{\dot{H}^s}^2 \leq \frac{C}{\epsilon^{\frac{s}{2}}} \| f \|_{H^s} \tag{12.2}$$

$$\| f_\epsilon \|_{L^2} \leq (1 + C\sqrt{\epsilon}) \| f \|_{L^2}, \tag{12.3}$$

where C depends only on Ω.

In addition, if f is non-negative in a neighborhood of $\partial\Omega$, then f_ϵ is non negative in $\Omega^\epsilon \setminus \Omega$.

Proof. **Step 1:** Flat case.

Assume for this step that for some $\delta > 0$

$$B(x,\delta) \bigcap \partial\Omega = \{y | y_d = 0\} \bigcap B(x,\delta). \tag{12.4}$$

We will use the notation

$$x = (\underline{x}, x_d), \tag{12.5}$$

where $\underline{x} \in \mathbf{R}^{d-1}$ and $x_d \in \mathbf{R}$. Let $x \in \partial\Omega$ and let

$$\underline{B}(x,\delta) = \{y | y_d = 0\} \bigcap B(x,\delta). \tag{12.6}$$

Let $\alpha > 0$ be such that

$$\underline{B}(x,\delta) \times (-\alpha, 0) \subset \Omega. \tag{12.7}$$

Define $\varphi : \underline{B}(x,\delta) \times (0,\alpha) \to \mathbf{R}$ by odd reflection,

$$\varphi(\underline{y}, y_d) = f(\underline{y}, -y_d). \tag{12.8}$$

Let $\mu \in C^\infty([0,\alpha], \mathbf{R}^+)$ be such that $\mu(0) = 1,\ \mu(\alpha) = 0$, and μ is decreasing. Consider now $\widehat{\varphi} : \underline{B}(x,\delta) \times (0,\alpha) \to \mathbf{R}$ defined as

$$\widehat{\varphi}(\underline{y}, y_d) = \varphi(\underline{y}, y_d)\mu(y_d). \tag{12.9}$$

Now we define the function $\varphi_\epsilon : \underline{B}(x,\delta) \times (0,\epsilon) \to \mathbf{R}$ as

$$\varphi_\epsilon(\underline{y}, y_d) = \widehat{\varphi}\left(\underline{y}, \frac{\alpha}{\epsilon} y_d\right). \tag{12.10}$$

We immediately get the estimate

$$\| \varphi_\epsilon \|_{L^2} \leq \sqrt{\frac{\epsilon}{\alpha}} \| f \|_{L^2} . \tag{12.11}$$

We also have the estimate

$$\| \varphi_\epsilon \|_{\dot{H}^s} \leq C \max\left(\left(\frac{\alpha}{\epsilon}\right)^{\frac{s}{2}}, 1\right) \| f \|_{H^1}, \tag{12.12}$$

where C depends only on Ω. Lastly, if f is non-negative in a neighborhood of $\partial\Omega$ consider the function

$$M\varphi_\epsilon = \max\left(\varphi_\epsilon, 0\right). \tag{12.13}$$

Then $M\varphi_\epsilon$ is positive, and $M\varphi_\epsilon$ agrees with f on $\{x_d = 0\}$. We also have that that

$$\begin{aligned} \| M\varphi_\epsilon \|_{\dot{H}^s} &\leq \| \varphi_\epsilon \|_{\dot{H}^s} \\ \| M\varphi_\epsilon \|_{L^2} &\leq \| \varphi_\epsilon \|_{L^2} . \end{aligned} \tag{12.14}$$

Step 2: General case.
Now we turn to the general case, where $\partial\Omega$ is not flat. Since by assumption $\partial\Omega$ is C^2, there exist finitely many balls $B(x_i, \epsilon_i)$ and C^2 diffeomorphisms $g_i : U_i \subset \mathbf{R}^{d-1} \to \mathbf{R}^d$ such that

$$g_i\left(U_i\right) = B(x_i, \epsilon_i) \bigcap \partial\Omega. \tag{12.15}$$

For any $\delta < \epsilon_*$, we can extend g_i to a C^1 diffeomorphism $\overline{g}_i : U_i \times (-\delta, \delta) \to V_i^\delta$, where

$$V_i^\delta = \{x + s\widehat{n}(x) | x \in B(x_i, \epsilon_i) \bigcap \partial\Omega, s \in (-\delta, \delta)\}. \tag{12.16}$$

We define $\overline{g}_i$ as

$$\overline{g}_i(\underline{x}, s) = g_i(\underline{x}) + s\nu(g_i(\underline{x})). \tag{12.17}$$

Now define for any $\epsilon < \epsilon_*$ the function $\varphi_\epsilon^i : U_i \times (-\epsilon, \epsilon)$ as in step 1, with $\alpha = \epsilon_*$. If f is non-negative in a neighborhood of $\partial\Omega$, define $M\varphi_\epsilon^i$ as in step 1.

Define the functions ϕ_ϵ^i as

$$\phi_\epsilon^i = \varphi_\epsilon^i \circ \overline{g}_i^{-1}. \tag{12.18}$$

If f is non-negative in a neighborhood of $\partial\Omega$, define the functions $M\phi_\epsilon^i$ as

$$M\phi_\epsilon^i = M\varphi_\epsilon^i \circ \overline{g}_i^{-1}. \tag{12.19}$$

Lastly, take a partition of unity q_i associated to $V_i^{\epsilon_*}$. Define the extension f_ϵ as

$$f_\epsilon(x) = \begin{cases} f(x) & \text{if } x \in \Omega \\ \sum_i (q_i \phi_i^\epsilon) & \text{if } x \in \bigcup V_i^\delta \\ 0 & \text{o.w.} \end{cases} \tag{12.20}$$

If f is non-negative in a neighborhood of $\partial\Omega$, define the extension Mf_ϵ as

$$Mf_\epsilon(x) = \begin{cases} f(x) & \text{if } x \in \Omega \\ \sum_i (q_i M\phi_i^\epsilon) & \text{if } x \in \bigcup V_i^\delta \\ 0 & \text{o.w.} \end{cases} \tag{12.21}$$

It is easy to check that f_ϵ, Mf_ϵ saitsfy the desired properties. □

We proceed to another lemma.

Lemma 12.2. *Let $\nu \in H^{-s}(\mathbf{R}^d)$. Assume that there exists a compact set Ω such that ν is nonpositive or nonnegative outside of Ω. Then there exists a compact set Ω_2 which contains Ω, a constant C, and a function $\phi \in H^s(\Omega_2)$ such that*

$$\int_{\Omega_2} \nu\phi \geq C \parallel \nu|_{\Omega_2} \parallel_{H^{-s}(\Omega_2)}, \tag{12.22}$$

$\parallel \phi \parallel_{H^s} = 1$, and ϕ is non-negative a neighborhood of $\partial\Omega_2$. Furthermore, C and Ω_2 depends only on Ω.

Proof. We assume WLOG that ν is positive outside of Ω and that $\partial\Omega \in C^2$. Otherwise we could find a compact set with C^2 boundary containing Ω. Let

$$\Omega^\epsilon = \{x \in \mathbf{R}^d | d(x, \Omega) \leq \epsilon\} \tag{12.23}$$

Take some $\epsilon > 0$. Note that

$$\parallel \nu|_{\Omega^\epsilon} \parallel_{H^{-s}(\Omega^\epsilon)} < \infty, \tag{12.24}$$

and hence, there exists some $\varphi \in H^s(\Omega^\epsilon)$ such that

$$\parallel \varphi \parallel_{H^s} = 1 \tag{12.25}$$

and

$$\int_{\Omega^\epsilon} \nu\varphi \geq \frac{1}{2} \parallel \nu|_{\Omega^\epsilon} \parallel_{H^{-s}(\Omega^\epsilon)} . \tag{12.26}$$

Consider now $\overline{\varphi} = \varphi|_\Omega$. By the extension lemma (Lemma 12.1), there exists an extension $\widetilde{\varphi}$ of $\overline{\varphi}$ such that

$$\text{supp}\,(\widetilde{\varphi}) \subset \Omega^\epsilon \tag{12.27}$$

and

$$\parallel \widetilde{\varphi} \parallel_{H^s} \leq C, \tag{12.28}$$

where C_ϵ depends on Ω and ϵ. Note that $\widetilde{\varphi}$ is 0 in a neighborhood of $\partial_{\Omega^\epsilon}$. Consider now

$$\phi = \max\{\varphi, \widetilde{\varphi}\}. \tag{12.29}$$

Then since $\phi \geq \widetilde{\varphi}$, we know that ϕ is non-negative in a neighborhood of $\partial\Omega^\epsilon$. We also know that

$$\begin{aligned} &\varphi(x) = \phi(x) \text{ for } x \in \Omega, \\ &\varphi(x) \leq \phi(x) \text{ for } x \in \Omega^\epsilon \setminus \Omega, \end{aligned} \tag{12.30}$$

which implies

$$\int_{\Omega^\epsilon} \nu\varphi \leq \int_{\Omega^\epsilon} \nu\phi, \tag{12.31}$$

since ν is positive outside of Ω. It can be shown that

$$\begin{aligned} \| \phi \|_{H^s} &\leq \| \varphi \|_{H^s} + \| \widetilde{\varphi} \|_{H^s} \\ &\leq 1 + C_\epsilon. \end{aligned} \tag{12.32}$$

Taking $\widehat{\phi} = \frac{\phi}{\|\phi\|_{H^s}}$, $C = \frac{1}{2(1+C_\epsilon)}$ and $\Omega_2 = \overline{\Omega}^\epsilon$ we obtain the result. □

We now prove Proposition 8.1, restated here.

Proposition 12.3. *Let $\nu \in H^{-s}(\mathbf{R}^d)$ and assume that there exists a compact set Ω such that ν is nonpositive or nonnegative outside of Ω. Then there exists a compact set Ω_1 which contains Ω, and a constant C such that*

$$\| \nu \|_{H^{-s}} \geq C \| \nu|_{\Omega_1} \|_{H^{-s}(\Omega_1)} . \tag{12.33}$$

Furthermore, C and Ω_1 depend only on Ω.

Proof. Again, without loss of generality we assume that ν is positive outside of Ω. Take some fixed $\epsilon > 0$. Then by Lemma 12.2 there exists a $\varphi \in H^s(\Omega^\epsilon)$ such that

1. $\| \varphi \|_{H^s} = 1$,
2. φ is positive in a neighborhood of $\partial\Omega^\epsilon$ and
3. $\int_{\Omega^\epsilon} \nu\varphi \geq C \| \nu|_{\Omega^\epsilon} \|_{H^{-s}(\Omega^\epsilon)}$, where C depends only on Ω and ϵ.

By Lemma 12.1, there exists an extension $\widehat{\varphi}$ of φ such that

1. The support of $\widehat{\varphi}$ is contained in $\overline{\Omega}^{1+\epsilon}$,
2. $\| \widehat{\varphi} \|_{\dot{H}^s} \leq C$, where C depends only on Ω and ϵ and
3. $\widehat{\varphi}$ is nonnegative in $\Omega^{1+\epsilon} \setminus \Omega^\epsilon$.

Since ν is positive outside of Ω and $\widehat{\varphi}$ is positive outside of Ω^ϵ, we have that

$$\int_{\Omega^\epsilon} \nu\varphi \leq \int_{\mathbf{R}^d} \nu\widehat{\varphi}. \tag{12.34}$$

Finally, we have that

$$\begin{aligned} \| \nu \|_{H^{-s}} &\geq \frac{1}{\| \widehat{\varphi} \|_{\dot{H}^s}} \int_{\mathbf{R}^d} \nu\widehat{\varphi} \\ &\geq \frac{1}{\| \widehat{\varphi} \|_{\dot{H}^s}} \int_{\Omega^\epsilon} \nu\varphi \\ &\geq C \| \nu|_{\Omega^\epsilon} \|_{H^{-s}(\Omega^\epsilon)}, \end{aligned} \tag{12.35}$$

where C depends only on Ω and ϵ. □

13 Appendix B: smearing for Riesz-type kernels

In this section, we prove Proposition 9.1. In order to do so, we need a fundamental result about regularization for Riesz-type kernels.

Lemma 13.1. *Let $d \geq 1$ and $0 < s < \min\{1, \frac{d}{2}\}$. Suppose that $X_N \in \mathbf{R}^{d \times N}$ is a pairwise distinct configuration and that $\mu \in \mathcal{P}(\mathbf{R}^d) \cap L^\infty(\mathbf{R}^d)$. Then there exists a constant C depending only on s, d, g such that for every $\eta_i < \min\{\frac{1}{2}, \frac{r_0}{2}\}$ we have*

$$\frac{1}{C}\left\|\frac{1}{N}\sum_{i=1}^N \delta_{x_i}^{(\eta_i)} - \widetilde{\mu}\right\|^2_{\dot{H}^{-s-\frac{m}{2}}} \leq \mathrm{F}_N(X_N, \mu) + \frac{1}{N^2}\sum_{i=1}^N G_{\eta_i}(0) + \frac{C}{N}\sum_{i=1}^N \left(\|\mu\|_{L^\infty}\left[\eta_i^{2s} + \eta_i^2\right]\right). \tag{13.1}$$

In this notation, $\delta_x^{(\eta)}$ is the uniform probability measure on $\partial B(x,\eta) \subset \mathbf{R}^{d+m}$, $\widetilde{\mu} = \mu\delta_{\mathbf{R}^d \times \{0\}}$ is a probability measure on $\mathbf{R}^{d+m}$, and $G_\eta = G * \delta_0^\eta$.

Proof. See [31, Proposition 2.2]. □

We now prove Proposition 9.1, restated here.

Proposition 13.2. *Let $\mu \in \mathcal{P}(\mathbf{R}^d)$ have compact support Ω and let $\alpha > s$. Then there exists a constant C, which depends on V, g, Ω, and α such that*

$$\|\mathrm{emp}_N - \mu\|_{C_*^{0,\alpha}} \leq N^{-\frac{\alpha}{d}} + C\left(\mathrm{F}_N(X_N,\mu) + C\|\mu\|_{L^\infty}N^{-\frac{2s}{d}}\right)^{\frac{1}{2}}. \tag{13.2}$$

Proof. By Lemma 9.2,

$$\|\mathrm{emp}_N - \mu\|_{C_*^{0,\alpha}} \leq 2\|(\mathrm{emp}_N - \mu)\mathbf{1}_\Omega\|_{C_*^{0,\alpha}}. \tag{13.3}$$

Note that

$$\|(\mathrm{emp}_N - \mu)\mathbf{1}_\Omega\|_{C_*^{0,\alpha}} = \int_{\mathbf{R}^d}(\mathrm{emp}_N - \mu)\mathbf{1}_\Omega\phi, \tag{13.4}$$

for some ϕ with compact support and such that

$$\|\phi\|_{C^{0,\alpha}} = 1. \tag{13.5}$$

Arguing as in [31], let $\overline{\phi} : \mathbf{R}^{d+m} \to \mathbf{R}$ be such that $\overline{\phi}(x,0) = \phi(x)$ for every x and

$$\begin{aligned}\|\overline{\phi}\|_{H^{s+\frac{m}{2}}} &\leq C\|\phi\|_{H^s}\\ |\overline{\phi}|_{\dot{C}^{0,\alpha}} &\leq C|\phi|_{\dot{C}^{0,\alpha}},\end{aligned} \tag{13.6}$$

for some C depending only on m and d. For example, take $\overline{\phi}(x,z) = \left(e^{|z|\Delta_{\mathbf{R}^d}}\phi\right)(x)$. We introduce the notation

$$\mathrm{emp}_N^\eta = \mathrm{emp}_N * \delta_0^{(\eta)}. \tag{13.7}$$

Now we can write

$$\begin{aligned}\int_{\mathbf{R}^d}\phi\mathbf{1}_\Omega\, d(\mathrm{emp}_N - \mu) &\leq \left|\int_{\mathbf{R}^{d+m}}\overline{\phi}\mathbf{1}_{\Omega\times B(0,1)}(\mathrm{emp}_N^\eta - \overline{\mu})\right|\\ &\quad+\left|\int_{\mathbf{R}^{d+m}}\overline{\phi}\mathbf{1}_{\Omega\times B(0,1)}\, d\left(\mathrm{emp}_N - \mathrm{emp}_N^\eta\right)\right|\\ &\leq \|\overline{\phi}\mathbf{1}_{\Omega\times B(0,1)}\|_{H^{s+\frac{m}{2}}}\left\|\left(\mathrm{emp}_N^\eta - \overline{\mu}\right)\mathbf{1}_{\Omega\times B(0,1)}\right\|_{H^{-s-\frac{m}{2}}(\Omega\times B(0,1))} + \eta^\alpha.\end{aligned} \tag{13.8}$$

Note that, since ϕ has compact support

$$\begin{aligned}\|\overline{\phi}\mathbf{1}_{\Omega\times B(0,1)}\|_{H^{s+\frac{m}{2}}} &\leq \|\phi\|_{H^s}\\ &\leq C\|\phi\|_{C^{0,\alpha}},\end{aligned} \tag{13.9}$$

where C depends on Ω. On the other hand, using the localization inequality (Proposition 8.1) and Lemma 13.1,

$$\begin{aligned}
&\left\| \left(\mathrm{emp}_N^\eta - \overline{\mu}\right) \mathbf{1}_{\Omega\times B(0,1)} \right\|_{H^{-s-\frac{m}{2}}(\Omega\times B(0,1))} \\
\leq & C \|\mathrm{emp}_N^\eta - \overline{\mu}\|_{\dot{H}^{-s-\frac{m}{2}}} \\
\leq & C \left(\mathrm{F}_N(X_N,\mu) + \frac{1}{N^2}\sum_{i=1}^N G_{\eta_i}(0) + \frac{C}{N}\sum_{i=1}^N \left(\|\mu\|_{L^\infty} \left[\eta_i^{2s} + \eta_i^2\right]\right)\right)^{\frac{1}{2}},
\end{aligned} \tag{13.10}$$

where C depends on V, g, Ω, and α. Taking $\eta = N^{-\frac{1}{d}}$, we obtain Equation (13.2). □

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