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Ray Knight theorems for spectrally negative Lévy processes*

Víctor Rivero† Jesús Contreras‡

Abstract

In this paper, we study the law of the local time processes $(L_T^x(X), x \in \mathbb{R})$ associated to a spectrally negative Lévy process X, in the cases $T = \tau_a^+$, the first passage time of X above $a > 0$ and $T = \tau(c)$, the first time it accumulates c units of local time at zero. We describe the branching like structure of local times and Poissonian constructions of them using excursion theory. The presence of jumps for X creates a type of excursions which can contribute simultaneously to local times of levels above and below a given reference point. This fact introduces dependency on local times, causing them to be non-Markovian. Nonetheless, the overshoots and undershoots of excursions will be useful to analyze this dependency. In both cases, local times are infinitely divisible and we give a description of the corresponding Lévy measures in terms of excursion measures. These are hence analogues in the spectrally negative Lévy case of the first and second Ray-Knight theorems, originally stated for the Brownian motion.

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1 Introduction

Let $X = (X_t, t \geq 0)$ be a real valued spectrally negative Lévy process, that is, a stochastic process with independent and stationary increments and no positive jumps. Denote by $S = (S_t, t \geq 0)$ its running supremum. Its Laplace transform exists, characterizes its law and can be expressed as

$$\mathbb{E}\left[e^{\lambda X_t}\right] = e^{t\Psi(\lambda)}, \quad t, \lambda \geq 0,$$

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†Centro de Investigación en Matemáticas A.C. Calle Jalisco s/n C.P. 36240, Guanajuato, Mexico.
E-mail: rivero@cimat.mx
‡Centro de Investigación en Matemáticas A.C. Calle Jalisco s/n C.P. 36240, Guanajuato, Mexico.
E-mail: jjcontreras@cimat.mx

where the function Ψ is called the Laplace exponent of X. Ψ can be expressed by the Lévy-Khintchine formula

$$\Psi(\lambda) = d\lambda + \frac{1}{2}\Sigma^2\lambda^2 + \int_{(-\infty,0)} (e^{\lambda x} - 1 - \lambda x 1_{\{-1<x\}})\Pi(dx),$$

where the triplet (d, Σ, Π) consists of $d, \Sigma \in \mathbb{R}$ and a $\sigma-$ finite measure Π over $(-\infty, 0)$ satisfying the condition $\int_{(-\infty,0)}(1 \wedge x^2)\Pi(dx) < \infty$. We assume w.l.o.g. that $\Psi(1) = 1$. We will denote by Φ the inverse of Ψ, which is the Laplace exponent of the inverse local time of X reflected at its running supremum process.

From now on, we consider X to be a SNLP such that:

(A) it is of unbounded variation (which is equivalent to $\Sigma \neq 0$ or $\int_{-1}^{0} |x|\Pi(dx) = \infty$),

and either

(B1) $X_t \to \infty$ a.s. as $t \to \infty$ (which is equivalent to the condition $\Psi'(0+) > 0$);

or

(B2) X oscillates as $t \to \infty$ (which is equivalent to the condition $\Psi'(0+) = 0$).

One can actually remove these conditions, but we will not tackle this task here. For deeper insight in the theory of Lévy processes, we refer to [1], [12] and [20] as our standard references.

We are interested in studying the local time process associated to X, which is here denoted by $(L_t^x(X), t \geq 0, x \in \mathbb{R})$. Essentially, one interprets $L_t^x(X)$ as the amount of time spent by X at level x on the interval $[0, t]$. Formally, $L_t^x(X)$ is defined as the a.s. limit

$$L_t^x(X) := \limsup_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_0^t 1_{\{|X_s - x| < \varepsilon\}} ds, \quad x \in \mathbb{R}, t \geq 0. \tag{1.1}$$

Under rather general conditions (see [1, Ch. V]), the convergence holds also in L^2 and uniformly over compact sets of t. Furthermore, local times satisfy the so-called occupation density formula, that is,

$$\int_0^T f(X_s)ds \overset{a.s.}{=} \int_{\mathbb{R}} f(x) L_T^x(X) dx,$$

for any $f \geq 0$ measurable and bounded and stopping times T.

Our aim is to describe the local time process $(L_T^x(X), x \in \mathbb{R})$, where T is a fixed stopping time. In general, local times indexed by the spatial variable are not easy to describe. Nonetheless, this has been a matter of research interest since early times of the theory of stochastic processes, perhaps originated by the pioneering works of Ray and Knight in the decade of 1960. In particular, the theory around the so-called isomorphism theorems, has proven to be one powerful tool to study properties of local times.

Ray and Knight completely characterized the law of the local times in the case X is a standard Brownian motion and T is either the first time X is above a positive level a or the first time it accumulates a certain amount of local time at zero. These results are known as the first and second Ray-Knight theorems, respectively, and they are expressed in terms of squared Bessel processes as follows.

Theorem 1.1 (First Ray-Knight theorem). *Let X be a Brownian motion issued from zero, $a > 0$ a fixed level and*

$$\tau_a^+ = \inf\{t > 0 : X_t > a\}$$

the first passage time above a. Then, *i) the process* $(L^{a-z}_{\tau_a^+}(X), z \in [0,a])$ *has the same law as a squared Bessel process of dimension 2 started from 0;*

ii) the process $(L^{a-z}_{\tau_a^+}(X), z \geq a)$ *has the same law as a squared Bessel process of dimension 0, issued from* $L^0_{\tau_a^+}(X)$*.*

Moreover, conditionally on $L^0_{\tau_a^+}(X)$ *the two parts are independent.*

Theorem 1.2 (Second Ray-Knight theorem)**.** *Let* X *be a Brownian motion issued from zero,* $c > 0$ *a constant and*

$$\tau(c) = \inf\{t > 0 : L^0_t(X) > c\}$$

the first time 0 *has accumulated* c *units of local time. Then, the process* $(L^y_{\tau(c)}(X), y \geq 0)$ *is distributed as a squared Bessel process of dimension* 0*, started from* c*. By symmetry of* X*, the law of the process* $(L^{-y}_{\tau(c)}(X), y \geq 0)$ *is also that of a squared Bessel process of dimension* 0 *started from* c *and it is independent from the first one.*

Further details on these results can be found for instance in [16] and [17]. Because of technical reasons and convenience on the narrative, we will present first the results concerning an analogue of the second Ray-Knight theorem and then the first. For Brownian motion, both theorems can be expressed using excursion theory (we refer to Section 2 for notation and more information on excursions). Indeed, if N_0 and $\overline{N}$ are the measures of the excursions away from 0 for X and the reflected process $S - X$ on the space $\mathcal{D} = D(0,\infty)$ of càdlàg paths, respectively, then local times up to $\tau(c)$ have the representation

$$L^y_{\tau(c)}(X) = \int_0^c \int_{\mathcal{D}} \ell(y) \widetilde{K}(ds, d\ell), \quad y \in \mathbb{R}, \tag{1.2}$$

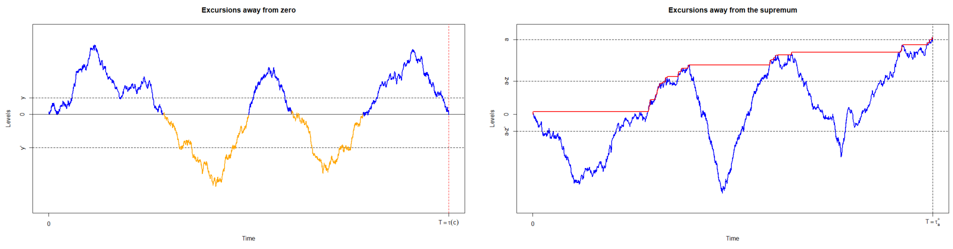
where $\widetilde{K}$ is a Poisson random measure related to N_0. The local time process up to τ_a^+ can be expressed as

$$L^{a-z}_{\tau_a^+}(X) = \int_0^{z\wedge a} \int_{\mathcal{D}} \ell(z-s) K(ds, d\ell), \quad z \geq 0, \tag{1.3}$$

where K is also a Poisson random measure but related to $\overline{N}$. Actually, as a consequence of Lévy's identity $(|X_t|, L^0_t(X))_{t\geq 0} \overset{(d)}{=} (S_t - X_t, S_t)_{t\geq 0}$ for Brownian motion, we have that $\overline{N} = 2N_0^+$, where N_0^+ is the restriction of N_0 to completely positive excursions. Hence, equations (1.2) and (1.3) can be written in terms of a single Poisson random measure. See Theorems 3.1 and 4.1 below for a proof of these representations in a more general setting.

On both Ray-Knight theorems, the properties of Brownian excursions play a key role to prove the independence of the two parts involved. Consider for instance the second Ray-Knight theorem, in which local times of levels above and below zero are independent. Since the Brownian motion has continuous paths, one can split the set of excursions away from zero into two disjoint sets: the excursions completely above zero, $\mathcal{E}_+$, and the ones completely below, $\mathcal{E}_-$ (see also Figure 1(a)). Observe that only excursions in $\mathcal{E}_+$ contribute to the local time of positive levels, and the same is true for $\mathcal{E}_-$ and local times of negative levels. So, given the fact that excursions form a Poisson point process, its restrictions to $\mathcal{E}_+$ and $\mathcal{E}_-$ are independent and this results in the independence of the processes $(L^y_{\tau(c)}(X), y \geq 0)$ and $(L^{-y}_{\tau(c)}(X), y \geq 0)$. This fact can also be used to explain the Markov property of local times. Indeed, conditionally on the information of the local time at a reference level, local times of levels above and below are independent.

Another important remark about both theorems is that a squared Bessel process of dimension 2 started from zero is in particular a continuous state branching process



(a) (b)

Figure 1: (a) Graphic representation of equation (1.2). Excursions away from 0 appear until 0 accumulates c units of local time and each one can contribute to the local time of levels $y > 0$ and $y' < 0$. (b) Graphic representation of equation (1.3). For the level $a - z$ with $z \in [0, a]$, its local time comes from the contribution of each excursion away from the supremum starting above that level, whilst for $a - z'$ with $z' \geq a$, all excursions can contribute.

(CSBP) with linear immigration $\psi(z) = 2z$ and branching mechanism $\phi(z) = z^2/2$. A squared Bessel process of dimension 0 started from c is also a CSBP with the same branching mechanism ϕ but no immigration. They satisfy the stochastic differential equations

$$Z_t = \int_0^t \sqrt{Z_s} d\beta_s + 2t, \quad t \geq 0,$$

and

$$Z_t = c + \int_0^t \sqrt{Z_s} d\beta_s, \quad t \geq 0,$$

respectively, where β is a Brownian motion. These facts reflect the branching nature of local times and bring interest on finding out if similar stochastic integral equations are satisfied by them in a more general setting.

We would like to explore if we can obtain analogues of the Ray-Knight theorems in the case in which the process does not have continuous trajectories, more specifically for SNLPs. One important issue here is a consequence of a result in [8]. N. Eisenbaum and H. Kaspi proved that if the local times of a process X have the Markov property, then X necessarily has continuous paths. This implies that for instance, if X is a Lévy process with negative jumps, the processes $(L^{a-z}_{\tau_a^+}(X), z \geq 0)$ and $(L^{y}_{\tau(c)}(X), y \in \mathbb{R})$ are not Markovian. Hence, trying to characterize the law of the local time process yields to a finer study of the structure of dependence between the local times at different levels. We will explore this via excursions. Additional to the sets $\mathcal{E}_+$ and $\mathcal{E}_-$, an excursion away from a point, say 0, can also be an element of a third set: $\mathcal{E}_\pm$ (see also the forthcoming Figure 2). This set consists of excursions starting above 0 and then jumping below, hence having the possibility to contribute to local times of both positive and negative levels. The positions $(\mathcal{O}_0, \mathcal{U}_0)$ of the excursion prior to and at the first passage time below 0 are called the overshoot and the undershoot, respectively, and will be relevant to describe the law of local times.

Recent efforts have been made in this direction for Lévy processes. In [13], B. Li and Z. Palmowski gave an expression for the Laplace transform of functionals of the form $\int_0^T f(X_t)dt$ in terms of generalized scale functions, where $T = \tau_a^+ \wedge \tau_b^-$ is the first exit time from the interval $[b, a]$. These functionals are strongly related to local times via the occupation density formula mentioned before. Later, in [14] B. Li and X. Zhou obtained joint Laplace transforms also in terms of generalized scale functions and they related the law of local times under a change of measure with permanental processes.

In 2023, V. Rivero and J. Contreras [4] extended Li and Palmowski's results for functionals also involving the supremum S of X. Actually, we can make use of the results there in order to gain some information on the local time process up to τ_a^+, $(L^{a-z}_{\tau_a^+}(X), z \geq 0)$. According to their notation, for a measurable and bounded function $f : \mathbb{R} \to \mathbb{R}_+$, the generalized scale function W_f is defined by

$$W_f(x,b) = W(x-b)\exp\left\{\int_b^x ds\overline{N}\left(1-e^{-\int_0^\zeta dr f(s-\mathbf{e}(r))}, H < s-b\right)\right\}, \quad x \geq b > -\infty. \tag{1.4}$$

It is known that when the second entry, say $v \in \mathbb{R}$, is fixed, $W_f(\cdot, v)$ is also a solution of the integral equation

$$W_f(u,v) = W(u-v) + \int_v^u dz W(u-z)f(z)W_f(z,v), \quad u \geq v,$$

up to a positive constant. In case the function f is constant, say $f \equiv q$, W_f coincides with the usual $q-$ scale function associated to X and when $f \equiv 0$ we will just write W (see [11] for further reference). In this paper, the function $\mathcal{W}_f : [0,\infty) \to \mathbb{R}_+$ defined by

$$\mathcal{W}_f(x) := \lim_{b\to-\infty}\frac{W_f(0,b)}{W_f(x,b)} = \exp\left\{-\int_0^x ds\overline{N}\left(1-e^{-\int_0^\zeta dr f(s-\mathbf{e}(r))}\right)\right\}, \quad x \geq 0, \tag{1.5}$$

will play an important role. See Sections 3 and 4 to observe its connections with the analogues of the Ray-Knight theorems in the spectrally negative case.

Finally, in [21], W. Xu explored the local times of a spectrally positive $\alpha-$ stable process Y up to $\tau(c)$. Since $X := -Y$ is spectrally negative, a consequence of their results is that conditioned on $\tau(c) < \infty$, the process $(L^{-y}_{\tau(c)}(X), y \geq 0)$ has the law of a non-Markovian branching system which they called *rough continuous state branching process*. This class of processes is characterized for being a weak solution to certain stochastic Volterra equations, and in particular $(L^{-y}_{\tau(c)}(X), y \geq 0)$ satisfies

$$Z_y = c(1-bW(y)) + \int_0^y\int_0^\infty\int_0^{Z_s}(W(y-s)-W(y-s-u))\widetilde{N}_\alpha(ds,du,dz),$$

where b is the drift of Y, W is its scale function and $\widetilde{N}_\alpha$ is certain compensated Poisson random measure. Note that this stochastic equation is more involved than those presented before for Bessel processes. We do not intend to tackle this point of view in this article and let it for future studies.

The content of the article is organized as follows. Section 2 is dedicated to introduce notation and recall some properties of excursions. In Sections 3 and 4 we state our main results. In Section 3, we perform a study of local times constituting an analogue of the second Ray-Knight theorem. Here, we provide a Poissonian construction for $(L^y_{\tau(c)}(X), y \in \mathbb{R})$, proving that it is infinitely divisible, giving its corresponding Lévy measure and emphasizing the importance of the overshoots and undershoots with respect to zero. Regarding the process $(L^{a-z}_{\tau_a^+}(X), z \geq 0)$, Section 4 contains an analogue of the first Ray-Knight theorem. We derive a Poissonian construction and describe its law as an infinitely divisible process, focusing on the joint law of local times under $\overline{N}$. Section 5 is based on the decomposition of the Lévy measure of an infinitely divisible process found in [7], applying the ideas there to the Lévy measures of the processes of local times. Finally, Section 6 contains some results which are useful for the main theorems and Section 7 compiles all the proofs.

2 Preliminaries on excursions

First, we deal with the excursions away from the supremum, or equivalently, the excursions away from 0 for $S - X$. Let us introduce the space $\mathcal{E}$ of positive right continuous paths with left limits and defined on an interval:

$$\mathcal{E} = \{\mathbf{e} : [0, \zeta] \to [0, \infty) \mid \zeta \in (0, \infty],\ \mathbf{e}((0, \zeta)) \subset (0, \infty) \text{ and } \mathbf{e} \text{ is càdlàg}\}.$$

This space is usually regarded as a subset of the more general space $\mathcal{D} = D(0, \infty)$ of càdlàg paths, so we might write one or the other depending on the context. For an element $\mathbf{e} \in \mathcal{E}$, the right endpoint of its interval of definition is called duration or lifetime and it is denoted by $\zeta(\mathbf{e})$. The supremum of $\mathbf{e}$ is called height and it is denoted by $H(\mathbf{e}) = \sup_{v \in [0,\zeta]} \mathbf{e}(v)$.

For a SNLP X, it is well known that one can take S as the local time at the supremum and that its right continuous inverse is the subordinator $(\tau_t^+, t \geq 0)$. For $t > 0$, such that $\tau_t^+ \neq \tau_{t-}^+ := \lim_{s \uparrow t} \tau_s^+$, the supremum is constant and equal to t on the interval $[\tau_{t-}^+, \tau_t^+]$. Therefore, we can define

$$\mathbf{e}_t(v) := (S - X)_{\tau_{t-}^+ + v}, \quad 0 \leq v \leq \tau_t^+ - \tau_{t-}^+,$$

the excursion of $S - X$ at local time t. In this case, $\mathbf{e}_t \in \mathcal{E}$ and actually, $\zeta(\mathbf{e}_t) = \tau_t^+ - \tau_{t-}^+$. In case $\tau_t^+ = \tau_{t-}^+$, one assigns $\mathbf{e}_t = \delta$, where $\delta \notin \mathcal{E}$ is an auxiliary state. See also [2] and [9] for more information.

A result of excursion theory (see for example [12, Th. 6.14]) states that there exists a measure space $(\mathcal{E}, \Sigma, \overline{N})$ such that Σ contains the sets of the form

$$\{\mathbf{e} \in \mathcal{E} : \zeta(\mathbf{e}) \in A, H(\mathbf{e}) \in B, \mathbf{e}(\zeta) \in C\},$$

where A, B, C are Borel sets on $\mathbb{R}$. Furthermore, if $\limsup_{t \to \infty} X_t = \infty$ a.s. (that is, iff $\Psi'(0+) \geq 0$), then $\{(t, \mathbf{e}_t) : t > 0, \mathbf{e}_t \neq \delta\}$ is a Poisson point process of intensity $ds \otimes \overline{N}(d\mathbf{e})$. For a deeper insight on this excursion measure, we refer to [3] and [6]. This fact explains our hypotheses **(B1)** and **(B2)** in Section 1. In the other case, if $\Psi'(0+) < 0$ one obtains a killed Poisson point process and thus similar tools are available to obtain the results. We have chosen not to deal with this case, as to adapt the identities we have obtained, we require more space and the paper is already rather long.

For excursions away from a point we have a similar situation, but we need a different subordinator. Taking as a reference a given point $y \in \mathbb{R}$, there exists an associated subordinator $\sigma^y = (\sigma_s^y, s \geq 0)$ defined by the right inverse of the local time as follows

$$\sigma_s^y = \inf\{t > 0 : L_t^y(X) > s\}, \quad s \geq 0.$$

To formally define the excursions away from a point, we work with σ^y. For each $s > 0$, σ_s^y corresponds to the first time the process X accumulates s units of local time at level y. We can define the excursions by looking at the constancy times of $L_\cdot^y(X)$, or equivalently, at the increasing times of σ^y. Indeed, for each $u \geq 0$ such that $\sigma_u^y > \sigma_{u-}^y := \lim_{s \uparrow u} \sigma_s^y$, we define the excursion by

$$\mathbf{e}_u^y(t) = X_{\sigma_{u-}^y + t}, \qquad 0 \leq t \leq \sigma_u^y - \sigma_{u-}^y,$$

and the quantity $\zeta(\mathbf{e}_u^y) := \sigma_u^y - \sigma_{u-}^y$ is called its length. We also denote its height by $H(\mathbf{e}_u^y) = \sup_{v \in [0,\zeta]} \mathbf{e}_u^y(v)$. The excursion process is a Poisson point process on the space $[0, \infty) \times \mathcal{E}^y$, where $\mathcal{E}^y$ is the space of càdlàg paths with lifetime, starting and ending at y, and the intensity is given by the product of the Lebesgue measure and the so-called Itô measure: $ds \otimes N_y(d\mathbf{e})$. Again, this space can be seen as a subspace of $\mathcal{D}$. More information on the excursion measure away from a point can be found in [18]. The previous paragraphs allow to use the tools from the theory of Poisson point processes, such as the compensation and exponential formulas, to perform computations related to excursions of X.

The following Proposition involving scale functions and quantities related to $\overline{N}$, will be useful in some of the results later.

Proposition 2.1. *Let $f : \mathbb{R} \to \mathbb{R}_+$ a measurable and bounded function. Then, the function $\mathcal{W}_f$ in* (1.5) *is well defined and it is a solution of the following equation involving the scale function W of X:*

$$F(x) = 1 - \int_0^\infty dz(W(x) - W(x-z))f(x-z)F(z), \quad x > 0. \tag{2.1}$$

Moreover, define

$$G_f(x) = \widehat{\mathbb{E}}_x \left[\exp\left\{ -\int_0^{\tau_0^-} ds f(x - X_s) \right\} \right], \quad x \geq 0,$$

and

$$g_f(x) = \overline{N}\left[1 - e^{-\int_0^\zeta dr f(x-\mathbf{e}(r))}\right] = \overline{N}\left[1 - e^{-\int_0^\infty dy f(x-y) L_\zeta^y(\mathbf{e})}\right], \quad x \in \mathbb{R}.$$

Then,

$$G_f(x) = \mathcal{W}_f(x) = \exp\left\{ -\int_0^x ds g_f(s) \right\},$$

and in particular we have the relation

$$\frac{d}{dx}(-\log G_f)(x) = g_f(x).$$

Notation We end this section introducing some further notation. Throughout this work, we will write $\mathbb{P}_x$ and $\mathbb{E}_x$ to denote the law of X started from $x \in \mathbb{R}$, and by $\widehat{\mathbb{P}}_x$ and $\widehat{\mathbb{E}}_x$ the law of the dual $\widehat{X}$ of X (we omit the subscripts when $x = 0$). Additionally, several excursion measures will appear, including $\overline{N}$, the excursion measure away from the supremum for X, and N_x and $\widehat{N}_x$, the excursion measures away from x for X and $\widehat{X}$, respectively. Concerning local times, we will indicate between parentheses the reference process for which local time is being measured, e.g., $L_t^x(X)$ for the local time at x up to time t for the process X. On the other hand, when making computations under the excursion measures, we will write ℓ_t^x for the local time at x up to time t of a generic excursion, and we refer to the corresponding excursion measure to identify the type of excursion involved and the conditions for it. For instance, $\widehat{N}_x(\ell_t^y \in dz, \tau_0^- < \zeta)$ stands for the law of the local time at y up to time t of an excursion away from x for $\widehat{X}$, for which its first passage time below zero occurs before its lifetime ζ. Finally, for any $\omega \in \mathbb{R}_+^{\mathbb{R}}$, we write $\omega = (\omega_-, \omega_+)$, with $\omega_- \in \mathbb{R}_+^{(-\infty,0]}$ given by $\omega_-(y) = \omega(y)$, $y \leq 0$ and $\omega_+ \in \mathbb{R}_+^{(0,\infty)}$ given by $\omega_+(z) = \omega(z)$, $z > 0$.

3 Second Ray-Knight theorem

Recall that a non-negative infinitely divisible process $\psi = (\psi_x, x \in E)$ is characterized by its Lévy measure μ, and as such it satisfies

$$\mathbb{E}\left[e^{-\int_{\mathbb{E}} f(x)\psi_x dx}\right] = \exp\left\{ -\int_{\mathbb{R}_+^{\mathbb{R}}} \left(1 - e^{-\int_{\mathbb{E}} f(x)\omega(x)dx}\right)\mu(d\omega) \right\},$$

for any non-negative, measurable and bounded function f. In [7], N. Eisenbaum provides a deeper understanding of the Lévy measure by decomposing it into two parts, essentially corresponding to the information of a process between the first and last visits to a point and the complement. In our case, we will explore this decomposition in Section 5.

We begin with the following theorem, which also holds for more general Markov processes with local times, but it is included here for sake of completeness. It establishes that the local times up to $\tau(c)$ are infinitely divisible and also provides a Poissonian representation of them.

Theorem 3.1. *Let X be a spectrally negative Lévy process and denote by N_0 the associated excursion measure away from zero for X. Assume X satisfies hypothesis* **(A)** *and* **(B2)***. Then, the local time process $(L^y_{\tau(c)}(X), y \in \mathbb{R})$ is infinitely divisible and its Lévy measure $\mu^{(c)}$ is given by*

$$\mu^{(c)}(d\omega) = cN_0(\ell^{\cdot}_{\zeta} \in d\omega),$$

that is, the image of the excursion measure away from zero N_0 under the function that maps an excursion into its local time process up to its lifetime ζ. Moreover, local time process admits the representation

$$L^y_{\tau(c)}(X) = \int_0^c \int_{\mathcal{D}} \ell(y)\widetilde{K}(ds, d\ell), \quad y \in \mathbb{R}, \tag{3.1}$$

where $\widetilde{K}$ is a Poisson random measure of intensity $ds \otimes \widetilde{M}(d\ell)$, $\widetilde{M}$ being the image of N_0 under the map that associates an excursion $\mathbf{e}$ its local time process up to its lifetime: $\mathbf{e} \mapsto \left(\ell^r_\zeta(\mathbf{e}), r \in \mathbb{R}\right)$.

This representation for local times is similar to that in [15, Ch. 6] for CSBP processes. In that context, the intensity of the Poisson random measure $\widetilde{K}$ is $ds \otimes Q_H$ and Q_H is a measure that has the information of both an entrance law and the transitions of a branching process and it is known as the Kuznetsov measure (see [5, Ch. XIX]). In our setting, $\widetilde{M}$ cannot be a Kuznetsov measure, since otherwise local times would be Markovian.

Our main purpose is to get a finer description of the Poissonian construction in (3.1) and provide information on $\mu^{(c)}$. For that end, we start by defining the concept of overshoot and undershoot of a path. These quantities tell us the relative position of the path prior and at the first passage time below a given level x. In general, for a càdlàg path Y having only negative jumps and a level x such that $Y_0 \geq x$, we denote by $(\mathcal{O}_x(Y),\mathcal{U}_x(Y))$ to the pair of values

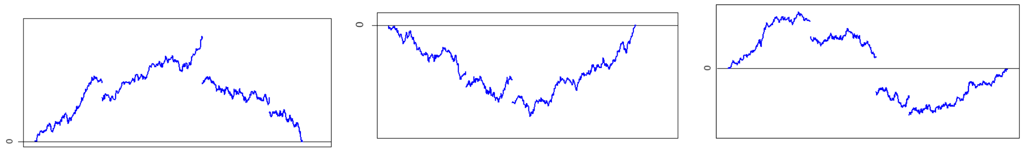
$$(\mathcal{O}_x(Y),\mathcal{U}_x(Y)) = (Y_{\tau^-_x(Y)-} - x, Y_{\tau^-_x(Y)} - x),$$

where $\tau^-_x(Y) = \inf\{t > 0 : Y_t < x\}$. Notice that if Y crosses down x continuously, both quantities are equal to zero. That is the reason we did not see their role in the Brownian motion case, since it has continuous paths. Nonetheless, since in this case the trajectories have negative jumps, they will arise naturally and will be really important for path decompositions.

According to [18], N_0 is now carried by the partition of the space of excursions into the sets $\mathcal{E}_+ \sqcup \mathcal{E}_- \sqcup \mathcal{E}_\pm$, which consist in completely positive, completely negative and mixed excursions, respectively. Observe that an excursion $\mathbf{e} \in \mathcal{E}_+$ contributes only to the local times of positive levels. Similarly, $\mathbf{e} \in \mathcal{E}_-$ only contributes to negative levels. But, unlike the Brownian motion case, when the process has one-sided jumps there is an additional kind of excursions, $\mathcal{E}_\pm$, which can add to the local time of both positive and negative levels. As a consequence, we cannot split the information of local times above and below a point into functions of disjoint sets of excursions, hence losing the independence mentioned for the Brownian motion in Theorems 1.1 and 1.2. This also translates to the absence of the Markov property for the process of local times.

A typical excursion $\mathbf{e} \in \mathcal{E}_\pm$ starts positive and then it jumps below 0, at time $\tau^-_0(\mathbf{e})$. After this time, since the path does not have positive jumps, it will creep upwards 0 and end at its lifetime $\zeta(\mathbf{e})$. Therefore, we can decompose $\mathbf{e}$ into the paths $\underleftarrow{\mathbf{e}}$ and $\underrightarrow{\mathbf{e}}$, where

$$\underleftarrow{\mathbf{e}}(t) = \mathbf{e}((\tau^-_0(\mathbf{e}) - t)-), \quad 0 \leq t \leq \tau^-_0(\mathbf{e})$$



(a) (b) (c)

Figure 2: Typical excursions away from zero in (a) $\mathcal{E}_+$, (b) $\mathcal{E}_-$ and (c) $\mathcal{E}_\pm$.

and

$$\underset{\rightarrow}{\mathbf{e}}(t) = \mathbf{e}(\tau_0^-(\mathbf{e}) + t), \quad 0 \le t \le \zeta(\mathbf{e}) - \tau_0^-(\mathbf{e}).$$

As we mentioned before, the position relative to 0 before and at first passage are called the overshoot and undershoot at 0 of the path, that is, $\mathcal{O}_0(\mathbf{e}) = \mathbf{e}(\tau_0^- -) > 0$ and $\mathcal{U}_0(\mathbf{e}) = \mathbf{e}(\tau_0^-) < 0$. By the strong Markov property at time τ_0^-, it turns out that $\underset{\leftarrow}{\mathbf{e}}$ and $\underset{\rightarrow}{\mathbf{e}}$ are conditionally independent given $(\mathcal{O}_0, \mathcal{U}_0)$. Moreover, the reversed path $\underset{\leftarrow}{\mathbf{e}}$ has the same law as the dual $\widehat{X}$ started from $\mathcal{O}_0$ and killed at its first hitting time of 0, denoted by $\widehat{\mathbb{E}}^0_{\mathcal{O}_0}$. On the other hand, $\underset{\rightarrow}{\mathbf{e}}$ has the same law as X started from $\mathcal{U}_0$ and killed at its first hitting time of 0, denoted by $\mathbb{E}^0_{\mathcal{U}_0}$. We can actually compute the joint law of $(\mathcal{O}_0, \mathcal{U}_0)$ in the set $\mathcal{E}_\pm$ under N_0, as can be read in Lemma 6.1. This helps to obtain the following refined version of Theorem 3.1.

Theorem 3.2. *Let X be a spectrally negative Lévy process satisfying* **(A)** *and* **(B2)***. Then, the Lévy measure $\mu^{(c)}$ associated to the local time process $(L^y_{\tau(c)}(X), y \in \mathbb{R})$ can be decomposed as*

$$\mu^{(c)}(d\omega) = cN_0(\ell^\cdot_\zeta \in d\omega) = cN_0(\ell^\cdot_\zeta \in d\omega, \mathcal{E}_+) + cN_0(\ell^\cdot_\zeta \in d\omega, \mathcal{E}_-) + cN_0(\ell^\cdot_\zeta \in d\omega, \mathcal{E}_\pm),$$

and when restricted to $\mathcal{E}_\pm$,

$$N_0(\ell^\cdot_\zeta \in d\omega, \mathcal{E}_\pm) = \int_{(0,\infty)} \int_{(-\infty,0)} db\Pi(du - b)\widehat{\mathbb{E}}^0_b \otimes \mathbb{E}^0_u(L^\cdot_\zeta \in d\omega);$$

where

$$\widehat{\mathbb{E}}^0_b \otimes \mathbb{E}^0_u(L^\cdot_\zeta \in d\omega) := \widehat{\mathbb{E}}_b\left(L^\cdot_{\tau_0^-}(X) \in d\omega_+\right)\mathbb{E}_u\left(L^\cdot_{\tau_0^+}(X) \in d\omega_-\right).$$

In particular, for any $f : \mathbb{R} \to [0,\infty)$ measurable and bounded, if we write $f = f_+ + f_-$, with $f_+ = f1_{(0,\infty)}$ and $f_- = f1_{(-\infty,0)}$, the Laplace transform of $\int_\mathbb{R} f(y)L^y_{\tau(c)}(X)dy$ is given by

$$\begin{aligned}
\mathbb{E}\left[e^{-\int_\mathbb{R} f(y)L^y_{\tau(c)}(X)dy}\right] &= \exp\left\{-cN_0\left[1 - e^{-\int_\mathbb{R} f(y)\ell^y_\zeta dy}\right]\right\} \\
&= \exp\left\{-cN_0\left[1 - e^{-\int_{(0,\infty)} f_+(y)\ell^y_\zeta dy}, \mathcal{E}_+\right]\right\} \\
&\times \exp\left\{-cN_0\left[1 - e^{-\int_{(-\infty,0)} f_-(y)\ell^y_\zeta dy}, \mathcal{E}_-\right]\right\} \\
&\times \exp\left\{-c\int_{(0,\infty)}\int_{(-\infty,0)} db\Pi(du - b)\left[1 - \mathcal{W}_{f_{+,b}}(b)\mathcal{W}_{f_{-,u}}(-u)\right]\right\},
\end{aligned}$$

where $\mathcal{W}_{f_{+,b}}$ and $\mathcal{W}_{f_{-,u}}$ are defined as in (1.5) *with $f_{+,b}(r) := f_+(-r + b)$ and $f_{-,u}(r) := f_-(r + u)$.*

In particular, since in Xu's $\alpha-$ stable case [21] there is no Brownian component (therefore N_0 is only carried by $\mathcal{E}_\pm$) and they only consider negative levels, the expression above in that case simplifies to

$$\begin{aligned}&\mathbb{E}\left[e^{-\int_{(-\infty,0)} f_-(y)L^y_{\tau(c)}(X)dy}\right]\\&=\exp\left\{-c\int_{(-\infty,0)} N_0\left(\mathcal{U}_0\in du,\mathcal{E}_\pm\right)\left[1-\mathbb{E}^0_u\left(e^{-\int_{(-\infty,0)} f_-(y)L^y_\zeta(X)dy}\right)\right]\right\}\\&=\exp\left\{-c\int_{(0,\infty)} db\int_{(-\infty,0)}\Pi(du-b)\left[1-\mathcal{W}_{f_-,u}(-u)\right]\right\}.\end{aligned}$$

The consideration on the overshoots and undershoots also allow to provide a further Poissonian construction involving them. Some results on transformations of Poisson point processes will be used and we refer to [10, Ch. 12] and [19, Ch. 4] for further details.

Suppose that X satisfies hypotheses **(A)** and **(B2)** and let us assume for now that it does not have a Brownian component. Then, all excursions of X away from 0 belong to the set $\mathcal{E}_\pm$. Using the additive property of local times, for any $y>0$, $L^y_{\tau(c)}(X)$ can be decomposed as the sum of the local time at y of each excursion $(s,\mathbf{e}^0_s)$ away from 0 up to local time c. Observe that, for a generic excursion $\mathbf{e}^0$, given its overshoot $b=\mathcal{O}_0(\mathbf{e}^0)$, the law of the reversed path $\underleftarrow{\mathbf{e}}^0$ under N_0 is $\widehat{\mathbb{E}}^0_b$, that is, the same law as $\widehat{X}$ started from b and killed at the first hitting time of zero. Therefore, conditionally on $\mathcal{O}_0(\mathbf{e}^0)$, this path can be decomposed again in excursions away from the infimum, say $\{(r,\underline{\mathbf{e}}_r):0<r<\mathcal{O}_0(\mathbf{e}^0)\}$, and express the local time at y in terms of the local time at $y-r$ of $\underline{\mathbf{e}}_r$. A similar computation can be performed for $L^z_{\tau(c)}(X)$, for any $z<0$, by decomposing the path $\underrightarrow{\mathbf{e}}^0$ into excursions away from the supremum, say $\{(v,\overline{\mathbf{e}}_v):\mathcal{U}_0(\mathbf{e}^0)<v<0\}$. Observe that the law $\widehat{\underline{N}}$ of the excursions away from the infimum for $\widehat{X}$ coincides with $\overline{N}$ because of duality and hence everything can be expressed in terms of this measure. Figure 3 shows a graphic representation of this decomposition.

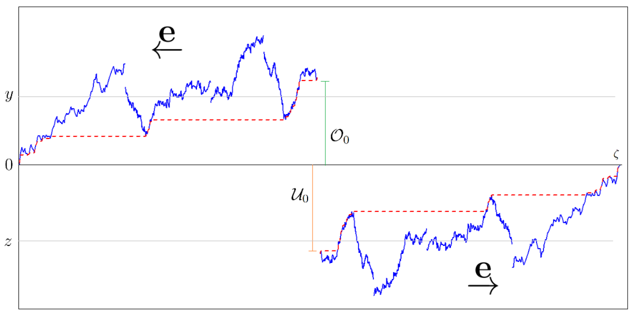


Figure 3: Representation of a typical excursion away from zero $\mathbf{e}\in\mathcal{E}_\pm$ split into $\underleftarrow{\mathbf{e}}$ and $\underrightarrow{\mathbf{e}}$. The reversed path $\underleftarrow{\mathbf{e}}$ is further decomposed into excursions away from the infimum which contribute to the local time of levels $y>0$ and $\underrightarrow{\mathbf{e}}$ is decomposed into excursions away from the supremum, contributing to the local time of levels $z<0$.

Repeating the above construction for each excursion away from 0, we have that the excursions away from zero can be seen as an atom in a marked Poisson point process that can be written as

$$\{(s,\mathcal{O}_0(\mathbf{e}^0_s),\mathcal{U}_0(\mathbf{e}^0_s),((r,\underline{\mathbf{e}}^s_r),0<r<\mathcal{O}_0(\mathbf{e}^0_s)),((v,\overline{\mathbf{e}}^s_v),\mathcal{U}_0(\mathbf{e}^0_s)<v<0)),s>0\}. \tag{3.2}$$

To be more formal, denote by $\mathcal{M}$ the space of Poisson random measures over

$(0,\infty)\times\mathcal{D}$. We define a Markovian kernel $\widehat{\kappa}$ from $(0,\infty)$ to $\mathcal{M}$ by

$$\widehat{\kappa}(b,dY)=\widehat{\mathbb{P}}\left(\sum_{0<r<b}\delta_{(r,\underline{\mathbf{e}}_r)}\in dY\right),\quad b>0, Y\in\mathcal{M}, \tag{3.3}$$

that is, the law of a random variable taking values in $\mathcal{M}$, which is given by a Poisson random measure with intensity $dr1_{\{r\in(0,b)\}}\overline{N}(\mathbf{e}\in d\omega)$. This kernel will be useful in Section 4 to establish the law of local times under $\overline{N}$. For a non-negative and measurable function $G:\mathcal{M}\to\mathbb{R}$, we write

$$\widehat{\kappa}(b,G):=\int_{\mathcal{M}}\widehat{\kappa}(b,dY)G(Y)=\widehat{\mathbb{E}}\left[G\left(\sum_{0<r<b}\delta_{(r,\underline{\mathbf{e}}_r)}\right)\right].$$

Furthermore, for later use, for any measurable functional $H:\mathbb{R}^{\mathbb{R}}\mapsto\mathbb{R}^+$ we define

$$\widehat{\kappa}\left(b,H\left(L^z_\zeta,z\in\mathbb{R}\right)\right):=\widehat{\mathbb{E}}\left[H\left(\sum_{0<r<b}\ell^{z-r}_\zeta(\underline{\mathbf{e}}_r),z\in\mathbb{R}\right)\right]. \tag{3.4}$$

One example of functional that will arise often in our calculations is, for any positive and measurable test function $f:\mathbb{R}\mapsto\mathbb{R}^+$ and any measurable set A,

$$H\left(L^z_\zeta,z\in\mathbb{R}\right)=\exp\left\{-\int_A f(z)L^z_\zeta dz\right\};$$

and for such a functional we have

$$\widehat{\kappa}\left(b,\exp\left\{-\int_A f(z)L^z_\zeta dz\right\}\right)=\widehat{\mathbb{E}}\left[\exp\left\{-\sum_{0<r<b}\int_A f(z)\ell^{z-r}_\zeta(\underline{\mathbf{e}}_r)dz\right\}\right].$$

Analogously, we define a kernel κ from $(-\infty,0)$ to $\mathcal{M}$ by

$$\kappa(u,dZ)=\mathbb{P}\left(\sum_{u<v<0}\delta_{(v,\overline{\mathbf{e}}_v)}\in dZ\right),\quad u<0, Z\in\mathcal{M}, \tag{3.5}$$

which corresponds to the law of a random variable on $\mathcal{M}$, which is a Poisson random measure of intensity $dv1_{\{v\in(u,0)\}}\overline{N}(\mathbf{e}\in d\omega)$.

Observe that, since the set $\{(s,\mathbf{e}^0_s):0<s\le c\}$ of excursions away from 0 form a Poisson point process, then the corresponding overshoots and undershoots

$$\{(s,(\mathcal{O}_0(\mathbf{e}^0_s),\mathcal{U}_0(\mathbf{e}^0_s))):0<s\le c\}$$

are also a Poisson point process, now with intensity $\widetilde{m}_\pm(ds,db,du):=ds\otimes N_0(\mathcal{O}_0\in db,\mathcal{U}_0\in du,\mathcal{E}_\pm)$. Therefore, the intensity of the marked process is given by

$$m_\pm(ds,db,du,dY,dZ)=\widetilde{m}_\pm(ds,db,du)\widehat{\kappa}(b,dY)\kappa(u,dZ).$$

Let $M_\pm(ds,db,du,dY,dZ)$ be a Poisson random measure on $(0,\infty)^2\times(-\infty,0)\times\mathcal{M}^2$ with intensity $m_\pm(ds,db,du,dY,dZ)$. By the above considerations, we can describe the local time process in terms of $M_\pm$. Moreover, since the generic Poisson random measures Y and Z can be written as $Y=\sum_{0<r<b}\delta_{(r,\underline{\mathbf{e}}_r)}$ and $Z=\sum_{u<v<0}\delta_{(v,\overline{\mathbf{e}}_v)}$, with an abuse of notation we can regard $M_\pm$ as a Poisson random measure over $(0,\infty)^2\times(-\infty,0)\times((0,\infty)\times\mathcal{D})^2$ with intensity $m_\pm(ds,db,du,dr,d\underline{\mathbf{e}},dv,d\overline{\mathbf{e}})$, where $(dr,d\underline{\mathbf{e}})$ and $(dv,d\overline{\mathbf{e}})$ are the atoms in (3.2). In order to provide separate expressions for local times of positive and negative levels, we denote by

$$M^1_\pm(ds,db,dr,d\mathbf{e})=M_\pm(ds,db,(-\infty,0),dr,d\mathbf{e},(0,\infty),\mathcal{D}),$$

the restriction of $M_\pm$ to the information on the overshoots and by

$$M^2_\pm(ds, du, dv, d\mathbf{e}) = M_\pm(ds, (0,\infty), du, (0,\infty), \mathcal{D}, dv, d\mathbf{e}),$$

the marginal related to the undershoots.

Now, if we remove the condition that X does not have a Brownian component, we additionally have to deal with excursions belonging to $\mathcal{E}_+$ and $\mathcal{E}_-$. Let $M_+(ds, d\mathbf{e})$ and $M_-(ds, d\mathbf{e})$ be Poisson random measures on $(0,\infty)\times\mathcal{D}$ with intensities $dsN_0(d\mathbf{e}, \mathcal{E}_+)$ and $dsN_0(d\mathbf{e}, \mathcal{E}_-)$, respectively. Since $\mathcal{E}_+$, $\mathcal{E}_-$ and $\mathcal{E}_\pm$ form a partition, the measures M_+, M_- and $M_\pm$ are independent. Hence, we obtain the following Poissonian representation for local times of positive and negative levels up to $\tau(c)$.

Theorem 3.3. *Let X be a SNLP satisfying* **(A)** *and* **(B2)***. Then, the following Poissonian representations hold:*

$$L^y_{\tau(c)}(X) = \int_0^c \int_{\mathcal{D}} \ell^y_{\zeta(\mathbf{e})}(\mathbf{e}) M_+(ds, d\mathbf{e}) + \int_0^c \int_0^\infty \int_0^{b\wedge y} \int_{\mathcal{D}} \ell^{y-r}_{\zeta(\mathbf{e})}(\mathbf{e})\, M^1_\pm(ds, db, dr, d\mathbf{e}), \quad (3.6)$$

for all $y>0$, and

$$L^z_{\tau(c)}(X) = \int_0^c \int_{\mathcal{D}} \ell^z_{\zeta(\mathbf{e})}(\mathbf{e}) M_-(ds, d\mathbf{e}) + \int_0^c \int_{-\infty}^0 \int_{u\vee z}^0 \int_{\mathcal{D}} \ell^{v-z}_{\zeta(\mathbf{e})}(\mathbf{e})\, M^2_\pm(ds, du, dv, d\mathbf{e}), \quad (3.7)$$

for all $z<0$.

Remark 3.4. Recall that in the case of continuous paths, the processes of local times $(L^y_{\tau(c)}(X), y\geq 0)$ and $(L^z_{\tau(c)}(X), z\leq 0)$ are independent without any conditioning, as in Theorem 1.2. In our case that is not true but as a consequence of the above Poissonian construction, if we condition to the whole Poisson point process of overshoots and undershoots at 0, the independence is recovered (notice that M_+ and M_- do not alter the conditional independence, since they are carried by disjoint sets of excursions).

4 First Ray-Knight theorem

As in the previous section, we start showing that local times up to τ^+_a are infinitely divisible and admit a Poissonian representation.

Theorem 4.1. *Let X be a spectrally negative Lévy process and denote by $\overline{N}$ the associated excursion measure away from zero for $S-X$. Assume X satisfies hypothesis* **(A)** *and* **(B1)** *or* **(B2)***. Then, the local time process $(L^{a-z}_{\tau^+_a}(X), z\geq 0)$ is infinitely divisible and its Lévy measure $\nu^{(a)}$ is given by*

$$\nu^{(a)}(d\omega) = \int_0^a \overline{N}(\ell^{\cdot - s}_\zeta 1_{\{\cdot - s>0\}} \in d\omega) ds,$$

where for each s, $\overline{N}(\ell^{\cdot-s}_\zeta 1_{\{\cdot-s>0\}} \in d\omega)$ is the image of $\overline{N}$ under the map that assigns to each excursion its local time process shifted by s. Moreover, local times admit the representation

$$L^{a-z}_{\tau^+_a}(X) = \begin{cases} \int_0^z \int_{\mathcal{D}} \ell(z-s) K(ds, d\ell), & z\in[0,a], \\ \int_0^a \int_{\mathcal{D}} \ell(z-s) K(ds, d\ell), & z\geq a, \end{cases} \quad (4.1)$$

where K is a Poisson random measure of intensity $ds\otimes M(d\ell)$, M being the image of $\overline{N}$ under the map that associates an excursion $\mathbf{e}$ to its local time process up to its lifetime: $\mathbf{e}\mapsto \Big(\ell^r_\zeta(\mathbf{e}), r\in\mathbb{R}\Big)$.

This representation is similar to that in [15, Ch. 6] for CB processes with linear immigration but, as in the case of $\tau(c)$, M cannot be a Kuznetsov measure. Nonetheless, certain branching-like structure can be recovered for local times, in the sense that we now explain (see also Figure 4 below). The local time of X at each level can be decomposed as the sum of the local time contributions of these excursions to that level. We view the excursions from level a downwards and think of them as individuals immigrating. The linear behavior of immigration comes from the fact that the supremum is linear on the local time scale. Since S takes values on $[0,a]$, there is no extra immigrants for negative levels, which explains the difference in the representation in Theorem 4.1 for levels in $[0,a]$ and levels in $(-\infty,0]$. For the branching part, if $0<x<y$, an excursion which has a contribution to the local time at $a-x$ will also have a contribution at $a-y$ if the excursion is deep enough. So, we can interpret the contribution of an excursion to $L^{a-y}_{\tau_a^+}(X)$ as a "mass branching" from $L^{a-x}_{\tau_a^+}(X)$, and because of the lack of positive jumps we can "track" the descendants of level $a-x$.

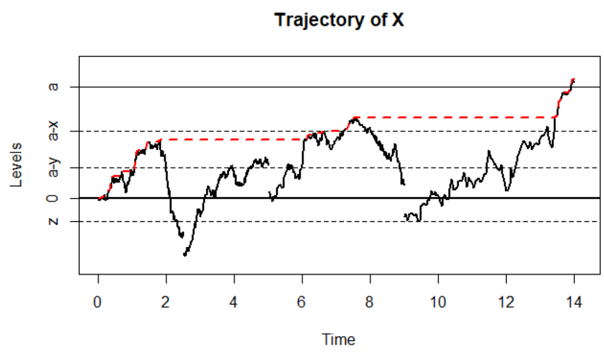


Figure 4: A path of X (bold line) and its supremum (dashed line) up to the time it surpasses level a.

We can get a better understanding of the Lévy measure $\nu^{(a)}$ by studying the law of local times under $\overline{N}$. This turns out to be difficult if one wants to consider all the levels but it is tractable when considering a positive reference level $x>0$, which can be made arbitrarily small.

Given $x>0$, an excursion $\mathbf{e}$ away from the supremum with height $H(\mathbf{e})<x$ has local time equal to zero at levels $y\geq x$. Hence, we will restrict ourselves to those excursions satisfying $H(\mathbf{e})>x$. On the event $H(\mathbf{e})>x$, we know that, by the absence of negative jumps under $\overline{N}$, the first hitting time $T_x(\mathbf{e})$ occurs before the lifetime $\zeta(\mathbf{e})$. Then, by the strong Markov property and the additivity of local times we can write

$$\ell^y_{\zeta(\mathbf{e})}(\mathbf{e})=\ell^y_{T_x(\mathbf{e})}(\mathbf{e})+\ell^y_{\zeta(\mathbf{e})}(\mathbf{e})\circ\theta_{T_x(\mathbf{e})},\quad y>x,$$

which splits the local time process for levels $y>x$ into two independent components: the information before hitting x and the information after, the latter being able to be written in terms of the excursions away from x. Both terms can be described in a Poissonian way, similar to the one in the previous section, as we will see next.

We begin with $(\ell^y_{T_x(\mathbf{e})}(\mathbf{e}),y>x)$. We only need to consider the event $\tau_x^+(\mathbf{e})<T_x(\mathbf{e})<\zeta(\mathbf{e})$, since on the event $\tau_x^+(\mathbf{e})=T_x(\mathbf{e})$, the accumulated local time up to $T_x(\mathbf{e})$ is zero for levels $y\geq x$. Therefore, the first overshoot of the excursion at x will play a key role (see Lemma 6.2 for its law under $\overline{N}$) and conditionally on it, we can use a similar decomposition of the path into excursions away from the infimum as in the previous section. This yields the following theorem.

Theorem 4.2. *Let X be a spectrally negative Lévy process satisfying* **(A)** *and* **(B1)** *or* **(B2)***, and $\overline{N}$ its excursion measure away from the supremum. For any measurable* *functional* $F : \mathbb{R}_+^{(x,\infty)} \to \mathbb{R}_+$ *with* $F(0) = 0$*, we have that*

$$\overline{N}(F(\ell^y_{T_x}, y > x), H > x) = \int_0^\infty \overline{N}(\mathcal{O}_x \in db, \tau_x^+ < T_x < \zeta)\widehat{\kappa}(b, F(L^{y-x}_\zeta, y > x)),$$

with $\widehat{\kappa}$ *as in* (3.4)*. In particular, for any* $f : (x,\infty) \to \mathbb{R}_+$ *measurable and bounded,*

$$\begin{aligned}
&\overline{N}\left(\exp\left\{-\int_x^\infty f(y)\ell^y_{T_x} dy\right\}, H > x\right) \\
&= \int_0^x d\ell \int_{(0,\infty)} \widehat{\Pi}(db + \ell) W(x-\ell) \left(\frac{W'(x-\ell)}{W(x-\ell)} - \frac{W'(x)}{W(x)}\right) \mathcal{W}_{f_{x,b}}(b),
\end{aligned}$$

where $\widehat{\Pi}$ *is given by* $\widehat{\Pi}(A) = \Pi(-A)$*,* $f_{x,b}(z) := f(x+b-z),\ z < b$ *and* $\mathcal{W}_{f_{x,b}}$ *as in* (1.5)*.*

Now focus on $(\ell^y_{\zeta(\mathbf{e})}(\mathbf{e}) \circ \theta_{T_x(\mathbf{e})}, y > x)$. Starting from time $T_x(\mathbf{e})$, we can decompose the path into excursions away from x. This bears some similarities with the situation of the second Ray-Knight Theorem in which we had "c" excursions away from 0, but in this case we have as many excursions away from x as the local time $\ell^x_{\zeta(\mathbf{e})}(\mathbf{e})$, which can be proved that is exponentially distributed with parameter $q^0_x := \widehat{N}_x(\tau_0^- < \zeta)$. This and the distribution of the overshoots of the excursions away from x (see Lemma 6.4) lead to the following theorem.

Theorem 4.3. *Let* X *be a spectrally negative Lévy process satisfying* **(A)** *and* **(B1)** *or* **(B2)** *and* $\overline{N}$ *its excursion measure away from the supremum. Denote* $q^0_x = \widehat{N}_x(\tau_0^- < \zeta)$*. Then,* $q^0_x = 1/W(x)$ *and* $L^x_{\tau_0^-}$ *is exponentially distributed with parameter* q^0_x *under* $\widehat{\mathbb{E}}_x$*. Moreover, for any measurable function* $f : (x,\infty) \to \mathbb{R}_+$*, we have that*

$$\begin{aligned}
&\overline{N}\left(\exp\left\{-\int_x^\infty f(y)\ell^y_\zeta \circ \theta_{T_x} dy\right\}, H > x\right) \\
&= \overline{N}(H > x)\widehat{\mathbb{E}}_x\left[\exp\left\{-L^x_{\tau_0^-}\widehat{N}_x\left(1 - e^{-\int_x^\infty f(y)\ell^y_\zeta dy}, \tau_0^- > \zeta\right)\right\}\right] \qquad (4.2)\\
&= \frac{W'(x)}{W(x)} \frac{1}{1 + W(x)\widehat{N}_x\left(1 - e^{-\int_x^\infty f(y)\ell^y_\zeta dy}, \tau_0^- > \zeta\right)}.
\end{aligned}$$

The term under $\widehat{N}_x$ *can be further decomposed into*

$$\begin{aligned}
\widehat{N}_x\left(1 - e^{-\int_x^\infty f(y)\ell^y_\zeta dy}, \tau_0^- > \zeta\right) &= \widehat{N}_x\left(1 - e^{-\int_x^\infty f(y)\ell^y_\zeta dy}, \mathcal{E}^x_+\right) \\
&\quad + \widehat{N}_x\left(1 - e^{-\int_x^\infty f(y)\ell^y_\zeta dy}, \tau_0^- > \zeta, \mathcal{E}^x_\pm\right),
\end{aligned}$$

where $\mathcal{E}^x_+$ *is the set of excursions which are completely above* x *and* $\mathcal{E}^x_\pm$ *are the excursions away from* x *which start negative and then jump above* x*. For this latter set, we have the following expression in terms of the law of the overshoot:*

$$\begin{aligned}
&\widehat{N}_x\left(1 - e^{-\int_x^\infty f(y)\ell^y_\zeta dy}, \tau_0^- > \zeta, \mathcal{E}^x_\pm\right) \\
&= \int_{(0,\infty)} \widehat{N}_x(\mathcal{O}_x \in db, \tau_0^- > \zeta, \mathcal{E}^x_\pm)\left[1 - \exp\left\{-\int_0^b ds \overline{N}\left(1 - e^{-\int_{s+x}^\infty f(y)\ell^{y-x-s}_\zeta dy}\right)\right\}\right] \\
&= \int_{(0,\infty)} \widehat{N}_x(\mathcal{O}_x \in db, \tau_0^- > \zeta, \mathcal{E}^x_\pm)\left[1 - \mathcal{W}_{f_{x,b}}(b)\right],
\end{aligned}$$

where $f_{x,b}(z) = f(x+b-z),\ z < b$ *and* $\mathcal{W}_{f_{x,b}}$ *as in* (1.5)*.*

Remark 4.4. Observe that the identity (4.2) is reminiscent of the branching property. Indeed, we can rewrite it as

$$\frac{\overline{N}\left(e^{-\int_x^\infty f(y)\ell_\zeta^y\circ\theta_{T_x}dy}, H > x\right)}{\overline{N}(H > x)} = \widehat{\mathbb{E}}_x\left[\exp\left\{-L^x_{\tau_0^-}\widehat{N}_x\left(1 - e^{-\int_x^\infty f(y)\ell_\zeta^y dy}, \tau_0^- > \zeta\right)\right\}\right].$$

The left hand side can be seen as an expected value conditioned to $H > x$, which is equivalent to having a positive amount of local time at x. And on the right hand side, we have that the contribution to the local time of each level y is coming from the $L^x_{\tau_0^-}$ "individuals" present at level x. In this case, the term corresponding to $\widehat{N}_x$ can be interpreted as the corresponding cumulant. Besides, if we consider a single level $y > x$, the Proposition 4.7 below provides an explicit expression for this cumulant.

It turns out that, without splitting the information at time $T_x(\mathbf{e})$ for local times of levels bigger than x, a similar property is satisfied. Actually, we can consider a functional of all levels above x as before or just consider a finite set of points, as the following proposition states.

Proposition 4.5. *Let X be a spectrally negative Lévy process satisfying* **(A)** *and* **(B1)** *or* **(B2)***, $\overline{N}$ its excursion measure away from the supremum and N_z, $\widehat{N}_z$ the excursion measures away from z for X and its dual $\widehat{X}$, respectively. Let $x > 0$ and $f : (x,\infty) \to \mathbb{R}_+$ a measurable and bounded function. Denote by*

$$u_x(f) := \widehat{N}_x\left(1 - e^{-\int_x^\infty f(y)\ell_\zeta^y dy}, \tau_0^- > \zeta\right) = N_0\left(1 - e^{-\int_x^\infty f(y)\ell_\zeta^{x-y} dy}, \tau_x^+ > \zeta\right).$$

Then,

$$\overline{N}\left(1 - e^{-\lambda\ell_\zeta^x - \int_x^\infty f(y)\ell_\zeta^y dy}\right) = \overline{N}\left(1 - e^{-(\lambda + u_x(f))\ell_\zeta^x - \int_x^\infty f(y)\ell_{T_x}^y dy}\right), \quad \lambda \geq 0.$$

Alternatively, if $0 < x < y_1, \ldots, y_n$ are n distinct points and

$$u_{x;y_1,\ldots,y_n}(\beta_1, \ldots, \beta_n) := \widehat{N}_x\left(1 - e^{-\beta_1\ell_\zeta^{y_1} - \cdots - \beta_n\ell_\zeta^{y_n}}, \tau_0^- > \zeta\right),$$

then, for any $\lambda, \beta_1, \ldots, \beta_n \geq 0$,

$$\overline{N}\left(1 - e^{-\lambda\ell_\zeta^x - \beta_1\ell_\zeta^{y_1} - \cdots - \beta_n\ell_\zeta^{y_n}}\right) = \overline{N}\left(1 - e^{-(\lambda + u_{x;y_1,\ldots,y_n}(\beta_1,\ldots,\beta_n))\ell_\zeta^x - \beta_1\ell_{T_x}^{y_1} - \cdots - \beta_n\ell_{T_x}^{y_n}}\right).$$

We end this section with explicit expressions of the law of the local time of a single point under $\overline{N}$ and $\widehat{N}_x$, which can be given in terms of scale functions.

Proposition 4.6. *Let X be a spectrally negative Lévy process satisfying* **(A)** *and* **(B1)** *or* **(B2)** *and $\overline{N}$ its excursion measure away from the supremum. Let*

$$u_y(\lambda) := \overline{N}\left(1 - e^{-\lambda\ell_\zeta^y}\right), \qquad y > 0,\ \lambda \geq 0.$$

This quantity can be expressed in terms of the scale function W as follows

$$u_y(\lambda) = \frac{\lambda W'(y)}{1 + \lambda W(y)}, \qquad \lambda \geq 0, y \geq 0.$$

Proposition 4.7. *Let X be a spectrally negative Lévy process satisfying* **(A)** *and* **(B1)** *or* **(B2)** *and $\widehat{N}_z$ the excursion measure away from z for $\widehat{X}$. Let us define, for any $x > 0$,*

$$v_{x,y}(\lambda) := \widehat{N}_x\left(\left(1 - e^{-\lambda\ell_\zeta^y}\right)1_{\{\tau_0^- > \zeta\}}\right), \qquad y \geq x,\ \lambda \geq 0.$$

Then,

$$v_{x,y}(\lambda) = \widehat{N}^0_x(H > y)\left(\frac{\lambda W(y-x)}{1+\lambda W(y-x)}\right), \qquad \lambda \geq 0, y \geq x,$$

where $\widehat{N}^0_x(\cdot) := \widehat{N}_x\left(\cdot; \tau_0^- > \zeta\right)$. *Moreover,*

$$\widehat{N}^0_x(H > y) = \frac{\sigma^2}{2}\frac{W'(y-x)}{W(y-x)} + \int_0^x \frac{W(x-z)}{W(x)}\left(\Pi(-\infty,-z) - \int_{-z-(y-x)}^{-z}\frac{W(u+z+y-x)}{W(y-x)}\Pi(du)\right)dz.$$

5 Decomposition of Lévy measures

Recall that a non-negative infinitely divisible process $\psi = (\psi_x, x \in E)$ is characterized by its Lévy measure μ. Given a reference state $h \in E$, Eisenbaum [7] also describes in their Theorem 1.2 a way to decompose μ into $\mu_h + \overline{\mu}_h$. The measure $\mu_h(d\omega) = 1_{\{\omega(h)=0\}}\mu(d\omega)$ is the Lévy measure of the process ψ conditioned to $\psi_h = 0$ and hence, the measure $\overline{\mu}_h(d\omega) = 1_{\{\omega(h)>0\}}\mu(d\omega)$ corresponds to the information of the process between successive visits to state h.

As an application, we provide some information on this decomposition of the Lévy measures $\mu^{(c)}$, relative to $(L^y_{\tau(c)}(X), y \in \mathbb{R})$ and $\nu^{(a)}$, corresponding to $(L^{a-z}_{\tau_a^+}(X), z \geq 0)$.

Consider first the measure $\mu^{(c)}$, which is given by $\mu^{(c)}(d\omega) = cN_0(\ell^{\cdot}_\zeta \in d\omega)$. For a positive level h, we can identify one of the components of the corresponding decomposition in terms of an exit problem (and hence in terms of scale functions), as can be seen in the next proposition.

Proposition 5.1. *Let $\mu^{(c)}$ be the Lévy measure of the process $(L^y_{\tau(c)}(X), y \in \mathbb{R})$ and $h > 0$ fixed. Decompose*

$$\mu^{(c)}(d\omega) = \mu^{(c)}_h(d\omega) + \overline{\mu}^{(c)}_h(d\omega) = 1_{\{\omega(h)=0\}}\mu^{(c)}(d\omega) + 1_{\{\omega(h)>0\}}\mu^{(c)}(d\omega).$$

Then,

$$\mu^{(c)}_h(d\omega) = cN_0\left[\ell^{\cdot}_\zeta \in d\omega, \tau_h^+ > \zeta\right] = cN_0\left[\ell^{\cdot}_\zeta \in d\omega, \tau_h^+ > \zeta, \mathcal{E}_+\right] + cN_0\left[\ell^{\cdot}_\zeta \in d\omega, \mathcal{E}_-\right] + c\int_{(0,h)}\int_{(-\infty,0)} db\Pi(du-b)\widehat{\mathbb{E}}^0_b \otimes \mathbb{E}^0_u(L^{\cdot}_\zeta \in d\omega, \tau_h^+ > \zeta),$$

where $\widehat{\mathbb{E}}^0_b \otimes \mathbb{E}^0_u(L^{\cdot}_\zeta \in d\omega, \tau_h^+ > \zeta) := \widehat{\mathbb{E}}_b\left(L^{\cdot}_{\tau_0^-} \in d\omega_+, \tau_h^+ > \tau_0^-\right)\mathbb{E}_u\left(L^{\cdot}_{\tau_0^+} \in d\omega_-\right)$. *In particular, if X does not have a Brownian component* $(\Sigma = 0)$, *then for any $f : \mathbb{R}_+ \to \mathbb{R}$ measurable and bounded,*

$$\int_{\mathbb{R}_+^{\mathbb{R}}}\left(1 - e^{-\int_0^\infty f(x)\omega(x)dx}\right)\mu^{(c)}_h(d\omega) = c\int_0^h db\Pi(-\infty,-b)\frac{W_{\widehat{f}}(-b,-h)}{W_{\widehat{f}}(0,-h)},$$

where $\widehat{f}(s) = f(-s)$ *and* $W_{\widehat{f}}$ *as in* (1.4).

Proof of Proposition 5.1. In this case, we know that $\mu^{(c)}(d\omega) = cN_0(\ell^{\cdot}_\zeta \in d\omega)$. Hence, for $h > 0$, we have that $\ell^h_\zeta = 0$ under N_0 if and only if the excursion e away from 0 does not reach level h. This means that positive excursions must not have height bigger than h and that the paths $\underleftarrow{\mathrm{e}}$ for mixed excursions have overshoots less than h and from this point, the reversed path must exit the interval $[0,h]$ by below. So we identify in this case $\mu_h^{(c)}$ as

$$\mu_h^{(c)}(d\omega) = cN_0\left[\ell_\zeta^\cdot \in d\omega, \tau_h^+ > \zeta\right] = cN_0\left[\ell_\zeta^\cdot \in d\omega, \tau_h^+ > \zeta, \mathcal{E}_+\right] + cN_0\left[\ell_\zeta^\cdot \in d\omega, \mathcal{E}_-\right] \\ + c\int_{(0,h)}\int_{(-\infty,0)} db\Pi(du-b)\widehat{\mathbb{E}}_b^0 \otimes \mathbb{E}_u^0(L_\zeta^\cdot \in d\omega, \tau_h^+ > \zeta).$$

In particular, if one only considers positive levels the last term simplifies to $\widehat{\mathbb{E}}_b(L_\zeta^\cdot \in d\omega_+, \tau_h^+ > \tau_0^-)$, which is the law of local times of $\widehat{X}$ issued from b and seen up to first time it exits $[0,h]$ from above. An expression for Laplace transforms of this exit problem is given in terms generalized scale functions (see (1.4)). Indeed, if f is measurable and bounded and has support on $\mathbb{R}_+$,

$$\begin{aligned}\widehat{\mathbb{E}}_b^0\left(e^{-\int_0^\infty f(x)L_\zeta^x dx}, \tau_h^+ > \zeta\right) &= \widehat{\mathbb{E}}_b\left(e^{-\int_0^\infty f(x)L_{\tau_0^-}^x dx}, \tau_h^+ > \tau_0^-\right) \\ &= \widehat{\mathbb{E}}_b\left(e^{-\int_0^{\tau_0^-} f(X_s)ds}, \tau_h^+ > \tau_0^-\right) \\ &= \mathbb{E}_{-b}\left(e^{-\int_0^{\tau_0^+} \widehat{f}(X_s)ds}, \tau_{-h}^- > \tau_0^+\right) \\ &= \frac{W_{\widehat{f}}(-b,-h)}{W_{\widehat{f}}(0,-h)}.\end{aligned}$$

Hence, if X does not have a Brownian component, we conclude that

$$\int_{\mathbb{R}_+^{\mathbb{R}}}\left(1 - e^{-\int_0^\infty f(x)\omega(x)dx}\right)\nu_h(d\omega) = c\int_0^h db\Pi(-\infty,-b)\frac{W_{\widehat{f}}(-b,-h)}{W_{\widehat{f}}(0,-h)}. \qquad \square$$

Now consider the Lévy measure $\nu^{(a)}$, which from Theorem 4.1 is given by $\nu^{(a)}(d\omega) = \int_0^a \overline{N}(\ell_\zeta^{\cdot-s}1_{\{\cdot-s>0\}} \in d\omega)ds$. It turns out that, for a fixed $h>0$, we can make use of the results of Theorems 4.2 and 4.3 to describe certain functionals of both $\nu_h^{(a)}$ and $\overline{\nu}_h^{(a)}$.

Proposition 5.2. *Let $\nu^{(a)}$ be the Lévy measure of the process $(L_{\tau_a^+}^{a-z}(X), z \geq 0)$ and $h>0$ fixed. Decompose*

$$\nu^{(a)}(d\omega) = \nu_h^{(a)}(d\omega) + \overline{\nu}_h^{(a)}(d\omega) = 1_{\{\omega(h)=0\}}\nu^{(a)}(d\omega) + 1_{\{\omega(h)>0\}}\nu^{(a)}(d\omega).$$

Then,

$$\nu_h^{(a)}(d\omega) = \int_0^{a\wedge h} ds\overline{N}(\ell_\zeta^{\cdot-s}1_{\{\cdot-s>0\}} \in d\omega, \tau_{h-s}^+ > \zeta) + \int_{a\wedge h}^a ds\overline{N}(\ell_\zeta^{\cdot-s}1_{\{\cdot-s>0\}} \in d\omega).$$

Moreover, if F is any non-negative, measurable and bounded functional,

$$\begin{aligned}\nu_h^{(a)}(F(\omega_y, y\geq h)) &= \int_0^a ds\overline{N}(F(\ell_{T_h}^{y-s}, y-s>h), H>h) \\ &= \int_0^a ds\int_{(0,\infty)}\overline{N}(\mathcal{O}_h \in db, \tau_h^+ < T_h < \zeta)\widehat{\kappa}(b, F(L_\zeta^{y-s-h}, y-s>h)),\end{aligned}$$

with $\widehat{\kappa}$ as in (3.4)*, and on the other hand,*

$$\overline{\nu}_h^{(a)}(F(\omega_y, y>h)) = \int_0^a ds\overline{N}(F(\ell_\zeta^{y-s}\circ\theta_{T_h}, y-s>h), H>h).$$

Proof of Proposition 5.2. In this case one has that $\ell_\zeta^{h-s} = 0$ if and only if either $s \geq h$ or $s < h$ and the excursion away from the supremum does not reach level $h - s$. Therefore, we obtain

$$\begin{aligned}\nu_h^{(a)}(d\omega) &= \nu^{(a)}(d\omega)\Big|_{\{\omega(h)=0\}} \\ &= \int_0^{a\wedge h} ds\overline{N}(\ell_\zeta^{\cdot-s}1_{\{\cdot-s>0\}} \in d\omega, \tau_{h-s}^+ > \zeta) + \int_{a\wedge h}^a ds\overline{N}(\ell_\zeta^{\cdot-s}1_{\{\cdot-s>0\}} \in d\omega),\end{aligned}$$

where the last term disappears if $h \geq a$. Recall that Theorem 4.2 gives an expression for the law of local times of levels bigger than a reference level $h > 0$ under $\overline{N}$. Since the process $(\ell_{T_h}^y, y > h)$ codes the information of the local times of levels bigger than h previous to its first hitting time, this implies we can actually use that result to provide information on $\nu_h^{(a)}$. Therefore, if F is any non-negative, measurable and bounded functional,

$$\begin{aligned}\nu_h^{(a)}(F(\omega_y, y \geq h)) &= \int_0^a ds\overline{N}(F(\ell_{T_h}^{y-s}, y - s > h), H > h) \\ &= \int_0^a ds \int_{(0,\infty)} \overline{N}(\mathcal{O}_h \in db, \tau_h^+ < T_h < \zeta)\widehat{\kappa}(b, F(L_\zeta^{y-s-h}, y - s > h)).\end{aligned}$$

Similarly, in Theorem 4.3 the process $(\ell_\zeta^y \circ \theta_{T_h}, y > h)$ encodes the information of the excursion from the first to the last visit to h, which implies we can recover information on the measure $\overline{\nu}_h^{(a)}$. Then, for any measurable, bounded and non-negative functional F,

$$\overline{\nu}_h^{(a)}(F(\omega_y, y > h)) = \int_0^a ds\overline{N}(F(\ell_\zeta^{y-s} \circ \theta_{T_h}, y - s > h), H > h). \qquad \square$$

In particular, if the functional on the above proposition is of the form $F((\omega_y, y > h)) = \exp\left\{-\int_h^\infty f(y)\omega_y dy\right\}$, we can write

$$\begin{aligned}&\nu_h^{(a)}(F(\omega_y, y \geq h)) \\ &\quad= \int_0^a ds \int_0^x d\ell \int_{(0,\infty)} \widehat{\Pi}(db + \ell)W(x - \ell)\left(\frac{W'(x - \ell)}{W(x - \ell)} - \frac{W'(x)}{W(x)}\right)\mathcal{W}_{f_{x,b,s}}(b),\end{aligned}$$

where $f_{x,b,s}(z) := f(x + b + s - z),\ z < s + b$. Similarly,

$$\overline{\nu}_h^{(a)}(F(\omega_y, y > h)) = \overline{N}(H > h)\int_0^a ds\widehat{\mathbb{E}}_h\left[\exp\left\{-L_{\tau_0^-}^h\widehat{N}_x\left(1 - e^{-\int_h^\infty f(s+y)\ell_\zeta^y dy}, \tau_0^- > \zeta\right)\right\}\right].$$

6 Auxiliary results

The next lemma provides the joint law of the overshoot and undershoot of an excursion under the measure N_0.

Lemma 6.1. *Let $(\mathcal{U}_0(\mathrm{e}), \mathcal{O}_0(\mathrm{e})) = (\mathrm{e}(\tau_0^-), \mathrm{e}(\tau_0^- -))$ be the undershoot and overshoot of an excursion e away from zero. Then,*

$$N_0\left(h(\mathcal{U}_0, \mathcal{O}_0), \mathcal{E}_\pm\right) = \int_0^\infty dz \int_{(-\infty,0)} \Pi(dy)h(z + y, z)1_{\{z+y<0\}}, \tag{6.1}$$

for any measurable and bounded function $h : (-\infty, 0) \times (0, \infty) \to \mathbb{R}_+$. Said otherwise,

$$N_0(\mathcal{U}_0 \in dy, \mathcal{O}_0 \in dz, \mathcal{E}_\pm) = dz\Pi(dy - z)1_{\{z>0\}}1_{\{y<0\}}. \tag{6.2}$$

Taking marginals in the above expression we get that, for any $f : (-\infty, 0) \to \mathbb{R}_+$ *and* $g : (0, \infty) \to \mathbb{R}_+$ *measurable and bounded,*

$$\begin{aligned} N_0\left(f(\mathcal{U}_0), \mathcal{E}_\pm\right) &= \int_0^\infty dz \int_{(-\infty,0)} \Pi(dy) f(z+y) 1_{\{z+y<0\}} \\ &= \int_0^\infty dz \int_{(-\infty,z)} \Pi(dy-z) f(y) 1_{\{y<0\}} \end{aligned}$$

and

$$N_0\left(g(\mathcal{O}_0), \mathcal{E}_\pm\right) = \int_0^\infty dz g(z) \Pi(-\infty, -z).$$

Proof of Lemma 6.1. Recall that $\mathcal{E}_\pm$ corresponds to the set of excursions for which $\{0 < \tau_0^- < \zeta\}$. Observe that this condition is fulfilled if and only if there exists an $s \in (0, \zeta)$ such that $\Delta_s := \mathbf{e}(s) - \mathbf{e}(s-) < 0$, $\mathbf{e}(s-) > 0$ and $\mathbf{e}(s-) + \Delta_s < 0$. Let us define

$$G_s(y) = h(\mathbf{e}(s-) + y, \mathbf{e}(s-)) 1_{\{y<0\}} 1_{\{\mathbf{e}(s-)>0\}} 1_{\{\mathbf{e}(s-)+y<0\}} 1_{\{s<\zeta\}}.$$

Then, we have the following identity

$$h(\mathbf{e}(\tau_0^-), \mathbf{e}(\tau_0^- -)) 1_{\{0<\tau_0^-<\zeta\}} = \sum_{0<s<\infty} G_s(\Delta_s) 1_{\{\Delta_s \neq 0\}}.$$

Indeed, by the observation above the indicator on the left hand side is non-zero if and only if there exists an $s \in (0, \infty)$ satisfying the previous conditions. Moreover, by the absence of positive jumps that s is unique and coincides with τ_0^-. Therefore, we can use the compensation formula under N_0 (see for instance [18, Eq. (18)]) to obtain

$$\begin{aligned} &N_0\left(h(\mathbf{e}(\tau_0^-), \mathbf{e}(\tau_0^- -)) 1_{\{0<\tau_0^-<\zeta\}}\right) \\ &\quad = N_0\left(\int_0^\infty ds \int_{(-\infty,0)} \Pi(dy) G_s(y)\right) \\ &\quad = N_0\left(\int_0^\infty ds \int_{(-\infty,0)} \Pi(dy) h(\mathbf{e}(s-) + y, \mathbf{e}(s-)) 1_{\{y<0\}} 1_{\{\mathbf{e}(s-)>0\}} 1_{\{\mathbf{e}(s-)+y<0\}} 1_{\{s<\zeta\}}\right) \\ &\quad = N_0\left(\int_0^\zeta ds 1_{\{\mathbf{e}(s-)>0\}} \int_{(-\infty,0)} \Pi(dy) h(\mathbf{e}(s-) + y, \mathbf{e}(s-)) 1_{\{\mathbf{e}(s-)+y<0\}}\right). \end{aligned}$$

Since the set of times $\{s : \mathbf{e}(s-) \neq \mathbf{e}(s)\}$ in which the excursion is discontinuous has Lebesgue measure zero, we can replace $\mathbf{e}(s-)$ by $\mathbf{e}(s)$. Therefore,

$$\begin{aligned} &= N_0\left(\int_0^\zeta ds 1_{\{\mathbf{e}(s)>0\}} \int_{(-\infty,0)} \Pi(dy) h(\mathbf{e}(s) + y, \mathbf{e}(s)) 1_{\{\mathbf{e}(s)+y<0\}}\right) \\ &= N_0\left(\int_0^\zeta ds 1_{\{\mathbf{e}(s)>0\}} \widetilde{h}(\mathbf{e}(s))\right), \end{aligned}$$

where $\widetilde{h}(z) = \int_{(-\infty,0)} \Pi(dy) h(z+y, z) 1_{\{z+y<0\}}$. Then, using the fact that $\Phi(0) = 0$ because of assumptions **(B1)** or **(B2)** and using identity (20) from [18] with $\lambda \downarrow 0$, we conclude that

$$\begin{aligned} N_0\left(h(\mathbf{e}(\tau_0^-),\mathbf{e}(\tau_0^- -))1_{\{0<\tau_0^-<\zeta\}}\right) &= N_0\left(\int_0^\zeta ds 1_{\{s<\tau_0^-\}}\widetilde{h}(\mathbf{e}(s))\right) \\ &= \int_0^\infty dz\widetilde{h}(z) \\ &= \int_0^\infty dz \int_{(-\infty,0)} \Pi(dy)h(z+y,z)1_{\{z+y<0\}} \\ &= \int_0^\infty dz \int_{(-\infty,z)} \Pi(dy-z)h(y,z)1_{\{y<0\}} \\ &= \int_0^\infty dz \int_{(-\infty,0)} \Pi(dy-z)h(y,z). \end{aligned}$$

The rest of the proof follows readily. □

The following lemma expresses the law of the overshoot with respect to a positive reference level under the measure $\overline{N}$.

Lemma 6.2. *For any $x>0$ and any measurable and bounded function $f:\mathbb{R}^2\to\mathbb{R}_+$,*

$$\overline{N}\left(\mathbf{e}(\tau_x^+)=x,\tau_x^+<\zeta\right)=\frac{\sigma^2}{2}\left[\frac{[W'(x)]^2}{W(x)}-W''(x)\right],$$

and

$$\begin{aligned} \overline{N}\left(f(\mathbf{e}(\tau_x^+ -),\mathbf{e}(\tau_x^+));\mathbf{e}(\tau_x^+)>x,\tau_x^+<\zeta\right) &= \int_0^x d\ell W(x-\ell)\left(\frac{W'(x-\ell)}{W(x-\ell)}-\frac{W'(x)}{W(x)}\right) \\ &\quad\times\int_{-\infty}^0 \Pi(dy)f(x-\ell,x-\ell-y)1_{\{0>\ell+y\}}. \end{aligned}$$

In particular, taking f to be a function of only the overshoot, $f(\mathbf{e}(\tau_x^+ -),\mathbf{e}(\tau_x^+)) = g(\mathbf{e}(\tau_x^+)-x)$, and with the notation $\widehat{\Pi}(A):=\Pi(-A)$, we have that

$$\begin{aligned} &\overline{N}\left(g(\mathbf{e}(\tau_x^+)-x);T_x<\tau_x^+<\zeta\right) \\ &\quad=\int_0^x d\ell W(x-\ell)\left(\frac{W'(x-\ell)}{W(x-\ell)}-\frac{W'(x)}{W(x)}\right)\int_0^\infty \widehat{\Pi}(dy)g(y-\ell)1_{\{y>\ell\}} \\ &\quad=\int_0^x d\ell W(x-\ell)\left(\frac{W'(x-\ell)}{W(x-\ell)}-\frac{W'(x)}{W(x)}\right)\int_0^\infty \widehat{\Pi}(dy+\ell)g(y)1_{\{y>0\}}, \end{aligned}$$

which provides the law of the overshoot with respect to x under $\overline{N}$.

Proof of Lemma 6.2. We will use here a result from L. Chaumont and R. Doney in [3] which tells us a way to approximate $\overline{N}$ as a certain limit involving $\mathbb{E}$. Indeed, if $g_\beta(z)=\frac{(1-e^{z\Phi(\beta)})}{\Phi(\beta)}$, $z\in\mathbb{R}$, using that in our case $\Phi(0)=0$ we have that

$$\lim_{\beta\downarrow 0} g_\beta(z):=g(z)=-z,\ z<0.$$

Their result states that

$$\lim_{z\to 0^-}\frac{1}{g(z)}\mathbb{E}_z(F,t<\tau_0^+)=\widehat{\overline{N}}(F,t<\zeta),$$

for any functional F up to time t and $\widehat{\overline{N}}$ being the excursion measure away from the supremum for $\widehat{X}$. (We notice that the cited result from [3] includes a multiplicative constant depending on the normalization of the local time, and in our case it equals 1 because we assumed that $\Psi(1) = 1 = \Phi(1)$.) This is translated to $\overline{N}$ as

$$\lim_{z\to 0^-} \frac{1}{g(z)} \widehat{\mathbb{E}}_{-z}(F, t < \tau_0^-) = \overline{N}(F, t < \zeta),$$

or equivalently,

$$\lim_{z\to 0^+} \frac{1}{z} \widehat{\mathbb{E}}_{z}(F, t < \tau_0^-) = \overline{N}(F, t < \zeta).$$

The so-called Kesten's identity (see for instance [11]):

$$\mathbb{P}_z\left(X_{\tau_0^-} = 0, \tau_0^- < \infty\right) = \frac{\sigma^2}{2}(W'(z) - \Phi(0)W(z)), \quad z > 0,$$

will also be useful for the result.

We will use the above facts for the following computation. We can calculate, for $x > 0$

$$\begin{aligned}
\overline{N}\left(\mathbf{e}(\tau_x^+) = x, \tau_x^+ < \zeta\right) &= \lim_{z\to 0^+} \frac{1}{z} \widehat{\mathbb{E}}_z\left(X_{\tau_x^+} = x, \tau_x^+ < \tau_0^-\right) \\
&= \lim_{z\to 0^+} \frac{1}{z} \mathbb{E}\left(X_{\tau_{z-x}^-} = z - x, \tau_{z-x}^- < \tau_z^+\right) \\
&= \lim_{z\to 0^+} \frac{1}{z} \left[\mathbb{E}\left(X_{\tau_{z-x}^-} = z - x, \tau_{z-x}^- < \infty\right)\right. \\
&\qquad \left. - \mathbb{E}\left(X_{\tau_{z-x}^-} = z - x, \tau_z^+ < \tau_{z-x}^-\right)\right].
\end{aligned}$$

Then, using Kesten's identity and the fact that $\Phi(0) = 0$, we get

$$\begin{aligned}
&\overline{N}\left(\mathbf{e}(\tau_x^+) = x, \tau_x^+ < \zeta\right) \\
&= \lim_{z\to 0^+} \frac{1}{z}\left[\frac{\sigma^2}{2} W'(x-z) - \mathbb{E}\left(\mathbb{P}_z\left(X_{\tau_{z-x}^-} = z - x, \tau_{z-x}^- < \infty\right), \tau_z^+ < \tau_{z-x}^-\right)\right] \\
&= \lim_{z\to 0^+} \frac{1}{z}\left[\frac{\sigma^2}{2} W'(x-z) - \mathbb{P}_x\left(X_{\tau_0^-} = 0, \tau_0^- < \infty\right)\frac{W(x-z)}{W(x)}\right] \\
&= \lim_{z\to 0^+} \frac{1}{z}\left[\frac{\sigma^2}{2} W'(x-z) - \frac{\sigma^2}{2} W'(x)\frac{W(x-z)}{W(x)}\right] \\
&= \frac{\sigma^2}{2}\lim_{z\to 0^+}\left[\frac{W'(x-z) - W'(x)}{z} + \frac{W'(x)}{z} - \frac{W'(x)}{W(x)}\frac{W(x-z) - W(x) + W(x)}{z}\right] \\
&= \frac{\sigma^2}{2}\lim_{z\to 0^+}\left[\frac{W'(x-z) - W'(x)}{z} - \frac{W'(x)}{W(x)}\frac{W(x-z) - W(x)}{z}\right] \\
&= \frac{\sigma^2}{2}\left[-W''(x) + \frac{W'(x)}{W(x)}W'(x)\right] \\
&= \frac{\sigma^2}{2}\left[\frac{[W'(x)]^2}{W(x)} - W''(x)\right],
\end{aligned}$$

where in the second line we used the strong Markov property at time τ_z^+ and the fact that $X_{\tau_z^+} = z$ because of the absence of positive jumps. We conclude that

$$\overline{N}\left(\mathbf{e}(\tau_x^+) = x, \tau_x^+ < \zeta\right) = \frac{\sigma^2}{2}\left[\frac{[W'(x)]^2}{W(x)} - W''(x)\right].$$

For the other identity, consider $g : \mathbb{R} \to \mathbb{R}_+$ measurable and bounded. We will use the fact that the potential of $\widehat{X}$ killed when it exits the closed interval $[0, x]$ for the first time is given in terms of scale functions, as can be seen in [12, Theorem 8.7]. Indeed, we have

$$\begin{aligned}
\widehat{U}^0 g(z) := \widehat{E}_z\left(\int_0^{\tau_x^+ \wedge \tau_0^-} dt g(X_t)\right) &= E_{-z}\left(\int_0^{\tau_{-x}^- \wedge \tau_0^+} dt g(-X_t)\right) \\
&= E_{x-z}\left(\int_0^{\tau_0^- \wedge \tau_x^+} dt g(-X_t + x)\right) \\
&= \int_0^x dv \left(\frac{W(x-z)}{W(x)} W(x-v) - W(x-z-v)\right) g(-v+x) \\
&= \int_0^x dv \left(\frac{W(x-z)}{W(x)} W(v) - W(v-z)\right) g(v).
\end{aligned}$$

This and the previous limit result of Chaumont and Doney allows to give an expression for the potential up to τ_x^+ under $\overline{N}$. Namely,

$$\begin{aligned}
\overline{N}\left(\int_0^{\tau_x^+} dt g(\mathbf{e}(t)), \tau_x^+ < \zeta\right) &= \lim_{z \to 0^+} \frac{1}{z}\left[\widehat{\mathbb{E}}_z\left(\int_0^{\tau_x^+ \wedge \tau_0^-} dt g(X_t)\right)\right] \\
&= \lim_{z \to 0^+} \frac{1}{z} \int_0^x dv \left(\frac{W(x-z)}{W(x)} W(v) - W(v-z)\right) g(v) \\
&= \int_0^x dv \left(W'(v) - \frac{W(v) W'(x)}{W(x)}\right) g(v),
\end{aligned}$$

where we have used that

$$\begin{aligned}
\frac{1}{z}\left(\frac{W(x-z)}{W(x)} W(v) - W(v-z)\right) &= \frac{1}{z}\left(\frac{W(x-z) - W(x) + W(x)}{W(x)} W(v) - W(v-z)\right) \\
&= \left(\frac{W(x-z) - W(x)}{z} \frac{W(v)}{W(x)} - \frac{W(v-z) - W(v)}{z}\right) \\
&\to W'(v) - \frac{W(v) W'(x)}{W(x)},
\end{aligned}$$

as $z \to 0^+$.

We will now use the compensation formula, as in the proof of Lemma 6.1, to compute the joint law $\overline{N}\left(f(\mathbf{e}(\tau_x^+ -), \mathbf{e}(\tau_x^+)), \mathbf{e}(\tau_x^+) > x, \tau_x^+ < \zeta\right)$ in terms of the jumps Δ_s of the excursion. Indeed, the event $\{\tau_x^+ < \zeta, \mathbf{e}(\tau_x^+) > x\}$ is equivalent to the existence of $s \in (0, \zeta)$ such that $\sup_{r \in (0,s)} \mathbf{e}(r) < x$ and $\mathbf{e}(s-) + \Delta_s > x$ (observe that such s is unique by definition). Define then

$$G_s(y) = f(\mathbf{e}(s-), \mathbf{e}(s-) + y) 1_{\{\sup_{(0,s)} \mathbf{e} < x\}} 1_{\{\mathbf{e}(s-) + y > x\}} 1_{\{s < \zeta\}}, \quad s > 0,$$

to write

$$f(\mathbf{e}(\tau_x^+ -), \mathbf{e}(\tau_x^+)), 1_{\{\mathbf{e}(\tau_x^+) > x, \tau_x^+ < \zeta\}} = \sum_{0 < s < \infty} G_s(\Delta_s) 1_{\{\Delta_s > 0\}}.$$

Using the compensation formula we get

$$\begin{aligned}
&\overline{N}\left(f(\mathbf{e}(\tau_x^+-),\mathbf{e}(\tau_x^+)),1_{\{\mathbf{e}(\tau_x^+)>x,\tau_x^+<\zeta\}}\right)=\overline{N}\left(\sum_{0<s<\infty}G_s(\Delta_s)1_{\{\Delta_s>0\}}\right)\\
&\quad=\overline{N}\left(\int_0^\infty ds\int_{(0,\infty)}\widehat{\Pi}(dy)G_s(y)\right)\\
&\quad=\overline{N}\left(\int_0^\infty ds\int_{(0,\infty)}\widehat{\Pi}(dy)f(\mathbf{e}(s-),\mathbf{e}(s-)+y)1_{\{\sup_{(0,s)}\mathbf{e}<x\}}1_{\{\mathbf{e}(s-)+y>x\}}1_{\{s<\zeta\}}\right)\\
&\quad=\overline{N}\left(\int_0^\zeta ds1_{\{\sup_{(0,s)}\mathbf{e}<x\}}\int_{(0,\infty)}\widehat{\Pi}(dy)f(\mathbf{e}(s),\mathbf{e}(s)+y)1_{\{\mathbf{e}(s)+y>x\}}\right)\\
&\quad=\overline{N}\left(\int_0^\zeta ds1_{\{\tau_x^+>s\}}\int_{(0,\infty)}\widehat{\Pi}(dy)f(\mathbf{e}(s),\mathbf{e}(s)+y)1_{\{\mathbf{e}(s)+y>x\}}\right)\\
&\quad=\overline{N}\left(\int_0^{\tau_x^+}dsg(\mathbf{e}(s)),\tau_x^+<\zeta\right),
\end{aligned}$$

where $g(v)=\int_{(0,\infty)}\widehat{\Pi}(dy)f(v,v+y)1_{\{v+y>x\}}$. Therefore, applying the previously obtained formula for the potential under $\overline{N}$, we conclude that

$$\begin{aligned}
&\overline{N}\left(f(\mathbf{e}(\tau_x^+-),\mathbf{e}(\tau_x^+)),1_{\{\mathbf{e}(\tau_x^+)>x,\tau_x^+<\zeta\}}\right)=\int_0^x dv\left(W'(v)-\frac{W'(x)W(v)}{W(x)}\right)g(v)\\
&=\int_0^x dv\left(W'(v)-\frac{W'(x)W(v)}{W(x)}\right)\int_{(-\infty,0)}\Pi(dy)f(v,v-y)1_{\{v-y>x\}}\\
&=\int_0^x d\ell\left(W'(x-\ell)-\frac{W'(x)W(x-\ell)}{W(x)}\right)\int_{(-\infty,0)}\Pi(dy)f(x-\ell,x-\ell-y)1_{\{x-\ell-y>x\}}\\
&=\int_0^x d\ell W(x-\ell)\left(\frac{W'(x-\ell)}{W(x-\ell)}-\frac{W'(x)}{W(x)}\right)\int_{(-\infty,0)}\Pi(dy)f(x-\ell,x-\ell-y)1_{\{0>\ell+y\}},
\end{aligned}$$

completing the proof. □

The next result is an auxiliary lemma to compute the law of the overshoot under $\widehat{N}_x$.

Lemma 6.3. *Let $x>0$, $q\geq 0$ and $f:\mathbb{R}\to\mathbb{R}_+$ measurable and bounded. Then,*

$$N_0\left(\int_0^{\tau_x^+(\mathbf{e})\wedge\tau_0^-(\mathbf{e})}e^{-qt}f(\mathbf{e}(t))dt\right)=\int_0^x\frac{W^{(q)}(x-y)}{W^{(q)}(x)}f(y)dy. \tag{6.3}$$

Proof of Lemma 6.3. Denote by $\tau(\mathbf{e})=\tau_x^+(\mathbf{e})\wedge\tau_0^-(\mathbf{e})$. From the Markov property at time $\varepsilon>0$, we have

$$\begin{aligned}
&N_0\left(\int_0^{\tau_x^+(\mathbf{e})\wedge\tau_0^-(\mathbf{e})}e^{-qt}f(\mathbf{e}(t))dt\right)\\
&=\lim_{\varepsilon\to0+}N_0\left(\int_\varepsilon^{\tau(\mathbf{e})}e^{-q\varepsilon}e^{-q(t-\varepsilon)}f(\mathbf{e}(t))dt,\varepsilon<\zeta\wedge\tau(\mathbf{e})\right)\\
&=\lim_{\varepsilon\to0+}N_0\left(\mathbb{E}_{\mathbf{e}(\varepsilon)}\left[\int_0^{\tau_x^+\wedge\tau_0^-}e^{-qt}f(X_t)dt\right],\varepsilon<\mathsf{e}_q\wedge\zeta\wedge\tau(\mathbf{e})\right),
\end{aligned}$$

where e_q is an independent exponential random variable of parameter q. On one hand, by the resolvent formula in [11, Theorem 2.7], we have that for any $z \in (0, x]$,

$$\begin{aligned}\mathbb{E}_z\left[\int_0^{\tau_x^+\wedge\tau_0^-} e^{-qt}f(X_t)dt\right] &= \int_0^x \left[\frac{W^{(q)}(x-y)}{W^{(q)}(x)}W^{(q)}(z) - W^{(q)}(z-y)\right]f(y)dy \\ &= W^{(q)}(z)\int_0^x \left[\frac{W^{(q)}(x-y)}{W^{(q)}(x)} - \frac{W^{(q)}(z-y)}{W^{(q)}(z)}1_{\{y\le z\}}\right]f(y)dy.\end{aligned}$$

Therefore,

$$\lim_{z\to 0+}\frac{1}{W^{(q)}(z)}\mathbb{E}_z\left[\int_0^{\tau_x^+\wedge\tau_0^-} e^{-qt}f(X_t)dt\right] = \int_0^x \frac{W^{(q)}(x-y)}{W^{(q)}(x)}f(y)dy,$$

as a consequence of the dominated convergence theorem, since $0 \le \frac{W^{(q)}(x-y)}{W^{(q)}(x)} \le 1$, f is bounded and $1_{\{y\le z\}} \downarrow 0$ as $z \downarrow 0$. On the other hand, from [18, Lemma 3], there is a function h_q such that

$$\lim_{z\to 0+}\frac{h_q(z)}{W^{(q)}(z)} = 1 - \frac{\sigma^2}{2}\Phi'(q)\Phi(q).$$

Actually, the result is for the quotient $h_q(z)/W(z)$, but it is also proven there that $W^{(q)}(z)/W(z) \to 1$ as $z \to 0$, which leads to the above limit. Additionally, from the proof of their Theorem 3 [18, pp. 97], the function h_q also satisfies

$$\lim_{\varepsilon\to 0+} N_0\left(h_q(\mathbf{e}(\varepsilon)), \varepsilon < \zeta\wedge\tau_0^-(\mathbf{e})\right) = 1 - \frac{\sigma^2}{2}\Phi'(q)\Phi(q).$$

Furthermore, from here we can also prove that

$$\lim_{\varepsilon\to 0+} N_0\left(h_q(\mathbf{e}(\varepsilon)), \varepsilon < \mathsf{e}_q\wedge\zeta\wedge\tau(\mathbf{e})\right) = 1 - \frac{\sigma^2}{2}\Phi'(q)\Phi(q),$$

since, for ε small, the N_0- measure of excursions that surpass level x in $[0,\varepsilon]$ is small and tends to 0 as $\varepsilon \to 0^+$ and also $1_{\{\mathsf{e}_q>\varepsilon\}} \uparrow 1$ as $\varepsilon \to 0^+$. Putting all the previous pieces together, and using that on the event $\varepsilon < \tau(\mathbf{e})$ the excursion is positive in $(0,\varepsilon]$, we conclude using a dominated convergence argument that

$$\begin{aligned}&N_0\left(\int_0^{\tau_x^+(\mathbf{e})\wedge\tau_0^-(\mathbf{e})} e^{-qt}f(\mathbf{e}(t))dt\right)\\ &= \lim_{\varepsilon\to 0+} N_0\left(\mathbb{E}_{\mathbf{e}(\varepsilon)}\left[\int_0^{\tau_x^+\wedge\tau_0^-} e^{-qt}f(X_t)dt\right], \varepsilon < \mathsf{e}_q\wedge\zeta\wedge\tau(\mathbf{e})\right)\\ &= \lim_{\varepsilon\to 0+} N_0\left(\frac{1}{W^{(q)}(\mathbf{e}(\varepsilon))}\mathbb{E}_{\mathbf{e}(\varepsilon)}\left[\int_0^{\tau_x^+\wedge\tau_0^-} e^{-qt}f(X_t)dt\right]\frac{W^{(q)}(\mathbf{e}(\varepsilon))}{h_q(\mathbf{e}(\varepsilon))}h_q(\mathbf{e}(\varepsilon))1_{\{\varepsilon<\mathsf{e}_q\wedge\zeta\wedge\tau(\mathbf{e})\}}\right)\\ &= \int_0^x \frac{W^{(q)}(x-y)}{W^{(q)}(x)}f(y)dy.\end{aligned}$$

□

This lemma gives the law of the overshoot under $\widehat{N}_x$ in terms of the scale function W and the jump measure Π.

Lemma 6.4. *Let $g : \mathbb{R} \to \mathbb{R}_+$ measurable and bounded and denote by $\widehat{\Pi}(dy) = \Pi(-dy)$. Then,*

$$\widehat{N}_x\left(g(\mathcal{O}_x), \tau_0^- > \zeta, \mathcal{E}_\pm^x\right) = \int_0^x dz\frac{W(x-z)}{W(x)}\int_{(0,\infty)}\widehat{\Pi}(dy+z)g(y).$$

Proof of Lemma 6.4. By space invariance and duality, we have that

$$\begin{aligned}\widehat{N}_x\left(g(\mathcal{O}_x), \tau_0^- > \zeta, \mathcal{E}_\pm^x\right) &= \widehat{N}_0\left(g(\mathcal{O}_0), \tau_{-x}^- > \zeta, \mathcal{E}_\pm\right)\\ &= N_0\left(g(-\mathcal{U}_0), \tau_x^+ > \zeta, \mathcal{E}_\pm\right),\end{aligned}$$

which is almost the law of the undershoot we have computed in Lemma 6.1 but with the additional condition $\{\tau_x^+ > \zeta\}$, which means that the excursion prior to τ_0^- does not go above x. Let $h(u,b) := g(-u)$. Mimicking the computations in the proof of the op. cit. lemma but now considering

$$G_s(y) = h(\mathbf{e}(s-)+y, \mathbf{e}(s-))1_{\{y<0\}}1_{\{\mathbf{e}(s-)\in(0,x)\}}1_{\{\mathbf{e}(s-)+y<0\}}1_{\{s<\zeta\}}1_{\{\sup_{r\in(0,s)}\mathbf{e}(r)<x\}},$$

we get that

$$\begin{aligned}&N_0\left(g(-\mathcal{U}_0), \tau_x^+ > \zeta, \mathcal{E}_\pm\right) = N_0\left(\int_0^\infty ds \int_{(-\infty,0)} \Pi(dy) G_s(y)\right)\\ &= N_0\left(\int_0^\zeta ds 1_{\{\mathbf{e}(s)\in(0,x)\}}1_{\{\sup_{r\in(0,s)}\mathbf{e}(r)<x\}}\int_{(-\infty,0)}\Pi(dy)h(\mathbf{e}(s)+y,\mathbf{e}(s))1_{\{\mathbf{e}(s)+y<0\}}\right)\\ &= N_0\left(\int_0^\zeta ds 1_{\{\mathbf{e}(s)\in(0,x)\}}1_{\{s<\tau_0^-\wedge\tau_x^+\}}\int_{(-\infty,0)}\Pi(dy)h(\mathbf{e}(s)+y,\mathbf{e}(s))1_{\{\mathbf{e}(s)+y<0\}}\right)\\ &= N_0\left(\int_0^{\tau_x^+\wedge\tau_0^-} ds 1_{\{\mathbf{e}(s)>0\}}\tilde{h}(\mathbf{e}(s))\right).\end{aligned}$$

Applying Lemma 6.3, we obtain the claimed identity. □

7 Proofs

We start by presenting the proof that the local time process $(L^y_{\tau(c)}, y \in \mathbb{R})$ is infinitely divisible and its Poissonian representation.

Proof of Theorem 3.1. Recall from Section 2 that

$$\sigma_s^0 = \inf\{t > 0 : L_t^0(X) > s\}, \quad s > 0,$$

is the right continuous inverse of the local time at zero. Then, for those times s such that $\sigma_s^0 > \sigma_{s-}^0 := \lim_{t\uparrow s}\sigma_t^0$, we define the excursion away from zero at local time s by

$$\mathbf{e}_s^0(u) = X_{\sigma_{s-}^0+u}, \quad u \in [0, \sigma_s^0 - \sigma_{s-}^0],$$

and the set of excursions $\{(s, \mathbf{e}_s^0) : s > 0\}$ is a Poisson point process of intensity $ds\otimes N_0(d\mathbf{e})$. Let $M(ds, d\mathbf{e})$ be the corresponding Poisson point measure. From the occupation formula, for any function f measurable and bounded we have

$$\int_{\mathbb{R}} L^y_{\tau(c)}(X) f(y) dy = \int_0^{\tau(c)} f(X_t) dt.$$

On the other hand, by regularity of X for $(-\infty, 0)$ and $(0,\infty)$ (which is a consequence of hypothesis **(A)**), we can decompose the integral on the right hand side into the contributions of each excursion away from zero as

$$\int_0^{\tau(c)} f(X_t)dt = \sum_{0<s<c} \int_{\sigma^0_{s-}}^{\sigma^0_s} f(X_t)dt = \sum_{0<s<c} \int_0^{\zeta(\mathbf{e}^0_s)} f(\mathbf{e}^0_s(t))dt$$
$$= \sum_{0<s<c} \int_{\mathbb{R}} \ell^y_{\zeta(\mathbf{e}^0_s)}(\mathbf{e}^0_s) f(y)dy$$
$$= \int_{\mathbb{R}} \left(\sum_{0<s<c} \ell^y_{\zeta(\mathbf{e}^0_s)}(\mathbf{e}^0_s) \right) f(y)dy,$$

where in the second in last line we used again the occupation density formula for the local times of each excursion. We can write the last line in terms of M as

$$\int_{\mathbb{R}} \ell^y_{\zeta(\mathbf{e}^0_s)}(\mathbf{e}^0_s) f(y)dy = \int_{\mathbb{R}} \left(\int_0^c \int_{\mathcal{D}} \ell^y_{\zeta(\mathbf{e})}(\mathbf{e}) M(ds, d\mathbf{e}) \right) f(y)dy.$$

Therefore,

$$\int_{\mathbb{R}} \ell^y_{\tau(c)} f(y)dy = \int_{\mathbb{R}} \left(\int_0^c \int_{\mathcal{D}} \ell^y_{\zeta(\mathbf{e})}(\mathbf{e}) M(ds, d\mathbf{e}) \right) f(y)dy,$$

and since this holds for any measurable and bounded function f, we conclude that

$$L^y_{\tau(c)}(X) = \int_0^c \int_{\mathcal{D}} \ell^y_{\zeta(\mathbf{e})}(\mathbf{e}) M(ds, d\mathbf{e}),$$

for a.e.-y. Then, the full identity follows using the right continuity in the space variable of local times. Finally, if $\widetilde{K}$ is the image of M under the function that maps e to its local times (hence with intensity $ds \otimes \widetilde{M}(d\ell)$ and $\widetilde{M}(d\ell)$ being the image of N_0 under the same map) then

$$L^y_{\tau(c)}(X) = \int_0^c \int_{\mathcal{D}} \ell(y) \widetilde{K}(ds, d\ell).$$

We now prove the infinite divisibility. From the strong Markov property of X and the fact that local times are additive, it follows that for any $n \geq 1$ we can split the local times into the information up to $\tau(c/n)$ and the information after as

$$L^y_{\tau(c)}(X) = L^y_{\tau(c/n)}(X) + L^y_{\tau(c)}(X) \circ \theta_{\tau(c/n)}, \quad y \in \mathbb{R}.$$

Since $X_{\tau(c/n)} = 0$, we can write $L^y_{\tau(c)}(X) \circ \theta_{\tau(c/n)}$ in terms of the process $(\tilde{X}_t := X_{\tau(c/n)+t}, t \geq 0)$ which is independent of the information up to $\tau(c/n)$ and has the same law of X. Indeed,

$$L^y_{\tau(c)}(X) \circ \theta_{\tau(c/n)} = L^y_{\widetilde{\tau}(\frac{n-1}{n}c)}(\widetilde{X}),$$

where $\widetilde{\tau}(\frac{n-1}{n}c)$ is the first time $\widetilde{X}$ accumulates $\frac{n-1}{n}c$ units of local time at 0. Hence,

$$L^y_{\tau(c)}(X) = L^y_{\tau(c/n)}(X) + L^y_{\widetilde{\tau}(\frac{n-1}{n}c)}(\widetilde{X}).$$

Repeating the argument for $\widetilde{X}$ and using induction, we conclude that we can write

$$L^y_{\tau(c)}(X) \overset{(d)}{=} \sum_{k=1}^n L^y_{\tau_k(c/n)}(X^{(k)}), \quad y \in \mathbb{R},$$

where $X^{(k)}$ are i.i.d. copies of X starting from zero and the stopping times $\tau_k(c/n) = \inf\left\{t > 0 : L^0_t(X^{(k)}) > c/n\right\}$ are the first times each copy accumulates c/n units of local time at 0. This proves that the local time process is infinitely divisible. We can identify the associated Lévy measure $\mu^{(c)}$, that is, a measure that satisfies

$$\mathbb{E}\left[e^{-\int_{\mathbb{R}} f(y)L^y_{\tau(c)}(X)dy}\right] = \exp\left\{-\int_{\mathbb{R}_+^{\mathbb{R}}}\left(1-e^{-\int_{\mathbb{R}} f(y)\omega(y)dy}\right)\mu^{(c)}(d\omega)\right\},$$

for any non-negative, measurable and bounded function f. Using the occupation formula one has that

$$\int_{\mathbb{R}} f(y)L^y_{\tau(c)}(X)dy = \int_0^{\tau(c)} f(X_s)ds,$$

and decomposing the integral on the right hand side into excursions away from 0 as before, from the exponential formula we obtain

$$\mathbb{E}\left[e^{-\int_0^{\tau(c)} f(X_s)ds}\right] = \exp\left\{-c\int_{\mathcal{D}}\left(1-e^{-\int_{\mathbb{R}} f(y)\ell^y_\zeta(\mathbf{e})dy}\right)N_0(d\mathbf{e})\right\}.$$

The latter yields the following identity for the Lévy measure of local times up to $\tau(c)$,

$$\mu^{(c)}(d\omega) = cN_0(\ell^{\cdot}_\zeta \in d\omega).$$ □

We now present the proof of the expression of the refined version of the Lévy measure $\mu^{(c)}$.

Proof of Theorem 3.2. For the representation of N_0 in the set $\mathcal{E}_\pm$ we just condition on the overshoot and undershoot $(\mathcal{O}_0, \mathcal{U}_0)$ of the excursion $\mathbf{e}$ and use the strong Markov property at time τ_0^- to get the conditional independence of the paths $\underleftarrow{\mathbf{e}}$ and $\underrightarrow{\mathbf{e}}$. Lemma 6.1 provides an expression for the joint density of $(\mathcal{O}_0, \mathcal{U}_0)$, proving the second expression. And for the last one, using the occupation formula and decomposing the path in excursions away from the supremum as in the proof of Proposition 2.1, one obtains for $u < 0$

$$\begin{aligned}\mathbb{E}^0_u\left(e^{-\int_{(-\infty,0)} f_-(y)L^y_\zeta dy}\right) &= \mathbb{E}_u\left(e^{-\int_0^{\tau_0^+} f_-(X_r)dr}\right)\\ &= \exp\left\{-\int_u^0 ds\overline{N}\left[1-e^{-\int_0^\zeta f_-(s-\mathbf{e}(r))dr}\right]\right\}\\ &= \mathcal{W}_{f_-,u}(-u).\end{aligned}$$

Proceeding analogously for $b > 0$, we obtain the desired expression for the Laplace transform. □

The next proof concerns the Poissonian representation of $(L^y_{\tau(c)}, y \in \mathbb{R})$ involving the overshoots and undershoots.

Proof of Theorem 3.3. We only show equation (3.6), as the proof of (3.7) is analogous. We will do it by considering Laplace transforms. Let $f : (0,\infty) \to \mathbb{R}_+$ measurable and bounded and call $G(y)$ the right hand side of (3.6). For an element $Y \in \mathcal{M}$, the space of Poisson random measures, we will denote by $\langle Y, F\rangle$ at the integral of a functional F with respect to Y. In particular, when F is the functional $\int_0^\infty f(y)\ell^{y-\cdot}_\zeta(\cdot)1_{\{y-\cdot>0\}}dy$, recalling that a Poisson random measure $Y \in \mathcal{M}$ is written in terms of its atoms $Y = \sum_{r>0}\delta_{(r,\underline{\mathbf{e}}_r)}$, we have that

$$\langle Y, F\rangle = \left\langle Y, \int_0^\infty f(y)\ell^{y-\cdot}_\zeta(\cdot)1_{\{y-\cdot>0\}}dy\right\rangle = \sum_{r>0}\int_0^\infty f(y)\ell^{y-r}_\zeta(\underline{\mathbf{e}}_r)1_{\{y-r>0\}}dy.$$

Using that M_+ and $M^1_\pm$ are independent we have that

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{-\int_0^\infty f(y)G(y)dy\right\}\right]\\
&=\mathbb{E}\left[\exp\left\{-\int_0^c\int_{\mathcal{D}}\left(\int_0^\infty f(y)\ell^y_\zeta(\mathbf{e})dy\right)M_+(ds,d\mathbf{e})\right\}\right]\\
&\times\mathbb{E}\left[\exp\left\{-\int_0^c\int_{(0,\infty)}\int_{\mathcal{M}}\left\langle Y,\left(\int_0^\infty f(y)\ell^{y-\cdot}_\zeta(\cdot)1_{\{y-\cdot>0\}}dy\right)\right\rangle M^1_\pm(ds,db,dY)\right\}\right].
\end{aligned}$$

Using the exponential formula, the first term equals

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{-\int_0^c\int_{\mathcal{D}}\left(\int_0^\infty f(y)\ell^y_\zeta(\mathbf{e})dy\right)M_+(ds,d\mathbf{e})\right\}\right]\\
&=\exp\left\{-\int_0^c\int_{\mathcal{D}}dsN_0(d\mathbf{e},\mathcal{E}_+)(1-e^{-\int_0^\infty f(y)\ell^y_\zeta dy})\right\}\\
&=\exp\left\{-cN_0\left(1-e^{-\int_0^\infty f(y)\ell^y_\zeta dy},\mathcal{E}_+\right)\right\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{-\int_0^c\int_{(0,\infty)}\int_{\mathcal{M}}\left\langle Y,\left(\int_0^\infty f(y)\ell^{y-\cdot}_\zeta(\cdot)1_{\{y-\cdot>0\}}dy\right)\right\rangle M^1_\pm(ds,db,dY)\right\}\right]\\
&=\exp\left\{-\int_0^c\int_{(0,\infty)}\int_{\mathcal{M}}dsN_0(\mathcal{O}_0\in db,\mathcal{E}_\pm)\widehat{\kappa}(b,dY)\left(1-e^{-\langle <Y,\int_0^\infty f(y)\ell^{y-\cdot}_\zeta(\cdot)1_{\{y-\cdot>0\}}dy\rangle}\right)\right\}\\
&=\exp\left\{-c\int_{(0,\infty)}db\Pi(-\infty,-b)\mathbb{E}\left(1-\exp\left\{-\sum_{0<r<b}\int_0^\infty f(y)\ell^{y-r}_\zeta(\underline{\mathbf{e}}_r)1_{\{y-r>0\}}dy\right\}\right)\right\}\\
&=\exp\left\{-c\int_{(0,\infty)}db\Pi(-\infty,-b)\left(1-\exp\left\{-\int_0^{b\wedge y}dr\overline{N}\left(1-e^{-\int_0^\infty f(y)\ell^{y-r}_\zeta dy}\right)\right\}\right)\right\}.
\end{aligned}$$

Therefore, comparing with the Laplace transform in Theorem 3.2, we conclude that

$$\mathbb{E}\left[\exp\left\{-\int_0^\infty f(y)G(y)dy\right\}\right]=\mathbb{E}\left[\exp\left\{-\int_0^\infty f(y)L^y_{\tau(c)}(X)dy\right\}\right],$$

which proves the Poissonian representation of local times. □

The following proof concerns the Poissonian representation and the infinite divisibility property of the local time process $(L^{a-z}_{\tau_a^+}(X),z\geq 0)$.

Proofs of Theorem 4.1. Let $M'(ds,d\mathbf{e})$ be the Poisson point measure of excursions away from the supremum and $f:\mathbb{R}^2\to\mathbb{R}_+$ measurable and bounded. Again, hypothesis **(A)** implies that 0 is regular for both $(-\infty,0)$ and $(0,\infty)$ and [12, Th.6.7] implies that the set of times in which X reaches new suprema on compact intervals has Lebesgue measure 0. Therefore, we have the following decomposition:

$$\begin{aligned}
\int_0^{\tau_a^+}f(S_t-X_t,X_t)dt&=\sum_{0<v<a}\int_{\tau^+_{v-}}^{\tau^+_v}f(S_t-X_t,X_t)dt\\
&=\sum_{0<v<a}\int_0^{\zeta(\mathbf{e}_v)}f(\mathbf{e}_v(t),v-\mathbf{e}_v(t))dt.
\end{aligned}$$

Applying the occupation density formula, we can express the latter quantity in terms of M' as

$$\begin{aligned}\sum_{0<v<a}\int_0^{\zeta(\mathbf{e}_v)} f(\mathbf{e}_v(t), v-\mathbf{e}_v(t))dt &= \sum_{0<v<a}\int_{\mathbb{R}} f(u, v-u)\ell^u_{\zeta_v(\mathbf{e})}(\mathbf{e})du \\ &= \sum_{0<v<a}\int_{[0,\infty)} f(u, v-u)\ell^u_{\zeta_v(\mathbf{e})}(\mathbf{e})du \\ &= \int_{[0,\infty)}\left(\int_0^a\int_{\mathcal{D}} f(u, s-u)\ell^u_{\zeta(\mathbf{e})}(\mathbf{e})M'(ds, d\mathbf{e})\right)du.\end{aligned}$$

Now we use Fubini's theorem and the change of variables $z = s-u$ to obtain

$$\begin{aligned}\int_0^{\tau_a^+} f(S_t - X_t, X_t)dt &= \int_0^a\int_{\mathcal{D}}\left(\int_{\mathbb{R}} f(s-z, z)\ell^{s-z}_{\zeta(\mathbf{e})}(\mathbf{e})1_{\{s-z\geq 0\}}dz\right)M'(ds, d\mathbf{e}) \\ &= \int_{(-\infty,a]}\left(\int_0^a\int_{\mathcal{D}}\ell^{s-z}_{\zeta(\mathbf{e})}(\mathbf{e})1_{\{s-z\geq 0\}}M'(ds, d\mathbf{e})\right)f(s-z, z)dz.\end{aligned}$$

On the other hand, from the occupation formula for X and taking f only depending on the second entry we have

$$\begin{aligned}\int_{(-\infty,a]} f(z)L^z_{\tau_a^+}(X)dz &= \int_0^{\tau_a^+} f(X_t)dt \\ &= \int_{(-\infty,a]}\left(\int_0^a\int_{\mathcal{D}}\ell^{s-z}_{\zeta(\mathbf{e})}(\mathbf{e})1_{\{s-z>0\}}M'(ds, d\mathbf{e})\right)f(z)dz,\end{aligned}$$

and since this holds for any test function f, we conclude that

$$L^z_{\tau_a^+}(X) = \int_0^a\int_{\mathcal{D}}\ell^{s-z}_{\zeta(\mathbf{e})}(\mathbf{e})1_{\{s-z>0\}}M'(ds, d\mathbf{e})$$

for almost every $z \in (-\infty, a]$. The last equality holds for all z by using the right continuity of local times.

To complete the proof we focus on the reversed process $L^{a-z}_{\tau_a^+}(X)$, with $z \geq 0$. Since Lebesgue measure is invariant under translations, the Poisson point measure $M(ds, d\mathbf{e}) := M'(a-ds, d\mathbf{e})$ for $s \in [0, a]$ has the same intensity as M restricted to $[0, a]$. Hence, we can write

$$\begin{aligned}L^{a-z}_{\tau_a^+}(X) &= \int_0^a\int_{\mathcal{D}}\ell^{s-a+z}_{\zeta(\mathbf{e})}(\mathbf{e})1_{\{s-a+z>0\}}M'(ds, d\mathbf{e}) \\ &= \int_0^a\int_{\mathcal{D}}\ell^{z-s}_{\zeta(\mathbf{e})}(\mathbf{e})1_{\{z-s>0\}}M(ds, d\mathbf{e}) \\ &= \begin{cases}\int_0^z\int_{\mathcal{D}}\ell^{z-s}_{\zeta(\mathbf{e})}(\mathbf{e})M(ds, d\mathbf{e}), & z\in[0,a], \\ \int_0^a\int_{\mathcal{D}}\ell^{z-s}_{\zeta(\mathbf{e})}(\mathbf{e})M(ds, d\mathbf{e}), & z\geq a,\end{cases} \\ &= \begin{cases}\int_0^z\int_{\mathcal{D}}\mathbf{e}(z-s)K(ds, d\mathbf{e}), & z\in[0,a], \\ \int_0^a\int_{\mathcal{D}}\mathbf{e}(z-s)K(ds, d\mathbf{e}), & z\geq a,\end{cases}\end{aligned}$$

which concludes the result.

We now show that local times are infinitely divisible. We split the information up to $\tau^+_{a/n}$ and after to obtain for any $n \geq 1$ that

$$L^x_{\tau_a^+}(X) = L^x_{\tau^+_{a/n}}(X) + L^x_{\tau_a^+}(X)\circ\theta_{\tau^+_{a/n}}, \quad x\in\mathbb{R}.$$

Since $X_{\tau^+_{a/n}} = a/n$, we can write $L^x_{\tau^+_a}(X)\circ\theta_{\tau^+_{a/n}}$ in terms of the process $(\widetilde{X}_t := X_{\tau_{a/n}+t} - a/n, t\ge 0)$, which is independent of the information up to $\tau^+_{a/n}$ and has the same law as X started from 0. Indeed,

$$L^x_{\tau^+_a}(X)\circ\theta_{\tau^+_{a/n}} = L^{x-a/n}_{\widetilde{\tau}^+_{\frac{n-1}{n}a}}(\widetilde{X}),\quad x\in\mathbb{R},$$

where $\widetilde{\tau}^+_{\frac{n-1}{n}a}$ is the first passage time above $\frac{n-1}{n}a$ for $\widetilde{X}$. Applying this argument inductively, we conclude that

$$L^x_{\tau^+_a}(X) \overset{(\mathrm{d})}{=} \sum_{k=1}^n L^{x-\frac{k-1}{n}a}_{\tau^{+,k}_{a/n}}(X^{(k)}),\quad x\in\mathbb{R},$$

where $X^{(k)}$ are i.i.d. copies of X started from 0 and $\tau^{+,k}_{a/n}$ are the corresponding first passage times above a/n. This implies the infinite divisibility of $(L^x_{\tau^+_a}(X), x\in\mathbb{R})$.

We now identify the corresponding Lévy measure. Indeed, using that

$$\int_0^\infty f(y)L^{a-y}_{\tau^+_a}(X)dy = \int_{-\infty}^a f(a-z)L^z_{\tau^+_a}(X)dz = \int_0^{\tau^+_a} f(a-X_s)ds$$

and decomposing the last integral into excursions away from the supremum, again from the exponential formula one gets

$$\begin{aligned}
\mathbb{E}\left[e^{-\int_0^{\tau^+_a} f(a-X_s)ds}\right] &= \exp\left\{-\int_0^a\int_{\mathcal{D}}\left(1-e^{-\int_0^\zeta f(a-s+\mathbf{e}(r))dr}\right)\overline{N}(d\mathbf{e})ds\right\}\\
&= \exp\left\{-\int_0^a\int_{\mathcal{D}}\left(1-e^{-\int_0^\infty f(a-s+z)\ell^z_\zeta(\mathbf{e})dz}\right)\overline{N}(d\mathbf{e})ds\right\}\\
&= \exp\left\{-\int_0^a\int_{\mathcal{D}}\left(1-e^{-\int_0^\infty f(s+z)\ell^z_\zeta(\mathbf{e})dz}\right)\overline{N}(d\mathbf{e})ds\right\}\\
&= \exp\left\{-\int_0^a\int_{\mathcal{D}}\left(1-e^{-\int_0^\infty f(y)\ell^{y-s}_\zeta(\mathbf{e})1_{\{y-s>0\}}dy}\right)\overline{N}(d\mathbf{e})ds\right\}.
\end{aligned}$$

Therefore, we can write

$$\mathbb{E}\left[e^{-\int_0^\infty f(y)L^{a-y}_{\tau^+_a}(X)dy}\right] = \exp\left\{-\int_{\mathbb{R}_+^{\mathbb{R}}}\left(1-e^{-\int_{\mathbb{R}} f(y)\omega(y)dy}\right)\nu^{(a)}(d\omega)\right\},$$

where

$$\nu^{(a)}(d\omega) = \int_0^a ds\overline{N}(\ell^{\cdot-s}_\zeta 1_{\{\cdot-s>0\}}\in d\omega).$$ □

The following two proofs show the law of local times prior to the first hitting time of a positive level x and between the first and last visits to it.

Proof of Theorem 4.2. For $x>0$, it might happen that $\tau^+_x(\mathbf{e}) = T_x(\mathbf{e})$, which implies that the excursion reaches x for the first time continuously and coming from below. In this case, $L^y_{T_x(\mathbf{e})}(\mathbf{e}) = 0$ for any $y>x$. Then, we only have to deal with the case $\tau^+_x(\mathbf{e}) < T_x(\mathbf{e}) < \zeta(\mathbf{e})$ in which the excursion goes above x for the first time by a jump, hence having a first strictly positive overshoot $\mathcal{O}_x(\mathbf{e})$ with respect to x. Conditionally on $\mathcal{O}_x(\mathbf{e}) = b>0$, by the strong Markov property the path between $\tau^+_x(\mathbf{e})$ and $T_x(\mathbf{e})$ has the law $\widehat{\mathbb{E}}^x_{x+b}$, which is the law of $\widehat{X}$ started from $x+b$ and killed when it reaches x for the first time. This, together with the expression for the law of the overshoots from Lemma 6.2 implies

$$\begin{aligned}\overline{N}\left[F(\ell^y_{T_x}, y > x), H > x\right] &= \overline{N}\left[F(\ell^y_{T_x}, y > x)1_{\{\tau^+_x < T_x < \zeta\}}\right] \\ &= \overline{N}\left[1_{\{\tau^+_x < T_x < \zeta\}}\widehat{\mathbb{E}}^x_{\mathbf{e}(\tau^+_x)}\left(F(L^y_{T_x}, y > x)\right)\right] \\ &= \int_{(0,\infty)} \overline{N}(\mathcal{O}_x \in db, \tau^+_x < T_x < \zeta)\widehat{\mathbb{E}}_b\left(F(L^{y-x}_{\tau^-_0}, y > x)\right),\end{aligned}$$

and for each $y > x$, the total local time $L^{y-x}_{\tau^-_0}(X)$ is decomposed under $\widehat{\mathbb{E}}_b$ in the contributions to it of the excursions away from the infimum $\{(r, \underline{\mathbf{e}}_r),\ r > 0\}$ as

$$L^{y-x}_{\tau^-_0}(X) = \sum_{0<r<b} L^{y-x-r}_{\zeta(\underline{\mathbf{e}}_r)}(\underline{\mathbf{e}}_r)1_{\{y-x>r\}}.$$

Hence,

$$\begin{aligned}&\overline{N}\left[F(\ell^y_{T_x}, y > x), H > x\right] \\ &= \int_{(0,\infty)} \overline{N}(\mathcal{O}_x \in db, \tau^+_x < T_x < \zeta)\widehat{\mathbb{E}}_b\left(F\left(\sum_{0<r<b} L^{y-x-r}_{\zeta(\mathbf{e}_r)}(\mathbf{e}_r)1_{\{y-x>r\}}, y > x\right)\right) \\ &= \int_{(0,\infty)} \overline{N}(\mathcal{O}_x \in db, \tau^+_x < T_x < \zeta)\widehat{\kappa}(b, F(L^{y-x}_{\zeta}, y > x)),\end{aligned}$$

proving the result. For the particular case, we have

$$\begin{aligned}&\overline{N}\left[\exp\left\{-\int_x^\infty f(y)\ell^y_{T_x}dy\right\}, H > x\right] \\ &= \overline{N}\left[1_{\{\tau^+_x < T_x < \zeta\}}\exp\left\{-\int_0^{T_x} f(\mathbf{e}(s))1_{\{\mathbf{e}(s)>x\}}ds\right\}\right] \\ &= \int_{(0,\infty)} \overline{N}(\mathcal{O}_x \in db, \tau^+_x < T_x < \zeta)\widehat{\mathbb{E}}^x_{x+b}\left(\exp\left\{-\int_0^\zeta f(X_s)1_{\{X_s>x\}}ds\right\}\right).\end{aligned}$$

For the expected value, we use duality to obtain

$$\widehat{\mathbb{E}}^x_{x+b}\left(\exp\left\{-\int_0^\zeta f(X_s)ds\right\}\right) = \mathbb{E}_0\left(\exp\left\{-\int_0^{\tau^+_b} f(x+b-X_s)ds\right\}\right)$$

and we decompose as usual the right hand side into excursions away from the supremum:

$$\mathbb{E}_0\left(\exp\left\{-\int_0^{\tau^+_b} f(x+b-X_s)ds\right\}\right) = \exp\left\{-\int_0^b ds\overline{N}\left[1 - e^{-\int_0^\zeta drf(x+b-s+\mathbf{e}(r))}\right]\right\}.$$

And by defining $f_{x,b}(z) := f(x+b-z), z < b$ we have that

$$\begin{aligned}\exp\left\{-\int_0^b ds\overline{N}\left[1 - e^{-\int_0^\zeta drf(x+b-s+\mathbf{e}(r))}\right]\right\} &= \exp\left\{-\int_0^b ds\overline{N}\left[1 - e^{-\int_0^\zeta drf_{x,b}(s-\mathbf{e}(r))}\right]\right\} \\ &= \mathcal{W}_{f_{x,b}}(b).\end{aligned}$$

□

Proof of Theorem 4.3. Equation (4.2) comes from the strong Markov Property. Since X has no positive jumps, paths under $\overline{N}$ have no negative jumps. Hence, in the event $H > x$ we necessarily have that $T_x(\mathbf{e}) < \zeta(\mathbf{e})$, that is, the excursion hits x before its lifetime. Conditioning on $T_x(\mathbf{e})$, in which we know that $\mathbf{e}(T_x) = x$, from the Markov property we have

$$\overline{N}\left(\exp\left\{-\int_x^\infty f(y)\ell_\zeta^y \circ \theta_{T_x} dy\right\}, H > x\right) = \overline{N}\left(\widehat{\mathbb{E}}_x\left[\exp\left\{-\int_x^\infty f(y)L^y_{\tau_0^-} dy\right\}\right], H > x\right)$$
$$= \overline{N}\left(H > x\right)\widehat{\mathbb{E}}_x\left[\exp\left\{-\int_x^\infty f(y)L^y_{\tau_0^-} dy\right\}\right].$$

From here, to compute the expected value we could proceed as in the last part of the previous proof, since by the occupation formula,

$$\widehat{\mathbb{E}}_x\left[\exp\left\{-\int_x^\infty f(y)L^y_{\tau_0^-} dy\right\}\right] = \mathbb{E}\left[\exp\left\{-\int_0^{\tau_x^+} f(x - X_s)1_{\{X_s<0\}} ds\right\}\right].$$

Nonetheless, in order to emphasize the role of the excursion overshoots and give insight in the branching-like structure of local times, we will next make the computation in a different way. Indeed, recall that under $\widehat{\mathbb{E}}_x$, the local time $L^x_{\tau_0^-}$ determines the number of excursions away from x that the dual process $\widehat{X}$ performs before passing below 0. Actually, we will verify that it follows an exponential distribution of parameter $\widehat{N}_x(\tau_0^- < \zeta)$. Hence, conditionally on this quantity, we will determine the value of the additive functional $\int_x^\infty f(y)L^y_{\tau_0^-} dy$ from the contributions of each of the excursions from x.

We begin with the distribution of $L^x_{\tau_0^-}$. For any $t > 0$, under $\widehat{\mathbb{E}}_x$, the event $\{L^x_{\tau_0^-} > t\}$ corresponds to the event in which none of the excursions away from x up to local time t has visited $(-\infty, 0)$. By denoting $\{(s, \widehat{\mathbf{e}}^x_s) : s > 0\}$ the Poisson point process of excursions away from x, with corresponding intensity measure $\widehat{N}_x$, we have

$$\begin{aligned}\widehat{\mathbb{P}}_x\left(L^x_{\tau_0^-} > t\right) &= \widehat{\mathbb{P}}_x\left(\#\left\{(s, \widehat{\mathbf{e}}^x_s) : 0 < s \le t,\ \tau_0^-(\widehat{\mathbf{e}}^x_s) < \zeta(\widehat{\mathbf{e}}^x_s)\right\} = 0\right)\\ &= e^{-t\widehat{N}_x\left(\tau_0^- < \zeta\right)}\\ &= e^{-q_x^0 t},\end{aligned}$$

which proves the assertion. We now compute q_x^0. Observe that, by duality, $\widehat{N}_x(\tau_0^- < \zeta) = N_0(\tau_x^+ < \zeta)$. Intersecting the latter event with the partition of the set of excursions away from 0, we obtain

$$\begin{aligned}N_0(\tau_x^+ < \zeta) &= N_0(\tau_x^+ < \zeta, \mathcal{E}_+) + N_0(\tau_x^+ < \zeta, \mathcal{E}_-) + N_0(\tau_x^+ < \zeta, \mathcal{E}_\pm)\\ &= N_0(\tau_x^+ < \zeta, \tau_0^- = \zeta) + N_0(\tau_x^+ < \zeta, \tau_0^- = 0) + N_0(\tau_x^+ < \zeta, 0 < \tau_0^- < \zeta)\\ &= N_0(\tau_x^+ < \tau_0^- = \zeta) + N_0(\tau_x^+ < \tau_0^- < \zeta).\end{aligned}$$

From the last identity on the proof of Theorem 3 in [18], there exists a constant c_+ (which is equal to 1 using the same argument as in the proof of Lemma 6.2) such that

$$N_0(\tau_x^+ < \tau_0^- < \zeta) = c_+\underline{N}(\tau_x^+ < \zeta, \mathbf{e}(\zeta-) > 0),$$

whilst from (ii) in the same theorem we have that

$$N_0(\tau_x^+ < \tau_0^- = \zeta) = c_+\underline{N}(\tau_x^+ < \zeta, \mathbf{e}(\zeta-) = 0).$$

Adding these two identities we obtain

$$q_x^0 = N_0(\tau_x^+ < \zeta) = \underline{N}(\tau_x^+ < \zeta),$$

and it is known that this last quantity equals $\frac{1}{W(x)}$ (see [1, Proposition 15, Ch. VII]).

Now, conditionally on $L^x_{\tau_0^-}$, we decompose the functional $\int_x^\infty f(y)L^y_{\tau_0^-}dy$ into the sum of the corresponding contribution of each excursion away from x: $(\widehat{\mathbf{e}}^x_s, 0 < s < L^x_{\tau_0^-})$. As pointed above, each one of these excursions satisfy $\tau_0^-(\widehat{\mathbf{e}}^x_s) > \zeta(\widehat{\mathbf{e}}^x_s)$. Using the exponential formula we obtain

$$\begin{aligned}\widehat{\mathbb{E}}_x\left[\exp\left\{-\int_x^\infty f(y)L^y_{\tau_0^-}dy\right\}\right] &= \int_0^\infty drq^0_x e^{-rq^0_x}\widehat{\mathbb{E}}_x\left[\exp\left\{-\sum_{0<s<r}\int_x^\infty f(y)\ell^y_\zeta(\widehat{\mathbf{e}}^x_s)dy\right\}\right]\\ &= \int_0^\infty drq^0_x e^{-rq^0_x}\exp\left\{-r\widehat{N}_x\left[1-e^{-\int_x^\infty f(y)\ell^y_\zeta dy}, \tau_0^- > \zeta\right]\right\}\\ &= \widehat{\mathbb{E}}_x\left[\exp\left\{-L^x_{\tau_0^-}\widehat{N}_x\left[1-e^{-\int_x^\infty f(y)\ell^y_\zeta dy}, \tau_0^- > \zeta\right]\right\}\right]\\ &= \frac{q^0_x}{q^0_x + \widehat{N}_x\left[1-e^{-\int_x^\infty f(y)\ell^y_\zeta dy}, \tau_0^- > \zeta\right]}\\ &= \frac{1}{1+W(x)\widehat{N}_x\left[1-e^{-\int_x^\infty f(y)\ell^y_\zeta dy}, \tau_0^- > \zeta\right]},\end{aligned}$$

which implies the first part of the theorem.

For the second part, we know that under $\widehat{N}_x$, excursions away from x are partitioned into those completely above x ($\mathcal{E}^x_+$), those completely below x ($\mathcal{E}^x_-$) and those starting below and then jumping above x ($\mathcal{E}^x_\pm$). Excursions in $\mathcal{E}^x_-$ do not contribute to the local times of levels bigger than x and hence we restrict to the other two sets. In $\mathcal{E}^x_+$, the condition $\tau_0^-(\mathbf{e}) > \zeta(\mathbf{e})$ is automatically fulfilled, so we can omit it. Finally, for an excursion in $\mathcal{E}^x_\pm$, there is an unique positive overshoot $\mathcal{O}_x(\mathbf{e})$ at time $\tau_x^+(\mathbf{e})$ and the part of the excursion contributing to levels $y > x$ is from $\tau_x^+(\mathbf{e})$ to $\zeta(\mathbf{e})$, which, conditionally on $\mathcal{O}_x(\mathbf{e}) = b$, we have seen that has the law $\widehat{\mathbb{E}}^x_{x+b}$ and therefore can be decomposed into excursions away from the infimum, as in the proof of Theorem 4.2. Using the computations there we have that

$$\begin{aligned}&\widehat{N}_x\left[1-e^{-\int_x^\infty f(y)\ell^y_\zeta dy}, \tau_0^- > \zeta, \mathcal{E}^x_\pm\right]\\ &= \int_0^\infty \widehat{N}_x(\mathcal{O}_x \in db, \tau_0^- > \zeta, \mathcal{E}^x_\pm)\widehat{\mathbb{E}}_{x+b}\left[1-e^{-\int_x^\infty f(y)L^y_{\tau_x^-}dy}\right]\\ &= \int_0^\infty \widehat{N}_x(\mathcal{O}_x \in db, \tau_0^- > \zeta, \mathcal{E}^x_\pm)\left[1-\exp\left\{-\int_0^b ds\overline{N}\left[1-e^{-\int_x^\infty f(y)\ell_\zeta^{y-x-s}1_{\{y-s>x\}}dy}\right]\right\}\right],\end{aligned}$$

which concludes the proof. □

Finally, the proofs of the proposition relative to the scale function $\mathcal{W}_f$ and the propositions from Section 6 can be found below.

Proof of Proposition 2.1. The function $\mathcal{W}_f$ from (1.5) is well defined, since

$$\begin{aligned}\lim_{b\to-\infty}\frac{W_f(0,b)}{W_f(x,b)} &= \lim_{b\to-\infty}\frac{W(-b)}{W(x-b)}\exp\left\{-\int_0^x ds\overline{N}\left(1-e^{-\int_0^\zeta drf(s-\mathbf{e}(r))}, H < s-b\right)\right\}\\ &= \lim_{b\to\infty}\frac{W(b)}{W(x+b)}\exp\left\{-\int_0^x ds\overline{N}\left(1-e^{-\int_0^\zeta drf(s-\mathbf{e}(r))}, H < s+b\right)\right\}.\end{aligned}$$

The first quotient equals $\mathbb{P}_0(\tau_x^+ < \tau_{-b}^-)$ and converges to $\mathbb{P}_0(\tau_x^+ < \infty)$ when $b \to \infty$, which is equal to one because of hypothesis **(B1)** or **(B2)**. And when $b \to \infty$, the condition $H < s+b$ inside the exponential is replaced by $H < \infty$ but again, since under **(B1)** or

(B2) we have $\limsup_{t\to\infty} X_t = \infty$, this implies that the excursions with infinite height have zero mass under $\overline{N}$.

Now let us check that G_f defined as in the statement coincides with $\mathcal{W}_f$ and solves equation (2.1). For $x > 0$, assumptions **(B1)** and **(B2)** imply that $\limsup_{t\to\infty} X_t = \infty$ and therefore $\tau_0^- < \infty$ $\widehat{\mathbb{P}}_x-$ a.s. Then, for $x > 0$,

$$\begin{aligned} G_f(x) &= \widehat{\mathbb{E}}_x \left[\exp\left\{ -\int_0^{\tau_0^-} ds f(x - X_s) \right\} \right] \\ &= 1 - \widehat{\mathbb{E}}_x \left[1 - \exp\left\{ -\int_0^{\tau_0^-} ds f(x - X_s) \right\} \right] \\ &= 1 - \widehat{\mathbb{E}}_x \left[\int_0^{\tau_0^-} dt f(x - X_t) \exp\left\{ -\int_t^{\tau_0^-} ds f(x - X_s) \right\} \right], \end{aligned}$$

where in the last line we use the fact that if $\widetilde{f}(t) := \exp\left\{ -\int_t^{\tau_0^-} ds f(x - X_s) \right\}$, $t \in [0, \tau_0^-]$, then $\widetilde{f}'(t) = f(x - X_t)\widetilde{f}(t)$ and $\widetilde{f}(\tau_0^-) - \widetilde{f}(0) = 1 - \exp\left\{ -\int_0^{\tau_0^-} ds f(x - X_s) \right\}$. Applying the Markov property at time t inside the expression in the last line we obtain

$$\begin{aligned} G_f(x) &= 1 - \widehat{\mathbb{E}}_x \left[\int_0^{\tau_0^-} dt f(x - X_t) \widehat{\mathbb{E}}_{X_t} \left[\exp\left\{ -\int_0^{\tau_0^-} ds f(x - X_s) \right\} \right] \right] \\ &= 1 - \widehat{\mathbb{E}}_x \left[\int_0^{\tau_0^-} dt f(x - X_t) G_f(X_t) \right], \end{aligned}$$

which can be written in terms of the potential $\widehat{U}^0$ of $\widehat{X}$ killed at τ_0^- as

$$G_f(x) = 1 - \int_0^\infty \widehat{U}^0(x, dz) f(x - z) G_f(z).$$

According to [12, Corollary 8.8], $\widehat{U}^0$ can also be expressed in terms of scale functions as

$$G_f(x) = 1 - \int_0^\infty dz (W(x) - W(x - z)) f(x - z) G_f(z).$$

This and the fact that $G_f(0) = 1$ because of the assumption of unbounded variation, implies the equation (2.1).

For the other part, we notice that

$$\begin{aligned} G_f(x) &= \widehat{\mathbb{E}}_x \left[\exp\left\{ -\int_0^{\tau_0^-} ds f(x - X_s) \right\} \right] = \mathbb{E}_{-x} \left[\exp\left\{ -\int_0^{\tau_0^+} ds f(x + X_s) \right\} \right] \\ &= \mathbb{E}_0 \left[\exp\left\{ -\int_0^{\tau_x^+} ds f(X_s) \right\} \right], \end{aligned}$$

and decomposing into excursions away from the supremum between 0 and x and using the exponential formula we get

$$\begin{aligned} G_f(x) &= \exp\left\{ -\int_0^x ds \overline{N} \left[1 - e^{-\int_0^\zeta du f(s - \mathbf{e}(u))} \right] \right\} \\ &= \exp\left\{ -\int_0^x ds g_f(s) \right\}, \end{aligned}$$

which also proves that $G_f(x) = \mathcal{W}_f(x)$. The relation $\frac{d}{dx}(-\log G_f)(x) = g_f(x)$ now follows from a simple differentiation of the latter equation. □

Proof of Proposition 4.5. We start by noticing that

$$\overline{N}\left(\left(1-e^{-\lambda\ell_\zeta^x-\int_x^\infty f(y)\ell_\zeta^y dy}\right)1_{\{H<x\}}\right)=0,$$

since for an excursion to accumulate local time at x and the levels above, it must have height bigger that x. Then, on the event $H>x$, applying the Markov property at time $T_x(\mathbf{e})$ we have that

$$\begin{aligned}\overline{N}\left(1-e^{-\lambda\ell_\zeta^x-\int_x^\infty f(y)\ell_\zeta^y dy}\right)&=\overline{N}\left(\left(1-e^{-\lambda\ell_\zeta^x-\int_x^\infty f(y)\ell_\zeta^y dy}\right)1_{\{H>x\}}\right)\\&=\overline{N}\left(\left(1-e^{-\int_x^\infty f(y)\ell_{T_x}^y dy}\right)1_{\{H>x\}}\right)\\&\quad+\overline{N}\left(e^{-\int_x^\infty f(y)\ell_{T_x}^y dy}\widehat{\mathbb{E}}_x\left[1-e^{-\lambda L^x_{\tau_0^-}-\int_x^\infty f(y)L^y_{\tau_0^-}dy}\right]1_{\{H>x\}}\right).\end{aligned}$$

It is known that under $\widehat{\mathbb{E}}_x$ the total local time at x up to the first passage time below 0, $L^x_{\tau_0^-}$, follows an exponential distribution of parameter $q_x=\widehat{N}_x(\tau_0^-<\zeta)$ (see the proof of Theorem 4.3). Also note that the total local time at levels $y_1,\dots y_n>x$, is the addition of the accumulated local time inside each excursion from x. This implies

$$\widehat{\mathbb{E}}_x\left[1-e^{-\lambda L^x_{\tau_0^-}-\int_x^\infty f(y)L^y_{\tau_0^-}dy}\right]=\widehat{\mathbb{E}}_x\left[1-e^{-\lambda L^x_{\tau_0^-}-\sum_{0<t<L^x_{\tau_0^-}}\int_x^\infty f(y)\ell_\zeta^y(\mathbf{e}_t^x)dy}\right].$$

Denote by $(\mathbf{e}_s^x(\tau_0^->\zeta_s),s>0)$ the excursions away from x that occur before τ_0^-. Then,

$$\begin{aligned}&\widehat{\mathbb{E}}_x\left[e^{-\lambda L^x_{\tau_0^-}-\int_x^\infty f(y)L^y_{\tau_0^-}}\right]\\&=\int_0^\infty dtq_xe^{-tq_x-\lambda t}\widehat{\mathbb{E}}_x\left(\exp\left\{-\sum_{0<s<t}\int_x^\infty f(y)\ell_\zeta^y(\mathbf{e}_s^x(\tau_0^->\zeta_s))dy\right\}\right)\\&=\int_0^\infty dtq_xe^{-tq_x-\lambda t}\exp\left\{-t\widehat{N}_x\left(1-e^{-\int_x^\infty f(y)\ell_\zeta^y dy},\tau_0^->\zeta\right)\right\}\\&=\int_0^\infty dtq_x\exp\left\{-t\left[q_x+\lambda+\widehat{N}_x\left(1-e^{-\int_x^\infty f(y)\ell_\zeta^y dy},\tau_0^->\zeta\right)\right]\right\},\end{aligned}$$

where the second equation holds by the exponential formula. Recalling that $u_x(f)=\widehat{N}_x\left(1-e^{-\int_x^\infty f(y)\ell_\zeta^y},\tau_0^->\zeta\right)=N_0\left(1-e^{-\int_x^\infty f(x-y)\ell_\zeta^y dy},\tau_x^+>\zeta\right)$, we conclude

$$\widehat{\mathbb{E}}_x\left[1-e^{-\lambda L^x_{\tau_0^-}-\int_x^\infty f(y)L_\zeta^y}\right]=\widehat{\mathbb{E}}_x\left[1-e^{-(\lambda+u_x(f))L^x_{\tau_0^-}}\right].$$

Using this, we get

$$\begin{aligned}\overline{N}\left(1-e^{-\lambda\ell_\zeta^x-\int_x^\infty f(y)\ell_\zeta^y dy}\right)&=\overline{N}\left(\left(1-e^{-\int_x^\infty f(y)\ell_{T_x}^y dy}\right)1_{\{H>x\}}\right)\\&\quad+\overline{N}\left(e^{-\int_x^\infty f(y)\ell_{T_x}^y dy}\widehat{\mathbb{E}}_x\left[1-e^{-\lambda L^x_{\tau_0^-}-\int_x^\infty f(y)L^y_{\tau_0^-}}\right]1_{\{H>x\}}\right)\\&=\overline{N}\left(\left(1-e^{-\int_x^\infty f(y)\ell_{T_x}^y dy}\right)1_{\{H>x\}}\right)\\&\quad+\overline{N}\left(e^{-\int_x^\infty f(y)L_{T_x}^y dy}\widehat{\mathbb{E}}_x\left[1-e^{-(\lambda+u_x(f))L^x_{\tau_0^-}}\right]1_{\{H>x\}}\right)\\&=\overline{N}\left(\left(1-e^{-(\lambda+u_x(f))\ell_\zeta^x\circ\theta_{T_x}-\int_x^\infty f(y)\ell_{T_x}^y dy}\right)1_{\{H>x\}}\right)\\&=\overline{N}\left(1-e^{-(\lambda+u_x(f))\ell_\zeta^x-\int_x^\infty f(y)\ell_{T_x}^y dy}\right),\end{aligned}$$

where the last expression follows from the fact that since an excursion does not accumulate local time at x prior to its first visit to it, then $\ell_\zeta^x(\mathbf{e}) \circ \theta_{T_x(\mathbf{e})} = \ell_\zeta^x(\mathbf{e})$. The result on the joint law of local times at n different points, $\ell_\zeta^{y_1}, \dots, \ell_\zeta^{y_n}$ is proved in the same way, just replacing everywhere the integral by the sum $\sum_{k=1}^n \beta_k \ell_\zeta^{y_k}$. □

Proof of Proposition 4.6. By the Markov property under $\overline{N}$ applied at time T_y

$$\begin{aligned} u_y(\lambda) &= \overline{N}\left(1 - e^{-\lambda \ell_\zeta^y}\right) = \overline{N}\left(\left(1 - e^{-\lambda \ell_\zeta^y}\right) 1_{\{H>y\}}\right) \\ &= \overline{N}(H>y) - \overline{N}\left(e^{-\lambda \ell_\zeta^y} 1_{\{H>y\}}\right) \\ &= \overline{N}(H>y) - \overline{N}(H>y)\,\widehat{\mathbb{E}}_y\left(e^{-\lambda L^y_{\tau_0^-}}\right) \\ &= \overline{N}(H>y) - \overline{N}(H>y)\,\widehat{\mathbb{E}}\left(e^{-\lambda L^0_{\tau_{-y}^-}}\right) \end{aligned}$$

As in the proof of Theorem 4.3, $L^0_{\tau_{-y}^-}$ follows an exponential distribution with parameter $\widehat{N}_0(\tau_{-y}^- < \zeta)$ under $\widehat{\mathbb{E}}$, where we have also calculated that

$$\widehat{N}_0(\tau_{-y}^- < \zeta) = N_0(\tau_y^+ < \zeta) = \frac{1}{W(y)}, \quad y > 0.$$

The rest of the calculation now follows from the fact that

$$\overline{N}(H>y) = \frac{W'(y)}{W(y)}, \quad y > 0,$$

(see [12, Lemma 8.2]). □

Proof of Proposition 4.7. Observe that $v_{x,y}(\lambda) = \widehat{N}_x^0\left(1 - e^{-\lambda \ell_\zeta^y}\right)$. Since the excursion does not accumulate local time if the path does not reach y, then

$$\widehat{N}_x^0\left(1 - e^{-\lambda \ell_\zeta^y}\right) = \widehat{N}_x^0\left[\left(1 - e^{-\lambda \ell_\zeta^y}\right) 1_{\{T_y<\zeta\}}\right].$$

From the Markov property at time $T_y(\mathbf{e})$,

$$\begin{aligned} v_{x,y}(\lambda) &= \widehat{N}_x^0\left[\left(1 - e^{-\lambda \ell_\zeta^y}\right) 1_{\{T_y<\zeta\}}\right] \\ &= \widehat{N}_x^0\left[1_{\{T_y<\zeta\}}\widehat{\mathbb{E}}_y\left(1 - e^{-\lambda L^y_{T_x}}\right)\right] \\ &= \widehat{N}_x^0\left[T_y<\zeta\right] - \widehat{N}_x^0\left[T_y<\zeta\right]\widehat{\mathbb{E}}_0\left(e^{-\lambda L^0_{T_{x-y}}}\right). \end{aligned}$$

Because of regularity and the absence of negative jumps, $T_{x-y} = \tau_{x-y}^-$ under $\widehat{\mathbb{E}}_0$. As in the previous proposition, $L^0_{\tau_{x-y}^-}$ follows an exponential distribution of parameter $\widehat{N}_0(\tau_{x-y}^- < \zeta) = N_0(\tau_{y-x}^+ < \zeta) = \frac{1}{W(y-x)}$, which implies $\widehat{\mathbb{E}}_0\left(e^{-\lambda L^0_{T_{x-y}}}\right) = \frac{1}{1+\lambda W(y-x)}$. Therefore,

$$v_{x,y}(\lambda) = \widehat{N}_x^0\left[T_y<\zeta\right]\frac{\lambda W(y-x)}{1+\lambda W(y-x)}.$$

Now, under $\widehat{N}_x^0$ the event $\{T_y(\mathbf{e}) < \zeta(\mathbf{e}), \tau_y^+(\mathbf{e}) > \zeta(\mathbf{e})\}$ has zero measure because of regularity of 0 for $(0,\infty)$. Since there are no negative jumps under $\widehat{N}_x^0$, we also have $\{\tau_y^+(\mathbf{e}) < \zeta(\mathbf{e})\} \subset \{T_y(\mathbf{e}) < \zeta(\mathbf{e})\}$. Hence,

$$\widehat{N}_x^0\left[T_y<\zeta\right] = \widehat{N}_x^0\left[T_y<\zeta, \tau_y^+ > \zeta\right] + \widehat{N}_x^0\left[T_y<\zeta, \tau_y^+ < \zeta\right] = \widehat{N}_x^0\left[\tau_y^+ < \zeta\right],$$

and therefore,

$$\widehat{N}_x^0\left[T_y<\zeta\right]=\widehat{N}_x\left[\tau_y^+<\zeta<\tau_0^-\right]=\widehat{N}_0\left[\tau_{y-x}^+<\zeta<\tau_{-x}^-\right]=N_0\left[\tau_{x-y}^-<\zeta<\tau_x^+\right].$$

To provide an expression for the last expression this in terms of W, we intersect the event inside N_0 with the partition $\mathcal{E}_+$, $\mathcal{E}_-$, $\mathcal{E}_\pm$. Excursions on $\mathcal{E}_+$ are completely above zero and therefore do not reach the negative level $x-y$. On $\mathcal{E}_-$ excursions are completely negative, and hence the condition $\tau_x^+(\mathbf{e})>\zeta(\mathbf{e})$ is fulfilled. Additionally, from Theorem 3 in [18] we know that N_0 on $\mathcal{E}_-$ is a multiple of $\widehat{\overline{N}}$, which is the pushforward of $\overline{N}$ under the map that sends each path to its negative. Therefore,

$$N_0\left[\tau_{x-y}^-<\zeta<\tau_x^+,\mathcal{E}_-\right]=N_0\left[\tau_{x-y}^-<\zeta,\mathcal{E}_-\right]=\frac{\sigma^2}{2}\widehat{\overline{N}}\left[\tau_{x-y}^-<\zeta\right]=\frac{\sigma^2}{2}\overline{N}\left[\tau_{y-x}^+<\zeta\right],$$

and using the fact that W is differentiable and $W'(z)=W(z)\overline{N}(H>z)$ (see [12, Lemma 8.2]), we conclude that

$$N_0\left[\tau_{x-y}^-<\zeta<\tau_x^+,\mathcal{E}_-\right]=\frac{\sigma^2}{2}\overline{N}\left[\tau_{y-x}^+<\zeta\right]=\frac{\sigma^2}{2}\overline{N}\left[H>y-x\right]=\frac{\sigma^2}{2}\frac{W'(y-x)}{W(y-x)}.$$

Finally, on $\mathcal{E}_\pm$ excursions start above zero, then jump below and die at the next hitting time at zero. Therefore,

$$\begin{aligned}N_0\left[\tau_{x-y}^-<\zeta<\tau_x^+,\mathcal{E}_\pm\right]&=N_0\left[\tau_{x-y}^-<\zeta<\tau_x^+,0<\tau_0^-<\zeta\right]\\&=N_0\left[\sup_{s\in(0,\tau_0^-]}\mathbf{e}(s)<x,\tau_{x-y}^-<\zeta,0<\tau_0^-<\zeta\right].\end{aligned}$$

From the Markov property at time $\tau_0^-(\mathbf{e})$, the excursion after the jump follows the same law as X started from $\mathbf{e}(\tau_0^-)$ and killed at the first passage time above zero. This fact together with a conditioning on the time the jump occurs implies that the expression on the right hand side of the display above equals

$$\begin{aligned}&N_0\left[1_{\left\{\sup_{s\in(0,\tau_0^-]}\mathbf{e}(s)<x\right\}}\mathbb{P}_{\mathbf{e}(\tau_0^-)}\left(\tau_{x-y}^-<\tau_0^+\right)\right]\\&=N_0\left[1_{\left\{\sup_{s\in(0,\tau_0^-]}\mathbf{e}(s)<x\right\}}1_{\{\mathbf{e}(\tau_0^--)>0,\mathbf{e}(\tau_0^-)<0\}}\mathbb{P}_{\mathbf{e}(\tau_0^-)}\left(\tau_{x-y}^-<\tau_0^+\right)\right]\\&=N_0\left[\int_0^\infty dt1_{\left\{\sup_{s\in(0,t]}\mathbf{e}(s)<x\right\}}1_{\{\mathbf{e}(t-)>0\}}\int_{-\infty}^0\Pi(du)1_{\{\mathbf{e}(t-)+u<0\}}\mathbb{P}_{\mathbf{e}(t-)+u}\left(\tau_{x-y}^-<\tau_0^+\right)\right]\\&=N_0\left[\int_0^\infty dt1_{\left\{\sup_{s\in(0,t]}\mathbf{e}(s)<x\right\}}1_{\{\mathbf{e}(t)>0\}}\int_{-\infty}^0\Pi(du)1_{\{\mathbf{e}(t)+u<0\}}\mathbb{P}_{\mathbf{e}(t)+u}\left(\tau_{x-y}^-<\tau_0^+\right)\right]\\&=\int_0^\infty dtN_0\left[1_{\left\{\sup_{s\in(0,t]}\mathbf{e}(s)<x\right\}}1_{\{\mathbf{e}(t)>0\}}h(\mathbf{e}(t))\right],\end{aligned}$$

where

$$h(z)=\int_{-\infty}^0\Pi(du)1_{\{z+u<0\}}\mathbb{P}_{z+u}\left(\tau_{x-y}^-<\tau_0^+\right)=\int_{-\infty}^{-z}\Pi(du)\mathbb{P}_{z+u}\left(\tau_{x-y}^-<\tau_0^+\right),\quad z>0.$$

Observe that if $u\le -z-(y-x)$ then $\mathbb{P}_{z+u}\left(\tau_{x-y}^-<\tau_0^+\right)=1$. On the other hand, if $-z-(y-x)<u<-z$ then the latter probability is just the exit problem from the interval $[x-y,0]$ starting from $z+u$, so $\mathbb{P}_{z+u}\left(\tau_{x-y}^-<\tau_0^+\right)=1-\frac{W(u+z+y-x)}{W(y-x)}$.

From the same computation as in the proof of Lemma 6.4, we know that

$$
\begin{aligned}
&N_0\left[1_{\left\{\sup_{s\in(0,t]}\mathbf{e}(s)<x\right\}}1_{\{\mathbf{e}(t)>0\}}h(\mathbf{e}(t))\right]\\
&=\int_0^x dze^{-\Phi(0)z}h(z)-\frac{1}{W(x)}\int_0^x dz\left(e^{-\Phi(0)z}W(x)-W(x-z)\right)h(z)\\
&=\int_0^x dz\frac{W(x-z)}{W(x)}h(z)\\
&=\int_0^x dz\frac{W(x-z)}{W(x)}\left(\Pi(-\infty,z)-\int_{-z-(y-x)}^{-z}\Pi(du)\frac{W(u+z+y-x)}{W(y-x)}\right).
\end{aligned}
$$

Putting the previous computations together we conclude that

$$
\begin{aligned}
\widehat{N}_x^0\left[T_y<\zeta\right]=&\frac{\sigma^2}{2}\frac{W'(y-x)}{W(y-x)}\\
&+\int_0^x dz\frac{W(x-z)}{W(x)}\left(\Pi(-\infty,-z)-\int_{-z-(y-x)}^{-z}\Pi(du)\frac{W(u+z+y-x)}{W(y-x)}\right).
\end{aligned}
$$

□

References

[1] Bertoin, J. *Lévy processes*, Cambridge Tracts in Mathematics, vol. 121, Cambridge University Press, Cambridge, 1996. MR1406564

[2] Blumenthal, R. M. *Excursions of Markov processes*, Probability and its Applications, Birkhäuser Boston, Inc., Boston, MA, 1992. MR1138461

[3] Chaumont, L. and Doney, R. A. *On Lévy processes conditioned to stay positive*, Electron. J. Probab. **10** (2005), no. 28, 948–961. MR2164035

[4] Contreras, J. and Rivero, V. *Generalized scale functions for spectrally negative Lévy processes*, ALEA Lat. Am. J. Probab. Math. Stat. **20** (2023), no. 1, 645–663. MR4567725

[5] Dellacherie, C. and Maisonneuve, B. and Meyer, P. A. *Processus de markov (fin). compléments de calcul stochastique*, Hermann, 1992.

[6] Duquesne, T. *Path decompositions for real Levy processes*, Ann. Inst. H. Poincaré Probab. Statist. **39** (2003), no. 2, 339–370. MR1962781

[7] Eisenbaum, N. *Decompositions of infinitely divisible nonnegative processes*, Electron. J. Probab. **24** (2019), Paper No. 109, 25. MR4017127

[8] Eisenbaum, N. and Kaspi, H. *A necessary and sufficient condition for the Markov property of the local time process*, Ann. Probab. **21** (1993), no. 3, 1591–1598. MR1235430

[9] Fitzsimmons, P. J. and Getoor, R. K. *Excursion theory revisited*, Illinois J. Math. **50** (2006), no. 1-4, 413–437. MR2247835

[10] Kallenberg, O. *Foundations of modern probability*, Probability Theory and Stochastic Modelling, vol. 99, Springer, Cham, [2021] ©2021, Third edition [of 1464694]. MR4226142

[11] Kuznetsov, A. and Kyprianou, A. and Rivero, V. *The theory of scale functions for spectrally negative Lévy processes*, Lévy matters II, Lecture Notes in Math., vol. 2061, Springer, Heidelberg, 2012, pp. 97–186. MR3014147

[12] Kyprianou, A. *Fluctuations of Lévy processes with applications*, second ed., Universitext, Springer, Heidelberg, 2014, Introductory lectures. MR3155252

[13] Li, B. and Palmowski, Z. *Fluctuations of omega-killed spectrally negative Lévy processes*, Stochastic Process. Appl. **128** (2018), no. 10, 3273–3299. MR3849809

[14] Li, B. and Zhou, X. *Local times for spectrally negative Lévy processes*, Potential Anal. **52** (2020), no. 4, 689–711. MR4091599

[15] Li, Z. *Continuous-state branching processes with immigration*, From probability to finance—lecture notes of BICMR Summer School on Financial Mathematics, Math. Lect. Peking Univ., Springer, Singapore, [2020] ©2020, pp. 1–69. MR4339413

[16] Mansuy, R. and Yor, M. *Aspects of Brownian motion*, Universitext, Springer-Verlag, Berlin, 2008. MR2454984

[17] Marcus, M. and Rosen, J. *Markov processes, Gaussian processes, and local times*, Cambridge Studies in Advanced Mathematics, vol. 100, Cambridge University Press, Cambridge, 2006. MR2250510

[18] Pardo, J. C. and Pérez, J. L. and Rivero, V. *The excursion measure away from zero for spectrally negative Lévy processes*, Ann. Inst. Henri Poincaré Probab. Stat. **54** (2018), no. 1, 75–99. MR3765881

[19] Resnick, S. *Adventures in stochastic processes*, Birkhäuser Boston, Inc., Boston, MA, 1992. MR1181423

[20] Sato, K.-I. *Lévy processes and infinitely divisible distributions*, Cambridge Studies in Advanced Mathematics, vol. 68, Cambridge University Press, Cambridge, 1999, Translated from the 1990 Japanese original. Revised by the author. MR1739520

[21] Xu, W. *Stochastic volterra equations for the local times of spectrally positive stable processes*, working paper or preprint, May 2021.

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