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Tail bounds for the O'Connell-Yor polymer

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Abstract

We derive upper and lower bounds for the right and left tails of the O'Connell-Yor polymer of the correct order of magnitude via probabilistic and geometric techniques in the moderate deviations regime. This result has not previously been obtained even by the methods of integrable probability. The inputs of our work are an identity for the generating function of a two-parameter model of Rains and Emrah-Janjigian-Seppäläinen, and the geometric techniques of Ganguly-Hegde and Basu-Ganguly-Hammond-Hegde. As an intermediate result we obtain strong tail estimates for the transversal fluctuation of the polymer path from the diagonal.

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1 Introduction

The O'Connell-Yor polymer (hereafter, the "OY polymer"), also known as the semi-discrete directed polymer, is a fundamental example of a directed polymer in a random environment in $1 + 1$ dimensions, a collection of models that are expected to lie in the Kardar-Parisi-Zhang (KPZ) universality class. It was introduced by O'Connell and Yor [23] as a positive temperature analog of the much-studied Brownian Last Passage percolation. These authors also studied a stationary version of the model, and showed that it possesses an invariance property found in certain queueing models, called the Burke property. This translates to a two-dimensional invariance property in the corresponding polymer and last passage percolation models. Starting with O'Connell [22], it was later discovered that the model has an even richer underlying integrable structure, a fact which ultimately enabled the verification of KPZ type asymptotics for the distribution of the normalized free energy for the first time in any polymer model, in the breakthrough work of Borodin, Corwin and Ferrari [5].

In parallel to the development of integrable probability, there has been an increasing interest in geometric and probabilistic methods to analyze the fluctuations of models

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expected to be in the KPZ class. In the case of the OY polymer, this line of research was initiated by Seppäläinen and Valkó [26], who first applied the coupling method of Balázs-Cator-Seppäläinen [3] to this model to obtain cube root scaling for the fluctuations at the level of the variance. See also [19] for the intermediate disorder case, where the variance of the environment (the "temperature parameter") is allowed to depend on the system size. In [20], the second author and Noack estimated the higher moments on a near-optimal scale. Although it features Gaussian integration by parts prominently, the method introduced in [20] in fact extends to discrete polymer models [21]. Most recently, the authors of the current article obtained upper and lower bounds for the right tail of the stationary OY polymer and the four integrable discrete polymer models [17] (see also the thesis of Xie containing similar results for three of the four discrete models [28]).

Geometric and probabilistic methods have generally not yet been able to provide as detailed information as those of integrable probability. For example, identifying asymptotic distributions without resorting to explicit formulas remains an outstanding challenge. Moreover, current implementations still require some modest integrable inputs like stationarity or the Burke property. However, the gap has been closing. We mention in this context the important recent results of Emrah, Janjigian, and Seppäläinen [7], who introduced a methodology for stationary models that allows them to obtain the exact right tail (including constants in the exponent) for exponential last passage percolation. See also [4, 8, 9, 17]. The reason these methods are of great interest is that they have the potential to be more robust than integrable methods under perturbations of parameters, including the initial data and, ultimately, the distribution of the underlying environment variables.

In this paper, we deal with a question that has attracted much recent attention, namely the tail behavior of models in the Kardar-Parisi-Zhang universality class. We complement our previous results on the right tails of the OY polymer with matching upper and lower bounds on the more delicate left tail. To the best of our knowledge, these results have not previously been obtained, even making full use of the integrable structure in the form of contour integrals for the distribution of the log partition function.

Limiting one-point distributions in the KPZ class, such as the Tracy-Widom and Baik-Rains distribution, exhibit characteristic super-exponential decay with specific exponents $\frac{3}{2}$ (for the right tail) and 3 (for the left tail). In many cases, KPZ models reproduce this tail behavior, at least in the moderate deviation range, in pre-limiting regimes. Moreover, for some models it is known that the tail exponents remain the same under perturbations of initial conditions and even the form of the model (see [17]). Tail behavior is a robust characteristic of the KPZ universality class, and the current work is thus a contribution towards a better understanding of this universality.

1.1 Definition of the model

The partition function of the OY polymer is defined by,

$$Z_{t,n} := \int_{0<s_1<\cdots<s_{n-1}<t} \exp\left(\sum_{i=1}^{n} B_i(s_i) - B_i(s_{i-1})\right) \mathrm{d}s_1 \dots \mathrm{d}s_{n-1}, \tag{1.1}$$

with the convention $s_0 = 0$ and $s_n = t$. Above, $\{B_i\}_i$ are a family of standard Brownian motions. Moriarty and O'Connell [18] calculated the limiting free energy density, for any $u > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \log Z_{un,n} = \theta u - \psi_0(\theta) \tag{1.2}$$

where,

$$\psi_k(\theta) := \frac{\mathrm{d}^{k+1}}{\mathrm{d}\theta^{k+1}} \log\left(\int_0^\infty s^{\theta-1}\mathrm{e}^{-s}\mathrm{d}s\right) \tag{1.3}$$

are the polygamma functions and θ is the unique solution to $\psi_1(\theta) = u$. In [17], we proved the estimate

$$\mathbb{P}\left[\log Z_{t,n} > n(\theta t - \psi_0(\theta)) + sn^{1/3}\right] \leq C\mathrm{e}^{-cs^{3/2}}, \qquad t = n\psi_1(\theta) \tag{1.4}$$

for some constants $c, C > 0$ and any $0 < s < cn^{2/3}$. In the present work, we will complement this upper bound for the right tail with a lower bound of matching order, as well as upper and lower bounds for the left tail. This is the content of the following theorem, which summarizes these statements.

Theorem 1.1. Let $\delta > 0$ and assume $\delta n \leq t \leq \delta^{-1} n$. There are $C, c > 0$ so that the following hold for n sufficiently large. Let θ satisfy $\psi_1(\theta) = t/n$. We have,

$$c\mathrm{e}^{-Cs^{3/2}} \leq \mathbb{P}\left[\log Z_{t,n} > (\theta t - n\psi_0(\theta)) + sn^{1/3}\right] \leq C\mathrm{e}^{-cs^{3/2}} \tag{1.5}$$

for all $0 < s < cn^{2/3}$. For $0 < s < cn^{2/3}/\log n$ we have,

$$\mathbb{P}\left[\log Z_{t,n} < (\theta t - n\psi_0(\theta)) - sn^{1/3}\right] \leq C\mathrm{e}^{-cs^3} \tag{1.6}$$

and for $0 < s < cn^{2/3}/(\log n)^2$ we have,

$$\mathbb{P}\left[\log Z_{t,n} < (\theta t - n\psi_0(\theta)) - sn^{1/3}\right] \geq c\mathrm{e}^{-Cs^3}. \tag{1.7}$$

The various estimates of the above theorem are proven in the following sections of the paper. The upper bound of (1.5) follows from Proposition 2.1 (a restatement of results of [17]), and the lower bound is proven in Section 9. The estimate (1.6) follows from Proposition 8.9. The estimate (1.7) follows from Theorem 6.3. As far as we are aware, these estimates have not been obtained even by the methods of integrable probability.

In addition to the moderate deviations tail estimates for exponential last passage percolation of Emrah, Janjigian and Seppäläinen [7] mentioned in the introduction, we also mention the related large deviations estimates for the OY polymer that were proven by Janjigian [12]. This work builds on the approach of [10] for the log gamma polymer. This corresponds to the regime $s = \mathcal{O}(n^{2/3})$ where the KPZ tail exponents are no longer expected to arise.

In Theorem 1.1 we assumed that t is of the same order as n. The case that t is much smaller than n corresponds to the *intermediate disorder* regime and has been studied in [1, 14, 19, 25]. Here, universality of the Tracy-Widom distribution has in fact been proven for some general, non-integrable directed polymers. We expect that with some modifications our methods could be used to study the OY polymer in the intermediate disorder regime, but deriving moderate deviations for the non-integrable models seems beyond our methods, which rely on stationarity properties enjoyed by only a few integrable models (see the discussion in [17] and the references therein).

1.2 Methodology

In the work [17] we considered stationary KPZ models (the stationary OY polymer is introduced in Section 2.1 below) and showed how two ingredients could be combined to yield a short and transparent proof of an upper bound for the left and right tails of the form $\mathrm{e}^{-cs^{3/2}}$ for the stationary models. These two ingredients are: (1) monotonicity and convexity of the models in the parameters defining the systems; and (2), a certain identity involving the moment generating function of a two-parameter version of the model. This identity was first derived by Rains [24] in the context of last passage percolation, but was recently re-introduced and used to great effect in the work of

Emrah-Janjigian-Seppäläinen [7]. We refer to this identity as the Rains-EJS identity, and is given in Proposition 2.6 in the context of the OY polymer. Due to the fact that the two-parameter model stochastically dominates the non-stationary model $\log Z_{t,n}$ considered here, the Rains-EJS identity in fact yields a short proof of the upper bound for the right tail, as indicated in [17].

The main contributions of this work are then the remaining estimates, the lower bound for the right tail and both bounds for the left tail. Our main inspiration here are the works of Ganguly-Hegde [8] and Basu-Ganguly-Hammond-Hegde [4] which consider general last passage models. In particular, the work [8] shows how under only concavity assumptions on the limit shape, one can "bootstrap" a weak tail estimate of the form $\mathrm{e}^{-c|s|^{\alpha}}$ to an estimate with the optimal exponents, using probabilistic and geometric techniques. Some of the techniques of [8] that we use rely on constructions that were first completed in [4]. The assumptions of [8] have been verified in only the most well-understood, integrable last passage models. Moreover, these constructions have only been carried for last passage models (the zero temperature version of polymers) and have not yet been considered for any polymer model.

The proof of the upper bound for the left tail of [8] relies on the construction of the "geodesic watermelon" of [4]. That is, the weight of the geodesic is compared to the total weight of a large number of disjoint, non-intersecting high weight paths. For lower bounds, one can further restrict the paths to lie in disjoint regions of the phase space, in order to take advantage of the spatial independence of the underlying environment. Our contribution here is to adapt this construction to the polymer case, by finding estimates for non-intersecting multi-path OY polymers.

An input required for this construction is a weak exponential upper bound for the left tail. This is an assumption of [8] but does not appear in the literature for our model (the work [17] gives only estimates for the right tail of the non-stationary model and estimates for both tails of the stationary models). The work [26] deduces variance estimates for the non-stationary model from the stationary one. By following the proof given there and inserting stronger estimates that we have derived for the stationary models using the techniques of [17], we are able to arrive at an initial upper bound of the form $\mathrm{e}^{-c|s|^{3/2}}$ for the left tail.

Given this as input, we then attempt to apply the construction of [4] to the semi-discrete polymer case. The main super-additivity property, that

$$Z_{(s,m),(t,n)} \geq Z_{(s,m),(u,p)} Z_{(u,p),(t,n)}$$

(here $Z_{p,q}$ is the partition function of all up-right paths from p to q) luckily still holds and is one of the main drivers of the various proofs. However, substantial difficulties are introduced by (a) the fact that we are at positive temperature, and so $\log Z_{t,n}$ can take negative values, and (b) the semi-discrete nature of the phase space. The latter difficulty is only seen once one attempts to prove transversal fluctuation estimates and will be discussed later.

Due to the fact that $\log Z_{t,n}$ can be negative, we are forced to substantially modify the construction of [4]. Whereas there, some terms can be simply dropped due to the fact that a weight is always non-negative in last passage percolation, here we have to introduce a dyadic sequence of branching steps, where polymer paths split in two, and then separate from each other. This branching phase is responsible for the logarithmic loss in the range of validity of (1.6).

An additional component of our work that is an input to both the upper and lower bounds for the left tail, is handling transversal fluctuations. The work [4] adapts an argument of Basu, Sidoravicious and Sly [6] which finds estimates for the geodesic weight of paths constrained to have large transversal fluctuation from the diagonal. We too adapt this argument; here the semidiscrete nature of the polymer space causes complications in the estimation of point-to-point polymer partition functions by the product of a point-to-line and line-to-point polymer partition function. However, the explicit form of the polymer partition function as well as Brownian deviation estimates allows for this sort of an estimate. The proof of the lower bound for the left tail requires iterating this kind of estimate a number of times that grows with n. This is one source of the logarithmic loss in the range of validity of (1.7). Modulo this difference, the transversal fluctuation estimates and proof of lower bound follow roughly the strategy of [8]. In particular, we rely on a version of the Harris-FKG inequality for the OY polymer (we provide a proof of a version sufficient for our purposes by approximation by discrete processes in an appendix).

One useful estimate that comes out of the treatment of transversal fluctuations that is worth separating from the rest of the paper is Corollary 5.8 which gives,

$$\mathbb{P}\left[Q_{n,t}[\mathrm{TF}(\gamma) > bn^{2/3}] > \mathrm{e}^{-cb^2n^{1/3}}\right] \leq C\mathrm{e}^{-cb^3} \tag{1.8}$$

for $b \leq n^{1/3}$, where $Q_{n,t}$ denotes the polymer Gibbs measure (defined in Section 2.2), γ is the up-right path formed by interpreting the jumps $\{s_i\}_i$ as the jumps of the up-right path γ taking integer values, and $\mathrm{TF}(\gamma)$ is the maximum distance of the path γ from the straight line connecting $(0,0)$ to (t,n). In particular, this is significantly stronger than the annealed estimate that was derived for the stationary polymer in [17] using only the Rains-EJS identity and monotonicity/convexity, as the current estimate bounds the entire polymer path, instead of only the deviation at a single point, and the estimate of [17] would see the $\mathrm{e}^{-cb^2n^{1/3}}$ factor replaced by the weaker e^{-cb^3}.

An additional wrinkle worth pointing out in the adaptation of [8] to polymer models is that the assumptions of [8] for the limit shape do not hold as written for our polymer model. For the last passage models the horizontal and vertical directions are interchangeable and so the derivative of the limit shape along the transverse direction at the diagonal vanishes; this is not the case for the OY polymer. The linear correction term must therefore be accounted for when considering point-to-line or line-to-line type polymers. We carry this out by instead introducing a "compensated" polymer; i.e., subtracting off the linear correction term.

Finally, as in [8], the lower bound for the right tail is a relatively straightforward consequence of super-additivity and convergence to the Tracy-Widom GUE distribution.

It is worth mentioning that all proofs except the lower bound for the right tail[1] make no use of integrable probability or exact formulas for the distribution of observables of the system, beyond the Burke property and stationarity (at one point we cite an estimate from [26] that is proven using the fact that Brownian LPP has the same distribution as the GUE; however the required tail estimate is also a consequence of the theory of Gaussian processes and does not require this connection – see [11]), and are probabilistic and geometric in nature. Overall, the key inputs are the Burke property of the stationary polymer, the Rains-EJS identity (which is in our setting a simple consequence of the Girsanov-Cameron-Martin formula) as well as the independence properties of the phase space combined with super-additivity of the polymer and concavity of the limit shape.

1.3 Notational conventions

For $a < b$ we set $[\![a,b]\!] := \{m \in \mathbb{Z} : a \leq m \leq b\}$. For nonnegative quantities $a(i), b(i)$ depending on a parameter i in an index set $\mathcal{I}$ (such as n in the definition of the polymer) we say that $a \asymp b$ if there are $c, C > 0$ so that $ca(i) \leq b(i) \leq Ca(i)$ for all $i \in \mathcal{I}$.

[1]The integrable input for the lower bound for the right tail is in fact only, roughly, that the distribution of the polymer (on the correct $n^{1/3}$ scale) is not asymptotically bounded above. Another alternative substitute would be a lower bound for the variance on the correct scale.

For $x = (n-m, n+m)$ we set,

$$\mathrm{ad}(x) := m, \tag{1.9}$$

(here ad stands for anti-diagonal). Since we are dealing with up-right paths, we will often need to refer to distance between points along the diagonal and anti-diagonal axes. The diagonal distance between the points $(0,0)$ and $(n,n)-(m,-m)$, for $|m| \leq n$ is n and their anti-diagonal displacement is $|m|$. We will often say that points lying on the line $\{(x,y) : x+y = 2\ell\}$ have height ℓ.

1.4 Organization

In Section 2 we collect various preliminary results from the literature about the OY polymer as well as its stationary version. In Section 3 we establish a suboptimal upper bound on the left tail of the form $\mathrm{e}^{-cs^{3/2}}$ that is used as an a-priori input to the remainder of our paper.

The lower bound for the left tail is carried out in Sections 4, 5 and 6. In more detail, in Section 4 we establish estimates for interval-to-interval polymers. In Section 5 we establish estimates for the partition function of polymers where the path is constrained to have a large transversal fluctuation. This also leads to an estimate of the quenched probability that a path has a large transversal fluctuation which is used later in the upper bound for the left tail. In Section 6 we use these elements to prove the lower bound on the left tail.

The upper bound on the left tail takes place in Sections 7 and 8. In Section 7 we establish estimates on the partition function of polymers where the path is constrained to not have a large transversal fluctuation. In Section 8 we use the watermelon construction of [4] to obtain the desired upper bound.

Finally in the short Section 9 we obtain a lower bound for the right tail via a short super-additivity argument.

2 Preliminaries

In this section we collect notation, definitions, and results from the literature useful for our work.

2.1 Stationary and non-stationary models

We will need to embed the OY polymer in a larger family of models. First, extend $\{B_i(s)\}_i$ to an infinite family of independent two-sided Brownian motions. We will take the convention that $B_i(0) = 0$ but in all of our definitions only increments arise and so this is irrelevant.

For $p, q \in \mathbb{R}^2$ we use the notation $p \leq q$ to denote that the inequality holds component-wise. For $(s,m) \leq (t,n)$ we introduce the point-to-point partition function:

$$Z_{(s,m),(t,n)} := \int_{s<s_m \cdots < s_{n-1}<t} \mathrm{e}^{\sum_{k=m}^{n} B_k(s_k) - B_k(s_{k-1})} \mathrm{d}s_m \ldots \mathrm{d}s_{n-1}, \tag{2.1}$$

where we use the convention $s_{m-1} = s$ and $s_n = t$. This convention will be used repeatedly throughout the paper without further comment when similar definitions arise. By convention we also set $Z_{(s,n),(t,n)} = \mathrm{e}^{B_n(t)-B_n(s)}$. Then $Z_{t,n} = Z_{(0,1),(t,n)}$. We will not use the notation $Z_{t,n}$ in the remainder of the paper. Note that we think of the x-axis as the time t coordinate and the y-axis as the spatial integer-valued coordinate.

We make one additional convention. If $p \leq q$ (with $p, q \in \mathbb{R} \times \mathbb{Z}$) does not hold, then set $Z_{p,q} = 0$, and $\log Z_{p,q} = -\infty$.

We will also have use for the following two-parameter version of the OY polymer,

$$Z_{t,n}^{(\eta,\theta)} := \int_{-\infty<s_0\cdots<s_{n-1}<t} \mathrm{e}^{B_0(s_0)-\eta(s_0)_-+\theta(s_0)_++\sum_{k=1}^n B_k(s_k)-B_k(s_{k-1})}\mathrm{d}s_0\ldots\mathrm{d}s_{n-1}, \quad (2.2)$$

where $\theta,\eta>0$. Here, $x_-=\max\{0,-x\}$ and $x_+=\max\{0,x\}$ denote the negative and positive part of x, respectively. We will also denote the special case $Z_{t,n}^{\theta}=Z_{t,n}^{(\theta,\theta)}$. In this case, $Z_{t,n}^{\theta}$ is stationary in a sense explained in Theorem 3.3 of [26]. In particular, this stationarity will be used in the proofs of Corollary 3.3 and Proposition 3.4 below.

We define now the free energy densities by

$$f_{t,n}^{\theta} := t\theta - n\psi_0(\theta), \qquad f_{t,n}=f_{t,n}^{\theta} \quad \text{with } \psi_1(\theta)=t/n. \quad (2.3)$$

The function $\psi_1(\theta)$ is strictly decreasing and satifies $\psi_1(0^+)=\infty$ and $\psi_1(\infty)=0$, and so the equation $\psi_1(\theta)=\kappa$ has a unique solution for any $\kappa>0$. By [19, (2.4)] we have

$$\mathbb{E}\left[\log Z_{t,n}^{\theta}\right]=f_{t,n}^{\theta}. \quad (2.4)$$

The first and second estimates in the following are from Theorem 6.3 and Corollary 7.4 of [17], respectively.

Proposition 2.1. Let $\delta>0$ be given and let $\delta n\le t\le\delta^{-1}n$ and let θ satisfy $\psi_1(\theta)=t/n$. There are $c,C>0$ so that for n sufficiently large,

$$\mathbb{P}\left[|\log Z_{t,n}^{\theta}-f_{t,n}^{\theta}|>sn^{1/3}\right]\le C\mathrm{e}^{-cs^{3/2}} \quad (2.5)$$

and

$$\mathbb{P}\left[\log Z_{(0,1),(t,n)}-f_{t,n}>sn^{1/3}\right]\le C\mathrm{e}^{-cs^{3/2}} \quad (2.6)$$

for $0<s<cn^{2/3}$.

We also have the following from [19, Theorem 1.1]

Lemma 2.2. Fix $\delta>0$ and let $\delta n\le t\le\delta^{-1}n$. There are $C,c>0$ so that for n sufficiently large,

$$\mathbb{E}\left[|\log Z_{(0,1),(t,n)}-f_{t,n}|\right]\ge cn^{1/3} \quad (2.7)$$

and

$$\mathbb{E}\left[|\log Z_{(0,1),(t,n)}-f_{t,n}|^2\right]\le Cn^{2/3} \quad (2.8)$$

Corollary 2.3. Let $\delta>0$ and assume $\delta n\le t\le\delta^{-1}n$. There is a $c_2>0$ and $\delta_1>0$ so that for n sufficiently large,

$$\mathbb{P}\left[\log Z_{(0,1),(t,n)}<f_{t,n}-c_2n^{1/3}\right]\ge\delta_1 \quad (2.9)$$

Proof. Let θ satisfy $\psi_1(\theta)=t/n$. Due to the deterministic inequality $Z_{(0,0),(t,n)}\le Z_{t,n}^{\theta}$ we have $\mathbb{E}[\log Z_{(0,1),(t,n)}]\le\mathbb{E}[\log Z_{t,n-1}^{\theta}]=f_{t,n}+\mathcal{O}(1)$, where we used (2.4). Let $X=\log Z_{(0,1),(t,n)}$. Let $2c_1>0$ so that $\mathbb{E}[|X-f_{t,n}|]\ge 2c_1n^{1/3}$. If $\mathbb{E}[(X-f_{t,n})_-]\ge c_1n^{1/3}$ then the claim follows, since we can use the inequality

$$c_1n^{1/3}\le\mathbb{E}[(X-f_{t,n})_-]\le c_3n^{1/3}+\mathbb{P}\left[X-f_{t,n}<-c_3n^{1/3}\right]^{1/2}\mathbb{E}[|X-f_{t,n}|^2]^{1/2} \quad (2.10)$$

to find the desired estimate after taking, say, $c_3=c_1/2$, after applying Lemma 2.2. Otherwise, assume $\mathbb{E}[(X-f_{t,n})_+]\ge c_1n^{1/3}$. Then, using $\mathbb{E}[X]\le f_{t,n}+C$ for some $C>0$ we have,

$$c_1n^{1/3}\le\mathbb{E}[(X-f_{t,n})_+]\le C+\mathbb{E}[(X-\mathbb{E}[X])_+]=C+\mathbb{E}[(X-\mathbb{E}[X])_-]\le 2C+\mathbb{E}[(X-f_{t,n})_-], \quad (2.11)$$

and we conclude as before. □

Finally we record here the super-additivity property of the polymer partition function, see, e.g., equation (6) of [12].

Lemma 2.4. We have that almost surely,

$$Z_{(s,m),(t+u,n+k)} \geq Z_{(s,m),(t,n)} Z_{(t,n),(t+u,n+k)} \tag{2.12}$$

for all $s \leq t, m \leq n$ and $k, u > 0$.

2.2 Gibbs measure, polymer paths

The partition function $Z_{(s,m),(t,n)}$ is the normalization constant in the following Gibbs measure on the simplex $\{(s_m, \ldots, s_{n-1}) \in \mathbb{R}^{n-m-1} : s_m < \cdots < s_{n-1}\}$ defined by

$$\begin{aligned} &Q_{(s,m),(t,n)}[(s_m, \ldots s_{n-1}) \in \mathcal{A}] \\ :=& \frac{1}{Z_{(s,m),(t,n)}} \int_{s<s_m \cdots < s_{n-1} < t} \mathbf{1}_{\{(s_m,\ldots s_{n-1}) \in \mathcal{A}\}} \mathrm{e}^{\sum_{k=m}^n B_k(s_k) - B_k(s_{k-1})} \mathrm{d}s_m \ldots \mathrm{d}s_{n-1} \end{aligned} \tag{2.13}$$

for Borel $A \subseteq \mathbb{R}^{n-1-m}$.

We will interpret the times $(s_m, \ldots, s_{n-1})$ as defining the jump times of a right continuous up-right polymer path $\gamma : [s,t] \to [\![m,n]\!]$ uniquely defined by $\gamma(s) = k, s \in (s_{k-1}, s_k)$. We could take γ to be left continuous instead, but this is immaterial.

Given some set of polymer paths $\mathcal{A}$ we will abuse notation and denote $Q_{(s,m),(t,n)}[\gamma \in \mathcal{A}]$ as the Gibbs probability that the polymer path defined by the jump times lies in the set $\mathcal{A}$. We will only take very simple $\mathcal{A}$ so there will be no measureability concerns.

In a similar fashion we denote the Gibbs measure associated to $Z^{\theta}_{t,n}$ by $Q^{\theta}_{t,n}$ and to $Z^{(\eta,\theta)}_{t,n}$ by $Q^{(\eta,\theta)}_{t,n}$. For sets of polymer paths or jump times we will use notation,

$$Z_{(s,m),(t,n)}[\gamma \in \mathcal{A}] := Z_{(s,m),(t,n)} Q_{(s,m),(t,n)}[\gamma \in \mathcal{A}] \tag{2.14}$$

to denote the partition function restricted to this set. Similar considerations apply to $Z^{\theta}_{n,t}$. Later, we will introduce several other modified or related partition functions Z, $\tilde{Z}$ etc., usually coming with some indices, superscripts or other decorations; they will always involve integrals over a simplex and then notation such as $Z[\gamma \in \mathcal{A}]$ or $\tilde{Z}[\gamma \in \mathcal{A}]$ always means to restrict the integrals defining the partition function at hand to the set $\mathcal{A}$.

2.3 Properties of limit shape

Note that by homogeneity, $f_{\kappa t,\kappa n} = \kappa f_{t,n}$. We will require the following concavity of the limit shape at the point $(1,1)$. We introduce the following two quantities for use throughout the paper,

$$\mu = f_{1,1}, \qquad \mathfrak{a} := \frac{\mathrm{d}}{\mathrm{d}w} f_{1-w,1+w}|_{w=0}. \tag{2.15}$$

Lemma 2.5. Let $\varepsilon > 0$. There are $c, C > 0$ and $\mathfrak{a} \in \mathbb{R}$ so that for $|w| < 1 - \varepsilon$ we have,

$$-Cw^2 \leq f_{1-w,1+w} - \mu - \mathfrak{a}w \leq -cw^2. \tag{2.16}$$

Consequently, for $|w| < (1-\varepsilon)n$ we have,

$$-Cw^2 n^{-1} \leq f_{n-w,n+w} - (\mu n + \mathfrak{a}w) \leq -cw^2 n^{-1}. \tag{2.17}$$

Proof. Let $g(x,y) := f_{x,y}$ and let $\theta = \theta(x,y)$ satisfy $\psi_1(\theta) = x/y$. Then, for the partial derivatives we have

$$g_x = \theta, \qquad g_y = -\psi_0(\theta), \tag{2.18}$$

as well as,

$$\theta_x = \frac{1}{\psi_2(\theta) y}, \qquad \theta_y = \frac{-x}{y^2 \psi_2(\theta)}. \tag{2.19}$$

For the Hessian of g we have,

$$\nabla^2 g = \begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix} = \frac{1}{y^3\psi_2(\theta)} \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix}. \tag{2.20}$$

Since $\psi_2(\theta) < 0$ we see that

$$(1,-1)\left(\nabla^2 g(x,y)\right)(1,-1)^T < -c \tag{2.21}$$

for some $c > 0$ and all $\{(x,y) \in \mathbb{R}^2 : 10 > x > \varepsilon, 10 > y > \varepsilon\}$. The claim follows from Taylor's theorem with integral remainder. □

2.4 Rains-EJS identity

Here we state an identity derived in a more general context in [16, Proposition 6.1]. It is the analog for the OY polymer of the identity of Rains and Emrah-Janjigian-Seppäläinen for last passage percolation.

Proposition 2.6. For any $\eta, \theta > 0$ we have

$$\mathbb{E}\left[\exp\left((\eta-\theta)\log Z_{t,n}^{(\eta,\theta)}\right)\right] = \exp\left(n(\psi_{-1}(\theta) - \psi_{-1}(\eta)) - \frac{1}{2}t\left(\theta^2 - \eta^2\right)\right). \tag{2.22}$$

2.5 Integer coordinates

In many places in our work we will implicitly round quantities so that they lie on the integer lattice $\mathbb{Z}^2$. This usually takes place when we consider points on lines $\{(x,y) : x+y = \ell, x, y \in \mathbb{Z}\}$. For example, a point $(a-b, a+b) \in \{(x,y) : x+y = \ell, x, y \in \mathbb{Z}\}$ where a and b are not necessarily integers should be understood as the point on this line closest to $(a-b, a+b)$. This is due to the fact that these coordinates will appear in arguments of the polymer partition function, e.g., $Z_{(0,0),(a-b,a+b)}$ which makes sense only if $a+b \in \mathbb{Z}$. This rounding convention does not affect proofs as the errors can be absorbed into the constants that arise in our estimates.

An additional example in which this occurs is when we divide n into k different segments n/k. For example we will want to relate $Z_{(0,0),(n,n)}$ to k copies of $Z_{(0,0),(n/k,n/k)}$. In order to do this, one should use some combination of $Z_{(0,0),(\lfloor n/k\rfloor,\lfloor n/k\rfloor)}$ and $Z_{(0,0),(\lceil n/k\rceil,\lceil n/k\rceil)}$, but we will ignore this in our proofs, as the modifications are trivial and only require tedious notation.

2.6 Rescaling

We will prove many of our theorems only along the diagonal $Z_{(0,0),(n,n)}$. Due to the continuous nature of the time variable, estimates for $Z_{(0,0),(t,n)}$ and $\delta n \le t \le \delta^{-1} n$, for some $\delta > 0$ may be reduced to $Z_{(0,0),(n,n)}$ by rescaling the time argument of the Brownian motions. This has the effect of changing the diffusivity constant of the family of the Brownian motions used in the definition of the model. Alternatively, one can introduce an inverse temperature $\beta > 0$ in the (1.1) and (2.1); i.e., changing the weight to $e^{\beta \sum_{k=m}^{n} B_k(s_k) - B_k(s_{k-1})}$ in (2.1). Since t is assumed to be of order n, the parameter $\beta > 0$ would satisfy $c \le \beta \le c^{-1}$ for some $c > 0$. All of our arguments that work on the diagonal $(t,n) = (n,n)$ do not use the fact that $\beta = 1$, and the introduction of the parameter $\beta > 0$ would not affect any of arguments. Simply, the constants would just need to be adjusted in our estimates. The limit shape $f_{t,n}$ would of course be rescaled in some fashion but the only property we use for it, Lemma 2.5, would still hold.

3 Weak bound for left tail

In this section we will make use of the quantity,

$$e_n(\theta, t) = t - n\psi_1(\theta) \tag{3.1}$$

which is the expectation of the first jump time s_0 with respect to the annealed measure $\mathbb{E}[Q^\theta_{n,t}[\cdot]]$, as can be seen by differentiating (2.4) wrt θ.

3.1 Jump estimates

In this section we derive tail estimates on the first jump time s_0 under the annealed measure $\mathbb{E}\, Q^\theta_{t,n}$. The work [17] derives estimates that are equivalent to a right tail bound. A left tail bound can be derived in much the same way. However, as the set-up in [17] at first appears slightly different from that considered here, we give all the details.

Proposition 3.1. Let $\delta > 0$ and let $\delta n \le t \le \delta^{-1} n$. Let θ_0 satisfy $e_n(\theta_0, t) = 0$. There are $C, c, \varepsilon > 0$ so that if $\theta_0 - \varepsilon < \eta < \theta_0$, then,

$$\mathbb{E}\left[Q^\eta_{t,n}[s_0 > 0]\right] \le C\mathrm{e}^{-cn(\theta_0-\eta)^3}. \tag{3.2}$$

Proof. We have, for any $0 < r < 1$ and $\lambda > 0$ and $\theta > \eta$,

$$\begin{aligned} Q^\eta_{t,n}[s_0 > 0] &\le Q^\eta_{t,n}[s_0 > 0]^r \le Q^{(\eta,\theta)}_{t,n}[s_0 > 0]^r \\ &= \left(\frac{Z^{(\eta,\theta)}_{t,n}[s_0 > 0]}{Z^{(\eta,\theta)}_{t,n}}\right)^r = \left(\frac{Z^{(\lambda,\theta)}_{t,n}[s_0 > 0]}{Z^{(\eta,\theta)}_{t,n}}\right)^r \\ &\le (Z^{(\lambda,\theta)}_{t,n})^r (Z^{(\eta,\theta)}_{t,n})^{-r} \end{aligned} \tag{3.3}$$

The first inequality follows from the fact that $x \le x^r$ for $0 \le x \le 1$. The second inequality follows from the fact that

$$\partial_y Q^{(x,y)}_{t,n}[s_0 > 0] = \mathrm{Cov}(\mathbf{1}_{\{s_0>0\}}, (s_0)_+) \ge 0 \tag{3.4}$$

where the covariance is with respect to $Q^{(x,y)}_{t,n}$. The equality in (3.4) is proven in Appendix C.5 by direct calculation, and the inequality is due to the general fact that the covariance of two increasing functions of a random variable is nonnegative.

Choose $4r = \theta_0 - \eta$, and $\lambda = \theta_0$ and $\theta = \eta + 2r$. Then we have by Proposition 2.6 and a Taylor expansion,

$$\begin{aligned} \mathbb{E}\left[(Z^{(\lambda,\theta)}_{t,n})^r (Z^{(\eta,\theta)}_{t,n})^{-r}\right]^2 &\le \mathbb{E}\left[\mathrm{e}^{2r \log Z^{(\theta_0,\theta)}_{t,n}}\right] \times \mathbb{E}\left[\mathrm{e}^{-2r\log Z^{(\eta,\theta)}_{t,n}}\right] \\ &= \exp\left(n(2\psi_{-1}(\theta) - \psi_{-1}(\theta_0) - \psi_{-1}(\eta) + \frac{t}{2}\left(\theta_0^2 + \eta^2 - 2\theta^2\right)\right) \\ &= \exp\left(8r^3 n\psi_2(\theta_0) + \mathcal{O}(Cr^4 n)\right). \end{aligned} \tag{3.5}$$

The terms quadratic in r in the Taylor expansion in the last line vanish due to the assumption that $e_n(\theta_0, t) = 0$ (the first order terms vanish due to the choice of θ). This yields the claim. ☐

Proposition 3.2. Let $\delta > 0$ and assume $\delta n \le t \le \delta^{-1} n$. Let θ_0 satisfy $e_n(\theta_0, t) = 0$. There are $C, c, \varepsilon > 0$ so that if $\theta_0 + \varepsilon > \eta > \theta_0$, then,

$$\mathbb{E}\left[Q^\eta_{t,n}[s_0 < 0]\right] \le C\mathrm{e}^{-cn(\eta-\theta_0)^3}. \tag{3.6}$$

Proof. We have, for any $\theta < \eta$ and $\lambda > 0$,

$$Q^{\eta}_{t,n}[s_0 < 0] \le Q^{(\theta,\eta)}_{t,n}[s_0 < 0] = \frac{Z^{(\theta,\eta)}_{t,n}[s_0 < 0]}{Z^{(\theta,\eta)}_{t,n}} \le \frac{Z^{(\theta,\lambda)}_{t,n}}{Z^{(\theta,\eta)}_{t,n}}. \tag{3.7}$$

Similar to the proof of the previous proposition, the first inequality follows from $\partial_x Q^{(xy)}_{t,n}[s_0 < 0] = -\operatorname{Cov}(\mathbf{1}_{\{s_0<0\}}, (s_0)_-) \le 0$.

Choose $4r = \eta - \theta_0$ and $\lambda = \theta_0$ and $\theta = \eta - 2r$. Then, by Proposition 2.6 and a Taylor expansion,

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{2r\log Z^{(\theta,\theta_0)}_{t,n}}\right]\mathbb{E}\left[\mathrm{e}^{-2r\log Z^{(\theta,\eta)}_{t,n}}\right]\\
&= \exp\left(n(-2\psi_{-1}(\theta) + \psi_{-1}(\eta) + \psi_{-1}(\theta_0)) + \frac{t}{2}(-\theta_0^2 - \eta^2 + 2\theta^2)\right)\\
&= \exp\left(8r^3 n\psi_2(\theta_0) + \mathcal{O}(Cr^4 n)\right)
\end{aligned} \tag{3.8}$$

This completes the proof in a similar manner to the previous result. □

Corollary 3.3. Let $\delta > 0$ and assume $\delta n \le t \le \delta^{-1} n$. There are $C, c > 0$ so that,

$$\mathbb{E}\left[Q^{\theta}_{n,t}[|s_0 - e_n(\theta,t)| > sn^{2/3}]\right] \le C\mathrm{e}^{-cs^3} \tag{3.9}$$

for $0 < s \le cn^{1/3}$.

Proof. By [26, Remark 3.1], we have the equality in distribution,

$$Q^{\theta}_{n,t}[s_0 > e_n(\theta,t) + sn^{2/3}] \stackrel{d}{=} Q^{\theta}_{n,t_1}[s_0 > 0] \tag{3.10}$$

where $t_1 = t - e_n(\theta,t) - sn^{2/3}$. We may assume that $t_1 \ge 0$ or else the claim is vacuous. Let θ_0 solve,

$$n\psi_1(\theta_0) = t_1 = t - (t - n\psi_1(\theta)) - sn^{2/3} \tag{3.11}$$

which is equivalent to,

$$n(\psi_1(\theta_0) - \psi_1(\theta)) = -sn^{2/3} \tag{3.12}$$

so that $\theta_0 - \theta \asymp sn^{-1/3}$, as long as $s \le cn^{1/3}$, some $c > 0$. We then apply Proposition 3.1 to $\mathbb{E}[Q^{\theta}_{n,t_1}[s_0 > 0]$ finding an estimate of $C\mathrm{e}^{-cs^3}$ as long as $s \le cn^{1/3}$, where $c > 0$ is taken sufficiently small to guarantee $\theta_0 - \theta \le \varepsilon$ where the $\varepsilon > 0$ is from the statement of Proposition 3.1.

For the other tail, we have

$$Q^{\theta}_{n,t}[s_0 < e_n(\theta,t) - sn^{2/3}] \stackrel{d}{=} Q^{\theta}_{n,t_2}[s_0 < 0] \tag{3.13}$$

where $t_2 = t - e_n(\theta,t) + sn^{2/3}$. Now let θ_0 solve,

$$n\psi_1(\theta_0) = t_2 = t - (t - n\psi_1(\theta)) + sn^{2/3} \tag{3.14}$$

so that

$$n(\psi_1(\theta_0) - \psi_1(\theta)) = sn^{2/3} \tag{3.15}$$

so that $\theta - \theta_0 \asymp sn^{-1/3}$. We then apply Proposition 3.2 and conclude in a similar manner to the other tail. □

3.2 Weak tail bound for non-stationary model

In this section we derive a sub-optimal tail estimate for the left tail of $Z_{(0,1),(t,n)}$ that will serve as an input for the rest of the paper.

The proof of the following is based on the proof of [19, Lemma 2.8]. Compared to that result, we have better estimates available for various quantities that arise, allowing us to conclude a better tail estimate than what was proven in that work.

Proposition 3.4. Let $\delta > 0$ and assume that $\delta n \leq t \leq \delta^{-1} n$. There are $c, C > 0$ so that for all $0 \leq b \leq cn^{2/3}$ we have,

$$\mathbb{P}\left[\log Z_{(0,1),(t,n)} - f_{t,n} < -bn^{1/3}\right] \leq C\mathrm{e}^{-cb^{3/2}}. \tag{3.16}$$

Proof. Let θ satisfy $\psi_1(\theta) = t/n$. By Proposition 2.1 it suffices to prove the estimate,

$$\mathbb{P}\left[\frac{Z^{\theta}_{t,n}}{Z_{(0,1),(t,n)}} \geq \mathrm{e}^{bn^{1/3}}\right] \leq C\mathrm{e}^{-cb^{3/2}}, \tag{3.17}$$

for some $c, C > 0$. Compared to [19], this is an improved version of the estimate (2.40) of that paper. In order to prove the above estimate we follow the proof of [19, Lemma 2.8] inserting our better estimates where appropriate. We may assume $b \geq 1$. Let $u = \sqrt{b}n^{2/3}$. Then,

$$\begin{aligned}\mathbb{P}\left[\frac{Z^{\theta}_{n,t}}{Z_{(0,1),(t,n)}} \geq \mathrm{e}^{bn^{1/3}}\right] &= \mathbb{P}\left[\frac{Z^{\theta}_{n,t}[|s_0| < u]}{Z_{(0,1),(t,n)}Q^{\theta}_{n,t}[|s_0| < u]} \geq \mathrm{e}^{bn^{1/3}}\right] \\ \leq \mathbb{P}\left[\frac{Z^{\theta}_{n,t}[|s_0| < u]}{Z_{(0,1),(t,n)}} \geq \frac{1}{2}\mathrm{e}^{bn^{1/3}}\right] &+ \mathbb{P}\left[Q^{\theta}_{n,t}[|s_0| < u] \leq 1/2\right]\end{aligned} \tag{3.18}$$

For the second probability, we have

$$\mathbb{P}\left[Q^{\theta}_{n,t}[|s_0| < u] \leq 1/2\right] = \mathbb{P}\left[Q^{\theta}_{n,t}[|s_0| \geq u] \geq 1/2\right] \leq C\mathrm{e}^{-cu^3 n^{-2}} = C\mathrm{e}^{-cb^{3/2}} \tag{3.19}$$

by Corollary 3.3. Note that we used that $e_n(\theta, t) = 0$ by our choice of θ. For the other term we estimate,

$$\begin{aligned}&\mathbb{P}\left[\frac{Z^{\theta}_{n,t}[|s_0| < u]}{Z_{(0,1),(t,n)}} \geq \frac{1}{2}\mathrm{e}^{bn^{1/3}}\right] \\ \leq &\mathbb{P}\left[\frac{Z^{\theta}_{n,t}[0 \leq s_0 < u]}{Z_{(0,1),(t,n)}} \geq \frac{1}{4}\mathrm{e}^{bn^{1/3}}\right] + \mathbb{P}\left[\frac{Z^{\theta}_{n,t}[-u \leq s_0 < 0]}{Z_{(0,1),(t,n)}} \geq \frac{1}{4}\mathrm{e}^{bn^{1/3}}\right]\end{aligned} \tag{3.20}$$

In order to estimate the first quantity, introduce the reverse system, $B^{(r)}_0(s) = -(B_n(t) - B_n(t-s))$ and $B^{(r)}_i(s) = B_{n-i}(t) - B_{n-i}(t-s)$. Denote using the superscript (r) the corresponding partition functions, Gibbs measures, etc., with respect to the reversed Brownian motions, $Z^{(r)}_{(s,1),(t,n)}$ and $Z^{\theta,(r)}_{n,t}$, etc. For example, $Z_{(s,1),(t,n)} = Z^{(r)}_{(0,0),(t-s,n-1)}$ for any $s \in (-\infty, t)$.

Before continuing we record here the following inequality (which is [19, (2.49)]),

$$\frac{Z^{\eta}_{t,n}[s_0 > 0]}{Z^{\eta}_{s,n}[s_0 > 0]} \geq \frac{Z_{(0,0),(t,n)}}{Z_{(0,0),(s,n)}} \geq \frac{Z^{\eta}_{t,n}[s_0 < 0]}{Z^{\eta}_{s,n}[s_0 < 0]} \tag{3.21}$$

which holds for any $0 < s < t$ and $\eta > 0$.

Now, let $\nu = c_1\sqrt{b}n^{-1/3}$ for $c_1 > 0$. Let $\lambda = \theta - \nu$. We have,

$$\begin{aligned}\frac{Z_{(s,1),(t,n)}}{Z_{(0,1),(t,n)}} &= \frac{Z^{(r)}_{(0,0),(t-s,n-1)}}{Z^{(r)}_{(0,0),(t,n-1)}} \leq \frac{Z^{\lambda,(r)}_{t-s,n-1}[s_0<0]}{Z^{\lambda,(r)}_{t,n-1}[s_0<0]} \\ &= \frac{Z^{\lambda,(r)}_{t-s,n-1}}{Z^{\lambda,(r)}_{t,n-1}}\frac{Q^{\lambda,(r)}_{t-s,n-1}[s_0<0]}{Q^{\lambda,(r)}_{t,n-1}[s_0<0]} =: \mathrm{e}^{Y^{(r)}_{n-1}(t-s,t)-\lambda s}\frac{Q^{\lambda,(r)}_{t-s,n-1}[s_0<0]}{Q^{\lambda,(r)}_{t,n-1}[s_0<0]} \\ &\leq \mathrm{e}^{Y^{(r)}_{n-1}(t-s,t)-\lambda s}\frac{1}{Q^{\lambda,(r)}_{t,n-1}[s_0<0]} \end{aligned}\tag{3.22}$$

where in the first inequality we used the second part of (3.21). Specifically, we applied this to the reversed system: that is, we used the second inequality of (3.21) but with superscripts $^{(r)}$ added to the partition functions.

Now, by definition, $Y^{(r)}_{n-1}(t-s,t) = \lambda s - \log Z^{\lambda,(r)}_{n-1,t} + \log Z^{\lambda,(r)}_{n-1,t-s}$. Therefore,

$$\begin{aligned} &\mathbb{P}\left[\frac{Z^{\theta}_{t,n}[0\leq s_0<u]}{Z_{(0,1),(t,n)}} \geq \frac{1}{4}\mathrm{e}^{bn^{1/3}}\right] = \mathbb{P}\left[\int_0^u \mathrm{e}^{-B_0(s)+\theta s}\frac{Z_{(s,1),(t,n)}}{Z_{(0,1),(t,n)}}\mathrm{d}s \geq \frac{1}{4}\mathrm{e}^{bn^{1/3}}\right] \\ \leq\ &\mathbb{P}\left[\int_0^t \frac{\mathrm{e}^{-B_0(s)+(\theta-\lambda)s+Y^{(r)}_{n-1}(t-s,t)}}{Q^{\lambda,(r)}_{n-1,t}[s_0<0]} \geq \frac{1}{4}\mathrm{e}^{n^{1/3}b}\right] \\ \leq\ &\mathbb{P}\left[Q^{\lambda,(r)}_{n-1,t}[s_0<0]\leq 1/2\right] + \mathbb{P}\left[\int_0^u \mathrm{e}^{-B(s)+Y^{(r)}_{n-1}(t-s,t)+\nu s}\mathrm{d}s \geq \frac{1}{8}\mathrm{e}^{n^{1/3}b}\right] \end{aligned}\tag{3.23}$$

For the first term on the RHS, we calculate,

$$\begin{aligned} e_{n-1}(\lambda,t) &\leq t - n\psi_1(\lambda) + C = (t-n\psi_1(\theta)) + n(\psi_1(\theta)-\psi_1(\lambda)) + C \\ &\leq -cn\nu + C \leq -cb^{1/2}n^{2/3}. \end{aligned}\tag{3.24}$$

Above, the first inequality follows directly from the definition (3.1) of $e_{n-1}(\lambda,t)$. The second inequality follows from the fact that $\psi_1(\theta) = t/n$ and that $(\psi_1'(x)) < 0$, and that $\theta = \lambda + \nu$. The last inequality holds for n large enough. Therefore by Corollary 3.3 we have,

$$\begin{aligned} &\mathbb{P}\left[Q^{\lambda,(r)}_{n-1,t}[s_0<0]\leq 1/2\right] = \mathbb{P}\left[Q^{\lambda,(r)}_{n-1,t}[s_0>0]\geq 1/2\right] \\ \leq &\mathbb{P}\left[Q^{\lambda,(r)}_{n-1,t}[s_0>b^{1/2}cn^{2/3}+e_{n-1}(\lambda,t)]\geq 1/2\right] \leq C\mathrm{e}^{-cb^{3/2}}. \end{aligned}\tag{3.25}$$

We turn now to the second term of (3.23). By the Burke property (Theorem 3.3 and Theorem 3.4 of [26]) we have that $s \mapsto Y^{(r)}_{n-1}(t-s,t)$ is a Brownian motion and by construction it is independent of $B_0(s)$. Choose $c_1 > 0$ sufficiently small so that $n^{1/3}b \geq 10\nu u$, and so $\frac{1}{8}\mathrm{e}^{n^{1/3}b} \geq \mathrm{e}^{3\nu u}$ for n sufficiently large. We therefore must bound,

$$\begin{aligned} &\mathbb{P}\left[\int_0^u \mathrm{e}^{\sqrt{2}B_0(s)+\nu s}\mathrm{d}s \geq \mathrm{e}^{3\nu u}\right] \leq \mathbb{P}\left[\int_{-\infty}^u \mathrm{e}^{\sqrt{2}B_0(s)+\nu s}\mathrm{d}s \geq \mathrm{e}^{3\nu u}\right] \\ \leq &\mathbb{P}\left[\mathrm{e}^{\sqrt{2}B_0(u)+\nu u} \geq \mathrm{e}^{2\nu u}\right] + \mathbb{P}\left[\int_{-\infty}^u \mathrm{e}^{\sqrt{2}(B_0(s)-B_0(u))+\nu(s-u)}\mathrm{d}s \geq \mathrm{e}^{\nu u}\right] \end{aligned}\tag{3.26}$$

The first probability is less than $C\mathrm{e}^{-c\nu^2 u} \leq C\mathrm{e}^{-cb^{3/2}}$. By Dufresne's identity, the integral in the second probability has the same distribution as the reciprocal of a Gamma (ν) random variable and so

$$\mathbb{P}\left[\mathrm{Gamma}(\nu) \leq \mathrm{e}^{-\nu u}\right] = \int_0^{\mathrm{e}^{-\nu u}} \frac{x^{\nu-1}\mathrm{e}^{-x}}{\Gamma(\nu)}\mathrm{d}x \leq \frac{\mathrm{e}^{-\nu^2 u}}{\nu\Gamma(\nu)} \leq C\mathrm{e}^{-\nu^2 u} \leq C\mathrm{e}^{-cb^{3/2}}. \tag{3.27}$$

Collecting the above, we see that,

$$\mathbb{P}\left[\frac{Z^{\theta}_{n,t}[0 \le s_0 < u]}{Z_{(0,1),(t,n)}} \ge \frac{1}{4}\mathrm{e}^{bn^{1/3}}\right] \le C\mathrm{e}^{-cb^{3/2}} \tag{3.28}$$

as desired. The second term of (3.20) is estimated similar to the first term. We instead choose $\lambda = \theta + \nu$ and use the first inequality of (3.21) to obtain,

$$\begin{aligned}\frac{Z^{\theta}_{n,t}[-u < s_0 < 0]}{Z_{(0,1),(t,n)}} &= \int_{-u}^{0} \mathrm{e}^{-B_0(s)+\theta s}\frac{Z_{(s,1),(t,n)}}{Z_{(0,1),(t,n)}}\mathrm{d}s \\ &\le \int_{-u}^{0} \frac{\mathrm{e}^{-B_0(s)-Y^{(r)}_{n-1}(t,t-s)-(\theta-\lambda)s}}{Q^{\lambda,(r)}_{n-1,t}[s_0 > 0]}\mathrm{d}s \end{aligned} \tag{3.29}$$

Everything else is identical. □

We record the above result, as well as the second estimate of Proposition 2.1, in a single corollary for easy reference.

Corollary 3.5. Fix $\delta > 0$ and assume $\delta n \le t \le \delta^{-1}n$. There are $C, c > 0$ so that

$$\mathbb{P}\left[|\log Z_{(0,1),(t,n)} - f_{t,n}| > sn^{1/3}\right] \le C\mathrm{e}^{-cs^{3/2}} \tag{3.30}$$

for all $0 \le s \le cn^{2/3}$.

4 Interval-to-interval estimates

In this section, we will consider interval-to-interval partition functions. However, the linear correction to the limit shape in Lemma 2.5 is large and must be compensated for. We therefore do not directly consider interval-to-interval partition functions and instead consider the following modified version.

First, for any $p \le q \in \mathbb{R} \times \mathbb{Z}$ we define the compensated partition function:

$$\tilde{Z}_{p,q} := Z_{p,q}\mathrm{e}^{\mathfrak{a}\,\mathrm{ad}(p-q)}. \tag{4.1}$$

Here, $\mathfrak{a}$ was defined in (2.15), and the anti-diagonal distance operator ad was defined in (1.9).

As stated above, this compensates the first order term in the correction to the free energy density of $Z_{p,q}$. For example,

$$\tilde{Z}_{(m,m)+(-i,i),(n,n)+(-j,j)} = Z_{(m,m)+(-i,i),(n,n)+(-j,j)}\mathrm{e}^{\mathfrak{a}i}\mathrm{e}^{-\mathfrak{a}j}. \tag{4.2}$$

We will need to consider various line segment-to-line segment partition functions (or interval-to-interval). We will only consider line segments with integer coordinates that are parallel to the anti-diagonal. We will use the notation ℓ to refer to these line segments, possibly decorated with subscripts or superscripts if we need to introduce a family of line segments. These line segments will always be of the form,

$$\ell := \{(i,i) - (j,-j) : j \in [\![a,b]\!]\} \tag{4.3}$$

for some $a \le b$ and $i \in \mathbb{Z}$. Then, for two line segments ℓ_1, ℓ_2 we define,

$$\tilde{Z}_{\ell_1,\ell_2} := \sum_{p\in\ell_1, q\in\ell_2} \tilde{Z}_{p,q}. \tag{4.4}$$

By considering a point to be a line segment of one point, this definition also includes interval-to-point and point-to-interval partition functions. Note that the quantity on the RHS may be identically 0 if $p \le q$ does not hold for any $(p,q) \in \ell_1 \times \ell_2$.

In what follows, we consider two line segments,

$$\ell_1 := \{(-i,i) : |i| \le l_1 n^{2/3}\}, \qquad \ell_2 := \{(n,n) - (w+j, -w-j) : |j| \le l_2 n^{2/3}\}. \tag{4.5}$$

for some w and $l_1, l_2 > 0$. The following is an analog of the first estimate of [4, Proposition 3.5], and is proven using a similar “stepping back strategy.”

Proposition 4.1. Let ℓ_1 and ℓ_2 be as above. Assume $|w| \le 1.5n$. There are constants $c_1 > 0$ and $s_0 > 0$, possibly depending on l_1, l_2 such that

$$\mathbb{P}\left[\log \tilde{Z}_{\ell_1,\ell_2} > \mu n - c_1 \frac{w^2}{n} + s n^{1/3}\right] \le \mathrm{e}^{-c_1 s^{3/2}} \tag{4.6}$$

for $s_0 \le s \le c_1 n^{2/3}$ and $n \ge s_0$. The same estimate holds also for $\tilde{Z}_{(0,0),\ell_2}$ and $\tilde{Z}_{\ell_1,(n-w,n+w)}$.

Proof. We do the case $l_1 = l_2 = 1$, the general case being similar. We may assume that $|w| \le n + 2n^{2/3}$ or else $\tilde{Z}_{\ell_1,\ell_2} = 0$ and the claim is trivial. Consider the points $p = -(n,n)$ and $q = (n-w, n+w) + (n,n)$. Let p_* and q_* be the points in ℓ_1 and ℓ_2, respectively, that satisfy

$$\max_{|i| \le n^{2/3}, |j| \le n^{2/3}} Z_{(-i,i),(n,n)-(w+j,-w-j)} \mathrm{e}^{\mathfrak{a} i} \mathrm{e}^{-\mathfrak{a}(w+j)} = \tilde{Z}_{p_*,q_*}. \tag{4.7}$$

Necessarily we have that $p_* \le q_*$. Then,

$$\log \tilde{Z}_{\ell_1,\ell_2} \le C \log(n) + \log\left(Z_{p_*,q_*} \mathrm{e}^{\mathfrak{a}\,\mathrm{ad}(p_*) - \mathfrak{a}\,\mathrm{ad}(q_*)}\right) \tag{4.8}$$

and,

$$\begin{aligned} \log\left(Z_{p_*,q_*} \mathrm{e}^{\mathfrak{a}\,\mathrm{ad}(p_*) - \mathfrak{a}\,\mathrm{ad}(q_*)}\right) \le& \log\left(Z_{p,q} \mathrm{e}^{\mathfrak{a}\,\mathrm{ad}(p-q)}\right) \\ &- \log\left(Z_{p,p_*} \mathrm{e}^{\mathfrak{a}\,\mathrm{ad}(p-p_*)}\right) - \log\left(Z_{q_*,q} \mathrm{e}^{\mathfrak{a}\,\mathrm{ad}(q_*-q)}\right) \end{aligned} \tag{4.9}$$

since $Z_{p,p_*} Z_{p_*,q_*} Z_{q_*,q} \le Z_{p,q}$. Now, p_* and q_* depend only on the Brownian increments $\{B_i(s) - B_i(-i) : -i \le s \le 2n - i\}_i$. The Brownian increments appearing in Z_{p,p_*} can be written in terms of $\{B_i(-i) - B_i(s) : s \le -i\}_i$, which are independent of the increments that p_* depends on. Therefore, conditional on p_*, the distribution of Z_{p,p_*} is simply that of a point-to-point OY polymer. A similar statement holds for $Z_{q_*,q}$. The height difference between p and p_* is n. The antidiagonal displacement between the two points is $\mathcal{O}(n^{2/3})$. Therefore, by Lemma 2.5 and Corollary 3.5 we have,

$$\mathbb{P}\left[|\log Z_{p,p_*} - (\mu n + \mathfrak{a}\,\mathrm{ad}(p_* - p))| > s n^{1/3}\right] \le \mathrm{e}^{-c s^{3/2}} \tag{4.10}$$

for all $n^{2/3} \ge s \ge s_0$, some $s_0 > 0$, as well as a similar estimate for $\log Z_{q_*,q}$. Therefore, if $n^{2/3} \ge s \ge s_0 + 1$ and n is large enough we have for any $c_1 > 0$ that,

$$\begin{aligned} \mathbb{P}\left[\log \tilde{Z}_{\ell_1,\ell_2} > \mu n - c_1 w^2 + 10 s n^{1/3}\right] \le& 2\mathrm{e}^{-c s^{3/2}} \\ &+ \mathbb{P}\left[\log\left(Z_{p,q} \mathrm{e}^{\mathfrak{a}\,\mathrm{ad}(p-q)}\right) > 3\mu n - c_1 w^2 + s n^{1/3}\right] \end{aligned} \tag{4.11}$$

The height difference between p and q is $3n$, and the anti-diagonal difference satisfies $|\,\mathrm{ad}(p-q)| = |w| \le n + 2n^{2/3}$. We may therefore apply Corollary 3.5 to conclude,

$$\mathbb{P}\left[\log Z_{p,q} > f_{q-p} + s n^{1/3}\right] \le \mathrm{e}^{-c s^{3/2}} \tag{4.12}$$

for s sufficiently large. We have also by Lemma 2.5 that

$$f_{q-p} \le 3\mu n + \mathfrak{a}\,\mathrm{ad}(q-p) - c_2 w^2 \tag{4.13}$$

some $c_2 > 0$. Taking $c_1 < c_2/5$ yields the claim. □

We also desire a lower bound. This is an analog of [8, Lemma 4.4], and is proven using a similar method.

Proposition 4.2. Let ℓ_1 and ℓ_2 be as above. Let $\delta_1 > 0$ and assume $|w| \leq (1-\delta_1)n$. There is a $\delta_2 > 0$ and $c_1 > 0$ so that for all n large enough,

$$\mathbb{P}\left[\log \tilde{Z}_{\ell_1,\ell_2} < \mu n - c_1 n^{1/3}\right] \geq \delta_2. \tag{4.14}$$

We will first prove the following preliminary statement, where $l_1 = l_2 = \varepsilon$ is taken to be small.

Lemma 4.3. Let $\delta_1 > 0$ and assume $|w| \leq (1-\delta_1)n$ and $l_1 = l_2 = \varepsilon > 0$ is small. There is a $\delta_2 > 0$, $c_1 > 0$ and $\varepsilon_0 > 0$ so that for all $0 < \varepsilon < \varepsilon_0$ we have for $n \geq n_0(\varepsilon)$ that,

$$\mathbb{P}\left[\log \tilde{Z}_{\ell_1,\ell_2} < \mu n - c_1 n^{1/3}\right] \geq \delta_2. \tag{4.15}$$

Proof. Let $p = -(\varepsilon^{3/2}n, \varepsilon^{3/2}n)$ and let $q = (n-w, n+w) + (\varepsilon^{3/2}n, \varepsilon^{3/2}n)$. Let p_* and q_* be the points maximizing the summand in the definition of $\tilde{Z}_{\ell_1,\ell_2}$ so that,

$$\log \tilde{Z}_{\ell_1,\ell_2} \leq C\log(n) + \log \tilde{Z}_{p_*,q_*}. \tag{4.16}$$

Now, the height difference of p and q is at least n, and the anti-diagonal displacement is $|w| \leq (1-\delta_1)n$. The height difference of p and p_* is $\varepsilon^{3/2}n$ and the anti-diagonal displacement is at most $\varepsilon n^{2/3} \leq (1-\delta_1)\varepsilon^{3/2}n$ for all n large enough, depending on ε.

Similar to the proof of Proposition 4.1 we have,

$$\log \tilde{Z}_{p_*,q_*} \leq \log \tilde{Z}_{p,q} - \log \tilde{Z}_{p,p_*} - \log \tilde{Z}_{q_*,q}. \tag{4.17}$$

Now by Lemma 2.5 we have,

$$\left| f_{p_*-p} - \mu\varepsilon^{3/2}n - \mathfrak{a}\,\mathrm{ad}(p_*-p)\right| \leq C\varepsilon^{1/2}n^{1/3}, \tag{4.18}$$

for some $C > 0$ independent of $\varepsilon > 0$. By the independence of p_* from the Brownian motion terms defining $\tilde{Z}_{p,p_*}$ we then have that for any $\delta_3 > 0$, there is an $M = M(\delta_3)$ so that

$$\mathbb{P}\left[|\log \tilde{Z}_{p,p_*} - \mu\varepsilon^{3/2}n| > M\varepsilon^{1/2}n^{1/3}\right] < \delta_3 \tag{4.19}$$

for all $n \geq n_0 = n_0(\varepsilon)$. We obtain a similar estimate for $\log \tilde{Z}_{q_*,q}$. On the other hand, by Corollary 2.3 there is a $\delta_4 > 0$ and $c_1 > 0$ so that for all n large enough that

$$\mathbb{P}\left[\log Z_{p,q} < f_{q-p} - c_1 n^{1/3}\right] > \delta_4. \tag{4.20}$$

Here we use that the fact that $|w| \leq (1-\delta_1)n$ implies that the anti-diagonal displacement of p and q is at most $(1-\delta_1)n$ but the height difference is at least n. By Lemma 2.5 we have,

$$f_{q-p} \leq (1+2\varepsilon^{3/2})n\mu + \mathfrak{a}\,\mathrm{ad}(q-p) \tag{4.21}$$

and so

$$\mathbb{P}\left[\log \tilde{Z}_{p,q} < (1+2\varepsilon^{3/2})n\mu - c_1 n^{1/3}\right] > \delta_4 \tag{4.22}$$

Choose now $\delta_3 < \delta_4/3$, which fixes M. Then choose $\varepsilon > 0$ so that $M\varepsilon^{1/2} < c_1/10$. Then for all $n \geq n_0(\varepsilon)$ we have,

$$\mathbb{P}\left[\log \tilde{Z}_{p_*,q_*} < n\mu - \frac{c_1}{2}n^{1/3}\right] > \frac{\delta_4}{3}. \tag{4.23}$$

This yields the claim. □

Proof of Proposition 4.2. Choose $\varepsilon > 0$ corresponding to $\delta_1/10$ from Lemma 4.3. Let us break up the line segments ℓ_1, ℓ_2 each into a family of smaller line segments $\{\ell_{3,k}\}_k$ and $\{\ell_{4,k}\}_k$. We can demand that the length of each of these smaller line segments is less than $\varepsilon n^{2/3}$, so that each family is of order ε^{-1}. Then we have that,

$$\log \tilde{Z}_{\ell_1,\ell_2} \le C|\log \varepsilon| + \max_{j,k} \log \tilde{Z}_{\ell_{3,j},\ell_{4,k}}. \tag{4.24}$$

Note that the anti-diagonal displacement $w_{j,k}$ between the midpoints of $\ell_{3,j}$ and $\ell_{4,k}$ satisfies $|w_{j,k}| \le (1-\delta_1)n + n^{2/3} \le (1-\delta_1/2)n$ for n large enough. Therefore, by Lemma 4.3 and the FKG inequality Proposition B.1 we have,

$$\mathbb{P}\left[\max_{j,k} \tilde{Z}_{\ell_{3,j},\ell_{4,k}} < n\mu - \frac{c_1}{2} n^{1/3}\right] \ge \prod_{j,k} \mathbb{P}\left[\log \tilde{Z}_{\ell_{3,j},\ell_{4,k}} < n\mu - c_1 n^{1/3}\right] \ge \delta \tag{4.25}$$

for some $\delta > 0$ and all n large enough. The claim now follows. □

Finally, we require the following point-to-long line segment estimate.

Proposition 4.4. Let $L = \{(n,n) - (k,-k), |k| \le n\}$. There is an $s_0 \ge 0$ so that for all $n^{2/3} \ge s \ge s_0$ we have,

$$\mathbb{P}\left[\log \tilde{Z}_{(0,0),L} > \mu n + s n^{1/3}\right] \le \mathrm{e}^{-cs^{3/2}} \tag{4.26}$$

Proof. We break up L into order $n^{1/3}$ line segments ℓ_i

$$\ell_i := \{(n,n) - (w_i + k, -w_i - k) : |k| \le n^{2/3}\} \tag{4.27}$$

with $w_i := i2n^{2/3}$. Then,

$$\log \tilde{Z}_{(0,0),L} \le C \log(n) + \max_i \log \tilde{Z}_{(0,0),\ell_i}. \tag{4.28}$$

By Proposition 4.1 we have for some $c_1 > 0$ that for all $s \ge s_0$,

$$\mathbb{P}\left[\log \tilde{Z}_{(0,0),\ell_i} > \mu n + s n^{1/3} - c_1 i^2 n^{1/3}\right] \le \mathrm{e}^{-c(s^{3/2}+|i|^{3/2})}. \tag{4.29}$$

The claim follows from a union bound. □

5 Transverse estimates

The goal of the present section is to estimate the behavior of the partition function restricted to polymer paths that have a large transversal fluctuation. This takes place over the course of several steps. In Section 5.1 we establish an estimate for polymer paths that have a large transversal fluctuation at their midpoint. This is accomplished by decomposing the partition function over such paths into the product of a point-to-line and line-to-point polymer, where the line has a large anti-diagonal displacement. For the purposes of the subsequent section, however, it will be necessary to establish this midpoint estimate for line-to-line polymers as well.

In Section 5.2 we use a dyadic scheme similar to [6] to establish the same estimate as the midpoint case as when the polymer path has a large transversal fluctuation about any point. This is Theorem 5.7 below. We then obtain Corollary 5.8, an estimate for the quenched probability that a polymer path has large transversal fluctuation.

5.1 Decomposition

Recall the notion of a polymer path γ associated to the jump times as in Section 2.2. For any $b>0$ we will let $\mathcal{A}_b^{(n)}$ denote the set of polymer paths that pass to the left of the point $(n/2,n/2)-(bn^{2/3},-bn^{2/3})$. That is, $\gamma\in\mathcal{A}_b^{(n)}$ if and only if $\gamma_{n/2-bn^{2/3}}>n/2+bn^{2/3}$ or equivalently, $s_{n/2+bn^{2/3}}<n/2-bn^{2/3}$. Note that this makes sense only if $n/2-bn^{2/3}\in[s,t]$. If $n/2-bn^{2/3}\notin[s,t]$, then we say that $\gamma\in\mathcal{A}_b^{(n)}$ iff the polymer path we get by extending $\gamma:\mathbb{R}\to[m,n]$ by setting it constant on the two intervals $(-\infty,s]$ and $[t,\infty)$ satisfies $\gamma_{n/2-bn^{2/3}}>n/2+bn^{2/3}$.

Now for any $a>0$, consider the line segments,

$$\ell_1^{(a)}:=\{(-i,i):|i|\le an^{2/3}\},\qquad \ell_2^{(a)}:=\{(n,n)-(i,-i):|i|\le an^{2/3}\} \tag{5.1}$$

and define,

$$\begin{aligned}\tilde{Z}^{(c)}_{n,a,b}&:=\tilde{Z}_{\ell_1^{(a)},\ell_2^{(a)}}[\mathcal{A}^{(n)}_{b+a}]\\ &=\sum_{|i|\le an^{2/3},|j|\le an^{2/3}}\tilde{Z}_{(-i,i),(n,n)-(j,-j)}[\mathcal{A}^{(n)}_{b+a}].\end{aligned} \tag{5.2}$$

Here, we use the superscript ${}^{(c)}$ to mean “constrained,” as in the polymer path is constrained to lie in the set $\mathcal{A}^{(n)}_{b+a}$. We now derive a decomposition of $\tilde{Z}^{(c)}_{n,a,b}$ as a product of line-to-line polymers.

Proposition 5.1. We have, for any $n\ge1$ and $a,b>0$,

$$\begin{aligned}\tilde{Z}^{(c)}_{n,a,b}=&\sum_{|i|,|j|\le an^{2/3}}\ \sum_{k>n/2+(a+b)n^{2/3}}^{n+j\wedge i-1}\tilde{Z}_{(-i,i),(n-k,k)}\tilde{Z}_{(n-k,k),(n,n)-(j,-j)}\\ &+\sum_{|i|,|j|\le an^{2/3}}\ \sum_{k\ge n/2+(a+b)n^{2/3}}^{n+j\wedge(i-1)}\int_{n-(k+1)}^{n-k}\Big\{\Big(Z_{(-i,i),(s_k,k)}e^{\mathfrak{a}(i+n/2-k)}\Big)\\ &\qquad\times\Big(Z_{(s_k,k+1),(n,n)-(j,-j)}e^{-\mathfrak{a}(j+n/2-k)}\Big)\Big\}ds_k.\end{aligned} \tag{5.3}$$

Proof. Let $p=(-i,i)$ and $q=(n,n)-(j,-j)$ be points in $\ell_1^{(a)},\ell_2^{(a)}$, respectively. We then decompose,

$$\begin{aligned}&Z_{p,q}[\mathcal{A}^{(n)}_{b+a}]\\ =\ &Z_{p,q}[\mathcal{A}^{(n)}_{b+a},s_i>n-i]+Z_{(p,q)}[\mathcal{A}^{(n)}_{b+a},s_i<n-i]\\ =\ &Z_{p,q}[\mathcal{A}^{(n)}_{b+a},s_i>n-i]+Z_{(p,q)}[\mathcal{A}^{(n)}_{b+a},n-(i+1)<s_i<n-i]\\ &+Z_{p,q}[\mathcal{A}^{(n)}_{b+a},s_i<n-(i+1)]\\ =\ &Z_{p,q}[\mathcal{A}^{(n)}_{b+a},s_i>n-i]+Z_{(p,q)}[\mathcal{A}^{(n)}_{b+a},n-(i+1)<s_i<n-i]\\ &+Z_{p,q}[\mathcal{A}^{(n)}_{b+a},s_{i+1}>n-(i+1),s_i<n-(i+1)]+Z_{p,q}[\mathcal{A}^{(n)}_{b+a},s_{i+1}<n-(i+1)]\\ &\ \dots\\ =\ &\sum_{k=i}^{n+j\wedge(i-1)}Z_{p,q}[\mathcal{A}^{(n)}_{b+a},s_{k-1}<n-k,s_k>n-k]\\ &+\sum_{k=i}^{n+j\wedge i-1}Z_{p,q}[\mathcal{A}^{(n)}_{b+a},n-(k+1)<s_k<n-k].\end{aligned} \tag{5.4}$$

Let us pause to state the geometric interpretation of each of the terms appearing above. We are decomposing the partition function according to where the path crosses the line $\{(x,y) : x+y=n\}$. The constraint $\{s_{k-1} < n-k, s_k > n-k\}$ implies that the path passes through the point $(n-k,k)$. The constraint $\{n-(k+1) < s_k < n-k\}$ implies that the path crosses the line $\{(x,y) : x+y=n\}$ in the interval $\{(n-k,k)-(s,s) : s \in (0,1)\}$. That is, the path jumps from level k to $k+1$ in the time interval $(n-(k+1), n-k)$.

As stated above, $\gamma \in \mathcal{A}^{(n)}_{a+b}$ if and only if $s_{n/2+(a+b)n^{2/3}} < n/2-(a+b)n^{2/3}$. Therefore, $Z_{p,q}[\mathcal{A}^{(n)}_{b+a}, s_{k-1} < n-k, s_k > n-k] = 0$ if $k \le n/2+(a+b)n^{2/3}$ and

$$Z_{p,q}[\gamma \in \mathcal{A}^{(n)}_{b+a}, s_{k-1} < n-k, s_k > n-k] = Z_{p,(n-k,k)} Z_{(n-k,k),q} \tag{5.5}$$

otherwise.

Similarly, since $s_k > n-(k+1) \implies s_{k+1} > n-(k+1)$ we see that $Z_{p,q}[\gamma \in \mathcal{A}^{(n)}_{b+a}, n-(k+1) < s_k < n-k] = 0$ if $k+1 \le n/2+(a+b)n^{2/3}$ and otherwise,

$$Z_{p,q}[\mathcal{A}^{(n)}_{b+a}, n-(k+1) < s_k < n-k] = \int_{n-(k+1)}^{n-k} Z_{p,(s_k,k)} Z_{(s_k,k+1),q} \mathrm{d}s_k. \tag{5.6}$$

By summation we conclude the proof. □

We require the following lemma.

Lemma 5.2. There is a $c>0$ so that, for any $0<\varepsilon<1$, and $r>1$ we have, for any $s<t$ and $m \le n$,

$$\mathbb{P}\left[\sup_{0\le u\le\varepsilon} Z_{(s,m),(t+u,n)} > \mathrm{e}^r Z_{(s,m),(t+\varepsilon,n)}\right] \le c^{-1}\mathrm{e}^{-cr^2\varepsilon^{-1}}, \tag{5.7}$$

as well as

$$\mathbb{P}\left[\sup_{0\le u\le\varepsilon} Z_{(s-u,m),(t,n)} > \mathrm{e}^r Z_{(s-\varepsilon,m),(t,n)}\right] \le c^{-1}\mathrm{e}^{-cr^2\varepsilon^{-1}}. \tag{5.8}$$

Proof. We have, if $n>m$,

$$\begin{aligned}
Z_{(s,m),(t+u,n)} &= \int_s^{t+u} Z_{(s,m),(w,n-1)} \mathrm{e}^{B_n(t+u)-B_n(w)} \mathrm{d}w \\
&= \mathrm{e}^{B_n(t+u)-B_n(t+\varepsilon)} \int_s^{t+u} Z_{(s,m),(w,n-1)} \mathrm{e}^{B_n(t+\varepsilon)-B_n(w)} \mathrm{d}w \\
&\le \mathrm{e}^{B_n(t+u)-B_n(t+\varepsilon)} \int_s^{t+\varepsilon} Z_{(s,m),(w,n-1)} \mathrm{e}^{B_n(t+\varepsilon)-B_n(w)} \mathrm{d}w \\
&= \mathrm{e}^{B_n(t+u)-B_n(t+\varepsilon)} Z_{(s,m),(t+\varepsilon,n)}.
\end{aligned} \tag{5.9}$$

The above inequality is by definition an equality in the case $m=n$. The first estimate then follows since,

$$\sup_{0<u<\varepsilon} B_n(t+u) - B_n(t+\varepsilon) \sim \sqrt{\varepsilon}|Z|, \tag{5.10}$$

where Z is a standard normal. The second estimate follows similarly using instead the identity

$$Z_{(s-u,m),(t,n)} = \int_{s-u}^{t} \mathrm{e}^{B_m(w)-B_m(s-u)} Z_{(w,m+1),(t,n)} \mathrm{d}w. \tag{5.11}$$

□

Introduce now the line segment $\ell^{(m)}$ as,

$$\ell^{(m)} := \{(n/2,n/2)-(k,-k) : 2an \ge k \ge (b+a)n^{2/3}\}. \tag{5.12}$$

Recall also the definitions of $\ell_1^{(a)}$ and $\ell_2^{(a)}$ above.

Proposition 5.3. Assume $a \geq 1$ and $1 \leq b \leq 2n$. There are $C, c > 0$ so that the following holds. For any $r > 1$ we have,

$$\log \tilde{Z}^{(c)}_{n,a,b} \leq \log \tilde{Z}_{\ell_1^{(a)}, \ell^{(m)}} + \log \tilde{Z}_{\ell^{(m)}, \ell_2^{(a)}} + C + r \tag{5.13}$$

with probability at least $1 - C(an)^3 \mathrm{e}^{-cr^2}$.

Proof. We will use the identity (5.3) of Proposition 5.1. For the sum on the first line of (5.3) we have,

$$\begin{aligned} &\sum_{|i|,|j| \leq an^{2/3}} \sum_{k > n/2+(a+b)n^{2/3}}^{n+j \wedge i-1} \tilde{Z}_{(-i,i),(n-k,k)} \tilde{Z}_{(n-k,k),(n,n)-(j,-j)} \\ &\leq \sum_{|i|,|j| \leq an^{2/3}} \left(\sum_{k > n/2+(a+b)n^{2/3}}^{n+j \wedge i-1} \tilde{Z}_{(-i,i),(n-k,k)} \right) \left(\sum_{k > n/2+(a+b)n^{2/3}}^{n+j \wedge i-1} \tilde{Z}_{(n-k,k),(n,n)-(j,-j)} \right) \\ &\leq \tilde{Z}_{\ell_1^{(a)}, \ell^{(m)}} \tilde{Z}_{\ell^{(m)}, \ell_2^{(a)}}. \end{aligned} \tag{5.14}$$

We now turn to the summation on the second and third line of (5.3). By Lemma 5.2 with $\varepsilon = 1$ we have

$$\sup_{n-(k+1) \leq s_k \leq n-k} Z_{(-i,i),(s_k,k)} Z_{(s_k,k+1),(n,n)-(j,-j)} \leq \mathrm{e}^r Z_{(-i,i),(n-k,k)} Z_{(n-(k+1),k+1),(n,n)-(j,-j)} \tag{5.15}$$

with probability at least $1 - \mathrm{e}^{-cr^2}$. The rest of the estimate follows similarly to the first line of (5.3) and a union bound. □

Proposition 5.4. There is a $c_1 > 0$ so that the following holds. Assume $n^{C_0} \geq a \geq 1$ for some $C_0 > 0$. There is a $b_0 > 0$, depending only on C_0 so that for all $n \geq b \geq b_0$ we have

$$\mathbb{P}\left[\log \tilde{Z}_{\ell_1^{(a)}, \ell^{(m)}} > \mu n/2 - c_1 b^2 n^{1/3}\right] \leq \mathrm{e}^{-c_1 b^3} \tag{5.16}$$

and,

$$\mathbb{P}\left[\log \tilde{Z}_{\ell^{(m)}, \ell_2^{(a)}} > \mu n/2 - c_1 b^2 n^{1/3}\right] \leq \mathrm{e}^{-c_1 b^3} \tag{5.17}$$

Proof. We prove only the first estimate, the second being similar. Let us divide $\ell_1^{(a)}$ into order a line segments $\ell_{1,i}$ each of length order $n^{2/3}$ with the ith midpoint at the point $(-an^{2/3}, an^{2/3}) + (i-1/2)(n^{2/3}, -n^{2/3})$. Divide $\ell^{(m)}$ into at most order n^{C_0+1} line segments $\ell_{2,j}$ of length order $n^{2/3}$ with midpoints $(n/2, n/2) + (a+b)(-n^{2/3}, n^{2/3}) - (j-1/2)(n^{2/3}, n^{2/3})$. Then,

$$\log \tilde{Z}_{\ell_1^{(a)}, \ell^{(m)}} \leq C \log(n) + \max_{i,j} \log \tilde{Z}_{\ell_{1,i}, \ell_{2,i}}. \tag{5.18}$$

We will use a union bound to bound the max on the RHS. Now, $\tilde{Z}_{\ell_{1,i}, \ell_{2,j}}$ is a line-to-line partition function and the anti-diagonal displacement between the midpoints of $\ell_{1,i}$ and $\ell_{2,i}$ satifies $w_{ij} \geq cn^{2/3}(i+j+b)$. Therefore, if b is sufficiently large we see by Proposition 4.1 that (note that if $|w_{ij}| \geq 1.5\frac{n}{2}$ then $\tilde{Z}_{\ell_{1,i}, \ell_{2,j}} = 0$)

$$\mathbb{P}\left[\log \tilde{Z}_{\ell_{1,i}, \ell_{2,j}} > \mu n/2 - c_2 b^2 n^{1/3}\right] \leq \mathrm{e}^{-c_2(b^3+i^3+j^3)} \tag{5.19}$$

The claim now follows from a union bound. □

The following is the analog of [4, Proposition 6.1].

Proposition 5.5. There are $c_2 > 0$ and $b_0 > 0$ so that for all $b \geq b_0$ and $n^{100} \geq a \geq 1$ we have,

$$\mathbb{P}\left[\log \tilde{Z}^{(c)}_{n,a,b} > \mu n - c_2 b^2 n^{1/3}\right] \leq \mathrm{e}^{-c_2 b^3} \tag{5.20}$$

Proof. We may assume that $b \le 10n^{1/3}$ or else $\tilde{Z}^{(c)}_{n,a,b} = 0$. Let $c_1 > 0$ be the constant from Proposition 5.4. By Proposition 5.3 we have with probability at least $1 - Cn^{1000}e^{-cb^4n^{2/3}}$ that,

$$\log \tilde{Z}^{(c)}_{n,a,b} \le \log \tilde{Z}_{\ell_1^{(a)},\ell^{(m)}} + \log \tilde{Z}_{\ell^{(m)},\ell_2^{(a)}} + c_1 b^2 n^{1/3}. \tag{5.21}$$

The probability of the complementary event satisfies $n^{1000}e^{-cb^4n^{2/3}} \le e^{-b^3}$ for n sufficiently large and $b \ge 1$. The claim now follows from a direct application of Proposition 5.4. ☐

Define now $\hat{A}^{(n)}_{b+a}$ to be the polymer paths that pass to the right of the point $(n/2, n/2) + ((a+b)n^{2/3}, -(a+b)n^{2/3})$. A similar proof to that given above establishes the following.

Proposition 5.6. There are $c_2 > 0$ and $b_0 > 0$ so that for all $b \ge b_0$ and $n^{100} \ge a \ge 1$ we have,

$$\mathbb{P}\left[\log \tilde{Z}_{\ell_1^{(a)},\ell_2^{(a)}}[\hat{A}^{(n)}_{b+a}] > \mu n - c_2 b^2 n^{1/3}\right] \le e^{-c_2 b^3} \tag{5.22}$$

5.2 Full estimate

For any polymer path γ we let $\mathrm{TF}(\gamma)$ be its transversal fluctuation, that is, the maximal distance of the path from the diagonal. The proof of the following result follows the proof of Theorem 11.1 of [6].

Theorem 5.7. There is a $c_1 > 0$ and $b_0, n_0 > 0$ so that for all $n^{1/3} \ge b \ge b_0$ and all $n \ge n_0$ we have

$$\mathbb{P}\left[\log Z_{(0,0),(n,n)}[\mathrm{TF}(\gamma) > bn^{2/3}] > \mu n - c_1 b^2 n^{1/3}\right] < e^{-c_1 b^3}. \tag{5.23}$$

Proof. Define the dyadic points,

$$S_j := \{k 2^{-j} n : 0 \le k \le 2^j\}. \tag{5.24}$$

Choose j_0 so that $2^{-j_0} n \in (0.5, 1] \times \frac{b}{10} n^{2/3}$. Define,

$$b_j := \frac{b}{M} \prod_{i=1}^{j-1} (1 + 2^{-i/3}), \qquad M = 2 \cdot \prod_{i=1}^{\infty} (1 + 2^{-i/3}). \tag{5.25}$$

Let T_j be the set of polymer paths that intersect all of the $2^j + 1$ line segments,

$$\ell_{k,j} := \{(k2^{-j}n, k2^{-j}n) + (-x, x) : |x| \le b_j n^{2/3}\}, \tag{5.26}$$

for $k = 0, 1, \dots 2^j$. A straightforward argument using that the paths are up-right shows that

$$T_{j_0} \subseteq \{\gamma : \mathrm{TF}(\gamma) \le bn^{2/3}\} \tag{5.27}$$

and so,

$$\{\gamma : \mathrm{TF}(\gamma) > bn^{2/3}\} \subseteq T^c_{j_0} = \bigcup_{j=1}^{j_0} T^c_j \cap T_{j-1} \tag{5.28}$$

where T_0 is by definition the set of all polymer paths. Therefore,

$$\log Z_{(0,0),(n,n)}[\mathrm{TF}(\gamma) > bn^{2/3}] \le C \log(n) + \max_{j \le j_0} \log Z_{(0,0),(n,n)}[T^c_j \cap T_{j-1}]. \tag{5.29}$$

We will use a union bound to estimate the max on the RHS. First, Propositions 5.5 and 5.6 immediately imply that

$$\mathbb{P}\left[\log Z_{(0,0),(n,n)}[T^c_1] > \mu n - c_1 b^2 n^{1/3}\right] \le e^{-c_1 b^3} \tag{5.30}$$

for some $c_1 > 0$. For $1 \le k \le 2^{j-1}$ we let $T^{(\pm)}_{jk}$ be the polymer paths that intersect the line segments $\ell_{k-1,j-1}$ and $\ell_{k,j-1}$ and pass either above or below $\ell_{2k-1,j}$ for $\pm = +$ and $\pm = -$, respectively. We have,

$$T_j^c \cap T_{j-1} \subseteq \bigcup_{k=1,\sigma\in\{+,-\}}^{2^j-1} T^{(\sigma)}_{jk}. \tag{5.31}$$

Since $2^{j_0} \le n$ we have

$$\log Z_{(0,0),(n,n)}[T_j^c \cap T_{j-1}] \le C\log(n) + \max_{k,\sigma\in\{+,-\}} \log Z_{(0,0),(n,n)}[T^{(\sigma)}_{jk}]. \tag{5.32}$$

We focus now on estimate the probability that $\log Z_{(0,0),(n,n)}[T^{(+)}_{jk}]$ is large. The argument for $\log Z_{(0,0),(n,n)}[T^{(-)}_{jk}]$ is similar and omitted.

We will decompose $\log Z_{(0,0),(n,n)}[T^{(+)}_{jk}]$ as the product of three partition functions: (i) a point-to-interval partition function; (ii) an interval-to-interval partition function of paths constrained to have large midpoint transversal fluctuations; (iii) an interval-to-point partition function. The key point is to estimate the second partition function using Proposition 5.5.

Set now $z_1 = (k-1)2^{-j+1}n$, $z_2 = k2^{-j+1}n$ and $z_0 = (2k-1)2^{-j}n$. These are the coordinates of the midpoints of the lines $\ell_{k-1,j-1}, \ell_{k,j-1}$ and $\ell_{2k-1,j}$, respectively.

We require some notation to furnish our decompositions. Let μ_i be the measure on $\mathbb{R}^2$ that is a sum of the δ functions at the points $\{(z_i, z_i) + (-m, m) : m \in \mathbb{Z}, |m| \le b_{j-1}n^{2/3}\}$ for $i = 1, 2$. Let ν_i be the measure on $\mathbb{R}^2$ that is a sum of $1d$ Lebesgue measures on the horizontal intervals $\{\{(x_i - m, x_i + m) - (s, 0) : 0 < s < 1\} : m \in \mathbb{Z}, -b_{j-1}n^{2/3} \le m < b_{j-1}n^{2/3}\}$ for $i = 1, 2$. Then, by a similar argument to Proposition 5.1 we have (see Section C.1 for a complete proof)

$$\begin{aligned}
&Z_{(0,0),(n,n)}[T^{(+)}_{jk}] = \int\int \tilde{Z}_{(0,0),p}\tilde{Z}_{p,q}[\mathcal{A}]\tilde{Z}_{q,(n,n)}\mathrm{d}\mu_1(p)\mathrm{d}\mu_1(q)\\
&+ \int\int \Big\{ \left(Z_{(0,0),(x_1,y_1)}\mathrm{e}^{-\mathfrak{a}\,\mathrm{ad}((x_1,y_1))}\right)\left(Z_{(x_1,y_1+1),(x_2,y_2)}[\mathcal{A}]\mathrm{e}^{\mathfrak{a}\,\mathrm{ad}((x_1-x_2,y_1-y_2))}\right)\\
&\quad\times\left(Z_{(x_2,y_2+1),(n,n)}\mathrm{e}^{\mathfrak{a}\,\mathrm{ad}((x_2,y_2))}\right)\Big\}\mathrm{d}\nu_1(x_1,y_1)\mathrm{d}\nu_2(x_2,y_2)\\
&+ \int\int \tilde{Z}_{(0,0),p}\left(Z_{p,(x,y)}[\mathcal{A}]\mathrm{e}^{\mathfrak{a}\,\mathrm{ad}(p-(x,y))}\right)\left(Z_{(x,y+1),(n,n)}\mathrm{e}^{\mathfrak{a}\,\mathrm{ad}((x,y))}\right)\mathrm{d}\mu_1(p)\mathrm{d}\nu_2(x,y)\\
&+ \int\int \left(Z_{(0,0),(x,y)}\mathrm{e}^{-\mathfrak{a}\,\mathrm{ad}((x,y))}\right)\left(Z_{(x,y+1),p}[\mathcal{A}]\mathrm{e}^{\mathfrak{a}\,\mathrm{ad}((x,y)-p)}\right)\tilde{Z}_{p,(n,n)}\mathrm{d}\nu_1(x,y)\mathrm{d}\mu_2(p),
\end{aligned} \tag{5.33}$$

where $\mathcal{A}$ is the set of polymer paths passing to the left of the point $(z_0 - b_j, z_0 + b_j)$. Let now ℓ_1 and ℓ_2 be the line segments,

$$\ell_i = \{(z_i, z_i) + (-m, m) : |m| \le b_{j-1}n^{2/3}\}, \qquad i = 1, 2. \tag{5.34}$$

Then by a similar argument to the proof of Proposition 5.3 (the details of which appear in Appendix C.4) we have that for any $\delta > 0$

$$\log Z_{(0,0),(n,n)}[T^{(+)}_{jk}] \le \delta b^2 n^{1/3} + C + \log \tilde{Z}_{(0,0),\ell_1} + \log \tilde{Z}_{\ell_1,\ell_2}[\gamma\in\mathcal{A}] + \log \tilde{Z}_{\ell_2,(n,n)}, \tag{5.35}$$

with probability at least $1 - Cn\mathrm{e}^{-c\delta^2 b^4 n^{2/3}}$. Now, set $r := z_2 - z_1 = 2^{-j+1}n \ge cn^{2/3}$. We see that,

$$\tilde{Z}_{\ell_1,\ell_2}[\gamma\in\mathcal{A}] \overset{d}{=} Z^{(c)}_{r,\tilde{a},\tilde{b}} \tag{5.36}$$

where $\tilde{a} = b_{j-1}n^{2/3}r^{-2/3} = 2^{2(j-1)/3}b_{j-1} \asymp 2^{2j/3}b$ and $\tilde{b} = (b_j - b_{j-1})n^{2/3}r^{-2/3} \asymp b2^{j/3}$. Then, by Proposition 5.5 we have (note that since $r \ge cn^{2/3}$ we have that $\tilde{a} \le r$ for sufficiently large n), as long as b is sufficiently large,

$$\mathbb{P}\left[\log\left(\tilde{Z}_{\ell_1,\ell_2}[\gamma \in \mathcal{A}]\right) > \mu r - c_2\tilde{b}^2 r^{1/3}\right] \le \mathrm{e}^{-c_2\tilde{b}^3} \le \mathrm{e}^{-cb^3 2^j} \tag{5.37}$$

and $c_2\tilde{b}^2 r^{1/3} \ge c_3 b^2 2^{j/3} n^{1/3}$ for some $c_3 > 0$. We take the δ introduced above in (5.35) to satisfy $\delta < c_3/10$. Since $cr \le z_1 \le n$ we have

$$\mathbb{P}\left[\log \tilde{Z}_{(0,0),\ell_1} > z_1\mu + \frac{c_3}{10}b^2 2^{j/3} n^{1/3}\right] \le \mathbb{P}\left[\log \tilde{Z}_{(0,0),\ell_1} > z_1\mu + \frac{c_3}{10}b^2 2^{j/3}(z_1)^{1/3}\right] \le \mathrm{e}^{-cb^3 2^j} \tag{5.38}$$

where we applied Proposition 4.4 in the second inequality (note that $b^2 2^{j/3} \le C(b/n^{1/3})(z_1)^{2/3}$ since $z_1 \ge r = 2^{-j+1}n$ and $2^{j_0} \le Cn^{1/3}/b$, and so the inequality is applicable). A similar estimate holds for $\log \tilde{Z}_{\ell_2,(n,n)}$. Therefore, we conclude that

$$\mathbb{P}\left[\log Z_{(0,0),(n,n)}[T^{(+)}_{jk}] > \mu n - \frac{c_3}{2}b^2 2^{j/3} n^{1/3}\right] \le Cn\mathrm{e}^{-cb^4 n^{2/3}} + C\mathrm{e}^{-cb^3 2^j} \le C\mathrm{e}^{-cb^3 2^j} \tag{5.39}$$

where we used $2^j \le n^{1/3}$ in the second inequality to simplify the estimate. Therefore, for sufficiently large n it holds for all j that,

$$\mathbb{P}\left[\log Z_{(0,0),(n,n)}[T^c_j \cap T_{j-1}] > \mu n - \frac{c_3}{3}b^2 n^{1/3}\right] \le C2^j\mathrm{e}^{-cb^3 2^j} \le C2^{-j}\mathrm{e}^{-cb^3}. \tag{5.40}$$

The claim now follows. □

We recall now the definition of the Gibbs measure in (2.13).

Corollary 5.8. There are $c, C > 0$ so that for all b sufficiently large,

$$\mathbb{P}\left[Q_{(0,0),(n,n)}[\mathrm{TF}(\gamma) > bn^{2/3}] > \mathrm{e}^{-cb^2 n^{1/3}}\right] \le C\mathrm{e}^{-cb^3} \tag{5.41}$$

and consequently,

$$\mathbb{E}\left[Q_{(0,0),(n,n)}[\mathrm{TF}(\gamma) > bn^{2/3}]\right] \le C\mathrm{e}^{-cb^3}. \tag{5.42}$$

Remark. In fact, the two estimates of Corollary 5.8 hold with $Q_{(0,0),(n,n)}$ replaced by $Q_{(0,0),(t,n)}$ uniformly in t satisfying $\delta n \le t \le \delta^{-1}n$ for any fixed $\delta > 0$. In this case, $\mathrm{TF}(\gamma)$ is the maximal distance of the polymer path from the line from $(0,0)$ to (t,n). The simplest way to deduce this more general estimate is to simply rescale the time variable using Brownian rescaling, and applying Corollary 5.8. The rescaled Gibbs measure will involve a family of Brownian motions whose diffusivity instead depends on the ratio t/n, but it is easy to see that the above proof applies to this model, at the cost of adjusting the constants. One could also easily modify the above proof to directly deal with the case of general (t,n), and this would only be somewhat more involved notationally. □

Proof. With probability at least $1 - C\mathrm{e}^{-cb^3}$ we have by Theorem 5.7,

$$Z_{(0,0),(n,n)}[\mathrm{TF}(\gamma) > bn^{2/3}] \le \mathrm{e}^{\mu n - c_1 b^2 n^{1/3}} \tag{5.43}$$

and by Corollary 3.5,

$$Z_{(0,0),(n,n)} \ge \mathrm{e}^{\mu n - \frac{c_1}{2}b^2 n^{1/3}}. \tag{5.44}$$

The claim follows since

$$Q_{(0,0),(n,n)}[\mathrm{TF}(\gamma) > bn^{2/3}] = Z_{(0,0),(n,n)}[\mathrm{TF}(\gamma) > bn^{2/3}](Z_{(0,0),(n,n)})^{-1}. \tag{5.45}$$

□

6 Lower bound for left tail

In this section we will prove a lower bound for the left tail of the OY polymer. Let $k \in \mathbb{Z}$ be sufficiently large. For $1 \le i, j \le k$ we let $v_{ij} \in \mathbb{Z}^2$ be,

$$v_{ij} := j\left(\frac{n}{k}, \frac{n}{k}\right) + \left(\left(i - \frac{k}{2}\right)\left(\frac{n}{k}\right)^{2/3}, -\left(i - \frac{k}{2}\right)\left(\frac{n}{k}\right)^{2/3}\right), \tag{6.1}$$

and let $I_{ij} \subseteq \mathbb{Z}^2$ be the line segment with endpoints $v_{i,j}$ and $v_{i+1,j}$ (i.e., I_{ij} is of the form (4.3)). Let $L_j = \bigcup_i I_{ij}$. Let $\mathcal{A}$ denote the set of polymer paths that intersect the convex hull of every L_j. If a path is not in $\mathcal{A}$, then its transversal fluctuation is at least $ck^{1/3}n^{2/3}$. Therefore by Theorem 5.7 we have that

$$\mathbb{P}\left[\log Z_{(0,0),(n,n)}[\mathcal{A}^c] \le \mu n - c_1 k^{2/3} n^{1/3}\right] \ge \frac{1}{2} \tag{6.2}$$

for some $c_1 > 0$ and all k sufficiently large. We now prove the following.

Proposition 6.1. For any $c_2 > 0$ there is a $c_3 > 0$ so that if $k \le c_3 n/(\log(n))^3$ then,

$$\begin{aligned} &\mathbb{P}\left[\log Z_{(0,0),(n,n)} \le \mu n - c_2 k^{2/3} n^{1/3}\right] \\ \ge\ &\mathbb{P}\left[\log Z_{(0,0),(n,n)}[\mathcal{A}^c] \le \mu n - 2c_2 k^{2/3} n^{1/3}\right] \times \prod_{j=1}^{k} \mathbb{P}\left[\log Z^{(j)} \le \mu(n/k) - 2c_2 k^{-1/3} n^{1/3}\right] \\ &- C\mathrm{e}^{-cn^5} \end{aligned} \tag{6.3}$$

Proof. We have $Z_{(0,0),(n,n)} = Z_{(0,0),(n,n)}[\mathcal{A}^c] + Z_{(0,0),(n,n)}[\mathcal{A}]$. We break up the partition function $Z_{(0,0),(n,n)}[\mathcal{A}]$ into a product of partition functions of L_j to L_{j+1} polymers. In order to do so we introduce the following measures. We let $\mu_{j,0}$ be the measure on $\mathbb{R}^2$ that is a sum of delta functions on the points of L_j. We let $\mu_{j,1}$ be the measure on $\mathbb{R}^2$ that is a sum of $1d$ Lebesgue measures[2] on the horizontal intervals of the form $\{p - (s,0) : 0 < s < 1\}_{p \in L_j}$, except for the interval corresponding to the top-left most point of L_j. Then, via similar calculations to Proposition 5.1 (see Appendix C.2 for a proof) we have,

$$\begin{aligned} &Z_{(0,0),(n,n)}[\mathcal{A}] \\ = &\sum_{\sigma \in \{0,1\}^{k-1}} \int \Bigg\{ Z_{(0,0),(x_1,y_1)} \mathrm{e}^{-\mathfrak{a}\,\mathrm{ad}((x_1,y_1))} \left(\prod_{j=1}^{k-2} Z_{(x_j, y_j+\sigma_j),(x_{j+1},y_{j+1})} \mathrm{e}^{\mathfrak{a}\,\mathrm{ad}((x_j - x_{j+1}),(y_j - y_{j+1}))} \right) \\ &\times Z_{(x_{k-1}, y_{k-1}+\sigma_{k-1}),(n,n)} \mathrm{e}^{\mathfrak{a}\,\mathrm{ad}((x_{k-1},y_{k-1}))} \Bigg\} \prod_{j=1}^{k-1} \mathrm{d}\mu_{j,\sigma_j}(x_j, y_j). \end{aligned} \tag{6.4}$$

We have the estimate,

$$\mathbb{P}\left[\sup_{i,j \in \mathbb{Z}, |i|,|j| \le n^{100}} \sup_{|u| \le n^{-9}} |B_i(jn^{-10}) - B_i(jn^{-10} + u)| > 1\right] \le C\mathrm{e}^{-cn^5} \tag{6.5}$$

for some $C, c > 0$. On the complement of the event on the LHS of (6.5), by the proof of Lemma 5.2 we see that (see Appendix C.3 for details),

$$\int_u^{u+1} Z_{(s,m),(w,p)} Z_{(w,p+1),(t,q)} \mathrm{d}w \le 10 \frac{1}{n^{10}} \sum_{j=1}^{n^{10}} Z_{(s,m),(u+jn^{-10},p)} Z_{(u+(j-1)n^{-10},p+1),(t,q)}. \tag{6.6}$$

[2]By a 1 d Lebesgue measure on a 1 d interval in $\mathbb{R}^2$ we simply mean the product of 1 d Lebesgue measure on an interval with a 1 d Dirac δ function.

Let now $\mu_{j,2}$ be the measure that is n^{-10} times the sum of delta functions located at the points $\{p-(mn^{-10},0) : p \in L_j, 0 \le m \le n^{10}\}$, except when p is the top left point of L_j. That is, $\mu_{j,2}$ is simply a discretization of $\mu_{j,1}$ to a fine mesh. Using (6.6) whenever there appears $d\mu_{j,1}$ in (6.4), we have

$$\begin{aligned} &Z_{(0,0),(n,n)}[\mathcal{A}] \\ \le &\sum_{\sigma\in\{0,1\}^{k-1}} \int \Big\{ Z_{(0,0),(x_1,y_1)} \mathrm{e}^{-\mathfrak{a}\,\mathrm{ad}((x_1,y_1))} \left(\prod_{j=1}^{k-2} Z_{(x_j,y_j+\sigma_j),(x_{j+1},y_{j+1})} \mathrm{e}^{\mathfrak{a}\,\mathrm{ad}((x_j-x_{j+1}),(y_j-y_{j+1}))} \right) \\ &\times Z_{(x_{k-1},y_{k-1}+\sigma_{k-1}),(n,n)} \mathrm{e}^{\mathfrak{a}\,\mathrm{ad}((x_{k-1},y_{k-1}))} \Big\} \left(\prod_{j=1}^{k-1} \mathrm{d}\mu_{j,2\sigma_j}(x_j,y_j) \right) \times C^k. \end{aligned} \tag{6.7}$$

That is, up to an overall factor of $\mathcal{O}(C^k)$ we can replace the appearance of $\mathrm{d}\mu_{j,1}$ by $\mathrm{d}\mu_{j,2}$. Then, using the fact that for nonnegative f,g we have (this is proven in Appendix C.6)

$$\int f(x)g(x)\mathrm{d}\mu_{j,2}(x) \le n^{10} \left(\int f(x)\mathrm{d}\mu_{j,2}(x) \right) \left(\int g(x)\mathrm{d}\mu_{j,2}(x) \right), \tag{6.8}$$

we find that on the event of (6.5) that

$$Z_{(0,0),(n,n)}[\mathcal{A}] \le (Cn)^{Ck} \prod_{j=1}^{k} Z^{(j)} \tag{6.9}$$

where,

$$Z^{(j)} := \sum_{\sigma\in\{0,1\}^2} \int \tilde{Z}_{(x_1,y_1+\sigma_1),(x_2,y_2)} \mathrm{d}\mu_{j-1,2\sigma_1}(x_1,y_1)\mathrm{d}\mu_{j,2\sigma_2}(x_2,y_2). \tag{6.10}$$

We conclude using the FKG inequality, Proposition B.1. □

We now turn to the proof of the following.

Proposition 6.2. There is a $c_1 > 0$ so that,

$$\mathbb{P}\left[\log Z^{(j)} \le \mu(n/k) - c_1 k^{-1/3} n^{1/3}\right] \ge \mathrm{e}^{-Ck} \tag{6.11}$$

for all k and n large enough, satisfying $k \le c_1 n/(\log(n))^3$.

Proof. Let $r = n/k$. Recall that I_{ij} is length $r^{2/3}$ and for each j there are k such intervals. We have,

$$\log Z^{(j)} \le C\log(k) + \max_{i_1,i_2} \log \hat{Z}_{I_{i_1,j-1},I_{i_2,j}}, \tag{6.12}$$

where $\hat{Z}_{I_{i_1,j-1},I_{i_2,j}}$ is the restriction of $Z^{(j)}$ to points lying near $I_{i_1,j-1}$ and $I_{i_2,j}$; that is, it involves the measures $\mu_{j-1,0}$ and $\mu_{j,0}$ restricted to the points in $I_{i_1,j-1}, I_{i_2,j}$ and the discretized intervals of the measures $\mu_{j-1,2}$ and $\mu_{j,2}$ whose right endpoint lies in $I_{i_1,j-1}$ and $I_{i_2,j}$ (except again, for the intervals whose right endpoint is the top left point of $I_{i_1,j-1}$ or $I_{i_2,j}$). Note that $\hat{Z}_{I_{i_1,j-1},I_{i_2,j}}$ is almost a line-to-line polymer, as in the definition (4.4), except that we have some extra discretized horizontal segments coming from the $\mathrm{d}\mu_{j,2}$. In a moment we will replace these discretized polymers by bonafide line-to-line polymers.

From (6.12) and the FKG inequality, Proposition B.1, we see that for any $c_2 > 0$ there is a $c_3 > 0$ so that

$$\mathbb{P}\left[\log Z^{(j)} \le \mu(n/k) - c_2 k^{-1/3} n^{1/3}\right] \ge \prod_{i_1=1}^{k} \prod_{i_2=1}^{k} \mathbb{P}\left[\log \hat{Z}_{I_{i_1,j-1},I_{i_2,j}} \le \mu(n/k) - 2c_2 r^{1/3}\right] \tag{6.13}$$

if $k \le c_3 n/(\log(n))^3$. We now wish to replace the discretized intervals by simple line-to-line polymers.

From the proof of Lemma 5.2 (i.e., the estimate (5.9) and the analog for times at the lower left endpoint) we have that for any $A \ge 1$ that,

$$\sup_{\{x_1 \in [a_1, a_1+1], x_2 \in [a_2-1, a_2]\}} \tilde{Z}_{(x_1,y_1),(x_2,y_2)} \le \mathrm{e}^A \tilde{Z}_{(a_1,y_1),(a_2,y_2)} \tag{6.14}$$

with probability at least $1 - \mathrm{e}^{-cA^2}$.

Therefore,

$$\log \hat{Z}_{I_{i_1,j-1}, I_{i_2,j}} \le C + A + \log \tilde{Z}_{I_{i_1,j-1}, I_{i_2,j}} \tag{6.15}$$

with probability at least $1 - Cr^2 \mathrm{e}^{-cA^2}$, the RHS defined as in (4.4). For $|i_1 - i_2| \le \frac{1}{2} r^{1/3}$ we have from Proposition 4.2 that,

$$\mathbb{P}\left[\log \tilde{Z}_{I_{i_1,j-1}, I_{i_2,j}} < \mu r - c_5 r^{1/3}\right] > \delta_2 \tag{6.16}$$

for some $\delta_2 > 0$. Taking $A = r^{1/10}$ so that $Cr^2 \mathrm{e}^{-cA^2} \le \frac{\delta_2}{2}$ for r sufficiently large, we see from (6.15) and (6.16) that for all r large,

$$\mathbb{P}\left[\log \hat{Z}_{I_{i_1,j-1}, I_{i_2,j}} < \mu r - \frac{c_5}{2} r^{1/3}\right] > \frac{\delta_2}{2}, \tag{6.17}$$

for $|i_1 - i_2| \le \frac{1}{2} r^{1/3}$. On the other hand, for $|i_1 - i_2| \ge M$, some $M > 0$ we see from Proposition 4.1 that

$$\mathbb{P}\left[\log \tilde{Z}_{I_{i_1,j-1}, I_{i_2,j}} < \mu r - c_6 (i_1 - i_2)^2 r^{1/3}\right] \ge 1 - \mathrm{e}^{-c_6 (i_1 - i_2)^3}. \tag{6.18}$$

Taking $A = (i_1 - i_2)^2 r^{1/10}$ we see from (6.15) and (6.18) that for r sufficiently large and after possibly increasing M that

$$\mathbb{P}\left[\log \hat{Z}_{I_{i_1,j-1}, I_{i_2,j}} < \mu r - \frac{c_6}{2} (i_1 - i_2)^2 r^{1/3}\right] \ge 1 - C\mathrm{e}^{-c(i_1 - i_2)^2}. \tag{6.19}$$

Therefore, we have for some $c_7 > 0$ and for any $M_1 \ge M$ that

$$\begin{aligned} &\prod_{i_1=1}^{k} \prod_{i_2=1}^{k} \mathbb{P}\left[\log \hat{Z}_{I_{i_1,j-1}, I_{i_2,j}} \le \mu n - c_7 r^{1/3}\right] \\ &\ge \left(\prod_{|i_1 - i_2| \le M_1} \frac{\delta_2}{2}\right) \left(\prod_{|i_1 - i_2| > M_1} (1 - C\mathrm{e}^{-c|i_1 - i_2|^2})\right) \ge c\left(\frac{\delta}{2}\right)^{kM_1} \end{aligned} \tag{6.20}$$

after setting M_1 possibly larger. This completes the proof. □

Theorem 6.3. There is a $c_1 > 0$ so that for any $1 \le \theta \le c_3 n^{2/3} (\log(n))^{-2}$, we have

$$\mathbb{P}\left[\log Z_{(0,0),(n,n)} \le \mu n - \theta n^{1/3}\right] \ge C\mathrm{e}^{-c_1 \theta^3}. \tag{6.21}$$

Proof. From Propositions 6.1, 6.2 and (6.2) we see that for $k \le c_2 n / \log(n)^3$ for some $c_2, c_1 > 0$ we have,

$$\mathbb{P}\left[\log Z_{(0,0),(n,n)} \le \mu n - c_1 k^{2/3} n^{1/3}\right] \ge c\mathrm{e}^{-Ck^2} - C\mathrm{e}^{cn^5} \ge c\mathrm{e}^{-Ck^2} \tag{6.22}$$

where we used that $k \le n$ in the second inequality. It remains to choose $k = C\theta^{3/2}$ for large $C > 0$. □

We remark that the source of the logarithmic loss in the estimate of Proposition 6.1 results from the n^{10} in (6.8). In order to avoid the loss when k is of order n, one would need to instead discretize to a mesh of around constant order. However, the probability in (6.5) needs to be less than e^{-cn^2} so this would require taking the 1 in the probability on the LHS of (6.5) larger. This would then introduce a larger error in (6.6) and so it seems a different approach would be needed. It may be possible to discretize later in the argument, after applying the FKG inequality by instead estimating the continuous integrals over x_j on the RHS of (6.4) by the maximum value they can take in the intervals. This would give a product form upper bound similar to (6.9) (with the definition of $Z^{(j)}$ modified to contain some maxima) to which the FKG inequality could be applied.

However, an additional source of the logarithmic loss in our lower bound for the left tail comes in the proof of Proposition 6.2, specifically from (6.12). It seems avoiding this source requires further investigation.

7 Constrained partition functions

For any $\ell > 0$ we define

$$Z^{\ell}_{(s,m),(t,n)} := Z_{(s,m),(t,n)}[\mathrm{TF}(\gamma) \leq \ell n^{2/3}]. \tag{7.1}$$

The constrained partition functions will be a useful tool in proving our upper bounds on the left tail in the next section. The goal of the section is to prove the following. It is similar to [4, Proposition 3.7].

Proposition 7.1. Let $\ell > 0$. Assume $\delta n \leq t \leq \delta^{-1} n$. There is a $C > 0, c > 0$, depending on ℓ and δ so that,

$$\mathbb{P}\left[\log Z^{\ell}_{(0,0),(t,n)} \leq f_{t,n} - u n^{1/3}\right] \leq C\mathrm{e}^{-cu} \tag{7.2}$$

for all $u \geq 1$ and $n \geq C$.

Proof. We break the proof into two different cases, depending on whether $u \geq n^{2/3}$ or not. First, let us assume that $u \leq C_0 n^{2/3}$. Set $J = u^{1/2}$. One can check the general inequality

$$Z^{\ell}_{(s,m),(v,p)} Z^{\ell}_{(v,p),(t,n)} \leq Z^{\ell}_{(s,m),(t,n)}. \tag{7.3}$$

Therefore,

$$\mathbb{P}\left[\log Z^{\ell}_{(0,0),(t,n)} \leq f_{t,n} - u n^{1/3}\right] \leq J\mathbb{P}\left[\log Z^{\ell}_{(0,0),(t',n')} < f_{t',n'} - J^{-2/3} u (n')^{1/3}\right] \tag{7.4}$$

where we set $n' = n/J$ and $t' = t/J$. We write now,

$$\log Z^{\ell}_{(0,0),(t',n')} = \log Z_{(0,0),(t',n')} + \log Q_{(0,0),(t',n')}[\mathrm{TF}(\gamma) \leq \ell J^{2/3} (n')^{2/3}]. \tag{7.5}$$

We have by Corollary 5.8 (and the remark immediately following it) that for some $C, c > 0$, (note that $u \leq C_0 n^{2/3}$ implies $J^{2/3} \leq C_1 (n')^{1/3}$, some $C_1 > 0$)

$$\begin{aligned} &\mathbb{P}\left[Q_{(0,0),(t',n')}[\mathrm{TF}(\gamma) > \ell J^{2/3}(n')^{2/3}] \leq C\mathrm{e}^{-cu}\right] \\ \geq &\mathbb{P}\left[Q_{(0,0),(t',n')}[\mathrm{TF}(\gamma) > \ell J^{2/3}(n')^{2/3}] \leq C'\mathrm{e}^{-c' J^{4/3}(n')^{1/3}}\right] \geq 1 - C''\mathrm{e}^{-c'' u}. \end{aligned} \tag{7.6}$$

In the first inequality we used that $J^{4/3}(n')^{1/3} \geq c_1 u$ for some $c_1 > 0$ due to our assumption that $u \leq C_0 n^{2/3}$. Note that we do not track the dependence on ℓ, absorbing it into the constants. Since,

$$Q_{(0,0),(t',n')}[\mathrm{TF}(\gamma) < \ell J^{2/3}(n')^{2/3}] = 1 - Q_{(0,0),(t',n')}[\mathrm{TF}(\gamma) \geq \ell J^{2/3}(n')^{2/3}] \tag{7.7}$$

we then find that,

$$\mathbb{P}\left[\log Q_{(0,0),(t',n')}[\mathrm{TF}(\gamma) < \ell J^{2/3}(n')^{2/3}] > -C\mathrm{e}^{-cu}\right] \geq 1 - C\mathrm{e}^{-cu}, \tag{7.8}$$

as long as $u \geq C_2$ for some $C_2 > 0$ (the case $1 \leq u \leq C_2$ is handled by adjusting the constants in the claimed estimate).

Therefore,

$$\begin{aligned} &\mathbb{P}\left[\log Z^{\ell}_{(0,0),(t',n')} < f_{t',n'} - J^{-2/3}u(n')^{1/3}\right] \\ \leq &\mathbb{P}\left[\log Z_{(0,0),(t',n')} < f_{t',n'} - J^{-2/3}u(n')^{1/3} + C\mathrm{e}^{-cu}\right] + C\mathrm{e}^{-cu} \\ \leq &C\mathrm{e}^{-cu^{3/2}J^{-1}} + C\mathrm{e}^{-cu} \leq C\mathrm{e}^{-cu}, \end{aligned} \tag{7.9}$$

where we used Corollary 3.5 in the second last inequality. This completes the proof for $u \leq C_0 n^{2/3}$. Now assume $u \geq C_0 n^{2/3}$. Take $J = C_1 n^{1/3}$. Then for $C_1 > 0$ sufficiently large, depending on $\ell > 0$, we have that $Z^{\ell}_{(0,0),(tJ^{-1},nJ^{-1})} = Z_{(0,0),(tJ^{-1},nJ^{-1})}$. Therefore,

$$\mathbb{P}\left[\log Z^{\ell}_{(0,0),(t,n)} \leq f_{t,n} - un^{1/3}\right] \leq J\mathbb{P}\left[\log Z_{(0,0),(c_1 t^{2/3}, c_2 n^{2/3})} < f_{c_1 t^{2/3}, c_2 n^{2/3}} - c_3 u\right]. \tag{7.10}$$

By [19, Lemma 2.9] we have, for C_0 sufficiently large,

$$\mathbb{P}\left[\log Z_{(0,0),(c_1 t^{2/3}, c_2 n^{2/3})} < f_{c_1 t^{2/3}, c_2 n^{2/3}} - c_3 u\right] \leq C\mathrm{e}^{-c'u}. \tag{7.11}$$

This yields the claim. □

The following is an elementary consequence of standard estimates of the tail of the maximum of Brownian motion.

Proposition 7.2. Suppose that there are $c, C > 0$ so that $c \leq n, t, \ell \leq C$. Then there is a $c_1 > 0$ so that,

$$\mathbb{P}\left[|\log Z^{\ell}_{(0,0),(n,t)}| > u\right] \leq (c_1)^{-1}\mathrm{e}^{-c_1 u^2}. \tag{7.12}$$

8 Watermelon construction

In this section, we show how to use the construction in Section 8 of the work [4] on the geodesic watermelon[3] in last passage percolation (LPP) to get the upper bound for the left tail. However, there are significant difficulties introduced by the fact that the log-polymer partition function can take negative values. Compared to [4], we are forced to introduce a "branching stage" in the construction below which results in a logarithmic loss in the range of our tail bounds compared to the last passage case.

Let us take $k = 2^{N_1}$ for some $N_1 > 0$ such that $k \leq c_0 n$, some $c_0 > 0$. We begin with an informal discussion and sketch of the methodology. The basic idea is to lower bound,

$$\log Z_{(0,0),(n,n)} = \frac{1}{k}\sum_{i=1}^{k} \log Z_{(0,0),(n,n)} \geq \frac{1}{k}\sum_{i=1}^{k} \log \hat{Z}^{(i)}_{(0,0),(n,n)} \tag{8.1}$$

where $\hat{Z}^{(i)}_{(0,0),(n,n)}$ is a carefully chosen constrained partition function. That is, $\hat{Z}^{(i)}_{(0,0),(n,n)}$ will be an integral over polymer paths with the same Brownian increment weights as $Z_{(0,0),(n,n)}$, however the integral will be only over paths obeying certain constraints. The constraints will be of the form that the paths have to pass through certain points in the (t, n) plane and lie within a certain distance of the straight line connecting consecutive points. There will be k distinct paths/constraints, temporarily indicated by the notation

[3] The name "watermelon" comes from the fact that the non-intersecting k-geodesics in LPP resemble a watermelon; see the figures in [4].

$\hat{Z}^{(i)}$, and the distinct paths will spend a good amount of time in disjoint regions of phase space.

We will make repeated use of inequalities such as

$$Z_{(s,m),(v,p)} Z_{(v,p),(t,n)} \le Z_{(s,m),(t,n)} \tag{8.2}$$

(the LHS being interpreted as the partition function of polymer paths on $[s,t]$ starting at m ending at n, constrained so that $\gamma_v = p$) and $Z^{\ell}_{m,n}(s,t) \le Z_{m,n}(s,t)$.

The constraints will imply that the paths spend significant time in disjoint regions of the square $\{(s,m) : 0 \le m \le n, 0 \le s \le n\}$; independence will then allow for the application of concentration estimates showing that,

$$\mathbb{P}\left[\frac{1}{k}\sum_{i=1}^{k} \log \hat{Z}^{(i)}_{(0,0),(n,n)} \le n\mu - k^{2/3} n^{1/3}\right] \le \mathrm{e}^{-ck^2}. \tag{8.3}$$

We now recall our terminology that is used in order to discuss the nature of the constraints. We will be breaking up the paths into segments that are constrained to pass through points located on lines of the form $\{x + y = 2\ell\}$. It is therefore convenient to use *height* to refer to distance along the diagonal – that is, points on the line $\{x + y = 2\ell\}$ will be said to be at height ℓ. A point of the form $(x + y, x - y)$ will be said to have *anti-diagonal displacement* y.

If the polymer paths are constrained to pass through two points (s,m) and (t,p) then typically we will constrain them to lie in *corridors* of some *width* $2w$. That is, the polymer paths will satisfy that the maximal distance of the path γ from the straight line connecting (s,m) and (t,n) will be less than w; that is, the paths lie within a region of width $2w$ centered on the straight line between the points (s,m) and (t,p). We will say that the corridor has height ℓ where the point $(t-s, n-m)$ lies on the line $\{(x,y) : x + y = 2\ell\}$ and anti-diagonal displacement z where $(t-s, n-m) = (\ell + z, \ell - z)$.

In general, the corridors we consider will be of height r, anti-diagonal displacement $\mathcal{O}(r^{2/3})$ and width $\mathcal{O}(r^{2/3})$. That is, the anti-diagonal displacement will not be too great compared to the corridor width.

A final useful concept is the notion of separation between adjacent paths. Generically, the k paths will be constrained to pass through some k points $\{p_i\}_i$ on a line $\{x + y = 2\ell\}$. We will use *separation* to refer to the distance along this line between consecutive points $\{p_i\}_i$.

The constraints on the polymer paths will be given as a series of five "phases." We will take six heights, $\{h_m\}_{m=0}^{5}$ with $h_0 = 0, h_5 = n$ and $h_1 \asymp k$, $h_2 = n/3 + \mathcal{O}(1)$, $h_3 = n - h_2$, $h_4 = n - h_1$. The mth phase will then refer to the constraints on the polymer paths as they pass between height h_{m-1} and h_m. The first three phases are called

(1) Branching phase

(2) Separation phase

(3) Middle phase

The fourth and fifth phases are just the reverse of the separation and branching phases, respectively. See Figure 1 for a schematic diagram of the five phases.

In the branching phase, all k paths begin at the point $(0,0)$ and follow the same trajectory. Alternately, they will carry out the following operations

(i) Every trajectory will split into two trajectories, each followed by half the paths following the parent trajectory

(ii) Double the separation between consecutive trajectories.

The outcome at the end of the branching phase will be k paths that all have an order 1 separation arrayed along the line $\{x + y = 2h_1\}$ where $h_1 \asymp k$. Note that the branching phase necessarily must contain separation steps, to avoid all the paths clustering in a small space. Note that our procedure contains two kinds of separation, which are distinct and of a somewhat different nature: the separation that takes place during the branching phase, and the separation phase separation. It is important not to confuse the two notions. In the branching phase, we will, for example, only seek to produce order 1 separation.

In the separation phase, the paths will increase their separation in a dyadic fashion from $\mathcal{O}(1)$ to finally $\asymp n^{2/3}k^{-2/3}$ at height h_2.

In the middle phase, the paths will continue along diagonal lines, maintaining the $n^{2/3}k^{-2/3}$ separation. The paths will be constrained to lie within $\mathcal{O}(n^{2/3}k^{-2/3})$ of diagonal lines so as that the weights are independent.

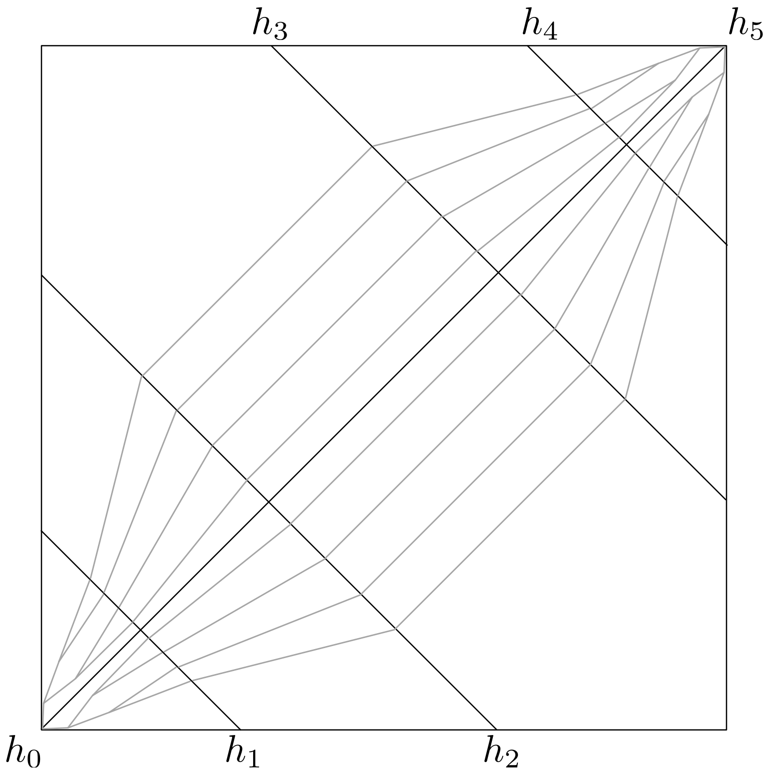

Figure 1: A schematic of the watermelon construction.

In summary, we will estimate,

$$k \log Z_{(0,0),(n,n)} \geq \sum_{m=1}^{5} \sum_{i=1}^{k} \log Z_{q_{i,m-1},q_{i,m}} \tag{8.4}$$

where $q_{i,m}$ are the points (to be determined) where the ith path intersects along the line $\{(x, y) : x + y = 2h_m\}$. We have $q_{i,0} = (0, 0)$ and $q_{i,5} = (n, n)$. In the next few subsections, we will further make constraints on the paths in each of the phases, seeking lower bounds for $\sum_{i=1}^{k} \log Z_{q_{i,m},q_{i,m-1}}$ for some fixed $m = 1, 2, 3$ (the cases $m = 4, 5$ omitted as they are similar to $m = 1, 2$).

8.1 Notational convention

In this section we will consider points with many subscripts. With the goal of readability we will let,

$$Z(p,q) := Z_{p,q} \tag{8.5}$$

and make similar conventions for other kinds of partition functions. We will also denote,

$$f(t,n) := f_{t,n}. \tag{8.6}$$

8.2 Branching phase

In this phase we will carry out an initial N branching steps, taking us to height $h_1 \asymp k$. That is, we will further specify constraints on the paths from $q_{i,0}$ to $q_{i,1}$ in order to lower bound the quantity,

$$\sum_{i=1}^{k} \log Z(q_{i,0}, q_{i,1}). \tag{8.7}$$

We now describe the constraints on each of the k paths.

First, every path passes from $(0,0)$ to the vertex $(10^5, 10^5)$. Set initially, $\hat{\ell}_0^{(1)} = 10^5$ and then $\ell_{j+1}^{(1)} = \hat{\ell}_j^{(1)} + 10^5$ for $j \geq 0$ and $\hat{\ell}_j^{(1)} = \ell_j^{(1)} + 10^5 \times 2^j$ for $j \geq 1$. Initially, all paths lie along the same trajectory. Then, for every $j \geq 0$,

(i) Between height $\hat{\ell}_j^{(1)}$ and $\ell_{j+1}^{(1)}$ each trajectory will branch into two trajectories, each followed by half the paths following the original trajectory

(ii) Between $\ell_{j+1}^{(1)}$ and $\hat{\ell}_{j+1}^{(1)}$, the separation between consecutive trajectories will increase by a factor of 2.

In particular, at height $\hat{\ell}_j^{(1)}$ and $\ell_j^{(1)}$, we will demand, using the super-addivity Lemma 2.4, that the paths intersect the two lines $\{(x,y) \in \mathbb{Z}^2 : x + y = 2\ell_j^{(1)}, 2\hat{\ell}_j^{(1)}\}$ at exactly 2^j distinct points that we will specify. This phase ends at $h_1 = \hat{\ell}_{N_1}^{(1)}$, after which there are $k = 2^{N_1}$ particles and they complete the separation step between $\ell_{N_1}^{(1)}$ and $\hat{\ell}_{N_1}^{(1)}$.

For $j \geq 1$, at height $\ell_j^{(1)}$ we consider the 2^j points, $m = 1, 2, \ldots, 2^j$,

$$\left(\ell_j^{(1)} - 10^4\left(\frac{2^j+1}{2} - m\right), \ell_j^{(1)} + 10^4\left(\frac{2^j+1}{2} - m\right)\right) =: \left(\ell_j^{(1)} - p_{mj}^{(1)}, \ell_j^{(1)} + p_{mj}^{(1)}\right) =: q_{mj}^{(1)} \tag{8.8}$$

and at height $\hat{\ell}_j^{(1)}$ the 2^j points, $m = 1, 2, \ldots, 2^j$,

$$\begin{aligned}\left(\hat{\ell}_j^{(1)} - 2\times 10^4\left(\frac{2^j+1}{2} - m\right), \hat{\ell}_j^{(1)} + 2\times 10^4\left(\frac{2^j+1}{2} - m\right)\right) &=: \left(\hat{\ell}_j^{(1)} - \hat{p}_{mj}^{(1)}, \hat{\ell}_j^{(1)} + \hat{p}_{mj}^{(1)}\right) \\ &=: \hat{q}_{mj}^{(1)}.\end{aligned} \tag{8.9}$$

At level j, the $k = 2^{N_1}$ paths are split into 2^j equally-sized blocks of size $2^{N_1 - j}$ so that the ith path passes through the points $q_{(i)_j,j}^{(1)}$ and $\hat{q}_{(i)_j,j}^{(1)}$ where,

$$(i)_j := \lceil i 2^{j - N_1} \rceil. \tag{8.10}$$

By convention we set $\hat{q}_{i,0}^{(1)} = (10^5, 10^5)$. Define now the points $q_{i,1}$ as the k points $\hat{q}_{(i)_{N_1},N_1}^{(1)} = \hat{q}_{i,N_1}^{(1)} =: q_{i,1}$ that have height $\hat{\ell}_{N_1}^{(1)} = h_1$. The above constraints are reflected in the inequality,

$$\sum_{i=1}^{k} \log Z(q_{i,0}, q_{i,1}) \geq k \log Z((0,0),(10^5,10^5)) \tag{8.11}$$

$$+ \sum_{j=1}^{N_1} \sum_{i=1}^{k} \log Z(\hat{q}^{(1)}_{(i)_{j-1},j-1}, q^{(1)}_{(i)_j,j}) \tag{8.12}$$

$$+ \sum_{j=1}^{N_1} \sum_{i=1}^{k} \log Z(q^{(1)}_{(i)_j,j}, \hat{q}^{(1)}_{(i)_j,j}). \tag{8.13}$$

Note also that the diagonal distance from $q^{(1)}_{i,j}$ to $\hat{q}^{(1)}_{i,j}$ is $10^5 \times 2^j$, and the anti-diagonal displacement is at most $\pm 10^4 \times 2^j$. Therefore, the slope of the line segment connecting these two points is positive, and bounded above and away from 0 uniformly in j and i. The point $\hat{q}^{(1)}_{i,j-1}$ will connect to points $q^{(1)}_{2i,j}$ and $q^{(1)}_{2i-1,j}$. The diagonal distance between these points is 10^5 and the anti-diagonal displacement is at most $\pm 10^4$, and so the lines connecting these points also has positive slope bounded above and away from 0.

Using this decomposition, we will prove the following over the rest of this section.

Proposition 8.1. There are $C, c > 0$ so that,

$$\mathbb{P}\left[\sum_{i=1}^{k} \log Z(q_{i,0}, q_{i,1}) \leq \mu k h_1 - C k^{5/3} n^{1/3}\right] \leq C \mathrm{e}^{-c k^{4/3} n^{2/3} (\log(k))^{-1}}. \tag{8.14}$$

The proof is split up into dealing with the two kinds of steps, the branching steps in (8.12) and the separation steps in (8.13).

8.2.1 Branching steps

We may rewrite the terms on the line (8.12) as,

$$\begin{aligned}\sum_{j=1}^{N_1} \sum_{i=1}^{k} \log Z(\hat{q}^{(1)}_{(i)_{j-1},j-1}, q^{(1)}_{(i)_j,j}) &= \sum_{j=1}^{N_1} 2^{N_1-j} \sum_{i=1}^{2^{j-1}} \log Z(\hat{q}^{(1)}_{i,j-1}, q^{(1)}_{2i-1,j}) + \log Z(\hat{q}^{(1)}_{i,j-1}, q^{(1)}_{2i,j}) \\ &\geq \sum_{j=1}^{N_1} 2^{N_1-j} \sum_{i=1}^{2^{j-1}} \log \hat{Z}^{(1)}(\hat{q}^{(1)}_{i,j-1}, q^{(1)}_{2i-1,j}) + \log \hat{Z}^{(1)}(\hat{q}^{(1)}_{i,j-1}, q^{(1)}_{2i,j}) \\ &=: \sum_{j=1}^{N_1} 2^{N_1-j} \sum_{i=1}^{2^{j-1}} Y^{(1)}_{ij} \end{aligned} \tag{8.15}$$

where we define $\hat{Z}^{(1)}(x,y)$ as the partition function of polymer paths from x to y constrained to lie within 10^3 of the straight line connecting x to y, and

$$Y^{(1)}_{ij} := \log \hat{Z}^{(1)}(\hat{q}^{(1)}_{i,j-1}, q^{(1)}_{2i-1,j}) + \log \hat{Z}^{(1)}(\hat{q}^{(1)}_{i,j-1}, q^{(1)}_{2i,j}) \tag{8.16}$$

Now, the collection $\{Y^{(1)}_{ij}\}_{ij}$ are mutually independent random variables and by Proposition 7.2 we have (since the width and height of the corridors involved are of constant order),

$$\mathbb{P}\left[|Y^{(1)}_{ij}| > u\right] \leq C \mathrm{e}^{-cu^2} \tag{8.17}$$

for some $C, c > 0$ and all $u \geq u_0$. The following follows from standard sub-Gaussian concentration results (see, e.g., [27, Section 2.5]).

Lemma 8.2. We have that for k sufficiently large,

$$\mathbb{P}\left[\left|2^{N_1}\sum_{j=1}^{N_1}2^{-j}\sum_{i=1}^{2^{j-1}}Y_{ij}^{(1)}\right|>k^{5/3}n^{1/3}\right]\leq 2\mathrm{e}^{-ck^{4/3}n^{2/3}}. \tag{8.18}$$

Proof. This estimate follows from a direct application of Proposition A.4, with $G_{ij}=Y_{ij}^{(1)}$ and $a_{ij}=2^{-j}$. The estimate (8.17) guarantees that the hypotheses are fulfilled. We calculate,

$$\|a\|_1=\sum_{j=1}^{N_1}\sum_{i=1}^{2^{j-1}}2^{-j}=\frac{N_1}{2}\leq C\log(k) \tag{8.19}$$

and

$$\|a\|_2^2=\sum_{j=1}^{N_1}\sum_{i=1}^{2^{j-1}}(2^{-j})^2\leq\sum_{j=1}^{N_1}2^{-j}\leq C. \tag{8.20}$$

Therefore, Proposition A.4 gives the estimate,

$$\mathbb{P}\left[\left|\sum_{j=1}^{N_1}\sum_{i=1}^{2^{j-1}}2^{-j}Y_{ij}^{(1)}\right|\geq C(\log(k)+t)\right]\leq C\mathrm{e}^{-ct^2}. \tag{8.21}$$

The claim now follows by taking $t=c_1k^{2/3}n^{1/3}$ for some small $c_1>0$. □

8.2.2 Order 1 separation steps

We begin by rewriting the terms (8.13) as,

$$\sum_{j=1}^{N_1}\sum_{i=1}^{k}\log Z(q_{(i)_j,j}^{(1)},\hat{q}_{(i)_j,j}^{(1)})=2^{N_1}\sum_{j=1}^{N_1}\frac{1}{2^j}\sum_{i=1}^{2^j}\log Z(q_{ij}^{(1)},\hat{q}_{ij}^{(1)}) \tag{8.22}$$

We would like to apply sub-Gaussian concentration to the random variables on the RHS. Our sub-Gaussian tail result is Proposition 7.2 which requires the rectangle on which we are considering the partition functions to be order 1. However, the height difference between the points $q_{ij}^{(1)}$ and $\hat{q}_{ij}^{(1)}$ is order 2^j. We therefore split the paths passing between the levels $\ell_j^{(1)}$ and $\hat{\ell}_j^{(1)}$ into 2^j further sub-levels.

For $s=0,1,\dots 2^j$ define $\hat{\ell}_{j,s}^{(1)}=\ell_j^{(1)}+10^5\times s$. Then, let $\hat{q}_{ij,s}^{(1)}$ be the point that is the intersection of the line $\{(x,y):x+y=2\hat{\ell}_{j,s}^{(1)}\}$ and the straight line segment connecting $q_{ij}^{(1)}$ and $\hat{q}_{ij}^{(1)}$. Then,

$$\begin{aligned}2^{N_1}\sum_{j=1}^{N_1}\frac{1}{2^j}\sum_{i=1}^{2^j}\log Z(q_{ij}^{(1)},\hat{q}_{ij}^{(1)})&\geq 2^{N_1}\sum_{j=1}^{N_1}\frac{1}{2^j}\sum_{i=1}^{2^j}\sum_{s=1}^{2^j}\log Z(\hat{q}_{ij,s-1}^{(1)},\hat{q}_{ij,s}^{(1)})\\&\geq 2^{N_1}\sum_{j=1}^{N_1}\frac{1}{2^j}\sum_{i=1}^{2^j}\sum_{s=1}^{2^j}\log\hat{Z}^{(1)}(\hat{q}_{ij,s-1}^{(1)},\hat{q}_{ij,s}^{(1)})\end{aligned} \tag{8.23}$$

where again, $\hat{Z}^{(1)}(x,y)$ denotes the polymer partition function of paths from x to y staying within 10^3 of the straight line connecting these points. Due to these constraints and the separation between consecutive points at each height $\hat{\ell}_{j,s}^{(1)}$ (i.e., the anti-diagonal distance between adjacent $\hat{q}_{ij,s}^{(1)}$ is at least 10^4) we see that the collection $\{\log\hat{Z}^{(1)}(\hat{q}_{ij,s-1}^{(1)},\hat{q}_{ij,s}^{(1)})\}_{ijs}$ are a family of independent random variables obeying,

$$\mathbb{P}\left[|\log\hat{Z}^{(1)}(\hat{q}_{ij,s-1}^{(1)},\hat{q}_{ij,s}^{(1)})|>u\right]\leq 2\mathrm{e}^{-cu^2}, \tag{8.24}$$

for some $c > 0$, where we applied Proposition 7.2. Therefore, again by sub-Gaussian concentration (see, e.g., [27, Section 2.5]), we derive the following.

Lemma 8.3. There is a $C_1 > 0$ so that,

$$\mathbb{P}\left[\left|2^{N_1}\sum_{j=1}^{N_1}\frac{1}{2^j}\sum_{i=1}^{2^j}\sum_{s=1}^{2^j}\log \hat{Z}^{(1)}(\hat{q}^{(1)}_{ij,s-1},\hat{q}^{(1)}_{ij,s})\right| > C_1 k^{5/3}n^{1/3}\right] \leq 2\mathrm{e}^{-ck^{4/3}n^{2/3}(\log(k))^{-1}}, \quad (8.25)$$

Proof. This estimate follows by a direct application of Proposition A.4. Let $G_{ijs} = \log \hat{Z}^{(1)}(\hat{q}^{(1)}_{ij,s-1},\hat{q}^{(1)}_{ij,s})$ and $a_{ijs} = 2^{-j}$. The estimate (8.24) guarantees that the hypotheses are fulfilled. We calculate,

$$\|a\|_1 = \sum_{j=1}^{N_1}\sum_{i=1}^{2^j}\sum_{s=1}^{2^j} 2^{-j} = \sum_{j=1}^{N_1} 2^j \leq C2^{N_1} \leq Ck \quad (8.26)$$

where we used that $2^{N_1} = k$. We also calculate,

$$\|a\|_2^2 = \sum_{j=1}^{N_1}\sum_{i=1}^{2^j}\sum_{s=1}^{2^j}(2^{-j})^2 \leq \sum_{j=1}^{N_1} C = CN_1 \leq C\log(k). \quad (8.27)$$

Therefore, from Proposition A.4 we conclude the estimate,

$$\mathbb{P}\left[\left|\sum_{j=1}^{N_1}\frac{1}{2^j}\sum_{i=1}^{2^j}\sum_{s=1}^{2^j}\log \hat{Z}^{(1)}(\hat{q}^{(1)}_{ij,s-1},\hat{q}^{(1)}_{ij,s})\right| > C(k+t)\right] \leq C\exp\left(-\frac{ct^2}{\log(k)}\right) \quad (8.28)$$

for all $t > 0$. The claim follows from the choice of $t = k^{2/3}n^{1/3}$ and the fact that $k \leq n$. ☐

8.2.3 Proof of Proposition 8.1

First, note that since $h_1 \leq Ck \leq Cn$, the term $\mu k h_1$ appearing in (8.14) can be absorbed into the term $k^{5/3}n^{1/3}$ at the expense of changing the constants. Now, in order to complete the proof of the Proposition, we use the lower bound of $\sum_i \log Z(q_{i,0}, q_{i,1})$ in terms of the three terms (8.11), (8.12) and (8.13). First, for the term (8.11), we have by Proposition 7.2 that,

$$\mathbb{P}\left[k\log Z((0,0),(10^5,10^5)) < -Ck^{5/3}n^{1/3}\right] \leq C\mathrm{e}^{-k^{4/3}n^{2/3}}. \quad (8.29)$$

Next, the terms (8.12) and (8.13) are handled by the estimates (8.18) and (8.25) that were proved over the course of the previous two subsections. ☐

8.3 Separation phase

In this phase, set first $\ell_0^{(2)} = h_1$, the endpoint of the previous phase. We then inductively define,

$$\ell_j^{(2)} = \ell_{j-1}^{(2)} + 10^5(2^j)^{3/2}k. \quad (8.30)$$

There will be N_2 levels where $2^{N_2} = 10^{-10}n^{2/3}k^{-2/3}(1+\mathcal{O}(1))$. We now define $h_2 := \ell^{(2)}_{N_2}$. Note that with this choice and the assumption $k \leq c_1 n$ some small $c_1 > 0$ we have that,

$$cn \leq h_2 \leq n/3 \quad (8.31)$$

The ith curve will intersect level $\ell_j^{(2)}$ at the point,

$$\left(\ell_j^{(2)} - p_{ij}^{(2)}, \ell_j^{(2)} + p_{ij}^{(2)}\right) =: q_{ij}^{(2)} \quad (8.32)$$

where,

$$p_{ij}^{(2)} = p_{i,j-1}^{(2)} + 2^j \left(\frac{k+1}{2} - i \right), \tag{8.33}$$

and $p_{i0}^{(2)} = 2 \times 10^4 \left(\frac{k+1}{2} - i \right)$ (i.e., the last points from the previous phase). With these definitions, the separation between consecutive points on level j is order 2^j,

$$C2^j \geq \mathrm{sep}_j := p_{i,j}^{(2)} - p_{i-1,j}^{(2)} \geq 2^j + 10^5. \tag{8.34}$$

On the other hand, the distance between levels $j-1$ and j is order $(2^j)^{3/2}k$. We would like to restrict the ith path to lie within a corrider of order 2^j around the straight line connecting $q_{i,j-1}^{(2)}$ and $q_{i,j}^{(2)}$ so that the partition functions corresponding to these paths are independent for different i, and apply Proposition 7.1. However, Proposition 7.1 is not directly applicable because the distance between levels $j-1$ and j is larger than $(2^j)^{3/2}$ by a factor of k. We therefore must split the path between consecutive levels $\ell_{j-1}^{(2)}$ and $\ell_j^{(2)}$ into k further sublevels. We will do this below momentarily.

Before doing this, let us remark that the antidiagonal displacement between the points that the ith path crosses on consecutive levels (i.e., between points $q_{i,j-1}^{(2)}$ and $q_{ij}^{(2)}$) is as much as,

$$p_{1j}^{(2)} - p_{1,j-1}^{(2)} = 2^j \left(\frac{k+1}{2} - 1 \right), \tag{8.35}$$

and that the diagonal separation between $q_{i,j-1}^{(2)}$ and $q_{i,j}^{(2)}$ is $10^5(2^j)^{3/2}k$ and so the slope of the straight line connecting this points is positive and bounded above and away from 0.

We now proceed to split path between consecutive levels $\ell_{j-1}^{(2)}$ and $\ell_j^{(2)}$ into k further sublevels,

$$\ell_{j,s}^{(2)} = \ell_{j-1}^{(2)} + 10^5 (2^j)^{3/2} s \tag{8.36}$$

for $s = 0, \ldots, k$. We then let

$$q_{ijs}^{(2)} = \left(\ell_{js}^{(2)} - p_{ijs}^{(2)}, \ell_{js}^{(2)} + p_{ijs}^{(2)} \right) \tag{8.37}$$

be the point that is on the intersection of the straight line connecting $q_{ij}^{(2)}$ and $q_{i,j+1}^{(2)}$ and the line $\{(x,y) : x + y = 2\ell_{j,s}^{(2)}\}$. Note that,

$$p_{ijs}^{(2)} = -p_{k+1-i,j,s}^{(2)} + \mathcal{O}(1) \tag{8.38}$$

the $\mathcal{O}(1)$ coming from the effect of the integer lattice. We now write,

$$\begin{aligned} \sum_{i=1}^{k} \log Z_{q_{i,1},q_{i,2}} &\geq \sum_{j=1}^{N_2} \sum_{i=1}^{k} \sum_{s=1}^{k} \log Z(q_{ij,s-1}^{(2)}, q_{ij,s}^{(2)}) \\ &\geq \sum_{j=1}^{N_2} \sum_{i=1}^{k} \sum_{s=1}^{k} \log \hat{Z}^{(2),j}(q_{ij,s-1}^{(2)}, q_{ij,s}^{(2)}) \end{aligned} \tag{8.39}$$

where we define $\hat{Z}^{(2),j}(x,y)$ to be the partition function of polymer paths from x to y restricted to the corrider of width $2^j + 10^4$ around the straight line connecting x to y.

Lemma 8.4. For some $c, C > 0$ we have,

$$\mathbb{P}\left[\sum_{j=1}^{N_2} \sum_{i=1}^{k} \sum_{s=1}^{k} \left(\log \hat{Z}^{(2),j}(q_{ij,s-1}^{(2)}, q_{ij,s}^{(2)}) - f(q_{ij,s}^{(2)} - q_{ij,s-1}^{(2)}) \right) \leq -Ck^{5/3}n^{1/3} \right] \leq \mathrm{e}^{-ck^2}. \tag{8.40}$$

Proof. Let $Y_{ijs} := f(q^{(2)}_{ij,s} - q^{(2)}_{ij,s-1}) - \log \hat{Z}^{(2),j}(q^{(2)}_{ij,s-1}, q^{(2)}_{ij,s})$. The height and width of the rectangle with opposite vertices $q_{ij,s}$ and $q_{ij,s-1}$ and sides parallel to the coordinate axes are both order $(2^j)^{3/2}$ by our earlier discussion of the diagonal and anti-diagonal separation of $q_{i,j-1}$ and $q_{i,j}$. With $r_j = (2^j)^{3/2}$ we see that the polymer paths are restricted to lie in a corridor of width $r_j^{2/3}$ and so Proposition 7.1 is applicable if j is sufficiently large. Therefore,

$$\mathbb{P}\left[Y_{ijs} \geq uC(r_j)^{1/3}\right] \leq \mathrm{e}^{-u} \tag{8.41}$$

for $u \geq u_0$ and j sufficiently large. For smaller j we instead can apply Proposition 7.2 to arrive at the same estimate.

We now wish to apply Proposition A.2. Note that the family of random variables Y_{ijs} are independent because the separation between points on level j is at least $2^j + 10^5$, as discussed above, and the paths are restricted to lie in corridors of width $2^j + 10^4$.

We therefore may apply Proposition A.2 with $a_{ijs}^{-1} := C(r_j)^{1/3}$. Then,

$$\nu := \sum_{ijs} \frac{1}{a_{ijs}} = Ck^2 \sum_{j=1}^{N_2} (r_j)^{1/3} = Ck^2 \sum_{j=1}^{N_2} (2^j)^{1/2} \asymp k^{5/3} n^{1/3} \tag{8.42}$$

by the choice $2^{N_2} \asymp n^{2/3} k^{-2/3}$, and

$$\min_{ijs} a_{ijs} \geq ck^{1/3} n^{-1/3}. \tag{8.43}$$

Therefore, Proposition A.2 implies,

$$\mathbb{P}\left[\sum_{ijs} Y_{ijs} \geq Ck^{5/3} n^{1/3}\right] \leq \mathrm{e}^{-ck^2} \tag{8.44}$$

as desired. □

Lemma 8.5. We have,

$$\left|(h_2 - h_1)\mu k - \sum_{j=1}^{N_2} \sum_{i=1}^{k} \sum_{s=1}^{k} f(q^{(2)}_{ij,s} - q^{(2)}_{ij,s-1})\right| \leq Ck^{5/3} n^{1/3}. \tag{8.45}$$

Proof. Writing $h_2 - h_1 = \sum_{j,s} (\ell^{(2)}_{j,s} - \ell^{(2)}_{j,s-1})$ we have,

$$\begin{aligned} &\sum_{j=1}^{N_2} \sum_{i=1}^{k} \sum_{s=1}^{k} f(q^{(2)}_{ij,s} - q^{(2)}_{ij,s-1}) - (h_2 - h_1)\mu k \\ &= \sum_{j=1}^{N_2} \sum_{s,i} \left(f(q^{(2)}_{ij,s} - q^{(2)}_{ij,s-1}) - \mu(\ell^{(2)}_{i,s} - \ell^{(2)}_{j,s-1}) \right). \end{aligned} \tag{8.46}$$

Let $r_{j,s} := (\ell^{(2)}_{i,s} - \ell^{(2)}_{j,s-1})$ and

$$\hat{p}_{i,j,s} = p^{(2)}_{ijs} - p^{(2)}_{ij,s-1} \tag{8.47}$$

so that,

$$q^{(2)}_{ij,s} - q^{(2)}_{ij,s-1} = (r_{j,s} - \hat{p}_{ijs}, r_{j,s} + \hat{p}_{ijs}). \tag{8.48}$$

Now we have

$$\hat{p}_{ijs} = -\hat{p}_{k+1-i,js} + \mathcal{O}(1) \tag{8.49}$$

as well as $|\hat{p}_{ijs}| \le C(r_{j,s})^{2/3}$. Therefore, for fixed s, j we have, by applying Lemma 2.5

$$\begin{aligned}\sum_{i=1}^{k} f(r_{j,s} - \hat{p}_{ijs}, r_{j,s} + \hat{p}_{ijs}) &= \sum_{i=1}^{k} r_{j,s} f(1 - r_{j,s}^{-1}\hat{p}_{ijs}, 1 + r_{j,s}^{-1}\hat{p}_{ijs}) \\ =& r_{j,s} \sum_{i=1}^{k/2} f(1 - r_{j,s}^{-1}\hat{p}_{ijs}, 1 + r_{j,s}^{-1}\hat{p}_{ijs}) + f(1 - r_{j,s}^{-1}\hat{p}_{k+1-i,js}, 1 + r_{j,s}^{-1}\hat{p}_{k+1-i,js}) \\ =& r_{j,s} k\mu + \sum_{i=1}^{k/2} \mathfrak{a}(\hat{p}_{ij,s} + \hat{p}_{k+1-i,js}) + \mathcal{O}(k(r_{j,s})^{1/3}) \\ =& r_{j,s} k\mu + \mathcal{O}(k(2^j)^{1/2})\end{aligned} \tag{8.50}$$

where in the last line we applied (8.49) as well as the fact that $r_{j,s} \le C(2^j)^{3/2}$. Therefore,

$$\left|(h_2 - h_1)\mu k - \sum_{j=1}^{N_2} \sum_{i=1}^{k} \sum_{s=1}^{k} f(q_{ij,s}^{(2)} - q_{ij,s-1}^{(2)})\right| \le C \sum_{j=1}^{N_2} k^2 (2^j)^{1/2} \le C k^{5/3} n^{1/3} \tag{8.51}$$

as desired. □

The previous two lemmas immediately give the following.

Proposition 8.6. There are $C, c > 0$ so that,

$$\mathbb{P}\left[\sum_{i=1}^{k} \log Z(q_{i,1}, q_{i,2}) \le \mu k(h_2 - h_1) - C k^{5/3} n^{1/3}\right] \le C \mathrm{e}^{-ck^2}. \tag{8.52}$$

8.4 Middle phase

At the end of the previous phase, the k paths intersect the line $\{(x, y) : x + y = 2h_2\}$ on the k points $q_{i,0}^{(3)}$ that have coordinates,

$$q_{i,0}^{(3)} := \left(h_2 - p_i^{(3)}, h_2 + p_i^{(3)}\right) \tag{8.53}$$

where

$$p_i^{(3)} := \left(2(2^{N_2} - 1) + 2 \times 10^4\right) \left(\frac{k+1}{2} - i\right). \tag{8.54}$$

Note that the separation between consecutive points along this line is of order $n^{2/3} k^{-2/3}$. Set now $\ell_0^{(3)} = h_2$ and

$$\ell_j^{(3)} = \ell_{j-1}^{(3)} + \frac{n - 2h_2}{k} \tag{8.55}$$

for $j = 1, \ldots, k$. We will demand that the ith curve passes through level ℓ_j at the point,

$$q_{ij}^{(3)} := \left(\ell_j^{(3)} - p_i^{(3)}, \ell_j^{(3)} + p_i^{(3)}\right). \tag{8.56}$$

We then bound

$$\begin{aligned}\sum_{i=1}^{k} \log Z_{q_{i,2}, q_{i,3}} &\ge \sum_{j=1}^{k} \sum_{i=1}^{k} \log Z(q_{i,j-1}^{(3)}, q_{i,j}^{(3)}) \\ &\ge \sum_{j=1}^{k} \sum_{i=1}^{k} \log \hat{Z}^{(3)}(q_{i,j-1}^{(3)}, q_{i,j}^{(3)})\end{aligned} \tag{8.57}$$

where $\hat{Z}^{(3)}(x, y)$ is the partition function of polymer paths starting at x and ending at y that stay within $cn^{2/3}k^{-2/3}$ of the straight line connecting x to y, for a small enough $c > 0$ so that the corridors of different paths are disjoint.

Lemma 8.7. There are $C, c > 0$ so that,

$$\mathbb{P}\left[\sum_{j=1}^{k}\sum_{i=1}^{k}\left(\log \hat{Z}^{(3)}(q^{(3)}_{i,j-1}, q^{(3)}_{i,j}) - f(q^{(3)}_{i,j} - q^{(3)}_{i,j-1})\right) < Ck^{5/3}n^{1/3}\right] \leq \mathrm{e}^{-ck^2} \tag{8.58}$$

and

$$(h_3 - h_2)k\mu = \sum_{j=1}^{k}\sum_{i=1}^{k} f(q^{(3)}_{i,j} - q^{(3)}_{i,j-1}) \tag{8.59}$$

Proof. Let $Y_{ij} := f(q^{(3)}_{i,j} - q^{(3)}_{i,j-1}) - \log \hat{Z}^{(3)}(q^{(3)}_{i,j-1}, q^{(3)}_{i,j})$. By the choice of the constraints, the random variables $\{Y_{ij}\}_{ij}$ are all independent. The distance between $q^{(3)}_{i,j-1}$ and $q^{(3)}_{i,j}$ is of order $r := (n/k)$ and the anti-diagonal displacement is 0. The polymer paths are restricted to lie in a corridor of width of order $(n/k)^{2/3} \leq Cr^{2/3}$ and so Proposition 7.1 is applicable. Therefore,

$$\mathbb{P}\left[Y_{ij} \geq Cr^{1/3}u\right] \leq \mathrm{e}^{-cu} \tag{8.60}$$

for $u \geq u_0$. So we may apply Proposition A.2 to the sum $\sum_{ij} Y_{ij}$. We have,

$$\nu \asymp k^2 r^{1/3} = k^{5/3}n^{1/3}, \qquad a_* \asymp r^{-1/3} = k^{1/3}n^{-1/3}, \tag{8.61}$$

and so,

$$\mathbb{P}\left[\sum_{ij} Y_{ij} \geq Ck^{5/3}n^{1/3}\right] \leq \mathrm{e}^{-ck^2}. \tag{8.62}$$

This completes the estimate of the lemma. The second statement follows from the fact that $f(q^{(3)}_{i,j} - q^{(3)}_{i,j-1}) = (\ell^{(3)}_j - \ell^{(3)}_{j-1})\mu$. □

We therefore obtain,

Proposition 8.8. There are $C > 0$ and $c > 0$ such that

$$\mathbb{P}\left[\sum_{i=1}^{k} \log Z(q_{i,2}, q_{i,3}) \leq (h_3 - h_2)\mu k - Ck^{5/3}n^{1/3}\right] \leq \mathrm{e}^{-ck^2}. \tag{8.63}$$

8.5 Tail bound

From all of the previous, we obtain the following.

Proposition 8.9. There is a $c_1 > 0$ and $c, C > 0$ so that for any $k = 2^N \leq c_1 n$ we have,

$$\mathbb{P}\left[\log Z_{(0,0),(n,n)} \leq n\mu - Ck^{2/3}n^{1/3}\right] \leq C\mathrm{e}^{-ck^2} + C\mathrm{e}^{-ck^{4/3}n^{2/3}\log(k)^{-1}} \tag{8.64}$$

Proof. We recall,

$$\log Z_{(0,0),(n,n)} - n\mu \geq \frac{1}{k}\sum_{m=1}^{5}\left(\sum_{i=1}^{k} \log Z(q_{i,m-1}, q_{i,m}) - (h_m - h_{m-1})n\mu\right). \tag{8.65}$$

The terms on the RHS with $m = 1, 2, 3$ were bounded by Propositions 8.1, 8.6 and 8.8, respectively. The terms with $m = 4, 5$ are treated by the mirror opposite constructions of the phases $m = 2, 1$, respectively. □

9 Lower bound for right tail

In this section we prove our lower bound for the right tail using the strategy of [8]. First, by convergence of $n^{-1/3}(\log Z_{(0,0),(n,n)} - \mu n)$ to a constant times a Tracy-Widom random variable [5, Theorem 1.3] and the unbounded support of this distribution we have that there is $c_1, \delta_1 > 0$ so that

$$\mathbb{P}\left[\log Z_{(0,0),(n,n)} \geq \mu n + c_1 n^{1/3}\right] \geq \delta_1. \tag{9.1}$$

That the Tracy-Widom distribution has unbounded support can be deduced from the explicit form of its distribution function (cdf). In fact, more is known [2, (25)]:

$$F_2(x) = 1 - \frac{1}{32\pi x^{3/2}} \exp\Big(-\frac{4}{3}x^{3/2}\Big)(1+o(1)), \quad x \to \infty.$$

Here, $F_2(x)$ is the cdf of the Tracy-Widom GUE distribution.

Fix some k. We have,

$$\log Z_{(0,0),(n,n)} \geq \sum_{j=1}^{k} \log Z_{\frac{j-1}{k}(n,n),\frac{j}{k}(n,n)} \tag{9.2}$$

and by independence,

$$\mathbb{P}\left[\log Z_{(0,0),(n,n)} \geq \mu n + u n^{1/3}\right] \geq \Big(\mathbb{P}\left[\log Z_{(0,0),(n/k,n/k)} \geq \mu n/k + u n^{1/3}/k\right]\Big)^k. \tag{9.3}$$

We take $k^{2/3} = u/c_1$ so that with $r = n/k$ we have,

$$\mathbb{P}\left[\log Z_{(0,0),(n/k,n/k)} \geq \mu n/k + u n^{1/3}/k\right] = \mathbb{P}\left[\log Z_{(0,0),(r,r)} \geq \mu r + c_1 r^{1/3}\right] \geq \delta_1 \tag{9.4}$$

for all r large enough. The lower bound of (1.5) follows.

A Concentration estimates

A.1 Sub-exponential random variables

We have the following, Theorem 5.1(i) of [13].

Proposition A.1. Let $W = \sum_{i=1}^n W_i$ where W_i are independent exponential random variables $W_i \sim \mathrm{Exp}(a_i)$. Let,

$$\nu = \mathbb{E}\, W = \sum_i \frac{1}{a_i}, \qquad a_* := \min_i a_i. \tag{A.1}$$

Then for $\lambda \geq 1$ we have,

$$\mathbb{P}\left[W \geq \lambda\nu\right] \leq \lambda^{-1} \mathrm{e}^{-a_*\nu(\lambda-1-\log\lambda)}. \tag{A.2}$$

With this we prove the following.

Proposition A.2. Let $\{Y_i\}_i$ be a collection of independent random variables such that for some θ_0 and $\{a_i\}_i$ we have,

$$\mathbb{P}\left[Y_i \geq \theta(a_i)^{-1}\right] \leq \mathrm{e}^{-\theta} \tag{A.3}$$

for all i and $\theta \geq \theta_0$. Define,

$$\nu = \sum_{i=1}^n \frac{1}{a_i}, \qquad a_* := \min_i a_i. \tag{A.4}$$

Then there are $C > 0$ and $c > 0$ so that,

$$\mathbb{P}\left[\sum_i Y_i \geq C\nu\right] \leq \mathrm{e}^{-c a_* \nu}. \tag{A.5}$$

Proof. There is a coupling of $\{Y_i\}_i$ to a family of mutually independent exponential random variables $X_i \sim \mathrm{Exp}(a_i)$ such that[4]

$$Y_i \leq X_i + a_i^{-1}\theta_0. \tag{A.6}$$

Taking $C \geq \theta_0 + 3$ we then see that,

$$\mathbb{P}\left[\sum_i Y_i \geq C\nu\right] \leq \mathbb{P}\left[\sum_i X_i \geq 3\nu\right] \leq \mathrm{e}^{-ca_*\nu} \tag{A.7}$$

where we applied Proposition A.1 in the last inequality with $\lambda = 3$. □

A.2 Sub-Gaussian random variables

For a random variable X we define the sub-Gaussian norm $\|X\|_{\psi_2}$ by

$$\|X\|_{\psi_2} := \inf\left\{K > 0 : \mathbb{P}\left[|X| > t\right] \leq 2\mathrm{e}^{-t^2/K^2}, \forall t > 0\right\} \tag{A.8}$$

For sub-Gaussian random variables we have the following, [27, Theorem 2.6.3].

Theorem A.3. There is a $c > 0$ so that the following holds. Let $\{X_i\}_{i=1}^N$ be mean-zero, independent sub-Gaussian random variables and let $K = \max_i \|X_i\|_{\psi_2}$. Let $a = (a_1, \dots, a_N) \in \mathbb{R}^N$. Then,

$$\mathbb{P}\left[\left|\sum_{i=1}^N a_i X_i\right| > t\right] \leq 2\exp\left(-\frac{ct^2}{K^2\|a\|_2^2}\right) \tag{A.9}$$

As an application we have the following.

Proposition A.4. Let $\{G_i\}_{i=1}^N$ be a family of independent random variables such that there are $C_0, c_0 > 0$ so that

$$\mathbb{P}\left[|G_i| > t\right] \leq C_0\mathrm{e}^{-c_0t^2} \tag{A.10}$$

for $|t| > C_0$. There are $C_1, c_1 > 0$ depending only on $C_0, c_0 > 0$ and not on N so that for any $a = (a_1, \dots a_N) \in \mathbb{R}^N$ we have,

$$\mathbb{P}\left[\left|\sum_{i=1}^N a_i G_i\right| > C_1\|a\|_1 + t\right] \leq C_1\exp\left(-\frac{c_1t^2}{\|a\|_2^2}\right) \tag{A.11}$$

Proof. Define $X_i := G_i - \mathbb{E}[G_i]$. Since $|\mathbb{E}[G_i]| \leq C$ for all i, we see that there is a $c > 0$ depending only on $c_0, C_0 > 0$ so that

$$\mathbb{P}\left[|X_i| > t\right] \leq 2\mathrm{e}^{-ct^2} \tag{A.12}$$

and so $K := \max_i \|X_i\|_{\psi_2}$ is bounded by a constant depending only on $c_0, C_0 > 0$. From Theorem A.3 we see that,

$$\mathbb{P}\left[\left|\sum_{i=1}^N a_i X_i\right| > t\right] \leq 2\exp\left(-\frac{c_1t^2}{\|a\|_2^2}\right) \tag{A.13}$$

for some $c_1 > 0$. On the other hand we have that

$$\left|\mathbb{E}\left[\sum_{i=1}^N a_i G_i\right]\right| \leq C_1\|a\|_1 \tag{A.14}$$

for some $C_1 > 0$ and so the claim follows. □

[4] This can be constructed, e.g., by setting $Y_i = F^{-1}(U_i)$ where F^{-1} is a generalized inverse of the CDF of Y and U_i are iid uniform $(0,1)$ random variables, and $a_iX_i = -\log U_i$.

B FKG inequality

In this section we prove a form of positive association ("the Harris-FKG inequality") for polymer partition functions. For $1 \le i \le N$ and $1 \le j \le M_i$, let X_{ij} be a random variable of the form

$$X_{ij} = \int_{a_{ij} < s_{m_{ij}} < \ldots s_{n_{ij}} < b_{ij}} \mathrm{e}^{\sum_{k=m_{ij}}^{n_{ij}} B_k(s_k) - B_k(s_{k-1})} \prod_{k=m_{ij}}^{n_{ij}-1} \mathbf{1}_{\{s_k \in I_{ijk}\}} \mathrm{d}s_{m_{ij}} \ldots \mathrm{d}s_{n_{ij}-1} \tag{B.1}$$

where the I_{ijk} are some intervals. Then, let

$$Z_i = \sum_{j=1}^{M_i} a_{ij} X_{ij} \tag{B.2}$$

where $a_{ij} > 0$.

Proposition B.1. Let the Z_i be as above. Let $A_i \in \mathbb{R}$. Then,

$$\mathbb{P}\left[\bigcap_{i=1}^{N} \{Z_i \le A_i\}\right] \ge \prod_{i=1}^{N} \mathbb{P}\left[Z_i < A_i\right] \tag{B.3}$$

Proof. Let $L \ge \max_{ij} |n_{ij}| + \max_{ij} |m_{ij}| + \max_{ij} |a_{ij}| + \max_{ij} |b_{ij}|$. Let $\{Y_{kl}\}_{(k,l)\in\mathbb{Z}^2}$ be a family of iid ± 1 random variables. Consider for every n the functions $\hat{B}_i^{(n)} : [-L, L] \to \mathbb{R}$ defined by

$$\hat{B}_i^{(n)}(t) := n^{1/2} \int_{-L}^{t} \sum_j Y_{ji} \mathbf{1}_{\{s \in (j/n, (j+1)/n)\}} \mathrm{d}s. \tag{B.4}$$

Then each $B_i^{(n)}(t)$ converges in the space $C([-L, L])$ equipped with the topology induced by the uniform norm $\|\cdot\|_\infty$ to Brownian motions $W_i(t)$ on $[-L, L]$ with $W_i(-L) = 0$. Viewing the random variables X_{ij} as functions $X_{ij}(\cdot) : C([-L, L])^{2L+1} \to \mathbb{R}$, we see that they are continuous with respect to the norm $\|\cdot\|_\infty$. This implies the joint convergence of the collection $\{Z_i(B^{(n)})\}_i$ to $\{Z_i(W)\}_i$. Here, $Z_i(\cdot) := \sum_{j=1}^{M_i} a_{ij} X_{ij}(\cdot)$ as in (B.2). However, this latter family has the same distribution as the original Z_i specified in (B.2) in terms of the original Brownian motions $B_i(t)$. By the Portmanteau theorem,

$$\limsup_{n\to\infty} \mathbb{P}\left[\bigcap_i \{Z_i(B^{(n)}) \le A_i\}\right] \le \mathbb{P}\left[\bigcap_i \{Z_i \le A_i\}\right] \tag{B.5}$$

and

$$\liminf_{n\to\infty} \prod_i \mathbb{P}\left[\{Z_i(B^{(n)}) < A_i\}\right] \ge \prod_i \mathbb{P}\left[\{Z_i < A_i\}\right]. \tag{B.6}$$

On the other hand, from the representation (B.4), we see that any increment $\hat{B}_i^{(n)}(t) - \hat{B}_i^{(n)}(s)$ is increasing under changing any Y_{ij} from -1 to $+1$. Therefore, by positive association (the Harris-FKG inequality for independent random variables, see, e.g., [15, Chaper II.2]),

$$\prod_i \mathbb{P}\left[\{Z_i(B^{(n)}) < A_i\}\right] \le \mathbb{P}\left[\bigcap_i \{Z_i(B^{(n)}) \le A_i\}\right] \tag{B.7}$$

for every n. The claim follows. □

C Miscellaneous proofs

In this section it will be useful to introduce the notation,

$$B_k(t, s) := B_k(t) - B_k(s) \tag{C.1}$$

for the Brownian increments.

C.1 Proof of (5.33)

Recall that $T_{jk}^{(+)}$ is the set of polymer paths intersectings the lines $\ell_i := \{(z_i, z_i) + (-m, m) : |m| \le b_{j-1} n^{2/3}\}$ for $i = 1, 2$ and passing above the line $\{(z_0, z_0) + (-m, m) : |m| \le b_j n^{2/3}\}$. Write $a = b_{j-1} n^{2/3}$ and $b = b_j n^{2/3}$. The event that the polymer path intersects the line ℓ_i can be written as the disjoint union of the sets (up to some sets of Lebesgue measure 0 which do not contribute to the partition function)

$$\left(\bigsqcup_{|m| \le a} \{s_{z_i+m} > z_i - m, s_{z_i+m-1} < z_i - m\} \right)$$
$$\bigsqcup \left(\bigsqcup_{-a \le m \le a-1} \{z_i - m - 1 < s_{z_i+m} < z_i - m\} \right) \tag{C.2}$$

and the event that the path passes above the line $\{(z_0, z_0) + (-m, m) : |m| \le b\}$ can be written as the event $\{s_{z_0+b} < z_0 - b\} =: \mathcal{A}$. Therefore,

$$Z_{(0,0),(n,n)}[T_{jk}^{(+)}]$$
$$= \sum_{|i|,|j| \le a} Z_{(0,0),(n,n)}[s_{z_1+i} > z_1 - i, s_{z_1+i-1} < z_1 - i, s_{z_2+j} > z_2 - j, s_{z_2+j-1} < z_2 - j, \mathcal{A}] \tag{C.3}$$
$$+ \sum_{-a \le i,j \le a-1} Z_{(0,0),(n,n)}[z_1 - i - 1 < s_{z_1+i} < z_1 - i, z_2 - j - 1 < s_{z_2+j} < z_2 - j, \mathcal{A}] \tag{C.4}$$
$$+ \sum_{\substack{|i| \le a \\ -a \le j \le a-1}} Z_{(0,0),(n,n)}[s_{z_1+i} > z_1 - i, s_{z_1+i-1} < z_1 - i, z_2 - j - 1 < s_{z_2+j} < z_2 - j, \mathcal{A}] \tag{C.5}$$
$$+ \sum_{\substack{-a \le i \le a-1 \\ |j| \le a}} Z_{(0,0),(n,n)}[z_1 - i - 1 < s_{z_1+i} < z_1 - i, s_{z_2+j} > z_2 - j, s_{z_2+j-1} < z_2 - j, \mathcal{A}] \tag{C.6}$$

The terms on the first, second, third and fourth lines above will be seen to give the first, second, third and fourth terms in (5.33), respectively. For the terms on the line (C.3), the set

$$\{s_{z_1+i} > z_1 - i, s_{z_1+i-1} < z_1 - i, s_{z_2+j} > z_2 - j, s_{z_2+j-1} < z_2 - j, s_{z_0+b} < z_0 - b\} \tag{C.7}$$

is empty unless $z_1 - i < z_0 - b$ and $z_2 + j > z_0 + b$. Since $a \le b$ and $|i| \le a$ the first inequality implies $z_0 + b > z_1 + i$. For the non-zero terms, group the terms in the integrand as,

$$\left(\mathbf{1}_{\{s_{z_1+i-1} < z_1 - i\}} e^{\sum_{k=0}^{z_1+i-1} B_k(s_k, s_{k-1}) + B_{z_1+i}(z_1 - i, s_{z_1+i-1})} \prod_{k=0}^{z_1+i-1} \mathrm{d}s_k \right)$$
$$\times \Bigg\{ \mathbf{1}_{\{s_{z_0+b} < z_0 - b\}} \mathbf{1}_{\{s_{z_1+i} > z_1 - i\}} \mathbf{1}_{\{s_{z_2+j-1} < z_2 - j\}}$$
$$\times e^{B_{z_1+i}(s_{z_1+i}, z_1 - i) + B_{z_2+j}(z_2 - j, s_{z_2+j-1}) + \sum_{k=z_1+i+1}^{z_2+j-1} B_k(s_k, s_{k-1})} \prod_{k=z_1+i}^{z_2+j-1} \mathrm{d}s_k \Bigg\}$$
$$\times \left(\mathbf{1}_{\{s_{z_2+j} > z_2 - j\}} e^{B_{z_2+j}(s_{z_2+j}, z_2 - j) + \sum_{k=z_2+j+1}^{n} B_k(s_k, s_{k-1})} \prod_{k=z_2+j}^{n-1} \mathrm{d}s_k \right) \tag{C.8}$$

so that,

$$\sum_{|i|,|j|\le a} Z_{(0,0),(n,n)}[s_{z_1+i} > z_1 - i, s_{z_1+i-1} < z_1 - i, s_{z_2+j} > z_2 - j, s_{z_2+j-1} < z_2 - j, \mathcal{A}]$$
$$= \sum_{|i|,|j|\le a} Z_{(0,0),(z_1-i,z_1+i)} Z_{(z_1-i,z_1+i),(z_2-j,z_2+j)}[\mathcal{A}] Z_{(z_2-j,z_2+j),(n,n)} \tag{C.9}$$

where the extra terms on the RHS that correspond to terms we argued were zero on the LHS are automatically 0 because $Z_{(z_1-i,z_1+i),(z_2-j,z_2+j)}[\mathcal{A}] = 0$ if either $z_1 - i \ge z_0 - b$ or $z_2 + j \le z_0 + b$.

For the terms on the line (C.4), the set

$$\{z_1 - i - 1 < s_{z_1+i} < z_1 - i, z_2 - j - 1 < s_{z_2+j} < z_2 - j, s_{z_0+b} < z_0 - b\} \tag{C.10}$$

is empty unless $z_1 - i \le z_0 - b$ and $z_2 + j > z_0 + b$. The first inequality implies $z_0 + b > z_1 + i$ since $b \ge a \ge i + 1$. For the non-zero terms on the line (C.4) we group the integrand as,

$$\Bigg\{ \left(\mathbf{1}_{\{s_{z_1+i-1} < s_{z_1+i}\}} e^{\sum_{k=0}^{z_1+i} B_k(s_k, s_{k-1})} \prod_{k=0}^{z_1+i-1} ds_k \right)$$
$$\times \left(\mathbf{1}_{\{s_{z_2+j+1} > s_{z_2+j}\}} e^{\sum_{k=z_2+j+1}^{n} B_k(s_k, s_{k-1})} \prod_{k=z_2+j+1}^{n-1} ds_k \right)$$
$$\times \left(\mathbf{1}_{\{s_{z_1+i+1} > s_{z_1+i}\}} \mathbf{1}_{\{s_{z_0+b} < z_0 + b\}} \mathbf{1}_{\{s_{z_2+j-1} < s_{z_2+j}\}} e^{\sum_{k=z_1+i+1}^{z_2+j} B_k(s_k, s_{k-1})} \prod_{k=z_1+i+1}^{z_2+j-1} ds_k \right) \Bigg\}$$
$$\mathbf{1}_{\{z_1-i-1 < s_{z_1+i} < z_1-i\}} \mathbf{1}_{\{z_2-j-1 < s_{z_2+j} < z_2-j\}} ds_{z_1+i} ds_{z_2+j}. \tag{C.11}$$

Therefore, the sum on the second line (C.4) equals,

$$\sum_{a \le i,j \le a-1} \int \Big\{ Z_{(0,0),(s_{z_1+i}, z_1+i)} Z_{(s_{z_1+i}, z_1+i+1),(s_{z_2+j}, z_2+j)}[\mathcal{A}] Z_{(s_{z_2+j}, z_2+j+1),(n,n)}$$
$$\times \mathbf{1}_{\{z_1-i-1 < s_{z_1+i} < z_1-i\}} \mathbf{1}_{\{z_2-j-1 < s_{z_2+j} < z_2-j\}} \Big\} ds_{z_1+i} ds_{z_2+j}, \tag{C.12}$$

where again, terms that were zero on the line (C.4) are also zero above. The remaining lines (C.5) and (C.6) are handled via highly similar arguments which are omitted. □

C.2 Proof of (6.4)

Let us label $L_i := \{(z_i, z_i) - (m, -m) : |m| \le a\}$. The event that the polymer path intersects the line L_i can be written as the disjoint union of the events (up to sets of Lebesgue measure 0 which do not contribute to the partition function),

$$\left(\bigsqcup_{|m| \le a} \{s_{z_i+m} > z_i - m, s_{z_i-m-1} < z_i - m\} \right)$$
$$\bigsqcup \left(\bigsqcup_{-a \le m \le a-1} \{z_i - m - 1 < s_{z_i+m} < z_i - m\} \right)$$
$$=: \left(\bigsqcup_{|m| \le a} \mathcal{A}_{m,i}^{(0)} \right) \bigsqcup \left(\bigsqcup_{-a \le m \le a-1} \mathcal{A}_{m,i}^{(1)} \right) =: \mathcal{A}_i^{(0)} \sqcup \mathcal{A}_i^{(1)}. \tag{C.13}$$

Therefore,

$$Z_{(0,0),(n,n)}[\mathcal{A}] = \sum_{\sigma \in \{0,1\}^{k-1}} Z_{(0,0),(n,n)} \left[\bigcap_{i=1}^{k-1} \mathcal{A}_i^{(\sigma_i)} \right]. \tag{C.14}$$

A single term in the summand on the RHS can be written,

$$Z_{(0,0),(n,n)}\left[\bigcap_{i=1}^{k-1}\mathcal{A}_i^{(\sigma_i)}\right]=\sum_{\substack{m_1,\dots m_k\\ -a\le m_i\le a-\sigma_i,\forall i}} Z_{(0,0),(n,n)}\left[\bigcap_{i=1}^{k-1}\mathcal{A}_{m_i,i}^{(\sigma_i)}\right]. \tag{C.15}$$

Each term on the RHS is non-zero only if $z_i+m_i\ge z_{i-1}+m_{i-1}+\sigma_{i-1}$ for $i=1,k$ with the convention $z_0+m_0+\sigma_0=0$ and $z_k+m_k+\sigma_k=n$. For such terms we have the decomposition,

$$Z_{(0,0),(n,n)}\left[\bigcap_{i=1}^{k-1}\mathcal{A}_{m_i,i}^{(\sigma_i)}\right] \tag{C.16}$$

$$=\int Z_{(0,0),(x_1,y_1)}Z_{(x_{k-1},y_{k-1}+\sigma_{k-1}),(n,n)}\prod_{i=1}^{k-2}Z_{(x_i,y_i+\sigma_i),(x_{i+1},y_{i+1})}\prod_{i=1}^{k-1}\mathrm{d}\xi_i(x_i,y_i) \tag{C.17}$$

where ξ_i is a delta function at (z_i-m_i,z_i+m_i) if $\sigma_i=0$ and 1 d Lebesgue measure on the interval $\{(z_i-m_i,z_i+m_i)-(s,0):0<s<1\}$ if $\sigma_i=1$. Note that the above identity extends to the excluded case where the LHS is 0 as so is the RHS by inspection (recall our convention $Z_{p,q}=0$ if the coordinate-wise ordering $p\le q$ does not hold). □

C.3 Proof of (6.6)

On the complement of the event of (6.5) we have by (5.9) that for $u+(j-1)n^{-10}\le w\le u+jn^{-10}$ that

$$Z_{(s,m),(w,p)}\le \mathrm{e}Z_{(s,m),(u+jn^{-10},p)}. \tag{C.18}$$

Similarly,

$$Z_{(w,p+1),(t,q)}\le \mathrm{e}Z_{(u+(j-1)n^{-10},p+1),(t,q)}. \tag{C.19}$$

The claim then follows. □

C.4 Proof of (5.35)

In order to prove this inequality we use the representation (5.33). There are several components. First, we clearly have,

$$\int\int \tilde Z_{(0,0),p}\tilde Z_{p,q}[\mathcal{A}]\tilde Z_{q,(n,n)}\mathrm{d}\mu_1(p)\mathrm{d}\mu_1(q) \tag{C.20}$$

$$\le\left(\int \tilde Z_{(0,0),p}\mathrm{d}\mu_1(p)\right)\left(\int \tilde Z_{p,q}[\mathcal{A}]\mathrm{d}\mu_1(p)\mathrm{d}\mu_2(q)\right)\left(\int \tilde Z_{q,(n,n)}\mathrm{d}\mu_2(q)\right) \tag{C.21}$$

since the measures μ_i are simply sums of delta functions. We now consider the second term of (5.33), the integral against $\mathrm{d}\nu_1\mathrm{d}\nu_2$. The event,

$$\mathcal{B}:=\bigcap_{k=-10n}^{10n}\left\{\sup_{|s_1|,|s_2|\le 10,i=1,2}|B_{z_i-k}(z_i-k+s_1)-B_{z_i-k}(z_i-k+s_2)|\le \delta b^2n^{1/3}\right\} \tag{C.22}$$

holds with probability at least $1-Cn\mathrm{e}^{-c\delta^2b^4n^{2/3}}$. On this event we see from (5.9) that for $s\in(0,1)$ and $|m|\le n$,

$$Z_{(0,0),(z_1-m-s,z_1+m)}\le \mathrm{e}^{\delta b^2n^{1/3}}Z_{(0,0),(z_i-m,z_i+m)} \tag{C.23}$$

as well as

$$Z_{(z_2-m-s,z_2+m+1),(n,n)}\le \mathrm{e}^{\delta b^2n^{1/3}}Z_{(z_i-m-1,z_i+m+1),(n,n)}. \tag{C.24}$$

For $|i|, |j| \le b_{j-1}n^{2/3}$ and $t_1, t_2 \in (0,1)$ the term,

$$Z_{(z_1-i-t_1, z_1+i+1),(z_2-j-t_2, z_2+j)}[\mathcal{A}] \tag{C.25}$$

is 0 unless $z_1 - i \le z_0 - b_j n^{2/3}$ (since $\mathcal{A} = \{s_{z_0+b_j n^{2/3}} < z_0 - b_j n^{2/3}\}$), which implies $z_0 + b_j n^{2/3} > z_1 + i + 2$ (since $b_j > b_{j-1} + 10$ and $i \le b_{j-1}n^{2/3}$). This term is also 0 unless $z_1 + j > z_0 + b_j n^{2/3}$. We then have the representation,

$$\begin{aligned}
&Z_{(z_1-i-t_1, z_1+i+1),(z_2-j-t_2, z_2+j)}[\mathcal{A}] \\
&= \int_{z_1-i-t_1 < s_{z_1+i+1} < s_{z_2+j-1} < z_2-j-t_2} \Big\{ Z_{(s_{z_1+i+1}, z_1+i+2),(s_{z_2+j-1}, z_2+j-1)}[\mathcal{A}_1] \mathbf{1}_{\{s_{z_2+j-1} \in \mathcal{A}_2\}} \\
&\times \mathrm{e}^{B_{z_1+i+1}(s_{z_1+i+1}, z_1-i-t_1) + B_{z_2+j}(z_2-j-t_2, s_{z_2+j-1})} \Big\} \mathrm{d}s_{z_1+1+i} \mathrm{d}s_{z_2+j-1}
\end{aligned} \tag{C.26}$$

where if $z_2 + j - 1 = z_0 + b_j n^{2/3}$ we have $\mathcal{A}_1$ is the whole polymer space and $\mathcal{A}_2 = (-\infty, z_0 - b_j n^{2/3})$ and otherwise $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{A}_2 = \mathbb{R}$. From this representation we conclude that on the event $\mathcal{B}$ that,

$$Z_{(z_1-i-t_1, z_1+i+1),(z_2-j-t_2, z_2+j)}[\mathcal{A}] \le \mathrm{e}^{2\delta b^2 n^{1/3}} Z_{(z_1-i-1, z_1+i+1),(z_2-j, z_2+j)}[\mathcal{A}] \tag{C.27}$$

in a similar manner to the proof of Lemma 5.2. By combining (C.23), (C.24) and (C.27) we see that the term on the second and third lines of (5.33) is bounded by

$$C\mathrm{e}^{4\delta b^2 n^{2/3}} \int\int \tilde{Z}_{(0,0),p} \tilde{Z}_{p,q}[\mathcal{A}] \tilde{Z}_{q,(n,n)} \mathrm{d}\mu_1(p) \mathrm{d}\mu_2(q). \tag{C.28}$$

The other terms of (5.33) can be estimated in a similar fashion. The claim follows after sending $\delta = c\delta$, for some $c > 0$. □

C.5 Proof of the equality in (3.4)

For simplicity of notation let us denote,

$$\begin{aligned}
W = W(s_1, \dots, s_{n-1}) &:= \mathrm{e}^{\sum_{k=1}^n B_k(s_k) - B_k(s_{k-1})} \\
\Delta_{t,n} &:= \{s_0, \dots, s_{n-1} : -\infty < s_0 < \cdots < s_{n-1} < t\}
\end{aligned} \tag{C.29}$$

recalling our convention $s_n = t$. Then,

$$\begin{aligned}
\partial_y Q_{t,n}^{(x,y)}[s_0 > 0] &= \partial_y \frac{\int_{\Delta_{t,n}} \mathbf{1}_{\{s_0>0\}} W \mathrm{e}^{B_0(s_0) - x(s_0)_- + y(s_0)_+} \mathrm{d}s_0 \dots \mathrm{d}s_{n-1}}{Z_{t,n}^{(x,y)}} \\
&= \frac{\int_{\Delta_{t,n}} (s_0)_+ \mathbf{1}_{\{s_0>0\}} W \mathrm{e}^{B_0(s_0) - x(s_0)_- + y(s_0)_+} \mathrm{d}s_0 \dots \mathrm{d}s_{n-1}}{Z_{t,n}^{(x,y)}} \\
&\quad - Q^{(x,y)}[s_0 > 0] \frac{\partial_y Z_{t,n}^{(x,y)}}{Z_{t,n}^{(x,y)}}.
\end{aligned} \tag{C.30}$$

If $\langle \cdot \rangle_{t,n}^{(x,y)}$ denotes expectation wrt the Gibbs measure $Q_{t,n}^{(x,y)}$, then the term on the second line equals $\langle \mathbf{1}_{\{s_0>0\}} (s_0)_+ \rangle_{t,n}^{(x,y)}$. For the term on the last line we have,

$$\partial_y Z_{t,n}^{(x,y)} = \int_{\Delta_{t,n}} (s_0)_+ W \mathrm{e}^{B_0(s_0) - x(s_0)_- + y(s_0)_+} \mathrm{d}s_0 \dots \mathrm{d}s_{n-1} \tag{C.31}$$

and so the last line of (C.30) equals $-\langle \mathbf{1}_{\{s_0>0\}} \rangle_{n,t}^{(x,y)} \langle (s_0)_+ \rangle_{n,t}^{(x,y)}$. This yields the equality in (3.4). □

C.6 Proof of inequality (6.8)

Let ν be a measure of the form

$$\nu = a \sum_{x \in I} \delta_x \tag{C.32}$$

where I is a finite set of points and $a > 0$. Then if f and g are non-negative we have,

$$\begin{aligned} \int f(x)g(x)\mathrm{d}\nu(x) &= a \sum_{x \in I} f(x)g(x) \\ &\le a \left(\sum_{x \in I} f(x) \right) \left(\sum_{x \in I} g(x) \right) = a^{-1} \left(\int f(x)\mathrm{d}\nu(x) \right) \left(\int g(x)\mathrm{d}\nu(x) \right). \end{aligned} \tag{C.33}$$

The inequality (6.8) follows from this. □

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