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SPDEs driven by standard symmetric α-stable cylindrical Lévy processes: Existence, Lyapunov functionals and Itô formula

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Abstract

We investigate several aspects of solutions to stochastic evolution equations in Hilbert spaces driven by a standard symmetric α-stable cylindrical noise. Similarly to cylindrical Brownian motion or Gaussian white noise, standard symmetric α-stable noise exists only in a generalised sense in Hilbert spaces. The main results of this work are the existence of a mild solution, long-term regularity of the solutions via Lyapunov functional approach, and an Itô formula for mild solutions to evolution equations under consideration. The main tools for establishing these results are Yosida approximations and an Itô formula for Hilbert space-valued semi-martingales where the martingale part is represented as an integral driven by cylindrical α-stable noise. While these tools are standard in stochastic analysis, due to the cylindrical nature of our noise, their application requires completely novel arguments and techniques.

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1 Introduction

Standard symmetric α-stable distributions are the natural generalisations of Gaussian distributions for modelling random perturbations of finite dimensional dynamical systems. They often meet various empirical requests, such as heavy tails, self-similarity and infinite variance, but are at the same time analytically tractable and well-understood. The importance of these models is reflected by the available vast literature on dynamical

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systems perturbed by random noises with α-stable distributions in various areas such as economics, biology etc.

In the infinite dimensional setting of modelling random perturbations of partial differential equations, much fewer results are known for systems perturbed by α-stable distributions. In fact, only in the random field approach, based on the seminal work by Walsh, one can find several publications on stochastic partial differential equations (SPDEs) driven by multiplicative α-stable noise, e.g. Mueller [27], Mytnik [28], and more recently Chong [7] and Chong et. al. [8]. However, in the semigroup approach, following the spirit of Da Prato and Zabczyk, one can find several results for equations only with additive driving noise distributed according to an α-stable law; see e.g. Brzeźniak and Zabczyk [6] and Riedle [34]. The only publication in the semigroup approach for multiplicative α-stable perturbation is Kosmala and Riedle [22], where however the assumptions are rather restrictive and do not correspond to the natural Lipschitz continuity and linear growth conditions. The lack of results in the semigroup approach is due to the fact that a random noise with a standard symmetric α-stable distribution does not exist as an ordinary Hilbert space-valued process but only in the generalised sense of Gel'fand and Vilenkin [13] or Segal [38].

In this work, we investigate several aspects of solutions to equations of the form

$$\mathrm{d}X(t) = \big(AX(t) + F(X(t))\big)\,\mathrm{d}t + G(X(t-))\,\mathrm{d}L(t), \tag{1.1}$$

where A is the generator of a C_0-semigroup in a separable Hilbert space H, the coefficients $F\colon H \to H$ and $G\colon H \to \mathcal{L}_2(U,H)$ are mappings with U being a separable Hilbert space, and L is a standard symmetric α-stable cylindrical process in U for $\alpha \in (1,2)$.

Analogously to the standard normal distribution, standard symmetric α-stable distributions in $\mathbb{R}^d$ can only be generalised to infinite dimensional spaces as cylindrical distributions. In particular, this means that the driving noise L in (1.1) exists only in the generalised sense; see Schwartz [37]. Since such processes do not attain values in the underlying Hilbert space, standard results for stochastic processes in infinite dimensional spaces are not applicable. Most notably, complications arise from the fact that while these processes are cylindrical semi-martingales, see Jakubowski et. al. [17], they do not enjoy a semi-martingale decomposition in a cylindrical sense, since semi-martingale decompositions are not invariant under linear transformations, see Jakubowski and Riedle [18, Re. 2.2]. Nevertheless, the problem of stochastic integration with respect to cylindrical Lévy processes was solved in Jakubowski and Riedle [18] by arguments avoiding the usual Lévy-Itô decomposition. This approach has been further developed for standard symmetric α-stable cylindrical process by two of us in Bodó and Riedle [5], which enables us to integrate predictable integrands and to derive a dominated convergence theorem for stochastic integrals.

This work comprises of 3 main results: the existence of a mild solution to Equation (1.1), a Lypunov functional approach for long-term regularity for solutions to Equation (1.1), and an Itô formula for mild solutions to Equation (1.1). The main tools for establishing these results are an Itô formula for Hilbert space-valued semi-martingales driven by standard symmetric α-stable cylindrical Lévy noise and a Yosida approximation of solutions to Equation (1.1). While these tools are standard in stochastic analysis, due to the cylindrical nature of our noise, their application in our setting requires completely novel arguments and techniques, which we highlight in the following.

A classical Itô formula for semi-martingales in Hilbert spaces is well known and easy to derive; see e.g. Metivier [26, Th. 27.2]. However, applying this formula often requires the identification of the martingale and bounded variation components of the process, which in the classical situation of a semi-martingale driven by an ordinary Hilbert space-valued process can easily be obtained via the semi-martingale decomposition of the driving process. Since in our case, the driving cylindrical process does not enjoy a semi-martingale decomposition, one needs to identify the martingale part of the stochastic integral process by carrying out a deep analysis of its jump structure.

The second major tool in our work is a Yosida approximation, which is an often-utilised device in the classical situation with an ordinary Hilbert space-valued process as driving noise; see e.g. Peszat and Zabczyk [31]. Convergence of the Yosida approximation is established by tightness arguments in the space $\mathcal{C}([0,T],L^p(\Omega,H))$ of p-th mean continuous Hilbert space-valued processes for any $p<\alpha$. It turns out that the space $\mathcal{C}([0,T],L^p(\Omega,H))$ is tailor-made for analysing equations driven by a standard symmetric α-stable cylindrical process. The observation that the solution is continuous in the above sense, despite having discontinuous paths, lies at the heart of this paper. To the best of our knowledge, we are the first to use this in the context of SPDEs driven by cylindrical stable noise.

These two tools, the Itô formula for semi-martingales driven by a standard symmetric α-stable cylindrical process and convergence of the Yosida approximation, enable us to establish the 3 main results of our work. For the existence result, we use tightness of the Yosida approximation to establish existence of a mild solution to Equation (1.1). In our setting, standard methods for establishing existence of a solution, such as fix point arguments or Grönwall's lemma are not applicable, since the integral operator with a standard symmetric α-stable integrator maps to a larger space than its domain; see Kosmala and Riedle [22] or Rosinski and Woyczynski [35].

By following the classical approach of Ichikawa in [15], we demonstrate the power of the established tools by investigating the long-term regularity of the mild solution to Equation (1.1) via the functional Lyapunov approach. The functional Lyapunov approach can be used to establish various regularity properties; in this work, we focus on exponential ultimate boundedness, but other quantitative properties can be investigated similarly. As the mild solution is not a semi-martingale, the derived Itô formula for semi-martingales cannot be applied directly. However, we successfully show that the Yosida approximations are semi-martingales, and thus the Itô formula can be applied to these, which immediately shows their exponential ultimate boundedness. It remains only to show that this boundedness property carries over to the limit, for which we establish convergence of the Markov generators in a suitable sense.

Mild solutions of SPDEs are not semi-martingales, and thus the classical Itô formula cannot be applied. This lack of a powerful tool is often circumvented by a specific Itô formula for mild solutions of SPDEs. One of the first versions of such an Itô formula for mild solutions can be found in Ichikawa [15] for the Gaussian case, and more recent versions in Da Prato et. al. [9] for the Gaussian case and in Alberverio et. al. [1] for the case of ordinary Lévy processes. In the last part of our work, we derive such an Itô formula for mild solutions of equation (1.1) driven by a standard symmetric α-stable cylindrical process.

We outline the structure of the paper. Selected preliminaries on standard symmetric α-stable cylindrical processes, integration with respect to them and underlying results on equations as well as the theory of predictable compensators are collected in Section 2. In Sections 3 and 4, we identify the predictable compensator and quadratic variation of the integral process. These observations lead us directly to the Itô formula for semi-martingales driven by a standard symmetric α-stable cylindrical process in Section 5. In Section 6, we prove existence of a mild solution under Lipschitz and boundedness conditions in the space of continuous functions, where the main result is formulated in Theorem 6.6. In Section 7, we establish conditions for exponential ultimate boundedness in Theorem 7.1. Finally, in Section 8, an Itô formula for mild solutions is proved.

2 Preliminaries

2.1 Standard symmetric α-stable cylindrical Lévy processes

Let U and H be separable Hilbert spaces with norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$. By $\overline{B}_H(r)$ we denote the closed ball in H with radius $r>0$ and, in the special case when $r=1$, we write $\overline{B}_H := \overline{B}_H(1)$. The space of Hilbert-Schmidt operators $\Phi\colon U\to H$ is denoted by $\mathcal{L}_2(U,H)$ and equipped with the norm $\|\cdot\|_{\mathcal{L}_2(U,H)}$.

Let S be a subset of U. For each $n\in\mathbb{N}$, elements $u_1,...,u_n\in S$ and Borel set $A\in\mathcal{B}(\mathbb{R}^n)$, we define

$$C(u_1,...,u_n;A) := \{u\in U : (\langle u,u_1\rangle,...,\langle u,u_n\rangle)\in A\}.$$

Such sets are called cylindrical sets with respect to S and the collection of all such cylindrical sets is denoted by $\mathcal{Z}(U,S)$. It is a σ-algebra if S is finite and otherwise an algebra. We write shortly $\mathcal{Z}(U)$ for $\mathcal{Z}(U,U)$.

A set function $\mu:\mathcal{Z}(U)\to[0,\infty]$ is called a cylindrical measure on $\mathcal{Z}(U)$ if for each finite subset $S\subseteq U$, the restriction of μ to the σ-algebra $\mathcal{Z}(U,S)$ is a σ-additive measure. A cylindrical measure is said to be a cylindrical probability measure if $\mu(U)=1$.

Let (Ω,Σ,P) be a complete probability space. We will denote by $L^0_P(\Omega,U)$ the space of equivalence classes of measurable functions $Y:\Omega\to U$ equipped with the topology of convergence in probability. A cylindrical random variable X in U is a linear and continuous mapping $X\colon U\to L^0_P(\Omega,\mathbb{R})$. It defines a cylindrical probability measure μ_X by

$$\mu_X\colon \mathcal{Z}(U)\to[0,1], \qquad \mu_X(Z) = P\big((Xu_1,\dots,Xu_n)\in A\big),$$

for cylindrical sets $Z=C(u_1,...,u_n;A)$. The cylindrical probability measure μ_X is called the *cylindrical distribution* of X. We define the characteristic function of the cylindrical random variable X by

$$\varphi_X\colon U\to\mathbb{C}, \qquad \varphi_X(u) = E\big[e^{iXu}\big].$$

Let $T\colon U\to H$ be a linear and continuous operator. By defining

$$TX\colon H\to L^0_P(\Omega,\mathbb{R}), \qquad (TX)h = X(T^*h),$$

we obtain a cylindrical random variable on H. In the special case when T is a Hilbert-Schmidt operator and hence 0-Radonifying by [40, Th. VI.5.2], it follows from [40, Pr. VI.5.3] that the cylindrical random variable TX is induced by a genuine random variable $Y:\Omega\to H$, that is $(TX)h=\langle Y,h\rangle$ for all $h\in H$.

A family $(L(t):t\geq 0)$ of cylindrical random variables $L(t)\colon U\to L^0_P(\Omega,\mathbb{R})$ is called a cylindrical $(\mathcal{F}_t)$-Lévy process if for each $n\in\mathbb{N}$ and $u_1,...,u_n\in U$, the stochastic process $\big((L(t)u_1,...,L(t)u_n\big) : t\geq 0\big)$ is an $(\mathcal{F}_t)$-Lévy process in $\mathbb{R}^n$ and the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions. We denote by $\mathcal{Z}_*(U)$ the collection

$$\big\{\{u\in U : (\langle u,u_1\rangle,...,\langle u,u_n\rangle)\in B\} : n\in\mathbb{N}, u_1,...,u_n\in U, B\in\mathcal{B}(\mathbb{R}^n\setminus\{0\})\big\}$$

of cylindrical sets, which forms an algebra of subsets of U. For fixed $u_1,...,u_n\in U$, let $\lambda_{u_1,...,u_n}$ be the Lévy measure of $\big((L(t)u_1,...,L(t)u_n) : t\geq 0\big)$. Define a function $\lambda\colon\mathcal{Z}_*(U)\to[0,\infty]$ by

$$\lambda(C) := \lambda_{u_1,...,u_n}(B) \quad \text{for} \quad C=\{u\in U : (\langle u,u_1\rangle,...,\langle u,u_n\rangle)\in B\},$$

for $B\in\mathcal{B}(\mathbb{R}^n)$. It is shown in [3] that λ is well defined. The set function λ is called the cylindrical Lévy measure of L.

In this paper, we restrict our attention to standard symmetric α-stable cylindrical Lévy processes for $\alpha \in (1,2)$, which we simply call α-stable cylindrical Lévy processes in the sequel. These are cylindrical Lévy processes with characteristic function $\phi_{L(t)}(u) = \exp(-t\,\|u\|^\alpha)$ for each $t \geq 0$ and $u \in U$. Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of U.

The Lévy measure λ of an α-stable cylindrical Lévy process for $\alpha \in (1,2)$ satisfies

$$\left(\lambda \circ \Phi^{-1}\right)(\overline{B}_H^c) \leq c_\alpha \,\|\Phi\|^\alpha_{\mathcal{L}_2(U,H)}\,, \quad \Phi \in \mathcal{L}_2(U,H) \tag{2.1}$$

for some $c_\alpha \in (0,\infty)$ depending only on α where $\overline{B}_H^c$ denotes the complement of the unit ball $\{h \in H : \|h\| < 1\}$; see [22, Le. 1]. This enables us to conclude the following technical Lemma:

Lemma 2.1. *Let λ be the cylindrical Lévy measure of an α-stable cylindrical Lévy process for $\alpha \in (1,2)$. For every $m \in \mathbb{N}$ there exists $d_\alpha^m < \infty$, depending only on α and m, such that*

$$\int_{\overline{B}_H(1/m)} ||h||^2 \,(\lambda\circ\Phi^{-1})(\mathrm{d}h) + \int_{\overline{B}_H(m)^c} ||h||\,(\lambda\circ\Phi^{-1})(\mathrm{d}h) \leq d_\alpha^m\,||\Phi||^\alpha_{\mathcal{L}_2(U,H)} \tag{2.2}$$

for all $\Phi \in \mathcal{L}_2(U,H)$. Moreover, we have $\lim_{m\to\infty} d_\alpha^m = 0$.

Proof. Let $m \in \mathbb{N}$ be fixed. We approximate the integrand of the first integral in (2.2) by

$$f_{m,n}\colon \overline{B}_H(\tfrac{1}{m}) \to \mathbb{R}, \qquad f_{m,n}(h) := \sum_{i=0}^{m2^n-1} \left(\frac{i}{m2^n}\right)^2 \mathbb{1}_{(\frac{i}{m2^n},\frac{i+1}{m2^n}]}(\|h\|).$$

Since $\lambda\circ\Phi^{-1}$ is a genuine α-stable measure in H, we have for each $r > 0$ that

$$(\lambda\circ\Phi^{-1})(\overline{B}_H^c(r)) = r^{-\alpha}(\lambda\circ\Phi^{-1})(\overline{B}_H^c); \tag{2.3}$$

see [23, Th. 6.2.7]. This enables us to conclude for each $n \in \mathbb{N}$ that

$$\begin{aligned}
&\int_{\overline{B}_H(1/m)} f_{m,n}(h)\,(\lambda\circ\Phi^{-1})(\mathrm{d}h)\\
&\qquad=\frac{1}{m^2}(\lambda\circ\Phi^{-1})(\overline{B}_H^c)\sum_{i=0}^{m2^n-1}\left(\frac{i}{2^n}\right)^2\left(\left(\frac{i}{m2^n}\right)^{-\alpha} - \left(\frac{i+1}{m2^n}\right)^{-\alpha}\right)\\
&\qquad=m^{\alpha-2}(\lambda\circ\Phi^{-1})(\overline{B}_H^c)\int_0^1\left(\sum_{i=0}^{m2^n-1}\left(\frac{i}{2^n}\right)^2\mathbb{1}_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(r)\right)\alpha r^{-(\alpha+1)}\,\mathrm{d}r.
\end{aligned}$$

The monotone convergence theorem implies

$$\lim_{n\to\infty}\int_{\overline{B}_H(1/m)} f_{m,n}(h)\,(\lambda\circ\Phi^{-1})(\mathrm{d}h) = m^{\alpha-2}\frac{\alpha}{2-\alpha}(\lambda\circ\Phi^{-1})(\overline{B}_H^c).$$

Since another application of the monotone convergence theorem shows

$$\lim_{n\to\infty}\int_{\overline{B}_H(1/m)} f_{m,n}(h)\,(\lambda\circ\Phi^{-1})(\mathrm{d}h) = \int_{\overline{B}_H(1/m)} \|h\|^2\,(\lambda\circ\Phi^{-1})(\mathrm{d}h),$$

we obtain from (2.1) that

$$\int_{\overline{B}_H(1/m)} \|h\|^2\,(\lambda\circ\Phi^{-1})(\mathrm{d}h) = m^{\alpha-2}\frac{\alpha}{2-\alpha}(\lambda\circ\Phi^{-1})(\overline{B}_H^c) \leq m^{\alpha-2}\frac{\alpha}{2-\alpha}c_\alpha\,\|\Phi\|^\alpha_{\mathcal{L}_2(U,H)}\,.$$

Applying similar arguments for the integrand in the second integral in (2.2) yields

$$\int_{\overline{B}_H(m)^c} ||h||\,(\nu\circ\Phi^{-1})(\mathrm{d}h) \leq m^{1-\alpha}\frac{\alpha}{\alpha-1}c_\alpha\,||\Phi||^\alpha_{\mathcal{L}_2(U,H)}\,,$$

for all $\Phi \in \mathcal{L}_2(U,H)$, which completes the proof as in (2.2) we can set

$$d_\alpha^m := (m^{\alpha-2} + m^{1-\alpha})c_\alpha \to 0, \quad m \to \infty. \tag{2.4}$$

□

If L is an α-stable cylindrical Lévy process and $T\colon H \to U$ a Hilbert-Schmidt operator, then the cylindrical random variable $TL(1)$ is induced by a genuine stable random variable on U with Lévy measure $\lambda \circ T^{-1}$. This Lévy measure depends continuously on T in the following way:

Lemma 2.2. *Let λ be the cylindrical Lévy measure of an α-stable cylindrical Lévy process for $\alpha \in (1,2)$. Then for each $r > 0$, the mapping $\Phi \mapsto \lambda \circ \Phi^{-1}|_{\overline{B}_H(r)^c}$ is continuous from $\mathcal{L}_2(U,H)$ to the space of Borel measures on $\overline{B}_H(r)^c$ equipped with the weak topology.*

Proof. If μ is a cylindrical probability measure and $(F_n)_{n\in\mathbb{N}}$ is a sequence converging to F in $\mathcal{L}_2(U,H)$ then $(\mu \circ F_n^{-1})_{n\in\mathbb{N}}$ converges weakly to $\mu \circ F^{-1}$ according to [5, Le. 2.1]. From this, the assertion follows from [29, Th. 5.5]. □

2.2 Stochastic integration

We briefly recall some facts on stochastic integration with respect to an α-stable cylindrical Lévy process L as introduced in [5]. A process $G\colon \Omega \times [0,T] \to \mathcal{L}_2(U,H)$ is called adapted and simple if it is of the form

$$G = \Phi_0 \mathbb{1}_{\{0\}} + \sum_{i=1}^{N} \Phi_i \mathbb{1}_{(t_{i-1},t_i]}, \tag{2.5}$$

where $N \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_N = T$ and Φ_i is an $\mathcal{F}_{t_{i-1}}$-measurable and $\mathcal{L}_2(U,H)$-valued random variable taking finitely many values. We denote by $\mathcal{S}_{\mathrm{adp}}^{\mathrm{HS}}$ the class of all adapted, simple processes. The integral process $\int_0^\cdot G\,\mathrm{d}L$ is defined as the sum of the Radonified increments

$$\int_0^t G\mathrm{d}L := \sum_{i=1}^{N} \Phi_i\big(L(t_i \wedge t) - L(t_{i-1}\wedge t)\big), \quad t \in [0,T]. \tag{2.6}$$

Here, $\Phi_i\big(L(t_i \wedge t) - L(t_{i-1}\wedge t)\big)$ is defined as the H-valued random variable satisfying

$$\big\langle \Phi_i\big(L(t_i \wedge t) - L(t_{i-1}\wedge t)\big), h\big\rangle = \big(L(t_i \wedge t) - L(t_{i-1}\wedge t)\big)(\Phi_i^* h) \qquad \text{for all } h \in H.$$

Let $\mathcal{S}_{\mathrm{adp}}^{1,\mathrm{op}}$ denote the class of adapted, simple $\mathcal{L}(H)$-valued processes bounded in the operator norm by 1 on $[0,T]$. An arbitrary predictable process $G\colon \Omega \times [0,T] \to \mathcal{L}_2(U,H)$ is stochastically integrable if there exists a sequence of adapted simple processes $(G_n)_{n\in\mathbb{N}} \subset \mathcal{S}_{\mathrm{adp}}^{\mathrm{HS}}$ such that:

(i) $(G_n)_{n\in\mathbb{N}}$ converges to G $P \otimes \mathrm{Leb}|_{[0,T]}$-almost everywhere,

(ii) $\displaystyle\lim_{m,n\to\infty} \sup_{\Gamma \in \mathcal{S}_{\mathrm{adp}}^{1,\mathrm{op}}} E\left[\left\|\int_0^T \Gamma(G_m - G_n)\,\mathrm{d}L\right\| \wedge 1\right] = 0$

In this case, $\int_0^\cdot G\,\mathrm{d}L$ is defined as the limit of $\int_0^\cdot G_n\,\mathrm{d}L$ in the topology of uniform convergence in probability on $[0,T]$.

It is shown in [5], that a predictable process G is stochastically integrable if and only it is an element of $L_P^0\big(\Omega, L_{\mathrm{Leb}}^\alpha\big([0,T], \mathcal{L}_2(G,H)\big)\big)$. It follows from [22, Co. 3] that for every $0 < p < \alpha$ we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left\|\int_0^t G\,\mathrm{d}L\right\|^p\right] \le e_{p,\alpha}\left(\mathbb{E}\left[\int_0^T \|G(t)\|_{\mathcal{L}_2(U,H)}^\alpha\,\mathrm{d}t\right]\right)^{p/\alpha}, \tag{2.7}$$

for every stochastically integrable predictable process G, where $e_{p,\alpha} = \frac{\alpha}{\alpha-p} e_{2,\alpha}^{p/\alpha}$ for some $e_{2,\alpha} \in (0,\infty)$ that depends only on α.

Lemma 2.3. *If G is a predictable stochastic process stochastically integrable with respect to the α-stable cylindrical Lévy process L for some $\alpha \in (1,2)$ then $\int_0^\cdot G \, \mathrm{d}L$ is a local martingale.*

Proof. Define the predictable stopping times $\tau_n = \inf \left\{ t > 0 : \int_0^t ||G(s)||^\alpha_{\mathcal{L}_2(U,H)} \, \mathrm{d}s > n \right\}$ for $n \in \mathbb{N}$. It follows from Proposition 4.22(ii) and Lemma 1.3 in [10] that for each $n \in \mathbb{N}$ there exists a sequence of adapted, simple processes $(G_{n,k})_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \mathbb{E}\left[\int_0^T \left|\left| G(s)\mathbb{1}_{[0,\tau_n]}(s) - G_{n,k}(s) \right|\right|^\alpha_{\mathcal{L}_2(U,H)} \, \mathrm{d}s \right] = 0. \tag{2.8}$$

Since inequality (2.7) guarantees for each $k, n \in \mathbb{N}$ that

$$\mathbb{E}\left[\sup_{0\le t\le T} \left|\left| \int_0^t G_{n,k} \, \mathrm{d}L \right|\right| \right] \le e_{1,\alpha} \left(\mathbb{E}\left[\int_0^T ||G_{n,k}(s)||^\alpha_{\mathcal{L}_2(U,H)} \, \mathrm{d}s \right] \right)^{1/\alpha} < \infty,$$

the same arguments as in [33, Th. I:51] show that the processes $\int_0^\cdot G_{n,k} \, \mathrm{d}L$ are martingales. Equation (2.8) shows that $\int_0^\cdot G\mathbb{1}_{[0,\tau_n]} \, \mathrm{d}L$ is a limit of martingales in $L^1(\Omega, H)$ by (2.7), and thus a martingale. Since standard arguments, e.g. [33, Th. I.12], establish

$$\left(\int_0^\cdot G \, \mathrm{d}L \right)^{\tau_n} = \int_0^\cdot G\mathbb{1}_{[0,\tau_n]} \, \mathrm{d}L \quad \text{a.s.,} \tag{2.9}$$

for the stopped integral process, the proof is completed. □

Theorem 2.4 (Stochastic Fubini Theorem)**.** *Let L be the standard symmetric α-stable cylindrical Lévy process for $\alpha \in (1,2)$. If $G\colon \Omega \times [0,T]^2 \to \mathcal{L}_2(U,H)$ is measurable, $G(t,\cdot)$ is predictable for every $t \in [0,T]$, and $\int_0^T \int_0^T \|G(t,s)\|^\alpha_{\mathcal{L}_2(U,H)} \, \mathrm{d}t \, \mathrm{d}s < \infty$ a.s. then it follows:*

(a) *$G(t,\cdot)$ is stochastically integrable for every $t \in [0,T]$ and $\int_0^T G(\cdot,s)\mathrm{d}L(s)$ is a.s. Bochner integrable;*

(b) *$G(\cdot,s)$ is a.s. Bochner integrable for every $s \in [0,T]$ and $\int_0^T G(t,\cdot) \, \mathrm{d}t$ is stochastically integrable;*

(c) $\displaystyle\int_0^T \left(\int_0^T G(t,s) \, \mathrm{d}t \right) \mathrm{d}L(s) = \int_0^T \left(\int_0^T G(t,s) \, \mathrm{d}L(s) \right) \mathrm{d}t$ *a.s.*

Proof. The proof is similar as in finite dimensions; see [43]. □

2.3 Random measures and compensators

In this section, we briefly recall some results on random measures and their compensators from [16, Ch. II].

Definition 2.5 (Random measure)**.** *A family $\mu = \{\mu(\omega) : \omega \in \Omega\}$ is called a random measure on $[0,T] \times H$ if $\mu(\omega)$ is a measure on $\mathcal{B}([0,T]) \otimes \mathcal{B}(H)$ for each $\omega \in \Omega$. It is said to be an integer-valued random measure if moreover, we have*

(i) $\mu(\{t\} \times H) \le 1$ *for all $t \in [0,T]$ P-a.s.;*

(ii) *μ takes values in $\mathbb{N} \cup \{\infty\}$ P-a.s.*

We denote by $\tilde{\mathcal{P}}$ (resp. $\tilde{\mathcal{O}}$) the *predictable* (resp. *optional*) σ-algebra on $\Omega \times [0,T] \times H$ and call a function $W : \Omega \times [0,T] \times H \mapsto \mathbb{R}$ *predictable* (resp. *optional*) if it is $\tilde{\mathcal{P}}$ (resp. $\tilde{\mathcal{O}}$) measurable.

If μ is a random measure and W is optional we define

$$\left(\int_0^t \int_H W(s,h)\,\mu(\mathrm{d}s,\mathrm{d}h)\right)(\omega) := \begin{cases} \int_0^t \int_H W(\omega,s,h)\,\mu(\omega)(\mathrm{d}s,\mathrm{d}h), & \text{if } \int_0^t \int_H |W(\omega,s,h)|\,\mu(\omega)(\mathrm{d}s,\mathrm{d}h) < \infty, \\ \infty, & \text{otherwise.} \end{cases}$$

A random measure μ is called *predictable* (resp. *optional*) if $(\int_0^t \int_H W(s,h)\,\mu(\mathrm{d}s,\mathrm{d}h) :\, t \in [0,T])$ is predictable (resp. optional) for every predictable (resp. optional) function W. An optional random measure μ is called σ-finite if there exists a sequence $(A_n)_{n\in\mathbb{N}} \subset \tilde{\mathcal{P}}$ with $\bigcup_{n=1}^\infty A_n = \Omega \times [0,T] \times H$, such that $\mathbb{E}\left[\int_0^T \int_H \mathbb{1}_{A_n}(s,h)\,\mu(\mathrm{d}s,\mathrm{d}h)\right] < \infty$ for each $n \in \mathbb{N}$.

For each σ-finite, optional measure μ on $[0,T] \times H$ there exists a predictable random measure ν on $\mathcal{B}([0,T]) \otimes \mathcal{B}(H)$ such that

$$\mathbb{E}\left[\int_0^t \int_H W(s,h)\,\mu(\mathrm{d}s,\mathrm{d}h)\right] = \mathbb{E}\left[\int_0^t \int_H W(s,h)\,\nu(\mathrm{d}s,\mathrm{d}h)\right] \tag{2.10}$$

for all $t \in [0,T]$, and any non-negative predictable function W. The measure ν is determined uniquely up to a set of probability zero by (2.10) and is called the *compensator* of μ; see [16, th. II.1.8].

If Y is an H-valued, adapted càdlàg process then the integer-valued random measure μ^Y characterised by

$$\mu^Y((0,t] \times B) = \sum_{0\le s\le t} \mathbb{1}_B(\Delta Y(s)), \quad t \in (0,T], B \in \mathcal{B}(H), 0 \notin B,$$

where $\Delta Y(s) := Y(s) - \lim_{h\searrow 0+} Y(s-h)$ for $s \in [0,T]$, is an optional and σ-finite random measure on $\mathcal{B}([0,T]) \otimes \mathcal{B}(H)$. Thus, its compensator exists which we denote by ν^Y.

Example 2.6. Let L be a genuine H-valued Lévy process with Lévy measure λ. Then the compensator ν^L of the jump measure μ^L is given as the extension of $\mu^L((s,t] \times B) = (t-s)\lambda(B)$, $0 \le s < t \le T$, $B \in \mathcal{B}(H)$ to $\mathcal{B}([0,T]) \otimes \mathcal{B}(H)$.

In the sequel, we will make use of another characterisation of compensators of jump-measures. We denote by $\mathcal{C}^+(H)$ the class of non-negative, continuous functions $k : H \to \mathbb{R}$ bounded on H and vanishing inside a neighbourhood of 0.

Proposition 2.7. *The compensator ν^Y of the jump-measure μ^Y of an H-valued càdlàg semimartingale Y is characterised by being predictable and satisfying either of the following:*

(i) *The process*

$$\left(\int_0^t \int_H k(h)\,\mu^Y(\mathrm{d}s,\mathrm{d}h) - \int_0^t \int_H k(h)\,\nu^Y(\mathrm{d}s,\mathrm{d}h) :\, t \in [0,T]\right)$$

is a local martingale for every $k \in \mathcal{C}^+(H)$.

(ii) *If W is predictable and the process*

$$\left(\int_0^t \int_H W(s,h)\,\mu^Y(\mathrm{d}s,\mathrm{d}h) :\, t \in [0,T]\right) \tag{2.11}$$

is locally integrable, then so is

$$\left(\int_0^t \int_H W(s,h)\,\nu^Y(\mathrm{d}s,\mathrm{d}h):\ t\in[0,T]\right)$$

and

$$\left(\int_0^t \int_H W(s,h)\,\mu^Y(\mathrm{d}s,\mathrm{d}h) - \int_0^t \int_H W(s,h)\,\nu^Y(\mathrm{d}s,\mathrm{d}h):\ t\in[0,T]\right)$$

is a local martingale.

Proof. The equivalence between (i) and (ii) follows by the same argument as in the proof of [16, Th. II.2.21.]. The fact that (ii) is an equivalent definition of the compensator is proved in [16, Th. II.1.8.]. □

Proposition 2.7 justifies the following standard notation: if W is predictable and (2.11) is locally integrable, we define the following local martingale

$$\int_0^t \int_H W(s,h)\,(\mu^Y-\nu^Y)(\mathrm{d}s,\mathrm{d}h) := \int_0^t \int_H W(s,h)\,\mu^Y(\mathrm{d}s,\mathrm{d}h) - \int_0^t \int_H W(s,h)\,\nu^Y(\mathrm{d}s,\mathrm{d}h)$$

for each $t\in[0,T]$.

3 Predictable compensator

For an α-stable cylindrical Lévy process L for some $\alpha\in(1,2)$ and a stochastically integrable predictable process G, we define the integral process $X=\int_0^\cdot G\mathrm{d}L$ and

$$\nu\left((0,t]\times B\right) := \int_0^t \left(\lambda\circ G(s)^{-1}\right)(B)\,\mathrm{d}s \qquad \text{for each } t\in(0,T], B\in\mathcal{B}(H) \text{ with } 0\notin\bar{B}. \tag{3.1}$$

The main result of this section is that ν extends to a random measure on $\mathcal{B}([0,T])\otimes\mathcal{B}(H)$ and that the extension is the predictable compensator of the jump measure of X. We will derive this result by a couple of Lemmata.

Lemma 3.1. *The set function ν defined in* (3.1) *is well defined and extends to a predictable random measure on $\mathcal{B}([0,T])\otimes\mathcal{B}(H)$. This extension is unique among the class of σ-finite random measures on $\mathcal{B}([0,T])\otimes\mathcal{B}(H)$ that assign* 0 *mass to the origin.*

Proof. Step 1: We show that for all open sets $B\subseteq H$ with $0\notin\bar{B}$ the process

$$f\colon\Omega\times[0,T]\to\mathbb{R},\quad f(\omega,t)=\left(\lambda\circ G(\omega,t)^{-1}\right)(B)$$

is predictable. Since the function $h\colon\mathcal{L}_2(U,H)\to\mathbb{R}$ defined by $h(\Phi)=(\lambda\circ\Phi^{-1})(B)$ is lower semicontinuous by Lemma 2.2 and the Portmonteau Theorem as the set B assumed to be open, h is measurable. Since $G\colon\Omega\times[0,T]\to\mathcal{L}_2(U,H)$ is predictable, it follows that $f=h\circ G$ is predictable.
Step 2: We show that f is predictable for all $B\in\mathcal{B}(H\setminus\{0\})$, which will immediately imply that (3.1) is almost surely well defined and predictable as it is then just an integral of a non-negative predictable process. We define

$$\mathcal{D}=\left\{B\in\mathcal{B}(H\setminus\{0\}): \lambda\circ G(\cdot,\cdot)^{-1}(B)\ \text{is predictable}\right\},$$

and claim that $\mathcal{D}$ is a λ-system. Continuity of measures implies that $H\setminus\{0\}\in\mathcal{D}$ since, for all $t\in(0,T]$ and $\omega\in\Omega$, we have

$$\left(\lambda\circ G(\omega,t)^{-1}\right)(H\setminus\{0\}) = \lim_{n\to\infty}\left(\lambda\circ G(\omega,t)^{-1}\right)\left(\bar{B}_H\left(1/n\right)^c\right),$$

where the right hand side is the limit of processes that are predictable by Step 1. If $B \in \mathcal{D}$ then $B^c \in \mathcal{D}$ since

$$\left(\lambda \circ G(\omega,t)^{-1}\right)(B^c) = \left(\lambda \circ G(\omega,t)^{-1}\right)(H \setminus \{0\}) - \left(\lambda \circ G(\omega,t)^{-1}\right)(B).$$

The collection $\mathcal{D}$ is closed under union of increasing sequences, which follows as above from continuity of measures and predictability of the pointwise limit. This concludes the proof of the claim that $\mathcal{D}$ is a λ-system.

We define the π-system.

$$\mathcal{I} = \{B \in \mathcal{B}(H \setminus \{0\}) : B \text{ is open}\}.$$

The family $\mathcal{I}$ is contained in $\mathcal{D}$, since for each $B \in \mathcal{D}$ we have

$$\left(\lambda \circ G(\omega,t)^{-1}\right)(B) = \lim_{n\to\infty} \left(\lambda \circ G(\omega,t)^{-1}\right)\left(B \cap \bar{B}_H(1/n)^c\right),$$

and the right-hand side is predictable by Step 1. The Dynkin π-λ theorem for sets, see e.g. [19, Th. 1.1] implies $\sigma(\mathcal{I}) \subseteq \mathcal{D}$, and thus $\mathcal{D} = \mathcal{B}(H \setminus \{0\})$.

Step 3: Let $\omega \in \Omega$ be such that $\int_0^T \|G(\omega,s)\|_{\mathcal{L}_2(U,H)}^\alpha \, \mathrm{d}s < \infty$. Equation (3.1) defines the set function $\nu(\omega)$ on the semi-ring

$$\mathcal{S} = \{(0,t] \times B : t \in [0,T] \text{ and } B \in \mathcal{B}(H \setminus \{0\})\}.$$

The set function $\nu(\omega)$ is σ-additive by its very definition and σ-finite, since for $n \in \mathbb{N}$ we have by (2.1) and (2.3) that

$$\begin{aligned}
\nu(\omega)\left((0,T] \times \overline{B}_H^c(1/n)\right) &= \int_0^T \left(\lambda \circ G(\omega,s)^{-1}\right)\left(\overline{B}_H^c(1/n)\right) \mathrm{d}s \\
&= n^\alpha \int_0^T \left(\lambda \circ G(\omega,s)^{-1}\right)\left(\overline{B}_H^c\right) \mathrm{d}s \\
&\leq n^\alpha c_\alpha \int_0^T \|G(\omega,s)\|_{\mathcal{L}_2(U,H)}^\alpha \, \mathrm{d}s < \infty.
\end{aligned}$$

Carathéodory's extension theorem, see e.g. [19, Th. 2.5], implies that the set function $\nu(\omega)$ extends uniquely to a measure on $\mathcal{B}([0,T]) \otimes \mathcal{B}(H \setminus \{0\})$ which we also denote by $\nu(\omega)$.

Step 4: It remains to show that ν is predictable. Applying the monotone class theorem as above shows that the process $\int_0^\cdot \left(\lambda \circ G(s)^{-1}\right)(B)\, \mathrm{d}s$ is predictable for each $B \in \mathcal{B}(H \setminus \{0\})$. Since

$$\int_0^\cdot \mathbb{1}_{(s,t]}(u)\mathbb{1}_A \left(\lambda \circ G(u)^{-1}\right)(B)\mathrm{d}u = \int_0^\cdot \left(\lambda \circ \left(\mathbb{1}_{(s,t]}(u)\mathbb{1}_A G(u)\right)^{-1}\right)(B)\mathrm{d}u,$$

it follows that the process $\int_0^\cdot \int_H W(u,h)\tilde{\nu}(\mathrm{d}u,\mathrm{d}h)(\cdot)$ is predictable for all functions $W = \mathbb{1}_{(s,t]}\mathbb{1}_A\mathbb{1}_B$ with $0 < s < t \leq T$, $A \in \mathcal{F}_s$ and $B \in \mathcal{B}(H \setminus \{0\})$. An application of the functional monotone class theorem (follows e.g. from [42, Th. 3.14]) extends this result to all predictable processes W on $\Omega \times [0,T] \times H$, which shows predictability of the random measure ν on $\mathcal{B}([0,T]) \otimes \mathcal{B}(H \setminus \{0\})$. Defining $\nu((s,t] \times \{0\}) := 0$ for any $0 \leq s < t \leq T$ extends ν to a predictable random measure on $\mathcal{B}([0,T]) \otimes \mathcal{B}(H)$. □

To show that the random measure ν characterised by (3.1) is the compensator of the jump-measure μ^X of the integral process X, we first consider the case when the integrand is an adapted, simple process.

Lemma 3.2. *Suppose that G is an adapted, simple process in $\mathcal{S}_{\mathrm{adp}}^{\mathrm{HS}}$. Then the random measure ν obtained in Lemma 3.1 is the predictable compensator of μ^X.*

Proof. Since Lemma 3.1 guarantees that ν is predictable, it remains to show (2.10), which by the functional monotone class theorem, e.g. [42, Th. 3.14], reduces to proving

$$\mathbb{E}\left[\mathbb{1}_A \sum_{s<u\le t} \mathbb{1}_B(\Delta X(u))\right] = \mathbb{E}\left[\mathbb{1}_A \int_s^t \left(\lambda \circ G(u)^{-1}\right)(B)\,\mathrm{d}u\right]$$

for any $0 < s < t \le T$, $A \in \mathcal{F}_s$ and $B \in \mathcal{B}(H)$ with $0 \notin \bar{B}$. Let G be of the form (2.5), and assume that the points of the partition contain s and t; otherwise these can be added. Then X takes the form (2.6), and it follows

$$\begin{aligned}\mathbb{1}_A \sum_{s<u\le t} \mathbb{1}_B(\Delta X(u)) &= \mathbb{1}_A \sum_{i=1}^N \sum_{s\le t_{i-1}<u\le t_i\le t} \mathbb{1}_B(\Delta G(t_i)L(u)) \\ &= \mathbb{1}_A \sum_{i=1}^N \sum_{s\le t_{i-1}<u\le t_i\le t} \mathbb{1}_B(\Delta \Phi_i L(u)).\end{aligned}$$

For each $i \in \{1,\dots,N\}$, the random variable Φ_i is of the form $\Phi_i = \sum_{j=1}^{m_i} \mathbb{1}_{A_{i,j}}\phi_{i,j}$ for some pairwise disjoint sets $A_{i,j} \in \mathcal{F}_{t_{i-1}}$ and $\phi_{i,j} \in \mathcal{L}_2(U,H)$ for $j \in \{1,\dots,m_i\}$. Since $0 \notin \bar{B}$, we have

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_A \sum_{t_{i-1}<u\le t_i} \mathbb{1}_B(\Delta \Phi_i L(u))\right] &= \sum_{j=1}^{m_i} \mathbb{E}\left[\mathbb{1}_{A\cap A_{i,j}} \sum_{t_{i-1}<u\le t_i} \mathbb{1}_B(\Delta \phi_{i,j} L(u))\right] \\ &= \sum_{j=1}^{m_i} (t_i - t_{i-1})\mathbb{E}\left[\mathbb{1}_{A\cap A_{i,j}} \left(\lambda \circ \phi_{i,j}^{-1}\right)(B)\right] \\ &= (t_i - t_{i-1})\mathbb{E}\left[\mathbb{1}_A \left(\lambda \circ \Phi_i^{-1}\right)(B)\right],\end{aligned}$$

because $A \cap A_{i,j} \in \mathcal{F}_{t_{i-1}}$ and the compensator of the jump measure of the Lévy process $\phi_{i,j}L$ in H is given by $\left(\lambda \circ \phi_{i,j}^{-1}\right)\,\mathrm{d}h\,\mathrm{d}t$ since its Lévy measure is $\left(\lambda \circ \phi_{i,j}^{-1}\right)$, see Example 2.6. □

Before we show that the result of Lemma 3.2 can be extended to general integrands, we need to prove some technical Lemmata. Recall the class of functions $\mathcal{C}^+(H)$ used in Proposition 2.7 (and defined just before) to determine the compensator.

Lemma 3.3. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of càdlàg functions $f_n\colon [0,T] \to H$ converging uniformly to $f\colon [0,T] \to H$. Then we have for any $k \in \mathcal{C}^+(H)$ that*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left| \sum_{0\le s\le t} k(\Delta f_n(s)) - \sum_{0\le s\le t} k(\Delta f(s)) \right| = 0. \tag{3.2}$$

Proof. Both sums in (3.2) are finite by the càdlàg property of f, f_n and since k vanishes inside a neighbourhood of 0. The assumed uniform convergence implies

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|\Delta f_n(t) - \Delta f(t)\| = 0. \tag{3.3}$$

Denoting $\mathrm{supp}(k) := \{h \in H : k(h) \neq 0\}$ and $\delta := \frac{1}{2}\mathrm{dist}(0, \mathrm{supp}(k))$, we obtain that $\mathrm{supp}(k)_\delta := \{h \in H, \mathrm{dist}(h, \mathrm{supp}(k)) < \delta\}$, is bounded away from zero, i.e. $0 \notin \overline{\mathrm{supp}(k)_\delta}$.

It follows that the set $D := \{t \in [0,T] : \Delta f(t) \in \text{supp}(k)_\delta\}$ is finite, which together with continuity of k and (3.3) implies

$$\lim_{n\to\infty} \sup_{t\in D} |k(\Delta f_n(t)) - k(\Delta f(t))| = 0. \tag{3.4}$$

Since (3.3) guarantees that there exists $n_0 \in \mathbb{N}$ such that we have $\Delta f_n(t) \notin \text{supp}(k)$ for all $n \geq n_0$ and $t \in [0,T] \setminus D$, we conclude from (3.4) for $n \geq n_0$ that

$$\begin{aligned}\sup_{t\in[0,T]} \left| \sum_{0\le s\le t} k(\Delta f_n(s)) - \sum_{0\le s\le t} k(\Delta f(s)) \right| &= \sup_{t\in[0,T]} \left| \sum_{s\in D\cap[0,t]} k(\Delta f_n(s)) - \sum_{s\in D\cap[0,t]} k(\Delta f(s)) \right| \\ &\le |D| \sup_{t\in D} |k(\Delta f_n(t)) - k(\Delta f(t))| \to 0, \quad n\to\infty.\end{aligned}$$

The proof is complete. □

Lemma 3.4. *Let $g_n, g \in L^\alpha_{\text{Leb}}([0,T], \mathcal{L}_2(U,H))$, $n \in \mathbb{N}$, be such that g_n converges to g in $L^\alpha_{\text{Leb}}([0,T], \mathcal{L}_2(U,H))$ and pointwise for almost every $s \in [0,T]$. Then we obtain for each $k \in \mathcal{C}^+(H)$ that*

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left| \int_0^t \int_H k(h) \, \left(\lambda \circ g_n(s)^{-1}\right) \, \mathrm{d}h \, \mathrm{d}s - \int_0^t \int_H k(h) \, \left(\lambda \circ g(s)^{-1}\right) \, \mathrm{d}h \, \mathrm{d}s \right| = 0.$$

Proof. Lemma 2.2 implies for almost each $s \in [0,T]$ and every $n \in \mathbb{N}$ that

$$\lim_{n\to\infty} \int_H k(h) \, \left(\lambda \circ g_n(s)^{-1}\right) \, \mathrm{d}h = \int_H k(h) \, \left(\lambda \circ g(s)^{-1}\right) \, \mathrm{d}h.$$

Since k is bounded and vanishes in a neighbourhood of 0, we conclude from inequality (2.1)

$$\int_H k(h) \, \left(\lambda \circ g_n(s)^{-1}\right) \, \mathrm{d}h \le c_{k,\alpha} \, ||g_n(s)||^\alpha_{\mathcal{L}_2(U,H)} \, ,$$

for a constant $c_{k,\alpha}$ independent of $s \in [0,T]$ and $n \in \mathbb{N}$. Since for each $t \in [0,T]$ we have

$$\lim_{n\to\infty} \int_0^t ||g_n(s)||^\alpha_{\mathcal{L}_2(U,H)} \, \mathrm{d}s = \int_0^t ||g(s)||^\alpha_{\mathcal{L}_2(U,H)} \, \mathrm{d}s,$$

the generalised Lebesgue's dominated convergence theorem, see e.g. [36, Th. 4.19], implies

$$\lim_{n\to\infty} \int_0^t \int_H k(h) \, \left(\lambda \circ g_n(s)^{-1}\right) \, \mathrm{d}h \, \mathrm{d}s = \int_0^t \int_H k(h) \, \left(\lambda \circ g(s)^{-1}\right) \, \mathrm{d}h \, \mathrm{d}s.$$

As the functions

$$t \mapsto \int_0^t \int_H k(h) \, \left(\lambda \circ g_n(s)^{-1}\right) \, \mathrm{d}h \, \mathrm{d}s$$

are continuous monotone and converge pointwise to a continuous limit on $[0,T]$, the convergence is uniform by [32, p. 81/127] (or deuxième théorème de Dini). □

Now we can prove the main result of this section.

Theorem 3.5. *Let L be an α-stable cylindrical Lévy process L for some $\alpha \in (1,2)$ and G a stochastically integrable predictable process. Then the predictable compensator ν^X of the jump measure μ^X of $X := \int_0^\cdot G \, \mathrm{d}L$ is characterised by* (3.1).

Proof. In light of Proposition 2.7, it suffices to show that the process M^k defined by

$$\left(M^k(t) := \int_0^t \int_H k(h)\,\mu^X(\mathrm{d}s,\mathrm{d}h) - \int_0^t \int_H k(h)\,\left(\lambda\circ G(s)^{-1}\right)\,\mathrm{d}h\,\mathrm{d}s,\; t\in[0,T]\right),$$

is a local martingale for any $k \in \mathcal{C}^+(H)$. Lemma 4.3 in [5] guarantees that there exists a sequence $(G_n)_{n\in\mathbb{N}}$ of adapted, simple processes in $\mathcal{S}_{\text{adp}}^{\text{HS}}$ converging both in $L^\alpha_{\text{Leb}}([0,T],L_2(U,H))$ a.s. and $P\otimes \text{Leb}|_{[0,T]}$ $-$ a.e. to G. Letting $X_n := \int_0^\cdot G_n \mathrm{d}L$ and denoting the jump-measure of X_n by μ^{X_n}, we define for each $k\in\mathcal{C}^+(H)$ and $n\in\mathbb{N}$ a process M_n^k by

$$\left(M_n^k(t) := \int_0^t \int_H k(h)\,\mu^{X_n}(\mathrm{d}s,\mathrm{d}h) - \int_0^t \int_H k(h)\,\left(\lambda\circ G_n(s)^{-1}\right)\,\mathrm{d}h\,\mathrm{d}s,\; t\in[0,T]\right).$$

Proposition 2.7 and Lemma 3.2 imply that M_n^k is a local martingale for all $n\in\mathbb{N}$. Since for each $n\in\mathbb{N}$ and $t\in[0,T]$ we have that $\mu^{X_n}(\{t\}\times H)\le 1$ almost surely, it follows that

$$\left\|\Delta\left(\int_0^t\int_H k(h)\,\mu^{X_n}(\mathrm{d}h,\mathrm{d}s)\right)\right\| \le \|k\|_\infty \quad \text{a.s.},$$

which shows $\left\|\Delta M_n^k(t)\right\| \le \|k\|_\infty$ a.s. for all $n\in\mathbb{N}$.

Almost sure uniform convergence of X_n and Lemma 3.3 guarantee that there exists an $\Omega_1\subseteq\Omega$ with $P(\Omega_1)=1$ such that, for all $\omega\in\Omega_1$, we have

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\left|\int_0^t\int_H k(h)\,\mu^{X_n(\omega)}(\mathrm{d}h,\mathrm{d}s) - \int_0^t\int_H k(h)\,\mu^{X(\omega)}(\mathrm{d}h,\mathrm{d}s)\right| = 0. \tag{3.5}$$

In the same way, by convergence of G_n both in $L^\alpha_{\text{Leb}}([0,T],L_2(U,H))$ a.s. and $P\otimes \text{Leb}|_{[0,T]}-\text{a.e.}$ and Lemma 3.4 there exists an $\Omega_2\subseteq\Omega$ with $P(\Omega_2)=1$ such that, for all $\omega\in\Omega_2$, we have

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\left|\int_0^t\int_H k(h)\,\left(\lambda\circ G_n(\omega,s)^{-1}\right)\,\mathrm{d}h\,\mathrm{d}s - \int_0^t\int_H k(h)\,\left(\lambda\circ G(\omega,s)^{-1}\right)\,\mathrm{d}h\,\mathrm{d}s\right| = 0. \tag{3.6}$$

Equations (3.5) and (3.6) show that M_n^k converges uniformly to M^k almost surely. As the jumps of M_n^k are a.s. uniformly bounded by $\|k\|_\infty$, we conclude from [16, Co. IX.1.19] that M^k is a local martingale and the proof is complete. □

4 Quadratic variation of the integral process

The quadratic covariation of two real-valued càdlàg semimartingales V_1 and V_2 starting from zero is the process $[V_1,V_2]$ defined by

$$[V_1,V_2](t) := V_1(t)V_2(t) - \int_0^t V_1(s-)\,\mathrm{d}V_2(s) - \int_0^t V_2(s-)\,\mathrm{d}V_1(s), \quad t\in[0,T].$$

When $V:=V_1=V_2$, we call the process $[V]:=[V,V]$ the quadratic variation of V. The continuous part of $[V]$ is defined by

$$[V]^c(t) = [V](t) - \sum_{0\le s\le t}(\Delta V(s))^2 \quad \text{for each } t\in[0,T]. \tag{4.1}$$

If $[V]^c = 0$ we say that V is purely discontinuous; see e.g. [33, Se. II.6].

The concept of quadratic variation is generalised for a càdlàg semimartingale Z with values in the separable Hilbert space H in [26, Se. 26]. Let $(f_i)_{i\in\mathbb{N}}$ denote an orthonormal basis of H. There exists a unique stochastic process $[[Z]]$ with values in the Hilbert-Schmidt tensor product of H satisfying

$$\langle [[Z]]\,, f_i \otimes f_j \rangle = [Z_i, Z_j] \quad \text{for all } i, j \in \mathbb{N},$$

where $\otimes$ denotes the tensor product and $Z_i(t) = \langle Z(t), f_i \rangle$ for $t \in [0,T]$ are the projection processes of Z; see [26, Se. 21.2] for brief introduction. The process $[[Z]]$ does not depend on the choice of the orthonormal basis $(f_i)_{i\in\mathbb{N}}$. The process $[[Z]]$ is called the tensor quadratic variation of Z and its continuous part $[[Z]]^c$ is defined by

$$\langle [[Z]]^c\,(t), f_i \otimes f_j \rangle = \langle [[Z]]\,(t), f_i \otimes f_j \rangle - \sum_{0\le s\le t} \Delta \left(Z^i(s) Z^j(s) \right) \quad \text{for all } t \in [0,T],\ i, j \in \mathbb{N}.$$

We say that Z is purely discontinuous if $[[Z]]^c = 0$.

Proposition 4.1. *Let L be an α-stable cylindrical Lévy process for some $\alpha \in (1,2)$ and G a stochastically integrable predictable process with values in $\mathcal{L}_2(U,H)$. Then the integral process $X := \int_0^\cdot G\,\mathrm{d}L$ is purely discontinuous.*

Proof. We proceed in three steps.
Step 1: Assume $H = \mathbb{R}$ and $U = \mathbb{R}^d$ for some $d \in \mathbb{N}$. In this case, L is a U-valued standard symmetric α-stable Lévy process, and therefore purely discontinuous; see e.g. [33, p. 71]. Pure discontinuity is preserved also for the integral process; see e.g. [16, Se. IX.5.5a] or [33, Th. II.29].
Step 2: Assume $H = \mathbb{R}$, but without any further restrictions on U. In that case, by the identification $U \simeq \mathcal{L}_2(U,\mathbb{R})$, the integrand G is a U-valued process satisfying

$$\int_0^T \|G(t)\|^\alpha \,\mathrm{d}t < \infty \quad \text{a.s.} \tag{4.2}$$

Fix an orthonormal basis $(f_k)_{k\in\mathbb{N}}$ in U and define for each $n \in \mathbb{N}$ the projection

$$\pi_n \colon U \to U, \qquad \pi_n(u) = \sum_{k=1}^n \langle u, f_k \rangle f_k.$$

Since the projection π_n is a Hilbert-Schmidt operator, there exists a U-valued Lévy process L_n with the property $\langle L_n, u \rangle = L(\pi_n^* u)$ for all $u \in U$. We define the approximations

$$X_n := \int_0^\cdot G \mathrm{d}L_n, \quad n \in \mathbb{N}.$$

Since L_n attains values in a finite-dimensional subspace and is a symmetric α-stable process by [34, Le. 2.4], it follows that X_n is purely discontinuous by *Step 1*.

Let M be a real-valued, continuous martingale and define for $k \in \mathbb{N}$ the stopping times

$$\tau_k = \inf \left\{ t > 0 : \int_0^t \|G(s)\|^\alpha \,\mathrm{d}s \ge k \right\} \wedge \inf \{ t > 0 : |M(t)| \ge k \} \wedge T.$$

It follows that $\tau_k \to T$ as $k \to \infty$ by (4.2). Since X_n is purely discontinuous, it follows from [16, Le. I.4.14] that $(X_n M)^{\tau_k}$ is a local martingale for each k, $n \in \mathbb{N}$. Since applying inequality (2.7) and equality (2.9) shows

$$\mathbb{E}\left[\sup_{0\le t\le T} \left| (X_n M)^{\tau_k}(t) \right| \right] = \mathbb{E}\left[\sup_{0\le t\le T} \left| M(t)^{\tau_k} \right| \left| \int_0^t \mathbb{1}_{[0,\tau_k]} G \,\mathrm{d}L_n \right| \right]$$

$$\le k\mathbb{E}\left[\int_0^{\tau_k} ||G(s)||^{\alpha}_{\mathcal{L}_2(U,H)}\,\mathrm{d}s\right] \le k^2 < \infty,$$

we obtain that $(X_nM)^{\tau_k}$ is a martingale by [33, Th. I:51].

Noting $\int_0^{\cdot} G\,\mathrm{d}L_n = \int_0^{\cdot} G\pi_n\,\mathrm{d}L$, inequality (2.7) and equality (2.9) establish for each $t \ge 0$ that

$$\begin{aligned}\mathbb{E}\left[|(X_nM - XM)^{\tau_k}(t)|\right] &\le k\mathbb{E}\left[\left|\int_0^t \mathbb{1}_{[0,\tau_k]}G(\pi_n - I)\,\mathrm{d}L\right|\right] \\ &\le e_{1,\alpha}k\left(\mathbb{E}\left[\int_0^T ||G(s)(\pi_n - I)||^{\alpha}_{\mathcal{L}_2(\mathrm{U},\mathrm{H})}\,\mathrm{d}s\right]\right)^{1/\alpha} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

It follows that the process $(XM)^{\tau_k}$ as a limit of martingales is itself a martingale. Since X is a local martingale according to Lemma 2.3 and M is an arbitrary real-valued continuous martingale, it follows from [16, Le. I.4.14] that X is purely discontinuous.

Step 3: For the general case, we fix an orthonormal basis $(e_i)_{i\in\mathbb{N}}$ in H and choose any $i, j \in \mathbb{N}$. Since $\langle X(t), e_i\rangle = \int_0^t G^*e_i\,\mathrm{d}L$ for every $t \ge 0$, the polarisation formula for real-valued covariation shows

$$\begin{aligned}\langle [[X]], e_i \otimes e_j\rangle &= \left[\int_0^{\cdot} G^*e_i\,\mathrm{d}L, \int_0^{\cdot} G^*e_j\,\mathrm{d}L\right] \\ &= \frac{1}{2}\left(\left[\int_0^{\cdot} G^*(e_i + e_j)\,\mathrm{d}L\right] - \left[\int_0^{\cdot} G^*e_i\,\mathrm{d}L\right] - \left[\int_0^{\cdot} G^*e_j\,\mathrm{d}L\right]\right).\end{aligned}$$

Linearity of the integral and binomial formula enable us to conclude

$$\begin{aligned}&\sum_{0\le s\le t} \Delta\langle X(s), e_i\rangle\langle X(s), e_j\rangle \\ &\quad= \sum_{0\le s\le t} \Delta\left(\int_0^s G^*e_i\,\mathrm{d}L\right)\left(\int_0^s G^*e_j\,\mathrm{d}L\right) \\ &\quad= \sum_{0\le s\le t} \frac{1}{2}\left(\Delta\left(\int_0^s G^*(e_i + e_j)\,\mathrm{d}L\right)^2 - \Delta\left(\int_0^s G^*e_i\,\mathrm{d}L\right)^2 - \Delta\left(\int_0^s G^*e_j\,\mathrm{d}L\right)^2\right).\end{aligned}$$

The very definition (4.1) of the continuous part leads us to

$$\begin{aligned}\langle [[X]]^c, e_i \otimes e_j\rangle &= \langle [[X]], e_i \otimes e_j\rangle - \sum_{0\le s\le t} \Delta\langle X(s), e_i\rangle\langle X(s), e_j\rangle \\ &= \frac{1}{2}\left(\left[\int_0^{\cdot} G^*(e_i + e_j)\,\mathrm{d}L\right]^c - \left[\int_0^{\cdot} G^*e_i\,\mathrm{d}L\right]^c - \left[\int_0^{\cdot} G^*e_j\,\mathrm{d}L\right]^c\right).\end{aligned}$$

Since *Step 2* guarantees that the processes $\int_0^{\cdot} G^*(e_i + e_j)\,\mathrm{d}L$, $\int_0^{\cdot} G^*e_i\,\mathrm{d}L$ and $\int_0^{\cdot} G^*e_i\,\mathrm{d}L$ are purely discontinuous, it follows that $\langle [[X]]^c, e_i \otimes e_j\rangle = 0$ for all $i, j \in \mathbb{N}$ which completes the proof. □

5 Strong Itô formula

In this section, we establish an Itô formula for processes that are given by a differential driven by a standard symmetric α-stable cylindrical Lévy process L for $\alpha \in (1, 2)$ and are of the form

$$dX(t) = F(t)\,\mathrm{d}t + G(t)\,\mathrm{d}L(t) \quad \text{for } t \in [0, T], \tag{5.1}$$

where $F\colon \Omega\times[0,T]\to H$, $G\colon \Omega\times[0,T]\to \mathcal{L}_2(U,H)$ are predictable and satisfy

$$\int_0^T \|F(t)\| + \|G(t)\|_{\mathcal{L}_2(U,H)}^{\alpha}\,\mathrm{d}t < \infty \quad \text{a.s.} \tag{5.2}$$

We denote by $\mathcal{C}_b^2(H)$ the space of continuous functions $f\colon H\to\mathbb{R}$ having bounded first and second Fréchet derivatives, which are denoted by Df and D^2f, respectively.

Theorem 5.1. *Let X be a stochastic process of the form* (5.1). *It follows for each $f\in\mathcal{C}_b^2(H)$ and $t\in[0,T]$ that*

$$\begin{aligned} f(X(t)) &= f(X(0)) + \int_0^t \langle Df(X(s-)),F(s)\rangle\,\mathrm{d}s + \int_0^t \langle G(s)^*Df(X(s-)),\cdot\rangle\,\mathrm{d}L(s) + M_f(t)\\ &\quad + \int_0^t\int_H \Big(f(X(s-)+g) - f(X(s-)) - \langle Df(X(s-)),g\rangle\Big)\,\big(\lambda\circ G(s)^{-1}\big)(\mathrm{d}g)\,\mathrm{d}s, \end{aligned}$$

where $M_f := \big(M_f(t) : t\in[0,T]\big)$ is a local martingale defined by

$$M_f(t) := \int_0^t\int_H \big(f(X(s-)+h) - f(X(s-)) - \langle Df(X(s-)),h\rangle\big)\,(\mu^X-\nu^X)(\mathrm{d}s,\mathrm{d}h).$$

Lemma 5.2. *Let λ be the cylindrical Lévy measure of an α-stable cylindrical Lévy process for $\alpha\in(1,2)$. Then we have for each $f\in\mathcal{C}_b^2(H)$, $h\in H$, and $\Phi\in\mathcal{L}_2(U,H)$ that*

$$\int_H \big|f(h+g)-f(h)-\langle Df(h),g\rangle\big|\,\big(\lambda\circ\Phi^{-1}\big)(\mathrm{d}g) \le d_\alpha^1\left(2\,||Df||_\infty + \frac{1}{2}\,\big|\big|D^2f\big|\big|_\infty\right)\|\Phi\|_{\mathcal{L}_2(U,H)}^{\alpha},$$

where d_α^1 is a constant depending only on α as defined in Inequality (2.2).

Proof. Taylor's remainder theorem in the integral form, see [2, Th. 5.8], and Inequality (2.2) imply

$$\begin{aligned} &\int_{\overline{B}_H} \big|f(h+g)-f(h)-\langle Df(h),g\rangle\big|\,\big(\lambda\circ\Phi^{-1}\big)(\mathrm{d}g)\\ &\quad= \int_{\overline{B}_H}\left|\int_0^1 \langle D^2f(h+\theta g)g,g\rangle(1-\theta)\,\mathrm{d}\theta\right|\,\big(\lambda\circ\Phi^{-1}\big)(\mathrm{d}g)\\ &\quad\le \big|\big|D^2f\big|\big|_\infty \int_{\overline{B}_H}\left(\int_0^1 \|g\|^2(1-\theta)\,\mathrm{d}\theta\right)\big(\lambda\circ\Phi^{-1}\big)(\mathrm{d}g)\\ &\quad= \frac{1}{2}\big\|D^2f\big\|_\infty \int_{\overline{B}_H}\|g\|^2\,\big(\lambda\circ\Phi^{-1}\big)(\mathrm{d}g)\\ &\quad\le d_\alpha^1\frac{1}{2}\big\|D^2f\big\|_\infty\|\Phi\|_{\mathcal{L}_2(U,H)}^{\alpha}. \end{aligned} \tag{5.3}$$

Similarly, Taylor's remainder theorem in the integral form and Inequality 2.2 show

$$\begin{aligned} \int_{\overline{B}_H^c} |f(h+g)-f(h)|\,\big(\lambda\circ\Phi^{-1}\big)(\mathrm{d}g) &= \int_{\overline{B}_H^c}\left|\int_0^1 \langle Df(h+\theta g),g\rangle d\theta\right|\,\big(\lambda\circ\Phi^{-1}\big)(\mathrm{d}g)\\ &\le ||Df||_\infty \int_{\overline{B}_H^c}\left(\int_0^1 ||g||\,\mathrm{d}\theta\right)\big(\lambda\circ\Phi^{-1}\big)(\mathrm{d}g)\\ &\le d_\alpha^1\,||Df||_\infty\,||\Phi||_{\mathcal{L}_2(U,H)}^{\alpha}. \end{aligned} \tag{5.4}$$

Another application of Inequality 2.2 shows

$$\begin{aligned} \int_{\overline{B}_H^c} |\langle Df(h),g\rangle|\,\big(\lambda\circ\Phi^{-1}\big)(\mathrm{d}g) &\le ||Df||_\infty \int_{\overline{B}_H^c} ||g||\,\big(\lambda\circ\Phi^{-1}\big)(\mathrm{d}g)\\ &\le d_\alpha^1\,||Df||_\infty\,||\Phi||_{\mathcal{L}_2(U,H)}^{\alpha}. \end{aligned} \tag{5.5}$$

Combining inequalities (5.3) to (5.5) completes the proof. □

Proof of Theorem 5.1. The stochastic process X given by (5.1) is purely discontinuous as it is the sum of a finite-variation process and a purely discontinuous process according to Proposition 4.1. The Itô formula in [26, Th. 27.2] takes for all $t \in [0,T]$ the form

$$\begin{aligned} \mathrm{d}f(X(t)) &= \langle Df(X(t-)), \cdot\rangle \,\mathrm{d}X(t) \\ &\quad + \int_H \big(f(X(t-)+h) - f(X(t-)) - \langle Df(X(t-)), h\rangle\big)\, \mu^X(\mathrm{d}t, \mathrm{d}h). \end{aligned} \tag{5.6}$$

One can show by approximating with simple integrands that

$$\langle Df(X(t-)), \cdot\rangle \,\mathrm{d}X(t) = \langle Df(X(t-)), F(t)\rangle \,\mathrm{d}t + \langle G(t)^* Df(X(t-)), \cdot\rangle \,\mathrm{d}L(t),$$

where both integrals are well defined since (5.2) guarantees

$$\begin{aligned} &\int_0^T |\langle Df(X(t-)), F(t)\rangle| + \|\langle G(t)^* Df(X(t-)), \cdot\rangle\|^\alpha_{\mathcal{L}_2(U,\mathbb{R})} \,\mathrm{d}t \\ &\qquad \le \|Df\|_\infty \int_0^T \|F(t)\| \,\mathrm{d}t + \|Df\|^\alpha_\infty \int_0^T \|G(t)\|^\alpha_{\mathcal{L}_2(U,H)} \,\mathrm{d}t < \infty \quad \text{a.s.} \end{aligned}$$

The definition of the compensator ν^X and Lemma 5.2 imply

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \int_H \big|f(X(s-)+h) - f(X(s-)) - \langle Df(X(s-)), h\rangle\big|\, \mu^X(\mathrm{d}s, \mathrm{d}h)\right] \\ &\quad = \mathbb{E}\left[\int_0^T \int_H \big|f(X(s-)+h) - f(X(s-)) - \langle Df(X(s-)), h\rangle\big|\, \nu^X(\mathrm{d}s, \mathrm{d}h)\right] \\ &\quad \le d_\alpha^1 \left(2\,||Df||_\infty + \frac{1}{2}\,||D^2 f||_\infty\right) \mathbb{E}\left[\int_0^T \|G(s)\|^\alpha_{\mathcal{L}_2(U,H)} \,\mathrm{d}s\right]. \end{aligned} \tag{5.7}$$

The stopping times $\tau_n := \inf\left\{t > 0 : \int_0^t ||G(s)||^\alpha_{\mathcal{L}_2(U,H)} \,\mathrm{d}s \ge n\right\} \wedge T$ satisfy $\tau_n \to T$ as $n \to \infty$ by (5.2). Since inequality (5.7) guarantees for all $n \in \mathbb{N}$ that

$$\mathbb{E}\left[\int_0^{T\wedge\tau_n} \int_H |f(X(s-)+h) - f(X(s-)) - \langle Df(X(s-)), h\rangle| \mu^X(\mathrm{d}s, \mathrm{d}h)\right] < \infty,$$

Proposition 2.7 shows that M_f is a local martingale. This concludes the proof, since the claimed formula is just a different form of (5.6). ☐

6 Mild solutions for stochastic evolution equations

We recall that U and H are separable Hilbert spaces with norms $\|\cdot\|$ and L is a standard symmetric α-stable cylindrical $(\mathcal{F}_t)$-Lévy process in U with $\alpha \in (1,2)$. In this section we consider the mild solution of the stochastic evolution equation:

$$\begin{aligned} \mathrm{d}X(t) &= \big(AX(t) + F(X(t))\big)\,\mathrm{d}t + G(X(t-))\,\mathrm{d}L(t) \qquad \text{for } t \in [0,T], \\ X(0) &= x_0, \end{aligned} \tag{6.1}$$

where A is a generator of a C_0-semigroup $(S(t))_{t\ge 0}$ in H, x_0 is an $\mathcal{F}_0$-measurable H-valued random variable, $F\colon H \to H$ and $G\colon H \to \mathcal{L}_2(U,H)$ are measurable mappings and $T > 0$.

Definition 6.1. *An H-valued predictable process X is a mild solution to* (6.1) *if*

$$X(t) = S(t)x_0 + \int_0^t S(t-s)F(X(s))\mathrm{d}s + \int_0^t S(t-s)G(X(s-))\mathrm{d}L(s) \quad \textit{for every } t \in [0,T].$$

We work under the following assumptions:

(A1) The C_0-semigroup $(S(t))_{t\geq 0}$ is compact, analytic and a semigroup of contractions and 0 is an element of the resolvent set of A.

(A2) The mapping F is Lipschitz and bounded, i.e. there exists $K_F \in (0,\infty)$ such that

$$\|F(h_1) - F(h_2)\| \leq K_F \|h_1 - h_2\|, \quad \|F(h)\| \leq K_F \tag{6.2}$$

for every $h_1, h_2, h \in H$.

(A3) The mapping G is Lipschitz and bounded, i.e. there exists $K_G \in (0,\infty)$ such that

$$\|G(h_1) - G(h_2)\|_{\mathcal{L}_2(U,H)} \leq K_G \|h_1 - h_2\|, \quad \|G(h)\|_{\mathcal{L}_2(U,H)} \leq K_G \tag{6.3}$$

for every $h_1, h_2, h \in H$.

(A4) The initial condition x_0 has finite p-th moment for every $p < \alpha$.

Remark 6.2. We shall use the notation $D^\delta := \mathrm{Dom}((-A)^\delta)$ for the domain of the fractional generator $(-A)^\delta$ for $\delta \in [0,1]$, and equip D^δ with the norm $||h||_\delta := \left||(-A)^\delta h\right||$. It follows from Assumption (A1) that the embedding of Hilbert spaces $D^\delta \hookrightarrow D^\gamma$ is dense and compact for every $0 \leq \gamma < \delta \leq 1$, cf. [4, Cor. 3.8.2].

Remark 6.3. Assumption (A1) implies, cf. [21, p. 289], that for every $\delta \geq 0$ there exists a $c_\delta \in (0,\infty)$ depending only on δ such that

$$\|S(t)\|_{\mathcal{L}(H,D^\delta)} \leq c_\delta t^{-\delta} \qquad \text{for every } t > 0. \tag{6.4}$$

Remark 6.4. By considering the cases $\|h_1 - h_2\| \leq 1$ and $\|h_1 - h_2\| > 1$ separately, we conclude from Assumptions (A2) and (A3) that there exist $K_F, K_G \in (0,\infty)$ such that for any $\beta \in (0,1)$ we have

$$\|F(h_1) - F(h_2)\| \leq K_F \|h_1 - h_2\|^\beta, \quad \|F(h)\| \leq K_F, \tag{6.5}$$

and

$$\|G(h_1) - G(h_2)\|_{\mathcal{L}_2(U,H)} \leq K_G \|h_1 - h_2\|^\beta, \quad \|G(h)\|_{\mathcal{L}_2(U,H)} \leq K_G, \tag{6.6}$$

for every $h_1, h_2, h \in H$.

Example 6.5. The most important example of Equation (6.1) is a non-linear heat equation. For this purpose, the generator A is chosen as the Laplace operator Δ and $H = L^2(\mathcal{O})$ for a bounded domain $\mathcal{O} \subseteq \mathbb{R}^d$ with smooth boundaries. Then the semigroup generated by A satisfies Condition (A1) above.

Simple examples of the coefficients F and G meeting Conditions (A2) and (A3) are provided by some diagonal operators along an arbitrary orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of H. Let $g_k\colon H \to U$ be functions satisfying that for each $k \in \mathbb{N}$ there exits $c_k > 0$ with

$$\|g_k(h_1) - g_k(h_2)\| \leq c_k \|h_1 - h_2\|, \qquad \|g_k(h)\| \leq c_k \qquad \text{for every } h_1, h_2, h \in H.$$

Assuming square-summability of $(c_k)_{k\in\mathbb{N}}$ enables us to define

$$G(h)u = \sum_{k=1}^{\infty} \langle g_k(h), u\rangle e_k \qquad \text{for all } h \in H, u \in U.$$

Since for all $h \in H$ we have

$$\|G(h)\|^2_{\mathcal{L}_2(U,H)} = \sum_{k=1}^{\infty} \|g_k(h)\|^2,$$

Condition (A3) is satisfied. This and other examples are extensively studied in [24].

In a similar way, an example for F can be constructed. Other possible examples for the coefficients F and G are Nemytskii or superposition operators.

The first main theorem of this article is the following existence result, which also includes properties on the path regularity of the solution.

Theorem 6.6. *Under the assumptions (A1)-(A4), there exists a mild solution X to* (6.1)*. The mild solution X is an element of $\mathcal{C}([0,T], L^p(\Omega,H))$ for every $p<\alpha$ and has càdlàg paths in H.*

We will obtain the solution to (6.1) by using the Yosida approximations. For this purpose, we define $R_n = n\,(n\mathrm{I}-A)^{-1}$ for $n\in\mathbb{N}$ and denote by X_n the mild solution to

$$\begin{aligned}\mathrm{d}X_n(t) &= \big(AX_n(t) + R_nF(X_n(t))\big)\,\mathrm{d}t + R_nG(X_n(t-))\,\mathrm{d}L(t),\\ X_n(0) &= R_nx_0.\end{aligned} \tag{6.7}$$

Before we establish existence of a mild solution to (6.7) we remark the following:

Remark 6.7. We recall that under Assumption (A1) we have for all $\delta\in[0,1]$ that

$$\|R_n\|_{\mathcal{L}(D^\delta)} \le 1, \quad n\in\mathbb{N}.$$

This follows from the fact, that if an operator commutes with A then it commutes with A^γ, see e.g. [14, Pr. 3.1.1], which enables us to conclude for every $n\in\mathbb{N}$ that

$$\|R_n\|_{\mathcal{L}(D^\gamma)} = \sup_{\|(-A)^\gamma h\|\le 1} \left\|n(n-A)^{-1}(-A)^\gamma h\right\| \le \sup_{\|h\|\le 1}\left\|n(n-A)^{-1}h\right\| = \|R_n\|_{\mathcal{L}(H)}.$$

Since $(S(t))_{t\ge0}$ is a contraction semigroup, Theorem 1.3.1 in [30] guarantees $\|R_n\|_{\mathcal{L}(H)}\le 1$ for all $n\in\mathbb{N}$.

Existence of the mild solution X_n to (6.7) is guaranteed by the following result, which is based on [22, Th. 12].

Lemma 6.8. *Under the Assumptions (A1), (A2) and (A3), there exists a mild solution to* (6.7) *with cádlág paths.*

Proof. We need to verify the 3 conditions in [22, Th. 12] which we denote by (B1), (B2) and (B3) in the following.

Contractivity of the semigroup and the fact that zero is in the resolvent set of the generator A is assumed directly. To show compactness of the embedding $D^1\subseteq H$, we observe that

$$\begin{aligned}\{h\in D^1 : \|h\|_1\le 1\} &= \{h\in D^1 : \|Ah\|\le 1\}\\ &= \left\{(-A)^{-1}h : h\in H \text{ and } \|Ah\|\le 1\right\}\\ &\subseteq \left\{(-A)^{-1}h\in H : \|h\|\le \left\|(-A)^{-1}\right\|\right\}.\end{aligned}$$

As $(-A)^{-1}$ is a compact operator on H according to [30, Co. 2.3.5], the set in the last line is compact in H, establishing compactness of the embedding $D^1\subseteq H$. Finally, by the analyticity of the semigroup we have by [30, Th. 2.5.2] that there exists $\tilde\omega\in(0,\frac{\pi}{2})$ such that $\{\lambda\in\mathbb{C} : |\arg\lambda| < \frac{\pi}{2}+\tilde\omega\}$ is contained in the resolvent set of A and thus

$$\{-\lambda\in\mathbb{C} : |\arg\lambda| < \tfrac{\pi}{2}+\tilde\omega\} = \{\lambda\in\mathbb{C} : \omega < |\arg\lambda| \le \pi\}$$

with $\omega=\frac{\pi}{2}-\tilde\omega$ is contained in the resolvent set of $-A$ and Condition (B1) is shown.
To verify Condition (B2), we conclude from contractivity of S and (A2), using similar arguments as in Remark 6.7, that, for any $n\in\mathbb{N}$, $\delta\in(0,1]$, $t>0$ and $h\in H$, we have

$$\|S(t)R_nF(h)\|_\delta \le \|S(t)\|_{\mathcal{L}(D^\delta)}\|R_n\|_{\mathcal{L}(H,D^\delta)}\|F(h)\| \le \|R_n\|_{\mathcal{L}(H,D^\delta)}K_F.$$

Similarly, by using (A3), we obtain

$$\|S(t)R_nG(h)\|_{\mathcal{L}_2(U,D^\delta)} \le \|S(t)\|_{\mathcal{L}(D^\delta)}\|R_n\|_{\mathcal{L}(H,D^\delta)}\|G(h)\|_{\mathcal{L}_2(U,H)} \le \|R_n\|_{\mathcal{L}(H,D^\delta)}K_G,$$

which verifies (B2).
By very similar arguments, we obtain, for any $n \in \mathbb{N}$, $\delta \in (0,1]$, $t > 0$ and $h_1, h_2 \in H$, that

$$\begin{aligned}\|S(t)R_n(F(h_1)-F(h_2))\|_\delta &\le \|R_n\|_{\mathcal{L}(H,D^\delta)} K_F \|h_1-h_2\|,\\ \|S(t)R_n(G(h_1)-G(h_2))\|_{\mathcal{L}_2(U,D^\delta)} &\le \|R_n\|_{\mathcal{L}(H,D^\delta)} K_G \|h_1-h_2\|,\end{aligned}$$

establishing (B3) and completing the proof. ☐

The solution to (6.1) will be constructed as a limit of X_n in $\mathcal{C}([0,T], L^p(\Omega,H))$ for an arbitrary but fixed $p < \alpha$. In the first three Lemmata, we establish relative compactness of the Yosida approximation $\{X_n : n \in \mathbb{N}\}$ in the space $\mathcal{C}([0,T], L^p(\Omega,H))$.

Lemma 6.9. *The set $\{X_n(t) : n \in \mathbb{N}\}$ is tight in H for every $t \in [0,T]$.*

Proof. The case $t = 0$ follows immediately from the strong convergence of R_n. For the case $t \in (0,T]$ we first prove that for every $1 \le q < \alpha$, and $0 \le \delta < 1/\alpha$ we have

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\|X_n(t)\|_\delta^q\right] < \infty. \tag{6.8}$$

Applying Hölder's inequality and inequality (2.7) shows for every $n \in \mathbb{N}$ that

$$\begin{aligned}\mathbb{E}\left[\|X_n(t)\|_\delta^q\right] \le 3^{q-1}\Bigg(&\mathbb{E}\left[\|S(t)R_nx_0\|_\delta^q\right] + t^{q-1}\mathbb{E}\left[\int_0^t \|S(t-s)R_nF(X_n(s))\|_\delta^q \,\mathrm{d}s\right]\\ &+e_{q,\alpha}\left(\mathbb{E}\left[\int_0^t \|S(t-s)R_nG(X_n(s))\|_{\mathcal{L}_2(U,D^\delta)}^\alpha \,\mathrm{d}s\right]\right)^{\frac{q}{\alpha}}\Bigg).\end{aligned}$$

Commutativity of S and R_n, Remark 6.3 and Remark 6.7 verify

$$\mathbb{E}\left[\|S(t)R_nx_0\|_\delta^q\right] \le c_\delta^q t^{-q\delta} \sup_{n\in\mathbb{N}} \|R_n\|_{\mathcal{L}(D^\delta)}^q \,\mathbb{E}\left[\|x_0\|^q\right] < \infty.$$

Assumption (A2) on boundedness of F together with Remark 6.3 and Remark 6.7 yield

$$\mathbb{E}\left[\int_0^t \|S(t-s)R_nF(X_n(s))\|_\delta^q \,\mathrm{d}s\right] \le c_\delta^q \frac{t^{1-q\delta}}{1-q\delta} \sup_{n\in\mathbb{N}} \|R_n\|_{\mathcal{L}(D^\delta)}^q K_F^q < \infty.$$

Similarly, Assumption (A3) on boundedness of G implies

$$\left(\mathbb{E}\left[\int_0^t \|S(t-s)R_nG(X_n(s))\|_{\mathcal{L}_2(U,D^\gamma)}^\alpha \,\mathrm{d}s\right]\right)^{\frac{q}{\alpha}} \le c_\delta^q \left(\frac{t^{1-\alpha\delta}}{1-\alpha\delta}\right)^{\frac{q}{\alpha}} \sup_{n\in\mathbb{N}} \|R_n\|_{\mathcal{L}(D^\delta)}^q K_G^q < \infty.$$

Combining the above estimates establishes (6.8), which in turn gives the statement of the Lemma. Indeed, choose any $\delta \in (0, 1/\alpha)$ and use Markov's inequality and (6.8) for $q = 1$ to obtain for each $N > 0$ that

$$\sup_{n\in\mathbb{N}} P\left(\|X_n(t)\|_\delta > N\right) \le \frac{c}{N}$$

for some constant $c \in (0,\infty)$. Since the embedding $D^\delta \hookrightarrow H$ is compact according to Remark 6.2, we obtain tightness of $\{X_n(t) : n \in \mathbb{N}\}$ by Prokhorov's theorem. ☐

Lemma 6.10. *The sequence $\{X_n(t) : n \in \mathbb{N}\}$ is relatively compact in $L_P^0(\Omega,H)$ for every $t \in [0,T]$.*

Proof. First, we impose the additional assumption that

$$T12^2 e_{2,\alpha}\,(K_F^\alpha + K_G^\alpha) < 1, \tag{6.9}$$

where K_F, K_G come from (6.5), (6.6) and $e_{2,\alpha}$ is defined just below (2.7). For $1 < p < \alpha$ and $m, n \in \mathbb{N}$ we estimate the p-th moment of the difference $X_m(t) - X_n(t)$ by

$$\begin{aligned}
\mathbb{E}\left[\|X_m(t) - X_n(t)\|^p\right] \le & 3^{p-1}\Bigg(\mathbb{E}\left[\|S(t)(R_m - R_n)x_0\|^p\right] \\
&+ \mathbb{E}\left[\left\|\int_0^t S(t-s)(R_mF(X_m(s)) - R_nF(X_n(s)))\mathrm{d}s\right\|^p\right] \\
&+ \mathbb{E}\left[\left\|\int_0^t S(t-s)(R_mG(X_m(s-)) - R_nG(X_n(s-)))\mathrm{d}L(s)\right\|^p\right]\Bigg) \\
\le & 6^{p-1}\Bigg(\mathbb{E}\left[\|S(t)(R_m - R_n)x_0\|^p\right] \\
&+ \mathbb{E}\left[\left\|\int_0^t S(t-s)(R_m - R_n)F(X_m(s))\mathrm{d}s\right\|^p\right] \\
&+ \mathbb{E}\left[\left\|\int_0^t S(t-s)R_n(F(X_m(s)) - F(X_n(s)))\mathrm{d}s\right\|^p\right] \\
&+ \mathbb{E}\left[\left\|\int_0^t S(t-s)(R_m - R_n)G(X_m(s-))\mathrm{d}L(s)\right\|^p\right] \\
&+ \mathbb{E}\left[\left\|\int_0^t S(t-s)R_n(G(X_m(s-)) - G(X_n(s-)))\mathrm{d}L(s)\right\|^p\right]\Bigg).
\end{aligned} \tag{6.10}$$

Furthermore, using Hölder's inequality, inequality (2.7), Remark 6.7, (A1) and estimates (6.5) and (6.6) we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left\|\int_0^t S(t-s)R_n(F(X_m(s)) - F(X_n(s)))\mathrm{d}s\right\|^p\right] \\
&\quad \le T^{1-p}\mathbb{E}\left[\int_0^t \|S(t-s)R_n(F(X_m(s)) - F(X_n(s)))\|^p\,\mathrm{d}s\right] \\
&\quad \le T^{p-\frac{p}{\alpha}}\left(\mathbb{E}\left[\int_0^t \|S(t-s)R_n(F(X_m(s)) - F(X_n(s)))\|^\alpha\,\mathrm{d}s\right]\right)^{\frac{p}{\alpha}} \\
&\quad \le T^{p-\frac{p}{\alpha}}K_F^p\left(\mathbb{E}\left[\int_0^t \|X_m(s) - X_n(s)\|^p\,\mathrm{d}s\right]\right)^{\frac{p}{\alpha}}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\left\|\int_0^t S(t-s)R_n(G(X_m(s-)) - G(X_n(s-)))\mathrm{d}L(s)\right\|^p\right] \\
&\quad \le e_{p,\alpha}\left(\mathbb{E}\left[\int_0^t \|S(t-s)R_n(G(X_m(s)) - G(X_n(s)))\|^\alpha\,\mathrm{d}s\right]\right)^{\frac{p}{\alpha}} \\
&\quad \le e_{p,\alpha}K_G^p\left(\mathbb{E}\left[\int_0^t \|X_m(s) - X_n(s)\|^p\,\mathrm{d}s\right]\right)^{\frac{p}{\alpha}},
\end{aligned}$$

which together with (6.10) yield

$$\begin{aligned}\mathbb{E}\left[\|X_m(t)-X_n(t)\|^p\right] \leq 6^{p-1}\Bigg(&\mathbb{E}\left[\|S(t)(R_m-R_n)x_0\|^p\right]\\&+\mathbb{E}\left[\left\|\int_0^t S(t-s)(R_m-R_n)F(X_m(s))\mathrm{d}s\right\|^p\right]\\&+T^{p-\frac{p}{\alpha}}K_F^p\left(\mathbb{E}\left[\int_0^t \|X_m(s)-X_n(s)\|^p\,\mathrm{d}s\right]\right)^{\frac{p}{\alpha}}\\&+\mathbb{E}\left[\left\|\int_0^t S(t-s)(R_m-R_n)G(X_m(s-))\mathrm{d}L(s)\right\|^p\right]\\&+e_{p,\alpha}K_G^p\left(\mathbb{E}\left[\int_0^t \|X_m(s)-X_n(s)\|^p\,\mathrm{d}s\right]\right)^{\frac{p}{\alpha}}\Bigg).\end{aligned}\tag{6.11}$$

If we define

$$\begin{aligned}u_{n,m,p}(t) &:= \left(\mathbb{E}\left[\|X_m(t)-X_n(t)\|^p\right]\right)^{\frac{\alpha}{p}}\\u^0_{n,m,p} &:= 5^{\frac{\alpha}{p}-1}6^{\frac{\alpha}{p}(p-1)}\Bigg(\sup_{t\in[0,T]}\|S(t)\|^\alpha_{\mathcal{L}(H)}\left(\mathbb{E}\left[\|(R_m-R_n)x_0\|^p\right]\right)^{\frac{\alpha}{p}}\\&\quad+\left(\sup_{t\in[0,T]}\mathbb{E}\left[\left\|\int_0^t S(t-s)(R_m-R_n)F(X_m(s))\mathrm{d}s\right\|^p\right]\right)^{\frac{\alpha}{p}}\\&\quad+\left(\sup_{t\in[0,T]}\mathbb{E}\left[\left\|\int_0^t S(t-s)(R_m-R_n)G(X_m(s-))\mathrm{d}L(s)\right\|^p\right]\right)^{\frac{\alpha}{p}}\Bigg)\\w_p &:= 5^{\frac{\alpha}{p}-1}6^{\frac{\alpha}{p}(p-1)}\left(T^{\alpha-1}K_F^\alpha+e_{p,\alpha}^{\frac{\alpha}{p}}K_G^\alpha\right)\end{aligned}$$

for $t\in[0,T]$, then after raising both sides of (6.11) to the power of α/p and simple algebraic steps we obtain

$$u_{n,m,p}(t)\leq u^0_{n,m,p}+w_p\int_0^t\left(u_{n,m,p}(s)\right)^{\frac{p}{\alpha}}\,\mathrm{d}s,$$

which in turn by Gronwall's inequality in [41, Th. 2] gives

$$u_{n,m,p}(t)\leq 2^{\frac{\alpha}{\alpha-p}-1}\left(u^0_{n,m,p}+\left(\frac{\alpha-p}{\alpha}tw_p\right)^{\frac{\alpha}{\alpha-p}}\right)\leq 2^{\frac{\alpha}{\alpha-p}}u^0_{n,m,p}+2^{\frac{\alpha}{\alpha-p}}\left(\frac{\alpha-p}{\alpha}tw_p\right)^{\frac{\alpha}{\alpha-p}}\tag{6.12}$$

If we show that

$$\lim_{p\to\alpha-}2^{\frac{\alpha}{\alpha-p}}\left(\frac{\alpha-p}{\alpha}tw_p\right)^{\frac{\alpha}{\alpha-p}}=\lim_{p\to\alpha-}\left(2\frac{\alpha-p}{\alpha}tw_p\right)^{\frac{\alpha}{\alpha-p}}=0,\tag{6.13}$$

and for any $1<p<\alpha$

$$\lim_{m,n\to\infty}u^0_{n,m,p}=0,\tag{6.14}$$

then by (6.12), for each $\epsilon\in(0,1)$ we can find $p^*\in(1,\alpha)$ such that

$$2^{\frac{\alpha}{\alpha-p^*}}\left(\frac{\alpha-p^*}{\alpha}tw_{p^*}\right)^{\frac{\alpha}{\alpha-p^*}}<\frac{\epsilon^{\alpha+1}}{2},$$

and an $N \in \mathbb{N}$ such that for all $m, n \geq N$

$$2^{\frac{\alpha}{\alpha - p^*}} u^0_{n,m,p^*} \leq \frac{\epsilon^{\alpha+1}}{2}.$$

Thus, for any $m, n \geq N$ we obtain

$$\begin{aligned} P\left(\|X_m(t) - X_n(t)\| \geq \epsilon\right) &\leq \frac{1}{\epsilon^{p^*}} \mathbb{E}\left[\|X_m(t) - X_n(t)\|^{p^*}\right] \\ &= \frac{1}{\epsilon^{p^*}} \left(u_{n,m,p^*}(t)\right)^{\frac{p^*}{\alpha}} \\ &\leq \epsilon^{\frac{p^*}{\alpha}(\alpha+1) - p^*} = \epsilon^{\frac{p^*}{\alpha}} \leq \sqrt{\epsilon}, \end{aligned}$$

which concludes the proof under the additional assumption (6.9).
Argument for (6.13)*:* Recall that $1 < p$ and $e_{p,\alpha} = \frac{\alpha}{\alpha - p} e_{2,\alpha}^{p/\alpha}$ for some $e_{2,\alpha} \in (0, \infty)$ independent of p. Thus, if p is sufficiently close to α we have $T^{\alpha-1} \leq e_{p,\alpha}^{\frac{\alpha}{p}}$ and estimate

$$\begin{aligned} \left(2\frac{\alpha - p}{p} t w_p\right)^{\frac{\alpha}{\alpha - p}} &= \left(2\frac{\alpha - p}{\alpha} t 5^{\frac{\alpha - p}{p}} 6^{\frac{\alpha}{p}(p-1)} \left(T^{\alpha-1} K_F^\alpha + e_{p,\alpha}^{\frac{\alpha}{p}} K_G^\alpha\right)\right)^{\frac{\alpha}{\alpha - p}} \\ &\leq 5^{\frac{\alpha}{p}} \left(2\frac{\alpha - p}{\alpha} t 6^2 \left(\frac{\alpha - p}{\alpha}\right)^{-\frac{\alpha}{p}} e_{2,\alpha}\left(K_F^\alpha + K_G^\alpha\right)\right)^{\frac{\alpha}{\alpha - p}} \\ &\leq 5^\alpha \left(\frac{\alpha - p}{\alpha}\right)^{-\frac{\alpha}{p}} \left(t 12^2 e_{2,\alpha}\left(K_F^\alpha + K_G^\alpha\right)\right)^{\frac{\alpha}{\alpha - p}} \\ &\leq 5^\alpha \left(\frac{\alpha - p}{\alpha}\right)^{-2} \left(T 12^2 e_{2,\alpha}\left(K_F^\alpha + K_G^\alpha\right)\right)^{\frac{\alpha}{\alpha - p}}. \end{aligned}$$

Thus, (6.13) follows from

$$\lim_{y \to \infty} y^2 \left(T 12^2 e_{2,\alpha}\left(K_F^\alpha + K_G^\alpha\right)\right)^y = 0$$

by L'Hospital's rule and (6.9).
Argument for (6.14)*:* Let $p \in (1, \alpha)$ be fixed. By strong convergence of R_n and Lebesgue's dominated convergence theorem we have

$$\lim_{m,n \to \infty} \mathbb{E}\left[\|(R_m - R_n)x_0\|^p\right] = 0. \tag{6.15}$$

Moreover, by Hölder's inequality, strong convergence of R_n, Lemma 9.1 and Lebesgue's dominated convergence theorem we have

$$\begin{aligned} &\lim_{m,n \to \infty} \left(\sup_{t \in [0,T]} \mathbb{E}\left[\left\|\int_0^t S(t-s)(R_m - R_n)F(X_m(s))\mathrm{d}s\right\|^p\right]\right) \\ &\leq T^{p-1} \sup_{t \in [0,T]} \|S(t)\|^p_{\mathcal{L}(H)} \lim_{m,n \to \infty} \left(\mathbb{E}\left[\int_0^T \|(R_m - R_n)F(X_m(s))\|^p \,\mathrm{d}s\right]\right) = 0, \end{aligned} \tag{6.16}$$

where the assumptions of Lemma 9.1 are satisfied by boundedness of F, see (6.2), and tightness of $\{F(X_n(s)), m \in \mathbb{N}\}$ implied by Lemma 6.9 and continuity of F. Similarly, using inequality (2.7) and (6.6), we obtain

$$\begin{aligned} &\lim_{m,n \to \infty} \left(\sup_{t \in [0,T]} \mathbb{E}\left[\left\|\int_0^t S(t-s)(R_m - R_n)G(X_m(s-))\mathrm{d}L(s)\right\|^p\right]\right) \\ &\leq e_{p,\alpha} \sup_{t \in [0,T]} \|S(t)\|^p_{\mathcal{L}(H)} \lim_{m,n \to \infty} \left(\mathbb{E}\left[\int_0^T \|(R_m - R_n)G(X_m(s))\|^\alpha \,\mathrm{d}s\right]\right)^{\frac{p}{\alpha}} = 0. \end{aligned} \tag{6.17}$$

To prove the general case without the assumption (6.9), we argue as follows. Fix a time

$$T_0 \in \left(0,\ \frac{1}{12^2 e_{2,\alpha}\left(K_F^\alpha + K_G^\alpha\right)}\right). \tag{6.18}$$

If $t \in [0, T_0]$, relative compactness in $L^0_P(\Omega, H)$ follows from the previous arguments and (6.18). On the other hand, when $t \in (T_0, 2T_0]$ we write

$$\begin{aligned} X_m(t) - X_n(t) = S\left(t - T_0\right) \Bigg(& S\left(T_0\right)(R_m - R_n)x_0 \\ & + \int_0^{T_0} S\left(T_0 - s\right)\left(R_m F(X_m(s) - R_n F(X_n(s))\right) \mathrm{d}s \\ & + \int_0^{T_0} S\left(T_0 - s\right)\left(R_m G(X_m(s-) - R_n G(X_n(s-))\right) \mathrm{d}L(s) \Bigg) \\ & + \int_{T_0}^t S(t-s)(R_m F(X_m(s) - R_n F(X_n(s))) \,\mathrm{d}s \\ & + \int_{T_0}^t S(t-s)(R_m G(X_m(s-) - R_n G(X_n(s-))) \,\mathrm{d}L(s) \\ = {} & S\left(t - T_0\right)\left(X_m\left(T_0\right) - X_n\left(T_0\right)\right) \\ & + \int_{T_0}^t S(t-s)(R_m F(X_m(s) - R_n F(X_n(s))) \,\mathrm{d}s \\ & + \int_{T_0}^t S(t-s)(R_m G(X_m(s-) - R_n G(X_n(s-))) \,\mathrm{d}L(s). \end{aligned}$$

Since our choice of T_0 implies that $\{S(t - T_0)X_n(T_0),\, n \in \mathbb{N}\}$ is relatively compact in $L^0_P(\Omega, H)$, and by Equation (6.18) we have $(t - T_0)\, 12^2 e_{2,\alpha}\left(K_F^\alpha + K_G^\alpha\right) < 1$, we can use the same argument as before to obtain relative compactness of $(X_n(t))_{n\in\mathbb{N}}$ in $L^0_P(\Omega, H)$ for each $t \in (T_0, 2T_0]$. Using a standard induction argument, we can now cover intervals of arbitrary length. This concludes the proof of the general case. □

We now step from relative compactness of $\{X_n(t) : n \in \mathbb{N}\}$ in $L^0_P(\Omega, H)$ for fixed time t to relative compactness of the processes $\{X_n : n \in \mathbb{N}\}$ using the Arzelà–Ascoli Theorem.

Lemma 6.11. *The collection $\{X_n : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{C}([0, T], L^p(\Omega, H))$ for any $0 < p < \alpha$.*

Proof. We consider the case $1 < p < \alpha$ as the case $p \le 1$ follows from the fact that relative compactness in $\mathcal{C}([0, T], L^p(\Omega, H))$ implies relative compactness in $\mathcal{C}([0, T], L^{p'}(\Omega, H))$ for $p > p'$. In light of the Arzelà–Ascoli Theorem, cf. e.g. [20, Th. 7.17]), it suffices to show that

(a) $\{X_n(t) : n \in \mathbb{N}\} \subset L^p(\Omega, H)$ is relatively compact for each $t \in [0, T]$;
(b) $\{X_n : n \in \mathbb{N}\} \subset \mathcal{C}([0, T], L^p(\Omega, H))$ is equicontinuous.

The claim in (a) follows from [12, Cor. 3.3] by Lemmata 6.9, 6.10 and the fact that Equation (6.8) with $\delta = 0$ and any $q \in (p, \alpha)$ implies via the Vallee-Poussin Theorem [11, Th. II.22] that the collection $\{X_n(t) : n \in \mathbb{N}\}$ is p-uniformly integrable and bounded in $L^p(\Omega, H)$. Hence, it remains only to prove (b). To that end, we take $t \in [0, T]$ and $h \in (0, T - t]$, and estimate

$$\|X_n(t+h) - X_n(t)\|^p$$

$$\leq 5^{p-1}\Bigg(\|(S(h)-I)\,S(t)R_n x_0\|^p + \left\| \int_t^{t+h} S(t+h-s)R_nF(X_n(s))\,\mathrm{d}s \right\|^p + \left\| \int_t^{t+h} S(t+h-s)R_nG(X_n(s-))\,\mathrm{d}L(s) \right\|^p + \left\| \int_0^t (S(h)-I)\,S(t-s)R_nF(X_n(s))\,\mathrm{d}s \right\|^p + \left\| \int_0^t (S(h)-I)\,S(t-s)R_nG(X_n(s-))\,\mathrm{d}L(s) \right\|^p \Bigg). \tag{6.19}$$

Commutativity of R_n and S and contractivity of S implies

$$E\left[\|(S(h)-I)\,S(t)R_n x_0\|^p\right] \leq \sup_{n\in\mathbb{N}} \|R_n\|^p_{\mathcal{L}(H)}\,\mathbb{E}\left[\|(S(h)-I)\,x_0\|^p\right]. \tag{6.20}$$

Applying Hölder's inequality, boundedness of F in Assumption (A2) and contractivity of S we get

$$\mathbb{E}\left[\left\| \int_t^{t+h} S(t+h-s)R_nF(X_n(s))\,\mathrm{d}s \right\|^p\right] \leq h^p \sup_{n\in\mathbb{N}} \|R_n\|^p_{\mathcal{L}(H)}\,K_F^p. \tag{6.21}$$

We conclude from Inequality (2.7) by using boundedness of G in Assumption (A3) and contractivity of S that

$$\mathbb{E}\left[\left\| \int_t^{t+h} S(t+h-s)R_nG(X_n(s-))\,\mathrm{d}L(s) \right\|^p\right] \tag{6.22}$$

$$\leq e_{p,\alpha}\left(\mathbb{E}\left[\int_t^{t+h} \|S(t+h-s)G(X_n(s))\|^\alpha_{\mathcal{L}_2(U,H)}\;\mathrm{d}s\right]\right)^{p/\alpha} \leq e_{p,\alpha} \sup_{n\in\mathbb{N}} \|R_n\|^p_{\mathcal{L}(H)}\,K_G^p h^{p/\alpha}.$$

It follows from Lemma 6.9 that $\{X_n(s) : n \in \mathbb{N}\}$ is tight in H for every $s \in [0,t]$. Lemma 9.1 implies

$$\lim_{h\searrow 0} \sup_{n\in\mathbb{N}} \mathbb{E}\left[\|(S(h)-I)\,S(t-s)R_nF(X_n(s))\|^p\right] = 0.$$

Lebesgue's dominated convergence theorem shows

$$\lim_{h\searrow 0} \int_0^t \sup_{n\in\mathbb{N}} \mathbb{E}\left[\|(S(h)-I)\,S(t-s)R_nF(X_n(s))\|^p\right]\,\mathrm{d}s = 0. \tag{6.23}$$

In the same way, after applying Inequality (2.7), we obtain from Lemma 9.1

$$\lim_{h\searrow 0} \sup_{n\in\mathbb{N}} \mathbb{E}\left[\left\| \int_0^t (S(h)-I)\,S(t-s)R_nG(X_n(s-))\,\mathrm{d}L(s) \right\|^p\right] = 0. \tag{6.24}$$

Applying (6.20) − (6.24) to Inequality (6.19) shows uniform continuity from the right. Similar arguments establish uniform continuity from the left, which proves (b), and thus completes the proof. □

Proof of Theorem 6.6. It is enough to consider the case $p \geq \alpha\beta$ where β is the Hölder exponent from (6.6). Lemma 6.11 guarantees that there is a subsequence $(n_k)_{k=1}^\infty$ such that

$$\lim_{k\to\infty} \sup_{t\in[0,T]} \mathbb{E}\left[||X_{n_k}(t) - Z(t)||^p\right] = 0 \tag{6.25}$$

for some $Z \in \mathcal{C}([0,T], L^p(\Omega, H))$. The proof will be complete if we show that Z is a mild solution to (6.1). Letting $c := 1 \wedge 3^{p-1}$, we conclude for each $k \in \mathbb{N}$ and $t \in [0,T]$ from Lipschitz continuity of F and Hölder continuity of G in (6.2) and (6.6) and contractivity of S by applying Hölder's inequality and Inequality (2.7) that

$$
\begin{aligned}
&\mathbb{E}\left[\left|\left|Z(t) - S(t)x_0 - \int_0^t S(t-s)F(Z(s))\,\mathrm{d}s - \int_0^t S(t-s)G(Z(s-))\,\mathrm{d}L(s)\right|\right|^p\right] \\
&\le c\Bigg(\mathbb{E}\left[||Z(t) - X_{n_k}(t)||^p\right] + \mathbb{E}\left[\left|\left|\int_0^t S(t-s)\big(F(Z(s)) - F(X_{n_k}(s))\big)\,\mathrm{d}s\right|\right|^p\right] \\
&\qquad\qquad + \mathbb{E}\left[\left|\left|\int_0^t S(t-s)\big(G(Z(s-)) - G(X_{n_k}(s-))\big)\,\mathrm{d}L(s)\right|\right|^p\right]\Bigg) \\
&\le c\Bigg(\mathbb{E}\left[||Z(t) - X_{n_k}(t)||^p\right] + T^{p-1}\mathbb{E}\left[\int_0^t \left|\left|S(t-s)\big(F(Z(s)) - F(X_{n_k}(s))\big)\right|\right|^p \,\mathrm{d}s\right] \\
&\qquad\qquad + e_{p,\alpha}\left(\mathbb{E}\left[\int_0^t \left|\left|S(t-s)\big(G(Z(s)) - G(X_{n_k}(s))\big)\right|\right|^\alpha_{\mathcal{L}_2(U,H)} \,\mathrm{d}s\right]\right)^{p/\alpha}\Bigg) \\
&\le c\Bigg(\mathbb{E}\left[\|Z(t) - X_{n_k}(t)\|^p\right] + T^{p-1}K_F^p \sup_{t\in[0,T]}\|S(t)\|^p_{\mathcal{L}(H)}\,\mathbb{E}\left[\int_0^t \|Z(s) - X_{n_k}(s)\|^p \,\mathrm{d}s\right] \\
&\qquad\qquad + e_{p,\alpha}K_G^p \sup_{t\in[0,T]}\|S(t)\|^p_{\mathcal{L}(H)}\left(\mathbb{E}\left[\int_0^t \|Z(s) - X_{n_k}(s)\|^{\alpha\beta}_{\mathcal{L}_2(U,H)} \,\mathrm{d}s\right]\right)^{p/\alpha}\Bigg) \\
&\le c\Bigg((1 + T^pK_F^p)\sup_{t\in[0,T]}\mathbb{E}\left[\|Z(t) - X_{n_k}(t)\|^p\right] \\
&\qquad\qquad + e_{p,\alpha}K_G^pT^{p/\alpha}\sup_{t\in[0,T]}\left(\mathbb{E}\left[\|Z(t) - X_{n_k}(t)\|^p\right]\right)^\beta\Bigg).
\end{aligned}
$$

As the last line tends to 0 as $k \to \infty$ by (6.25), it follows that Z is a mild solution to (6.1).

It remains to establish that Z has càdlàg paths, but this follows immediately from the following corollary as X_n has càdlàg paths. ☐

At the end of this section, we present a stronger convergence result for Yosida approximations that not only completes the proof of Theorem 6.6 but also turns out to be useful in applications as will be seen in the following sections.

Corollary 6.12. *For all* $0 < p < \alpha$ *there exists a subsequence* $(X_{n_k})_{k\in\mathbb{N}}$ *of the Yosida approximations, which converges to a solution to* (6.1) *both in* $\mathcal{C}([0,T], L^p(\Omega, H))$ *and uniformly on* $[0,T]$ *almost surely.*

Proof. Lemma 6.11 enables us to choose a subsequence $(X_n)_{n\in\mathbb{N}}$ of the Yosida approximations which converges in $\mathcal{C}([0,T], L^p(\Omega, H))$ to the mild solution X. To prove almost sure convergence, we fix an arbitrary $r > 0$ and estimate

$$
\begin{aligned}
P\left(\sup_{t\in[0,T]}\|X(t) - X_n(t)\| > r\right) &\le P\left(\sup_{t\in[0,T]}\|S(t)(I - R_n)x_0\| > \frac{r}{3}\right) \\
&\quad + P\left(\sup_{t\in[0,T]}\left\|\int_0^t S(t-s)\big(F(X(s)) - R_nF(X_n(s))\big)\,\mathrm{d}s\right\| > \frac{r}{3}\right) \\
&\quad + P\left(\sup_{t\in[0,T]}\left\|\int_0^t S(t-s)\big(G(X(s-)) - R_nG(X_n(s-))\big)\,\mathrm{d}L(s)\right\| > \frac{r}{3}\right).
\end{aligned}
\tag{6.26}
$$

For the following arguments, we define $m := \sup_{t\in[0,T]} \|S(t)\|_{\mathcal{L}(H)}$. As $I - R_n$ converges to zero strongly as $n \to \infty$ we obtain

$$P\left(\sup_{t\in[0,T]} \|S(t)(I-R_n)x_0\| > \frac{r}{3}\right) \le P\left(m\,\|(I-R_n)x_0\| > \frac{r}{3}\right) \to 0.$$

For estimating the second term in (6.26), we apply Markov's inequality and Lipschitz continuity of F in (A2) to obtain

$$\begin{aligned}
&P\left(\sup_{t\in[0,T]}\left\|\int_0^t S(t-s)\big(F(X(s))-R_nF(X_n(s))\big)\,\mathrm{d}s\right\| > \tfrac{r}{3}\right)\\
&\le P\left(m\int_0^T \|F(X(s))-R_nF(X_n(s))\|\,\mathrm{d}s > \tfrac{r}{3}\right)\\
&\le P\left(\int_0^T \|(I-R_n)F(X(s))\|\,\mathrm{d}s > \tfrac{r}{6m}\right) + P\left(\int_0^T \|R_n(F(X(s))-F(X_n(s)))\|\,\mathrm{d}s > \tfrac{r}{6m}\right)\\
&\le \tfrac{6m}{r}\mathbb{E}\left[\int_0^T \|(I-R_n)F(X(s))\|\,\mathrm{d}s\right] + \tfrac{6m}{r}\mathbb{E}\left[\int_0^T \|R_n(F(X(s))-F(X_n(s)))\|\,\mathrm{d}s\right]\\
&\le \tfrac{6m}{r}\mathbb{E}\left[\int_0^T \|(I-R_n)F(X(s))\|\,\mathrm{d}s\right] + \tfrac{6m}{r}TK_F\sup_{n\in\mathbb{N}}\|R_n\|_{\mathcal{L}(H)}\sup_{t\in[0,T]}\mathbb{E}\left[\|X(t)-X_n(t)\|\right].
\end{aligned}$$

We conclude from the last inequality by Lebesgue's dominated convergence theorem and convergence of X_n to X in $\mathcal{C}([0,T],L^1(\Omega,H))$ that

$$\lim_{n\to\infty} P\left(\sup_{t\in[0,T]}\left\|\int_0^t S(t-s)(F(X(s))-R_nF(X_n(s)))\,\mathrm{d}s\right\| > \frac{r}{3}\right) = 0.$$

To estimate the last term in (6.26), we apply the dilation theorem for contraction semigroups, see [39, Th. I.8.1]): there exists a C_0-group $(\hat{S}(t))_{t\in\mathbb{R}}$ of unitary operators $\hat{S}(t)$ on a larger Hilbert space $\hat{H}$ in which H is continuously embedded satisfying $S(t) = \pi\hat{S}(t)i$ for all $t \ge 0$, where π is the projection from $\hat{H}$ to H and i is the continuous embedding of H into $\hat{H}$. Thus, if we denote $m = \sup_{t\in[0,T]}\left\|\pi\hat{S}(t)\right\|_{\mathcal{L}(\hat{H},H)}$, $k = \sup_{s\in[-T,0]}\left\|\hat{S}(s)i\right\|_{\mathcal{L}(H,\hat{H})}$, we may estimate using Markov's inequality, Inequality (2.7) and Hölder continuity of G in (6.6)

$$\begin{aligned}
&P\left(\sup_{t\in[0,T]}\left\|\int_0^t S(t-s)\left(G(X(s-))-R_nG(X_n(s-))\right)\mathrm{d}L(s)\right\| > \frac{r}{3}\right)\\
&\le P\left(\sup_{t\in[0,T]}\left\|\int_0^t S(t-s)\left(I-R_n\right)G(X(s-))\mathrm{d}L(s)\right\| > \frac{r}{6}\right)\\
&\qquad + P\left(\sup_{t\in[0,T]}\left\|\int_0^t S(t-s)R_n\left(G(X(s-))-G(X_n(s-))\right)\mathrm{d}L(s)\right\| > \frac{r}{6}\right)\\
&= P\left(\sup_{t\in[0,T]}\left\|\int_0^t \pi\hat{S}(t-s)i\left(I-R_n\right)G(X(s-))\mathrm{d}L(s)\right\| > \frac{r}{6}\right)\\
&\qquad + P\left(\sup_{t\in[0,T]}\left\|\int_0^t \pi\hat{S}(t-s)iR_n\left(G(X(s-))-G(X_n(s-))\right)\mathrm{d}L(s)\right\| > \frac{r}{6}\right)\\
&= P\left(\sup_{t\in[0,T]}\left\|\pi\hat{S}(t)\int_0^t \hat{S}(-s)i\left(I-R_n\right)G(X(s-))\mathrm{d}L(s)\right\| > \frac{r}{6}\right)
\end{aligned}$$

$$
\begin{aligned}
&\quad + P\left(\sup_{t\in[0,T]}\left\|\pi\hat{S}(t)\int_0^t \hat{S}(-s)iR_n\left(G(X(s-))-G(X_n(s-))\right)\mathrm{d}L(s)\right\|>\frac{r}{6}\right)\\
&\le P\left(\sup_{t\in[0,T]}\left\|\int_0^t \hat{S}(-s)i\left(I-R_n\right)G(X(s-))\mathrm{d}L(s)\right\|_{\hat{H}}>\frac{r}{6m}\right)\\
&\quad + P\left(\sup_{t\in[0,T]}\left\|\int_0^t \hat{S}(-s)iR_n\left(G(X(s-))-G(X_n(s-))\right)\mathrm{d}L(s)\right\|_{\hat{H}}>\frac{r}{6m}\right)\\
&\le \frac{6m}{r}\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t \hat{S}(-s)i\left(I-R_n\right)G(X(s-))\mathrm{d}L(s)\right\|_{\hat{H}}\right]\\
&\quad + \frac{6m}{r}\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t \hat{S}(-s)iR_n\left(G(X(s-))-G(X_n(s-))\right)\mathrm{d}L(s)\right\|_{\hat{H}}\right]\\
&\le e_{1,\alpha}\frac{6m}{r}\left(\mathbb{E}\left[\int_0^T \left\|\hat{S}(-s)i\left(I-R_n\right)G(X(s-))\right\|^{\alpha}_{\mathcal{L}_2(U,\hat{H})}\mathrm{d}s\right]\right)^{1/\alpha}\\
&\quad + e_{1,\alpha}\frac{6m}{r}\left(\mathbb{E}\left[\int_0^T \left\|\hat{S}(-s)iR_n\left(G(X(s-))-G(X_n(s-))\right)\right\|^{\alpha}_{\mathcal{L}_2(U,\hat{H})}\mathrm{d}s\right]\right)^{1/\alpha}\\
&\le e_{1,\alpha}\frac{6m}{r}k\left(\mathbb{E}\left[\int_0^T \left\|\left(I-R_n\right)G(X(s-))\right\|^{\alpha}_{\mathcal{L}_2(U,\hat{H})}\mathrm{d}s\right]\right)^{1/\alpha}\\
&\quad + e_{1,\alpha}\frac{6m}{r}kK_G T^{1/\alpha}\left(\sup_{t\in[0,T]}\mathbb{E}\left[\|(X(t))-X_n(t)\|^{\alpha\beta}\right]\right)^{1/\alpha}.
\end{aligned}
$$

We conclude from the last inequality by Lebesgue's dominated convergence, strong convergence of R_n to I, boundedness G in (6.6) and convergence of X_n to X in $\mathcal{C}([0,T],L^{\alpha\beta}(\Omega,H))$ that

$$
\lim_{n\to\infty} P\left(\sup_{t\in[0,T]}\left\|\int_0^t S(t-s)(G(X(s-))-R_nG(X_n(s-)))\,\mathrm{d}L(s)\right\|>\frac{r}{3}\right)=0,
$$

We have shown that all the terms on the right hand side of (6.26) converge to zero as n tends to infinity which gives uniform convergence of X_n to X in probability on $[0,T]$. This concludes the proof, since uniform convergence in probability implies the existence of a desired subsequence. □

7 Moment boundedness for evolution equations

In this section, we investigate stability properties of the solution for the stochastic evolution equation (6.1) by applying the Itô's formula derived in Theorem 5.1. More precisely, we shall derive conditions on the coefficients such that the mild solution X is ultimately exponentially bounded in the p-th moment, that is there exist constants $m_1, m_2, m_3 > 0$ such that

$$
\mathbb{E}\left[\|X(t)\|^p\right] \le m_1 e^{-tm_2}\mathbb{E}\left[\|x_0\|^p\right]+m_3 \quad \text{for all } t\ge 0.
$$

Recall that $\mathcal{C}_b^2(H)$ denotes the space of continuous real-valued functions defined on H with bounded first and second Fréchet derivatives. In what follows, our goal is to derive a Lyapunov-type criterion using the following operator on $\mathcal{C}_b^2(H)$:

$$
\begin{aligned}
\mathcal{L}f(h) &= \langle Df(h), Ah+F(h)\rangle\\
&\quad + \int_H \left(f(h+g)-f(h)-\langle Df(h),g\rangle\right)\left(\lambda\circ G(h)^{-1}\right)(\mathrm{d}g), \quad h\in D^1 \qquad (7.1)
\end{aligned}
$$

for $f \in \mathcal{C}_b^2(H)$. Note that the right hand side of (7.1) is well defined by Lemma 5.2. We can now state the main result of this section, the following general moment boundedness criterion.

Theorem 7.1. *Let $p \in (0,1)$ be fixed and V be a function in $\mathcal{C}_b^2(H)$ satisfying for some constants $\beta_1, \beta_2, \beta_3, k_1, k_3 > 0$ the inequalities*

$$\beta_1 \|h\|^p - k_1 \le V(h) \le \beta_2 \|h\|^p \quad \text{for all } h \in H, \tag{7.2}$$

$$\mathcal{L}V(h) \le -\beta_3 V(h) + k_3 \quad \text{for all } h \in D^1. \tag{7.3}$$

Then the solution X to (6.1) is exponentially ultimately bounded in the p-th moment:

$$\mathbb{E}\left[\|X(t)\|^p\right] \le \frac{\beta_2}{\beta_1} e^{-\beta_3 t} \mathbb{E}\left[\|x_0\|^p\right] + \frac{1}{\beta_1}\left(k_1 + \frac{k_3}{\beta_3}\right).$$

Before we prove Theorem 7.1 we demonstrate its application by deriving conditions for moment boundedness in terms of the coefficients of Equation (6.1).

Corollary 7.2. *Suppose that there exists $\epsilon > 0$ such that*

$$\langle Ah + F(h), h\rangle \le -\epsilon \|h\|^2 \quad \text{for all } h \in D^1,$$

then the solution to (6.1) is exponentially ultimately bounded in the p-th moment for every $p \in (0,1)$.

Proof. Fix $p \in (0,1)$ and let ζ be a function in $\mathcal{C}^2([0,\infty))$ satisfying $\zeta(x) = x^{p/2}$ for $x \ge 1$ and $\zeta(x) \le 1$ for $x < 1$. By defining $V(h) = \zeta(\|h\|^2)$ for all $h \in H$, we obtain $V \in \mathcal{C}_b^2(H)$ and

$$V(h) = \|h\|^p \quad \text{for all } h \in \overline{B}_H^c \quad \text{and} \quad 0 \le V(h) \le 1 \quad \text{for all } h \in \overline{B}_H.$$

It follows that (7.2) holds with $\beta_1 = \beta_2 = k_1 = 1$. We show that (7.3) also holds. By the definition of V, it follows for each $h \in D^1 \cap \overline{B}_H^c$ that

$$\langle DV(h), Ah + F(h)\rangle = p\|h\|^{p-2}\langle h, Ah + F(h)\rangle \le -\epsilon p \|h\|^p = -\epsilon p V(h).$$

For $h \in D^1 \cap \overline{B}_H$, one obtains by boundedness of F in Assumption (A2) that

$$\langle DV(h), Ah + F(h)\rangle \le \|DV\|_\infty \left(\|A\|_{\mathcal{L}(D^1)} + K_F\right).$$

Since Lemma 5.2 together with boundedness of G in Assumption (A3) implies for each $h \in H$ that

$$\int_H \big(V(h+g) - V(h) - \langle DV(h), g\rangle\big)\, \big(\lambda \circ G^{-1}(h)\big)(\mathrm{d}g) \le d_\alpha^1 \left(2\|DV\|_\infty + \tfrac{1}{2}\left\|D^2 V\right\|_\infty\right) K_G^\alpha,$$

we have verified Condition (7.3). □

In the remaining of this section, we prove Theorem 7.1 using the Yosida approximations established in the previous sections. For this purpose, let X_n denote the mild solution to the approximating equations (6.7) for each $n \in \mathbb{N}$. We may assume due to Corollary 6.12, by passing to a subsequence if necessary, that X_n converges to the solution X of (6.1) uniformly almost surely on $[0,T]$. In what follows, we will routinely pass on to a subsequence without changing the indices.

Proposition 7.3. *The mild solution X_n of (6.7) is a strong solution attaining values in D^1, that is, for each $t \in [0,T]$, it satisfies*

$$X_n(t) = R_n x_0 + \int_0^t \big(AX_n(s) + R_n F(X_n(s))\big)\,\mathrm{d}s + \int_0^t R_n G(X_n(s-))\,\mathrm{d}L(s).$$

Proof. Our argument will follow closely the proof of [1, Th.2]. As mild solution, X_n satisfies

$$X_n(t) = S(t)R_nx_0 + \int_0^t S(t-s)R_nF(X_n(s))\,\mathrm{d}s + \int_0^t S(t-s)R_nG(X_n(s-))\,\mathrm{d}L(s). \quad (7.4)$$

The process X_n is $(\mathcal{F}_t)$-measurable with càdlàg paths and attains values in D^1. First, we obtain from (7.4) by interchanging integrals and $A \in \mathcal{L}(D^1)$ for $t \geq 0$ that

$$\begin{aligned} AX_n(t) = AS(t)R_nx_0 + \int_0^t AS(t-s))R_nF(X_n(s))\,\mathrm{d}s \\ + \int_0^t AS(t-s)R_nG(X_n(s-))\,\mathrm{d}L(s). \end{aligned} \quad (7.5)$$

Each term on the right hand side of (7.5) is almost surely Bochner integrable. Indeed, integrability of the first term is immediate from the uniform boundedness principle. For the second term, boundedness of F in Condition (A2) and commutativity of S and R_n implies

$$\begin{aligned} &\int_0^t \int_0^s \|AS(s-r)R_nF(X_n(r))\|_1 \,\mathrm{d}r\,\mathrm{d}s \\ &\quad \leq \|A\|_{\mathcal{L}(D^1)} \|R_n\|_{\mathcal{L}(H,D^1)} \int_0^t \int_0^s \|S(s-r)F(X_n(s))\| \,\mathrm{d}r\,\mathrm{d}s < \infty \quad \text{a.s.} \end{aligned}$$

Almost sure Bochner integrability of the stochastic integral in (7.5) follows from boundedness of G in Assumption(A3), commutativity of S and R_n, and Theorem 2.4 via the estimate

$$\begin{aligned} &\mathbb{E}\left[\int_0^t \int_0^s \|AS(s-r)R_nG(X_n(r))\|^\alpha_{\mathcal{L}_2(U,D^1)} \,\mathrm{d}r\,\mathrm{d}s\right] \\ &\quad \leq \|A\|^\alpha_{\mathcal{L}(D^1)} \|R_n\|^\alpha_{\mathcal{L}(H,D^1)}\, \mathbb{E}\left[\int_0^t \int_0^s \|S(s-r)G(X_n(r))\|^\alpha_{\mathcal{L}_2(U,H)} \,\mathrm{d}r\,\mathrm{d}s\right] < \infty. \end{aligned}$$

Integrating both sides of (7.5) results in the equality

$$\begin{aligned} \int_0^t AX_n(s)\,\mathrm{d}s = \int_0^t AS(s)R_nx_0\,\mathrm{d}s + \int_0^t \int_0^s AS(s-r)R_nF(X_n(r))\,\mathrm{d}r\,\mathrm{d}s \\ + \int_0^t \int_0^s AS(s-r)R_nG(X_n(r-))\,\mathrm{d}L(r)\,\mathrm{d}s. \end{aligned}$$

Applying Fubini's theorems, see Theorem 2.4 for the stochastic version, and the equality $\int_0^t AS(s)R_nh\,\mathrm{d}s = S(t)R_nh - R_nh$ for all $h \in H$ enable us to conclude

$$\begin{aligned} \int_0^t AX_n(s)\,\mathrm{d}s &= \int_0^t AS(s)R_nx_0\,\mathrm{d}s + \int_0^t \int_r^t AS(s-r)R_nF(X_n(r))\,\mathrm{d}s\,\mathrm{d}r \\ &\quad + \int_0^t \int_r^t AS(s-r)R_nG(X_n(r-))\,\mathrm{d}s\,\mathrm{d}L(r) \\ &= S(t)R_nx_0 - R_nx_0 + \int_0^t S(t-r)R_nF(X_n(r))\,\mathrm{d}r - \int_0^t R_nF(X_n(r))\,\mathrm{d}r \\ &\quad + \int_0^t S(t-r)R_nG(X_n(r-))\,\mathrm{d}L(r) - \int_0^t R_nG(X_n(r-))\,\mathrm{d}L(r) \\ &= X_n(t) - R_nx_0 - \int_0^t R_nF(X_n(r))\,\mathrm{d}r - \int_0^t R_nG(X_n(r-))\,\mathrm{d}L(r), \end{aligned}$$

which verifies X_n as a strong solution to (6.1). □

We denote by $\mathcal{L}_n$ the usual generator associated with the Yosida approximations X_n, $n \in \mathbb{N}$, defined for $f \in \mathcal{C}_b^2(H)$ and $h \in D^1$ by

$$\begin{aligned}\mathcal{L}_n f(h) &= \langle Df(h), Ah + R_n F(h)\rangle \\ &\quad + \int_H \big(f(h + R_n g) - f(h) - \langle Df(h), R_n g\rangle\big) \big(\lambda \circ G(h)^{-1}\big)(\mathrm{d}g). \end{aligned} \tag{7.6}$$

The right hand side of (7.6) is well defined by Lemma 5.2. Recall that the counterpart to $\mathcal{L}_n$ for the mild solution X denoted by $\mathcal{L}$ was introduced in (7.1). The generators $\mathcal{L}_n$ and $\mathcal{L}$ are related by the following crucial convergence result.

Lemma 7.4. *Let $(X_n)_{n\in\mathbb{N}}$ be solutions of* (6.7) *which a.s. converges uniformly to the solution of* (6.1)*. It follows for each $f \in \mathcal{C}_b^2(H)$ that*

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T \Big|\mathcal{L}_n f(X_n(s)) \ - \mathcal{L} f(X_n(s))\Big| \,\mathrm{d}s\right] = 0.$$

Proof. Denoting $\lambda_{G(h)} := \lambda \circ G(h)^{-1}$ for each $h \in D^1$, we obtain

$$\begin{aligned} &|\mathcal{L}f(h) - \mathcal{L}_n f(h)| \le \|Df\|_\infty \|(I - R_n)\, F(h)\| \\ &+ \int_{\overline{B}_H} \big|f(h+g) - f(h+R_n g) - \langle Df(h), (I-R_n)\, g\rangle\big|\, \lambda_{G(h)}(\mathrm{d}g) \\ &+ \int_{\overline{B}_H^c} \big|f(h+g) - f(h+R_n g)\big|\, \lambda_{G(h)}(\mathrm{d}g) + \int_{\overline{B}_H^c} \big|\langle Df(h), (I-R_n)\, g\rangle\big|\, \lambda_{G(h)}(\mathrm{d}g). \end{aligned} \tag{7.7}$$

Taylor's remainder theorem in the integral form implies

$$\begin{aligned} &\int_{\overline{B}_H} \big|f(h+g) - f(h+R_n g) - \langle Df(h), (I-R_n)\, g\rangle\big|\, \lambda_{G(h)}(\mathrm{d}g) \\ &\quad \le \int_{\overline{B}_H} \int_0^1 \big|\langle D^2 f(h + \theta\,(I-R_n)\, g)\,(I-R_n)\, g, (I-R_n)\, g\rangle (1-\theta)\,\big|\,\mathrm{d}\theta\, \lambda_{G(h)}(\mathrm{d}g) \\ &\quad \le \frac{1}{2}\left\|D^2 f\right\|_\infty \int_{\overline{B}_H} \|(I-R_n)\, g\|^2\ \lambda_{G(h)}(\mathrm{d}g). \end{aligned}$$

In the same way, we obtain

$$\int_{\overline{B}_H^c} \big|f(h+g) - f(h+R_n g)\big|\, \lambda_{G(h)}(\mathrm{d}g) \le \|Df\|_\infty \int_{\overline{B}_H^c} \|(I-R_n)\, g\|\ \lambda_{G(h)}(\mathrm{d}g).$$

and also

$$\int_{\overline{B}_H^c} \big|\langle Df(h), (I-R_n)\, g\rangle\big|\, \lambda_{G(h)}(\mathrm{d}g) \le \|Df\|_\infty \int_{\overline{B}_H^c} \|(I-R_n)\, g\|\ \lambda_{G(h)}(\mathrm{d}g).$$

Applying the last three estimates to (7.7) and taking expectation on both sides, it follows from Inequality (2.2) and for each $n \in \mathbb{N}$ that

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \big|\mathcal{L}_n f(X_n(s)) - \mathcal{L} f(X_n(s))\big|\,\mathrm{d}s\right] \\ &\le \|Df\|_\infty\, \mathbb{E}\left[\int_0^T \|(I-R_n)\, F(X_n(s))\|\ \mathrm{d}s\right] + c\mathbb{E}\left[\int_0^T \|(I-R_n)\, G(X_n(s))\|_{\mathcal{L}_2(U,H)}^\alpha\ \mathrm{d}s\right], \end{aligned}$$

where $c := d_\alpha^1 \big(2\,\|Df\|_\infty + \frac{1}{2}\left\|D^2 f\right\|_\infty\big)$.

To complete the proof, it remains to show that both

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T ||(I-R_n)\, F(X_n(s))||\,\mathrm{d}s\right] = 0, \tag{7.8}$$

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T ||(I-R_n)\, G(X_n(s))||^{\alpha}_{\mathcal{L}_2(U,H)}\,\mathrm{d}s\right] = 0. \tag{7.9}$$

Let $t \in [0,T]$ be arbitrary but fixed, and recall that we chose $X_n(t)$ almost surely convergent and thus $\{X_m(t)(\omega) : m \in \mathbb{N}\} \subset H$ is compact for almost all $\omega \in \Omega$. Strong convergence of $I - R_n$ to zero, see [30, Le. 1.3.2], continuity of F and G and the fact that continuous mapping converging pointwise to a continuous mapping converge uniformly over compacts together imply for each $t \in [0,T]$ that, almost surely, we obtain

$$\lim_{n\to\infty} ||(I-R_n)\, F(X_n(t))|| \le \lim_{n\to\infty} \sup_{m\in\mathbb{N}} ||(I-R_n)\, F(X_m(t))|| = 0,$$
$$\lim_{n\to\infty} ||(I-R_n)\, G(X_n(t))||^{\alpha}_{\mathcal{L}_2(U,H)} \le \lim_{n\to\infty} \sup_{m\in\mathbb{N}} ||(I-R_n)\, G(X_m(t))||^{\alpha}_{\mathcal{L}_2(U,H)} = 0.$$

Since the boundedness conditions in (A2) and (A3) guarantee

$$||(I-R_n)\, F(X_n(t))|| \le \left(\sup_{n\in\mathbb{N}} ||I-R_n||_{\mathcal{L}(H)}\right) K_F \quad \text{a.s.},$$
$$||(I-R_n)\, G(X_n(t))||^{\alpha}_{\mathcal{L}_2(U,H)} \le \left(\sup_{n\in\mathbb{N}} ||I-R_n||^{\alpha}_{\mathcal{L}(H)}\right) K_G^{\alpha} \quad \text{a.s.}$$

an application of Lebesgue's dominated convergence theorem verifies (7.8) and (7.9), which completes the proof. □

Proof of Theorem 7.1. Let $(X_n)_{n\in\mathbb{N}}$ be the solutions of (6.7). Because of Corollary 6.12, we can assume that $(X_n)_{n\in\mathbb{N}}$ converges uniformly to the solution of (6.1) a.s. Proposition 7.3 enables us to apply the Itô formula in Theorem 5.1 to X_n, which results in

$$\begin{aligned} V(X_n(t)) = V(X_n(0) &+ \int_0^t \mathcal{L}_n V(X_n(s))\mathrm{d}s + \int_0^t \langle G(X_n(s-))^* R_n^* DV(X_n(s-)), \cdot\rangle\, \mathrm{d}L(s) \\ &+ \int_0^t \int_H V(X_n(s-)+h) - V(X_n(s-)) - \langle DV(X_n(s-)), h\rangle (\mu^{X_n} - \nu^{X_n})\,(\mathrm{d}s, \mathrm{d}h) \end{aligned} \tag{7.10}$$

almost surely for all $t \ge 0$. Applying the product formula to the real-valued semimartingale $V(X_n(\cdot))$ and the function $t \mapsto e^{\beta_3 t}$ and taking expectations on both sides of (7.10) shows

$$e^{\beta_3 t}\mathbb{E}\,[V(X_n(t))] = \mathbb{E}\,[V(X_n(0))] + \mathbb{E}\left[\int_0^t e^{\beta_3 s}\,(\beta_3 V(X_n(s)) + \mathcal{L}_n V(X_n(s)))\,\mathrm{d}s\right]. \tag{7.11}$$

Here, we used the fact that the last two integrals in (7.10) define martingales, and thus have expectation zero. This follows from the observation that they are local martingales according to Lemma 2.3 and Theorem 5.1 and are uniformly bounded. The latter is guaranteed by the boundedness of G in (A3), since

$$\begin{aligned} &\mathbb{E}\left[\int_0^t ||\langle G(X_n(s))^* R_n^* DV(X_n(s)), \cdot\rangle||^{\alpha}_{\mathcal{L}_2(U,\mathbb{R})}\,\mathrm{d}s\right] \\ &\quad = \mathbb{E}\left[\int_0^t ||G(X_n(s))^* R_n^* DV(X_n(s))||^{\alpha}\,\mathrm{d}s\right] \le ||R_n||^{\alpha}_{\mathcal{L}(H)}\, ||DV||^{\alpha}_{\infty}\, T K_G^{\alpha} < \infty, \end{aligned}$$

and similarly, by using Lemma 5.2,

$$\mathbb{E}\left[\int_0^t \int_H |V(X_n(s-)+h) - V(X_n(s-)) - \langle DV(X_n(s-),h\rangle|\,\nu^{X_n}(\mathrm{d}s,\mathrm{d}h)\right]$$
$$\leq d_\alpha^1 \left(2\,||DV||_\infty + \frac{1}{2}\,\left|\left|D^2V\right|\right|_\infty\right) ||R_n||_{\mathcal{L}(H)}^\alpha\, \mathbb{E}\left[\int_0^t ||G(X_n(s))||_{\mathcal{L}_2(U,H)}^\alpha\ \mathrm{d}s\right] < \infty.$$

The first term on the right hand side in (7.11) is finite since

$$\mathbb{E}\left[V(X_n(0))\right] \leq \beta_2\, \|R_n\|_{\mathcal{L}(H)}^p\, \mathbb{E}\left[\|x_0\|^p\right] < \infty.$$

The same holds for the second term, which can be shown using the same arguments as in the proof of Lemma 7.4. By applying Inequality (7.3) to (7.11), we conclude

$$e^{\beta_3 t}\mathbb{E}\left[[V(X_n(t))\right] \leq \mathbb{E}\left[V(X_n(0))\right] + \mathbb{E}\left[\int_0^t e^{\beta_3 s}\Big(- \mathcal{L}V(X_n(s)) + \mathcal{L}_nV(X_n(s)) + k_3\Big)\mathrm{d}s\right]$$
$$\leq \mathbb{E}\left[V(X_n(0))\right] + e^{\beta_3 T}\mathbb{E}\left[\int_0^t \Big|\mathcal{L}_nV(X_n(s)) - \mathcal{L}V(X_n(s))\Big|\mathrm{d}s\right] + \frac{k_3}{\beta_3}\left(e^{\beta_3 t} - 1\right).$$

Lemma 7.4 together with Fatou's lemma implies

$$\mathbb{E}\left[V(X(t))\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[V(X_n(t))\right] \leq e^{-\beta_3 t}\mathbb{E}\left[V(x_0)\right] + \frac{k_3}{\beta_3}.$$

Applying Assumption (7.2) completes the proof. □

8 Mild Itô formula

In this section, we prove an Itô formula for mild solutions of Equation (6.1) and mappings $f \in \mathcal{C}_b^2(H)$ such that the second derivative $D^2 f$ is not only continuous but satisfies

$$\lim_{n\to\infty} \|g_n - g\| = 0 \implies \lim_{n\to\infty} \sup_{h\in\overline{B}_H} \left\|D^2 f(g_n + h) - D^2 f(g+h)\right\|_{\mathcal{L}(H)} = 0. \tag{8.1}$$

The subspace of all these functions is denoted by $\mathcal{C}_{b,u}^2(H)$.

Theorem 8.1 (Itô formula for mild solutions)**.** *A mild solution X of* (6.1) *satisfies for each $f \in \mathcal{C}_{b,u}^2(H)$ and $t \geq 0$ that*

$$\begin{aligned} f(X(t)) = f(x_0) &+ \int_0^t \langle G(X(s-)^* Df(X(s-))), \cdot\rangle\, \mathrm{d}L(s) \qquad (8.2)\\ &+ \int_0^t \int_H \big(f(X(s-)+h) - f(X(s-)) - \langle Df(X(s-)),h\rangle\big)\,\left(\mu^X - \nu^X\right)(\mathrm{d}s,\mathrm{d}h)\\ &+ \lim_{n\to\infty}\left(\int_0^t \langle Df(X_n(s)), AX_n(s)\rangle\,\mathrm{d}s\right) + \int_0^t \langle Df(X(s)), F(X(s))\rangle\,\mathrm{d}s\\ &+ \int_0^t \int_H \big(f(X(s)+h) - f(X(s)) - \langle Df(X(s)),h\rangle\big)\,\left(\lambda\circ G(X(s))^{-1}\right)(\mathrm{d}h)\,\mathrm{d}s, \end{aligned}$$

where the limit is taken in $L_P^0(\Omega,\mathbb{R})$.

Remark 8.2. Note that while X may not be a semimartingale, the compensated measure $\mu^X - \nu^X$ in (8.2) still exists as X is both adapted and càdlàg; see [16, Chap. II].

Remark 8.3. Unlike in similar situation with the driving noise being Gaussian e.g. in [25] we do not identify the limit in (8.2) as then the imposed assumptions on f are very restrictive. In many applications (see e.g. [1]), it is enough to identify some bound on

$$\lim_{n\to\infty}\left(\int_0^t \langle Df(X_n(s)), AX_n(s)\rangle \,\mathrm{d}s\right)$$

which leads to natural assumptions on the generator A.

We divide the proof of the above theorem in some technical lemmas. To simplify the notation, we introduce the function $T_f\colon H\times H\to\mathbb{R}$ for $f\in\mathcal{C}^2_{b,u}(H)$ defined by

$$T_f(g,h)=f(g+h)-f(g)-\langle Df(g),h\rangle,\quad g,h\in H.$$

Lemma 8.4. *Let λ be the cylindrial Lévy measure of L. It follows for every $f\in\mathcal{C}^2_b(H)$, $\phi\in\mathcal{L}_2(U,H)$ and $g,h\in H$ that*

$$\int_H |T_f(g,b)-T_f(h,b)|\left(\lambda\circ\phi^{-1}\right)(\mathrm{d}b)$$
$$\le 2d^1_\alpha \|\phi\|^\alpha_{\mathcal{L}_2(U,H)}\left(\sup_{b\in\overline{B}_H}\left\|D^2f(g+b)-D^2f(h+b)\right\|_{\mathcal{L}(H)}+\left\|D^2f\right\|_\infty\|g-h\|\right).$$

Proof. Taylor's remainder theorem in the integral form and Inequality (2.2) imply

$$\begin{aligned}
&\int_{\overline{B}_H}|T_f(g,b)-T_f(h,b)|\,\left(\lambda\circ\phi^{-1}\right)(\mathrm{d}b)\\
&\quad=\int_{\overline{B}_H}\left|\int_0^1\langle (D^2f(g+\theta b)-D^2f(h+\theta b))b,b\rangle(1-\theta)\,\mathrm{d}\theta\right|\,\left(\lambda\circ\phi^{-1}\right)(\mathrm{d}b)\\
&\quad\le\frac12\sup_{b\in\overline{B}_H}\left\|D^2f(g+b)-D^2f(h+b)\right\|_{\mathcal{L}(H)}\int_{\overline{B}_H}\|b\|^2\,\left(\lambda\circ\phi^{-1}\right)(\mathrm{d}b)\\
&\quad\le\frac12 d^1_\alpha\left(\sup_{b\in\overline{B}_H}\left\|D^2f(g+b)-D^2f(h+b)\right\|_{\mathcal{L}(H)}\right)\|\phi\|^\alpha_{\mathcal{L}_2(U,H)}. \qquad (8.3)
\end{aligned}$$

A similar argument yields

$$\begin{aligned}
&\int_{\overline{B}^c_H}|T_f(g,b)-T_f(h,b)|\,\left(\lambda\circ\phi^{-1}\right)(\mathrm{d}b)\\
&\quad\le\int_{\overline{B}^c_H}\left|\int_0^1\langle Df(g+\theta b)-Df(h+\theta b),b\rangle\,\mathrm{d}\theta\right|\,\left(\lambda\circ\phi^{-1}\right)(\mathrm{d}b)\\
&\qquad\quad+\int_{\overline{B}^c_H}|\langle Df(g)-Df(h),b\rangle|\,\left(\lambda\circ\phi^{-1}\right)(\mathrm{d}b)\\
&\quad\le\left(\sup_{b\in H}\|Df(g+b)-Df(h+b)\|+\|Df(g)-Df(h)\|\right)\int_{\overline{B}^c_H}\|b\|\,\left(\lambda\circ\phi^{-1}\right)(\mathrm{d}b)\\
&\quad\le 2d^1_\alpha\left\|D^2f\right\|_\infty\|g-h\|\,\|\phi\|^\alpha_{\mathcal{L}_2(U,H)}. \qquad (8.4)
\end{aligned}$$

Combining Inequalities (8.3) and (8.4) completes the proof. □

Lemma 8.5. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of càdlàg processes in H which converges to a process X both in $\mathcal{C}([0,T],L^p(\Omega,H))$ and uniformly on $[0,T]$ almost surely. Then it follows for any $f\in\mathcal{C}^2_{b,u}(H)$ and $t\in[0,T]$ that*

$$\lim_{n\to\infty}\int_0^t\int_H T_f(X_n(s-),h)\,\mu^{X_n}(\mathrm{d}s,\mathrm{d}h)=\int_0^t\int_H T_f(X(s-),h)\,\mu^X(\mathrm{d}s,\mathrm{d}h)\qquad \textit{in } L^0_P(\Omega,\mathbb{R}).$$

Proof. Theorem 3.5 guarantees for each $n \in \mathbb{N}$ that

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t \int_H \big(T_f(X_n(s-),h) - T_f(X(s-),h)\big)\,\mu^{X_n}(\mathrm{d}s,\mathrm{d}h)\right|\right] \\
&\quad \le \mathbb{E}\left[\int_0^t \int_H \big|T_f(X_n(s-),h) - T_f(X(s-),h)\big|\,\mu^{X_n}(\mathrm{d}s,\mathrm{d}h)\right] \\
&\quad = \mathbb{E}\left[\int_0^t \int_H \big|T_f(X_n(s),h) - T_f(X(s),h)\big|\,\nu^{X_n}(\mathrm{d}s,\mathrm{d}h)\right] \\
&\quad = \mathbb{E}\left[\int_0^t \int_H \big|T_f(X_n(s),h) - T_f(X(s),h)\big|\,\left(\lambda \circ (R_n G(X_n(s-)))^{-1}\right)(\mathrm{d}h)\,\mathrm{d}s\right]. \qquad (8.5)
\end{aligned}$$

Since Remark 6.7 guarantees $c := 2d_\alpha^1 K_G^\alpha \sup_{n\in\mathbb{N}} \|R_n\|_{\mathcal{L}(H)}^\alpha < \infty$, we obtain from Lemma 8.4 for $P\otimes$Leb-a.a. $(\omega,s)\in\Omega\times[0,T]$ that

$$\begin{aligned}
&\int_H |T_f(X_n(s)(\omega),h) - T_f(X(s)(\omega),h)|\,\left(\lambda\circ(R_nG(X_n(s-)(\omega)))^{-1}\right)(\mathrm{d}h) \\
&\le c\Bigg(\sup_{b\in\overline{B}_H} \left\|D^2 f(X_n(s)(\omega)+b) - D^2 f(X(s)(\omega)+b)\right\|_{\mathcal{L}(H)} \\
&\qquad\qquad + \left\|D^2 f\right\|_\infty \|f(X_n(s)(\omega)) - f(X(s)(\omega))\|\Bigg).
\end{aligned}$$

Since f satisfies (8.1), Lebesgue's dominated convergence theorem implies

$$\lim_{n\to\infty} E\left[\left|\int_0^t \int_H \big(T_f(X_n(s-),h) - T_f(X(s-),h)\big)\,\mu^{X_n}(\mathrm{d}s,\mathrm{d}h)\right|\right] = 0. \qquad (8.6)$$

For the next step, fix $\epsilon,\epsilon' > 0$, and use for any $m,n\in\mathbb{N}$ the decomposition

$$\begin{aligned}
&\left(\left|\int_0^t \int_H T_f(X(s-),h)\left(\mu^{X_n}(\mathrm{d}s,\mathrm{d}h) - \mu^X(\mathrm{d}s,\mathrm{d}h)\right)\right| > \epsilon\right) \\
&\quad \le P\left(\left|\int_0^t \int_{\overline{B}_H(1/m)} T_f(X(s-),h)\,\mu^{X_n}(\mathrm{d}s,\mathrm{d}h)\right| > \frac{\epsilon}{3}\right) \\
&\qquad\quad + P\left(\left|\int_0^t \int_{\overline{B}_H(1/m)} T_f(X(s-),h)\,\mu^{X}(\mathrm{d}s,\mathrm{d}h)\right| > \frac{\epsilon}{3}\right) \\
&\qquad\quad + P\left(\left|\int_0^t \int_{\overline{B}_H(1/m)^c} T_f(X(s-),h)\,\left(\mu^{X_n}(\mathrm{d}s,\mathrm{d}h) - \mu^X(\mathrm{d}s,\mathrm{d}h)\right)\right| > \frac{\epsilon}{3}\right). \qquad (8.7)
\end{aligned}$$

Since Taylor's remainder theorem in the integral form gurantees that $|T_f(X(s-),h)| \le \frac{1}{2}\left\|D^2 f\right\|_\infty \|h\|^2$ for all $h\in H$, we obtain by applying Theorem 3.5 and Inequality (2.2) that

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t \int_{\overline{B}_H(1/m)} T_f(X(s-),h)\,\mu^{X_n}(\mathrm{d}s,\mathrm{d}h)\right|\right] \\
&\le \frac{1}{2}\left\|D^2 f\right\|_\infty \mathbb{E}\left[\int_0^t \int_{\overline{B}_H(1/m)} \|h\|^2 \left(\lambda\circ(R_nG(X_n(s)))^{-1}\right)(\mathrm{d}h)\mathrm{d}s\right] \le d_\alpha^m K_G^\alpha \frac{1}{2}\left\|D^2 f\right\|_\infty T.
\end{aligned}$$

Since the last line is independent of $n\in\mathbb{N}$ and $d_\alpha^m \to 0$ as $m\to\infty$ according to Inequality (2.2), Markov's inequality implies that there exists $m_1\in\mathbb{N}$ such that for all $m\ge m_1$ and all $n\in\mathbb{N}$

$$P\left(\left|\int_0^t \int_{\overline{B}_H(1/m)} T_f(X(s-),h)\,\mu^{X_n}(\mathrm{d}s,\mathrm{d}h)\right| > \frac{\epsilon}{3}\right) \le \epsilon'. \qquad (8.8)$$

Exactly the same arguments establish that for all $m \geq m_1$

$$P\left(\left|\int_0^t \int_{\overline{B}_H(1/m)} T_f(X(s-),h)\, \mu^X(\mathrm{d}s,\mathrm{d}h)\right| > \frac{\epsilon}{3}\right) \leq \epsilon'. \tag{8.9}$$

For the last term in (8.7) we calculate for each $m, n \in \mathbb{N}$ that

$$\begin{aligned}
&\int_0^t \int_{\overline{B}_H(1/m)^c} T_f(X(s-),h)\, \left(\mu^{X_n}(\mathrm{d}s,\mathrm{d}h) - \mu^X(\mathrm{d}s,\mathrm{d}h)\right) \\
&= \sum_{0\leq s\leq t} T_f(X(s-),\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X_n(s)) \\
&\quad - \sum_{0\leq s\leq t} T_f(X(s-),\Delta X(s))\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X(s)) \\
&= \sum_{0\leq s\leq t} T_f(X(s-),\Delta X_n(s))\left(\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X_n(s)) - \mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X(s))\right) \\
&\quad + \sum_{0\leq s\leq t} \left(T_f(X(s-),\Delta X_n(s)) - T_f(X(s-),\Delta X(s))\right)\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X(s)).
\end{aligned} \tag{8.10}$$

For estimating the first term in the last line, we use the equality $\mathbb{1}_A(x) - \mathbb{1}_A(y) = \mathbb{1}_A(x)\mathbb{1}_{A^c}(y) - \mathbb{1}_{A^c}(x)\mathbb{1}_A(y)$. For the first term, resulting from the application of this identity, we conclude from Taylor's remainder theorem in the integral form that

$$\begin{aligned}
&\left|\sum_{0\leq s\leq t} T_f\left(X(s-),\Delta X_n(s)\right)\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X(s))\right| \\
&\leq \sum_{0\leq s\leq t} \left|T_f(X(s-),\Delta X_n(s))\right|\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X(s)) \\
&\leq \frac{1}{2}\left\|D^2 f\right\|_\infty \sum_{0\leq s\leq t} \|\Delta X_n(s)\|^2\, \mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X(s)) \\
&\leq c_f \sum_{0\leq s\leq t}\left(\|\Delta X(s)\|^2 + \|\Delta X_n(s) - \Delta X(s)\|^2\right)\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X(s)),
\end{aligned} \tag{8.11}$$

where we used the notation $c_f := \left\|D^2 f\right\|_\infty$. Applying Theorem 3.5 and using the boundedness assumption on G in (A3) result in

$$\begin{aligned}
\mathbb{E}\left[\sum_{0\leq s\leq t}\|\Delta X(s)\|^2\, \mathbb{1}_{\overline{B}_H(1/m)}(\Delta X(s))\right] &= \mathbb{E}\left[\int_0^t \int_{\overline{B}_H(1/m)} \|h\|^2\, \mu^X(\mathrm{d}s,\mathrm{d}h)\right] \\
&= \mathbb{E}\left[\int_0^t \int_{\overline{B}_H(1/m)} \|h\|^2\, \left(\lambda\circ G(X(s-))^{-1}\right)(\mathrm{d}h,\mathrm{d}s)\right] \leq d_\alpha^m T K_G^\alpha.
\end{aligned}$$

Since $d_\alpha^m \to 0$ as $m \to \infty$, Markov's inequality implies that there exists $m_2 \in \mathbb{N}$ with $m_2 \geq m_1$ such that for all $m \geq m_2$ and all $n \in \mathbb{N}$

$$P\left(\sum_{0\leq s\leq t}\|\Delta X(s)\|^2\, \mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X(s)) \geq \frac{\epsilon}{24 c_f}\right) \leq \frac{\epsilon'}{8}. \tag{8.12}$$

In the remaining part of the proof fix $m \geq m_2 \geq m_1$ such that (8.12) is satisfied. There exists $n_1 \in \mathbb{N}$ such that the set $A_n := \{\sup_{s\in[0,t]}\|\Delta X_n(s) - \Delta X(s)\| \leq \frac{1}{2m}\}$ satisfies

$P(A_n^c) \le \frac{\epsilon'}{16}$ for all $n \ge n_1$. If $\sup_{s\in[0,t]} \|\Delta X_n(s)\| \ge \frac{1}{m}$ then we obtain on A_n for $n \ge n_1$ that

$$\sup_{s\in[0,t]} \|\Delta X(s)\| \ge -\sup_{s\in[0,t]} \|\Delta X(s) - \Delta X_n(s)\| + \sup_{s\in[0,t]} \|\Delta X_n(s)\| \ge \frac{1}{2m}.$$

Consequently, we obtain for all $n \ge n_1$ that

$$\begin{aligned} &P\left(\sum_{0\le s\le t} \|\Delta X_n(s) - \Delta X(s)\|^2 \mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X(s)) \ge \frac{\epsilon}{24c_f}\right) \\ &\qquad \le P\left(\sum_{0\le s\le t} \|\Delta X_n(s) - \Delta X(s)\|^2 \mathbb{1}_{\overline{B}_H(1/2m)^c}(\Delta X(s)) \ge \frac{\epsilon}{24c_f}\right) + \frac{\epsilon'}{16}. \end{aligned}$$

Since X has only finitely many jumps in $\overline{B}_H(1/2m)^c$ on $[0,t]$ and $\Delta X_n(s)$ converges to $\Delta X(s)$ for all $s \in [0,t]$, there exists n_2 such that for all $n \ge n_2$

$$P\left(\sum_{0\le s\le t} \|\Delta X_n(s) - \Delta X(s)\|^2 \mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X(s)) \ge \frac{\epsilon}{24c_f}\right) \le \frac{\epsilon'}{8}.$$

Applying this together with (8.12) to Inequality (8.11) proves that for $m \ge m_2$ there exists n_2 such that for all $n \ge n_2$

$$P\left(\sum_{0\le s\le t} T_f\left(X(s-), \Delta X_n(s)\right)\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X(s)) \ge \frac{\epsilon}{12}\right) \le \frac{\epsilon'}{4}. \tag{8.13}$$

As ΔX_n converges to ΔX uniformly on $[0,T]$ almost surely we obtain that for almost all $\omega \in \Omega$ we have

$$\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X_n(s)(\omega))\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X(s)(\omega)) = 0$$

if n is large enough. Therefore

$$\lim_{n\to\infty}\left(\sum_{0\le s\le t} T_f(X(s-), \Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X(s))\right) = 0 \quad \text{a.s.}$$

and thus we obtain that there exists n_3 such that for all $n \ge n_3$ we have

$$P\left(\sum_{0\le s\le t} T_f(X(s-), \Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)}(\Delta X_n(s))\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X(s)) \ge \frac{\epsilon}{12}\right) \le \frac{\epsilon'}{4}.$$

Combining this with (8.13) shows that for $m \ge m_2$ and $n \ge \max\{n_2, n_3\}$ we have

$$P\left(\sum_{0\le s\le t} T_f(X(s-), \Delta X_n(s))\left(\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X_n(s)) - \mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X(s))\right) \ge \frac{\epsilon}{6}\right) \le \frac{\epsilon'}{2}. \tag{8.14}$$

Since X has only finitely many jumps in $\overline{B}_H(1/m)^c$ on $[0,t]$ and $\Delta X_n(s)$ converges to $\Delta X(s)$ for all $s \in [0,t]$, there exits n_4 such that for all $n \ge n_4$

$$P\left(\sum_{0\le s\le t} \left(T_f(X(s-), \Delta X_n(s)) - T_f(X(s-), \Delta X(s))\right)\mathbb{1}_{\overline{B}_H(1/m)^c}(\Delta X(s)) \ge \frac{\epsilon}{6}\right) \le \frac{\epsilon'}{2}.$$

Applying this together with (8.14) to (8.10) shows

$$P\left(\int_0^t \int_{\overline{B}_H(1/m)^c} T_f(X(s-),h)\,\left(\mu^{X_n}(\mathrm{d}s,\mathrm{d}h)-\mu^X(\mathrm{d}s,\mathrm{d}h)\right|\geq \frac{\epsilon}{3}\right)\leq \epsilon'. \tag{8.15}$$

By applying Equations (8.8), (8.9) and (8.15) to (8.7), the proof is now complete. □

Proof of Theorem 8.1. Let $(X_n)_{n\in\mathbb{N}}$ be the solutions to (6.7). According to Corollary 6.12, we can assume that $(X_n)_{n\in\mathbb{N}}$ converges both in $\mathcal{C}([0,T],L^p(\Omega,H))$ and uniformly on $[0,T]$ almost surely to the mild solution X, which has càdlàg paths. Since X_n is a strong solution to (6.7) according to Proposition 7.3, the Itô formula in Theorem 5.1 implies for all $t\geq 0$ and $n\in\mathbb{N}$ that

$$\begin{aligned} f(X_n(t)) = f(R_n x_0) &+ \int_0^t \langle G(X_n(s-))^* R_n^* Df(X_n(s-)),\cdot\rangle\,\mathrm{d}L(s) \\ &+ \int_0^t \langle Df(X_n(s)), AX_n(s)\rangle\,\mathrm{d}s + \int_0^t \langle Df(X_n(s)), R_nF(X_n(s))\rangle\,\mathrm{d}s \\ &+ \int_0^t\int_H \Big(f(X_n(s-)+h)-f(X_n(s-))-\langle Df(X_n(s-)),h\rangle\Big)\,\mu^{X_n}(\mathrm{d}s,\mathrm{d}h). \end{aligned} \tag{8.16}$$

Continuity of f shows that $f(X_n(t))\to f(X(t))$ and $f(R_nx_0)\to f(x_0)$ a.s. Inequality (2.7) implies for the first integral in (8.16) that

$$\begin{aligned} &\mathbb{E}\left[\left\|\int_0^t \langle G(X_n(s-))^* R_n^* Df(X_n(s-)),\cdot\rangle\,\mathrm{d}L(s) - \int_0^t \langle G(X(s-))^* Df(X(s-)),\cdot\rangle\,\mathrm{d}L(s)\right\|\right] \\ &\quad\leq e_{1,\alpha}\left(\mathbb{E}\left[\int_0^t \|G(X_n(s))^* R_n^* Df(X_n(s)) - G(X(s))^* Df(X(s))\|^\alpha_{\mathcal{L}_2(U,\mathbb{R})}\,\mathrm{d}s\right]\right)^{1/\alpha}, \end{aligned}$$

which tends to zero by a similar argument as in the proof of Lemma 7.4. It follows in $L^1_P(\Omega,\mathbb{R})$ that

$$\lim_{n\to\infty}\int_0^t \langle G(X_n(s-))^* R_n^* Df(X_n(s-)),\cdot\rangle\,\mathrm{d}L(s) = \int_0^t \langle G(X(s-))^* Df(X(s-)),\cdot\rangle\,\mathrm{d}L(s).$$

Lemma 8.5 implies in $L^0_P(\Omega,\mathbb{R})$ that

$$\begin{aligned} &\lim_{n\to\infty}\int_0^t\int_H \left(f(X_n(s-)+h)-f(X_n(s-))-\langle Df(X_n(s-)),h\rangle\right)\mu^{X_n}(\mathrm{d}s,\mathrm{d}h) \\ &\quad= \int_0^t\int_H \left(f(X(s-)+h)-f(X(s-))-\langle Df(X(s-)),h\rangle\right)\mu^{X}(\mathrm{d}s,\mathrm{d}h). \end{aligned}$$

Convergence of $(X_n)_{n\in\mathbb{N}}$ and Lebesgue's dominated convergence theorem yields

$$\lim_{n\to\infty}\int_0^t \langle Df(X_n(s)), R_nF(X_n(s))\rangle\,\mathrm{d}s = \int_0^t \langle Df(X(s)), F(X(s))\rangle\,\mathrm{d}s \quad \text{a.s.}$$

As all terms in (8.16) converge in $L^0_P(\Omega,\mathbb{R})$, it follows that the remaining term

$$\int_0^t \langle Df(X_n(s)), AX_n(s)\,\rangle \mathrm{d}s$$

also converges in $L^0_P(\Omega,\mathbb{R})$, which completes the proof. □

9 Appendix

Lemma 9.1. *Let V be a separable Hilbert space with the norm $\|\cdot\|_V$ and let $A_m \in \mathcal{L}(V)$ be a sequence of operators converging strongly to 0. If $(B_n)_{n\in\mathbb{N}}$ is a tight sequence of uniformly bounded V-valued random variables, then it follows for all $p > 0$ that*

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} \mathbb{E}\left[\|A_m B_n\|_V^p\right] = 0.$$

Proof. Let $\epsilon > 0$ be fixed. Our assumptions guarantee that there exists a constant $c > 0$ such that $\sup_{n,m\in\mathbb{N}} \|A_m B_n\|_V^p \le c$ a.s. and a compact set $K_\epsilon \subseteq V$ satisfying $P(B_n \notin K_\epsilon) < \frac{\epsilon}{c}$ for every $n \in \mathbb{N}$. Since continuous mapping converging to zero converges uniformly on compacts there exists $m_1 \in \mathbb{N}$ such that for all $m \ge m_1$ we have

$$\sup_{n\in\mathbb{N}} \int_{\{B_n\in K_\epsilon\}} \|A_m B_n(\omega)\|_V^p \; P(\mathrm{d}\omega) < \epsilon.$$

It follows for all $n \in \mathbb{N}$ and $m \ge m_1$ that

$$\mathbb{E}\left[\|A_m B_n\|_V^p\right] \le \int_{\{B_n\in K_\epsilon\}} \|A_m B_n(\omega)\|_V^p \; P(\mathrm{d}\omega) + \int_{\{B_n\notin K_\epsilon\}} \|A_m B_n(\omega)\|_V^p \; P(\mathrm{d}\omega) \le 2\epsilon,$$

which completes the proof. □

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