Electronic Journal of Probability

Electron. J. Probab. **29** (2024), article no. 43, 1–23.
ISSN: 1083-6489 https://doi.org/10.1214/24-EJP1101

Moderate deviations for lattice gases with mixing conditions*

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Abstract

We study moderate deviations for additive functionals of stochastic lattice gases (Kawasaki dynamics for the Ising model). Under a mixing condition, we prove that the additive functional of any local functions satisfies a moderate deviation principle. The main tool is the logarithmic Sobolev inequality obtained by Yau.

Keywords: additive functional; moderate deviation; logarithmic Sobolev inequality; mixing condition; stochastic lattice gas.
MSC2020 subject classifications: 60K35; 60J55; 60F10; 82C22.
Submitted to EJP on January 14, 2023, final version accepted on February 22, 2024.

1 Introduction

Let E be a Polish space and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, P_x)$ be a cádlág E-valued Markov process with generator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$, semigroup $\{P_t, t \geq 0\}$ and invariant probability measure μ. Let $L_t = \frac{1}{t}\int_0^t \delta_{X_s} ds$ denote the empirical distribution of the Markov process. Let $\mathcal{M}_1(E)$ be the space of probability measures on E and for any $\nu \in \mathcal{M}_1(E)$, denote by $P_\nu(\cdot) = \int_E \nu(dx)P_x(\cdot)$. The law of large numbers in this context is the statement that $L_t \to \mu$ in probability on the space $\mathcal{M}_1(E)$. The corresponding large deviation principle is that for any Borel set in $A \in \mathcal{M}_1(E)$,

$$-\inf_{\nu\in A^o} I(\nu) \leq \liminf_{t\to\infty} \frac{1}{t}\log P_\mu(L_t \in A), \ \ \limsup_{t\to\infty} \frac{1}{t}\log P_\mu(L_t \in A) \leq -\inf_{\nu\in \bar{A}} I(\nu), \tag{1.1}$$

where

$$I(\nu) = -\inf_{u\in D(\mathcal{A}),\inf_{x\in E} u(x)>0} \int_E \frac{\mathcal{A}u}{u} d\nu, \ \ \nu \in \mathcal{M}(E),$$

*Supported by the National Natural Science Foundation of China (Nos. 11971361, 11731012, 12371275) and the Natural Sciences and Engineering Research Council of Canada.

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is the Donsker-Varadhan entropy, A^o and $\bar{A}$ denote the interior and the closure of the set A, respectively (cf. [6], [25]). In this paper, we often use $\nu(f)$ or $\langle f, \nu\rangle$ to denote the integral $\int_E f d\nu$ for a measurable function and a measure ν on E.

The moderate deviation principle refers to asymptotics intermediate between the central limit theorem and the large deviation principle. Let $a(t)$, $t \geq 0$ be such that

$$\lim_{t\to\infty} a(t)/\sqrt{t} = \infty, \qquad \lim_{t\to\infty} a(t)/t = 0. \tag{1.2}$$

Then the moderate deviation principle (cf. [27]) is to say that for any Borel set in $A \in \mathcal{M}_b(E)$ which is the space of all signed measures on E,

$$\begin{aligned} -\inf_{\nu\in A^o} Q(\nu) \leq \liminf_{t\to\infty} \frac{t}{a^2(t)} \log P_\mu\left(\frac{t}{a(t)}(L_t-\mu)\in A\right), \\ \limsup_{t\to\infty} \frac{t}{a^2(t)} \log P_\mu\left(\frac{t}{a(t)}(L_t-\mu)\in A\right) \leq -\inf_{\nu\in\bar{A}} Q(\nu), \end{aligned} \tag{1.3}$$

where

$$Q(\nu) := \sup_{f\in L_0^2(\mu)}\left\{\nu(f) - \frac{1}{2}\mu(f(U+U^*)f)\right\},\ L_0^2(\mu) := \{f;\ \mu(f^2)<\infty, \mu(f)=0\},$$

$Uf = \int_0^\infty P_t(f-\mu(f))dt$ is the potential operator and U^* is the adjoint of U in $L_0^2(\mu)$ (cf. [27]). Note that if the process is reversible, then

$$I(\nu) = D\left(\sqrt{d\nu/d\mu}\right),\ Q(\nu) = \frac{1}{4}D(d\nu/d\mu),$$

where $D(f) = -\mu(f\mathcal{A}f)$ is the Dirichlet form associated with $\mathcal{A}$.

The moderate deviations have been proved under the assumption of geometric ergodicity, or spectral gap (cf. [2], [10], [18], [27], also see [3] and references therein). Let us give an explanation under the spectral gap for reversible case. Suppose that our Markov process is reversible Feller process and has a spectral gap. For any bounded continuous function V with $\mu(V) = 0$, let $\lambda(V)$ denote the principal eigenvalue of the perturbation operator $\mathcal{A}+V$. Then by the perturbation theory of operators (cf. [22], also see Corollary A.1 in [10]),

$$\begin{aligned} &\lim_{t\to\infty} \frac{t}{a^2(t)} \log E_\mu\left(\exp\left\{\frac{a(t)}{t}\int_0^t V(X_s)ds\right\}\right) \\ &= \lim_{t\to\infty} \frac{t^2}{a^2(t)}\lambda\left(\frac{a(t)}{t}V\right) = \mu(VUV) = \sup_{f\in L_0^2(\mu)}\left\{\mu(Vf) - D(f)/4\right\}, \end{aligned}$$

where $E_\mu(\cdot)$ denotes the expectation with respect to P_μ, i.e., the process with initial distribution μ. The above eqation is equivalent to the moderate deviation principle (1.3) by Varadhan's lemma (cf. Theorem 4.3.1 in [5]) and the abstract Gärtner-Ellis Theorem (cf. Corollary 4.5.27 in [5]).

In this article we consider the conservative (lattice gas) dynamics for the Ising model. These Markov processes are not irreducible and each one among them has infinite invariant probability measures. Specially, the processes do not have the geometric ergodicity, and the spectral gap property. Some techniques, such as the regenerative decomposition method based on the minorization and drift conditions (cf. [1], [2], [13]) and the perturbation method of operators (cf. [10], [18], [27]) cannot be used. We will develop new techniques to study the moderate deviations for these models. Our approach is based on some methods and techniques in hydrodynamic limits (cf. [12], [15], [16], [26]). We use the equivalence of ensembles and the logarithmic Sobolev inequality in [28] to study the upper bound of the moderate deviations. The lower bound is completed by a suitable measure transformation(see Lemma 8.18 and Lemma 8.21 in [24] for large deviations). Our method can be applied to particle system models that have the same type of logarithmic Sobolev inequality as the lattice gas model, such as, zero-range processes, Ginzberg-Laudau models,

The paper is organized as follows. We recall the lattice gas models and state the main result in Section 2. Some exponential moment estimates for the Gibbs measures are given in Section 3. The proofs of the upper bound and the lower bound are presented in Section 4 and Section 5, respectively.

2 Lattice gas model and main result

In this section, let us first recall the Gibbs state, the canonical Gibbs state and the lattice gas model, including some notations, the mixing condition, the equivalence of ensembles and the logarithmic Sobolev inequality(cf. [11], [21], [26], [28]), and then state the main result.

2.1 Lattice gas model

For given a domain Λ in $\mathbb{Z}^d$, let $|\Lambda|$ denote the cardinality of the set Λ, and let $\partial\Lambda$ denote the boundary of Λ, i.e.

$$\partial\Lambda = \{y \in \mathbb{Z}^d \setminus \Lambda;\ \mathrm{dist}(y,\Lambda) := \inf_{x\in\Lambda} |x-y| = 1\},$$

where $|x-y| = \max_{1\le i\le d} |x^i - y^i|$ is the lattice distance. Consider the state space $E = \{0,1\}^{\mathbb{Z}^d}$. A function $f : E \to \mathbb{R}$ is said to be local if there exists a non-negative integer m such that $f(\eta) = f(\xi)$ whenever $\eta(x) = \xi(x)$ for every $x \in \mathbb{Z}^d$ with $|x| \le m$. Let $f : E \to \mathbb{R}$ be a local function. For a finite set $A \subseteq \mathbb{Z}^d$, we say that $\mathrm{supp}(f) \subseteq A$ if $f(\eta) = f(\xi)$ whenever $\eta(x) = \xi(x)$ for any $x \in A$, that is, $\mathrm{supp}(f)$ is the smallest subset of $\mathbb{Z}^d$ such that f only depends on variables in that set. Let $\mathcal{C}$ be the set of all local functions on E. For a real function f on E, define $\|f\|_\infty = \sup_{\eta\in E} |f(\eta)|$. For a vector $\eta \in E$ and two sets $A \subset B \subset \mathbb{Z}^d$, $\eta^A \in E_A := \{0,1\}^A$ denotes the vector $\{\eta(x), x \in A\}$, and we can write $\eta^B = (\eta^A, \eta^{B\setminus A})$.

For a bounded domain Λ in $\mathbb{Z}^d$, the Hamiltonian $H_{\Lambda,\omega}$: $\{0,1\}^\Lambda \ni \eta = \{\eta(x), x \in \Lambda\} \to H_{\Lambda,\omega}(\eta)$ with boundary condition $\omega = \{\omega(x), x \in \partial\Lambda\} \in \{0,1\}^{\partial\Lambda}$ is given by

$$H_{\Lambda,\omega}(\eta) = \sum_{\substack{x,y\in\Lambda\\|x-y|=1}} J(\eta(x),\eta(y)) + \sum_{\substack{x\in\Lambda, y\notin\Lambda\\|x-y|=1}} J(\eta(x),\omega(y)), \tag{2.1}$$

where $J(\eta,\zeta) = -\beta(2\eta-1)(2\zeta-1)$, $\eta,\zeta \in \{0,1\}$, $\beta > 0$ denotes inverse temperature. The grand canonical Gibbs state with chemical potential λ and boundary condition ω is the probability measure on $E_\Lambda := \{0,1\}^\Lambda$ given by

$$\mu^{gc}_{\Lambda,\omega,\lambda}(\eta) = \frac{1}{Z^{gc}_{\Lambda,\omega,\lambda}} \exp\left\{-H_{\Lambda,\omega}(\eta) + \lambda \sum_{x\in\Lambda} \eta(x)\right\},\ \eta \in E_\Lambda, \tag{2.2}$$

where $\lambda \in \mathbb{R}$, $Z^{gc}_{\Lambda,\omega,\lambda}$ is the normalizing constant (the partition function) to make it a probability measure, i.e.,

$$Z^{gc}_{\Lambda,\omega,\lambda} = \sum_{\eta\in E_\Lambda} \exp\left\{-H_{\Lambda,\omega}(\eta) + \lambda \sum_{x\in\Lambda} \eta(x)\right\}.$$

The canonical Gibbs state is the probability measure $\mu^c_{\Lambda,\omega,n}$ given by conditioning $\mu^{gc}_{\Lambda,\omega,\lambda}$ on $E_{\Lambda,n} := \{\eta \in E_\Lambda; \sum_{i\in\Lambda} \eta(i) = n\}$, that is,

$$\mu^c_{\Lambda,\omega,n}(\eta^\Lambda) = \mu^{gc}_{\Lambda,\omega,\lambda}(\{\eta^\Lambda\}\big|\bar\eta_\Lambda = n/|\Lambda|) = \frac{1}{Z^c_{\Lambda,\omega,n}} \exp\{-H_{\Lambda,\omega}(\eta)\} I_{\{\bar\eta_\Lambda = n/|\Lambda|\}}, \tag{2.3}$$

where $\bar\eta_\Lambda = \frac{1}{|\Lambda|}\sum_{i\in\Lambda}\eta(i)$, and $Z^c_{\Lambda,\omega,n}$ is the canonical partition function defined by

$$Z^c_{\Lambda,\omega,n} = \sum_{\eta\in E_{\Lambda,n}} \exp\{-H_{\Lambda,\omega}(\eta)\}.$$

Note that since $\bar\eta_\Lambda$ is fixed, the right side of (2.3) is independent of λ. We can replace $\mu^{gc}_{\Lambda,\omega,\lambda}$ by $\mu^{gc}_{\Lambda,\omega} := \mu^{gc}_{\Lambda,\omega,0}$.

For any $\Lambda \subset \mathbb{Z}^d$, set $\mathcal{F}_\Lambda = \sigma(\eta(x), x\in\Lambda)$, and for any bounded domain Λ in $\mathbb{Z}^d$, denote by $\mathcal{G}_{\Lambda^c} = \sigma(\mathcal{F}_{\Lambda^c}, \bar\eta_\Lambda) := \sigma(\mathcal{F}_{\Lambda^c}\cup\sigma(\bar\eta_\Lambda))$. Then for bounded domains $\Lambda_1 \subset \Lambda_2$,

$$\mathcal{F}_{\Lambda_2^c} \subset \mathcal{F}_{\Lambda_1^c}, \;\; \mathcal{G}_{\Lambda_2^c} \subset \mathcal{G}_{\Lambda_1^c}.$$

It is known (cf. Section 1 in [11]) that the grand Gibbs state satisfies the consistency condition:

$$\begin{aligned}&\sum_{\eta^{\Lambda_2\setminus\Lambda_1}=\zeta^{\Lambda_2\setminus\Lambda_1}} \mu^{gc}_{\Lambda_2,\omega,\lambda}(\eta^{\Lambda_2})\mu^{gc}_{\Lambda_1,(\zeta^{\partial\Lambda_1\cap\Lambda_2},\omega^{\partial\Lambda_1\setminus\Lambda_2}),\lambda}(\zeta^{\Lambda_1})\\ &= \mu^{gc}_{\Lambda_2,\omega,\lambda}(\zeta), \;\; \zeta\in E_{\Lambda_2},\end{aligned} \tag{2.4}$$

and the grand canonical Gibbs state satisfies the following consistency condition:

$$\begin{aligned}&\sum_{\eta^{\Lambda_2\setminus\Lambda_1}=\zeta^{\Lambda_2\setminus\Lambda_1},\bar\eta_{\Lambda_1}=\bar\zeta_{\Lambda_1}} \mu^{c}_{\Lambda_2,\omega,n}(\eta^{\Lambda_2})\mu^{c}_{\Lambda_1,(\zeta^{\partial\Lambda_1\cap\Lambda_2},\omega^{\partial\Lambda_1\setminus\Lambda_2}),|\Lambda_1|\bar\zeta_{\Lambda_1}}(\zeta^{\Lambda_1})\\ &= \mu^{c}_{\Lambda_2,\omega,n}(\zeta),, \;\; \zeta\in E_{\Lambda_2,n}.\end{aligned} \tag{2.5}$$

For any $f, g\in\mathcal{C}$, the correlation $\mu^{gc}_{\Lambda,\omega,\lambda}[f;g]$ is defined by

$$\mu^{gc}_{\Lambda,\omega,\lambda}[f;g] = \mu^{gc}_{\Lambda,\omega,\lambda}(fg) - \mu^{gc}_{\Lambda,\omega,\lambda}(f)\mu^{gc}_{\Lambda,\omega,\lambda}(g).$$

The following Assumption A is the mixing conditions in [26],

Assumption A. *Let $\mu^{gc}_{\Lambda,\omega,\lambda}$ denote a Gibbs measure on Λ with boundary condition ω and chemical potential λ, and let $\rho = \mu^{gc}_{\Lambda,\omega,\lambda}(\bar\eta_\Lambda)$ denote the density. Then there are constants C_1, C_2 and $C_3 \geq R+1$ such that for any $f, g\in\mathcal{C}$, we have*

$$\left|\mu^{gc}_{\Lambda,\omega,\lambda}[f;g]\right| \leq C_1\rho(1-\rho)\exp\{-C_2\mathrm{dist}(\mathrm{supp}(f),\mathrm{supp}(g))\|f\|_\infty\|g\|_\infty\}, \tag{2.6}$$

provided that the diameters of $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$ are bounded by C_3, where $\mathrm{dist}(A,B) = \min_{x\in A, y\in B}|x-y|$. Note that the constants are independent of the size of the cube, the chemical potential and the boundary condition.

It is known that the mixing condition (2.6) holds for β sufficiently small (i.e, high temperature) since the mixing condition is implied by the Dobrushin-Shlosman mixing condition which is always valid at high temperature (see [26], [28]). By the first lemma in Section 10 of [26], the mixing condition (2.6) yields the following mixing condition.

Mixing conditions in [28]. *Let V be a local function. There exist constants $C(V)$ and $\gamma > 0$ such that for any cube Λ,*

$$\left|\mu^{gc}_{\Lambda,\omega_1,\lambda}(V) - \mu^{gc}_{\Lambda,\omega_2,\lambda}(V)\right| \leq C(V)|A_{\omega_1,\omega_2}|\exp\{-\gamma\;\mathrm{dist}(A_{\omega_1,\omega_2},\mathrm{supp}(V))\}, \tag{2.7}$$

where for any two configurations $\omega_1, \omega_2 \in \{0,1\}^{\partial\Lambda}$*,*

$$A_{\omega_1,\omega_2} = \{x \in \partial\Lambda \mid \omega_1(x) \neq \omega_2(x)\}.$$

Under the mixing condition (2.7), the following results are proved in [28]:

Equivalence of ensembles. *Suppose that the mixing condition* (2.7) *holds for all* λ*. Let* V *be a local function. Then there exists* $C(V) < \infty$ *such that for any cube* Λ *and any* $\omega \in \partial\Lambda$*, if* λ *and* n *are chosen such that* $\mu^{gc}_{\Lambda,\omega,\lambda}(\bar{\eta}_\Lambda) = \mu^c_{\Lambda,\omega,n}(\bar{\eta}_\Lambda) = \frac{n}{|\Lambda|}$*, then we have*

$$\left|\mu^{gc}_{\Lambda,\omega,\lambda}(V) - \mu^c_{\Lambda,\omega,n}(V)\right| \leq C(V)|\Lambda|^{-1}. \tag{2.8}$$

Logarithmic Sobolev inequality. *Suppose that the mixing condition* (2.7) *holds for all* λ*. Let* Λ *be a cube of side length* L*. There exists a constant* C *independent of* L*,* n *and* ω *such that for any* $f \geq 0$ *with* $\mu^c_{\Lambda,\omega,n}(f) = 1$*,*

$$\mu^c_{\Lambda,\omega,n}(f \log f) \leq CL^2 D_{\Lambda,\omega,n}(\sqrt{f}), \tag{2.9}$$

where

$$D_{\Lambda,\omega,n}(\sqrt{f}) = \mu^c_{\Lambda,\omega,n}\left(\sum_{\substack{x,y\in\Lambda\\|x-y|=1}} (\sqrt{f(\eta^{xy})} - \sqrt{f(\eta)})^2\right). \tag{2.10}$$

The equilibrium lattice gas dynamics. Next, let us introduce the equilibrium lattice gas dynamics with density $\rho \in (0,1)$. For $\rho \in (0,1)$, if Λ_L, $L \geq 1$ are cubes of side length $2L$ centred at the origin and λ_L, n_L, ω_L are chosen such that

$$\mu^{gc}_{\Lambda_L,\omega_L,\lambda_L}(\bar{\eta}_{\Lambda_L}) = \mu^c_{\Lambda_L,\omega_L,n_L}(\bar{\eta}_{\Lambda_L}) = \frac{n_L}{(2L+1)^d} \to \rho,$$

then $\{\mu^{gc}_{\Lambda_L,\omega_L,\lambda_L}, L \geq 1\}$ and $\{\mu^c_{\Lambda_L,\omega_L,n_L}, L \geq 1\}$ are tight since E is compact. Thus, there exist a probability measure μ_ρ on E and a subsequence $\{L_k, k \geq 1\}$ such that

$$\mu^c_{\Lambda_{L_k},\omega_{L_k},n_{L_k}} \to \mu_\rho \quad \text{weakly as } k \to \infty.$$

Then under the mixing condition (2.7), we use (2.8) to obtain $\mu^{gc}_{\Lambda_{L_k},\omega_{L_k},\lambda_{L_k}} \to \mu_\rho$ weakly as $k \to \infty$.

Now, by (2.4) and (2.5), for any $V \in \mathcal{C}$, $\Lambda \supset \mathrm{supp}(V)$, the following Dobrushin-Lanford-Ruelle (DLR) equations hold:

$$\int_E \mu_\rho(d\eta)\mu^{gc}_{\Lambda,\eta^{\partial\Lambda},\lambda}(V) = \mu_\rho(V), \tag{2.11}$$

and

$$\int_E \mu_\rho(d\eta)\mu^c_{\Lambda,\eta^{\partial\Lambda},|\Lambda|\bar{\eta}_\Lambda}(V) = \mu_\rho(V). \tag{2.12}$$

That is, μ_ρ is a Gibbs measure with the Hamiltonian $H_{\Lambda,\omega}$ and the chemical potential λ, and it is also a canonical Gibbs measure corresponding to the Hamiltonian $H_{\Lambda,\omega}$.

For any $L \geq 1$, $\zeta \in E_{\Lambda_L}$, let ζ' be obtained from ζ by a permutation of the sites in Λ_L. Then by (2.12),

$$\begin{aligned}\mu_\rho(\{\xi \in E;\ \xi^{\Lambda_L} = \zeta'\}) &= \int_E \mu_\rho(d\eta)\mu^c_{\Lambda_L,\eta^{\partial\Lambda},|\Lambda_L|\bar{\zeta}_{\Lambda_L}}(\{\xi^{\Lambda_L} \in E_{\Lambda_L};\ \xi^{\Lambda_L} = \zeta'\}) \\ &= \int_E \mu_\rho(d\eta)\mu^c_{\Lambda_L,\eta^{\partial\Lambda},|\Lambda_L|\bar{\zeta}_{\Lambda_L}}(\{\xi^{\Lambda_L} \in E_{\Lambda_L};\ \xi^{\Lambda_L} = \zeta\}) \\ &= \mu_\rho(\{\xi \in E;\ \xi^{\Lambda_L} = \zeta\}).\end{aligned}$$

Thus, μ_ρ is a symmetric probability measure on E.

Next, we show that μ_ρ is the unique weak limit of the sequences $\{\mu^{gc}_{\Lambda_L,\omega_L,\lambda_L}, L\geq 1\}$ and $\{\mu^{c}_{\Lambda_L,\omega_L,n_L}, L\geq 1\}$. Firstly, for given $V\in\mathcal{C}$, choose L_V large enough such that for $\omega_1,\omega_2\in\{0,1\}^{\partial\Lambda_L}$,

$$|A_{\omega_1,\omega_2}|\leq 2d^2(2d+1),\quad \mathrm{dist}(A_{\omega_1,\omega_2},\mathrm{supp}(V))\leq L/2.$$

Then by (2.7),

$$\lim_{L\to\infty}\sup_{\omega_1,\omega_2\in\{0,1\}^{\partial\Lambda_L}}\left|\mu^{gc}_{\Lambda_L,\omega_1,\lambda}(V)-\mu^{gc}_{\Lambda_L,\omega_2,\lambda}(V)\right|=0.$$

Now, assume that $\tilde{\mu}_\rho$ is other weak limit of the sequences $\{\mu^{gc}_{\Lambda_L,\omega_L,\lambda_L}, L\geq 1\}$. Then (2.11) is also holds for $\tilde{\mu}_\rho$. By (2.11), we have that for any $V\in\mathcal{C}$,

$$\begin{aligned}|\mu_\rho(V)-\tilde{\mu}_\rho(V)|&=\left|\int_E\int_E\mu_\rho(d\omega_1)\tilde{\mu}_\rho(d\omega_2)\left(\mu^{gc}_{\Lambda_L,\omega_1^{\partial\Lambda_l},\lambda}(V)-\mu^{gc}_{\Lambda_L,\omega_2^{\partial\Lambda_L},\lambda}(V)\right)\right|\\&\leq\sup_{\omega_1,\omega_2\in\{0,1\}^{\partial\Lambda_L}}\left|\mu^{gc}_{\Lambda_L,\omega_1,\lambda}(V)-\mu^{gc}_{\Lambda_L,\omega_2,\lambda}(V)\right|\to 0 \text{ as } L\to\infty.\end{aligned}$$

Therefore, $\mu_\rho=\tilde{\mu}_\rho$.

Let η_t, $t\geq 0$ be the equilibrium lattice gas dynamics with density ρ. More precisely, η_0 is distributed according to μ_ρ and the dynamics is a cádlág E-valued Markov process with the generator $\mathcal{A}$ acting on local functions by

$$\mathcal{A}f(\eta)=\sum_{\substack{x,y\in\mathbb{Z}^d\\|x-y|=1}}(1+\exp\{-(H(\eta^{xy})-H(\eta))\})(f(\eta^{xy})-f(\eta)),\tag{2.13}$$

where

$$(\eta^{xy})(z)=\begin{cases}\eta(z), & z\neq x,y,\\ \eta(x), & z=y,\\ \eta(y), & z=x.\end{cases}$$

Note that $H(\eta^{xy})-H(\eta)$ does not depend on Λ and λ as long as Λ is taken large enough, so H can equal any $H_{\Lambda,\omega}$ for any such Λ.

The Markov process $\{\eta_t,\ t\geq 0\}$ exists and has the Feller property (cf. [21]). Let $\{P_t, t\geq 0\}$ be its operator semigroup, and let $\mathcal{D}(\mathcal{A})$ denote the domain of the generator $\mathcal{A}$ on $L^2(\mu_\rho)$, where as usually,

$$L^2(\mu_\rho)=\left\{f:E\to\mathbb{R};\ \langle f,f\rangle_{\mu_\rho}:=\int_E f^2d\mu_\rho<\infty\right\}.$$

For any $f,g\in\mathcal{C}$, take $L\geq 1$ such that $\mathrm{supp}(f)\cup\mathrm{supp}(g)\subset\Lambda_L$. Then for any cube Λ of side length bigger than $2L+3$ centred at the origin, by (2.12) and the symmetry of μ_ρ, we have that

$$\begin{aligned}&\langle g,-\mathcal{A}f\rangle_{\mu_\rho}\\&=-\sum_{\substack{x,y\in\Lambda\\|x-y|=1}}\int_E\mu_\rho(d\eta)(1+\exp\{-(H_{\Lambda,\eta^{\partial\Lambda}}(\eta^{xy})-H_{\Lambda,\eta^{\partial\Lambda}}(\eta))\})(f(\eta^{xy})-f(\eta))g(\eta)\\&=-\sum_{\substack{x,y\in\Lambda\\|x-y|=1}}\int_E\mu_\rho(d\eta)(Z^c_{\Lambda,\eta^{\partial\Lambda},|\Lambda|\bar{\eta}^\Lambda})^{-1}\sum_{\bar{\zeta}^\Lambda=\bar{\eta}^\Lambda}\left(e^{-H_{\Lambda,\eta^{\partial\Lambda}}(\zeta)}+e^{-H_{\Lambda,\eta^{\partial\Lambda}}(\zeta^{xy})}\right)(f(\zeta^{xy})-f(\zeta))g(\zeta)\\&=-\sum_{\substack{x,y\in\Lambda\\|x-y|=1}}\left(\int_E\mu_\rho(d\eta)(f(\eta^{xy})-f(\eta))g(\eta)+\int_E\mu_\rho(d\eta)(f(\eta)-f(\eta^{xy}))g(\eta^{xy})\right)\\&=\sum_{\substack{x,y\in\Lambda\\|x-y|=1}}\int_E\mu_\rho(d\eta)(f(\eta^{xy})-f(\eta))(g(\eta^{xy})-g(\eta)),\end{aligned}$$

where the second equation is due to (2.12), and the third is obtained by (2.12) and the symmetry of μ_ρ. In particular, $\langle g, \mathcal{A}f\rangle_{\mu_\rho} = \langle \mathcal{A}g, f\rangle_{\mu_\rho}$. Therefore, $\{\eta_t, t \geq 0\}$ is reversible with respect to μ_ρ, and the Dirichlet form associated to $\mathcal{A}$ is

$$D(f) = \sum_{|x-y|=1} \int (f(\eta^{xy}) - f(\eta))^2 \mu_\rho(d\eta), \;\; \mathcal{D} := \{f \in L^2(\mu_\rho);\; D(f) < \infty\}. \tag{2.14}$$

2.2 Main result

Let us introduce first the spaces $\mathcal{H}_1$ and $\mathcal{H}_{-1}$ associated with the operator $\mathcal{A}$ (cf. Section 2.2 in [17]). Consider the semi-norm $\|\cdot\|_1$ defined on $\mathcal{C}$ by

$$\|f\|_1^2 = \langle f, -\mathcal{A}f\rangle_{\mu_\rho}. \tag{2.15}$$

Two elements f and g in $\mathcal{C}$ are considered the same if $\|f-g\|_1 = 0$. Let $\mathcal{H}_1$ denote the completion of $\mathcal{C}$ with respect to the semi-norm $\|\cdot\|_1$. For any $f \in L^2(\mu)$, define

$$\|f\|_{-1}^2 = \sup_{g \in \mathcal{C}} \{2\langle f, g\rangle_{\mu_\rho} - \|g\|_1^2\}. \tag{2.16}$$

Let $\mathcal{H}_{-1}$ be the completion of $\big\{f \in L^2(\mu);\; \|f\|_{-1} < \infty\big\}$ with respect to norm $\|\cdot\|_{-1}$. Then for any $f \in \mathcal{C}$,

$$\|\mathcal{A}f\|_{-1}^2 = \|f\|_1^2,$$

and $\mathcal{H}_{-1}$ is the closure of $\{\mathcal{A}f;\; f \in \mathcal{C}\}$.

For any $V \in L^2(\mu_\rho)$, let $\mu_\rho(V)$ denote the mean of V with respect to μ_ρ, i.e.,

$$\mu_\rho(V) = \int V d\mu_\rho. \tag{2.17}$$

For a mean zero function V, define

$$\sigma_\rho^2(V) = 2\int_0^\infty \langle P_t V, V\rangle_{\mu_\rho} dt. \tag{2.18}$$

Then

$$\sigma_\rho^2(V) = 2\langle V, -\mathcal{A}V\rangle_{\mu_\rho} = 2\|V\|_1.$$

Now, let us state our main result in this paper.

Theorem 2.1. Suppose that the mixing condition (2.6) holds. Let $d \geq 3$ and $\rho \in (0,1)$ and let $a(t)$, $t \geq 0$ be a positive function satisfying (1.2). Let $(\Omega, \mathcal{F}, \{\eta_t\}_{t\geq 0}, P_\rho)$ be a Markov process with initial distribution μ_ρ and with the operator $\mathcal{A}$, i.e., under P_ρ, $\{\eta_t$, $t \geq 0\}$ is the equilibrium lattice gas dynamics with density ρ. For any local function V with mean zero, if $V \in \mathcal{H}_{-1}$, then

$$\left\{P_\rho\left(\frac{1}{a(t)}\int_0^t V(\eta_s)ds \in \cdot\right),\; t > 0\right\}$$

satisfies the large deviation principle with speed $\frac{a^2(t)}{t}$ and with rate function $I^V(z)$ defined by

$$I^V(z) = \frac{z^2}{2\sigma_\rho^2(V)}, \;\; z \in \mathbb{R}. \tag{2.19}$$

That is, for any closed set F in $\mathbb{R}$,

$$\limsup_{t\to\infty} \frac{t}{a^2(t)} \log P_\rho\left(\frac{1}{a(t)}\int_0^t V(\eta_s)ds \in F\right) \leq -\inf_{z\in F} I^V(z), \tag{2.20}$$

and for any open set G in $\mathbb{R}$,

$$\liminf_{t\to\infty} \frac{t}{a^2(t)} \log P_\rho\left(\frac{1}{a(t)}\int_0^t V(\eta_s)ds \in G\right) \geq -\inf_{z\in G} I^V(z). \tag{2.21}$$

3 Exponential moment estimates for Gibbs measures

In this section, we present some exponential moment estimates for the Gibbs measures. Let us first introduce some notations. For any positive integer L, let Λ_L denote a cube of side length $2L$ centered at the origin and define

$$\bar{\eta}^L = \frac{1}{(2L+1)^d} \sum_{x\in\Lambda_L} \eta(x) \tag{3.1}$$

and

$$\mu_L(\cdot) = \mu_\rho(\bar{\eta}^L \in \cdot). \tag{3.2}$$

We also define the regular conditional probability:

$$\mu_{L,q}(\cdot) = \mu_{L,\eta^{\partial\Lambda_L},q}(\cdot) = \mu_\rho(\cdot|\mathcal{F}_{\Lambda_L^c}, \bar{\eta}^L = q) = \mu^c_{\Lambda_L,\eta^{\partial\Lambda_L},q(2L+1)^d}(\cdot), \tag{3.3}$$

where $\mu^c_{\Lambda_L,\eta^{\partial\Lambda_L},q(2L+1)^d}$ is the canonical Gibbs state defined by (2.3), and $\mathcal{F}_{\Lambda_L^c}=\sigma(\eta(x), x \in \Lambda_L^c)$. By the definition of the canonical Gibbs state, $\mu_{L,\eta^{\partial\Lambda_L},q} : E \times \mathcal{F}_{\Lambda_L} \ni (\eta, A) \to \mu^c_{\Lambda_L,\eta^{\partial\Lambda_L},q(2L+1)^d}(A) \in [0,1]$, is $\mathcal{F}_{\partial\Lambda_L}$-measurable in η for each $A \subset E_{\Lambda_L}$, and for each $\eta^{\partial\Lambda_L} \in E_{\partial\Lambda_L}$,

$$\mu_{L,q}(A) = \mu_{L,\eta^{\partial\Lambda_L},q}(A) = \frac{1}{Z^c_{\Lambda,\eta^{\partial\Lambda_L},q(2L+1)^d}} \sum_{\zeta\in A} \exp\left\{-H_{\Lambda,\eta^{\partial\Lambda_L}}(\zeta)\right\} I_{\{\bar{\zeta}^L=q\}}, \quad A \subset E_{\Lambda_L},$$

is a probability measure on E_{Λ_L}. Set

$$\mu_{L,\bar{\eta}^L}(A) = \mu_{L,\eta^{\partial\Lambda_L},\bar{\eta}^L}(A) = \mu_{L,\eta^{\partial\Lambda_L},q}(A)\Big|_{q=\bar{\eta}^L}, \quad A \subset E_{\Lambda_L}.$$

Then for μ_ρ-a.s. η,

$$\mu_{L,\eta^{\partial\Lambda_L},\bar{\eta}^L}(A) = \mu_\rho(A|\mathcal{G}_{\Lambda_L^c}),$$

where $\mathcal{G}_{\Lambda_L^c} = \sigma(\mathcal{F}_{\Lambda_L^c}, \bar{\eta}^L)$, and $\mu_{L,\eta^{\partial\Lambda_L},\bar{\eta}^L} : E \times \mathcal{F}_{\Lambda_L} \ni (\eta, A) \to \mu^c_{\Lambda_L,\eta^{\partial\Lambda_L},\bar{\eta}^L(2L+1)^d}(A) \in [0,1]$, is $\mathcal{F}_{\partial\Lambda_L} \vee \sigma(\bar{\eta}^L)$-measurable in η for each $A \subset E_{\Lambda_L}$, and a probability measure on E_{Λ_L} for each $\eta^{\partial\Lambda_L} \in E_{\partial\Lambda_L}$.

Remark 3.1. (1). Note that the definition makes sense only for $q = i/(2L+1)^d$ with i an integer, so we will always assume this in the sequel though we will not state it explicitly.

(2). In order to simplify notation, we drop the subscript ω in $\mu_{L,q}$ and $\mu_{L,\bar{\eta}^L}$ when the boundary $\eta^{\partial\Lambda_L}$ is fixed.

For a local function V, and $L \geq 1$ such that $\Lambda_L \supset \mathrm{supp}(V)$ and $\mathrm{dist}(\partial\Lambda_L, \mathrm{supp}(V)) \geq L/2$, for each fixed boundary $\omega \in E_{\partial\Lambda_L}$, we define

$$\psi^{gc}_{L,V}(y) = \psi^{gc}_{L,\omega,V}(y) = \mu^{gc}_{\Lambda_L,\omega,\lambda}(V), \quad y \in (0,1), \tag{3.4}$$

where λ is chosen such that $\mu^{gc}_{\Lambda_L,\omega,\lambda}(\bar{\eta}^L) = y$, and set

$$\psi_{L,V}(q) = \psi_{L,\omega,V}(q) = \mu_{L,\omega,q}(V), \quad \psi_{L,V}(\bar{\eta}^L) = \psi_{L,\omega,V}(\bar{\eta}^L) = \mu_{L,\omega,\bar{\eta}^L}(V). \tag{3.5}$$

Then by the precise representation of $\mu_{L,q}$, it is known that $\psi_{L,\eta^{\partial\Lambda_L},V}(\bar{\eta}^L)$ is $\mathcal{F}_{\partial\Lambda_L} \vee \sigma(\bar{\eta}^L)$-measurable. In order to estimate the exponential moment of $\psi_{L,\eta^{\partial\Lambda_L},V}(\bar{\eta}^L)$ under $\mu_{2L,q}$, let us first compute $\mu_{2L,q}\left(\psi_{L,\eta^{\partial\Lambda_L},V}(\bar{\eta}^L)\right)$. By $\psi_{L,V}(\bar{\eta}^L) = \mu_\rho(V|\mathcal{G}_{\Lambda_L^c})$, $\mu_{2L,q}(V) = \mu_\rho(V|\mathcal{G}_{\Lambda_{2L}^c})|_{\bar{\eta}^{2L}=q}$ and $\mathcal{G}_{\Lambda_{2L}^c} \subset \mathcal{G}_{\Lambda_L^c}$, we have that for μ_ρ-a.s. ω,

$$\mu_{2L,\omega,q}\left(\psi_{L,\eta^{\partial\Lambda_L},V}(\bar{\eta}^L)\right) = \mu_\rho(\mu_\rho(V|\mathcal{G}_{\Lambda_L^c})|\mathcal{G}_{\Lambda_{2L}^c})|_{\bar{\eta}^{2L}=q} = \mu_\rho(V|\mathcal{G}_{\Lambda_{2L}^c})|_{\bar{\eta}^{2L}=q} = \psi_{2L,\omega,V}(q).$$

Therefore, for μ_ρ-a.s. ω,

$$\mu_{2L,\omega,q}\left(\psi_{L,\eta^{\partial\Lambda_L},V}(\bar{\eta}^L)\right) = \psi_{2L,\omega,V}(q). \tag{3.6}$$

Next, we give some exponential moment estimates for the Gibbs measures.

Lemma 3.1. Suppose that the mixing condition (2.6) holds. Let V be a local function with mean zero. For $L \geq 1$ such that $\Lambda_L \supset \mathrm{supp}(V)$ and $\mathrm{dist}(\partial\Lambda_L, \mathrm{supp}(V)) \geq L/2$, set

$$W_L(\eta) = \psi_{L,\eta^{\partial\Lambda_L},V}(\bar{\eta}^L) - \psi_{2L,\eta^{\partial\Lambda_{2L}},V}(\bar{\eta}^{2L}).$$

Then there exist constants $0 < B = B_V < \infty$, $\epsilon \in (0,\infty)$ such that for any $|\theta| \leq \epsilon L^{d/2-1}$, $L \geq 1$, $q \in (0,1)$,

$$\log \mu_{2L,q}\left(\exp\left\{\theta L^{1+d/2} W_L\right\}\right) \leq B\theta^2 L^2. \tag{3.7}$$

Proof. Define

$$g_L(y) = g_{L,\omega}(y) = \psi^{gc}_{L,\omega,V}(y) - \psi^{gc}_{L,\omega,V}(q), \quad q^c_L = q^c_L = \mu_{2L,q}(\bar{\eta}^L).$$

Then under the mixing condition (2.7), applying Theorem 2.2 in [28] to V and $-V$, there exists a positive constant A_1 independent of λ, ω, L, such that

$$g_L(q) = 0, \quad |g'_L(y)| \leq A_1, \quad |g''_L(y)| \leq A_1, \quad \text{for all } y \in (0,1).$$

Then, under $\bar{\eta}^{2L} = q$, we can write

$$W_L(\eta) = g_L(\bar{\eta}^L) + \left(\psi_{L,\eta^{\partial\Lambda_L},V}(\bar{\eta}^L) - \psi^{gc}_{L,\eta^{\partial\Lambda_L},V}(\bar{\eta}^L)\right) - \left(\psi_{2L,\eta^{\partial\Lambda_{2L}},V}(q) - \psi^{gc}_{L,\eta^{\partial\Lambda_L},V}(q)\right).$$

By **Lemma** A.4 in Appendix (the second lemma in Section 10, [28]), there exists a constant $0 < A_2 < \infty$ such that

$$|g_L(q^c_L)| \leq A_1 |q^c_L - q| \leq A_2 L^{-d}.$$

Thus,

$$\begin{aligned}\mu_{2L,q}(|g_L(\bar{\eta}^L) - g_L(q)|^2) \leq& 2\mu_{2L,q}(|g_L(\bar{\eta}^L) - g_L(q^c_L)|^2) + 2A_2^2 L^{-2d} \\ \leq& 2A_1^2 \mu_{2L,q}(|\bar{\eta}^L - q^c_L|^2) + 2A_2^2 L^{-2d}\end{aligned}$$

By (3.6),

$$\psi_{2L,\eta^{\partial\Lambda_{2L}},V}(q) - \psi^{gc}_{L,\eta^{\partial\Lambda_L},V}(q) = \int \left(\psi_{L,\zeta^{\partial\Lambda_L},V}(\bar{\zeta}^L) - \psi^{gc}_{L,\eta^{\partial\Lambda_L},V}(q)\right) \mu_{2L,\eta^{\partial\Lambda_{2L}},q}(d\zeta^{\Lambda_{2L}}).$$

We can write

$$\begin{aligned}&\psi_{L,\zeta^{\partial\Lambda_L},V}(\bar{\zeta}^L) - \psi^{gc}_{L,\eta^{\partial\Lambda_L},V}(q) \\ =&\psi_{L,\zeta^{\partial\Lambda_L},V}(\bar{\zeta}^L) - \psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(\bar{\zeta}^L) + \left(\psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(\bar{\zeta}^L) - \psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(q)\right) \\ &+ \left(\psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(q) - \psi^{gc}_{L,\eta^{\partial\Lambda_L},V}(q)\right).\end{aligned}$$

By the equivalence of ensembles, there exists a constant $0 < A_{3,1} < \infty$ such that

$$\left|\psi^{gc}_{L,V}(\bar{\eta}^L) - \psi_{L,V}(\bar{\eta}^L)\right| \leq A_{3,1} L^{-d}, \quad \left|\psi_{L,\zeta^{\partial\Lambda_L},V}(\bar{\zeta}^L) - \psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(\bar{\zeta}^L)\right| \leq A_{3,1} L^{-d}.$$

Since $\mathrm{dist}(\partial\Lambda_L, \mathrm{supp}(V)) \geq L/2$, by the mixing condition (2.7), there exist positive constants $0 < A_{3,2}, A_{3,3} A_{3,4} < \infty$ such that

$$\left|\psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(q) - \psi^{gc}_{L,\eta^{\partial\Lambda_L},V}(q)\right| \leq A_{3,2} L^{d-1} e^{-A_{3,3} L} \leq A_{3,4} L^{-d}.$$

Set $\tilde{q}^c_L = \mu_{2L,\eta^{\partial\Lambda_{2L}},q}(\bar{\zeta}^L)$. Then

$$\begin{aligned}&\mu_{2L,q}\left(\int \left(\psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(\bar{\zeta}^L) - \psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(q)\right)^2 \mu_{2L,\eta^{\partial\Lambda_{2L}},q}(d\zeta^{\Lambda_{2L}})\right) \\ \leq& 2\mu_{2L,q}\left(\int \left(\psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(\bar{\zeta}^L) - \psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(\tilde{q}^c_L)\right)^2 \mu_{2L,\eta^{\partial\Lambda_{2L}},q}(d\zeta^{\Lambda_{2L}})\right) + 2A_2^2 L^{-2d} \\ \leq& 2A_1^2 \mu_{2L,q}\left(\int |\bar{\zeta}^L - \tilde{q}^c_L|^2 \mu_{2L,\eta^{\partial\Lambda_{2L}},q}(d\zeta^{\Lambda_{2L}})\right) + 2A_2^2 L^{-2d}.\end{aligned}$$

Thus, there exists a constant $0 < A_4 < \infty$ such that

$$\mu_{2L,q}(|W_L|^2) \leq A_4\mu_{2L,q}(|\bar{\eta}^L - q_L^c|^2) + A_4L^{-2d} + A_4\mu_{2L,q}\left(\int |\bar{\zeta}^L - \tilde{q}_L^c|^2 \mu_{2L,\eta^{\partial\Lambda_{2L}},q}(d\zeta^{\Lambda_{2L}})\right).$$

Applying (A.4) in Appendix to $g(x) = x$, then there exist constants $0 < A_5 < \infty$, $\epsilon \in (0,\infty)$ such that for any $|\theta| \leq \epsilon L^{d/2-1}$, for any $q \in (0,1)$,

$$\log \mu_{2L,q}\left(\exp\left\{\theta L^{1+d/2}\left(\bar{\eta}^L - q_L^c\right)\right\}\right) \leq A_5|\theta|^2L^2.$$

Therefore, for any $|\theta| \leq CL^{-(1+d/2)}$, where $C \in (0, \min\{1, 1/\|V\|\}/8)$, and $\|V\| = \sup_{\eta\in E}|V(\eta)|$,

$$\frac{1}{8}\theta^2L^{2+d}\mu_{2L,q}((\bar{\eta}^L - q_L^c)^2) \leq \log\left(1 + \frac{1}{4}\theta^2L^{2+d}\mu_{2L,q}((\bar{\eta}^L - q_L^c)^2)\right) \leq A_5\theta^2L^2.$$

The same proof yields

$$\frac{1}{8}\theta^2L^{2+d}\int |\bar{\zeta}^L - \tilde{q}_L^c|^2\mu_{2L,\eta^{\partial\Lambda_{2L}},q}(d\zeta^{\Lambda_{2L}}) \leq A_5\theta^2L^2.$$

Therefore,

$$\mu_{2L,q}(|W_L|^2) \leq 2A_4A_5L^{-d} + A_4L^{-2d}.$$

Now, by $\mu_{2L,q}(W_L) = 0$, and using the Taylor expansion and (3.6), there exists a constant $0 < A_6 < \infty$ such that for any $|\theta| \leq CL^{-(1+d/2)}$, for any $q \in (0,1)$,

$$\begin{aligned}\mu_{2L,q}\left(\exp\left\{\theta L^{1+d/2}W_L\right\}\right) &= 1 + \sum_{k=2}^{\infty}\frac{(\theta L^{1+d/2})^k}{k!}\mu_{2L,q}\left((W_L)^k\right)\\ &\leq 1 + L^{2+d}\theta^2\mu_{2L,q}(|W_L|^2)\sum_{k=0}^{\infty}\frac{(2C\|V\|)^k}{(k+2)!}\\ &\leq 1 + A_6\theta^2L^2.\end{aligned}$$

Thus, (3.7) holds for $|\theta| \leq CL^{-(1+d/2)}$.

Next, let us assume that $|\theta| \geq CL^{-(1+d/2)}$. By the Hölder inequality, for any $|\theta| \leq \epsilon L^{d/2-1}$, $L \geq 1$, $q \in (0,1)$,

$$\begin{aligned}&\log\mu_{2L,q}\left(\exp\left\{\theta L^{1+d/2}W_L\right\}\right)\\ \leq&\frac{1}{7}\log\mu_{2L,q}\left(\exp\left\{7\theta L^{1+d/2}\left(g_L(\bar{\eta}^L) - g_L(q_L^c)\right)\right\}\right) + \frac{1}{7}\log\mu_{2L,q}\left(\exp\left\{7\theta L^{1+d/2}g_L(q_L^c)\right\}\right)\\ &+\frac{1}{7}\log\mu_{2L,q}\left(\exp\left\{7\theta L^{1+d/2}\left(\psi_{L,\eta^{\partial\Lambda_L},V}(\bar{\eta}^L) - \psi^{gc}_{L,\eta^{\partial\Lambda_L},V}(\bar{\eta}^L)\right)\right\}\right)\\ &+\frac{1}{7}\log\mu_{2L,q}\left(\exp\left\{7\theta L^{1+d/2}\int\left(\psi_{L,\zeta^{\partial\Lambda_L},V}(\bar{\zeta}^L) - \psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(\bar{\zeta}^L)\right)\mu_{2L,\eta^{\partial\Lambda_{2L}},q}(d\zeta^{\Lambda_{2L}})\right\}\right)\\ &+\frac{1}{7}\log\mu_{2L,q}\left(\exp\left\{7\theta L^{1+d/2}\int\left(\psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(\bar{\zeta}^L) - \psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(q)\right)\mu_{2L,\eta^{\partial\Lambda_{2L}},q}(d\zeta^{\Lambda_{2L}})\right\}\right)\\ &+\frac{1}{7}\log\mu_{2L,q}\left(\exp\left\{7\theta L^{1+d/2}\int\left(\psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(\bar{\zeta}^L) - \psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(\tilde{q}_L^c)\right)\mu_{2L,\eta^{\partial\Lambda_{2L}},q}(d\zeta^{\Lambda_{2L}})\right\}\right)\\ &+\frac{1}{7}\log\mu_{2L,q}\left(\exp\left\{7\theta L^{1+d/2}\int\psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(\tilde{q}_L^c)\mu_{2L,\eta^{\partial\Lambda_{2L}},q}(d\zeta^{\Lambda_{2L}})\right\}\right).\end{aligned}$$

By **Lemma** A.3 in Appendix, there exist constants $0 < A_7 < \infty$, $\epsilon \in (0,\infty)$ such that for any $|\theta| \leq \epsilon L^{d/2-1}$, for any $q \in (0,1)$,

$$\log\mu_{2L,q}\left(\exp\left\{7\theta L^{1+d/2}\left(g_L(\bar{\eta}^L) - g_L(q_L^c)\right)\right\}\right) \leq A_7\theta^2L^2, \tag{3.8}$$

and

$$
\begin{aligned}
&\log \mu_{2L,q}\left(\exp\left\{7\theta L^{1+d/2}\int\left(\psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(\bar{\zeta}^L)-\psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(\tilde{q}^c_L)\right)\mu_{2L,\eta^{\partial\Lambda_{2L}},q}(d\zeta^{\Lambda_{2L}})\right\}\right)\\
\le&\log \mu_{2L,q}\left(\int\exp\left\{7\theta L^{1+d/2}\left(\psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(\bar{\zeta}^L)-\psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(\tilde{q}^c_L)\right)\right\}\mu_{2L,\eta^{\partial\Lambda_{2L}},q}(d\zeta^{\Lambda_{2L}})\right)\\
\le&A_7\theta^2L^2.
\end{aligned}
$$

By **Lemma** A.4 in Appendix, there exists a constant $0<A_8<\infty$ such that

$$|g_L(q^c_L)|\le A_1|q^c_L-q|\le A_2L^{-d}\le A_8\theta^2L^2 \text{ for any } |\theta|\ge CL^{-(1+d/2)},$$

and

$$\left|\psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(\tilde{q}^c_L)\right|\le A_8\theta^2L^2 \text{ for any } |\theta|\ge CL^{-(1+d/2)}.$$

By the equivalence of ensembles, there exists a constant $0<A_9<\infty$ such that

$$\left|\psi^{gc}_{L,V}(\bar{\eta}^L)-\psi_{L,V}(\bar{\eta}^L)\right|\le A_3L^{-d}\le A_9\theta^2L^2 \text{ for any } |\theta|\ge CL^{-(1+d/2)},$$

and

$$\left|\psi_{L,\zeta^{\partial\Lambda_L},V}(\bar{\zeta}^L)-\psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(\bar{\zeta}^L)\right|\le A_3L^{-d}\le A_9\theta^2L^2 \text{ for any } |\theta|\ge CL^{-(1+d/2)}.$$

Since $\mathrm{dist}(\partial\Lambda_L,\mathrm{supp}(V))\ge L/2$, by the mixing condition (2.7), there exists a constant $0<A_{10}<\infty$ such that

$$\left|\psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(q)-\psi^{gc}_{L,\eta^{\partial\Lambda_L},V}(q)\right|\le A_{3,2}L^{-d}\le A_{10}\theta^2L^2 \text{ for any } |\theta|\ge CL^{-(1+d/2)}.$$

Therefore, there exist constants $0<A_{11}<\infty$, $\epsilon\in(0,\infty)$ such that for any $CL^{-(1+d/2)}\le|\theta|\le\epsilon L^{d/2-1}$, for any $q\in(0,1)$,

$$
\begin{aligned}
&\max\left\{\log\mu_{2L,q}\left(\exp\left\{7\theta L^{1+d/2}g_L(q^c_L)\right\}\right),\right.\\
&\left.\quad\log\mu_{2L,q}\left(\exp\left\{7\theta L^{1+d/2}\int\psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(\tilde{q}^c_L)\mu_{2L,\eta^{\partial\Lambda_{2L}},q}(d\zeta^{\Lambda_{2L}})\right\}\right)\right\}\\
\le&A_{11}L^2\theta^2,
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
&\max\left\{\log\mu_{2L,q}\left(\exp\left\{7\theta L^{1+d/2}\left(\psi_{L,\eta^{\partial\Lambda_L},V}(\bar{\eta}^L)-\psi^{gc}_{L,\eta^{\partial\Lambda_L},V}(\bar{\eta}^L)\right)\right\}\right),\right.\\
&\quad\log\mu_{2L,q}\left(\exp\left\{7\theta L^{1+d/2}\int\left(\psi_{L,\zeta^{\partial\Lambda_L},V}(\bar{\zeta}^L)-\psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(\bar{\zeta}^L)\right)\mu_{2L,\eta^{\partial\Lambda_{2L}},q}(d\zeta^{\Lambda_{2L}})\right\}\right),\\
&\left.\quad\log\mu_{2L,q}\left(\exp\left\{7\theta L^{1+d/2}\int\left(\psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(\bar{\zeta}^L)-\psi^{gc}_{L,\zeta^{\partial\Lambda_L},V}(q)\right)\mu_{2L,\eta^{\partial\Lambda_{2L}},q}(d\zeta^{\Lambda_{2L}})\right\}\right)\right\}\\
\le&A_{10}L^2\theta^2.
\end{aligned}
\tag{3.10}
$$

Thus, (3.7) holds for any $CL^{-(1+d/2)}\le|\theta|\le\epsilon L^{d/2-1}$. □

Lemma 3.2. Suppose that the mixing condition (2.6) holds. Let V be a local function with mean zero. Then there exist constants $0<C_V<\infty$, $\epsilon\in(0,\infty)$ such that for any $|\theta|\le\epsilon L^{d/2-1}$, $L\ge1$ with $\Lambda_L\supset\mathrm{supp}(V)$ and $\mathrm{dist}(\partial\Lambda_L,\mathrm{supp}(V))\ge L/2$,

$$\mu_\rho\left(\exp\left\{\theta L^{1+d/2}\psi_{L,V}(\bar{\eta}^L)\right\}\right)\le\exp\{C_V\theta^2L^2\}. \tag{3.11}$$

Proof. For $j \geq 0$, set

$$W_{L_j}(\eta) = \psi_{L_j,V}(\bar{\eta}^{L_j}) - \psi_{2L_j,V}(\bar{\eta}^{2L_j}), \quad \text{where } L_j = 2^j L.$$

Noting $\psi_{2L_j,V}(\bar{\eta}^{2L_j}) \to \mu_\rho(V) = 0$ μ_ρ-$a.s.$ as $j \to \infty$, we have that

$$\psi_{L,V}(\bar{\eta}^L) = \sum_{j=0}^{\infty} W_{L_j}(\eta).$$

Since μ_ρ is the canonical Gibbs measure, for any $j \geq 0$,

$$\mu_\rho\left(\exp\left\{\theta L_j^{1+d/2} W_{L_j}(\eta)\right\}\right) = \mu_\rho\left(\mu_{2L_j,\bar{\eta}^{2L_j}}\left(\exp\left\{\theta L_j^{1+d/2} W_{L_j}(\eta)\right\}\right)\right).$$

By **Lemma** 3.1, there exist constants $0 < B_V < \infty$, $\epsilon \in (0,\infty)$ such that for any $|\theta| \leq \epsilon L^{d/2-1}$, $j \geq 0$, $q \in (0,1)$,

$$\mu_{2L_j,\bar{\eta}^{2L_j}}\left(\exp\left\{\theta L_j^{1+d/2} W_{L_j}(\eta)\right\}\right) \leq B_V \theta^2 L_j^2.$$

Thus, by the Hölder inequality and $d \geq 3$, for any $|\theta| \leq \epsilon L^{d/2-1}$, $L \geq 1$,

$$\begin{aligned}
&\log \mu_\rho\left(\exp\left\{\theta L^{1+d/2} \psi_{L,V}(\bar{\eta}^L)\right\}\right) \\
\leq &\sum_{j=0}^{\infty} 2^{-j(1+d/2)} \log \mu_\rho\left(\exp\left\{\theta L_j^{1+d/2} W_{L_j}(\eta)\right\}\right) \\
\leq &\sum_{j=0}^{\infty} 2^{-j(d/2-1)} B_V \theta^2 L^2 \\
\leq &B_V \theta^2 L^2/(1 - 2^{-d/2+1}).
\end{aligned}$$

and so, (3.11) holds for $C_V := B_V/(1 - 2^{-d/2+1})$. □

4 Upper bound

In this section, we prove the upper bound of the moderate deviation principle. The main tool is the logarithmic Sobolev inequality obtained by Yau. Our proof is based on the variational formulae of the relative entropy, and the principal eigenvalue of Feynman-Kac semigroup. For the sake of convenience, let us first introduce the two variational formulae (cf. [5], [6], [15], [24], [25].

The variational formula of relative entropy. Let $\mu \in \mathcal{M}_1(E)$. The relative entropy $H(\nu,\mu)$ of $\nu \in \mathcal{M}_1(E)$ with respect to μ is defined by

$$H(\nu,\mu) = \begin{cases} \displaystyle\int_E \left(\frac{d\nu}{d\mu} \log \frac{d\nu}{d\mu}\right) d\mu & \text{if } \nu \ll \mu, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.1}$$

where $C_b(E)$ is the space of all bounded continuous functions on E. The relative entropy $H(\nu,\mu)$ has the following Donsker-Varadhan variational formula (cf. [25]):

$$H(\nu,\mu) = \sup\left\{\nu(f) - \log \mu(e^f), \quad f \in C_b(E)\right\}. \tag{4.2}$$

The variational formula of the principal eigenvalue of Feynman-Kac semigroup. Let X_t, $t \geq 0$ be a reversible Markov process with the Feller property, and let $\mu \in \mathcal{M}_1(E)$ be its reversible measure. Let $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ and $\{P_t, t \geq 0\}$ be its generator and semigroup on $L^2(\mu)$, respectively. Let $V: [0,\infty)\times E \ni (t,x) \to V(t,x) \in \mathbb{R}$ be a bounded continuous function. It is known that (cf. Appendix 1 [15], Section 7 in [24])

$$\mathcal{A}_t = \mathcal{A} + V_t,$$

is also a symmetric operator on $L^2(\mu)$, where $V_t(x) := V(t,x)$, and the semigroup of bounded operators $\{P_t^V, t \geq 0\}$ associated with the generator $\mathcal{A}_t$ can be represented by

$$P_t^V f(x) := E_x\left(\exp\left\{\int_0^t V(s,X_s)ds\right\} f(X_t)\right),\ f \in C_b(E),$$

where $E_x(\cdot)$ denotes the expectation with respect to the process starting at x. $\{P_t^V; t \geq 0\}$ is called Feynman-Kac semigroup. Let $\lambda_t(V)$ be the principal eigenvalue of the operator P_t^V. Then

$$\lambda_t(V) = \sup\left\{\mu(V_t\phi^2) - \mu(-\phi\mathcal{A}\phi);\ \mu(|\phi|^2) \leq 1\right\}, \tag{4.3}$$

and

$$E_\mu\left(e^{\int_0^t V(s,X_s)ds}\right) \leq e^{\int_0^t \lambda_s(V)ds}. \tag{4.4}$$

Lemma 4.1. Suppose that the mixing condition (2.6) holds. Let $d \geq 3$ and let V be a local function with mean zero. Then for any $\alpha \in \mathbb{R}$,

$$\limsup_{L\to\infty}\limsup_{t\to\infty} \frac{t}{a^2(t)} \log E_\rho\left(\exp\left\{\frac{a(t)}{t}\alpha\int_0^t \psi_{L,V}(\bar{\eta}_s^L)ds\right\}\right) \leq 0. \tag{4.5}$$

Proof. For $L \geq 1$ such that $\Lambda_L \supset \mathrm{supp}(V)$ and $\mathrm{dist}(\partial\Lambda_L, \mathrm{supp}(V)) \geq L/2$, as the proof in **Lemma** 3.2, we can write

$$\psi_{L,V}(\bar{\eta}^L) = \sum_{j=0}^\infty W_{L_j}(\eta),$$

where

$$W_{L_j}(\eta) = \psi_{L_j,V}(\bar{\eta}^{L_j}) - \psi_{2L_j,V}(\bar{\eta}^{2L_j}), \quad \text{and } L_j = 2^j L.$$

Thus, by the Hölder inequality, and the definition of the canonical Gibbs measure,

$$\begin{aligned}
&\frac{t}{a^2(t)} \log E_\rho\left(\exp\left\{\frac{a(t)}{t}\alpha\int_0^t \psi_{L,V}(\bar{\eta}_s^L)ds\right\}\right)\\
\leq& \sum_{j=0}^\infty L_j^{1-d/2} \times \frac{t}{a^2(t)} \log E_\rho\left(\exp\left\{\frac{a(t)}{t}\alpha L_j^{d/2-1}\int_0^t W_{L_j}(\eta_s)ds\right\}\right)\\
=& \sum_{j=0}^\infty L_j^{1-d/2} \times \frac{t}{a^2(t)} \log \mu_\rho\left(E_{\mu_{2L_j,\bar{\eta}^{2L_j}}}\left(\exp\left\{\frac{a(t)}{t}\alpha L_j^{d/2-1}\int_0^t W_{L_j}(\eta_s)ds\right\}\right)\right).
\end{aligned}$$

Since V is bounded, by the Feynman-Kac formula and the variational formula of the principal eigenvalue of $\mathcal{A} + \frac{a(t)}{t}\alpha 2^{j(d/2-1)} W_{L_j}$ (see (4.3) and (4.4)),

$$\begin{aligned}
&\log E_{\mu_{2L_j,q}}\left(\exp\left\{\frac{a(t)}{t}\alpha L_j^{d/2-1}\int_0^t W_{L_j}(\eta_s)ds\right\}\right)\\
\leq& t \sup_{0\leq f\in\mathcal{F}_{2L_j}, \mu_{2L_j,q}(f)=1}\left\{\frac{a(t)}{t}\alpha L_j^{d/2-1}\mu_{2L_j,q}(W_{L_j}f) - D_{2L_j,q}(\sqrt{f})\right\},
\end{aligned}$$

where $\mathcal{F}_{2L_j} = \sigma(\eta(x), x \in \Lambda_{2L_j})$,

$$D_{2L_j,q}(\sqrt{f}) = \mu_{2L_j,q}\left(\sum_{x,y\in\Lambda_{2L_j},|x-y|=1}(\sqrt{f(\eta^{xy})} - \sqrt{f(\eta)})^2\right).$$

By the logarithmic Sobolev inequality,

$$H_{2L_j,q}(f) \le CL_j^2 D_{2L_j,q}(\sqrt{f}), \ \ 0 \le f \in \mathcal{F}_{2L_j}, \ \mu_{2L_j,q}(f) = 1,$$

where

$$H_{2L_j,q}(f) = \mu_{2L_j,q}(f \log f).$$

By the variational formula of the relative entropy (see (4.2)), for any $\gamma > 0$,

$$\begin{aligned}\mu_{2L_j,q}(W_{L_j} f) \le& \frac{1}{\gamma} \log \mu_{2L_j,q} \left(\exp\left\{\gamma W_{L_j}\right\}\right) + \frac{H_{2L_j,q}(f)}{\gamma} \\ \le& \frac{1}{\gamma} \log \mu_{2L_j,q} \left(\exp\left\{\gamma W_{L_j}\right\}\right) + \frac{CL_j^2 D_{2L_j,q}(\sqrt{f})}{\gamma}.\end{aligned}$$

For any $0 \le f \in \mathcal{F}_{2L_j}$ such that $\mu_{2L_j,q}(f) = 1$ and $D_{2L_j,q}(\sqrt{f}) \ne 0$, we choose $\gamma = L_j^{d/2+1}\sqrt{D_{2L_j,q}(\sqrt{f})}$, then

$$\begin{aligned}\mu_{2L_j,q}(W_{L_j} f) \le& \frac{1}{\gamma} \log \mu_{2L_j,q} \left(\exp\left\{\gamma W_{L_j}\right\}\right) + \frac{H_{2L_j,q}(f)}{\gamma} \\ \le& \frac{1}{\gamma} \log \mu_{2L_j,q} \left(\exp\left\{L_j^{d/2+1}\sqrt{D_{2L_j,q}(\sqrt{f})} W_{L_j}\right\}\right) \\ &+ CL_j^{1-d/2}\sqrt{D_{2L_j,q}(\sqrt{f})}.\end{aligned}$$

By **Lemma** 3.1, there exist constants $0 < B < \infty$, $\epsilon \in (0,1)$ such that for any

$$L_j^{1-d/2}\sqrt{D_{2L_j,q}(\sqrt{f})} \le \epsilon,$$

and $q \in (0,1)$,

$$\log \mu_{2L_j,q} \left(\exp\left\{L_j^{d/2-1}\sqrt{D_{2L_j,q}(\sqrt{f})} W_{L_j}\right\}\right) \le BL_j^{1-d/2}\sqrt{D_{2L_j,q}(\sqrt{f})},$$

and so

$$\mu_{2L_j,q}(W_{L_j} f) \le (C+B) L_j^{1-d/2}\sqrt{D_{2L_j,q}(\sqrt{f})}.$$

By a density argument, the last inequality holds for any $0 \le f \in \mathcal{F}_{2L_j}$ with $\mu_{2L_j,q}(f) = 1$.

Since $W_{L_j}, j \ge 1$ are uniformly bounded, there exists a constant A such that

$$\begin{aligned}&\frac{a(t)}{t}\alpha L_j^{d/2-1}\mu_{2L_j,q}(W_{L_j} f) - D_{2L_j,q}(\sqrt{f}) \\ \le& L_j^{d/2-1}\left(\frac{a(t)}{t}\alpha\mu_{2L_j,q}(W_{L_j} f) - L_j^{1-d/2} D_{2L_j,q}(\sqrt{f})\right) I_{\{L_j^{1-d/2} D_{2L_j,q}(\sqrt{f}) \le \frac{a(t)}{t}|\alpha| A\}}.\end{aligned}$$

Since $\frac{a(t)}{t} \to 0$ as $t \to \infty$, for any given α, there exists $t_\alpha > 0$, such that for all $t \ge t_\alpha$,

$$I_{\{L_j^{1-d/2} D_{2L_j,q}(\sqrt{f}) \le \frac{a(t)}{t}|\alpha| A\}} \le I_{\{L_j^{1-d/2} D_{2L_j,q}(\sqrt{f}) \le \epsilon^2\}} \le I_{\{L_j^{1-d/2}\sqrt{D_{2L_j,q}(\sqrt{f})} \le \epsilon\}}.$$

Therefore

$$\begin{aligned}&\sup_{0 \le f \in \mathcal{F}_{2L_j}, \mu_{2L_j,q}(f)=1}\left\{\frac{a(t)}{t}\alpha L_j^{d/2-1}\mu_{2L_j,q}(W_{L_j} f) - D_{2L_j,q}(\sqrt{f})\right\} \\ \le& \sup_{0 \le f \in \mathcal{F}_{2L_j}, \mu_{2L_j,q}(f)=1}\left\{\frac{a(t)}{t}\alpha (C+B)\sqrt{D_{2L_j,q}(\sqrt{f})} - D_{2L_j,q}(\sqrt{f})\right\} \\ \le& \frac{a^2(t)}{4t^2}\alpha^2 (C+B)^2.\end{aligned}$$

Thus, noting $d \geq 3$, we have

$$\begin{aligned}&\limsup_{t\to\infty}\frac{t}{a^2(t)}\log E_\rho\left(\exp\left\{\frac{a(t)}{t}\alpha\int_0^t\psi_{L,V}(\bar{\eta}_s^L)ds\right\}\right)\\ \leq&\frac{1}{4}\alpha^2(C+B)^2L^{1-d/2}\sum_{j=0}^{\infty}2^{j(1-d/2)}\to 0,\end{aligned}$$

as $L\to\infty$. □

The following lemma is an extension of Theorem 1.1 in [15] Appendix 3 (also Lemma 3.2 in [14]). The proof is similar to that of Theorem 1.1 in [15]. For convenience, we give its proof.

Lemma 4.2. Let $(E,\mathcal{E},\mu)$ be a probability space and set $L^2(\mu) := \{f\in\mathcal{E}; \|f\|_2^2=\mu(f^2)<\infty\}$. Let $-\mathcal{A}$ be a nonnegative definite symmetric operator on $L^2(\mu)$, which has 0 as a simple eigenvalue with corresponding eigenfunction space $\mathcal{U}_0\subset L^2(\mu)$ and $1\in\mathcal{U}_0$, and second smallest eigenvalue $\gamma>0$, i.e., the spectral gap of magnitude γ:

$$\gamma=\inf_{\|f\|_2\leq 1}\frac{\langle f-\pi_{\mathcal{U}_0}(f),-\mathcal{A}(f-\pi_{\mathcal{U}_0}(f))\rangle_\mu}{\|f-\pi_{\mathcal{U}_0}(f)\|_2^2}>0,$$

where $\pi_{\mathcal{U}_0}$ is the projection from $L^2(\mu)$ to $\mathcal{U}_0$. Let V be an essentially bounded function with $\mu(V)=0$, and denote by λ_ϵ the principal eigenvalue of $\mathcal{A}+\epsilon(V-\pi_{\mathcal{U}_0}(V))$ given by the variational formula

$$\lambda_\epsilon=\sup_{\|f\|_2\leq 1}\langle f,(\mathcal{A}+\epsilon(V-\pi_{\mathcal{U}_0}(V))f\rangle_\mu.$$

Then

$$0\leq\lambda_\epsilon\leq\frac{\epsilon^2}{1-2\epsilon\|V\|_\infty\gamma^{-1}}\langle V-\pi_{\mathcal{U}_0}(V),(-\mathcal{A})^{-1}(V-\pi_{\mathcal{U}_0}(V))\rangle_\mu.$$

Proof. Fix $\epsilon>0$. Denote by $\mathcal{A}_\epsilon=\mathcal{A}+\epsilon(V-\pi_{\mathcal{U}_0}(V))$ for simplicity. Note that

$$\lambda_\epsilon=\sup_{\|f\|_2\leq 1}\langle f-\pi_{\mathcal{U}_0}(f),(\mathcal{A}+\epsilon(V-\pi_{\mathcal{U}_0}(V))(f-\pi_{\mathcal{U}_0}(f))\rangle_\mu.$$

Therefore, there exists a sequence of functions $\{G_{\epsilon,n},n\geq 1\}$ of $L^2(\mu)$ such that

$$G_{\epsilon,n}\perp\mathcal{U}_0,\quad \|G_{\epsilon,n}\|_{L^2(\mu)}=1\quad\text{and}\quad\lim_{n\to\infty}\langle G_{\epsilon,n},\mathcal{A}_\epsilon G_{\epsilon,n}\rangle=\lambda_\epsilon.$$

Notice that $\langle G_{\epsilon,n},(\lambda_\epsilon-\mathcal{A}_\epsilon)G_{\epsilon,n}\rangle\to 0$, as $n\uparrow\infty$ by definition of $G_{\epsilon,n}$. By $1\in\mathcal{U}_0$, we also have $\mu(G_{\epsilon,n})=0$.

By definition of $G_{\epsilon,n}$, we have

$$\lambda_\epsilon=\langle G_{\epsilon,n},(\lambda_\epsilon-\mathcal{A}_\epsilon)G_{\epsilon,n}\rangle+\langle G_{\epsilon,n},\mathcal{A}_\epsilon G_{\epsilon,n}\rangle:=I_1+I_2.$$

By the Schwarz inequality we have that for any positive C

$$\begin{aligned}I_2=&\epsilon 2\mu((V-\pi_{\mathcal{U}_0})(G_{\epsilon,n}-1))+\epsilon\mu((V-\pi_{\mathcal{U}_0})(G_{\epsilon,n}-1)^2)-\langle G_{\epsilon,n},-\mathcal{A}G_{\epsilon,n}\rangle\\ \leq&\epsilon\bigg(\frac{\epsilon}{C}\langle(-\mathcal{A})^{-1}(V-\pi_{\mathcal{U}_0}),V-\pi_{\mathcal{U}_0}\rangle+\frac{C}{\epsilon}\langle(-\mathcal{A})(G_{\epsilon,n}-1),(G_{\epsilon,n}-1)\rangle\\ &+\mu((V-\pi_{\mathcal{U}_0})(G_{\epsilon,n}-1)^2)\bigg)-\langle G_{\epsilon,n},\mathcal{A}G_{\epsilon,n}\rangle\\ \leq&\frac{\epsilon^2}{C}\langle(-\mathcal{A})^{-1}(V-\pi_{\mathcal{U}_0}),V-\pi_{\mathcal{U}_0}\rangle+(C-1)\langle G_{\epsilon,n},-\mathcal{A}G_{\epsilon,n}\rangle+2\epsilon\|V\|_\infty.\end{aligned}$$

By our choice of $G_{\epsilon,n}$, $1 = \mu(G_{\epsilon,n})^2 \leq \gamma^{-1}\langle G_{\epsilon,n}, -\mathcal{A}G_{\epsilon,n}\rangle$. Recollecting all previous estimates we obtain that λ_ϵ is bounded above by

$$\langle G_{\epsilon,n}, (\lambda_\epsilon - \mathcal{A}_\epsilon)G_{\epsilon,n}\rangle + \frac{\epsilon^2}{C}\langle(-\mathcal{A})^{-1}(V-\pi_{\mathcal{U}_0}), V-\pi_{\mathcal{U}_0}\rangle - (1-C-2\|V\|\epsilon\gamma^{-1})\langle G_{\epsilon,n}, -\mathcal{A}G_{\epsilon,n}\rangle.$$

Now we take $C = 1-2\|V\|\epsilon\gamma^{-1}$. Noting $\langle G_{\epsilon,n}, (\lambda_\epsilon - \mathcal{A}_\epsilon)G_{\epsilon,n}\rangle \to 0$ as $n \uparrow \infty$, we conclude the proof of the upper bound by $n \uparrow \infty$..

The lower bound follows from the variational formula for λ_ϵ by taking $f = 1$. □

Lemma 4.3. Suppose that the mixing condition (2.6) holds. Let $d \geq 3$ and let $V \in \mathcal{H}_{-1}$ be a local function with mean zero. Then for any $\alpha \in \mathbb{R}$,

$$\limsup_{L\to\infty}\limsup_{t\to\infty}\frac{t}{a^2(t)}\log E_\rho\left(\exp\left\{\frac{a(t)}{t}\int_0^t \alpha\left(V(\eta_s)-\psi_{L,V}(\eta_s)\right)ds\right\}\right) \leq \frac{\alpha^2\sigma_\rho^2(V)}{2}, \tag{4.6}$$

where $\sigma_\rho^2(V) = 2\langle V, -\mathcal{A}V\rangle_{\mu_\rho}$.

Proof. Without loss of generality, assume $\mathrm{supp}(V) \subset \Lambda_L$ and $\mathrm{dist}(\partial\Lambda_L, \mathrm{supp}(V)) \geq L/2$. Set $\mathcal{G}_L = \sigma(\eta(x), x \in \Lambda_L)$. Then by Feynman-Kac formula and the variational formula of the principal eigenvalue of $\mathcal{A} + V - \psi_{L,V}$, for t large enough,

$$\begin{aligned}&E_\rho\left(\exp\left\{\frac{a(t)}{t}\int_0^t \alpha(V(\eta_s)-\psi_{L,V}(\eta_s))ds\right\}\right)\\ \leq&\exp\left\{t\sup_{f\geq 0,\mu_\rho(f)=1}\left\{\frac{a(t)}{t}\alpha\mu_\rho((V-\psi_{L,V})f) - D(\sqrt{f})\right\}\right\}.\end{aligned}$$

For $f \geq 0$ with $\mu_\rho(f) = 1$, set $f_L(\eta) = \mathbb{E}_\rho(f|\mathcal{G}_L)$. Then $0 \leq f \in \mathcal{G}_L$, $\mu_\rho(f) = 1$,

$$\mu_\rho((V-\psi_{L,V})f) = \mu_\rho((V-\psi_{L,V})f_L).$$

Noting that $f \to D(\sqrt{f})$ is a convex function, we also get

$$D(\sqrt{f}) \geq D(\sqrt{f_L}) \geq D_L(\sqrt{f_L}),$$

where

$$D_L(\sqrt{f}) = \mu_\rho\left(\sum_{\substack{x,y\in\Lambda_L\\|x-y|=1}}(\sqrt{f(\eta^{xy})}-\sqrt{f(\eta)})^2\right). \tag{4.7}$$

Thus,

$$\begin{aligned}&E_\rho\left(\exp\left\{\frac{a(t)}{t}\int_0^t \alpha(V(\eta_s)-\psi_{L,V}(\eta_s))ds\right\}\right)\\ \leq&\exp\left\{t\sup_{0\leq f\in\mathcal{G}_L,\mu_\rho(f)=1}\left\{\frac{a(t)}{t}\alpha\mu_\rho((V-\psi_{L,V})f) - D_L(\sqrt{f})\right\}\right\}.\end{aligned}$$

Now we consider the lattice gas dynamics corresponding to the Dirichlet form $D_L(f)$. Noting that $\mu_\rho((f-\psi_{L,f})^2) = \mu_\rho\left(\mu_{L,\bar{\eta}^L}((f-\psi_{L,f})^2)\right)$, by the logarithmic Sobolev inequality, the following Poincaré inequality holds:

$$\mu_\rho((f-\psi_{L,f})^2) \leq CL^2\int\mu_\rho(d\eta)D_{L,\eta^{\partial\Lambda_L},\bar{\eta}^L}(f) = CL^2D_L(f).$$

The process has a spectral gap of magnitude γ_L that depends on L. Therefore, by the previous lemma,

$$\begin{aligned}&\limsup_{t\to\infty}\frac{t}{a^2(t)}\log E_\rho\left(\exp\left\{\frac{a(t)}{t}\alpha\int_0^t\left(V(\eta_s)-\psi_{L,V}(\eta_s)\right)ds\right\}\right)\\ \le&\limsup_{t\to\infty}\frac{\alpha^2}{1-CL^2a(t)/t}\langle(V-\psi_{L,V}),-\mathcal{A}_L^{-1}(V-\psi_{L,V})\rangle_{\mu_\rho}\\ \le&\alpha^2\langle(V-\psi_{L,V}),-\mathcal{A}_L^{-1}(V-\psi_{L,V})\rangle_{\mu_\rho},\end{aligned}$$

where $\mathcal{A}_L$ is the generator corresponding to the Dirichlet form $D_L(f)=\langle f,-\mathcal{A}_L\rangle_{\mu_\rho}$. Note that

$$\begin{aligned}\langle(V-\psi_{L,V}),-\mathcal{A}_L^{-1}(V-\psi_{L,V})\rangle_{\mu_\rho}=&\sup_{g\in\mathcal{G}_L,\mu_\rho(g^2)<\infty}\{2\mu_\rho((V-\psi_{L,V})g)-D_L(g)\}\\ =\sup_{g\in\mathcal{G}_L,\mu_\rho(g^2)=1}&\frac{(\mu_\rho((V-\psi_{L,V})g))^2}{D_L(g)}.\end{aligned}$$

Since $f\to D_L(f)$ is lower-semicontinuous in $L^2(\mu_\rho)$, there exists $g_L\in\mathcal{G}_L$ with $\mu_\rho(g_L^2)=1$ such that

$$\langle(V-\psi_{L,V}),-\mathcal{A}_L^{-1}(V-\psi_{L,V})\rangle_{\mu_\rho}=\frac{(\mu_\rho((V-\psi_{L,V})g_L))^2}{D_L(g_L)}.$$

Choose subsequence $\{g_{L_n},n\ge1\}$ such that g_{L_n} weakly convergences g in $L^2(\mu_\rho)$ and

$$\limsup_{L\to\infty}\langle(V-\psi_{L,V}),-\mathcal{A}_L^{-1}(V-\psi_{L,V})\rangle_{\mu_\rho}=\lim_{n\to\infty}\frac{(\mu_\rho((V-\psi_V^{L_n})g_{L_n}))^2}{D_L(g_{L_n})}.$$

By equivalence of ensembles, we have that

$$\psi_{L_n,V}(\eta)\to0\ \ \mu_\rho-\text{a.s. and }L^2(\mu_\rho),$$

and so

$$\lim_{n\to\infty}\mu_\rho((V-\psi_{L_n,V})g_{L_n})=\mu_\rho(Vg).$$

Therefore

$$\begin{aligned}&\limsup_{L\to\infty}\limsup_{t\to\infty}\frac{t}{a^2(t)}\log E_\rho\left(\exp\left\{\frac{a(t)}{t}\int_0^t\alpha\left(V(\eta_s)-\psi_{L,V}(\eta_s)\right)ds\right\}\right)\\ &\le\alpha^2\limsup_{L\to\infty}\langle(V-\psi_{L,V}),-\mathcal{A}_L^{-1}(V-\psi_{L,V})\rangle_{\mu_\rho}\\ &\le\frac{(\mu_\rho(Vg))^2}{D(g)}\alpha^2\le\alpha^2\langle V,-\mathcal{A}^{-1}V\rangle_{\mu_\rho}=\frac{\alpha^2\sigma_\rho^2(V)}{2}.\end{aligned}$$

□

Theorem 4.1. Suppose that the mixing condition (2.6) holds. Let $d\ge3$ and let $V\in\mathcal{H}_{-1}$ be a local function with mean zero. Then for any $\alpha\in\mathbb{R}$,

$$\limsup_{t\to\infty}\frac{t}{a^2(t)}\log E_\rho\left(\exp\left\{\frac{a(t)}{t}\alpha\int_0^tV(\eta_s)ds\right\}\right)\le\frac{\alpha^2\sigma_\rho^2(V)}{2}.\tag{4.8}$$

In particular, for any closed set $F\subset\mathbb{R}$,

$$\limsup_{t\to\infty}\frac{t}{a^2(t)}\log P_\rho\left(\frac{1}{a(t)}\int_0^tV(\eta_s)ds\in F\right)\le-\inf_{x\in F}\frac{x^2}{2\sigma_\rho^2(V)}.\tag{4.9}$$

Proof. By the Gartner-Ellis theorem (cf. Theorem II.2 in [8]), we can obtain (4.9) from (4.8). Next, let us prove (4.9). For any $p > 1$ with $1/p + 1/q = 1$, by Hölder's inequality, the term inside the limit is bounded above by

$$\frac{t}{pa^2(t)} \log E_\rho \left(\exp\left\{ \frac{a(t)}{t} p\alpha \int_0^t (V(\eta_s) - \psi_{2^m,V}(\eta_s))\, ds \right\} \right)$$
$$+ \frac{t}{qa^2(t)} \log E_\rho \left(\exp\left\{ \frac{a(t)}{t} q\alpha \int_0^t \psi_{2^m,V}(\eta_s) \right\} \right).$$

Letting $t \to \infty$, by **Lemma** 4.1 and **Lemma** 4.3, we obtain

$$\limsup_{t\to\infty} \frac{t}{a^2(t)} \log E_\rho \left(\exp\left\{ \frac{a(t)}{t} \alpha \int_0^t V(\eta_s) ds \right\} \right) \le \frac{p\alpha^2 \sigma_\rho^2(V)}{2},$$

which implies (4.8) by letting $p \downarrow 1$. □

5 Lower bound

In this section, we prove the lower bound of the moderate deviation principle. The proof is based on a suitable measure transformation (see Lemma 8.18 and Lemma 8.21 in [24] for large deviations).

Let $d \ge 3$ and let $V \in \mathcal{H}_{-1}$ be a local function with mean zero. Suppose that the mixing condition holds for all λ. For any local function f with mean zero, and $t > 0$ with $\frac{a(t)}{t}\|f\|_\infty < 1$, set $u_t^f = \sqrt{1 + \frac{a(t)}{t} f}$. Then $\mu_\rho((u_t^f)^2) = 1$, and (cf. Lemma 8.18 in [24]),

$$M_s^{u_t^f} := \frac{u_t^f(\eta_s)}{u_t^f(\eta_0)} \exp\left\{ \int_0^s \frac{-\mathcal{A}u_t^f}{u_t^f}(\eta_v) dv \right\},\ s \ge 0,$$

is a martingale with expectation one. Define $P_x^{u_t^f}(A) = E_x(\mathbf{1}_A M_s^{u_t^f})$, $A \in \mathcal{F}_s$. Then $P_x^{u_t^f}$ is a Markov process with reversible invariant measure $d\mu^t := (u_t^f)^2 d\mu_\rho$. Let us call the corresponding stationary process $P^t = P_{\mu^t}^{u_t^f}$, the corresponding generator $\mathcal{A}^t$ and Dirichlet form D^t.

Lemma 5.1. Suppose that the mixing condition (2.6) holds. Let $d \ge 3$ and let $g \in \mathcal{H}_{-1}$ be a local function with mean zero. Then for any $\alpha \in \mathbb{R}$,

$$\limsup_{t\to\infty} \frac{t}{a^2(t)} \log E^t \left(\exp\left\{ \frac{a(t)}{t} \alpha \int_0^t (g(\eta_s) - \mu^t(g))) ds \right\} \right) \le \frac{\alpha^2 \sigma_\rho^2(g)}{2}, \tag{5.1}$$

where E^t denotes the expectation corresponding to P^t. In particular, for any $r > 0$,

$$\limsup_{t\to\infty} \frac{t}{a^2(t)} \log P^t \left(\left| \frac{1}{a(t)} \int_0^t g(\eta_s) ds - \mu_\rho(gf) \right| \ge r \right) \le -\frac{r^2}{2\sigma_\rho^2(g)}. \tag{5.2}$$

Proof. Let $\mathbf{P}_s^t,\ s \ge 0$ denote the semigroup of $\mathcal{A}^t$. Using the Taylor expansion, we can get

$$\begin{aligned}
\frac{\mathcal{A}u_t^f}{u_t^f} =& \frac{\sqrt{1 + (a(t)/t)f}\mathcal{A}\sqrt{1 + (a(t)/t)f}}{1 + (a(t)/t)f} \\
=& \sqrt{1 + (a(t)/t)f}\mathcal{A}\sqrt{1 + (a(t)/t)f} \left(1 - \frac{a(t)}{t} f + O\left(\frac{a^2(t)}{t^2} \right) \right) \\
=& \frac{1}{2}\frac{a(t)}{t}\mathcal{A}f - \frac{1}{4}\frac{a^2(t)}{t^2} f\mathcal{A}f - \frac{1}{8}\frac{a^2(t)}{t^2}\mathcal{A}f^2 + O\left(\frac{a^3(t)}{t^3} \right).
\end{aligned} \tag{5.3}$$

Note that $\mu_\rho(\mathcal{A}f)=0$ and $\mu_\rho(\mathcal{A}f^2)=0$. By the definition of the martingale $M_t^{u_t^f}$ and (5.3), we have that for any local function g with $\mu_\rho(g)=0$, and any $\alpha\in\mathbb{R}$,

$$\begin{aligned}
&\frac{t}{a^2(t)}\log E^t\left(\exp\left\{\frac{a(t)}{t}\alpha\int_0^t(g(\eta_s)-\mu^t(g)))ds\right\}\right)\\
&=\frac{t}{a^2(t)}\log E_\rho\left(M_t^{u_t^f}\exp\left\{\frac{a(t)}{t}\alpha\int_0^t g(\eta_s)ds-\frac{a^2(t)}{t}\alpha\mu_\rho(gf)\right\}\right)\\
&\le\frac14\mu_\rho(f\mathcal{A}f)-\alpha\mu_\rho(gf)+\frac{t}{a^2(t)}\log E_\rho\Bigg(\exp\left\{\frac{a(t)}{t}\int_0^t\left(\alpha g(\eta_s)-\frac12\mathcal{A}f(\eta_s)\right)ds\right.\\
&\qquad\left.+\frac14\frac{a^2(t)}{t^2}\int_0^t\left(f(\eta_s)\mathcal{A}f(\eta_s)(\eta_s)-\mu_\rho(f\mathcal{A}f)\right)ds\right\}\Bigg)+O\left(\frac{a(t)}{t}\right).
\end{aligned}\tag{5.4}$$

For any $p>1$ with $1/p+1/q=1$, by Hölder's inequality, the second term in the last inequality has the following estimate:

$$\begin{aligned}
&\frac{t}{a^2(t)}\log E_\rho\Bigg(\exp\left\{\frac{a(t)}{t}\int_0^t\left(\alpha g(\eta_s)-\frac12\mathcal{A}f(\eta_s)\right)ds\right.\\
&\qquad\left.+\frac14\frac{a^2(t)}{t^2}\int_0^t\left(f(\eta_s)\mathcal{A}f(\eta_s)(\eta_s)-\mu_\rho(f\mathcal{A}f)\right)ds\right\}\Bigg)\\
&\le\frac{t}{pa^2(t)}\log E_\rho\left(\exp\left\{\frac{pa(t)}{t}\int_0^t\left(\alpha g(\eta_s)-\frac12\mathcal{A}f(\eta_s)\right)ds\right\}\right)\\
&\quad+\frac{t}{qa^2(t)}\log E_\rho\left(\exp\left\{\frac14\frac{qa^2(t)}{t^2}\int_0^t\left(f(\eta_s)\mathcal{A}f(\eta_s)(\eta_s)-\mu_\rho(f\mathcal{A}f)\right)ds\right\}\right).
\end{aligned}$$

By (4.8),

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{t}{pa^2(t)}\log E_\rho\left(\exp\left\{\frac{pa(t)}{t}\int_0^t\left(\alpha g(\eta_s)-\frac12\mathcal{A}f(\eta_s)\right)ds\right\}\right)\\
&\le\frac{p\sigma_\rho^2\left(\alpha g-\frac12\mathcal{A}f\right)}{2}\\
&=\frac{p}{2}\left(\alpha^2\sigma_\rho^2(g)-2\alpha\mu_\rho(gf)+\frac12\mu_\rho(f\mathcal{A}f)\right),
\end{aligned}$$

and for any $\epsilon>0$,

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{t}{qa^2(t)}\log E_\rho\left(\exp\left\{\frac14\frac{qa^2(t)}{t^2}\left|\int_0^t\left(f(\eta_s)\mathcal{A}f(\eta_s)(\eta_s)-\mu_\rho(f\mathcal{A}f)\right)ds\right|\right\}\right)\\
&\le\limsup_{t\to\infty}\frac{t}{qa^2(t)}\log E_\rho\left(\exp\left\{\frac{a(t)}{t}\epsilon\left|\int_0^t\left(f(\eta_s)\mathcal{A}f(\eta_s)(\eta_s)-\mu_\rho(f\mathcal{A}f)\right)ds\right|\right\}\right)\\
&\le\frac{\epsilon^2\sigma_\rho^2(f\mathcal{A}f)}{2},
\end{aligned}$$

which implies

$$\limsup_{t\to\infty}\frac{t}{qa^2(t)}\log E_\rho\left(\exp\left\{\frac14\frac{qa^2(t)}{t^2}\left|\int_0^t\left(f(\eta_s)\mathcal{A}f(\eta_s)(\eta_s)-\mu_\rho(f\mathcal{A}f)\right)ds\right|\right\}\right)=0.$$

Thus, letting $t\to\infty$ in (5.4), we obtain

$$\limsup_{t\to\infty}\frac{t}{pa^2(t)}\log E^t\left(\exp\left\{\frac{pa(t)}{t}\alpha\int_0^t(g(\eta_s)-\mu^t(g))ds\right\}\right)\le\frac{p\alpha^2\sigma_\rho^2(g)}{2}.$$

This yields (5.1) by letting $p\to1$. Therefore, for any $r>0$, (5.2) also holds. □

Theorem 5.1. Suppose that the mixing condition (2.6) holds. Let $d \geq 3$ and let $V \in \mathcal{H}_{-1}$ be a local function with mean zero. Then for any open set $G \subset \mathbb{R}$,

$$\liminf_{t\to\infty} \frac{t}{a^2(t)} \log P_\rho\left(\frac{1}{a(t)}\int_0^t V(\eta_s)ds \in G\right) \geq -\inf_{z\in G} \frac{z^2}{2\sigma_\rho^2(V)}. \tag{5.5}$$

Proof. For $0 \neq z \in G$ fixed, choose $0 < \delta_0 < \sigma_\rho^2(V)/2$ such that the closed ball $B(z,\delta_0) := \{y; |z-y| \leq \delta_0\} \subset G$. Since

$$\frac{\sigma_\rho^2(V)}{2} = \|V\|_{-1} = \sup_{f\in\mathcal{C}} \left\{2\langle V,f\rangle_{\mu_\rho} - \langle f, -\mathcal{A}f\rangle_{\mu_\rho}\right\} = \sup_{g\in\mathcal{C},\mu_\rho(g)=0,\mu_\rho(g^2)=1} \frac{(\mu_\rho(Vg))^2}{\langle g,-\mathcal{A}g\rangle_{\mu_\rho}},$$

for any $0 < \delta < \delta_0$, there exists $g \in \mathcal{C}$ with $\mu_\rho(g) = 0$ and $\mu_\rho(g^2) = 1$ such that

$$\sigma_\rho^2(V) - \delta \leq \frac{2(\mu_\rho(Vg))^2}{\langle g,-\mathcal{A}g\rangle_{\mu_\rho}}.$$

In particular, we have $(\mu_\rho(Vg))^2 > 0$. Now, we take $f = zg/\mu_\rho(Vg)$. Then

$$f \in \mathcal{C},\ \mu_\rho(f) = 0,\ \mu_\rho(Vf) = z,\ \text{ and }\ \sigma_\rho^2(V) - \delta \leq \frac{2z^2}{\langle f,-\mathcal{A}f\rangle_{\mu_\rho}}.$$

For each $0 < \delta < \delta_0$, set

$$A_t(\delta) = \left\{\left|\frac{1}{t}\int_0^t f\mathcal{A}f(\eta_s)ds - \mu_\rho(f\mathcal{A}f)\right| < \delta\right\},$$

$$B_t(\delta) = \left\{\left|\frac{1}{a(t)}\int_0^t \mathcal{A}f(\eta_s)ds - \mu_\rho(f\mathcal{A}f)\right| < \delta\right\},$$

and

$$C_t(\delta) = \left\{\left|\frac{1}{a(t)}\int_0^t V(\eta_s)ds - \mu_\rho(Vf))\right| \leq \delta\right\}.$$

Then, by **Lemma** 5.1, for any $\delta > 0$,

$$\lim_{t\to\infty} P^t(A_t(\delta)\cap B_t(\delta)\cap C_t(\delta)) = 1.$$

Thus

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{t}{a^2(t)} \log P_\rho(G)\\
\geq&\liminf_{t\to\infty} \frac{t}{a^2(t)} \log P_\rho(C_t(\delta))\\
\geq&\liminf_{t\to\infty} \frac{t}{a^2(t)} \log \int_{A_t(\delta)\cap B_t(\delta)\cap C_t(\delta)} \frac{u_t^f(\eta_0)}{u_t^f(\eta_t)} e^{\int_0^t \frac{\mathcal{A}u_t^f}{u_t^f}(\eta_s)ds} dP^t\\
\geq&-\mu_\rho(f\mathcal{A}f)/2 + \mu_\rho(f\mathcal{A}f)/4 - 2\delta + \liminf_{t\to\infty} \frac{t}{a^2(t)} \log P^t(A_t(\delta)\cap B_t(\delta)\cap C_t(\delta))\\
=&-\frac{1}{4}\langle f,-\mathcal{A}f\rangle - 2\delta\\
\geq&\frac{z^2}{2(\sigma_\rho^2(V)-\delta)} - 2\delta.
\end{aligned}$$

Letting $\delta \to 0$, we obtain

$$\liminf_{t\to\infty} \frac{t}{a^2(t)} \log P_\rho(G) \geq -\frac{z^2}{2\sigma_\rho^2(V)}.$$

□

A Some moment estimates of the Gibbs measures

For the sake of convenience, we introduce some results for moment estimates of the Gibbs measures in [26] and [28].

Firstly, let us recall **Lemma** 5.2 in [28].

Lemma A.1 (see Lemma 5.2 in [28])**.** Let ν be a probability measure on $(\Omega, \mathcal{F})$. Suppose g is a real measurable function on $(\Omega, \mathcal{F})$ satisfying

$$\sup_{\omega\in\Omega} |g(\omega) - \nu(g)| \le L^{-d/2}.$$

Then there exist a positive constant A such that for any $L \ge 1$, $\beta > 0$,

$$\frac{1}{\beta L^d} \log \nu\left(\exp\left\{\beta L^d (g - \nu(g))\right\}\right) \le A\beta. \tag{A.1}$$

Let g be a smooth function on $(0, 1)$ with

$$|g'(y)| \le C, \ \ |g''(y)| \le C \text{ for all } y \in (0, 1). \tag{A.2}$$

Define

$$\phi(y) = g(y) - g'(q_L^c)(y - q_L^c) - g(q_L^c), \ \ y \in (0, 1),$$

where

$$q_L^c = \mu_{2L,q}\left(\bar{\eta}^L\right).$$

Note that

$$\Lambda_{2L} \setminus \Lambda_L = \text{ a union of cubes of side length } 2L.$$

Applying **Theorem** 5.1 in [28] to $\mu_{2L,q}$, $U = \Lambda_L$, and $\pm V$, then there exist constants $0 < A < \infty$, $\epsilon \in (0, \infty)$ such that for any $0 < \beta \le \epsilon$, for any $q \in (0, 1)$,

$$\frac{1}{\beta L^d} \log \mu_{2L,q}\left(\exp\left\{\pm\beta L^d \phi(\bar{\eta}^L)\right\}\right) \le A\beta.$$

Set $\beta = \theta L^{-d/2+1}$. Then for any $0 < \theta \le \epsilon L^{d/2-1}$, for any $q \in (0, 1)$,

$$\log \mu_{2L,q}\left(\exp\left\{\pm\theta L^{1+d/2} \phi(\bar{\eta}^L)\right\}\right) \le A\theta^2 L^2.$$

Thus, the following exponential moment estimate holds.

Lemma A.2 (cf. Theorem 5.1 in [28])**.** Let g be a smooth function on $(0, 1)$ with (A.2). Assume that the mixing condition (2.7) holds. Then there exist constants $0 < A < \infty$, $\epsilon \in (0, \infty)$ such that for any $|\theta| \le \epsilon L^{d/2-1}$, for any $q \in (0, 1)$,

$$\log \mu_{2L,q}\left(\exp\left\{\theta L^{1+d/2} \phi(\bar{\eta}^L)\right\}\right) \le A\theta^2 L^2. \tag{A.3}$$

Lemma A.3 (cf. (5.20) and (5.21) in [28])**.** Let g be a smooth function on $(0, 1)$ with (A.2). Assume that the mixing condition (2.7) holds. Then there exist constants $0 < A < \infty$, $\epsilon \in (0, \infty)$ such that for any $|\theta| \le \epsilon L^{d/2-1}$, for any $q \in (0, 1)$,

$$\log \mu_{2L,q}\left(\exp\left\{\theta (2L+1)^{1+d/2} \left(g(\bar{\eta}^L) - g(q_L^c)\right)\right\}\right) \le A|\theta|^2 L^2, \tag{A.4}$$

and

$$\log \mu_{2L,q}\left(\exp\left\{\theta (2L+1)^{1+d/2} g'(q_L^c)\left(\bar{\eta}^L - q_L^c\right)\right\}\right) \le A|\theta|^2 L^2. \tag{A.5}$$

Proof. By (5.20) and (5.21) in [28], there exist constants $0 < A < \infty$, $\epsilon \in (0,\infty)$ such that for any $|\beta| \le \epsilon$, for any $q \in (0,1)$,

$$\frac{1}{(2L+1)^d} \log \mu_{2L,q} \left(\exp\left\{\beta(2L+1)^d \left(g(\bar{\eta}^L) - g(q_L^c)\right)\right\}\right) \le A|\beta|^2, \tag{A.6}$$

and

$$\frac{1}{(2L+1)^d} \log \mu_{2L,q} \left(\exp\left\{\beta(2L+1)^d g'(q_L^c) \left(\bar{\eta}^L - q_L^c\right)\right\}\right) \le A|\beta|^2. \tag{A.7}$$

Now, take $\beta = \theta L^{-d/2+1}$ in (A.6) and (A.7). Then for any $0 < \theta \le \epsilon L^{d/2-1}$, for any $q \in (0,1)$, (A.4) and (A.5) hold. $\square$

The following lemma is the second lemma in Section 10 of [26] which gives a comparison of the mean of particles at two different points.

Lemma A.4 (cf. Section 10 of [26])**.** Suppose that the mixing condition (2.6) holds. Then for any $s > 0$, there exists a constant C_s such that

$$\sup_{\omega\in E} \sup_{y\in(0,1)} \left|\mu^c_{\Lambda_L,\omega,y(2L+1)^d}(\eta_0 - \eta_e)\right| \le C_s L^{-s},$$

for any $e \in \Lambda_L$. In particular, there exists a constant C_d such that

$$\sup_{\omega\in E} \sup_{y\in(0,1)} \left|\mu^c_{\Lambda_{2L},\omega,y(4L+1)^d}(\bar{\eta}^L) - y\right| \le C_d L^{-d}.$$

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Acknowledgments. The authors are very grateful to the anonymous referees for their helpful comments and suggestions.