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Asymptotic capacity of the range of random walks on free products of graphs

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Abstract

In this article we prove existence of the asymptotic capacity of the range of random walks on free products of graphs. In particular, we will show that the asymptotic capacity of the range is almost surely constant and strictly positive. Furthermore, we provide a central limit theorem for the capacity of the range and show that it varies real-analytically in terms of finitely supported probability measures of constant support.

Keywords: capacity; free product; random walk; range; central limit theorem; analyticity.
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1 Introduction

Suppose we are given finite or countable sets V_1 and V_2 with $|V_i| \geq 2$, $i \in \{1,2\}$, and distinguished vertices $o_1 \in V_1$ and $o_2 \in V_2$. The free product of V_1 and V_2 is given by $V := V_1 * V_2$, the set of all finite words $x_1 \dots x_n$ over the alphabet $\big(V_1 \setminus \{o_1\}\big) \cup \big(V_2 \setminus \{o_2\}\big)$ such that no two consecutive letters x_j and x_{j+1} arise from the same $V_i \setminus \{o_i\}$, $i \in \{1,2\}$. Furthermore, let P_1 and P_2 be transition matrices on V_1 and V_2. Consider now a time-homogeneous, transient Markov chain $(X_n)_{n\in\mathbb{N}_0}$ starting at the empty word o whose transition matrix arises as a convex combination of some versions of P_1 and P_2 shifted to V (see Section 2.2 for the exact definition). For better visualization, we may think of graphs $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}$ with vertex sets V_1, V_2 and V such that there is an oriented edge connecting the vertices x and y if and only if the corresponding transition probability (w.r.t. P_1, P_2 and P) of going from x to y in one step is strictly positive.

For $A \subseteq V$, denote by $S_A := \inf\{m \in \mathbb{N} \mid X_m \in A\} \in [1,\infty]$ the first returning time to A. The capacity of the set A is then defined as

$$\mathrm{Cap}(A) := \sum_{x\in A} \mathbb{P}\big[S_A = \infty \mid X_0 = x\big].$$

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The number $\mathrm{Cap}(A)$ is a measure for the size and density of the set A and can be regarded as a mathematical analogue of the ability of A to hold electrical charge. In particular, the denser the set A the heavier it is to never visit A again when starting at an “inner” point which has many elements of A in its neighbourhood. In this article we are interested to study the asymptotic behaviour of the capacity of the range at time $n \in \mathbb{N}$ as $n \to \infty$. Recall that the range of the random walk $(X_n)_{n\in\mathbb{N}_0}$ at time n is given by the set

$$\mathbf{R}_n := \{X_0, X_1, \dots, X_n\},$$

the set of vertices visited up to time n. The *asymptotic capacity of the range of the random walk* $(X_n)_{n\in\mathbb{N}_0}$ *on* V is given by

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{Cap}(\mathbf{R}_n),$$

provided the limit exists. The aim of this article is to show that the asymptotic capacity exists, is almost surely constant and strictly positive. In particular, we link the asymptotic capacity with the rate of escape of the underlying random walk (see (3.11) in the proof of Theorem 1.1). Moreover, we will provide a central limit theorem for the capacity of the range and show that it varies real-analytically, when the transition probabilities depend on finitely many parameters.

The range of random walks has been studied in great variety in the past. Let us outline some main results. Dvoretzky and Erdös [9] proved a strong law of large numbers for the range of simple random walk on $\mathbb{Z}^d$, $d \geq 2$. This result was generalized to arbitrary random walks on $\mathbb{Z}^d$, $d \geq 1$, by Spitzer [26]. Jain and Orey [16] proved a central limit theorem for the range on $\mathbb{Z}^d$. For simple random walk on regular trees with $N+1$ branches, Chen, Yan and Zhou [8] calculated that the asymptotic range is given by $(N-1)/N$. For random walks on finitely generated groups with identity e, Guivarc'h [15] provided the nice formula

$$\lim_{n\to\infty} \frac{|\mathbf{R}_n|}{n} = 1 - \mathbb{P}[\exists n \in \mathbb{N} : X_n = e \mid X_0 = e].$$

The capacity of the range of random walks has been studied mainly on $\mathbb{Z}^d$ and groups. Jain and Orey [16] proved existence of the asymptotic capacity of the range for random walks on the integer lattice $\mathbb{Z}^d$, $d \geq 3$, where the asymptotic capacity is strictly positive if and only if $d \geq 5$. Lawler [19] gave estimates for intersection probabilities of random walks which in turn allow to estimate the capacity of the range. More recently, some topics have been studied which are related to the capacity of the range of random walks and have given some momentum to the study of the capacity, e.g., Sznitman [28] studied random interlacements, Asselah and Schapira [1] investigated the geometry of random walks under localization constraints. For the capacity of the range of random walks on $\mathbb{Z}^d$, Asselah, Schapira and Sousi [2, 3] proved a central limit theorem. Further results are due to Chang [7] and Schapira [24].

Recently, Mrazović, Sandrić and Šebek [22] proved that the asymptotic capacity of the range of symmetric simple random walks on finitely generated groups exists. They applied Kingman's subadditive theorem for proving existence of the asymptotic capacity of the range, and they also provided a central limit theorem for the asymptotic capacity. The goal of this paper is to go beyond group-invariant random walks and to derive analogous statements on existence of the asymptotic capacity of the range. In our case of general free products we have no group operation on V, and therefore we can *not* apply Kingman's subadditive ergodic theorem. Thus, existence of the asymptotic capacity of the range is not guaranteed a-priori. Since the asymptotic capacity of the range is an important random walk characteristic number, studying existence for random walks on general free products deserves its own right.

Free products form an important class of (graph) structures whose importance is due to Stallings' Splitting Theorem (see Stallings [27]): a finitely generated group Γ has more than one (geometric) end if and only if Γ admits a non-trivial decomposition as a free product by amalgamation (amalgam) or as an HNN extension over a finite subgroup; see, e.g., Lyndon and Schupp [20] for more information on amalgams and HNN extensions. We recall that free products are amalgams over the trivial subgroup. In this article we consider free products of graphs which form a generalization of free products of groups. In particular, we have no underlying group-invariant random walk and the reasoning of the group case cannot be applied which makes it necessary to follow other approaches. Let us note that free products can – at least to some extent – also be used to model random walks on regular languages, First-In First-Out queues or stacks.

Random walks on free products have been studied to a great extent throughout the last decades. Asymptotic behaviour of return probabilities of random walks on free products has been studied, e.g., by Gerl and Woess [10], Woess [29], Sawyer [23], Cartwright and Soardi [6], Lalley [17, 18] and Candellero and G. [5]. Explicit formula for the drift and the asymptotic entropy have been computed by Mairesse and Mathéus [21] for random walks on free products of finite groups, while G. [12, 13] calculated different formulas for both characteristic numbers for random walks on free products of graphs. The spectral radius of random walks on some classes of free products of graphs has been studied by Shi et al. [25]. Finally, the range of random walks on (general) free products has been studied in G. [14], where existence of the limit $\lim_{n\to\infty} \mathbf{R}_n/n$ has been proven, including a central limit theorem. This work will serve as a main reference for the current article, but the current article will go far beyond the scope of [14].

The aim of this article is to go a step beyond the group case when studying the asymptotic capacity of the range of random walks. Throughout this paper we make the basic assumption that the Green function $G(z) := \sum_{n\geq 0} \mathbb{P}[X_n = o | X_0 = o] z^n$, $z \in \mathbb{C}$, has radius of convergence $\mathscr{R}$ strictly bigger than 1. This is equivalent to

$$\varrho := \limsup_{n\to\infty} \mathbb{P}[X_n = o | X_0 = o]^{\frac{1}{n}} < 1.$$

This assumption ensures transience of the underlying random walk $(X_n)_{n\in\mathbb{N}_0}$ and excludes degenerate cases. Our main result guarantees existence of the asymptotic capacity of the range of $(X_n)_{n\in\mathbb{N}_0}$:

Theorem 1.1. *Assume that $\varrho < 1$. Then there exists a constant $\mathfrak{c} \in (0,1]$ such that*

$$\lim_{n\to\infty} \frac{\mathrm{Cap}(\mathbf{R}_n)}{n} = \mathfrak{c} \quad \textit{almost surely.}$$

We also provide a central limit theorem w.r.t. the capacity of the range:

Theorem 1.2. *Assume that $\varrho < 1$. Furthermore, assume that there are $x_0 \in V$ and $\kappa \in \mathbb{N}$ such that $\mathbb{P}[X_\kappa = x_0 | X_0 = x_0] > 0$. Then the asymptotic capacity satisfies the following central limit theorem:*

$$\frac{\mathrm{Cap}(\mathbf{R}_n) - n \cdot \mathfrak{c}}{\sigma \cdot \sqrt{n}} \xrightarrow{d} N(0,1),$$

where $\sigma^2 > 0$ is given by (4.7).

The assumption of existence of $x_0 \in V$ and $\kappa \in \mathbb{N}$ with $\mathbb{P}[X_\kappa = x_0 | X_0 = x_0] > 0$ excludes once again degenerate cases, where a central limit theorem might not exist, see Remark 4.7 for a counterexample.

We remark that Theorem 1.1 and 1.2 formulate new results for random walks on free products of groups, since in [22] only symmetric simple random walk on finitely generated groups were considered.

The third main result shows that the capacity of the range varies real-analytically in terms of probability measures of constant support. Here, we assume that there are finitely many values $p_1, \dots, p_d \in (0,1)$, $d \in \mathbb{N}$, such that all strictly positive single-step transition probabilities $p(x,y)$, $x,y \in V$, satisfy $p(x,y) \in \{p_1, \dots, p_d\}$. If we let vary the values $p_1, \dots, p_d$ among all positive values which still allow well-defined random walks on V, we may then regard $\mathfrak{c}$ as a function in $(p_1, \dots, p_d)$. Then:

Theorem 1.3. *Assume that $\varrho < 1$. Suppose that the single step transition probabilities of the random walk $(X_n)_{n \in \mathbb{N}_0}$ take only $d \in \mathbb{N}$ many non-negative values $p_1, \dots, p_d \in (0,1)$. Then the mapping*

$$(p_1, \dots, p_d) \mapsto \mathfrak{c} = \mathfrak{c}(p_1, \dots, p_d)$$

varies real-analytically.

We note that, for random walks on groups beyond free products, it is unknown whether the asymptotic capacity of the range varies real-analytically. In particular, this problem is open in the case of $\mathbb{Z}^d$ for $d \geq 2$. This underlines the innovative character of Theorem 1.3. The proof heavily involves generating function techniques, which requires a deep understanding of the interplay of the generating functions on V and on the free factors V_i.

The paper is organized as follows: in Section 2 we give a short introduction to free products on which we define a natural class of random walks, and we introduce some basic notation. In Section 3 we derive existence of the asymptotic capacity of the range for random walks on free products, while in Section 4 the proposed central limit theorem is proven. In Section 5 we prove the real-analytic behaviour of $\mathfrak{c}$.

2 Random walks on free products

2.1 Free products of graphs

Let V_1 and V_2 be finite or countable sets with at least two elements. We assume that $V_1 \cap V_2 = \emptyset$ and we exclude the case $|V_1| = |V_2| = 2$; see Remark 2.5. For each $i \in \mathcal{I} := \{1,2\}$, we select a distinguished element o_i of V_i, which we call the "root" of V_i. On each V_i consider a random walk with transition matrix $P_i = (p_i(x,y))_{x,y \in V_i}$. The corresponding n-step transition probabilities are denoted by $p_i^{(n)}(x,y)$, where $x,y \in V_i$. Since only those elements of V_i will be of interest, which can be reached from o_i, we may assume w.l.o.g. that, for every $i \in \mathcal{I}$ and every $x \in V_i$, there exists some $n_x \in \mathbb{N}$ such that $p_i^{(n_x)}(o_i, x) > 0$. Furthermore, for sake of simplicity, we assume $p_i(x,x) = 0$ for every $i \in \mathcal{I}$ and $x \in V_i$; this assumption can be lifted without restrictions but the general proof would reduce the readability of the proofs; see [14, Section 6].

For better visualization, we may think of rooted graphs $\mathcal{X}_i$ with vertex sets V_i and roots o_i such that there is an oriented edge $x \to y$ if and only if $p_i(x,y) > 0$.

For $i \in \mathcal{I}$, set $V_i^\times := V_i \setminus \{o_i\}$ and $V_*^\times := V_1^\times \cup V_2^\times$. The *free product of* V_1 *and* V_2 is given by the set

$$V := V_1 * V_2 := \left\{ x_1 x_2 \dots x_n \,\middle|\, n \in \mathbb{N}, x_j \in V_*^\times, x_j \in V_k^\times \Rightarrow x_{j+1} \notin V_k^\times \right\} \cup \{o\}, \tag{2.1}$$

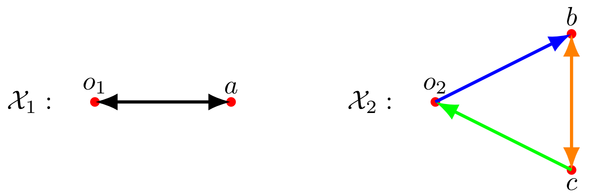
the set of all finite words over the alphabet $V_*^\times$ such that no two consecutive letters come from the same $V_i^\times$, where o describes the empty word. Note that $V_i^\times \subseteq V$ and we may consider o_i as the "empty word" of V_i. Throughout this paper we will use the representation in (2.1) for elements in V.

Observe that there is a natural partial composition law on V: if $u = u_1 \dots u_m \in V$ and $v = v_1 \dots v_n \in V$ with $u_m \in V_i^\times$, $i \in \mathcal{I}$, and $v_1 \notin V_i^\times$, then $uv \in V$ stands for their concatenation as words. In particular, we set $uo_i := u$ for all $i \in \mathcal{I}$ and $o_i u := u$; hence, o_i is also identified with the empty word o. Since concatenation of words is only partially defined, concatenation is *not* a group operation on V; in particular, standard arguments from the group case like Kingman's subadditive ergodic theorem (as used in [22]) can *not* be applied directly.

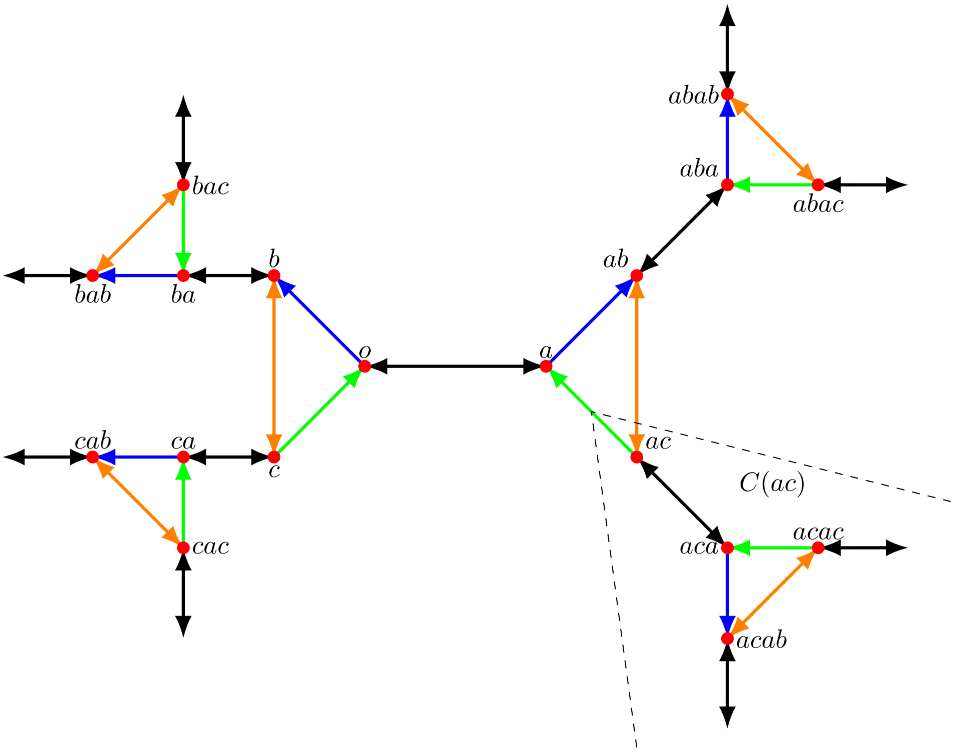
The *word length* of a word $u = u_1 \dots u_m$ is defined as $\|u\| := m$. Additionally, we set $\|o\| := 0$. The *type* $\delta(u)$ of u is defined to be $i \in \mathcal{I}$ if $u_m \in V_i^\times$; we set $\delta(o) := 0$.

The set V can again be identified as the vertex set of a graph $\mathcal{X}$ which is constructed inductively as follows: take copies of $\mathcal{X}_1$ and $\mathcal{X}_2$ and glue them together at their roots to one single common root, which becomes o; inductively, at each vertex $v = v_1 \dots v_k$ with $v_k \in V_i$ added in the step before attach a copy of $\mathcal{X}_j$, $j \in \mathcal{I} \setminus \{i\}$, where v is identified with o_j from the new copy of $\mathcal{X}_j$. Then $\mathcal{X}$ is the *free product of the graphs* $\mathcal{X}_1$ *and* $\mathcal{X}_2$. The underlying graph structure of free products allows us to define paths: a *path* of length $n \in \mathbb{N}$ in $\mathcal{X}$ is a sequence of vertices $(z_0, z_1, \dots, z_n)$ in V such that there is an oriented edge in $\mathcal{X}$ from z_{i-1} to z_i for each $i \in \{1, \dots, n\}$. Recall that, for each $x \in V$, there is a path from o to x by construction of V and the assumption made at the beginning of this subsection.

Example 2.1. Consider the sets $V_1 = \{o_1, a\}$ and $V_2 = \{o_2, b, c\}$ equipped with the following graph structure:



The graph $\mathcal{X}$ of the free product $V_1 * V_2$ has then the following structure:



The tree-like graph structure of free products motivates the following definition: the *cone* rooted at $x \in V$ is given by the set

$$C(x) := \{y \in V \mid y \text{ has prefix } x\}.$$

In particular, for all $y \in C(x)$, each path from o to y has to pass through x. Moreover, we have $C(o) = V$. E.g., in Example 2.1 we have $C(ac) = \{ac, aca, acab, acac, \dots\}$, the set of all words inside the dashed cone.

2.2 Random walks on free products

We now construct a natural random walk on V arising from P_1 and P_2. For this purpose, we lift the transition matrices P_1 and P_2 to transition matrices $\bar{P}_i = \big(\bar{p}_i(x,y)\big)_{x,y\in V}$, $i \in \mathcal{I}$, on V: if $x \in V$ with $\delta(x) \neq i$ and $v, w \in V_i$, then $\bar{p}_i(xv, xw) := p_i(v,w)$. Otherwise, we set $\bar{p}_i(x,y) := 0$. Choose $\alpha \in (0,1)$. Then we define a new transition matrix P on V by

$$P = \alpha \cdot \bar{P}_1 + (1-\alpha) \cdot \bar{P}_2,$$

which governs a nearest neighbour random walk on $\mathcal{X}$. We may interpret the random walk as follows: if the random walker stands at some vertex $x \in V$ with $\delta(x) = i \in \mathcal{I}$, he first tosses a coin and afterwards – in dependence of the outcome of the coin toss – he either performs one step within the copy of $\mathcal{X}_i$ to which x belongs according to $\bar{P}_i$ *or* one step into the new copy of $\mathcal{X}_j$, $j \in \mathcal{I} \setminus \{i\}$, attached at x according to $\bar{P}_j$. The sequence of random variables $(X_n)_{n\in\mathbb{N}_0}$ with $X_0 := o$ describes the random walk on V governed by P, where X_n denotes the random walker's position at time $n \in \mathbb{N}_0$. For $x, y \in V$, the correspondung single and n-step transition probabilities are denoted by $p(x,y)$ and $p^{(n)}(x,y)$. Thus, P governs a nearest neighbour random walk on the graph $\mathcal{X}$, where P arises from a convex combination of the nearest neighbour random walks on the graphs $\mathcal{X}_1$ and $\mathcal{X}_2$. This definition ensures that every path $(w_0, \dots, w_n)$ in $\mathcal{X}$ has strictly positive probability $\mathbb{P}\big[X_1 = w_1, \dots, X_n = w_n \mid X_0 = w_0\big] > 0$ to be performed. We use the notation $\mathbb{P}_x[\,\cdot\,] := \mathbb{P}[\,\cdot \mid X_0 = x]$ for $x \in V$.

The spectral radius at o is defined as

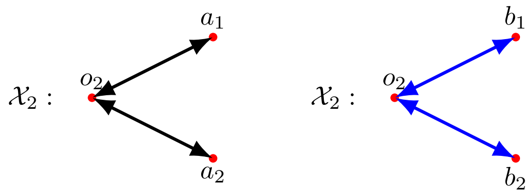
$$\varrho := \limsup_{n\to\infty} p^{(n)}(o,o)^{1/n}.$$

As a *basic assumption* throughout this paper we assume that

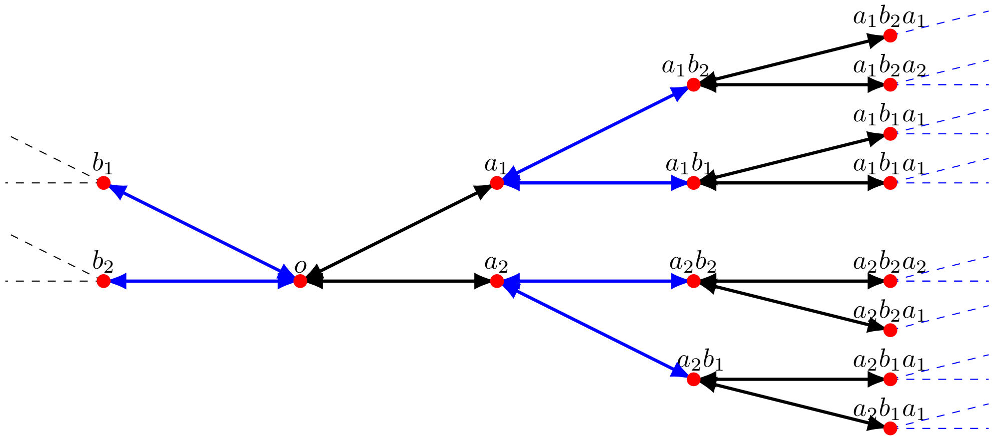
$$\varrho < 1.$$

Equivalently, the Green function $G(o,o|z) := \sum_{n\geq 0} p^{(n)}(o,o)\, z^n$, $z \in \mathbb{C}$, has radius of convergence $\mathscr{R}$ strictly bigger than 1. This assumption implies *transience* of the random walk governed by P and excludes degenerate cases; in particular, the recurrent case $|V_1| = |V_2| = 2$ is excluded, see Remark 2.5. If one out of P_1 and P_2 is not irreducible, then $\varrho < 1$ (easy to check!). If, e.g., P_1 and P_2 govern irreducible and reversible random walks, then $\varrho < 1$; see [30, Theorem 10.3]. Note that it is possible to construct null-recurrent random walks with $\mathscr{R} = 1$ and $|V_1| > 2 = |V_2|$:

Example 2.2. Consider the sets $V_1 = \{o_1, a_1, a_2\}$ and $V_2 = \{o_2, b_1, b_2\}$ equipped with the following graph structure:



The graph $\mathcal{X}$ of the free product $V_1 * V_2$ has then the following structure:



Set $\alpha_1 = \alpha_2 = \frac{1}{2}$. Now it is easy to see that the process $(\|X_n\|)_{n\in\mathbb{N}_0}$ is an irreducible, null-recurrent random walk on N_0, which in turn implies that $(X_n)_{n\in\mathbb{N}_0}$ is null-recurrent. Hence, $R = 1$.

Denote by V_∞ the set of *infinite* words $y_1y_2y_3\dots$ over the alphabet $V_*^\times$ such that no two consecutive letters arise from the same $V_i^\times$. For $x \in V$ and $y \in V_\infty$, denote by $x \wedge y$ the common prefix of maximal length of x and y. In [12, Proposition 2.5] it is shown that the random walk $(X_n)_{n\in\mathbb{N}_0}$ converges to some V_∞-valued random variable X_∞ in the sense that the length of the common prefix of X_n and X_∞ tends to infinity almost surely. In other words, $\lim_{n\to\infty} \|X_n \wedge X_\infty\| = \infty$ almost surely. We will make use of these results in the proofs later. For more details, we refer to [12].

Important random walk characteristic numbers are given by the *rate of escape* (or *drift*) and the *asymptotic range* of random walks. In [12, Theorem 3.3] it was shown that there exists a strictly positive number $\ell \in (0,1]$, the *rate of escape w.r.t. the word length (or block length)* of $(X_n)_{n\in\mathbb{N}_0}$, such that

$$\ell = \lim_{n\to\infty} \frac{\|X_n\|}{n} \quad \text{almost surely.}$$

The set of vertices visited by the random walk $(X_n)_{n\in\mathbb{N}_0}$ until time n is given by

$$\mathbf{R}_n := \{X_0, X_1, \dots, X_n\}.$$

The recent paper [14, Theorem 1.1] has proven existence of a strictly positive number $\mathfrak{r} \in (0,1]$, the *(asymptotic) range* of the random walk $(X_n)_{n\in\mathbb{N}_0}$, such that

$$\mathfrak{r} = \lim_{n\to\infty} \frac{|\mathbf{R}_n|}{n} \quad \text{almost surely.} \tag{2.2}$$

While ℓ measures the speed at which the random walk escapes to infinity, $\mathfrak{r}$ measures the speed at which new vertices are visited and it serves as a measure for how much of the graph is explored by the random walk. Both characteristic numbers will come into play later.

We mention two final remarks. The following equation will be important, which states that probabilities of paths within a cone depend only on their relative location to the cone's root:

Lemma 2.3. *Let $n \in \mathbb{N}$ and $w \in V$ and $w_1, \dots, w_n \in C(w)$. Write $w_i = ww_i'$ for $i \in \{1, \dots, n\}$. Then:*

$$\mathbb{P}_w[X_1 = w_1, \dots, X_n = w_n] = \mathbb{P}_o[X_1 = w_1', \dots, X_n = w_n'].$$

Proof. See [14, Lemma 3.2]. □

Due to the structure of the free product and Lemma 2.3 it is easy to verify that, for all $i \in \mathcal{I}$ and all $x \in V \setminus \{o\}$ with $\delta(x) = i$, the probability

$$\xi_i := \mathbb{P}\big[\exists n \in \mathbb{N} : X_n \notin C(x) \mid X_0 = x\big] \tag{2.3}$$

does not depend on x. In [12, Lemma 2.3] it is shown that $\xi_i < 1$ for each $i \in \mathcal{I}$.

2.3 Asymptotic capacity of the range

We recall the definition of the capacity of a subset $A \subseteq V$. For $A \subseteq V$, the stopping time of the first return into the set A is defined as

$$S_A := \inf\{m \in \mathbb{N} \mid X_m \in A\} \in \mathbb{N} \cup \{\infty\}.$$

The *capacity* of the set A is then defined as

$$\mathrm{Cap}(A) := \sum_{x \in A} \mathbb{P}_x\big[S_A = \infty\big].$$

This number is a mathematical analogue from physics of the ability of A to store an electrical charge.

If there exists a constant $\mathfrak{c} \in [0,1]$ such that

$$\lim_{n\to\infty} \frac{\mathrm{Cap}(\mathbf{R}_n)}{n} = \mathfrak{c} \quad \text{almost surely,}$$

then we call $\mathfrak{c}$ the *asymptotic capacity of the range* of the random walk $(X_n)_{n\in\mathbb{N}_0}$ on V. It measures the asymptotic increase (per unit of time) of the capacity of the range. The goal of this article is to prove existence of the asymptotic capacity of the range of random walks on free products; see Theorem 1.1.

Existence of the asymptotic capacity of the range of symmetric random walks on finitely generated groups was shown in [22] with the help of Kingman's subadditive ergodic theorem. Since we have only a partial composition law on V and therefore no group operation on V, we cannot apply the reasoning from the group case (in particular, Kingman's subadditive ergodic theorem cannot be applied). This was the starting point for the present article to study existence of the asymptotic capacity of the range for general free products of graphs, which form an important class of graphs. Important pre-work has been done in the article [14], which will serve as a base reference for our proofs.

The following little lemma will be very helpful in the proof of Theorem 1.1:

Lemma 2.4. *For any finite sets $A, B \subset V$ with $A \subseteq B$, we have*

$$\mathrm{Cap}(B) - \mathrm{Cap}(A) \leq |B \setminus A|.$$

Proof. Let $A, B \subset V$ be finite sets with $A \subseteq B$. Then $S_B = \infty$ implies $S_A = \infty$, which implies $\mathbb{P}_x\big[S_B = \infty\big] \leq \mathbb{P}_x\big[S_A = \infty\big]$ for $x \in A$. This yields:

$$\begin{aligned} \mathrm{Cap}(B) - \mathrm{Cap}(A) &= \sum_{x\in B} \mathbb{P}_x\big[S_B = \infty\big] - \sum_{x \in A} \mathbb{P}_x\big[S_A = \infty\big] \\ &= \sum_{x \in B\setminus A} \underbrace{\mathbb{P}_x\big[S_B = \infty\big]}_{\leq 1} + \sum_{x\in A} \Big(\underbrace{\mathbb{P}_x\big[S_B = \infty\big] - \mathbb{P}_x\big[S_A = \infty\big]}_{\leq 0}\Big) \leq |B \setminus A|. \end{aligned}$$

□

Finally, we explain briefly why we may exclude the case $|V_1| = |V_2| = 2$ in the following sections:

Remark 2.5. If $V_1 = \{o_1, a\}$ and $V_2 = \{o_2, b\}$, then $V = V_1 * V_2$ becomes the free product $\big(\mathbb{Z}/(2\mathbb{Z})\big) \times \big(\mathbb{Z}/(2\mathbb{Z})\big)$ and the random walk on V is recurrent. This yields $\mathfrak{r} = \lim_{n\to\infty} |\mathbf{R}_n|/n = 0$. Since $\mathrm{Cap}(\mathbf{R}_n) \leq |\mathbf{R}_n|$, we obtain in this case $\mathfrak{c} = 0$.

3 Existence of the asymptotic capacity

In this section we will prove existence of the asymptotic capacity of the range of the random walk $(X_n)_{n\in\mathbb{N}_0}$ on V. For this purpose, we introduce exit times which track the random walk's path to infinity; compare, e.g., with [12, 14]. Some concepts of [14] will be crucial for our proofs, but the reasoning in the current article goes far beyond the scope of [14].

Denote by $X_n^{(k)}$ the projection of X_n onto the first k letters. The *exit times* are defined as follows: set $\mathbf{e}_0 := 0$, and for $k \geq 1$:

$$\begin{aligned}
\mathbf{e}_k &:= \inf\big\{m > 0 \mid \forall n \geq m : X_n^{(k)} \text{ is constant}\big\} \\
&= \inf\big\{m > 0 \mid \|X_m\| = k, \forall n \geq m : X_n \in C(X_m)\big\}.
\end{aligned}$$

That is, the random time $\mathbf{e}_k$ denotes the first instant of time from which onwards the random walk remains in the cone $C(X_{\mathbf{e}_k})$. In [12, Proposition 2.5] it is shown that $\|X_n\| \to \infty$ almost surely as $n \to \infty$, yielding $\mathbf{e}_k < \infty$ almost surely and $\mathbf{e}_{k+1} > \mathbf{e}_k$ for all $k \in \mathbb{N}$. This motivates to consider only those random walk trajectories such that $\mathbf{e}_k < \infty$ for all $k \in \mathbb{N}_0$: denote by Ω_0 the set of all random walk trajectories $\omega = (x_0, x_1, \dots) \in V^{\mathbb{N}_0}$ such that $p(x_i, x_{i+1}) > 0$ for all $i \in \mathbb{N}_0$ and $\lim_{n\to\infty} \|x_n\| = \infty$; the latter property implies $\mathbf{e}_k(\omega) < \infty$ for all $k \in \mathbb{N}_0$, and we have $\mathbb{P}(\Omega_0) = 1$.

Remark 3.1. Note that the exit times used in [14, Section 3.1] were informally (and supposed to be also formally) defined as above, but the formal definition in that article was erroneous. This does not affect the results in [14], since our definition above was used in the proofs of [14].

We collect some useful results: by [12, Proposition 3.2, Theorem 3.3], we have

$$\ell = \lim_{n\to\infty} \frac{\|X_n\|}{n} = \lim_{k\to\infty} \frac{k}{\mathbf{e}_k} \in (0,1] \quad \text{almost surely.} \tag{3.1}$$

For $n \in \mathbb{N}_0$, set

$$\mathbf{k}(n) := \max\{k \in \mathbb{N}_0 \mid \mathbf{e}_k \leq n\},$$

that is, $\mathbf{e}_{\mathbf{k}(n)}$ is the maximal exit time at time n. In [14, Eq. (3.2)] it is shown that

$$\lim_{n\to\infty} \frac{\mathbf{e}_{\mathbf{k}(n)}}{n} = 1 \quad \text{almost surely.} \tag{3.2}$$

If $X_{\mathbf{e}_k} = g_1 \dots g_k$, then we set $\mathbf{W}_k := g_k$.

Since the random walk enters finally the cone $C(X_{\mathbf{e}_1})$ and stays therein, then finally enters $C(X_{\mathbf{e}_2}) \subset C(X_{\mathbf{e}_1})$ and stays therein for forever and so on, the idea is now to construct a partition of $\mathbf{R}_n$ into disjoint subsets of

$$C(X_{\mathbf{e}_k}) \setminus \big(C(X_{\mathbf{e}_{k+1}}) \cup \{X_{\mathbf{e}_k}\}\big), \quad k \leq \mathbf{k}(n),$$

and some remaining parts. This allows us to decompose $\mathrm{Cap}(\mathbf{R}_n)$ according to the partition of $\mathbf{R}_n$ which in turn will enable us to control the asymptotic increase of $\mathrm{Cap}(\mathbf{R}_n)$ as $n \to \infty$.

We start by decomposing $\mathbf{R}_{\mathbf{e}_k}$, $k \in \mathbb{N}$. For this purpose, we define several random sets and quantities in the following. Denote by $C_0 \subset V$ the random set of words which start with a letter in $V_{\delta(X_{\mathbf{e}_1})}$ including o. Let

$$\begin{aligned} \mathcal{R}_0^{(I)} &:= \mathbf{R}_{\mathbf{e}_1} \cap \Big(C_0 \setminus \big(C(X_{\mathbf{e}_1}) \cup \{o\}\big)\Big), \\ \text{and for } k \geq 1: \qquad \mathcal{R}_k^{(I)} &:= \mathbf{R}_{\mathbf{e}_{k+1}} \cap \Big(C(X_{\mathbf{e}_k}) \setminus \big(C(X_{\mathbf{e}_{k+1}}) \cup \{X_{\mathbf{e}_k}\}\big)\Big) \end{aligned}$$

be the “interior part of the range” between two consecutive exit time points. Set

$$\begin{aligned} \mathcal{R}_0 &:= \mathcal{R}_0^{(I)} \cup \big\{o, X_{\mathbf{e}_1}\big\}, \\ \text{and for } k \geq 1: \qquad \mathcal{R}_k &:= \mathcal{R}_k^{(I)} \cup \big\{X_{\mathbf{e}_k}, X_{\mathbf{e}_{k+1}}\big\}. \end{aligned}$$

Observe that we have established the following disjoint decomposition of $\mathbf{R}_{\mathbf{e}_k}$, $k \geq 1$:

$$\mathbf{R}_{\mathbf{e}_k} = \big(\mathbf{R}_{\mathbf{e}_1} \cap \overline{C_0}\big) \ \cup \ \bigcup_{i=0}^{k-1} \big(\mathcal{R}_i \setminus \{X_{\mathbf{e}_{i+1}}\}\big) \ \cup \ \big(\mathbf{R}_{\mathbf{e}_k} \cap C(X_{\mathbf{e}_k})\big). \tag{3.3}$$

We remark that, from time $\mathbf{e}_1$ on, the set $\mathbf{R}_{\mathbf{e}_1} \cap \overline{C_0}$ does not change any more, since every path from $C(X_{\mathbf{e}_1})$ to $\overline{C_0}$ has to leave $C(X_{\mathbf{e}_1})$, but the random walk does not leave $C(X_{\mathbf{e}_1})$ anymore after time $\mathbf{e}_1$.

The next step is to decompose $\mathrm{Cap}(\mathbf{R}_{\mathbf{e}_k})$ according to the decomposition of $\mathbf{R}_{\mathbf{e}_k}$. In the following we will evaluate random variables and random sets at arbitrary $\omega \in \Omega_0$ in order to ensure clarity of notations. Observe that, for $\omega \in \Omega_0$,

$$\mathrm{Cap}\big(\mathbf{R}_{\mathbf{e}_k}(\omega)\big) = \sum_{x \in \mathbf{R}_{\mathbf{e}_k}(\omega)} \mathbb{P}_x\big[S_{\mathbf{R}_{\mathbf{e}_k}(\omega)} = \infty\big]. \tag{3.4}$$

We define for $k \in \mathbb{N}_0$ and $\omega \in \Omega_0$:

$$\mathcal{C}_k^{(I)}(\omega) := \sum_{x \in \mathcal{R}_k^{(I)}(\omega)} \mathbb{P}_x\big[S_{\mathcal{R}_k(\omega)} = \infty\big]$$

and set

$$\begin{aligned} \mathcal{C}_0(\omega) \ &:= \ \mathcal{C}_0^{(I)}(\omega) + \mathbb{P}_o\big[S_{\mathcal{R}_0(\omega)} = \infty, \forall n \geq 1 : X_n \in C_0(\omega)\big] \\ &\qquad + \mathbb{P}_{X_{\mathbf{e}_1}}\big[S_{\mathcal{R}_0(\omega)} = \infty, \forall n \geq 1 : X_n \notin C\big(X_{\mathbf{e}_1}(\omega)\big)\big], \\ \text{for } k \geq 1: \ \mathcal{C}_k(\omega) \ &:= \ \mathcal{C}_k^{(I)}(\omega) + \mathbb{P}_{X_{\mathbf{e}_k}(\omega)}\big[S_{\mathcal{R}_k(\omega)} = \infty, \forall n \geq 1 : X_n \in C\big(X_{\mathbf{e}_k}(\omega)\big)\big] \\ &\qquad + \mathbb{P}_{X_{\mathbf{e}_{k+1}}(\omega)}\big[S_{\mathcal{R}_k(\omega)} = \infty, \forall n \geq 1 : X_n \notin C\big(X_{\mathbf{e}_{k+1}}(\omega)\big)\big]. \end{aligned}$$

Additionally, we define

$$\mathcal{C}_0^*(\omega) := \sum_{x \in \mathbf{R}_{\mathbf{e}_1}(\omega) \cap \overline{C_0(\omega)}} \mathbb{P}_x\big[S_{\mathbf{R}_{\mathbf{e}_1}(\omega)} = \infty\big] + \mathbb{P}_o\big[S_{\mathbf{R}_{\mathbf{e}_1}(\omega)} = \infty, \forall n \geq 1 : X_n \notin C_0(\omega)\big]$$

and

$$\begin{aligned} \mathcal{O}_k(\omega) \ &:= \sum_{x \in \mathbf{R}_{\mathbf{e}_k}(\omega) \cap C\left(X_{\mathbf{e}_k}(\omega)\right) \setminus \{X_{\mathbf{e}_k}(\omega)\}} \mathbb{P}_x\big[S_{\mathbf{R}_{\mathbf{e}_k}(\omega)} = \infty\big] \\ &\qquad + \mathbb{P}_{X_{\mathbf{e}_k}(\omega)}\big[S_{\mathbf{R}_{\mathbf{e}_k}(\omega)} = \infty, \forall n \geq 1 : X_n \in C\big(X_{\mathbf{e}_k}(\omega)\big)\big]. \end{aligned}$$

The decomposition of $\mathbf{R}_{\mathbf{e}_k}$ in (3.3) leads to the following decomposition of $\mathrm{Cap}(\mathbf{R}_{\mathbf{e}_k})$, which will be one of the the main keys for the proofs later:

Proposition 3.2. *For all $k \in \mathbb{N}$,*

$$\mathrm{Cap}(\mathbf{R}_{\mathbf{e}_k}) = \mathcal{C}_0^* + \sum_{i=0}^{k-1} \mathcal{C}_i + \mathcal{O}_k \quad \textit{almost surely.} \tag{3.5}$$

Proof. Let $k \in \mathbb{N}$, and take $\omega \in \Omega_0$ such that $\lim_{n\to\infty} \|X_n(\omega)\| = \infty$. We check that the summands on the right hand sides of (3.5) (evaluated at ω) and (3.4) are the same by using the decomposition of $\mathbf{R}_{\mathbf{e}_k}$ in (3.3). Let $x \in \mathcal{R}_i^{(I)}(\omega)$ for some $i \in \{0, 1, \ldots, k-1\}$, that is, $x \in C\big(X_{\mathbf{e}_i}(\omega)\big) \setminus \{X_{\mathbf{e}_i}(\omega)\}$ but $x \notin C\big(X_{\mathbf{e}_{i+1}}(\omega)\big)$. Then $\mathbb{P}_x\big[S_{\mathbf{R}_{\mathbf{e}_k}(\omega)} = \infty\big]$ is already determined by $\mathcal{R}_i(\omega)$ via

$$\mathbb{P}_x\big[S_{\mathbf{R}_{\mathbf{e}_k}(\omega)} = \infty\big] = \mathbb{P}_x\big[S_{\mathcal{R}_i(\omega)} = \infty\big],$$

since every path from x to $\mathbf{R}_{\mathbf{e}_k}(\omega) \setminus \mathcal{R}_i(\omega)$ has to pass through either $X_{\mathbf{e}_i}(\omega) \in \mathcal{R}_i(\omega)$ or $X_{\mathbf{e}_{i+1}}(\omega) \in \mathcal{R}_i(\omega)$. Therefore, in this case the summand $\mathbb{P}_x\big[S_{\mathbf{R}_{\mathbf{e}_k}(\omega)} = \infty\big]$ is – by definition of $\mathcal{C}_i$ – counted in $\mathcal{C}_i^{(I)}(\omega)$, and thus in $\mathcal{C}_i(\omega)$.

If $x = X_{\mathbf{e}_i}(\omega)$ for some $i \in \{0, 1, \ldots, k-1\}$, then

$$\begin{aligned}
\mathbb{P}_x\big[S_{\mathbf{R}_{\mathbf{e}_k}(\omega)} = \infty\big] &= \mathbb{P}_x\big[S_{\mathbf{R}_{\mathbf{e}_k}(\omega)} = \infty, \forall n \geq 1 : X_n \notin C\big(X_{\mathbf{e}_i}(\omega)\big)\big] \\
&\quad + \mathbb{P}_x\big[S_{\mathbf{R}_{\mathbf{e}_k}(\omega)} = \infty, \forall n \geq 1 : X_n \in C\big(X_{\mathbf{e}_i}(\omega)\big)\big] \\
&= \mathbb{P}_x\big[S_{\mathcal{R}_{i-1}(\omega)} = \infty, \forall n \geq 1 : X_n \notin C\big(X_{\mathbf{e}_i}(\omega)\big)\big] \\
&\quad + \mathbb{P}_x\big[S_{\mathcal{R}_i(\omega)} = \infty, \forall n \geq 1 : X_n \in C\big(X_{\mathbf{e}_i}(\omega)\big)\big],
\end{aligned}$$

since every path from $C\big(X_{\mathbf{e}_i}(\omega)\big) \setminus \{X_{\mathbf{e}_i}(\omega)\}$ to its complement (and vice versa) has to pass through $X_{\mathbf{e}_i}(\omega)$. The first summand on the rightmost hand side is counted in $\mathcal{C}_{i-1}(\omega)$, while the second summand is counted in $\mathcal{C}_i(\omega)$.

Consider now the case $x \in \mathbf{R}_{\mathbf{e}_1}(\omega) \cap \overline{C_0(\omega)}$. Since $\mathbf{R}_{\mathbf{e}_k} \setminus \mathbf{R}_{\mathbf{e}_1} \subset C(X_{\mathbf{e}_1}) \subset C_0$ and every path from $x \in \overline{C_0(\omega)}$ to $\mathbf{R}_{\mathbf{e}_k} \setminus \mathbf{R}_{\mathbf{e}_1}$ has to pass through $X_{\mathbf{e}_1} \in \mathbf{R}_{\mathbf{e}_1}$, we have

$$\mathbb{P}_x\big[S_{\mathbf{R}_{\mathbf{e}_k}(\omega)} = \infty\big] = \mathbb{P}_x\big[S_{\mathbf{R}_{\mathbf{e}_1}(\omega)} = \infty\big].$$

This probability is counted in $\mathcal{C}_0^*(\omega)$. An analogous argument gives in the case $x = o$:

$$\begin{aligned}
&\mathbb{P}_o\big[S_{\mathbf{R}_{\mathbf{e}_k}(\omega)} = \infty, \forall n \geq 1 : X_n \notin C_0(\omega)\big] = \mathbb{P}_o\big[S_{\mathbf{R}_{\mathbf{e}_1}(\omega)} = \infty, \forall n \geq 1 : X_n \notin C_0(\omega)\big] \\
\text{and}\quad &\mathbb{P}_o\big[S_{\mathbf{R}_{\mathbf{e}_k}(\omega)} = \infty, \forall n \geq 1 : X_n \in C_0(\omega)\big] = \mathbb{P}_o\big[S_{\mathcal{R}_0(\omega)} = \infty, \forall n \geq 1 : X_n \in C_0(\omega)\big].
\end{aligned}$$

This implies

$$\begin{aligned}
&\mathbb{P}_o\big[S_{\mathbf{R}_{\mathbf{e}_k}(\omega)} = \infty\big] \\
= \; &\mathbb{P}_o\big[S_{\mathbf{R}_{\mathbf{e}_k}(\omega)} = \infty, \forall n \geq 1 : X_n \notin C_0(\omega)\big] + \mathbb{P}_o\big[S_{\mathbf{R}_{\mathbf{e}_k}(\omega)} = \infty, \forall n \geq 1 : X_n \in C_0(\omega)\big] \\
= \; &\mathbb{P}_o\big[S_{\mathbf{R}_{\mathbf{e}_1}(\omega)} = \infty, \forall n \geq 1 : X_n \notin C_0(\omega)\big] + \mathbb{P}_o\big[S_{\mathcal{R}_0(\omega)} = \infty, \forall n \geq 1 : X_n \in C_0(\omega)\big];
\end{aligned}$$

the first summand on the rightmost hand side is counted in $\mathcal{C}_0^*(\omega)$, while the second one is counted in $\mathcal{C}_0(\omega)$.

If $x \in \mathbf{R}_{\mathbf{e}_k}(\omega) \cap C\big(X_{\mathbf{e}_k}(\omega)\big) \setminus \{X_{\mathbf{e}_k}(\omega)\}$, then $\mathbb{P}_x\big[S_{\mathbf{R}_{\mathbf{e}_k}(\omega)} = \infty\big]$ is counted in $\mathcal{O}_k(\omega)$; in the case $x = X_{\mathbf{e}_k}(\omega)$, we get with an analogous argument as above that

$$\begin{aligned}
&\mathbb{P}_{X_{\mathbf{e}_k}(\omega)}\big[S_{\mathbf{R}_{\mathbf{e}_k}(\omega)} = \infty\big] \\
= \; &\mathbb{P}_{X_{\mathbf{e}_k}(\omega)}\big[S_{\mathbf{R}_{\mathbf{e}_k}(\omega)} = \infty, \forall n \geq 1 : X_n \notin C\big(X_{\mathbf{e}_k}(\omega)\big)\big] \\
&+ \mathbb{P}_{X_{\mathbf{e}_k}(\omega)}\big[S_{\mathbf{R}_{\mathbf{e}_k}(\omega)} = \infty, \forall n \geq 1 : X_n \in C\big(X_{\mathbf{e}_k}(\omega)\big)\big]
\end{aligned}$$

$$= \mathbb{P}_{X_{\mathbf{e}_k}(\omega)}\big[S_{\mathcal{R}_{k-1}(\omega)} = \infty, \forall n \geq 1 : X_n \notin C\big(X_{\mathbf{e}_k}(\omega)\big)\big] \\ +\mathbb{P}_{X_{\mathbf{e}_k}(\omega)}\big[S_{\mathbf{R}_{\mathbf{e}_k}(\omega)} = \infty, \forall n \geq 1 : X_n \in C\big(X_{\mathbf{e}_k}(\omega)\big)\big],$$

that is, the first summand on the rightmost hand side is counted in $\mathcal{C}_{k-1}(\omega)$, while the second one is counted in $\mathcal{O}_k(\omega)$.

Since we have compared all summands in (3.4) and (3.5), we have proven the proposed equality. □

The following two corollaries will be needed in the proof of Proposition 3.6.

Corollary 3.3.

$$\lim_{k\to\infty} \frac{\mathcal{C}_0 + \mathcal{C}_0^*}{k} = 0 \quad \textit{almost surely.}$$

Proof. Since $\mathcal{C}_0+\mathcal{C}_0^* \leq |\mathbf{R}_{\mathbf{e}_1}| \leq \mathbf{e}_1+1 < \infty$ almost surely, the claim follows immediately. □

Corollary 3.4.

$$\lim_{k\to\infty} \frac{\mathcal{O}_k}{k} = 0 \quad \textit{almost surely.}$$

Proof. Define $\mathbf{O}_k := \big|\mathbf{R}_{\mathbf{e}_k} \cap C(X_{\mathbf{e}_k})\big|$. By [14, Corollary 3.13], we have $\mathbf{O}_k/k \to 0$ almost surely as $k \to \infty$; we note that in [14, Corollary 3.13] the common prefix of the elements in $\mathbf{R}_{\mathbf{e}_k} \cap C(X_{\mathbf{e}_k})$ are cancelled which obviously does not affect the cardinality of this set. Since $0 \leq \mathcal{O}_k \leq \mathbf{O}_k$, the claim follows. □

In order to track the range between two consecutive exit times we introduce the following random functions on V: for $k \in \mathbb{N}$, define

$$\psi_k : V \to \{0,1\},\ x \mapsto \begin{cases} 1, & \text{if } \mathbf{W}_1 \dots \mathbf{W}_{k-1} x \in \mathbf{R}_{\mathbf{e}_k}, \\ 0, & \text{otherwise.} \end{cases}$$

That is, ψ_k describes the elements visited by the random walk in $C(X_{\mathbf{e}_{k-1}})$ up to time $\mathbf{e}_k$, where the common first $k-1$ letters are deleted. A main key is the following:

Proposition 3.5. *The stochastic process $(\mathbf{W}_k, \psi_k)_{k\in\mathbb{N}}$ forms an homogeneous, irreducible positive-recurrent Markov chain.*

Proof. See [14, Propositions 3.4 & 3.10]. □

This fact will play a crucial role in the following proposition which will serve as a key ingredient in the proof of Theorem 1.1:

Proposition 3.6. *Assume that $|V_2| \geq 3$. Then there exists a real number $\bar{\mathfrak{c}} > 0$ such that*

$$\lim_{k\to\infty} \frac{\mathrm{Cap}(\mathbf{R}_{\mathbf{e}_k})}{k} = \bar{\mathfrak{c}} \quad \textit{almost surely.}$$

Proof. In view of Proposition 3.2, Corollaries 3.3 and 3.4 it suffices to show existence of a real number $\bar{\mathfrak{c}} > 0$ such that

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k-1} \mathcal{C}_i = \bar{\mathfrak{c}} \quad \text{almost surely.}$$

First, we claim that $\mathcal{C}_i$ can be rewritten as a function in $(\mathbf{W}_i, \psi_i)$ and $(\mathbf{W}_{i+1}, \psi_{i+1})$ for all $i \in \mathbb{N}$: an analogous argument as in the proof of Proposition 3.2 yields for all $\omega \in \Omega_0$ and $x \in \mathcal{R}_k^{(I)}(\omega)$ that

$$\mathbb{P}_x\big[S_{\mathbf{R}_{\mathbf{e}_k}(\omega)} = \infty\big] = \mathbb{P}_x\big[S_{\mathcal{R}_k(\omega)} = \infty\big] \tag{3.6}$$

and

$$\mathbb{P}_{X_{\mathbf{e}_k}(\omega)}\big[S_{\mathbf{R}_{\mathbf{e}_k}(\omega)}=\infty\big] \;=\; \mathbb{P}_{X_{\mathbf{e}_k}(\omega)}\big[S_{\mathcal{R}_{k-1}(\omega)}=\infty, \forall m\geq 1: X_m\notin C\big(X_{\mathbf{e}_k}(\omega)\big)\big] + \mathbb{P}_{X_{\mathbf{e}_k}(\omega)}\big[S_{\mathcal{R}_k(\omega)}=\infty, \forall m\geq 1: X_m\in C\big(X_{\mathbf{e}_k}(\omega)\big)\big]; \tag{3.7}$$

with the help of Lemma 2.3 (remove the first $k-1$ letters of $X_{\mathbf{e}_k}(\omega)$ and in each $v\in\mathcal{R}_{k-1}(\omega)$, $w\in\mathcal{R}_k(\omega)$) it is easy to check that the above probabilities on the right hand sides of (3.6) and (3.7) can be rewritten in terms of $\mathbf{W}_i(\omega),\mathbf{W}_{i+1}(\omega),\psi_i(\omega),\psi_{i+1}(\omega)$, that is, $\mathcal{C}_i$ can be formulated as a function in $\mathbf{W}_i,\mathbf{W}_{i+1},\psi_i,\psi_{i+1}$. Letting π be the equilibrium of the homogeneous, positive-recurrent Markov chain $\big((\mathbf{W}_k,\psi_k),(\mathbf{W}_{k+1},\psi_{k+1})\big)_{k\in\mathbb{N}}$, we can apply the ergodic theorem for positive recurrent Markov chains and obtain

$$\lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k-1}\mathcal{C}_i=\int\mathcal{C}_1\,d\pi=:\bar{\mathfrak{c}},$$

where the integral is well-defined since $\mathcal{C}_1\geq 0$.

We now show that the integral is finite. To this end, assume that $\int\mathcal{C}_1\,d\pi=\infty$. This implies together with (3.1) and (3.2):

$$\frac{1}{n}\sum_{j=1}^{\mathbf{k}(n)}\mathcal{C}_j=\underbrace{\frac{\mathbf{e}_{\mathbf{k}(n)}}{n}}_{\to 1}\underbrace{\frac{\mathbf{k}(n)}{\mathbf{e}_{\mathbf{k}(n)}}}_{\to\ell}\underbrace{\frac{1}{\mathbf{k}(n)}\sum_{j=1}^{\mathbf{k}(n)}\mathcal{C}_j}_{\to\infty}\xrightarrow{n\to\infty}\infty \text{ almost surely.}$$

On the other hand side, we have

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{\mathbf{k}(n)}\mathcal{C}_j\leq\limsup_{n\to\infty}\frac{1}{n}\cdot|\mathbf{R}_n|\leq 1 \text{ almost surely,}$$

which yields a contradiction. Therefore, $\bar{\mathfrak{c}}=\int\mathcal{C}_1\,d\pi<\infty$.

Finally, it remains to show that $\bar{\mathfrak{c}}>0$. For this purpose, take $g_1\in V_1\setminus\{o_1\}$ and $g_2\in V_2\setminus\{o_2\}$ with $p_1(o_1,g_1)>0$ and $p_2(o_2,g_2)>0$. Choose now any $\bar{g}_2\in V_2\setminus\{o_2,g_2\}$ such that $p(g_2,\bar{g}_2)>0$ or $p^{(2)}(g_2,\bar{g}_2)\geq p(g_2,o_2)\cdot p(o_2,\bar{g}_2)>0$; recall that this choice of $\bar{g}_2$ is possible since $|V_2|\geq 3$, due to non-existence of loops and stochasticity of P_2. For any $M\subseteq V$, denote by $\mathbb{1}_M:V\to\{0,1\}$ the indicator function w.r.t. M, that is, $\mathbb{1}_M(x)=1$ for $x\in V$ if and only if $x\in M$. Consider now the event that the random walk's first step goes from o to g_1, followed by a step to g_1g_2 and staying inside the cone $C(g_1g_2)$ afterwards. This event has positive probability to occur, namely $p(o,g_1)\cdot p(g_1,g_1g_2)\cdot(1-\xi_2)>0$. Moreover, this event is a subevent of

$$\mathcal{W}:=\big[(\mathbf{W}_1,\psi_1)=(g_1,\mathbb{1}_{\{o,g_1\}}),(\mathbf{W}_2,\psi_2)=(g_2,\mathbb{1}_{\{o,g_2\}})\big].$$

On the event $\mathcal{W}$, the term

$$\big(p(g_2,\bar{g}_2)+p(g_2,o)\cdot p(o,\bar{g}_2)\big)\cdot(1-\xi_2)>0$$

contributes to $\mathcal{C}_1$, yielding

$$\bar{\mathfrak{c}}=\int\mathcal{C}_1\,d\pi\geq\big(p(g_2,\bar{g}_2)+p(g_2,o)\cdot p(o,\bar{g}_2)\big)\cdot(1-\xi_2)\cdot\pi\big((g_1,\mathbb{1}_{\{o,g_1\}}),(g_2,\mathbb{1}_{\{o,g_2\}})\big)>0.$$

This finishes the proof. □

Recall that the assumption $|V_2|\geq 3$ in the last proposition is just stated for completeness, since the case $|V_1|=|V_2|=2$ was excluded at the beginning of Subsection 2.1. Finally, we can prove:

Proof of Theorem 1.1. By Proposition 3.6 and (3.1),

$$\lim_{k\to\infty} \frac{\mathrm{Cap}(\mathbf{R}_{\mathbf{e}_k})}{\mathbf{e}_k} = \lim_{k\to\infty} \frac{\mathrm{Cap}(\mathbf{R}_{\mathbf{e}_k})}{k} \frac{k}{\mathbf{e}_k} = \bar{\mathfrak{c}} \cdot \ell \quad \text{almost-surely.} \tag{3.8}$$

Furthermore, by Lemma 2.4 and $\mathbf{R}_{\mathbf{e}_{\mathbf{k}(n)}} \subseteq \mathbf{R}_n \subseteq \mathbf{R}_{\mathbf{e}_{\mathbf{k}(n)+1}}$, we have:

$$\begin{aligned} \mathrm{Cap}\big(\mathbf{R}_n\big) - \mathrm{Cap}\big(\mathbf{R}_{\mathbf{e}_{\mathbf{k}(n)}}\big) &\leq \big|\mathbf{R}_n \setminus \mathbf{R}_{\mathbf{e}_{\mathbf{k}(n)}}\big| \leq \big|\mathbf{R}_{\mathbf{e}_{\mathbf{k}(n)+1}} \setminus \mathbf{R}_{\mathbf{e}_{\mathbf{k}(n)}}\big|, & (3.9)\\ \mathrm{Cap}\big(\mathbf{R}_{\mathbf{e}_{\mathbf{k}(n)+1}} - \mathrm{Cap}\big(\mathbf{R}_n\big)\big) &\leq \big|\mathbf{R}_{\mathbf{e}_{\mathbf{k}(n)+1}} \setminus \mathbf{R}_n\big| \leq \big|\mathbf{R}_{\mathbf{e}_{\mathbf{k}(n)+1}} \setminus \mathbf{R}_{\mathbf{e}_{\mathbf{k}(n)}}\big|. & (3.10) \end{aligned}$$

Moreover, we have

$$\begin{aligned} &\frac{\big|\mathbf{R}_{\mathbf{e}_{\mathbf{k}(n)+1}} \setminus \mathbf{R}_{\mathbf{e}_{\mathbf{k}(n)}}\big|}{\mathbf{e}_{\mathbf{k}(n)}} = \frac{\big|\mathbf{R}_{\mathbf{e}_{\mathbf{k}(n)+1}}\big| - \big|\mathbf{R}_{\mathbf{e}_{\mathbf{k}(n)}}\big|}{\mathbf{e}_{\mathbf{k}(n)}} \\ \overset{(2.2),(3.1)}{=} & \underbrace{\frac{\big|\mathbf{R}_{\mathbf{e}_{\mathbf{k}(n)+1}}\big|}{\mathbf{e}_{\mathbf{k}(n)+1}}}_{\to\mathfrak{r}} \underbrace{\frac{\mathbf{e}_{\mathbf{k}(n)+1}}{\mathbf{k}(n)+1}}_{\to 1/\ell} \underbrace{\frac{\mathbf{k}(n)+1}{\mathbf{k}(n)}}_{\to 1} \underbrace{\frac{\mathbf{k}(n)}{\mathbf{e}_{\mathbf{k}(n)}}}_{\to\ell} - \underbrace{\frac{\big|\mathbf{R}_{\mathbf{e}_{\mathbf{k}(n)}}\big|}{\mathbf{e}_{\mathbf{k}(n)}}}_{\to\mathfrak{r}} \xrightarrow{n\to\infty} 0 \quad \text{almost surely.} \end{aligned}$$

Therefore, the bounds in (3.9) and (3.10) yield:

$$\begin{aligned} \limsup_{n\to\infty} \frac{\mathrm{Cap}(\mathbf{R}_n) - \mathrm{Cap}(\mathbf{R}_{\mathbf{e}_{\mathbf{k}(n)}})}{\mathbf{e}_{\mathbf{k}(n)}} &\leq 0 \quad \text{and} \\ \limsup_{n\to\infty} \frac{\mathrm{Cap}(\mathbf{R}_{\mathbf{e}_{\mathbf{k}(n)+1}}) - \mathrm{Cap}(\mathbf{R}_n)}{\mathbf{e}_{\mathbf{k}(n)}} &\leq 0 \quad \text{almost surely.} \end{aligned}$$

Together with (3.8) the above inequalities imply

$$\begin{aligned} \limsup_{n\to\infty} \frac{\mathrm{Cap}(\mathbf{R}_n)}{\mathbf{e}_{\mathbf{k}(n)}} &= \limsup_{n\to\infty} \frac{\mathrm{Cap}(\mathbf{R}_n) - \mathrm{Cap}(\mathbf{R}_{\mathbf{e}_{\mathbf{k}(n)}})}{\mathbf{e}_{\mathbf{k}(n)}} + \underbrace{\frac{\mathrm{Cap}(\mathbf{R}_{\mathbf{e}_{\mathbf{k}(n)}})}{\mathbf{e}_{\mathbf{k}(n)}}}_{\to\bar{\mathfrak{c}}\cdot\ell} \leq \bar{\mathfrak{c}} \cdot \ell, \\ \liminf_{n\to\infty} \frac{\mathrm{Cap}(\mathbf{R}_n)}{\mathbf{e}_{\mathbf{k}(n)}} &= \liminf_{n\to\infty} \frac{\mathrm{Cap}(\mathbf{R}_n) - \mathrm{Cap}(\mathbf{R}_{\mathbf{e}_{\mathbf{k}(n)+1}})}{\mathbf{e}_{\mathbf{k}(n)}} + \underbrace{\frac{\mathrm{Cap}(\mathbf{R}_{\mathbf{e}_{\mathbf{k}(n)+1}})}{\mathbf{e}_{\mathbf{k}(n)}}}_{\to\bar{\mathfrak{c}}\cdot\ell} \geq \bar{\mathfrak{c}} \cdot \ell, \end{aligned}$$

that is, we have shown that

$$\lim_{n\to\infty} \frac{\mathrm{Cap}(\mathbf{R}_n)}{\mathbf{e}_{\mathbf{k}(n)}} = \bar{\mathfrak{c}} \cdot \ell \quad \text{almost surely.}$$

Finally, we obtain the proposed convergence statement with (3.2):

$$\frac{\mathrm{Cap}(\mathbf{R}_n)}{n} = \underbrace{\frac{\mathrm{Cap}(\mathbf{R}_n)}{\mathbf{e}_{\mathbf{k}(n)}}}_{\to\bar{\mathfrak{c}}\cdot\ell} \cdot \underbrace{\frac{\mathbf{e}_{\mathbf{k}(n)}}{n}}_{\to 1} \xrightarrow{n\to\infty} \mathfrak{c} := \bar{\mathfrak{c}} \cdot \ell > 0 \quad \text{almost surely.} \tag{3.11}$$

□

Remark 3.7. The capacity of the range of random walks on $\mathbb{Z}^d$ is linked with hitting probabilities from afar. In the case of random walks on free products we have some different behaviour which we want to discuss briefly.

For simple random walk on $\mathbb{Z}^d$, a well-known formula for the capacity of a finite set $A \subset \mathbb{Z}^d$ is given by

$$\mathrm{Cap}(A) = \lim_{\|x\|_2\to\infty} \frac{\mathbb{P}_x[S_A < \infty]}{G(x, 0|1)},$$

where $G(x,0|1)=\sum_{n\geq 0} p_d^{(n)}(x,0)$ is the Green function w.r.t. simple random walk on $\mathbb{Z}^d$ with $p_d^{(n)}(x,0)$ denoting the n-step transition probabilities of walking from $x\in\mathbb{Z}^d$ to 0.

In the case of random walks on free products we have a similar, but different connection between the hitting probabilities from afar and the probabilities $\mathbb{P}_x[S_A=\infty]$. For any $x\in V$, we obtain by decomposing according to the last visit of any finite set $R\subset V$ (recall transience of our random walks) the following last passage decomposition for the random walk on the free product V:

$$\mathbb{P}_x[S_R<\infty]=\sum_{y\in R} G(x,y|1)\cdot \mathbb{P}_y[S_R=\infty].$$

Dividing this equation by $G(x,o|1)$ gives

$$\frac{\mathbb{P}_x[S_R<\infty]}{G(x,o|1)}=\sum_{y\in R}\frac{G(x,y|1)}{G(x,o|1)}\cdot\mathbb{P}_y[S_R=\infty]\overset{(5.3)}{=}\sum_{y\in R}\frac{F(x,y|1)}{F(x,o|1)}\cdot\frac{G(y,y|1)}{G(o,o|1)}\cdot\mathbb{P}_y[S_R=\infty].$$

The quotients $\frac{F(x,y|1)}{F(x,o|1)}$ can be simplified as in the proof of Proposition 5.6. E.g., if $x=x_1\dots x_m$, $y=y_1\dots y_n\in V$ and if the common prefix of x and y of maximal length is given by $x_1\dots x_k=y_1\dots y_k$, where $k<\min\{m,n\}$, then $\frac{F(x,y|1)}{F(x,o|1)}$ simplifies to

$$\frac{F(x,y|1)}{F(x,o|1)}=\frac{\prod_{i=k+1}^m F(x_i,o|1)\cdot F(x_1\dots x_{k+1},y|1)}{\prod_{i=1}^m F(x_i,o|1)}=\frac{F(x_1\dots x_{k+1},y|1)}{\prod_{i=1}^k F(x_i,o|1)}.$$

Similar simplifications can be performed if $k=m$ or $k=$n. In contrast to $\mathrm{Cap}(R)$, where the summands $\mathbb{P}_y[S_R=\infty]$ have weight 1 for all $y\in R$, these summands now get the weights $\frac{G(x,y|1)}{G(x,o|1)}$, which depend only on the maximal common prefix of x and y, but not on how far away x is located from R.

4 Central limit theorem for the capacity of the range

In this section we will prove the Central Limit Theorem 1.2. To this end, we will follow a similar basic reasoning as in [14, Section 4] with the introduction of regeneration times whose properties we will use in our proofs. However, several non-straightforward, essential extensions to the present setting are required. Throughout this section we assume that there exists $x_0\in V$ and $\kappa\in\mathbb{N}$ such that $p^{(\kappa)}(x_0,x_0)>0$, which excludes degenerate cases; see Remark 4.7. This assumption will be needed in the proofs of Lemma 4.5 and Theorem 1.2. In particular, we still assume $\varrho<1$, which excludes the recurrent case $|V_1|=|V_2|=2$.

For $x\in V$, denote by

$$T_x:=\inf\{m\in\mathbb{N}_0\mid X_m=x\},$$

the stopping time of the first visit to x. Choose and fix now for the rest of this section any $\mathfrak{g}\in V_1^\times$. In the following we stop the Markov chain $(\mathbf{W}_k,\psi_k)_{k\in\mathbb{N}}$ at those intermediate random times k when $X_{\mathbf{e}_k}$ ends with letter $\mathfrak{g}$ (that is, when $\mathbf{W}_k=\mathfrak{g}$) and when the vertex $X_{\mathbf{e}_k}$ is hit for the first time at time $\mathbf{e}_k$ (that is, when $\mathbf{e}_k=T_{X_{\mathbf{e}_k}}$). More formally, define the random times

$$\begin{aligned}\tau_0 &:= \inf\{m\in\mathbb{N}\mid \mathbf{W}_m=\mathfrak{g},\mathbf{e}_m=T_{X_{\mathbf{e}_m}}\},\\ \forall k\geq 1:\ \tau_k &:= \inf\{m>\tau_{k-1}\mid \mathbf{W}_m=\mathfrak{g},\mathbf{e}_m=T_{X_{\mathbf{e}_m}}\};\end{aligned}$$

compare with [14, Section 4]. Recall from Proposition 3.5 that $(\mathbf{W}_k,\psi_k)_{k\in\mathbb{N}_0}$ is positive recurrent; hence, the event $[\mathbf{W}_k=\mathfrak{g}]$ occurs for infinitely many indices k with probability 1. Each time when the random walk $(X_n)_{n\in\mathbb{N}_0}$ visits a word $w\in V$ ending with $\mathfrak{g}$ for the first time, the random walk has strictly positive probability to remain in $C(w)$ for forever (namely with probability $1-\xi_1>0$ which is independent of w; see end of Subsection 2.2). Positive recurrence of $(\mathbf{W}_k,\psi_k)_{k\in\mathbb{N}_0}$ together with a standard geometric argument yields that $\tau_k<\infty$ almost surely for all $k\in\mathbb{N}_0$. For $k\in\mathbb{N}_0$, set

$$\mathbf{T}_k := \mathbf{e}_{\tau_k}$$

and define for $i\in\mathbb{N}$

$$\begin{aligned}\widetilde{\mathcal{C}}_i &:= \sum_{j=\tau_{i-1}}^{\tau_i-1}\mathcal{C}_j,\\ \mathcal{D}_i &:= \widetilde{\mathcal{C}}_i-\mathfrak{c}\cdot(\mathbf{T}_i-\mathbf{T}_{i-1}).\end{aligned}$$

The following proposition will play an important role in the proofs later:

Proposition 4.1. $(\mathbf{T}_i-\mathbf{T}_{i-1})_{i\in\mathbb{N}}$ *is an i.i.d. sequence of random variables. Furthermore,* $\mathbf{T}_0$ *and* $\mathbf{T}_i-\mathbf{T}_{i-1}$*,* $i\in\mathbb{N}$*, have exponential moments.*

Proof. See [14, Prop. 4.2, Lemma 4.3, Prop. 4.5] □

Moreover, we have the following important inequality:

Lemma 4.2. *For all* $\omega\in\Omega_0$*,*

$$\widetilde{\mathcal{C}}_1(\omega)\leq \mathbf{T}_1(\omega)-\mathbf{T}_0(\omega)+1.$$

Proof. Let $\omega\in\Omega_0$. Then:

$$\begin{aligned}\widetilde{\mathcal{C}}_1(\omega) &= \underbrace{\sum_{j=\tau_0}^{\tau_1-1} C_j^{(I)}(\omega)}_{\leq \mathbf{e}_{\tau_1}(\omega)-\mathbf{e}_{\tau_0}(\omega)-\left(\tau_1(\omega)-\tau_0(\omega)\right)} +\mathbb{P}_{X_{\mathbf{e}_j}(\omega)}\big[S_{\mathcal{R}_j(\omega)}=\infty,\forall n\geq 1: X_n\in C\big(X_{\mathbf{e}_j}(\omega)\big)\big]\\
&\qquad +\mathbb{P}_{X_{\mathbf{e}_{j+1}}(\omega)}\big[S_{\mathcal{R}_j(\omega)}=\infty,\forall n\geq 1: X_n\notin C\big(X_{\mathbf{e}_j+1}(\omega)\big)\big]\\
&\leq \mathbf{T}_1(\omega)-\mathbf{T}_0(\omega)-\big(\tau_1(\omega)-\tau_0(\omega)\big)\\
&\qquad +\sum_{j=\tau_0+1}^{\tau_1-1}\underbrace{P_{X_{\mathbf{e}_j}(\omega)}\left[\begin{matrix}S_{\mathcal{R}_j(\omega)}=\infty,\\ \forall n\geq 1: X_n\in C\big(X_{\mathbf{e}_j}(\omega)\big)\end{matrix}\right]+\mathbb{P}_{X_{\mathbf{e}_j}(\omega)}\left[\begin{matrix}S_{\mathcal{R}_j(\omega)}=\infty,\\ \forall n\geq 1: X_n\notin C\big(X_{\mathbf{e}_j}(\omega)\big)\end{matrix}\right]}_{\leq \mathbb{P}_{X_{\mathbf{e}_j}(\omega)}[S_{\mathcal{R}_j(\omega)}=\infty]\leq 1}\\
&\qquad +P_{X_{\mathbf{e}_{\tau_0}}(\omega)}\left[\begin{matrix}S_{\mathcal{R}_{\tau_0}(\omega)}=\infty,\\ \forall n\geq 1: X_n\in C\big(X_{\mathbf{e}_{\tau_0}}(\omega)\big)\end{matrix}\right]+\mathbb{P}_{X_{\mathbf{e}_{\tau_1}}(\omega)}\left[\begin{matrix}S_{\mathcal{R}_{\tau_1}(\omega)}=\infty,\\ \forall n\geq 1: X_n\notin C\big(X_{\mathbf{e}_{\tau_1}}(\omega)\big)\end{matrix}\right]\\
&\leq \mathbf{T}_1(\omega)-\mathbf{T}_0(\omega)-\big(\tau_1(\omega)-\tau_0(\omega)\big)+\big(\tau_1(\omega)-\tau_0(\omega)-1\big)+2=\mathbf{T}_1(\omega)-\mathbf{T}_0(\omega)+1.\end{aligned}$$

□

In the following we collect some basic properties of the sequence $(\mathcal{D}_i)_{i\in\mathbb{N}_0}$.

Lemma 4.3. $\mathrm{Var}(\mathcal{D}_1)<\infty$*.*

Proof. Since $\mathfrak{c}>0$ and $\widetilde{\mathcal{C}}_1\geq 0$ we have

$$-\mathfrak{c}\cdot(\mathbf{T}_1-\mathbf{T}_0)\leq \mathcal{D}_1\leq \widetilde{\mathcal{C}}_1.$$

Lemma 4.2 together with $\mathfrak{c}\leq 1$ yields:

$$\big|\mathcal{D}_1\big|\leq \max\big\{\widetilde{\mathcal{C}}_1,\mathfrak{c}\cdot(\mathbf{T}_1-\mathbf{T}_0)\big\}\leq \mathbf{T}_1-\mathbf{T}_0+1 \quad \text{almost surely.}$$

Since $\mathbf{T}_1-\mathbf{T}_0$ has exponential moments, we get $\mathcal{D}_1\in\mathcal{L}_2$, that is, $\mathrm{Var}(\mathcal{D}_1)<\infty$. □

The reasoning in the proof of the following proposition is analogously to the proof of [14, Proposition 4.5]. Nonetheless, we give a proof since the proposition is another key ingredient in the proof of Theorem 1.2.

Proposition 4.4. $(\mathcal{D}_i)_{i\in\mathbb{N}}$ *is an i.i.d. sequence of random variables.*

Proof. We introduce some notation in order to decompose some set of paths accordingly. Let $i,m\in\mathbb{N}$, $j\in\mathbb{N}_0$ and $z\in\mathbb{R}$. For $x_0\in V$ with $\mathbb{P}[X_{\mathbf{T}_j}=x_0]>0$, denote by $\mathcal{P}^{(1)}_{j,x_0,m}$ the set of all paths $(o,w_1,\dots,w_m=x_0)\in V^{m+1}$ of length m such that

$$\mathbb{P}\Big[X_1=w_1,\dots,X_{m-1}=w_{m-1},X_m=x_0,\mathbf{T}_j=m\Big]>0,$$

that is, each path in $\mathcal{P}^{(1)}_{j,x_0,m}$ allows to generate $\mathbf{T}_j$ with $X_{\mathbf{T}_j}=x_0$ at time m with positive probability. In particular, there exists such a path due to choice of x_0 with $\mathbb{P}[X_{\mathbf{T}_j}=x_0]>0$.

Furthermore, for $x_0\in V$ with $\mathbb{P}[X_{\mathbf{T}_{i-1}}=x_0]>0$, denote by $\mathcal{P}^{(2)}_{i,x_0,n,z}$ the set of paths $(x_0,y_1,\dots,y_n)\in V^{n+1}$ of length $n\in\mathbb{N}$ such that

$$\mathbb{P}\Big[\exists t\in\mathbb{N}: X_t=x_0,X_{t+1}=y_1,\dots,X_{t+n}=y_n,\mathbf{T}_{i-1}=t,\mathbf{T}_i=t+n,\mathcal{D}_i=z\Big]>0,$$

that is, each path in $\mathcal{P}^{(2)}_{i,x_0,n,z}$ allows to generate $X_{\mathbf{T}_{i-1}}=x_0$, $\mathbf{T}_i-\mathbf{T}_{i-1}=n$ and $\mathcal{D}_i=z$. In particular, we have $y_i\in C(x_0)$, that is, we can write $y_i=x_0y_i'$.

First, we show that $(\mathcal{D}_i)_{i\in\mathbb{N}}$ is a sequence of identically distributed random variables. By decomposing all paths until time $\mathbf{T}_i$ into the part until time $\mathbf{T}_{i-1}$ and into the part between the random times $\mathbf{T}_{i-1}$ and $\mathbf{T}_i$ we obtain with Lemma 2.3:

$$\begin{aligned}
\mathbb{P}[\mathcal{D}_i=z] &= \sum_{\substack{x_0\in V:\\ \mathbb{P}[X_{\mathbf{T}_{i-1}}=x_0]>0}} \mathbb{P}\big[X_{\mathbf{T}_{i-1}}=x_0,\mathcal{D}_i=z\big]\\
&= \sum_{\substack{x_0\in V:\\ \mathbb{P}[X_{\mathbf{T}_{i-1}}=x_0]>0}} \sum_{\substack{n\in\mathbb{N},\\ (x_0,x_0y_1',\dots,x_0y_n')\in\mathcal{P}^{(2)}_{i,x_0,n,z}}} \mathbb{P}\left[\substack{X_{\mathbf{T}_{i-1}}=x_0,\\ X_{\mathbf{T}_{i-1}+1}=x_0y_1',\dots,X_{\mathbf{T}_{i-1}+n}=x_0y_n',\\ \forall l\geq 1: X_l\in C(x_0y_n')}\right]\\
&= \sum_{\substack{x_0\in V:\\ \mathbb{P}[X_{\mathbf{T}_{i-1}}=x_0]>0}} \sum_{m\geq 1} \sum_{(o,w_1,\dots,w_m)\in\mathcal{P}^{(1)}_{i-1,x_0,m}} \mathbb{P}\big[X_1=w_1,\dots,X_m=w_m\big]\\
&\quad\cdot\sum_{n\geq 1}\sum_{(x_0,x_0y_1',\dots,x_0y_n')\in\mathcal{P}^{(2)}_{i,x_0,n,z}} \mathbb{P}_{x_0}\big[X_1=x_0y_1',\dots,X_n=x_0y_n'\big]\\
&\quad\cdot\mathbb{P}_{x_0y_n'}\big[\forall l\geq 1: X_l\in C(x_0y_n')\big]\\
&\overset{L.\ 2.3}{=} \sum_{\substack{x_0\in V:\\ \mathbb{P}[X_{\mathbf{T}_{i-1}}=x_0]>0}} \sum_{m\geq 1} \sum_{(o,w_1,\dots,w_m)\in\mathcal{P}^{(1)}_{i-1,x_0,m}} \mathbb{P}\big[X_1=w_1,\dots,X_m=w_m\big]\\
&\quad\cdot\sum_{n\geq 1}\sum_{(x_0,x_0y_1',\dots,x_0y_n')\in\mathcal{P}^{(2)}_{i,x_0,n,z}} \mathbb{P}_{\mathfrak{g}}\big[X_1=\mathfrak{g}y_1,\dots,X_n=\mathfrak{g}y_n\big]\cdot\big(1-\xi_1\big).
\end{aligned}$$

Note that each path $(x_0,x_0y_1',\dots,x_0y_n')\in\mathcal{P}^{(2)}_{i,x_0,n,z}$ lies completely in the cone $C(x_0)$ and that x_0 ends with letter $\mathfrak{g}$. Therefore, there is a natural 1-to-1 correspondence between paths in $\mathcal{P}^{(2)}_{i,x_0,n,z}$ and $\mathcal{P}^{(2)}_{1,\mathfrak{g},n,z}$ established by the vertex-wise shift $C(x_0)\ni x_0g\mapsto\mathfrak{g}g\in C(\mathfrak{g})$, that is, we can map

$$\mathcal{P}^{(2)}_{i,x_0,n,z}\ni(x_0,x_0y_1',\dots,x_0y_n')\mapsto(\mathfrak{g},\mathfrak{g}y_1',\dots,\mathfrak{g}y_n')\in\mathcal{P}^{(2)}_{1,\mathfrak{g},n,z},$$

which is a bijective and measure preserving mapping according to Lemma 2.3. At this point recall once again that x_0 ends with letter $\mathfrak{g}$ by choice of x_0 with $\mathbb{P}[X_{\mathbf{T}_{j-1}} = x_0] > 0$, which ensures that the words $\mathfrak{g}y_i'$ are well-defined. Furthermore, we have

$$\begin{aligned} &\sum_{\substack{x_0\in V:\\ \mathbb{P}[X_{\mathbf{T}_{i-1}}=x_0]>0}} \sum_{\substack{m\geq 1,\\ (o,w_1,\dots,w_m)\in\mathcal{P}^{(1)}_{i-1,x_0,m}}} \mathbb{P}\big[X_1=w_1,\dots,X_m=w_m\big]\cdot \underbrace{(1-\xi_1)}_{=\mathbb{P}_{w_m}\left[\forall n\geq 1: X_n\in C(w_m)\right]} \\ = &\sum_{\substack{x_0\in V:\\ \mathbb{P}[X_{\mathbf{T}_{i-1}}=x_0]>0}} \mathbb{P}\big[X_{\mathbf{T}_{i-1}}=x_0\big] = \mathbb{P}[\mathbf{T}_{i-1}<\infty]=1. \end{aligned}$$

Therefore,

$$\mathbb{P}[\mathcal{D}_i = z] = \sum_{n\geq 1} \sum_{(\mathfrak{g},y_1,\dots,y_n)\in\mathcal{P}^{(2)}_{1,\mathfrak{g},n,z}} \mathbb{P}_{\mathfrak{g}}\big[X_1=y_1,\dots,X_n=y_n\big]. \tag{4.1}$$

Since the probabilities on the right hand side do not depend on i any more, we have proven that the $\mathcal{D}_i$'s all have the same distribution.

For the proof of independence of the sequence $(\mathcal{D}_i)_{i\in\mathbb{N}}$, we only show independence of $\mathcal{D}_1$ and $\mathcal{D}_2$; the general proof follows completely analogously, is however very lengthy and therefore we omit it. Let $d_1, d_2 \in \mathbb{R}$. We make decomposition according to the values of $\mathbf{T}_0, \mathbf{T}_1, \mathbf{T}_2$ and $X_{\mathbf{T}_0}, X_{\mathbf{T}_1}, X_{\mathbf{T}_2}$ and obtain:

$$\begin{aligned} &\mathbb{P}[\mathcal{D}_1=d_1,\mathcal{D}_2=d_2] \\ = &\sum_{\substack{x_0\in V:\\ \mathbb{P}[X_{\mathbf{T}_0}=x_0]>0}} \sum_{m,n_1,n_2\geq 1} \sum_{\substack{(o,w_1,\dots,w_m)\in\mathcal{P}^{(1)}_{0,x_0,m},\\ (x_0,y_1,\dots,y_{n_1})\in\mathcal{P}^{(2)}_{1,x_0,n_1,d_1},\\ (y_{n_1},z_1,\dots,z_{n_2})\in\mathcal{P}^{(2)}_{2,y_{n_1},n_2,d_2}}} \mathbb{P}\left[\begin{array}{c} X_1=w_1,\dots,X_m=w_m,\\ X_{m+1}=y_1,\dots,X_{m+n_1}=y_{n_1},\\ X_{m+n_1+1}=z_1,\dots,\\ X_{m+n_1+n_2}=z_{n_2},\\ \forall l\geq 1: X_l\in C(z_{n_2}) \end{array}\right] \\ = &\sum_{\substack{x_0\in V:\\ \mathbb{P}[X_{\mathbf{T}_0}=x_0]>0}} \sum_{m\geq 1} \sum_{(o,w_1,\dots,w_m)\in\mathcal{P}^{(1)}_{0,x_0,m}} \mathbb{P}\big[X_1=w_1,\dots,X_m=w_m\big] \\ &\cdot \sum_{n_1\geq 1} \sum_{(x_0,y_1,\dots,y_{n_1})\in\mathcal{P}^{(2)}_{1,x_0,n_1,d_1}} \mathbb{P}_{x_0}\big[X_1=y_1,\dots,X_{n_1}=y_{n_1}\big] \\ &\cdot \sum_{n_2\geq 1} \sum_{(y_{n_1},z_1,\dots,z_{n_2})\in\mathcal{P}^{(2)}_{2,y_{n_1},n_2,d_2}} \mathbb{P}_{y_{n_1}}\big[X_1=z_1,\dots,X_{n_2}=z_{n_2}\big] \\ &\cdot \mathbb{P}_{z_{n_2}}\big[\forall l\geq 1: X_l\in C(z_{n_2})\big]. \end{aligned} \tag{4.2}$$

For the last equation we recall the equation $\mathbb{P}_{z_{n_2}}\big[\forall k \geq 1 : X_k \in C(z_{n_2})\big] = 1-\xi_1$. Similarily, decomposing according to the values of $\mathbf{T}_1$ and $X_{\mathbf{T}_1}$, we get

$$\begin{aligned} 1 &= \mathbb{P}[\mathbf{T}_1<\infty] \\ &= \sum_{\substack{z_0\in V:\\ \mathbb{P}[X_{\mathbf{T}_1}=z_0]>0}} \sum_{m_1\geq 1} \sum_{(o,w_1,\dots,w_{m_1})\in\mathcal{P}^{(1)}_{1,z_0,m_1}} \mathbb{P}\left[\begin{array}{c} X_1=w_1,\dots,X_{m_1}=w_{m_1},\\ \forall l>m: X_l\in C(z_0) \end{array}\right] \\ &= \sum_{\substack{z_0\in V:\\ \mathbb{P}[X_{\mathbf{T}_1}=z_0]>0}} \sum_{m_1\geq 1} \sum_{(o,w_1,\dots,w_{m_1})\in\mathcal{P}^{(1)}_{1,z_0,m_1}} \mathbb{P}\big[X_1=w_1,\dots,X_{m_1}=w_{m_1}\big]\cdot(1-\xi_1). \end{aligned} \tag{4.3}$$

Now observe that, for $z_0 \in V$ with $\mathbb{P}[X_{\mathbf{T}_1} = z_0] > 0$, the mapping

$$\mathcal{P}^{(2)}_{2,y_{n_1},n_2,d_2} \ni (y_{n_1}, z_1,\dots,z_{n_2}) \mapsto (z_0, z_0z_1',\dots,z_0z_{n_2}') \in \mathcal{P}^{(2)}_{1,z_0,n_2,d_2},$$

where $z_i := y_{n_1} z_i'$ for $i \in \{1, \dots, n_2\}$, is measure-preserving (see Lemma 2.3), that is,

$$\mathbb{P}_{y_{n_1}}[X_1 = y_{n_1} z_1', \dots, X_{n_2} = y_{n_1} z_{n_2}'] = \mathbb{P}_{z_0}[X_1 = z_0 z_1', \dots, X_{n_2} = z_0 z_{n_2}']. \tag{4.4}$$

Furthermore, we remark that

$$\begin{aligned}
\mathbb{P}[\mathcal{D}_1 = d_1] &= \sum_{\substack{x_0 \in V: \\ \mathbb{P}[X_{\mathbf{T}_0} = x_0] > 0}} \sum_{m \geq 1} \sum_{(o, w_1, \dots, w_m) \in \mathcal{P}^{(1)}_{0, x_0, m}} \mathbb{P}\big[X_1 = w_1, \dots, X_m = w_m\big] \\
&\quad \cdot \sum_{n_1 \geq 1} \sum_{(x_0, y_1, \dots, y_{n_1}) \in \mathcal{P}^{(2)}_{1, x_0, n_1, d_1}} \mathbb{P}_{x_0}\big[X_1 = y_1, \dots, X_{n_1} = y_{n_1}\big] \cdot (1 - \xi_1), \\
\mathbb{P}[\mathcal{D}_2 = d_2] &= \sum_{\substack{z_0 \in V: \\ \mathbb{P}[X_{\mathbf{T}_1} = z_0] > 0}} \sum_{m_1 \geq 1} \sum_{(o, w_1, \dots, w_{m_1}) \in \mathcal{P}^{(1)}_{1, z_0, m_1}} \mathbb{P}\big[X_1 = w_1, \dots, X_{m_1} = w_{m_1}\big] \\
&\quad \cdot \sum_{n_2 \geq 1} \sum_{(z_0, z_1, \dots, z_{n_2}) \in \mathcal{P}^{(2)}_{2, z_0, n_2, d_2}} \mathbb{P}_{z_0}\big[X_1 = z_1, \dots, X_{n_2} = z_{n_2}\big] \cdot (1 - \xi_1).
\end{aligned}$$

The required independence equation follows now from (4.2) with the help of (4.3) and (4.4):

$$\begin{aligned}
& \mathbb{P}[\mathcal{D}_1 = d_1, \mathcal{D}_2 = d_2] \\
= & \sum_{\substack{x_0 \in V: \\ \mathbb{P}[X_{\mathbf{T}_0} = x_0] > 0}} \sum_{m \geq 1} \sum_{(o, w_1, \dots, w_m) \in \mathcal{P}^{(1)}_{0, x_0, m}} \mathbb{P}\big[X_1 = w_1, \dots, X_m = w_m\big] \\
& \cdot \sum_{n_1 \geq 1} \sum_{(x_0, y_1, \dots, y_{n_1}) \in \mathcal{P}^{(2)}_{1, x_0, n_1, d_1}} \mathbb{P}_{x_0}\big[X_1 = y_1, \dots, X_{n_1} = y_{n_1}\big] \\
& \underbrace{\cdot \sum_{\substack{z_0 \in V: \\ \mathbb{P}[X_{\mathbf{T}_1} = z_0] > 0}} \sum_{m_1 \geq 1} \sum_{(o, \bar{w}_1, \dots, \bar{w}_{m_1}) \in \mathcal{P}^{(1)}_{1, z_0, m_1}} \mathbb{P}\big[X_1 = \bar{w}_1, \dots, X_{m_1} = \bar{w}_{m_1}\big] \cdot (1 - \xi_1)}_{=1} \\
& \cdot \sum_{n_2 \geq 1} \sum_{(y_{n_1}, z_1, \dots, z_{n_2}) \in \mathcal{P}^{(2)}_{2, y_{n_1}, n_2, d_2}} \mathbb{P}_{y_{n_1}}\big[X_1 = z_1, \dots, X_{n_2} = z_{n_2}\big] \cdot (1 - \xi_1) \\
\overset{(4.4)}{=} & \Bigg(\sum_{\substack{x_0 \in V: \\ \mathbb{P}[X_{\mathbf{T}_0} = x_0] > 0}} \sum_{m \geq 1} \sum_{(o, w_1, \dots, w_m) \in \mathcal{P}^{(1)}_{0, x_0, m}} \mathbb{P}\big[X_1 = w_1, \dots, X_m = w_m\big] \\
& \cdot \sum_{n_1 \geq 1} \sum_{(x_0, y_1, \dots, y_{n_1}) \in \mathcal{P}^{(2)}_{1, x_0, n_1, d_1}} \mathbb{P}_{x_0}\big[X_1 = y_1, \dots, X_{n_1} = y_{n_1}\big] \cdot (1 - \xi_1) \Bigg) \\
& \cdot \Bigg(\sum_{\substack{z_0 \in V: \\ \mathbb{P}[X_{\mathbf{T}_1} = z_0] > 0}} \sum_{m_1 \geq 1} \sum_{(o, \bar{w}_1, \dots, \bar{w}_{m_1}) \in \mathcal{P}^{(1)}_{1, z_0, m_1}} \mathbb{P}\big[X_1 = \bar{w}_1, \dots, X_{m_1} = \bar{w}_{m_1}\big] \\
& \cdot \sum_{n_2 \geq 1} \sum_{(z_0, z_1, \dots, z_{n_2}) \in \mathcal{P}^{(2)}_{2, z_0, n_2, d_2}} \mathbb{P}_{z_0}\big[X_1 = z_1, \dots, X_{n_2} = z_{n_2}\big] \cdot (1 - \xi_1) \Bigg) \\
= & \ \mathbb{P}[L_1 = d_1] \cdot \mathbb{P}[L_2 = d_2].
\end{aligned}$$

This finishes the proof of the proposition. □

The next step is to consider those times $\mathbf{T}_m$ which occur until time $n \in \mathbb{N}$. For this purpose, define for $n \in \mathbb{N}_0$

$$\mathbf{t}(n) := \sup\big\{m \in \mathbb{N}_0 \mid \mathbf{T}_m \leq n\big\}.$$

In [14, (4.8)] it is shown that

$$\frac{\mathbf{T}_{\mathbf{t}(n)}}{\mathbf{t}(n)} \xrightarrow{n\to\infty} \mathbb{E}\big[\mathbf{T}_1-\mathbf{T}_0\big] \quad \text{almost surely.} \tag{4.5}$$

Lemma 4.5. *We have:*

1. $\mathfrak{c} = \dfrac{\mathbb{E}\big[\widetilde{\mathcal{C}}_1\big]}{\mathbb{E}[\mathbf{T}_1-\mathbf{T}_0]}$ *almost surely.*
2. $\mathbb{E}[\mathcal{D}_1]=0$.
3. *Assume that there are* $\bar{g}_0\in V$ *and* $\kappa\in\mathbb{N}$ *such that* $\mathbb{P}[X_\kappa=\bar{g}_0\mid X_0=\bar{g}_0]>0$. *Then:* $\mathrm{Var}(\mathcal{D}_1)>0$.

Proof.

1. Due to Lemma 4.2 together with existence of exponential moments of $\mathbf{T}_1-\mathbf{T}_0$ (see Proposition 4.1), we have $\mathbb{E}\big[\widetilde{\mathcal{C}}_1\big]<\infty$. Now we obtain, completely analogously to the proof of Proposition 3.6 (replace the exit times $\mathbf{e}_i$ by $\mathbf{T}_i$),

$$\lim_{n\to\infty}\frac{\mathrm{Cap}(\mathbf{R}_{\mathbf{T}_k})}{k}=\mathbb{E}\big[\widetilde{\mathcal{C}}_1\big] \quad \text{almost surely.}$$

   Together with (4.5) we obtain

$$\mathfrak{c}=\lim_{n\to\infty}\frac{\mathrm{Cap}(\mathbf{R}_{\mathbf{T}_{\mathbf{t}(n)}})}{\mathbf{T}_{\mathbf{t}(n)}}=\lim_{n\to\infty}\frac{\mathrm{Cap}(\mathbf{R}_{\mathbf{T}_{\mathbf{t}(n)}})}{\mathbf{t}(n)}\frac{\mathbf{t}(n)}{\mathbf{T}_{\mathbf{t}(n)}}=\frac{\mathbb{E}\big[\widetilde{\mathcal{C}}_1\big]}{\mathbb{E}[\mathbf{T}_1-\mathbf{T}_0]} \quad \text{almost surely.}$$

2. This follows now immediately from (i) with

$$\mathbb{E}\big[\widetilde{\mathcal{C}}_1\big]=\mathfrak{c}\cdot\mathbb{E}[\mathbf{T}_1-\mathbf{T}_0].$$

   and by definition of $\mathcal{D}_1$:

$$\mathbb{E}[\mathcal{D}_1]=\mathbb{E}\big[\widetilde{\mathcal{C}}_1\big]-\mathfrak{c}\cdot\mathbb{E}[\mathbf{T}_1-\mathbf{T}_0]=0.$$

3. It suffices to show that $\mathcal{D}_1=\widetilde{\mathcal{C}}_1-\mathfrak{c}\cdot(\mathbf{T}_1-\mathbf{T}_0)$ is not almost surely constant by constructing two paths of different length but which visit the same vertices between time $\mathbf{T}_0$ and $\mathbf{T}_1$. Assume now for a moment that $\bar{g}_0\in V_2^\times$ and take now any path inside $C(\mathfrak{g})$ from $\mathfrak{g}$ to $\mathfrak{g}\bar{g}_0\mathfrak{g}$ which visits $\mathfrak{g}\bar{g}_0$ twice, say

$$\Pi_1:=(\mathfrak{g},g_1,\dots,g_{j-1},\mathfrak{g}\bar{g}_0,g_{j+1},\dots,g_{j+k-1},\mathfrak{g}\bar{g}_0,g_{j+k+1},\dots,g_{j+k+l-1},\mathfrak{g}\bar{g}_0\mathfrak{g}).$$

   Consider also a second path, where we add another loop at $\mathfrak{g}\bar{g}_0$ as follows:

$$\begin{aligned}\Pi_2 \quad := \quad & (\mathfrak{g},g_1,\dots,g_{j-1},\mathfrak{g}\bar{g}_0,g_{j+1},\dots,g_{j+k-1},\mathfrak{g}\bar{g}_0,\\ & g_{j+1},\dots,g_{j+k-1},\mathfrak{g}\bar{g}_0,g_{j+k+1},\dots,g_{j+k+l-1},\mathfrak{g}\bar{g}_0\mathfrak{g}).\end{aligned}$$

   Both paths visit the same elements of V, but have different lengths. Then a trajectory in $V^{\mathbb{N}_0}$ starting at o, which goes on a shortest path to $\mathfrak{g}$, followed by Π_1 and stays afterwards inside $C(\mathfrak{g}\bar{g}_0\mathfrak{g})$ has strictly positive probability to occur and it leads to $\mathbf{T}_1-\mathbf{T}_0=k+l$ and to some value of $\widetilde{\mathcal{C}}_1=c_1\in\mathbb{R}$. Analogously, a trajectory in $V^{\mathbb{N}_0}$ starting at o, which goes on a shortest path to $\mathfrak{g}$, followed by Π_2 and stays afterwards inside $C(\mathfrak{g}\bar{g}_0\mathfrak{g})$ has also strictly positive probability to occur and it leads to $\mathbf{T}_1-\mathbf{T}_0=2k+l$, but to the same value of $\widetilde{\mathcal{C}}_1=c_1$. Hence, both paths lead to different realizations of $\mathcal{D}_1$ due to $\mathfrak{c}>0$. That is, there are $d_1,d_2\in\mathbb{R}$, $d_1\neq d_2$, such that $\mathbb{P}[\mathcal{D}_1=d_1],\mathbb{P}[\mathcal{D}_1=d_2]>0$, providing $\mathrm{Var}(\mathcal{D}_1)>0$.

The cases $\bar{g}_0\in V_1^\times$ and $\bar{g}_0=o$ can be handled completely analogously, and the case $\bar{g}_0\in V\setminus(V_1^*\cup V_2^*)$ can be easily traced back to the case $\bar{g}_0\in V_1\cup V_2$. ☐

For $k \in \mathbb{N}$, set

$$\mathfrak{S}_k := \sum_{i=1}^{k} \mathcal{D}_i \quad \text{and} \quad \widetilde{\mathfrak{S}}_k := \sum_{i=1}^{k} \widetilde{\mathcal{C}}_i.$$

Proposition 4.6. *We have:*

$$\frac{\mathrm{Cap}(\mathbf{R}_n) - \widetilde{\mathfrak{S}}_{\mathbf{t}(n)}}{\sqrt{n}} \xrightarrow{\mathbb{P}} 0.$$

Proof. Let $\omega \in \Omega_0$. Observe that $\mathbf{R}_{\mathbf{e}_j}(\omega) \subseteq \mathbf{R}_{\mathbf{e}_{\mathbf{t}(n)}}(\omega) \subseteq \mathbf{R}_n(\omega)$ for $j \in \{1, \dots, \mathbf{t}(n)\}$. If $x \in \mathcal{R}_k^{(I)}(\omega)$, $k \in \{1, \dots, \mathbf{e}_{\mathbf{t}(n)-1}\}$, then every path from x to $C\big(X_{\mathbf{e}_{k+1}}(\omega)\big)$ (or vice versa) has to pass through $X_{\mathbf{e}_{k+1}}(\omega)$; therefore, for $k \in \{1, \dots, \mathbf{e}_{\mathbf{t}(n)-1}\}$,

$$\begin{aligned}
\mathbb{P}_x\big[S_{\mathcal{R}_k(\omega)} = \infty\big] &= \mathbb{P}_x\big[S_{\mathbf{R}_n(\omega)} = \infty\big],\\
\mathbb{P}_{X_{\mathbf{e}_k}(\omega)}\left[\begin{array}{c} S_{\mathcal{R}_{k-1}(\omega)} = \infty, \\ \forall n \geq 1 : X_n \notin C\big(X_{\mathbf{e}_k}(\omega)\big) \end{array}\right] &= \mathbb{P}_{X_{\mathbf{e}_k}(\omega)}\left[\begin{array}{c} S_{\mathbf{R}_n(\omega)} = \infty, \\ \forall n \geq 1 : X_n \notin C\big(X_{\mathbf{e}_k}(\omega)\big) \end{array}\right],\\
\mathbb{P}_{X_{\mathbf{e}_k}(\omega)}\left[\begin{array}{c} S_{\mathcal{R}_k(\omega)} = \infty, \\ \forall n \geq 1 : X_n \in C\big(X_{\mathbf{e}_k}(\omega)\big) \end{array}\right] &= \mathbb{P}_{X_{\mathbf{e}_k}(\omega)}\left[\begin{array}{c} S_{\mathbf{R}_n(\omega)} = \infty, \\ \forall n \geq 1 : X_n \in C\big(X_{\mathbf{e}_k}(\omega)\big) \end{array}\right].
\end{aligned}$$

Comparing the summands in $\mathrm{Cap}\big(\mathbf{R}_n(\omega)\big)$ and $\widetilde{\mathfrak{S}}_{\mathbf{t}(n)}(\omega)$ and using the above equations we get for n large enough:

$$\begin{aligned}
\mathrm{Cap}\big(\mathbf{R}_n(\omega)\big) - \widetilde{\mathfrak{S}}_{\mathbf{t}(n)}(\omega) &= \sum_{x \in \mathbf{R}_n(\omega) \cap C\left(X_{\mathbf{e}_{\mathbf{t}(n)}}(\omega)\right) \setminus \{X_{\mathbf{e}_{\mathbf{t}(n)}}(\omega)\}} \mathbb{P}_x\big[S_{\mathbf{R}_n(\omega)} = \infty\big]\\
&\quad + \mathbb{P}_{X_{\mathbf{e}_{\mathbf{t}(n)}}(\omega)}\Big[S_{\mathbf{R}_n(\omega)} = \infty, \forall n \geq 1 : X_n \in C\big(X_{\mathbf{e}_{\mathbf{t}(n)}}(\omega)\big)\Big]\\
&\quad + \sum_{x \in R_{\mathbf{T}_0}(\omega) : x \notin C\left(X_{\mathbf{T}_0}(\omega)\right)} \mathbb{P}_x\big[S_{\mathbf{R}_{\mathbf{T}_0}(\omega)} = \infty\big]\\
&\quad + \mathbb{P}_{X_{\mathbf{T}_0}(\omega)}\Big[S_{\mathbf{R}_{\mathbf{T}_0}(\omega)} = \infty, \forall n \geq 1 : X_n \notin C\big(X_{\mathbf{T}_0}(\omega)\big)\Big]\\
&\leq \mathbf{T}_{\mathbf{t}(n)+1}(\omega) - \mathbf{T}_{\mathbf{t}(n)}(\omega) + \mathbf{T}_0(\omega) + 2.
\end{aligned}$$

In particular, the first equation shows that $\mathrm{Cap}\big(\mathbf{R}_n(\omega)\big) - \widetilde{\mathfrak{S}}_{\mathbf{t}(n)}(\omega) \geq 0$. Recall from Proposition 4.1 that $\mathbf{T}_i - \mathbf{T}_{i-1}$, $i \in \mathbb{N}$, are i.i.d. and that $\mathbf{T}_0$ and $\mathbf{T}_i - \mathbf{T}_{i-1}$ have exponential moments. For any $\varepsilon > 0$ and $n \in \mathbb{N}$, we obtain then:

$$\begin{aligned}
&\mathbb{P}\Big[\mathrm{Cap}(\mathbf{R}_n) - \widetilde{\mathfrak{S}}_{\mathbf{t}(n)} > \varepsilon\sqrt{n}, \mathbf{t}(n) \geq 1\Big]\\
\leq\ &\mathbb{P}\Big[\mathbf{T}_{\mathbf{t}(n)+1} - \mathbf{T}_{\mathbf{t}(n)} + \mathbf{T}_0 + 2 > \varepsilon\sqrt{n}, \mathbf{t}(n) \geq 1\Big]\\
\leq\ &\mathbb{P}\Big[\mathbf{T}_{\mathbf{t}(n)+1} - \mathbf{T}_{\mathbf{t}(n)} + \mathbf{T}_0 > \frac{\varepsilon}{2}\sqrt{n}, \mathbf{t}(n) \geq 1\Big] + \mathbb{P}\Big[2 > \frac{\varepsilon}{2}\sqrt{n}, \mathbf{t}(n) \geq 1\Big]\\
\leq\ &\mathbb{P}\Big[\exists k \in \{1, \dots, n\} : \mathbf{T}_{k+1} - \mathbf{T}_k + \mathbf{T}_0 > \frac{\varepsilon}{2}\sqrt{n}\Big] + \mathbb{P}\Big[4 > \varepsilon\sqrt{n}, \mathbf{t}(n) \geq 1\Big]\\
\leq\ &\mathbb{P}\Big[\exists k \in \{1, \dots, n\} : \mathbf{T}_{k+1} - \mathbf{T}_k > \frac{\varepsilon}{4}\sqrt{n}\Big] + \mathbb{P}\Big[\mathbf{T}_0 > \frac{\varepsilon}{4}\sqrt{n}\Big] + \mathbb{P}\Big[\begin{smallmatrix} 4 > \varepsilon\sqrt{n}, \\ \mathbf{t}(n) \geq 1 \end{smallmatrix}\Big]\\
\overset{\text{Prop. 4.1}}{\leq}\ &n \cdot \mathbb{P}\Big[\mathbf{T}_1 - \mathbf{T}_0 > \frac{\varepsilon}{4}\sqrt{n}\Big] + \mathbb{P}\Big[\mathbf{T}_0 > \frac{\varepsilon}{4}\sqrt{n}\Big] + \mathbb{P}\Big[4 > \varepsilon\sqrt{n}, \mathbf{t}(n) \geq 1\Big]\\
\leq\ &n \cdot \mathbb{P}\Big[(\mathbf{T}_1 - \mathbf{T}_0)^4 > \frac{\varepsilon^4}{4^4} n^2\Big] + \mathbb{P}\Big[\mathbf{T}_0 > \frac{\varepsilon}{4}\sqrt{n}\Big] + \mathbb{P}\Big[4 > \varepsilon\sqrt{n}, \mathbf{t}(n) \geq 1\Big]\\
\leq\ &n \cdot \frac{\mathbb{E}\big[(\mathbf{T}_1 - \mathbf{T}_0)^4\big]}{\frac{\varepsilon^4}{4^4} n^2} + \frac{\mathbb{E}[\mathbf{T}_0]}{\frac{\varepsilon}{4}\sqrt{n}} + \mathbb{P}\Big[4 > \varepsilon\sqrt{n}, \mathbf{t}(n) \geq 1\Big] \xrightarrow{n \to \infty} 0.
\end{aligned}$$

The last inequality applied Markov's Inequality twice. Since $\mathbf{t}(n) \to \infty$ almost surely as $n \to \infty$, we have proven the claim. ☐

Proof of Theorem 1.2. By Billingsley [4, Theorem 14.4], we get the following convergence in distribution:

$$\frac{\mathfrak{S}_{\mathbf{t}(n)}}{\sqrt{\mathrm{Var}(\mathcal{D}_1)}\cdot\sqrt{\mathbf{t}(n)}} \xrightarrow{\mathrm{d}} N(0,1).$$

In [14, p. 398] it is shown that

$$\frac{n}{\mathbf{t}(n)} \xrightarrow{n\to\infty} \mathbb{E}[\mathbf{T}_1-\mathbf{T}_0] \quad \text{almost surely.}$$

An application of the Lemma of Slutsky yields

$$\frac{\mathfrak{S}_{\mathbf{t}(n)}}{\sqrt{n}} = \frac{\mathfrak{S}_{\mathbf{t}(n)}}{\sqrt{\mathrm{Var}(\mathcal{D}_1)}\sqrt{\mathbf{t}(n)}}\frac{\sqrt{\mathbf{t}(n)}}{\sqrt{n}}\sqrt{\mathrm{Var}(\mathcal{D}_1)} \xrightarrow{\mathrm{d}} N(0,\sigma^2), \tag{4.6}$$

where

$$\sigma^2 := \frac{\mathrm{Var}(\mathcal{D}_1)}{\mathbb{E}[\mathbf{T}_1-\mathbf{T}_0]} = \frac{\mathbb{E}\big[\big(\widetilde{\mathcal{C}}_1-\mathfrak{c}\cdot(\mathbf{T}_1-\mathbf{T}_0)\big)^2\big]}{\mathbb{E}[\mathbf{T}_1-\mathbf{T}_0]} > 0. \tag{4.7}$$

The next goal is to show that

$$\frac{\big(\mathrm{Cap}(\mathbf{R}_n)-n\cdot\mathfrak{c}\big)-\mathfrak{S}_{\mathbf{t}(n)}}{\sqrt{n}} \xrightarrow{\mathbb{P}} 0.$$

In order to prove this convergence we note that

$$\mathfrak{S}_{\mathbf{t}(n)} = \widetilde{\mathfrak{S}}_{\mathbf{t}(n)} - \big(\mathbf{T}_{\mathbf{t}(n)}-\mathbf{T}_0\big)\cdot\mathfrak{c}.$$

This equation together with $\mathrm{Cap}(\mathbf{R}_n)-\widetilde{\mathfrak{S}}_{\mathbf{t}(n)}\geq 0$ almost surely (see proof of Proposition 4.6) and $n\geq \mathbf{T}_{\mathbf{t}(n)}-\mathbf{T}_0$ yields:

$$\begin{aligned}
&\mathbb{P}\Big[\Big|\big(\mathrm{Cap}(\mathbf{R}_n)-n\cdot\mathfrak{c}\big)-\mathfrak{S}_{\mathbf{t}(n)}\Big| > \varepsilon\cdot\sqrt{n}, \mathbf{t}(n)\geq 1\Big]\\
\leq\ &\mathbb{P}\left[\begin{matrix}\big|\mathrm{Cap}(\mathbf{R}_n)-\widetilde{\mathfrak{S}}_{\mathbf{t}(n)}\big| > \frac{\varepsilon}{2}\cdot\sqrt{n},\\ \mathbf{t}(n)\geq 1\end{matrix}\right] + \mathbb{P}\left[\begin{matrix}\mathfrak{c}\cdot\big(n-(\mathbf{T}_{\mathbf{t}(n)}-\mathbf{T}_0)\big) > \frac{\varepsilon}{2}\cdot\sqrt{n},\\ \mathbf{t}(n)\geq 1\end{matrix}\right]\\
\leq\ &\underbrace{\mathbb{P}\left[\begin{matrix}\mathrm{Cap}(\mathbf{R}_n)-\widetilde{\mathfrak{S}}_{\mathbf{t}(n)} > \frac{\varepsilon}{2}\cdot\sqrt{n},\\ \mathbf{t}(n)\geq 1\end{matrix}\right]}_{(*)} + \underbrace{\mathbb{P}\left[\begin{matrix}\mathfrak{c}\cdot\big(n-(\mathbf{T}_{\mathbf{t}(n)}-\mathbf{T}_0)\big) > \frac{\varepsilon}{2}\cdot\sqrt{n},\\ \mathbf{t}(n)\geq 1\end{matrix}\right]}_{(**)}.
\end{aligned}$$

By Proposition 4.6, $(*)$ tends to 0 as $n\to\infty$. Furthermore, in [14, p. 398] it is shown that $(**)$ tends also to 0 as $n\to\infty$. Hence, since $\mathbf{t}(n)\to\infty$ almost surely,

$$\mathbb{P}\Big[\Big|\big(\mathrm{Cap}(\mathbf{R}_n)-n\cdot\mathfrak{c}\big)-\mathfrak{S}_{\mathbf{t}(n)}\Big| > \varepsilon\cdot\sqrt{n}\Big] \xrightarrow{n\to\infty} 0. \tag{4.8}$$

Another application of the Lemma of Slutsky together with (4.6) and (4.8) yields the proposed central limit theorem:

$$\frac{\mathrm{Cap}(\mathbf{R}_n)-n\cdot\mathfrak{c}}{\sigma\cdot\sqrt{n}} \xrightarrow{d} N(0,1). \qquad \square$$

Remark 4.7. In this section we have assumed that there are $x_0\in V$ and $\kappa\in\mathbb{N}$ such that $p^{(\kappa)}(x_0,x_0)>0$. We present an example in which this assumption does not hold and where we have $\mathrm{Var}(\mathcal{D}_1)=0$ such that a central limit theorem becomes redundant.

Let $V_1=\{o_1=g_0,g_1,g_2,\dots\}$, $V_2=\{o_2=h_0,h_1,h_2,\dots\}$ be infinite, but countable sets, and set the transition probabilities $p_1(g_n,g_{n+1}):=1$, $p_2(h_n,h_{n+1}):=1$ for $n\in\mathbb{N}_0$.

Set $\alpha := \frac{1}{2}$. It is easy to check that $\mathbf{R}_n(\omega) = n+1$ and $\mathbb{P}_x[S_{\mathbf{R}_n(\omega)} = \infty] = \frac{1}{2}$ for all $\omega \in \Omega_0$ and $x \in \mathbf{R}_n(\omega) \setminus \{X_n(\omega)\}$. This implies $\mathfrak{c} = \frac{1}{2}$.

Furthermore, set $\mathfrak{g} := g_1$. Then:

$$\begin{aligned}
\widetilde{\mathcal{C}}_1(\omega) &= \big(\mathbf{T}_1(\omega) - \mathbf{T}_0(\omega) - 1\big) \cdot \frac{1}{2} \\
&\quad + \mathbb{P}_{X_{\mathbf{T}_0(\omega)}}\left[\substack{S_{\mathcal{R}_{\tau_0}(\omega)} = \infty, \\ \forall n \geq 1 : X_n \in C\left(X_{\mathbf{T}_0(\omega)}\right)}\right] + \mathbb{P}_{X_{\mathbf{T}_1(\omega)}}\left[\substack{S_{\mathcal{R}_{\tau_1}(\omega)} = \infty, \\ \forall n \geq 1 : X_n \notin C\left(X_{\mathbf{T}_1(\omega)}\right)}\right] \\
&= \big(\mathbf{T}_1(\omega) - \mathbf{T}_0(\omega) - 1\big) \cdot \frac{1}{2} + \frac{1}{2} + 0 = \big(\mathbf{T}_1(\omega) - \mathbf{T}_0(\omega)\big) \cdot \frac{1}{2} \\
&= \big(\mathbf{T}_1(\omega) - \mathbf{T}_0(\omega)\big) \cdot \mathfrak{c},
\end{aligned}$$

that is, $\mathcal{D}_1 = 0$ almost surely, implying $\mathrm{Var}(\mathcal{D}_1) = 0$. A central limit theorem is redundant in this example.

5 Analyticity of the asymptotic capacity

In this section we prove that $\mathfrak{c}$ varies real-analytically if the transition matrix of the underlying random walk depends on finitely many parameters only. For this purpose, we fix graphs $\mathcal{X}_1, \mathcal{X}_2$ (arising from any given transition matrices P_1, P_2) from Section 2 and set $E_i := \{(x,y) \in V_i^2 \mid p_i(x,y) > 0\}$ for $i \in \mathcal{I}$, the set of oriented edges of $\mathcal{X}_i$. We assume from now on that each non-negative single-step transition probability of the random walk on V takes one out of finitely many parameters $p_1, \dots, p_d$, $d \in \mathbb{N}$, which take values in $(0,1)$. That is, if $p(x,y) > 0$ for $x, y \in V$ then $p(x,y) = p_j$ for some $j \in \{1, \dots, d\}$.

The idea is now to vary the values of the parameters slightly such that we still obtain a well-defined random walk on V, and to study the behaviour of $\mathfrak{c}$ as a function in $(p_1, \dots, p_d)$. For this purpose, let $\eta : E_1 \cup E_2 \to \{p_1, \dots, p_d\}$ be a mapping. Then a parameter vector $\underline{p} := (p_1, \dots, p_d) \in (0,1)^d$ gives rise to a *well-defined random walk* on V if

$$\begin{aligned}
\forall x \in V_1 : \sum_{y \in V_1 : (x,y) \in E_1} \eta(x,y) + \sum_{z \in V_2 : (o_2, z) \in E_2} \eta(o_2, z) &= 1 \quad \text{and} \\
\forall x \in V_2 : \sum_{y \in V_2 : (x,y) \in E_2} \eta(x,y) + \sum_{z \in V_1 : (o_1, z) \in E_1} \eta(o_1, z) &= 1.
\end{aligned}$$

In other words, the probabilities of the outgoing edges at each $x \in V_1 \cup V_2$ must sum up to 1. Then $\alpha := \sum_{y \in V_1 : (o_1, y) \in E_1} \eta(o_1, y)$, and for $x_1, y_1 \in V_1$, $x_2, y_2 \in V_2$

$$p_1(x_1, y_1) := \begin{cases} \frac{\eta(x_1, y_1)}{\alpha}, & \text{if } (x_1, y_1) \in E_1, \\ 0, & \text{otherwise}, \end{cases} \quad \text{and} \quad p_2(x_2, y_2) := \begin{cases} \frac{\eta(x_2, y_2)}{1-\alpha}, & \text{if } (x_2, y_2) \in E_2, \\ 0, & \text{otherwise}. \end{cases}$$

Denote by

$$\mathcal{P} := \left\{ \underline{p} = (p_1, \dots, p_d) \in (0,1)^d \mid \underline{p} \text{ defines a well-defined random walk on } V \right\}$$

the set of parameter vectors which allow well-defined random walks of constant support induced by $E_1 \cup E_2$. Observe that each random walk defined by any $\underline{p} \in \mathcal{P}$ leads to the same transition graph $\mathcal{X}$. Moreover, we may then regard $\mathfrak{c}$ as a mapping

$$\mathfrak{c} : \mathcal{P} \to [0,1], (p_1, \dots, p_d) \mapsto \mathfrak{c} = \mathfrak{c}(p_1, \dots, p_d).$$

We want to show that $\mathfrak{c}(p_1, \dots, p_d)$ varies real-analytically in $(p_1, \dots, p_d)$, that is, one can extend $\mathfrak{c}(p_1, \dots, p_d)$ as a multivariate power series in $(p_1, \dots, p_d)$ in a neighbourhood of any $\underline{p}_0 \in \mathcal{P}$. To this end, we use the formula from Lemma 4.5.(i) given by

$$\mathfrak{c} = \lim_{n\to\infty} \frac{\mathrm{Cap}(\mathbf{R}_n)}{n} = \frac{\mathbb{E}[\widetilde{\mathcal{C}}_1]}{\mathbb{E}[\mathbf{T}_1 - \mathbf{T}_0]} \quad \text{almost surely.} \tag{5.1}$$

Hence, it suffices to show that both numerator and denominator vary real-analytically in $(p_1,\dots,p_d)$. We already have the following result:

Lemma 5.1. *The mapping*

$$\mathcal{P} \ni (p_1,\dots,p_d) \mapsto \mathbb{E}[\mathbf{T}_1 - \mathbf{T}_0]$$

varies real-analytically in $(p_1,\dots,p_d)$*.*

Proof. See [14, Lemma 5.1]. □

Thus, it remains to show that also $\mathbb{E}[\widetilde{\mathcal{C}}_1]$ varies real-analytically.

Remark 5.2. The idea is to construct a power series in the form $T(z) = \sum_{n\geq 0} a_n z^n$ such that $\mathbb{E}[\widetilde{\mathcal{C}}_1] = T(1)$, and that a_n is a sum of monomials of the form $a(n_1,\dots,n_d)\cdot p_1^{n_1}\cdot\ldots\cdot p_d^{n_d}$ with $n_1,\dots,n_d, a(n_1,\dots,n_d) \in \mathbb{N}_0$, $n_1+\ldots+n_d = n$. If $T(z)$ has radius of convergence bigger than 1, then there exists $\delta > 0$ small enough such that

$$\infty > T(1+\delta) = \sum_{n\geq 0} a_n\cdot(1+\delta)^n = \sum_{n\geq 0} \sum_{\substack{n_1,\dots,n_d\in\mathbb{N}_0:\\ n_1+\ldots+n_d=n}} a(n_1,\dots,n_d)\big(p_1(1+\delta)\big)^{n_1}\cdot\ldots\cdot\big(p_d(1+\delta)\big)^{n_d}.$$

From this follows analyticity of $\mathbb{E}[\widetilde{\mathcal{C}}_1]$ in a neighbourhood of any $(p_1,\dots,p_d) \in \mathcal{P}$, provided existence of $T(z)$.

Before we can show that such a power series $T(z)$ as in Remark 5.2 exists, we have to introduce further generating functions and we will prove essential properties of them in the following subsection.

5.1 Uniform bounds of some generating functions

We introduce further generating functions, summarize some essential properties and prove some important uniform bounds in this subsection. For finite $R \subset V$, $x,y \in V$ and $z \in \mathbb{C}$, set

$$\begin{aligned}
G(x,y|z) &:= \sum_{n\geq 0} \mathbb{P}_x[X_n = y] z^n \quad \text{(Green function)},\\
U(x,y|z) &:= \sum_{n\geq 1} \mathbb{P}_x[S_y = n] z^n,\\
U(x,R|z) &:= \sum_{n\geq 1} \mathbb{P}_x[S_R = n] z^n,\\
\overline{U}(x,R|z) &:= 1 - \sum_{n\geq 1} \mathbb{P}_x[S_R = n] z^n = 1 - U(x,R|z).
\end{aligned}$$

In particular, we have $\overline{U}(x,R|1) = \mathbb{P}_x[S_R = \infty]$. Moreover, define the *first visit generating function* by

$$F(x,y|z) = \sum_{n\geq 0} \mathbb{P}_x[T_y = n]\, z^n, \text{ where } T_y := \inf\{m \in \mathbb{N}_0 \mid X_m = y\},$$

and the *last visit generating function* is defined by

$$L(x,y|z) = \sum_{n\geq 0} \mathbb{P}_x\big[X_n = y, \forall k < n : X_k \neq x\big]\, z^n.$$

If $x \neq y$, then $F(x,y|z) = U(x,y|z)$. Analogously, we denote by F_i and L_i, $i \in \mathcal{I}$, the associated first visit and last visit generating functions of the random walks on V_i governed by P_i. If all paths from $x \in V$ to $y \in V$ have to pass through $w \in V$, then

$$F(x,y|z) = F(x,w|z) \cdot F(w,y|z) \quad \text{and} \quad L(x,y|z) = L(x,w|z) \cdot L(w,y|z); \tag{5.2}$$

these equations are obtained by conditioning with respect to the first/last visit of w, which must be visited before finally walking to y (see, e.g. Woess [31, Prop. 1.43]). Similarily, one can easily show that

$$G(x,y|z) = F(x,y|z) \cdot G(y,y|z) \text{ and } G(x,y|z) = G(x,x|z) \cdot L(x,y|z); \tag{5.3}$$

see, e.g. Woess [31, Thm. 1.38]. Furthermore, there are probability generating functions

$$\xi_i(z) := \sum_{n \geq 1} \mathbb{P}[S_{V_i^\times} = n] z^n, \ i \in \mathcal{I},$$

such that for all $x, y \in V_i$

$$F(x,y|z) = F_i\big(x,y|\xi_i(z)\big) \quad \text{and} \quad L(x,y|z) = L_i\big(x,y|\xi_i(z)\big). \tag{5.4}$$

In particular, $\xi_i(z)$ converges for all $|z| < \mathscr{R}$ and we have $0 < \xi_i(1) < 1$; see [30, Proposition 9.18 (c)], [11, Proposition 2.7] and [12, Lemma 2.3]. Obviously, $\xi_i(1) = \xi_i$; compare with (2.3). Due to the tree-like structure of $\mathcal{X}$, we have further important identities: for each $w \in V$ and $g \in V_1^\times \cup V_2^\times$ with $g \notin V_{\delta(w)}$, we have

$$\begin{aligned} F(wg,w|z) &= F(g,o|z) = F_{\delta(g)}\big(g, o_{\delta(g)}|\xi_{\delta(g)}(z)\big) \text{ and} \\ L(w,wg|z) &= L(o,g|z) = L_{\delta(g)}\big(o_{\delta(g)}, g|\xi_{\delta(g)}(z)\big); \end{aligned} \tag{5.5}$$

these equations follow directly from the fact that $F(wg,w|z)$ and $L(w,wg|z)$ consider only paths inside the cone $C(w)$, the rest follows with Lemma 2.3.

In the following we will derive uniform upper bounds for some important generating functions.

Lemma 5.3. *There exists* $\varrho_0 > 1$ *such that*

$$\sup_{x \in V_1^\times, y \in V_2^\times} L(o,x|\varrho_0) L(o,y|\varrho_0) < 1.$$

Proof. For $z \in \mathbb{C}$ and $i \in \mathcal{I}$, define

$$\mathcal{L}_i^+(z) := \sum_{x \in V_i^\times} L(o,x|z) \overset{(5.4)}{=} \sum_{x \in V_i^\times} L_i\big(o_i, x|\xi_i(z)\big).$$

Since $\xi_i(z)$ is continuous and monotonously increasing for real $z > 0$, $\xi_i(1) < 1$ and $\xi_i(z)$ has radius of convergence of at least $\mathscr{R}$, there exists some $\bar{\varrho}_0 \in (1, \mathscr{R})$ such that $\xi_1(\bar{\varrho}_0) < 1$ and $\xi_2(\bar{\varrho}_0) < 1$, implying

$$\mathcal{L}_i^+(\bar{\varrho}_0) \leq \sum_{x \in V_i^\times} \sum_{n \geq 1} p_i^{(n)}(o_i, x) \cdot \xi_i(\bar{\varrho}_0)^n \leq \sum_{n \geq 1} \xi_i(\bar{\varrho}_0)^n < \frac{1}{1 - \xi_i(\bar{\varrho}_0)} < \infty,$$

where $p_i^{(n)}(x,y)$, $x, y \in V_i$, denotes the n-step transition probabilities of the random walk on V_i governed by P_i. Consequently, there are at most finitely many $x \in V_i^\times$ with $L(o,x|\bar{\varrho}_0) \geq 1$, and we have

$$\sup_{x \in V_1^\times \cup V_2^\times} L(o,x|\bar{\varrho}_0) \leq \max\left\{\frac{1}{1 - \xi_1(\bar{\varrho}_0)}, \frac{1}{1 - \xi_2(\bar{\varrho}_0)}\right\} < \infty.$$

Choose now any $x\in V_1^\times$ and $y\in V_2^\times$ and set $w_k:=xy\dots xy$, where xy is repeated $k\in\mathbb{N}$ times. By decomposition of all paths from o to w_k with respect to the last visit of o and w_k before finally staying in $C(w_k)\setminus\{w_k\}$, we obtain for all $k\in\mathbb{N}$:

$$\begin{aligned}
1 \quad &\geq \quad \mathbb{P}\big[X_\infty \text{ starts with prefix } w_k\big]\\
&= \quad G(o,w_k|1)\cdot \mathbb{P}_{w_k}\big[\forall n\geq 1: X_n\in C(w_k)\setminus\{w_k\}\big]\\
&\overset{(5.3)}{=} \quad G(o,o|1)\cdot L(o,w_k|1)\cdot \mathbb{P}_{w_k}\big[\forall n\geq 1: X_n\in C(w_k)\setminus\{w_k\}\big]\\
&\overset{(5.2),(5.5)}{=} \quad G(o,o|1)\cdot L(o,x|1)^k\cdot L(o,y|1)^k\cdot \underbrace{\mathbb{P}_{w_k}\big[\forall n\geq 1: X_n\in C(w_k)\setminus\{w_k\}\big]}_{\geq \mathbb{P}[X_1\in V_1^\times,\forall n>1:X_n\notin V_1]}\\
&\geq \quad G(o,o|1)\cdot L(o,x|1)^k\cdot L(o,y|1)^k\cdot \alpha\cdot \big(1-\xi_1(1)\big)>0.
\end{aligned}$$

In the last step we have used that $1-\xi_1(1)$ is the probability that $V_1^\times$ is not visited any more when starting at any state in $V_1^\times$. Since k can be chosen arbitrarily large, we must have $L(o,x|1)\cdot L(o,y|1)\leq 1$ for all $x\in V_1^\times$, $y\in V_2^\times$. Assume now for a moment that $L(o,x|1)\cdot L(o,y|1)=1$ would hold for some $x\in V_1^\times$ and $y\in V_2^\times$. If $|V_1|>2$, then there is some $x_0\in V_1^\times\setminus\{x\}$ such that we obtain analogously as above for every $j\in\mathbb{N}$:

$$\begin{aligned}
&\mathbb{P}\big[X_\infty \text{ starts with prefix } w_jx_0\big]\\
\geq\ & G(o,o|1)\cdot L(o,x|1)^j\cdot L(o,y|1)^j\cdot L(o,x_0|1)\cdot(1-\alpha)\cdot\big(1-\xi_2(1)\big)\\
=\ & G(o,o|1)\cdot L(o,x_0|1)\cdot(1-\alpha)\cdot\big(1-\xi_2(1)\big)=:C_{x_0}>0.
\end{aligned}$$

If we choose k large enough, we get a contradiction due to

$$\begin{aligned}
1 \quad &\geq \quad \mathbb{P}\big[X_\infty \text{ starts with a prefix in } \{w_1x_0,\dots,w_kx_o\}\big]\\
&= \quad \sum_{j=1}^k \mathbb{P}\big[X_\infty \text{ starts with prefix } w_jx_0\big]\geq k\cdot C_{x_0}>1.
\end{aligned}$$

Therefore, we must have $L(o,x|1)\cdot L(o,y|1)<1$ for all $x\in V_1^\times$, $y\in V_2^\times$. If $|V_1|=2$, then we must have $|V_2|>2$ and the reasoning follows analogously by exchanging the roles of x and y.

As mentioned above there are at most finitely many $x\in V_1^\times$ with $L(o,x|\bar\varrho_0)\geq 1$. For each such $x\in V_1^\times$ there are also at most finitely many $y\in V_2^\times$ with $L(o,y|\bar\varrho_0)\geq L(o,x|\bar\varrho_0)^{-1}$. Analogously, there are finitely many $y\in V_2^\times$ with $L(o,y|\bar\varrho_0)\geq 1$ and finitely many $x\in V_1^\times$ with $L(o,x|\bar\varrho_0)\geq L(o,y|\bar\varrho_0)^{-1}$.

Since $L(o,x|1)\cdot L(o,y|1)<1$ and by continuity of the involved generating functions, there exists $\varrho_0\in(1,\bar\varrho_0)$ such that

$$L(o,x|\varrho_0)\cdot L(o,y|\varrho_0)<1$$

for all $x\in V_1^\times$ and $y\in V_2^\times$. This finishes the proof. □

From the last lemma follows immediately that there exists $\varrho_0>1$ such that

$$\sup_{x\in V_1\cup V_2} L(o,x|\varrho_0)<\infty. \tag{5.6}$$

We will use this fact in the next lemma.

Lemma 5.4. *There exists $\varrho_0>1$ such that*

$$\sup_{x\in V,y\in C(x)} L(x,y|\varrho_0)<\infty. \tag{5.7}$$

Proof. From Lemma 5.3 follows existence of $\varrho_0>1$ with

$$\sup_{x\in V_1^\times, y\in V_2^\times} L(o,x|\varrho_0)L(o,y|\varrho_0)<1.$$

Now take any $x\in V$ and $y\in C(x)\setminus\{x\}$, that is, we can rewrite y as $y=xy_1\dots y_k$ in the form of (2.1). Then:

$$L(x,y|\varrho_0) \overset{\text{Lemma 2.3}}{=} L(o,y_1\dots y_k|\varrho_0) \overset{(5.2),(5.4)}{=} \prod_{j=1}^{k} L_{\delta(y_i)}\big(o_{\delta(y_i)},y_i|\xi_{\delta(y_i)}(\varrho_0)\big)$$

$$= \prod_{j=1}^{\lfloor k/2\rfloor} \underbrace{\Big(L_{\delta(y_{2i-1})}\big(o_{\delta(y_{2i-1})},y_{2i-1}|\xi_{\delta(y_{2i-1})}(\varrho_0)\big)L_{\delta(y_{2i})}\big(o_{\delta(y_{2i})},y_{2i}|\xi_{\delta(y_{2i})}(\varrho_0)\big)\Big)}_{<1 \text{ (by Lemma 5.3)}}\cdot L_0,$$

where $L_0:=L(o,y_k|\varrho_0)=L_{\delta(y_k)}\big(o_{\delta(y_k)},y_k|\xi_{\delta(y_k)}(\varrho_0)\big)$, if k is odd, and $L_0:=1$, if k is even. In both cases, since $L(w,w|\varrho_0)=1$ for all $w\in V$, (5.6) yields.

$$L(x,y|\varrho_0)\le \sup_{x\in V_1\cup V_2} L(o,x|\varrho_0)<\infty. \qquad \square$$

We show another uniform upper bound for some family of Green functions:

Lemma 5.5. *There exists $\varrho_1\in(1,\mathscr{R})$ such that*

$$\sup\Big\{G(wx,wy|\varrho_1)\,\Big|\, w\in V, x,y\in V_i^\times \text{ with } i\in\mathcal{I}\setminus\{\delta(w)\}\Big\}<\infty. \tag{5.8}$$

Proof. Let $w\in V$ and $x,y\in V_i^\times$, where $i\in\mathcal{I}\setminus\{\delta(w)\}$. For $z\in\mathbb{C}$, define

$$\begin{aligned}\check F(wx,wy|z) &:= \sum_{n\ge 0}\mathbb{P}_{wx}\big[T_{wy}=n,\forall j<n: X_j\neq w\big]\,z^n,\\ \check G(wy,wy|z) &:= \sum_{n\ge 0}\mathbb{P}_{wx}\big[X_n=wy,\forall j<n: X_j\neq w\big]\,z^n.\end{aligned}$$

Then, by distinguishing whether w is visited on a path from wx to wy or not, we obtain

$$\begin{aligned} G(wx,wy|z) &= F(wx,w|z)\cdot G(w,wy|z)+\check F(wx,wy|z)\cdot\check G(wy,wy|z)\\ &\overset{(5.3)}{=} F(wx,w|z)\cdot G(w,w|z)\cdot L(w,wy|z)+\check F(wx,wy|z)\cdot\check G(wy,wy|z)\\ &\overset{(5.5)}{=} F_i\big(x,o_i|\xi_i(z)\big)\cdot G(w,w|z)\cdot L_i\big(o_i,y|\xi_i(z)\big)+\check F(wx,wy|z)\cdot\check G(wy,wy|z). \end{aligned}\tag{5.9}$$

From [14, Lemma 3.6] follows existence of $\bar\varrho_1\in(1,\mathscr{R})$ such that

$$\sup_{v\in V} G(v,v|\bar\varrho_1)<\infty.$$

Choose $\varrho_1\in(1,\bar\varrho_1)$ such that $\xi_j(\varrho_1)<1$ for all $j\in\mathcal{I}$. Then $F_i\big(x,o_i|\xi_i(\varrho_1)\big)\le 1$. Furthermore,

$$L_i\big(o_i,y|\xi_i(\varrho_1)\big)\le\sum_{n\ge1}\xi_i(\varrho_1)^n<\frac{1}{1-\xi_i(\varrho_1)}<\infty.$$

Moreover,

$$\begin{aligned}\check F(wx,wy|\varrho_1) &= \sum_{n\ge0}\mathbb{P}_{wx}\big[T_{wy}=n,\forall j<n:X_j\neq w\big]\,\varrho_1^n,\\ &\overset{\text{Lemma 2.3}}{=} \sum_{n\ge0}\mathbb{P}_{x}\big[T_{y}=n,\forall j<n:X_j\neq o\big]\,\varrho_1^n,\end{aligned}$$

$$\begin{aligned} &\le \sum_{n\geq 0} \mathbb{P}_x\big[T_y=n\big]\, \varrho_1^n = F(x,y|\varrho_1)\\ &= F_i\big(x,y|\xi_i(\varrho_1)\big) \leq \frac{1}{1-\xi_i(\varrho_1)} < \infty. \end{aligned}$$

Since $\check{G}(wy,wy|\varrho_1) \leq G(wy,wy|\varrho_1)$, we have shown – in view of Equation (5.9) – that the values $G(wx,wy|\varrho_1)$ are uniformly bounded. □

Proposition 5.6. *There exists a real number $\varrho > 1$ such that*

$$\sup_{x,y\in V} G(x,y|\varrho) < \infty.$$

Proof. Let $x,y \in V$ and write $x = x_1 \dots x_m$ and $y = y_1 \dots y_n$ in the form of (2.1), where $m,n \in \mathbb{N}_0$. Let $k \in \mathbb{N}_0$ be maximal such that $w := x_1 \dots x_k = y_1 \dots y_k$ (if $x_1 \neq y_1$ then $k = 0$), that is, $x_1 \dots x_k$ is the common prefix of x and y of maximal length. First, we consider the case $k < \min\{m,n\}$. Observe that each path from x to y has to pass through wx_{k+1} and wy_{k+1}.Therefore, we obtain for all real $z \in (0,\mathscr{R})$:

$$\begin{aligned}
&G(x,y|z)\\
\overset{(5.3)}{=}\ &F(x,wx_{k+1}|z)\cdot G(wx_{k+1},y|z)\\
\overset{(5.3)}{=}\ &F(x,wx_{k+1}|z)\cdot G(wx_{k+1},wx_{k+1}|z)\cdot L(wx_{k+1},y|z)\\
\overset{(5.2)}{=}\ &F(x,wx_{k+1}|z)\cdot G(wx_{k+1},wx_{k+1}|z)\cdot L(wx_{k+1},wy_{k+1}|z)\cdot L(wy_{k+1},y|z)\\
\overset{(5.3)}{=}\ &F(x,wx_{k+1}|z)\cdot G(wx_{k+1},wy_{k+1}|z)\cdot L(wy_{k+1},y|z)\\
\overset{(5.2)}{=}\ &\prod_{i=k+2}^{m} F(x_1\dots x_i, x_1\dots x_{i-1}|z)\cdot G(wx_{k+1},wy_{k+1}|z)\cdot L(wy_{k+1},y|z)\\
\overset{(5.5)}{=}\ &\prod_{i=k+2}^{m} F_{\delta(x_i)}\big(x_i,o_{\delta(x_i)}|\xi_{\delta(x_i)}(z)\big)\cdot G(wx_{k+1},wy_{k+1}|z)\cdot L(wy_{k+1},y|z). \qquad (5.10)
\end{aligned}$$

Let $\varrho_0 \in (1,\mathscr{R})$ be a real number satisfying (5.7) and let $\varrho_1 \in (1,\mathscr{R})$ satisfying (5.8). Furthermore, choose any $\bar{\varrho}_1 \in (1,\mathscr{R})$ such that $\sup_{v\in V} G(v,v|\bar{\varrho}_1) < \infty$, which exists due to [14, Lemma 3.6]. Now take any $\varrho \in \big(1,\min\{\varrho_0,\varrho_1,\bar{\varrho}_1\}\big)$ such that $\xi_i(\varrho) < 1$ for all $i \in \mathcal{I}$. Then:

$$\prod_{i=k+2}^{m} F_{\delta(x_i)}\big(x_i,o_{\delta(x_i)}|\xi_{\delta(x_i)}(\varrho)\big) < 1.$$

By Lemma 5.5, $G(wx_{k+1},wy_{k+1}|\varrho)$ is uniformly bounded. Moreover, since $y \in C(wy_{k+1})$, Lemma 5.4 guarantees that $L(wy_{k+1},y|\varrho)$ is also uniformly bounded. In view of (5.10) we have proven the claim in the case $k < \min\{m,n\}$.

If $k = m < n$, we have $x = w$, $y \in C(x)$ and

$$G(x,y|\varrho) \overset{(5.3)}{=} G(x,x|\varrho)\cdot L(x,y|\varrho).$$

The claim follows now directly from $\varrho < \min\{\varrho_0,\bar{\varrho}_1\}$ and Lemma 5.4.

Similarily, in the case $k = n < m$, we have $y = w$ and

$$G(x,y|z) = \prod_{i=k+1}^{m} F_{\delta(x_i)}\big(x_i,o_{\delta(x_i)}|\xi_{\delta(x_i)}(z)\big)\cdot G(y,y|z).$$

The claim follows now directly from $\varrho < \bar{\varrho}_1$ and $\xi_i(\varrho) < 1$ for all $i \in \mathcal{I}$.

Finally, if $k = m = n$, then $x = y$ and the claim follows immediately from the choice of $\varrho < \bar{\varrho}_1$. This finishes the proof. □

Proposition 5.7. *There exists $\varrho > 1$ such that, for all finite $R \subset V$ and for all $x \in V$,*

$$U(x, R|\varrho) < \infty. \tag{5.11}$$

In particular, there exists a finite constant $M_0 > 0$ such that

$$U(x, R|\varrho) \leq |R| \cdot M_0 \quad \textit{for all finite } R \subset V \textit{ and for all } x \in V.$$

Proof. Let $R \subset V$ be finite and $x \in V$. Choose $\varrho > 1$ which satisfies Proposition 5.6. Then:

$$\begin{aligned}
U(x, R|\varrho) &= \sum_{n\geq 1} \mathbb{P}_x[S_R = n]\varrho^n = \sum_{n\geq 1} \mathbb{P}_x\big[X_1, \ldots, X_{n-1} \notin R, X_n \in R\big]\varrho^n \\
&= \sum_{n\geq 1}\sum_{y\in R} \mathbb{P}_x\big[X_1, \ldots, X_{n-1} \notin R, X_n = y\big]\varrho^n \\
&\leq \sum_{n\geq 1}\sum_{y\in R} \mathbb{P}_x\big[X_1, \ldots, X_{n-1} \neq y, X_n = y\big]\varrho^n \\
&= \sum_{y\in R}\sum_{n\geq 1} \mathbb{P}_x[T_y = n]\varrho^n = \sum_{y\in R} U(x, y|\varrho) \leq |R| \cdot \sup_{x_1,x_2\in V} U(x_1, x_2|\varrho).
\end{aligned}$$

Obviously, $U(x_1, x_2|\varrho) \leq G(x_1, x_2|\varrho)$ for all $x_1, x_2 \in V$. From Proposition 5.6 follows now

$$U(x, R|\varrho) \leq |R| \cdot \sup_{x_1,x_2\in V} U(x_1, x_2|\varrho) \leq |R| \cdot \sup_{x_1,x_2\in V} G(x_1, x_2|\varrho) < \infty.$$

Setting $M_0 := \sup_{x_1,x_2\in V} G(x_1, x_2|\varrho) < \infty$ yields the second part of the claim. □

An almost immediate consequence is the following:

Corollary 5.8. *$\overline{U}(x, R|z)$ has radius of convergence strictly bigger than 1.*

Proof. Let $\varrho > 1$ be a suitable real number which satisfies (5.11). Pringsheim's Theorem yields that $U(x, R|z)$ has radius of convergence of at least ϱ. □

5.2 Analyticity of $\mathbb{E}[\widetilde{\mathcal{C}}_1]$

In this subsection we prove the real-analytic behaviour of $\mathbb{E}[\widetilde{\mathcal{C}}_1]$ as a function in $(p_1, \ldots, p_d) \in \mathcal{P}$. Define

$$\widetilde{\mathcal{R}}_1 := \bigcup_{j=\tau_0}^{\tau_1 - 1} \mathcal{R}_j = \mathbf{R}_{\mathbf{T}_1} \cap \Big(C(X_{\mathbf{T}_0}) \setminus \big(C(X_{\mathbf{T}_1}) \setminus \{X_{\mathbf{T}_1}\}\big)\Big), \quad \widetilde{\mathcal{R}}_1^{(I)} := \widetilde{\mathcal{R}}_1 \setminus \{X_{\mathbf{T}_0}, X_{\mathbf{T}_1}\}.$$

Then for all $\omega \in \Omega_0$:

$$\begin{aligned}
\widetilde{\mathcal{C}}_1(\omega) &= \sum_{x\in\widetilde{\mathcal{R}}_1^{(I)}\setminus\{X_{\mathbf{T}_0}, X_{\mathbf{T}_1}\}} \mathbb{P}_x[S_{\widetilde{\mathcal{R}}_1(\omega)} = \infty] + \mathbb{P}_{X_{\mathbf{T}_0}(\omega)}\left[\substack{S_{\widetilde{\mathcal{R}}_1(\omega)}=\infty,\\ \forall n\geq 1: X_n \in C\left(X_{\mathbf{T}_0}(\omega)\right)}\right] \\
&\quad + \mathbb{P}_{X_{\mathbf{T}_1}(\omega)}\left[\substack{S_{\widetilde{\mathcal{R}}_1(\omega)}=\infty,\\ \forall n\geq 1: X_n \notin C\left(X_{\mathbf{T}_1}(\omega)\right)}\right];
\end{aligned} \tag{5.12}$$

compare with the calculations in Section 3. The next step is to construct a "normalized version" of $\widetilde{\mathcal{R}}_1$ and $X_{\mathbf{T}_1}$ by removing the common prefix $X_{\mathbf{T}_0}$: set

$$\widetilde{\mathcal{R}}_{\text{norm}} := \big\{w \in V \,\big|\, X_{\mathbf{T}_0} w \in \widetilde{\mathcal{R}}_1\big\}.$$

Moreover, define

$$V_{\mathfrak{g}} := \big\{v_1 \ldots v_k \in V \mid k \in \mathbb{N}, v_k = \mathfrak{g}\big\},$$

the set of words in V which end up with the letter $\mathfrak{g}$ with no further occurrence of this letter except at the end; these are the possible values of $X_{\mathbf{T}_0}$. Furthermore, define

$$V_{2,\mathfrak{g}} := \{v_1 \dots v_k \in V \mid k \in \mathbb{N}, v_1 \in V_2^\times, v_k = \mathfrak{g}\},$$

the set of words in V which start with a letter in $V_2^\times$ and end up with the letter $\mathfrak{g}$ with no further occurrence of $\mathfrak{g}$ except at the end; then $X_{\mathbf{T}_1}$ has the form $w_1 w_2$, where $w_1 \in V_\mathfrak{g}$ and $w_2 \in V_{2,\mathfrak{g}}$. The increment (a "normalized version" of $X_{\mathbf{T}_1}$) between $X_{\mathbf{T}_0} = w_1$ and $X_{\mathbf{T}_1} = w_1 w_2$ is given by

$$\mathbf{i} := w_2.$$

By definition, $\mathbf{i}$ takes values in $V_{2,\mathfrak{g}}$, while $\widetilde{\mathcal{R}}_{\text{norm}}$ takes almost surely values in a set

$$\mathcal{W} := \{R \subset V \,\big|\, \mathbb{P}[\widetilde{\mathcal{R}}_{\text{norm}} = R] > 0\}.$$

For all $x_0 \in V_\mathfrak{g}$ and $R \in \mathcal{W}$, the set $x_0 R := \{x_0 w | w \in R\}$ is well-defined since all words in $\mathcal{W}$ start with a letter in V_2 while x_0 ends with the letter $\mathfrak{g} \in V_1$.

In a next step we introduce further generating functions which will play a crucial role in our proof. Recall that $\alpha = \sum_{w \in V_1} p_i(o_i, w)$. Define for finite $R \in \mathcal{W}$, $x \in V_{2,\mathfrak{g}}$, and $z \in \mathbb{C}$

$$\begin{aligned}
g(x, R|z) &:= \sum_{m \geq 1} \mathbb{P}\big[\widetilde{\mathcal{R}}_{\text{norm}} = R, \mathbf{i} = x, \mathbf{T}_1 - \mathbf{T}_0 = m\big] z^m, \\
U_0(\mathfrak{g}, \mathfrak{g}R|z) &:= \sum_{n \geq 1} \mathbb{P}_\mathfrak{g}\big[S_{\mathfrak{g}R} = n, \forall m \leq n : X_m \in C(\mathfrak{g})\big]\, z^n, \\
\overline{U}_0(\mathfrak{g}, \mathfrak{g}R|z) &:= (1 - \alpha) \cdot z - U_0(\mathfrak{g}, \mathfrak{g}R|z), \\
U_1(\mathfrak{g}x, \mathfrak{g}R|z) &:= \sum_{n \geq 1} \mathbb{P}_{\mathfrak{g}x}\big[S_{\mathfrak{g}R} = n, \forall m \in \{1, \dots, n-1\} : X_m \notin C(\mathfrak{g}x)\big]\, z^n, \\
\overline{U}_1(\mathfrak{g}x, \mathfrak{g}R|z) &:= \alpha \cdot z - U_1(\mathfrak{g}x, \mathfrak{g}R|z).
\end{aligned}$$

Observe that $g(x, R|1) = \mathbb{P}[\widetilde{\mathcal{R}}_{\text{norm}} = R, \mathbf{i} = x]$ and, for real $z > 0$,

$$g(x, R|z) \leq \sum_{m \geq 1} \mathbb{P}\big[\mathbf{T}_1 - \mathbf{T}_0 = m\big] z^m =: \mathbb{T}(z).$$

Note that the power series $\mathbb{T}(z)$ has radius of convergence strictly bigger than 1 due to existence of exponential moments of $\mathbf{T}_1 - \mathbf{T}_0$; see Proposition 4.1. This together with Proposition 5.7 yields that there exists some $r_1 > 1$ such that, for all finite $R \in \mathcal{W}$, all $w \in V$, $x \in V_{2,\mathfrak{g}}$ and for all real $z \in (0, r_1)$, $U(w, R|z) < \infty$ and $g(x, R|z) < \infty$.

Furthermore, we have:

Lemma 5.9. *For all $R \in \mathcal{W}$ and $x \in V_{2,\mathfrak{g}}$ with $\mathbb{P}\big[\widetilde{\mathcal{R}}_{norm} = R, \mathbf{i} = x\big] > 0$,*

1. $\overline{U}_0(\mathfrak{g}, \mathfrak{g}R|1) = \mathbb{P}_\mathfrak{g}\big[S_{\mathfrak{g}R} = \infty, \forall n \geq 1 : X_n \in C(\mathfrak{g})\big]$.
2. $\overline{U}_1(\mathfrak{g}x, \mathfrak{g}R|1) = \mathbb{P}_{\mathfrak{g}x}\big[S_{\mathfrak{g}R} = \infty, \forall n \geq 1 : X_n \notin C(\mathfrak{g}x)\big]$.

Proof. Let $R \in \mathcal{W}$ and $x \in V_{2,\mathfrak{g}}$ with $\mathbb{P}\big[\widetilde{\mathcal{R}}_{\text{norm}} = R, \mathbf{i} = x\big] > 0$. Then $o, x \in R$ by definition of $\widetilde{\mathcal{R}}_{\text{norm}}$ and $\mathbf{i}$.

1. In the following we consider only trajectories $\omega = (w_0 = \mathfrak{g}, w_1, \dots) \in V^{\mathbb{N}_0}$ starting at $\mathfrak{g}$ with $p(w_i, w_{i+1}) > 0$ for all $i \in \mathbb{N}_0$, that is, we condition on the event $[X_0 = \mathfrak{g}]$. We claim:

$$\begin{aligned}
A_0 &:= \{X_1 \in C(\mathfrak{g})\} \setminus \bigcup_{n \geq 1} \{S_{\mathfrak{g}R} = n, \forall m \leq n : X_m \in C(\mathfrak{g})\} \\
&= \{S_{\mathfrak{g}R} = \infty, \forall n \geq 1 : X_n \in C(\mathfrak{g})\} =: B_0.
\end{aligned}$$

Indeed, if $\omega \in A_0$, then $X_1(\omega) \in C(\mathfrak{g})$ and we either must have $S_{\mathfrak{g}R}(\omega) = \infty$ with $w_i \in C(\mathfrak{g})$ for all $i \in \mathbb{N}_0$ (because otherwise we would have $S_{\mathfrak{g}R}(\omega) < \infty$ since $\mathfrak{g} \in \mathfrak{g}R$) or there exists some $n \in \mathbb{N}$ with $S_{\mathfrak{g}R}(\omega) = n$ and $X_m \notin C(\mathfrak{g})$ for some $m \leq n$; in the latter case $\mathfrak{g} \in \mathfrak{g}R$ must be visited before leaving $C(\mathfrak{g})$ implying $S_{\mathfrak{g}R}(\omega) < n$, a contradiction. Consequently, $\omega \in B_0$. Vice versa, if $\omega \in B_0$, then we have $X_1(\omega) \in C(\mathfrak{g})$ and ω is obviously contained in the set A_0.

The above equation of sets implies:

$$\begin{aligned}
& \mathbb{P}_{\mathfrak{g}}\big[S_{\mathfrak{g}R} = \infty, \forall n \geq 1 : X_n \in C(\mathfrak{g})\big] \\
= \ & \mathbb{P}_{\mathfrak{g}}\big[X_1 \in C(\mathfrak{g})\big] - \sum_{n\geq 1} \mathbb{P}_{\mathfrak{g}}\big[S_{\mathfrak{g}R} = n, \forall m \leq n : X_m \in C(\mathfrak{g})\big] \\
= \ & (1-\alpha) - U_0(\mathfrak{g}, \mathfrak{g}R|1) = \overline{U}_0(\mathfrak{g}, \mathfrak{g}R|1).
\end{aligned}$$

2. The proof works analogously to the first part. In the following we consider only trajectories $\omega = (w_0 = \mathfrak{g}x, w_1, \dots) \in V^{\mathbb{N}_0}$ starting at $\mathfrak{g}x$ with $p(w_i, w_{i+1}) > 0$ for all $i \in \mathbb{N}_0$, that is, we condition on the event $[X_0 = \mathfrak{g}x]$. We claim:

$$\begin{aligned}
A_1 \ &:= \ \big\{X_1 \notin C(\mathfrak{g}x)\big\} \setminus \bigcup_{n\geq 1} \big\{S_{\mathfrak{g}R} = n, \forall m \in \{1, \dots, n-1\} : X_m \notin C(\mathfrak{g}x)\big\} \\
&= \ \big\{S_{\mathfrak{g}R} = \infty, \forall n \geq 1 : X_n \notin C(\mathfrak{g}x)\big\} =: B_1.
\end{aligned}$$

Indeed, if $\omega \in A_1$, then $X_1(\omega) \notin C(\mathfrak{g}x)$. If we would have some $n \in \mathbb{N}$ with $S_{\mathfrak{g}R}(\omega) = n$ and $m \in \{1, \dots, n-1\}$ with $X_m \in C(\mathfrak{g}x)$, then there is some $m' \leq m$ with $X_{m'} = \mathfrak{g}x \in \mathfrak{g}R$, a contradiction to $S_{\mathfrak{g}R}(\omega) = n > m'$. Therefore, we must have $S_{\mathfrak{g}R}(\omega) = \infty$ with $w_i \notin C(\mathfrak{g}x)$ for all $i \in \mathbb{N}$ (because otherwise we would have $S_{\mathfrak{g}R}(\omega) < \infty$ due to $\mathfrak{g}x \in \mathfrak{g}R$)). Hence, $\omega \in B_1$. Vice versa, if $\omega \in B_1$, then we have $X_1(\omega) \notin C(\mathfrak{g}x)$ and ω is obviously contained in the set A_1.

The above equation of set implies:

$$\begin{aligned}
& \mathbb{P}_{\mathfrak{g}x}\big[S_{\mathfrak{g}R} = \infty, \forall n \geq 1 : X_n \notin C(\mathfrak{g}x)\big] \\
= \ & \mathbb{P}_{\mathfrak{g}x}\big[X_1 \notin C(\mathfrak{g}x)\big] - \sum_{n\geq 1} \mathbb{P}_{\mathfrak{g}x}\big[S_{\mathfrak{g}R} = n, \forall m \in \{1, \dots, n-1\} : X_m \notin C(\mathfrak{g}x)\big] \\
= \ & \alpha - U_1(\mathfrak{g}x, \mathfrak{g}R|1) = \overline{U}_1(\mathfrak{g}x, \mathfrak{g}R|1). \qquad \square
\end{aligned}$$

We will rewrite $\mathbb{E}[\widetilde{\mathcal{C}}_1]$ with the help of the above introduced generating functions: for $z \in \mathbb{C}$, set

$$\begin{aligned}
\mathcal{E}^{(I)}(z) \ &:= \ \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} \sum_{x\in R\setminus\{o,x_1\}} \overline{U}(x, R|z) \cdot g(x_1, R|z), \\
\mathcal{E}^{(0)}(z) \ &:= \ \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} \overline{U}_0(\mathfrak{g}, \mathfrak{g}R|z) \cdot g(x_1, R|z), \\
\mathcal{E}^{(1)}(z) \ &:= \ \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} \overline{U}_1(\mathfrak{g}x_1, \mathfrak{g}R|z) \cdot g(x_1, R|z), \\
\mathcal{E}(z) \ &:= \ \mathcal{E}^{(I)}(z) + \mathcal{E}^{(0)}(z) + \mathcal{E}^{(1)}(z).
\end{aligned}$$

The following proposition plays a key role in the proofs later:

Proposition 5.10.

$$\mathbb{E}[\widetilde{\mathcal{C}}_1] = \mathcal{E}(1).$$

Proof. Recall once again that we can shift paths $\big(C(X_{\mathbf{T}_0}), v_1, \dots, v_m\big) \in C(X_{\mathbf{T}_0})^{m+1}$ in a measure-preserving way to paths in $C(\mathfrak{g})$ by substituting the common prefix $X_{\mathbf{T}_0}$ with $\mathfrak{g}$; compare with Lemma 2.3. By (5.12), we can then rewrite $\mathbb{E}[\widetilde{\mathcal{C}}_1]$ in the following way:

$$
\begin{aligned}
&\mathbb{E}[\widetilde{\mathcal{C}}_1]\\
= & \sum_{\substack{R\in\mathcal{W},\\ x_0\in V_{\mathfrak{g}}, x_1\in V_{2,\mathfrak{g}}}} \mathbb{P}\big[\widetilde{\mathcal{R}}_{\mathrm{norm}} = R, X_{\mathbf{T}_0} = x_0, X_{\mathbf{T}_1} = x_0x_1\big]\\
&\cdot\Bigg(\sum_{x\in x_0R\setminus\{x_0,x_0x_1\}} \mathbb{P}_x[S_{x_0R}=\infty] + \mathbb{P}_{x_0}\left[\substack{S_{x_0R}=\infty,\\ \forall n\geq 1: X_n\in C(x_0)}\right] + \mathbb{P}_{x_0x_1}\left[\substack{S_{x_0R}=\infty,\\ \forall n\geq 1: X_n\notin C(x_0x_1)}\right]\Bigg)\\
\overset{\text{L.2.3}}{=} & \sum_{\substack{R\in\mathcal{W},\\ x_0\in V_{\mathfrak{g}}, x_1\in V_{2,\mathfrak{g}}}} \mathbb{P}\big[\widetilde{\mathcal{R}}_{\mathrm{norm}} = R, X_{\mathbf{T}_0} = x_0, X_{\mathbf{T}_1} = x_0x_1\big]\\
&\cdot\Bigg(\sum_{x\in R\setminus\{o,x_1\}} \mathbb{P}_x[S_R=\infty] + \mathbb{P}_{\mathfrak{g}}\left[\substack{S_{\mathfrak{g}R}=\infty,\\ \forall n\geq 1: X_n\in C(\mathfrak{g})}\right] + \mathbb{P}_{\mathfrak{g}x_1}\left[\substack{S_{\mathfrak{g}R}=\infty,\\ \forall n\geq 1: X_n\notin C(\mathfrak{g}x_1)}\right]\Bigg)\\
= & \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} \mathbb{P}\big[\widetilde{\mathcal{R}}_{\mathrm{norm}} = R, \mathbf{i} = x_1\big]\\
&\cdot\Bigg(\sum_{x\in R\setminus\{o,x_1\}} \mathbb{P}_x[S_R=\infty] + \mathbb{P}_{\mathfrak{g}}\left[\substack{S_{\mathfrak{g}R}=\infty,\\ \forall n\geq 1: X_n\in C(\mathfrak{g})}\right] + \mathbb{P}_{\mathfrak{g}x_1}\left[\substack{S_{\mathfrak{g}R}=\infty,\\ \forall n\geq 1: X_n\notin C(\mathfrak{g}x_1)}\right]\Bigg)\\
= & \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} \sum_{m\geq 1} \mathbb{P}\big[\widetilde{\mathcal{R}}_{\mathrm{norm}} = R, \mathbf{i} = x_1, \mathbf{T}_1-\mathbf{T}_0 = m\big]\\
&\cdot\Bigg(\sum_{x\in R\setminus\{o,x_1\}} \overline{U}(x,R|1) + \overline{U}_0(\mathfrak{g},\mathfrak{g}R|1) + \overline{U}_1(\mathfrak{g}x_1,\mathfrak{g}R|1)\Bigg)\\
= & \mathcal{E}^{(I)}(1) + \mathcal{E}^{(0)}(1) + \mathcal{E}^{(1)}(1) = \mathcal{E}(1). \qquad (5.13)
\end{aligned}
$$

□

Set

$$
\begin{aligned}
\mathcal{E}_0^{(I)}(z) &:= \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} \sum_{x\in R\setminus\{o,x_1\}} U(x,R|z)g(x_1,R|z) \quad \text{and}\\
\mathcal{E}_i^{*}(z) &:= \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} (|R|-2)^i g(x_1,R|z) \quad \text{for } i\in\{0,1\}.
\end{aligned}
$$

In the following proofs we will make use of the observation that $\mathbf{T}_1-\mathbf{T}_0 = m\in\mathbb{N}$ implies $|\widetilde{\mathcal{R}}_{\mathrm{norm}}| = |\widetilde{\mathcal{R}}_1| \leq m+1 \leq 2m$. We have:

Lemma 5.11. *The power series $\mathcal{E}_i^{*}(z)$, $i\in\{0,1\}$, have radii of convergence strictly bigger than* 1.

Proof. Let $z\in(0,r_1)$. Then:

$$
\begin{aligned}
\mathcal{E}_1^{*}(z) &= \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} (|R|-2)\cdot g(x_1,R|z) \leq \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} |R|\cdot g(x_1,R|z)\\
&= \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} \sum_{m\geq 1} |R|\cdot \mathbb{P}\left[\substack{\widetilde{\mathcal{R}}_{\mathrm{norm}}=R, \mathbf{i}=x_1,\\ \mathbf{T}_1-\mathbf{T}_0=m}\right]\cdot z^m \leq \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} \sum_{m\geq 1} 2m\cdot \mathbb{P}\left[\substack{\widetilde{\mathcal{R}}_{\mathrm{norm}}=R, \mathbf{i}=x_1,\\ \mathbf{T}_1-\mathbf{T}_0=m}\right]\cdot z^m\\
&= 2\cdot\sum_{m\geq 1} m\cdot\mathbb{P}\big[\mathbf{T}_1-\mathbf{T}_0=m\big]z^m = 2z\cdot\mathbb{T}'(z) < \infty,
\end{aligned}
$$

where finiteness follows from existence of exponential moments of $\mathbf{T}_1 - \mathbf{T}_0$ (see Proposition 4.1). Thus, $\mathcal{E}_1^*(z)$ has radius of convergence strictly bigger than 1. The same holds for $\mathcal{E}_0^*(z)$ since $|\widetilde{\mathcal{R}}_{\mathrm{norm}}| \geq 2$ almost surely and

$$\mathcal{E}_0^*(z) = \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} g(x_1,R|z) \leq \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} |R|\cdot g(x_1,R|z) < \infty. \qquad \square$$

Now we are able to prove:

Proposition 5.12. *$\mathcal{E}^{(I)}(z)$ has radius of convergence strictly bigger than 1.*

Proof. Let $z \in (0, r_1)$. By Proposition 5.7, there is some constant M_0 such that, for all finite $R \subset V$ and every $x \in R$, $U(x,R|z) \leq |R| \cdot M_0$. This yields:

$$\begin{aligned}
\mathcal{E}_0^{(I)}(z) &= \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} \sum_{x\in R\setminus\{o,x_1\}} U(x,R|z) g(x_1,R|z) \\
&\leq \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} \sum_{x\in R\setminus\{o,x_1\}} |R|\cdot M_0 \cdot g(x_1,R|z) \\
&\leq M_0 \cdot \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} |R|^2 \cdot g(x_1,R|z) \\
&\leq M_0 \cdot \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} \sum_{m\geq 1} |R|^2 \cdot \mathbb{P}\left[\substack{\widetilde{\mathcal{R}}_{\mathrm{norm}}=R, \mathbf{i}=x_1,\\ \mathbf{T}_1-\mathbf{T}_0=m}\right] \cdot z^m \\
&\leq 4M_0 \cdot \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} \sum_{m\geq 1} m^2 \cdot \mathbb{P}\left[\substack{\widetilde{\mathcal{R}}_{\mathrm{norm}}=R, \mathbf{i}=x_1,\\ \mathbf{T}_1-\mathbf{T}_0=m}\right] \cdot z^m \\
&= 4M_0 \cdot \sum_{m\geq 1} m^2 \cdot \mathbb{P}\big[\mathbf{T}_1 - \mathbf{T}_0 = m\big] \cdot z^m \\
&= 4M_0 z^2 \cdot \mathbb{T}''(z) + 4M_0 z \cdot \mathbb{T}'(z) < \infty,
\end{aligned}$$

where finiteness follows once again from existence of exponential moments of $\mathbf{T}_1 - \mathbf{T}_0$. Hence, $\mathcal{E}_0^{(I)}(z)$ has radius of convergence strictly bigger than 1. Finally, we get the proposed claim with Lemma 5.11 due the equation

$$\mathcal{E}^{(I)}(z) = \mathcal{E}_1^*(z) - \mathcal{E}_0^{(I)}(z). \qquad \square$$

Furthermore:

Proposition 5.13. *$\mathcal{E}^{(0)}(z)$ and $\mathcal{E}^{(1)}(z)$ have radii of convergence strictly bigger than 1.*

Proof. The proof works analogously to Lemma 5.11. By Proposition 5.7, we have $U_0(x,R|z) \leq U(x,R|z) \leq |R| \cdot M_0$. Let $z \in (0, r_1)$. Then:

$$\begin{aligned}
\mathcal{E}_0^{(0)}(z) &:= \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} U_0(\mathfrak{g}, \mathfrak{g}R|z)\cdot g(x_1,R|z) \leq \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} |R|\cdot M_0 \cdot g(x_1,R|z) \\
&= M_0 \cdot \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} \sum_{m\geq 1} |R| \cdot \mathbb{P}\left[\substack{\widetilde{\mathcal{R}}_{\mathrm{norm}}=R, \mathbf{i}=x_1,\\ \mathbf{T}_1-\mathbf{T}_0=m}\right] \cdot z^m \\
&\leq M_0 \cdot \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} \sum_{m\geq 1} 2m \cdot \mathbb{P}\left[\substack{\widetilde{\mathcal{R}}_{\mathrm{norm}}=R, \mathbf{i}=x_1,\\ \mathbf{T}_1-\mathbf{T}_0=m}\right] \cdot z^m \\
&= 2M_0 \cdot \sum_{m\geq 1} m \cdot \mathbb{P}\big[\mathbf{T}_1 - \mathbf{T}_0 = m\big] z^m = 2M_0 z \cdot \mathbb{T}'(z) < \infty.
\end{aligned}$$

Since $\mathcal{E}^{(0)}(z)=(1-\alpha)z\mathcal{E}_0^*(z)-\mathcal{E}_0^{(0)}(z)$ and by Lemma 5.11, we have shown that $\mathcal{E}^{(0)}(z)$ has radius of convergence strictly bigger than 1.

Moreover, since

$$\mathcal{E}_0^{(1)}(z):=\sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} U_1(\mathfrak{g}x_1,\mathfrak{g}R|z)\cdot g(x_1,R|z)\leq M_0\cdot \sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} |R|\cdot g(x_1,R|z)<\infty,$$

the same calculus as above and Lemma 5.11 show that $\mathcal{E}^{(1)}(z)=\alpha z\mathcal{E}_0^*(z)-\mathcal{E}_0^{(1)}(z)$ has also radius of convergence strictly bigger than 1. □

Corollary 5.14. *$\mathcal{E}(z)$ has radius of convergence strictly bigger than* 1.

Proof. This follows immediately from Propositions 5.12, 5.13 and by definition of $\mathcal{E}(z)$. □

Now we have constructed a power series $\mathcal{E}(z)$ having radius of convergence strictly bigger than 1 and satisfying $\mathcal{E}(1)=\mathbb{E}[\widetilde{\mathcal{C}}_1]$. It remains to show that the coefficients of z^m, $m\in\mathbb{N}$, in $\mathcal{E}(z)$ have the form as requested in Remark 5.2, that is, the coefficients of z^m are sums of monomials of the form $a(n_1,\dots,n_d)\cdot p_1^{n_1}\cdot\ldots\cdot p_d^{n_d}$ with $n_1,\dots,n_d,a(n_1,\dots,n_d)\in\mathbb{N}_0$ and $n_1+\ldots+n_d=m$. Once this is shown, we have proven analyticity of $\mathbb{E}[\widetilde{\mathcal{C}}_1]$ in $(p_1,\dots,p_d)$ according to Remark 5.2.

Proposition 5.15. *For all $R\in\mathcal{W}$, $x\in V_{2,\mathfrak{g}}$ and $m\in\mathbb{N}$, the probability $\mathbb{P}\big[{\widetilde{\mathcal{R}}_{norm}=R,\mathbf{i}=x, \atop \mathbf{T}_1-\mathbf{T}_0=m}\big]$ can be rewritten in the form*

$$\mathbb{P}\Big[{\widetilde{\mathcal{R}}_{norm}=R,\mathbf{i}=x, \atop \mathbf{T}_1-\mathbf{T}_0=m}\Big]=\sum_{\substack{n_1,\dots,n_d\in\mathbb{N}:\\ n_1+\ldots+n_d=m}} a_{R,x,m}(n_1,\dots,n_d)\cdot p_1^{n_1}\cdot\ldots\cdot p_d^{n_d},\quad a_{R,x,m}(n_1,\dots,n_d)\in\mathbb{N}_0.$$

Proof. For $R\in\mathcal{W}$, $x_0\in V_{\mathfrak{g}}$, $x\in V_{2,\mathfrak{g}}$ and $m\in\mathbb{N}$, write $\mathcal{Q}_{x_0,x,R,m}$ for the set of paths $(x_0,w_1,\dots,w_m=x_0x)\in V^{m+1}$ such that

$$\mathbb{P}\big[X_{\mathbf{T}_0}=x_0,X_{\mathbf{T}_0+1}=w_1,\dots,X_{\mathbf{T}_0+m}=w_m,\mathcal{R}_1=x_0R,\mathbf{T}_1-\mathbf{T}_0=m\big]>0.$$

By decomposing according to the values of $\mathbf{T}_0$ and $X_{\mathbf{T}_0}$ we obtain with Lemma 2.3 (by shifting paths inside $C(w),w\in V_{\mathfrak{g}}$ to paths inside $C(\mathfrak{g})$ by replacing the common prefix w with $\mathfrak{g}$):

$$\begin{aligned}
&\mathbb{P}\big[\widetilde{\mathcal{R}}_{\text{norm}}=R,\mathbf{i}=x,\mathbf{T}_1-\mathbf{T}_0=m\big]\\
=\;&\sum_{x_0\in V_{\mathfrak{g}}}\sum_{k\geq 1}\mathbb{P}\big[\widetilde{\mathcal{R}}_1=x_0R,\mathbf{T}_0=k,\mathbf{T}_1-\mathbf{T}_0=m,X_{\mathbf{T}_0}=x_0,X_{\mathbf{T}_1}=x_0x\big]\\
=\;&\sum_{x_0\in V_{\mathfrak{g}}}\sum_{k\geq 1}\sum_{(x_0,w_1,\dots,w_m)\in Q_{x_0,x,R,m}}\mathbb{P}\left[{X_k=x_0,\forall j<k:X_j\notin C(x_0) \atop {X_{k+1}=w_1,\dots,X_{k+m-1}=w_{m-1},X_{k+m}=x_0x, \atop \forall t\geq 1:X_{k+m+t}\in C(x_0x)}}\right]\\
=\;&\sum_{x_0\in V_{\mathfrak{g}}}\sum_{k\geq 1}\mathbb{P}\big[X_k=x_0,\forall j<k:X_j\notin C(x_0)\big]\\
&\quad\cdot\sum_{(x_0,w_1,\dots,w_m)\in Q_{x_0,x,R,m}}\mathbb{P}_{x_0}\big[X_1=w_1,\dots,X_{m-1}=w_{m-1},X_m=x_0x\big]\\
&\quad\cdot\underbrace{\mathbb{P}_{x_0x}\big[\forall t\geq 1:X_t\in C(x_0x)\big]}_{=(1-\xi_1)\text{ (since }\delta(x_0x)=\delta(\mathfrak{g})=1)}\\
\overset{\text{L. 2.3}}{=}\;&\underbrace{\sum_{x_0\in V_{\mathfrak{g}}}\sum_{k\geq 1}\underbrace{\mathbb{P}\big[X_k=x_0,\forall j<k:X_j\notin C(x_0)\big]\cdot(1-\xi_1)}_{=\mathbb{P}[X_{\mathbf{T}_0}=x_0,\mathbf{T}_0=k]}}_{=\mathbb{P}[\mathbf{T}_0<\infty]=1}
\end{aligned}$$

$$\cdot \sum_{(\mathfrak{g},w_1,\ldots,w_m)\in Q_{\mathfrak{g},x,R,m}} \mathbb{P}_{\mathfrak{g}}\big[X_1=w_1,\ldots,X_{m-1}=w_{m-1},X_m=w_m\big]$$
$$= \sum_{(\mathfrak{g},w_1,\ldots,w_m)\in Q_{\mathfrak{g},x,R,m}} \mathbb{P}_{\mathfrak{g}}\big[X_1=w_1,\ldots,X_{m-1}=w_{m-1},X_m=w_m\big].$$

In the latter sum, each summand is obviously a product of factors $p_1,\ldots,p_d$ of degree m. That is, we have shown that $\mathbb{P}\big[\widetilde{\mathcal{R}}_{\mathrm{norm}}=R,\mathbf{i}=x,\mathbf{T}_1-\mathbf{T}_0=m\big]$ can be rewritten as a sum of multivariate monomials in $p_1,\ldots,p_d$ of degree m. □

Lemma 5.16. *The coefficient of z^m, $m\in\mathbb{N}$, in $\mathcal{E}_0^{(I)}(z)$ can be rewritten as a sum of multivariate monomials in $(p_1,\ldots,p_d)$ of degree m.*

Proof. Let $m\in\mathbb{N}$. Then the coefficient of z^m in $\mathcal{E}_0^{(I)}(z)$ is given by

$$\sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} \sum_{x\in R\setminus\{o,x_1\}} \sum_{k=1}^{m-1} \mathbb{P}_x[S_R=k]\cdot\mathbb{P}\Big[\substack{\widetilde{\mathcal{R}}_{\mathrm{norm}}=R,\mathbf{i}=x_1,\\ \mathbf{T}_1-\mathbf{T}_0=m-k}\Big]. \tag{5.14}$$

Observe that, for all $k\in\mathbb{N}$, $\mathbb{P}_x[S_R=k]$ can obviously be written in the form

$$\sum_{n_1,\ldots,n_d\in\mathbb{N}:n_1+\cdots+n_d=k} b(n_1,\ldots,n_d)\cdot p_1^{n_1}\cdot\ldots\cdot p_d^{n_d},$$

where $b(n_1,\ldots,n_d)\in\mathbb{N}_0$. By Proposition 5.15, the same representation as a sum of multivariate monomials of degree $m-k$ holds for the probability $\mathbb{P}_x\Big[\substack{\widetilde{\mathcal{R}}_{\mathrm{norm}}=R,\mathbf{i}=x_1,\\ \mathbf{T}_1-\mathbf{T}_0=m-k}\Big]$ for all $R\in\mathcal{W}$, $x_1\in V_{2,\mathfrak{g}}$ and $k\in\{1,\ldots,m-1\}$. In view of (5.14) the claim is proven. □

Lemma 5.17. *The coefficient of z^m, $m\in\mathbb{N}$, in $\mathcal{E}_i^*(z)$, $i\in\{0,1\}$, can be rewritten as a sum of multivariate monomials in $(p_1,\ldots,p_d)$ of degree m.*

Proof. The coefficient of z^m, $m\in\mathbb{N}$, in $\mathcal{E}_i^*(z)$, $i\in\{0,1\}$, is given by

$$\sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} (|R|-2)^i\cdot\mathbb{P}_x\Big[\widetilde{\mathcal{R}}_{\mathrm{norm}}=R,\mathbf{i}=x_1,\mathbf{T}_1-\mathbf{T}_0=m\Big].$$

The claim follows now immediately from Proposition 5.15. □

Lemma 5.18. *The coefficient of z^m, $m\in\mathbb{N}$, in $\mathcal{E}^{(0)}(z)$ can be rewritten as a sum of multivariate monomials in $(p_1,\ldots,p_d)$ of degree m.*

Proof. For $z\in\mathbb{C}$, set

$$\mathcal{E}_0^{(0)}(z):=\sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} U_0(\mathfrak{g},\mathfrak{g}R|z)\cdot g(x_1,R|z).$$

Then $\mathcal{E}^{(0)}(z)=(1-\alpha)z\mathcal{E}_0^*(z)-\mathcal{E}_0^{(0)}(z)$. By Lemma 5.17, it suffices to show that the coefficients of $\mathcal{E}_0^{(0)}(z)$ have the requested form. The coefficient of z^m, $m\in\mathbb{N}$, in $\mathcal{E}_0^{(0)}(z)$ is given by

$$\sum_{\substack{R\in\mathcal{W},\\ x_1\in V_{2,\mathfrak{g}}}} \sum_{k=1}^{m-1} \mathbb{P}_{\mathfrak{g}}\big[S_{\mathfrak{g}R}=k,\forall m\leq k: X_m\in C(\mathfrak{g})\big]\cdot\mathbb{P}\Big[\substack{\widetilde{\mathcal{R}}_{\mathrm{norm}}=R,\mathbf{i}=x_1,\\ \mathbf{T}_1-\mathbf{T}_0=m-k}\Big]. \tag{5.15}$$

Obviously, for all $k \in \mathbb{N}$, $\mathbb{P}_{\mathfrak{g}}\big[S_{\mathfrak{g}R} = k, \forall m \leq k : X_m \in C(\mathfrak{g})\big]$ can be written in the form

$$\sum_{n_1,\dots,n_d \in \mathbb{N}: n_1+\dots+n_d=k} b(n_1,\dots,n_d) \cdot p_1^{n_1} \cdot \ldots \cdot p_d^{n_d}, \quad b(n_1,\dots,n_d) \in \mathbb{N}_0.$$

The claim follows now directly with Proposition 5.15 and the fact that $1-\alpha$ is a sum of some values out of $p_1,\dots,p_d$. ☐

Lemma 5.19. *The coefficient of z^m, $m \in \mathbb{N}$, in $\mathcal{E}^{(1)}(z)$ can be rewritten as a sum of multivariate monomials in $(p_1,\dots,p_d)$ of degree m.*

Proof. For $z \in \mathbb{C}$, set

$$\mathcal{E}_0^{(1)}(z) := \sum_{\substack{R \in \mathcal{W},\\ x_1 \in V_{2,\mathfrak{g}}}} U_1(\mathfrak{g}x_1, \mathfrak{g}R|z) \cdot g(x_1, R|z).$$

Then $\mathcal{E}^{(1)}(z) = \alpha z \mathcal{E}_0^*(z) - \mathcal{E}_0^{(1)}(z)$. By Lemma 5.17, it suffices to show that the coefficients of $\mathcal{E}_0^{(1)}(z)$ have the requested form. The coefficient of z^m, $m \in \mathbb{N}$, in $\mathcal{E}_0^{(1)}(z)$ is given by

$$\sum_{\substack{R \in \mathcal{W},\\ x_1 \in V_{2,\mathfrak{g}}}} \sum_{k=1}^{m-1} \mathbb{P}_{\mathfrak{g}x_1}\big[S_{\mathfrak{g}R} = k, \forall m \in \{1,\dots,k-1\} : X_m \notin C(\mathfrak{g}x_1)\big] \cdot \mathbb{P}\left[\substack{\widetilde{\mathcal{R}}_{\mathrm{norm}}=R, \mathbf{i}=x_1,\\ \mathbf{T}_1-\mathbf{T}_0=m-k}\right]. \quad (5.16)$$

Obviously, for all $k \in \mathbb{N}$, $\mathbb{P}_{\mathfrak{g}x_1}\big[S_{\mathfrak{g}R} = k, \forall m \in \{1,\dots,k-1\} : X_k \notin C(\mathfrak{g}x_1)\big]$ can be written in the form

$$\sum_{n_1,\dots,n_d \in \mathbb{N}: n_1+\dots+n_d=k} b(n_1,\dots,n_d) \cdot p_1^{n_1} \cdot \ldots \cdot p_d^{n_d}, \quad b(n_1,\dots,n_d) \in \mathbb{N}_0.$$

The claim follows now directly with Proposition 5.15 and the fact that α is a sum of some values out of $p_1,\dots,p_d$. ☐

Finally, we have proven the remaining part that the coefficients of $\mathcal{E}(z)$ have the form as requested in Remark 5.2:

Proposition 5.20. *The coefficient of z^m, $m \in \mathbb{N}$, in $\mathcal{E}(z)$ can be rewritten as a sum of multivariate monomials in $(p_1,\dots,p_d)$ of degree m.*

Proof. The claim follows now immediately from Lemmas 5.16, 5.17, 5.18 and 5.19. ☐

Proof of Theorem 1.3. In view of formulas (5.1) and (5.13) the claim follows now directly with Lemma 5.1, Corollary 5.14 and Proposition 5.20 together with the explanations in Remark 5.2. ☐

5.3 Open problems

Finally, we formulate some open questions in connection with the analyticity results in Section 5:

Remark 5.21.

- Does the asymptotic capacity vary real-analytically for nearest-neighbour (or finite-range) random walks on $\mathbb{Z}^d$ for $d \geq 2$?
- For which classes of finitely generated groups do finite-range random walks vary real-analytically in terms of probability measures of constant support?

References

[1] A. Asselah and B. Schapira, *Moderate deviations for the range of a transient random path: path concentration*, Ann. Sci. Éc. Norm. Supér. (4) **50** (2017), no. 3, 755–786.

[2] A. Asselah, B. Schapira, and P. Sousi, *Capacity of the range of random walk on* $\mathbb{Z}^d$, Transactions of the American Mathematical Society **370** (2018), no. 11, 7627–7645.

[3] A. Asselah, B. Schapira, and P. Sousi, *Capacity of the range of random walk on* $\mathbb{Z}^4$, Ann. Probab. **47** (2019), no. 3, 1447–1497.

[4] P. Billingsley, *Convergence of probability measures*, Wiley, 1999.

[5] E. Candellero and L. Gilch, *Phase transitions for random walk asymptotics on free products of groups*, Random Structures & Algorithms **40** (2009), no. 2, 150–181.

[6] D.I. Cartwright and P.M. Soardi, *A local limit theorem for random walks on the cartesian product of discrete groups*, Boll. Un. Math. Ital. A VII **1** (1987), no. 1, 107–115.

[7] Y. Chang, *Two observations of the capacity of the range of simple random walks on* $\mathbb{Z}^3$ *and* $\mathbb{Z}^4$, Electron. Commun. Probab. **22** (2017), no. 25, 1–9.

[8] M. Chen, S. Yan and Y. Zhou, *The range of random walks on trees and related trapping problem*, Acta Mathematicae Applicatae Sinica **13** (1997), no. 1, 1–16.

[9] A. Dvoretzky and P. Erdös, *Some problems on random walks in space*, Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability 1950, University of California Press, Berkeley and Los Angeles (1951), 353–367.

[10] P. Gerl and W. Woess, *Local limits and harmonic functions for nonisotropic random walks on free groups*, Probab. Theory Rel. Fields **71** (1986), 341–355.

[11] L. Gilch, *Rate of Escape of Random Walks*, PhD thesis, University of Technology Graz (Austria) (2007).

[12] L. Gilch, *Rate of escape of random walks on free products*, J. Aust. Math. Soc. **83** (2007), no. I, 31–54.

[13] L. Gilch, *Asymptotic entropy of random walks on free products*, Electron. J. Probab. **16** (2011), 76–105.

[14] L. Gilch, *Range of random walks on free products*, Stochastic Processes and their Applications **149** (2022), 369–403.

[15] Y. Guivarc'h, *Sur la loi des grands nombres et le rayon spectral d'une marche aléatoire*, Astérisque **74** (1980), 47–98.

[16] N. C. Jain and S. Orey, *On the range of random walk*, Israel J. Math. **6** (1968), 373–380.

[17] S.P. Lalley, *Finite range random walk on free groups and homogeneous trees*, Ann. Probab. **21** (1993), no. 4, 2087–2130.

[18] S.P. Lalley, *Algebraic systems of generating functions and return probabilities for random walks*, Dynamics and Randomness II (Dordrecht), Springer Netherlands, 2004, pp. 81–122.

[19] G.F. Lawler, *Intersections of random walks*, Probability and its Applications, Birkhäuser Boston, 1991.

[20] R. Lyndon and P. Schupp, *Combinatorial group theory*, Springer-Verlag, 1977.

[21] J. Mairesse and F. Mathéus, *Random walks on free products of cyclic groups*, J. London Math. Soc. **75** (2007), no. 1, 47–66.

[22] R. Mrazović, N. Sandrić, and S. Šebek, *Capacity of the range of random walks on groups*, to appear in Kyoto J. of Mathematics (2021).

[23] S. Sawyer, *Isotropic random walks in a tree*, Zeitschrift f. Wahrscheinlichkeitstheorie Verw. Geb. **42** (1978), 279–292.

[24] B. Schapira, *Capacity of the range in dimension* 5, Ann. Probab. **48** (2020), 2988–3040.

[25] Z. Shi, V. Sidoravicius, H. Song, L. Wang, and K. Xiang, *On spectral radius of biased random walks on infinite graphs*, arXiv, https://arxiv.org/abs/1805.01611 (2018).

[26] F. Spitzer, *Principles of random walk*, 2nd edition ed., Springer-Verlag, New York-Heidelberg, 1976.

[27] J. Stallings, *Group theory and three-dimensional manifolds*, a James K. Whittemore Lecture in Mathematics given at Yale University, 1969 ed., Yale Mathematical Monographs, Yale University Press, New Haven, Conn.-London, 1971.

[28] A.-S. Sznitman, *Vacant set of random interlacements and percolation*, Ann. of Math. (2) **171** (2010), no. 3, 2039–2087.

[29] W. Woess, *Nearest neighbour random walks on free products of discrete groups*, Boll. Un. Mat. Ital. **5-B** (1986), 961–982.

[30] W. Woess, *Random walks on infinite graphs and groups*, Cambridge University Press, 2000.

[31] W. Woess, *Denumerable Markov Chains*, EMS Press (2009).

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