Electron. Commun. Probab. **29** (2024), article no. 12, 1–6.
https://doi.org/10.1214/24-ECP584
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Asymptotics of the rate function in the large deviation principle for sums of independent identically distributed random variables

Iosif Pinelis*

Abstract

Let Λ^* be the rate function in the large deviation principle for the sums $X_1+\dots+X_n$ of independent identically distributed random variables $X_1, X_2, \dots$. It is shown that $\Lambda^*(x) \sim -\ln \mathsf{P}(X_1 \ge x)$ (as $x \to \infty$) if and only if $\ln \mathsf{P}(X_1 \ge x) \sim L_0(x)$ for some concave function L_0. The main ingredient of the proof is the general, explicit expression of a suitable quasi-minimizer in $t \ge 0$ of the Bernstein–Chernoff upper bound $e^{-tx}\mathsf{E}e^{tX_1}$ on $\mathsf{P}(X_1 \ge x)$, which is amenable to analysis and, at the same time, is close enough to a true minimizer.

Keywords: large deviation principle (LDP); rate function; sums of independent random variables; asymptotics; log-concavity.
MSC2020 subject classifications: 60F10.
Submitted to ECP on December 21, 2023, final version accepted on March 4, 2024.

Let $X, X_1, X_2, \dots$ be i.i.d random variables, and let $S_n := X_1 + \dots + X_n$ for natural n. By a large deviation principle (LDP) – see e.g. [3, Corollary 2.2.19], for all real x

$$\ell(x) := \lim_{n\to\infty} \frac{1}{n} \ln \mathsf{P}(S_n \ge nx) = -\inf_{y\in[x,\infty)} \Lambda^*(y), \tag{1}$$

where

$$\Lambda^*(y) := \sup_{t\in\mathbb{R}} (ty - \Lambda(t)) \quad \text{and} \quad \Lambda(t) := \ln \mathsf{E}e^{tX}, \tag{2}$$

so that Λ^* is the Fenchel–Legendre transform of Λ. In particular, for each real x the limit $\ell(x)$ in (1) always exists (and is ≤ 0), but may take the value $-\infty$. We let $\ln 0 := -\infty$.

For a formulation of a general LDP and corresponding historical notes, see e.g. [3, Sections 1.2 and 1.3]. The LDP allows one to find the asymptotics of the logarithm of small probabilities of regular enough sets. Under additional assumptions, it is in some cases possible to find the asymptotics of small probabilities themselves; see e.g. [5, Chapter VIII] and [1, 4]; these results go back to Cramér (1938) [2].

What can be said about the asymptotics of $\ell(x)$ (as $x \to \infty$)?

Before answering this question in the main result of this note, let us first do some light cleaning:

Proposition 1. *If $\Lambda(t) = \infty$ for all real $t > 0$, then $\Lambda^*(y) \to 0$ as $y \to \infty$.*

*Michigan Technological University, United States of America. E-mail: ipinelis@mtu.edu

Proof. If $\Lambda(t) = \infty$, not only for all real $t > 0$, but also for all real $t < 0$, then, by the definition of $\Lambda^*(y)$ in (2), $\Lambda^*(y) = 0$ for all real y, so that the conclusion of Proposition 1 holds.

It remains to consider the case when $\Lambda(t_0) < \infty$ for some real $t_0 < 0$. The condition that $\Lambda(t) = \infty$ for all real $t > 0$ and the definition of $\Lambda^*(y)$ imply

$$\Lambda^*(y) = \sup_{t \le 0}(ty - \Lambda(t)). \tag{3}$$

Since the function Λ is convex and $> -\infty$ and $\Lambda(0) = 0$, it follows that Λ is real-valued on the interval $[t_0, 0]$. So, on the interval $(t_0, 0)$ the function Λ has a nondecreasing (say) right derivative Λ'. (In this case, it is easy to see, using dominated convergence, that Λ is differentiable on $(t_0, 0)$.) So, for each $s \in (t_0, 0)$ and all real t we have $\Lambda(t) \ge \Lambda(s) + \Lambda'(s)(t - s) = \Lambda(s) - \Lambda'(s)s + \Lambda'(s)t$. Hence, in view of (3),

$$\Lambda^*(y) \le -\Lambda(s) + \Lambda'(s)s + \sup_{t \le 0} t(y - \Lambda'(s)) = -\Lambda(s) + \Lambda'(s)s$$

if $y \ge \Lambda'(s)$, which implies that

$$\limsup_{y \to \infty} \Lambda^*(y) \le -\Lambda(s) + \Lambda'(s)s, \tag{4}$$

for each $s \in (t_0, 0)$. By dominated convergence, $\Lambda(s) \to \Lambda(0) = 0$ as $s \uparrow 0$. Also, since Λ' is nondecreasing and real-valued on the interval $(t_0, 0)$, there exists $\lim_{s \uparrow 0} \Lambda'(s) \in (-\infty, \infty]$. It follows that $\limsup_{s \uparrow 0} \Lambda'(s)s \le 0$ and hence $\limsup_{s \uparrow 0}(-\Lambda(s) + \Lambda'(s)s) \le 0$. So, by (4), $\limsup_{y \to \infty} \Lambda^*(y) \le 0$. On the other hand, by (say) (3), $\Lambda^*(y) \ge 0t - \Lambda(0) = 0$ for all real y. This completes the proof of Proposition 1. $\square$

It follows from Proposition 1 and the nonnegativity of Λ^* (mentioned at the end of the proof of Proposition 1) that, if $\Lambda(t) = \infty$ for all real $t > 0$, then, by (1), $\ell(x) = 0$ for all real $x \ge 0$.

Excluding this trivial case, we will have

$$\Lambda(t_1) < \infty \text{ for some real } t_1 > 0, \tag{5}$$

and then, by (2), $\Lambda^*(y) \ge t_1 y - \Lambda(t_1) \to \infty$ as $y \to \infty$. Also, the function Λ^* is convex, being the supremum of affine functions. So, the function Λ^* is increasing in a neighborhood of ∞ and hence, by (1), for all large enough real $x > 0$

$$\ell(x) = -\Lambda^*(x). \tag{6}$$

So, in the "nontrivial" case, the asymptotics of ℓ reduces to that of the rate function Λ^*.

In a few cases, Λ^* is an elementary function, and then easily analyzed. E.g., if X has the standard normal distribution, then $\Lambda^*(x) = x^2/2$ and hence

$$\ell(x) = -\Lambda^*(x) \sim \ln \mathsf{P}(X \ge x). \tag{7}$$

If now X has the standard exponential distribution (with mean 1), then $\Lambda^*(x) = x - 1 - \ln x \sim x$ and hence (7) holds again. Here and in what follows, all asymptotic relations are understood to hold as $x \to \infty$, unless specified otherwise. The other relations involving x are understood to hold eventually – that is, for all large enough real $x > 0$.

These examples suggest that, in the "nontrivial" case (when (5) holds), the asymptotics

$$\ell(x) = -\Lambda^*(x) \sim L(x) := \ln q(x), \tag{8}$$

where

$$q(x) := \mathsf{P}(X \geq x),$$

may be somewhat common – provided, of course, that the tail function q varies regularly enough.

More specifically, since the function Λ^* is convex, it is obvious that, for (8) to hold, it is necessary that the function L be asymptotically equivalent to *some* log-concave function. Remarkably, this necessary condition turns out to be sufficient as well:

Theorem 2. *For* (8) *to hold (as $x \to \infty$), it is necessary and sufficient that*

$$L(x) \sim L_0(x) \tag{9}$$

for some real-valued concave function L_0.

Remark 3. The relation $a(x) \sim b(x)$ between two expressions with values in the extended real line is understood here as $a(x)/b(x) \to 1$. In particular, $a(x) \sim b(x)$ implies that eventually the value of $|b(x)|$ is not in the set $\{0, \infty\}$, and hence the value of $|a(x)|$ is not in the set $\{0, \infty\}$.

Proof of Theorem 2. **Part 1: Necessity:** Letting $L_0 := -\Lambda^*$ in a neighborhood of ∞ and recalling that the function Λ^* is convex, we see that, for (8) to hold it is indeed necessary that (9) be true for some real-valued concave function L_0.

Part 2: Sufficiency: Assume that (9) holds for some real-valued concave function L_0. We have to show that then (8) holds as well.

The key here will be a particular choice (made in (21)) of a quasi-maximizer t_x of $tx - \Lambda(t)$ in t such that the expression $t_x x - \Lambda(t_x)$ is

(i) easy enough to analyze and, at the same time,

(ii) close enough to $\Lambda^*(x)$ (cf. the definition of Λ^* in (2)).

Since $L(x) \to -\infty$ (as $x \to \infty$), (9) impies that $L_0(x) \to -\infty$. So, in view of the concavity of L_0, for some $\tau \in [-\infty, 0)$ we have $L_0(x)/x \to \tau$ and hence $L(x)/x \to \tau$, which in turn implies (5) and therefore (6). That is, eventually

$$\ell(x) = -\Lambda^*(x) = \ln \inf_{t \in \mathbb{R}} Q(x,t), \tag{10}$$

where

$$Q(x,t) := e^{-tx} M(t) \quad \text{and} \quad M(t) := \mathsf{E} e^{tX}. \tag{11}$$

It also follows that there exists $\mu := \mathsf{E}X \in [-\infty, \infty)$. So, by Jensen's inequality, for any real $\nu > \mu$ and all $x > \nu$ and all $t < 0$ we have $Q(x,t) \geq e^{t(\nu - x)} > 1 = Q(x,0)$. Therefore, eventually

$$\ell(x) = \ln \inf_{t \geq 0} Q(x,t). \tag{12}$$

By the Markov–Bernstein–Chernoff inequality, $q(x) \leq Q(x,t)$ for all real $t \geq 0$. So, by (12), eventually

$$\ell(x) \geq L(x). \tag{13}$$

By the definition of $M(t)$ in (11), for real $t>0$,

$$
\begin{aligned}
M(t)=\mathsf{E}e^{tX}&=-\int_{-\infty}^{\infty}dq(u)e^{tu}\\
&=-\int_{-\infty}^{\infty}dq(u)\int_{-\infty}^{u}t\,dv\,e^{tv}\\
&=-\int_{-\infty}^{\infty}t\,dv\,e^{tv}\int_{[v,\infty)}dq(u)\\
&=t\int_{-\infty}^{\infty}dv\,e^{tv}q(v)\\
&=I_1+I_2,
\end{aligned}
\tag{14}
$$

where

$$I_1:=t\int_{-\infty}^{x-A}dv\,e^{tv}q(v)\le t\int_{-\infty}^{x-A}dv\,e^{tv}=e^{t(x-A)}, \tag{15}$$

$$I_2:=t\int_{x-A}^{\infty}dv\,e^{tv}e^{-g(v)}, \tag{16}$$

A is a real number, depending on A (which latter is to be specified later), and

$$g(v):=-\ln q(v)=-L(v).$$

Let

$$g_0(v):=-L_0(v).$$

The function g_0 is convex, (strictly) increasing (to ∞), and $<\infty$ in a neighborhood of ∞ – say on the interval

$$[a,\infty)$$

for some real a. So, $g_0'>0$ on (a,∞), where g_0' is (say) the right derivative of the convex function g_0. In what follows, by default $x\in(a,\infty)$.

Take now any $c\in(0,1)$ and let

$$A:=A(x):=\frac{cg_0(x)}{g_0'(x)}, \tag{17}$$

so that eventually $A>0$. Note also that, by the convexity of g_0 on $[a,\infty)$, we have $(x-a)g_0'(x)\ge g_0(x)-g_0(a)$ (for $x\in(a,\infty)$), that is, $xg_0'(x)-g_0(x)\ge ag_0'(x)-g_0(a)$, so that

$$x-A=(1-c)x+c\frac{xg_0'(x)-g_0(x)}{g_0'(x)}\ge(1-c)x+c\frac{ag_0'(x)-g_0(a)}{g_0'(x)}. \tag{18}$$

Also, again by the convexity of g_0 on $[a,\infty)$, there exists $\lim_{x\to\infty}g_0'(x)\in(0,\infty]$. So, by (18), $x-A\to\infty$ and hence wlog $x-A\ge a$, which will be assumed by default in the sequel.

Let now

$$k(x):=\inf_{v\in[x-A,\infty)}\frac{g(v)}{g_0(v)}, \tag{19}$$

so that, in view of (9) and because $x-A\to\infty$, we have

$$k(x)\to1. \tag{20}$$

Now comes the crucial point, which is choosing the value of t as follows:

$$t_x:=k(x)g_0'(x)-\frac{1}{A}=\frac{ck(x)g_0(x)-1}{A}. \tag{21}$$

Using the convexity of g_0 on $[a,\infty)$ again, we have

$$g_0(v) \ge h_x(v) := g_0(x) + g_0'(x)(v-x) \tag{22}$$

for real $v \ge x - A$.

Letting $t = t_x$ in the rest of the proof and using (16), (19), (22), (21), (20), and (9), we have

$$\begin{aligned} e^{-tx} I_2 &= t\int_{x-A}^{\infty} dv\, e^{t(v-x)} e^{-g(v)} \\ &\le t\int_{x-A}^{\infty} dv\, e^{t(v-x)} e^{-k(x)g_0(v)} \\ &\le t\int_{x-A}^{\infty} dv\, e^{t(v-x)} e^{-k(x)h_x(v)} \\ &= e^{-k(x)g_0(x)+1}(ck(x)g_0(x)-1) \\ &= e^{-k(x)g_0(x)(1+o(1))} = q(x)^{1+o(1)}. \end{aligned} \tag{23}$$

Also, by (15), (21), and (20),

$$e^{-tx} I_1 \le e^{-tA} = e^{-ck(x)g_0(x)+1} = q(x)^{c+o(1)}.$$

In view of (20) and because $c \in (0,1)$ was arbitrary, we now get

$$e^{-tx} I_1 \le q(x)^{1+o(1)}; \tag{24}$$

alternatively, everywhere above we can replace the constant $c \in (0,1)$ by a variable $c(x)$ such that $c(x) \to 1$ but $(1-c(x))x \to \infty$. It follows from (12), (11), (14), (24), (23), and the definition of $L(x)$ in (8) that

$$\ell(x) \le \ln\big(q(x)^{1+o(1)}\big) \sim \ln q(x) = L(x). \tag{25}$$

Finally, (8) follows from (13) and (25).

This completes the sufficiency part of the proof as well. $\square$

Remark 4. In view of (21) and (17), it appears that, for large $x > 0$,

$$\tilde t_x := g_0'(x) - \frac{g_0'(x)}{g_0(x)} = g_0'(x)\Big(1 - \frac{1}{g_0(x)}\Big) \tag{26}$$

should in general be a good approximation to a maximizer of $tx - \Lambda(t)$ in real t – cf. (2). The expression for $\tilde t_x$ in (26) is obtained from the expression for t_x in (21) by replacing there $k(x)$ by 1 (cf. (20)) and c by 1 (cf. the sentence containing formula (24)). Possibly, this approximation can be used in other contexts. More transparently, t_x and $\tilde t_x$ can be described as quasi-minimizers in $t \ge 0$ of the Bernstein–Chernoff upper bound (BCub) $e^{-tx}\mathsf{E}e^{tX}$ on $\mathsf{P}(X \ge x)$, which are amenable to general analysis and, at the same time, are close enough to a true minimizer of the BCub.

In particular, if X has the standard exponential distribution and $g_0 = g = -\ln q$ (so that g_0 is convex), then $\tilde t_x = 1 - 1/x$ is exactly equal, for all real $x > 0$, to the maximizer of $tx - \Lambda(t)$ in real t. If X has the standard normal distribution and again $g_0 = g = -\ln q$ (so that g_0 is convex), then

$$\tilde t_x = \frac{\varphi(x)}{1-\Phi(x)}\Big(1 + \frac{1}{\ln(1-\Phi(x))}\Big) = x\Big(1 - \frac{1+o(1)}{x^2}\Big), \tag{27}$$

where φ and Φ are, respectively, the p.d.f. and the c.d.f. of the standard normal distribution, whereas the maximizer of $tx - \Lambda(t)$ in real t in this case is x. $\square$

References

[1] R. R. Bahadur and R. Ranga Rao, *On deviations of the sample mean*, Ann. Math. Statist. **31** (1960), 1015–1027. MR117775

[2] Harald Cramér, *On a new limit theorem in probability theory (translation of 'Sur un nouveau théorème-limite de la théorie des probabilités')*, (2022), arXiv:1802.05988 [math.HO].

[3] Amir Dembo and Ofer Zeitouni, *Large deviations techniques and applications*, second ed., Applications of Mathematics (New York), vol. 38, Springer-Verlag, New York, 1998. MR1619036

[4] V. V. Petrov, *On the probabilities of large deviations for sums of independent random variables*, Theor. Probability Appl. **10** (1965), 287–298. MR0185645

[5] V. V. Petrov, *Sums of independent random variables*, Springer-Verlag, New York, 1975, Translated from the Russian by A. A. Brown, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 82. MR0388499