Electron. Commun. Probab. **29** (2024), article no. 14, 1–10.
https://doi.org/10.1214/24-ECP581
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

On the spine of the two-particle Fleming-Viot process driven by Brownian motion[*][†]

Krzysztof Burdzy[‡] Tvrtko Tadić[§]

Abstract

The spine of the two-particle Fleming-Viot process driven by Brownian motion is not a Bessel-3 process.

Keywords: Fleming-Viot type process; Brownian motion; Bessel process; logarithmic transformation.
MSC2020 subject classifications: 60G17; 60J65; 60J80.
Submitted to ECP on June 12, 2023, final version accepted on March 4, 2024.

1 Introduction

We start with an informal outline of the main idea of this note. A more detailed review, including history and citations, will be presented later in the introduction.

A Fleming-Viot process is a process with a branching structure (but not a branching process according to terminology adopted in the literature on branching processes). Under very mild assumptions, it has a unique spine. When the number of individuals in the population is very large, the distribution of the spine is expected to be very close to the distribution of the driving process conditioned on survival forever. There is an example showing that the distribution of the spine may be different from the distribution of the driving process conditioned on survival forever. The published example is rather artificial so we present in this note a different example illustrating the same claim. Our new example is more natural in the sense that it is based on a model examined in a number of papers on Fleming-Viot processes.

1.1 Literature review

Fleming-Viot-type processes were originally defined in [6]. In this model, there is a population of fixed size. Every individual moves independently from all other individuals according to the same Markovian transition mechanism, in a domain with a boundary. When an individual hits the boundary, the indvidual is killed and an individual chosen randomly (uniformly) from the survivors splits into two individuals and the process continues in this manner. The question of whether the process can be continued for all

*Supported in part by Simons Foundation Grants 506732 and 928958.
†This article was first posted with the incorrect name of author Tvrtko Tadić. The metadata were corrected on 26 March 2024.
‡University of Washington, Seattle, USA. E-mail: burdzy@uw.edu
§Microsoft Corporation, Redmond, USA. E-mail: tvrtko@math.hr

times was addressed in [6, 5, 4, 13]. All of these papers studied, among other processes, Fleming-Viot processes driven by Brownian motion.

A very special case of a Fleming-Viot process is when there are only two individuals driven by Brownian motion on $[0, \infty)$ and 0 plays that role of the boundary; this model was studied in [5, 13]. In particular, it was shown in both papers that this process has infinite lifetime.

Every Fleming-Viot process has a unique spine, i.e. a trajectory inside the branching tree that never hits the boundary of the domain where the process is confined; this was proved under strong assumptions in [13, Thm. 4] and later in full generality in [3].

It was proved in [3] that if the state space is finite and the number of individuals in the population goes to infinity then the distributions of spine processes converge to the distribution of the driving Markov process conditioned on survival forever.

Weak convergence of the spine to the driving process conditioned never to be killed was proven when the driving process is a normally reflected diffusion in a compact domain with soft killing in Corollary 5.3 of a preprint [17], with the extra results on convergence of the branching rate to twice the rate of a generic particle and of the side branches to the critical branching process proven in Corollary 6.5.

The connection between the Fleming-Viot particle system and the Fleming-Viot process from population genetics was recently published in [18], an article based on a part of [17]. Moreover in Section 5 of [18], the same connection for the Fleming-Viot particle system driven by Brownian motion with hard killing is established when the domain is bounded and C^∞. In a forthcoming article by O. Tough, this connection will be used to prove convergence of the spine.

The rate of convergence of the distribution of the Fleming-Viot process driven by a general Markov process to the quasi-stationary distribution was investigated in [19].

In [3], an example was given of a Fleming-Viot process driven by a Markov process on a three-element state space such that one of the elements plays the role of the boundary, the population consists of two individuals and the distribution of the spine is not equal to the distribution of the driving Markov process conditioned on survival forever. A Markov process with a three-element state space seems to be a rather artificial example in the context of Fleming-Viot models. We will show that the spine of the Fleming-Viot process with two individuals driven by Brownian motions on $[0, \infty)$ has a spine with a distribution different from the distribution of Brownian motion conditioned to stay positive, i.e. the distribution of the 3-dimensional Bessel process. The point of this note is to show that proving that the spine does not have the distribution of the 3-dimensional Bessel process is somewhat tricky. In hindsight, this does not seems to be difficult because our proof is quite elementary. Nevertheless, our previous attempts in [7, 8] generated some new results but failed to show the difference. In a sense that will be made more precise later on in the paper, the spine is quite “close” to a 3-dimensional Bessel process and therefore it is quite hard to distinguish the two. The problem has been open for some time and while the solution is not really difficult we now have a better understanding as to why the spine is nevertheless similar to a 3-dimensional Bessel process in certain respects.

2 Model and main result

We will now define a Fleming-Viot process and other elements of the model. Informally, the process consists of two independent Brownian particles starting at the same point in $(0, \infty)$. At the time when one of them hits 0, it is killed and the other one branches into two particles. The new particles start moving as independent Brownian motions and the scheme is repeated.

2.1 Notation and definitions

On the formal side, let $(W_1(t) : t \geq 0)$ and $(W_2(t) : t \geq 0)$ be two independent Brownian motions starting from $W_1(0) = W_2(0) = 1$. Let

$$\begin{aligned} &T_0 = 0, \\ &Y_0 = 1, \\ &\tau_j = \inf\{t \geq 0 : W_j(t) = 0\}, \qquad j = 1, 2, \\ &T_1 = \min(\tau_1, \tau_2), \\ &Y_1 = \max(W_1(T_1), W_2(T_1)), \end{aligned}$$

and for $k \geq 2$,

$$\begin{aligned} &T_k = \inf\{t > T_{k-1} : \min(W_1(t) - W_1(T_{k-1}) + Y_{k-1}, W_2(t) - W_2(T_{k-1}) + Y_{k-1}) = 0\}, \\ &Y_k = \max(W_1(T_k) - W_1(T_{k-1}) + Y_{k-1}, W_2(T_k) - W_2(T_{k-1}) + Y_{k-1}). \end{aligned}$$

It follows from [4, Thm. 5.4] or [13, Thm. 1] that, a.s.,

$$T_k \to \infty. \tag{2.1}$$

Hence, for any $t \geq 0$ we can find j such that $t \in [T_{j-1}, T_j)$. Then we set

$$\mathcal{V}(t) = (V_1(t), V_2(t)) = (W_1(t) - W_1(T_{j-1}) + Y_{j-1}, W_2(t) - W_2(T_{j-1}) + Y_{j-1}). \tag{2.2}$$

This completes the definition of $\{\mathcal{V}(t), t \geq 0\}$, an example of a Fleming-Viot process.

Let $J_t = J(t)$ denote the spine, i.e. $J_t = V_1(t)$ for $t \in [T_{k-1}, T_k)$ if $V_1(T_k-) > V_2(T_k-) = 0$. If the last condition fails, we let $J_t = V_2(t)$ for $t \in [T_{k-1}, T_k)$.

Note that $J(T_k) = Y_k$ for all $k \geq 1$.

Recall that a d-dimensional Bessel process X_t is defined by

$$dX_t = dB_t + \frac{d-1}{2X_t} dt, \tag{2.3}$$

where B is Brownian motion; see [14, Sect. 3.3 C]. It is well known that Brownian motion on $[0, \infty)$ conditioned to never hit 0 has the transition probabilities of the 3-dimensional Bessel process; this theorem was first proved in [10].

2.2 Main result

Theorem 2.1. *The distributions of* $\{J_t, 0 \leq t < \infty\}$ *and the 3-dimensional Bessel process* $\{X_t, 0 \leq t < \infty\}$ *starting from 1 are singular with respect to each other.*

We will explicitly define an event that has a strictly positive probability according to the first distribution but not according to the second one, and vice versa.

We will now review two attempts to prove Theorem 2.1 that failed.

The following version of the Law of the Iterated Logarithm was proved in [8].

Theorem 2.2. *Almost surely,*

$$\limsup_{n \to \infty} \frac{Y_n}{\sqrt{2T_n \log \log T_n}} = 1. \tag{2.4}$$

The Law of the Iterated Logarithm stated in (2.4) is the same as that for the 3-dimensional Bessel process (see [16]), which has the same distribution as the one-dimensional Brownian motion conditioned not to hit 0. Hence, Theorem 2.2 does not eliminate the possibility that the spine J_t has the distribution of Brownian motion conditioned not to hit 0.

In this note, we will prove the following result.

Theorem 2.3. *For every $u > 0$,*

$$\frac{1}{u}\log \mathbb{P}\left(\inf_{s\geq 0}\log X(s) < -u\right) = -1 = \lim_{t\to\infty}\frac{1}{t}\log \mathbb{P}\left(\inf_{n\geq 0}\log J(T_n) < -t\right).$$

The proof of Theorem 2.3 is based on explicit computations. We know from the Wiener-Hopf theory (see, for example, [2]), that, asymptotically, the minimum of a random walk with a positive drift is less than $-t$ with probability $ce^{-\gamma t}$. A continuous version of this result applies to $\{\log X(s), s \geq 0\}$ with $c = \gamma = 1$ as shown in Lemma 3.2.

The general message from Theorems 2.2 and 2.3 is that it is hard to distinguish between the spine and the 3-dimensional Bessel process by studying "extreme" behavior of the two processes. The proof of Theorem 2.1 will be based on the analysis of the processes on the "logarithmic scale."

3 Bessel processes

Let

$$\rho(t) = \int_0^t \frac{1}{X_s^2}ds.$$

Lemma 3.1. *If X is three-dimensional Bessel process with $X_0 = 1$ and B is Brownian motion with $B_0 = 0$ then $\{\log X_{\rho^{-1}(t)}, t \geq 0\}$ has the same distribution as the process $\left\{B_t + \frac{1}{2}t, t \geq 0\right\}$.*

Proof. Recall the stochastic differential equation (2.3) defining Bessel processes. Let $f(x) = \log x$. Then $f'(x) = 1/x$ and $f''(x) = -1/x^2$. Let $A_t = f(X_t)$. Then by the Ito formula

$$\begin{aligned} dA_t = df(X_t) &= \frac{1}{X_t}dB_t + \left(\frac{d-1}{2X_t}\cdot\frac{1}{X_t} - \frac{1}{2}\cdot\frac{1}{X_t^2}\right)dt = \frac{1}{X_t}dB_t + \frac{d-2}{2X_t^2}dt \\ &= e^{-A_t}dB_t + \frac{d-2}{2}e^{-2A_t}dt. \end{aligned}$$

For the 3-dimensional Bessel process, i.e. when $d = 3$, the formula is

$$dA_t = e^{-A_t}dB_t + \frac{1}{2}e^{-2A_t}dt.$$

We see that the process A_t is a time change of the process $B_t + \frac{1}{2}t$, if we use the clock

$$\rho(t) = \int_0^t e^{-2A_s}ds = \int_0^t \frac{1}{X_s^2}ds.$$

In other words, $\{A_{\rho^{-1}(t)}, t \geq 0\} = \{\log X_{\rho^{-1}(t)}, t \geq 0\}$ has the same distribution as the process $\left\{B_t + \frac{1}{2}t, t \geq 0\right\}$. □

Lemma 3.2. *Suppose that $X = \{X(t) : t \geq 0\}$ is the 3-dimensional Bessel process with $X(0) = 1$. Let $M = \inf_{t\geq 0}\log X(t)$. Then $-M$ has the exponential distribution with mean 1.*

Proof. Time change does not affect the distribution of the infimum of a process, hence, by Lemma 3.1, M has the same distribution as $\min_{t\geq 0}\left(B_t + \frac{1}{2}t\right)$. According to [14, Sect. 3.3, Exercise 5.9], $-M$ is exponential with mean 1. □

4 Logarithmic transformation of Fleming-Viot process

We will use the complex representation $V_1(t)+iV_2(t)$ of the process $\mathcal{V}(t) = (V_1(t), V_2(t))$ defined in (2.2). We apply the complex mapping $z \mapsto \log z$ to this process so that it is transformed into a process in the strip $D := \{(x, y) : 0 < y < \pi/2\}$ (see Fig. 1). Consider the following "clocks,"

$$\phi(t) = \int_0^t \frac{1}{|\mathcal{V}(s)|^2} ds,$$
$$\sigma(t) = \int_0^t \frac{1}{J(s)^2} ds.$$

It follows from conformal invariance of two-dimensional Brownian motion (see [15, Thm. V (2.5)]) that the process $Z(t) = (Z_1(t), Z_2(t)) := \log \mathcal{V}(\phi^{-1}(t))$ is two-dimensional Brownian motion jumping from the boundary of D to an appropriate point in D every time it exits D. Let $R_1, R_2, \dots$ be the times of jumps of Z, and let $R_0 = 0$.

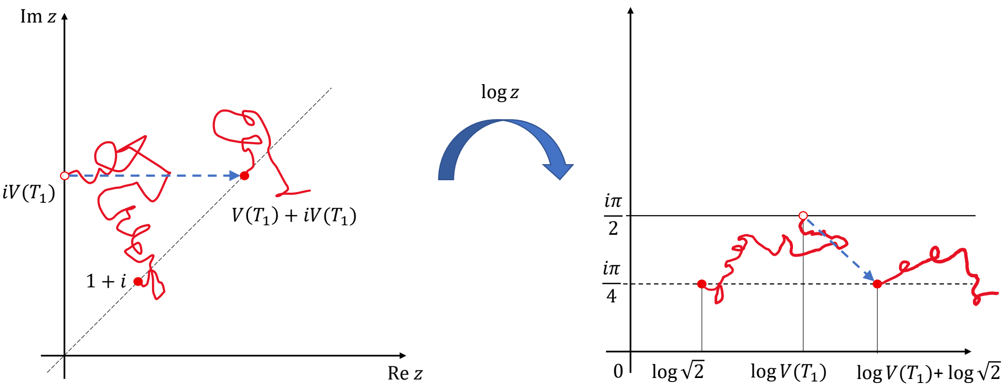

Figure 1: The logarithmic transformation of the Fleming-Viot process.

Lemma 4.1. *The process* $\{\log J(T_n), n \geq 0\}$ *is a random walk, such that* $\log J(T_0) = \log J_0 = 0$*, and satisfying*

$$\log J(T_n) = \log J(T_{n-1}) + \log \sqrt{2} + K_n, \qquad n \geq 1, \tag{4.1}$$

where $\{K_n, n \geq 1\}$ *is an i.i.d. sequence. The distribution of* K_n *is that of* $Z_1(R_1-)$*. We also have*

$$\lim_{n\to\infty} \frac{\log J(T_n)}{n} = \log \sqrt{2}. \tag{4.2}$$

Proof. A jump takes Z from $(Z_1(R_k-), Z_2(R_k-)) \in \partial D$, i.e. the point at which Z exits D, to $\left(\log \sqrt{2} + Z_1(R_k-), \pi/4\right)$.

Brownian motions driving Z_1 and Z_2 between jumps are independent of each other. The times R_k are the times when Z_2 exits $[0, \pi/2]$. Hence, random variables $\{R_k, k \geq 0\}$ are independent of the Brownian motion B_t driving $Z_1(t)$. The time R_1 is the exit time from the interval $[0, \pi/2]$ for a Brownian motion starting at $\pi/4$ and independent of B. Note that

$$\log J(T_n) - \log J(T_{n-1}) = \log \sqrt{2} + Z_1(R_n-) - Z_1(R_{n-1}).$$

The first two claims of the lemma follow from independence of $\{R_n, n \geq 1\}$ and B, and the fact that $R_n - R_{n-1} \stackrel{d}{=} R_1$.

It is easy to see that $\mathbb{E}\,|Z_1(R_n-) - Z_1(R_{n-1})| < \infty$. Hence, by symmetry, $\mathbb{E}(Z_1(R_n-) - Z_1(R_{n-1})) = 0$. Thus we can use the law of large numbers and (4.1) to obtain (4.2). □

Theorem 4.2. *There exists a $c \in (0, \infty)$ such that*

$$\lim_{t\to\infty} e^t\, \mathbb{P}\left(\inf_{n\geq 0} \log J(T_n) < -t\right) = c.$$

Proof. Since $\{\log J(T_n), n \geq 0\}$ is a random walk, the function

$$t \to \mathbb{P}\left(\inf_{n\geq 0} \log J(T_n) < -t\right) = \mathbb{P}\left(\sup_{n\geq 0}(-\log J(T_n)) > t\right)$$

satisfies the Wiener-Hopf equation; see [2, point 2, bottom of page 191] for general overview, and [12, Theorem 3.1]. It follows from these references that there exists a $c \in (0, \infty)$ such that

$$\lim_{t\to\infty} e^{\gamma t}\, \mathbb{P}\left(\inf_{n\geq 0} \log J(T_n) < -t\right) = \lim_{t\to\infty} e^{\gamma t}\, \mathbb{P}\left(\sup_{n\geq 0}(-\log J(T_n)) > t\right) = c, \tag{4.3}$$

where γ is the positive solution to the equation

$$\mathbb{E}\left[e^{\gamma(-\log J(T_1))}\right] = \mathbb{E}[J(T_1)^{-\gamma}] = 1.$$

It follows from [13, (6.21)] that

$$\mathbb{P}(J(T_1) \in dy) = \frac{2}{\pi}\left[\frac{1}{(1-y)^2+1} - \frac{1}{(1+y)^2+1}\right] dy.$$

It is not difficult to check that $\mathbb{E}[J(T_1)^{-1}] = 1$. Hence $\gamma = 1$ and, therefore, the theorem follows from (4.3). □

Proof of Theorem 2.3. The theorem follows from Lemma 3.2 and Theorem 4.2. □

5 Comparing the spine and 3-dimensional Bessel process

Lemma 5.1. *(i) We have*

$$\alpha := \mathbb{E}\int_0^{T_1} \frac{1}{J_t^2} dt = \frac{8C}{\pi} - \log(2) \approx 1.63934, \tag{5.1}$$

where $C \approx 0.915966$ is the Catalan's constant.

(ii) The random variables

$$\int_{T_n}^{T_{n+1}} \frac{1}{J_t^2} dt, \qquad n \geq 0, \tag{5.2}$$

are i.i.d.

Proof. (i) We will switch between complex and real notation and write $z = x + iy$, $|z| = \sqrt{x^2 + y^2}$. Let $A \subset \mathbb{C}$ denote the first quadrant.

Given a connected open set in the complex plain whose complement has non-zero capacity, and a base point inside the domain, the Green function is the unique function, up to a multiplicative constant, that has a pole at the base point, is positive and harmonic outside the base point, and vanishes on the boundary of the domain (see [1, Chap. 6, Sec. 5.2]). We will use this characterization to find the Green function for A. The characterization also shows that the Green function is conformally invariant.

We will argue that

$$G(z) = \frac{1}{\pi} \log \left| \frac{z^2 + 2i}{z^2 - 2i} \right| \tag{5.3}$$

is the Green function in A with a pole at $1 + i$. It is elementary to check that $G(z)$ has a pole at $1 + i$ and vanishes on the real and imaginary axes. It is harmonic outside $1 + i$ because it is the real part of an analytic function. The function $G(z)$ is positive in A by the maximum principle.

The normalization $1/\pi$ in (5.3) is probabilistic, i.e. the integral of thus normalized Green function is equal to the expected lifetime of Brownian motion starting from $1 + i$ and killed upon exiting A. This normalization can be checked directly for a disc. It applies to other domains by conformal invariance.

The process $\{J_t, 0 \le t < T_1\}$ has the same distribution as $\{W_2(t), 0 \le t \le T_1\}$ conditioned on $\{\tau_1 < \tau_2\}$. Hence we will estimate the expectation in (5.1) assuming that (W_1, W_2) is conditioned to exit A through the vertical axis. This process is a Doob's transform, or an h-process, where h is harmonic in A with boundary values 1 on the vertical axis and 0 on the horizontal axis (see [11, Part 2, Chap. X] or [9, Ch. 11] for the theory of h-processes). The only harmonic function with these boundary values is $h(z) = (2/\pi) \arg(z) = (2/\pi) \arctan(y/x)$.

If $p_t(u, v)$ denotes the transition density for (W_1, W_2) then the transition density for the h-process is $p_t(u, v)h(v)/h(u)$. Hence, the Green function for (W_1, W_2) conditioned by h is

$$\int_0^\infty p_t(1 + i, z)h(z)/h(1 + i)dt = 2h(z) \int_0^\infty p_t(1 + i, z)dt = 2h(z)G(z).$$

Therefore,

$$\begin{aligned}
\mathbb{E} \int_0^{T_1} \frac{1}{J_t^2} dt &= \int_A 2G(z)h(z)\frac{1}{y^2} dz = \int_A 2G(x + iy)h(x + iy)\frac{1}{y^2} dxdy \\
&= \int_A 2G(x + iy)(2/\pi) \arctan(y/x)\frac{1}{y^2} dxdy \\
&= \int_A \frac{4}{\pi} G(x + iy) \arctan(y/x) \frac{x^2 + y^2}{y^2} \frac{1}{x^2 + y^2} dxdy \\
&= \int_A \frac{4}{\pi} G(x + iy) \arctan(y/x) \frac{1 + (y/x)^2}{(y/x)^2} \frac{1}{x^2 + y^2} dxdy.
\end{aligned}$$

Next we will change variables. Informally speaking, we will apply the complex function $z \to \log z$. In terms of real coordinates, we take

$$\begin{aligned}
(x, y) &= (e^r \cos\theta, e^r \sin\theta), \\
drd\theta &= \frac{1}{x^2 + y^2} dxdy.
\end{aligned}$$

Note that $y/x = \tan\theta$ and $\arctan(y/x) = \theta$. Let $G_*(r,\theta) = G(x+iy)$. The function G_* is the Green function in the strip $A_* := \{(r,\theta) : 0 < r < \pi/2\}$ with a pole at $(\sqrt{2}, \pi/4)$ by conformal invariance of the Green function. We obtain

$$\begin{aligned}
&\int_A \frac{4}{\pi} G(x+iy) \arctan(y/x) \frac{1+(y/x)^2}{(y/x)^2} \frac{1}{x^2+y^2} dxdy \\
&= \int_{A_*} \frac{4}{\pi} G_*(r,\theta)\theta \frac{1+\tan^2\theta}{\tan^2\theta} drd\theta = \int_0^{\pi/2} \frac{4}{\pi}\theta \frac{1+\tan^2\theta}{\tan^2\theta} \int_{-\infty}^{\infty} G_*(r,\theta) drd\theta.
\end{aligned}$$

The function $G_1(\theta) := \int_{-\infty}^{\infty} G_*(r,\theta)dr$ is the Green function for the one-dimensional Brownian motion starting from $\pi/4$ and killed upon exiting $(0,\pi/2)$. The function $G_1(\theta)$ is a one-dimensional harmonic function so it has to be linear, except at the base point, with zero boundary conditions. Hence,

$$G_1(\theta) = \begin{cases} \theta & \text{for } 0 < \theta < \pi/4, \\ \pi/2 - \theta & \text{for } \pi/4 < \theta < \pi/2. \end{cases}$$

Note that $G_1(\theta)$ is properly normalized, i.e. $\int_0^{\pi/2} G_1(\theta)d\theta = \pi^2/16$. In other words, the integral of the thus normalized Green function is equal to the expected exit time, known to be $\pi^2/16$, from $(0,\pi/2)$ for one-dimensional Brownian motion starting from $\pi/4$ (see [14, Problem 8.14, p. 100]).

We obtain

$$\begin{aligned}
&\int_0^{\pi/2} \frac{4}{\pi}\theta \frac{1+\tan^2\theta}{\tan^2\theta} \int_{-\infty}^{\infty} G_*(r,\theta) drd\theta \\
&= \int_0^{\pi/4} \frac{4}{\pi}\theta \frac{1+\tan^2\theta}{\tan^2\theta} \theta d\theta + \int_{\pi/4}^{\pi/2} \frac{4}{\pi}\theta \frac{1+\tan^2\theta}{\tan^2\theta} (\pi/2-\theta) d\theta \\
&= \left[\frac{4C}{\pi} - \frac{\pi}{4} + \log(2)\right] + \left[\frac{1}{4}\left(\frac{16C}{\pi} + \pi - \log(256)\right)\right] = \frac{8C}{\pi} - \log(2) \approx 1.63934,
\end{aligned}$$

where $C \approx 0.915966$ is the Catalan's constant. The exact values of the integrals were computed using Mathematica. The numerical value was confirmed by numerical calculations (Riemann sum approximation).

(ii) By Brownian scaling and the strong Markov property applied at T_n,

$$\left\{ \frac{\mathcal{V}(T_n + tV_1^2(T_n))}{V_1(T_n)}, t \in [0, (T_{n+1} - T_n)/V_1^2(T_n)) \right\}, \quad n \geq 1,$$

are i.i.d. For more details see [8, Lemma 7.10]. This implies that

$$\left\{ \frac{J(T_n + tV_1^2(T_n))}{V_1(T_n)}, t \in [0, (T_{n+1} - T_n)/V_1^2(T_n)) \right\}, \quad n \geq 1,$$

are i.i.d., and so are

$$\int_{T_n}^{T_{n+1}} \frac{1}{J_t^2} dt = \int_0^{T_{n+1}-T_n} \frac{1}{J^2(T_n+t)} dt = \int_0^{(T_{n+1}-T_n)/V_1^2(T_n)} \frac{V_1^2(T_n)}{J^2(T_n + sV_1^2(T_n))} ds.$$

□

Remark 5.2. We have $\log\sqrt{2} \approx 0.346574 < 0.81967 \approx \alpha/2$.

Proof of Theorem 2.1. For a continuous process $\{H_t, t \geq 0\}$ taking values in $(0,\infty)$, set

$$\psi(t) = \int_0^t \frac{1}{H_s^2} ds,$$

and let ψ^{-1} denote the inverse function. Let

$$F(H) = \left\{ \lim_{t\to\infty} \frac{\log H_{\psi^{-1}(t)}}{t} = \frac{1}{2} \right\}.$$

By Lemma 3.1, $\{\log X_{\rho^{-1}(t)}, t \geq 0\}$ has the same distribution as the process $\left\{B_t + \frac{1}{2}t, t \geq 0\right\}$. Hence, a.s.,

$$\lim_{t\to\infty} \frac{\log X_{\rho^{-1}(t)}}{t} = \lim_{t\to\infty} \frac{B_t + \frac{1}{2}t}{t} = 1/2.$$

In other words, the event $F(H) = F(X)$ has probability 1 if $H_t = X_t$ is the three-dimensional Bessel process with $X_0 = 1$.

Suppose that $\{M_n, n \geq 1\}$ is a sequence of random variables defined on the same probability space as H_t but not necessarily related to the process H_t in any way, for example, M_n does not have to be adapted to the filtration generated by H_t. If $F(H)$ holds and the event

$$\left\{ \lim_{n\to\infty} M_n/n = \alpha \right\}$$

also occurs then,

$$\lim_{n\to\infty} \frac{\log H_{\psi^{-1}(M_n)}}{n} = \lim_{n\to\infty} \frac{\log H_{\psi^{-1}(M_n)}}{M_n/\alpha} = \frac{\alpha}{2}. \tag{5.4}$$

Let $U_n = \sigma(T_n)$, recall (5.1), and use Lemma 5.1 (ii) and the law of large numbers to see that, a.s.,

$$\lim_{n\to\infty} U_n/n = \alpha.$$

By (4.2) and Remark 5.2, a.s.,

$$\lim_{n\to\infty} \frac{\log J(\sigma^{-1}(U_n))}{n} = \lim_{n\to\infty} \frac{\log J(T_n)}{n} = \log \sqrt{2} \neq \frac{\alpha}{2}. \tag{5.5}$$

In view of (5.4), this implies that the event $F(H) = F(J)$ has probability 0 if $H_t = J_t$ and $J_0 = 1$. This proves that the distributions of J and X are mutually singular. □

Remark 5.3. Theorem 2.1 compares the distributions of the processes J and X starting from 1 but it is clear that Theorem 2.1 holds for any initial distributions of J and X.

References

[1] Lars V. Ahlfors, *Complex analysis*, third ed., International Series in Pure and Applied Mathematics, McGraw-Hill Book Co., New York, 1978, An introduction to the theory of analytic functions of one complex variable. MR0510197

[2] Søren Asmussen, *A probabilistic look at the Wiener-Hopf equation*, SIAM Rev. **40** (1998), no. 2, 189–201 (electronic). MR1624253

[3] Mariusz Bieniek and Krzysztof Burdzy, *The distribution of the spine of a Fleming-Viot type process*, Stochastic Process. Appl. **128** (2018), no. 11, 3751–3777. MR3860009

[4] Mariusz Bieniek, Krzysztof Burdzy, and Sam Finch, *Non-extinction of a Fleming-Viot particle model*, Probab. Theory Related Fields **153** (2012), no. 1-2, 293–332. MR2925576

[5] Mariusz Bieniek, Krzysztof Burdzy, and Soumik Pal, *Extinction of Fleming-Viot-type particle systems with strong drift,* Electron. J. Probab. **17** (2012), no. 11, 15. MR2878790

[6] Krzysztof Burdzy, Robert Hołyst, and Peter March, *A Fleming-Viot particle representation of the Dirichlet Laplacian*, Comm. Math. Phys. **214** (2000), no. 3, 679–703. MR1800866

[7] Krzysztof Burdzy, Bartosz Kołodziejek, and Tvrtko Tadić, *Inverse exponential decay: stochastic fixed point equation and ARMA models*, Bernoulli **25** (2019), no. 4B, 3939–3977. MR4010978

[8] Krzysztof Burdzy, Bartosz Kołodziejek, and Tvrtko Tadić, *Stochastic fixed-point equation and local dependence measure*, Ann. Appl. Probab. **32** (2022), no. 4, 2811–2840. MR4474520

[9] Kai Lai Chung and John B. Walsh, *Markov processes, Brownian motion, and time symmetry*, second ed., Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 249, Springer, New York, 2005. MR2152573

[10] J. L. Doob, *Conditional Brownian motion and the boundary limits of harmonic functions*, Bull. Soc. Math. France **85** (1957), 431–458. MR0109961

[11] J. L. Doob, *Classical potential theory and its probabilistic counterpart*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 262, Springer-Verlag, New York, 1984. MR0731258

[12] P. Embrechts and N. Veraverbeke, *Estimates for the probability of ruin with special emphasis on the possibility of large claims*, Insurance Math. Econom. **1** (1982), no. 1, 55–72. MR652832

[13] Ilie Grigorescu and Min Kang, *Immortal particle for a catalytic branching process*, Probab. Theory Related Fields **153** (2012), no. 1-2, 333–361. MR2925577

[14] Ioannis Karatzas and Steven E. Shreve, *Brownian motion and stochastic calculus*, second ed., Graduate Texts in Mathematics, vol. 113, Springer-Verlag, New York, 1991. MR1121940

[15] Daniel Revuz and Marc Yor, *Continuous martingales and Brownian motion*, third ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 293, Springer-Verlag, Berlin, 1999. MR1725357

[16] Tokuzo Shiga and Shinzo Watanabe, *Bessel diffusions as a one-parameter family of diffusion processes*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **27** (1973), 37–46. MR0368192

[17] Oliver Tough, *Scaling limit of the fleming-viot multi-colour process*, (2021), Preprint. Math ArXiv. arXiv:2110.05049. MR4666298

[18] Oliver Tough, *Scaling limit of the Fleming-Viot multicolor process*, Ann. Probab. **51** (2023), no. 6, 2345–2386. MR4666298

[19] Denis Villemonais, *General approximation method for the distribution of Markov processes conditioned not to be killed*, ESAIM Probab. Stat. **18** (2014), 441–467. MR3333998

Acknowledgments. We are grateful to Don Marshall and Jan Swart for the most useful advice. We thank the referees for finding and correcting a significant error in the first version of this article and for many suggestions for improvement.