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Large deviations for the longest alternating and the longest increasing subsequence in a random permutation avoiding a pattern of length three

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Abstract

We calculate the large deviations for the length of the longest alternating subsequence and for the length of the longest increasing subsequence in a uniformly random permutation that avoids a pattern of length three. We treat all six patterns in the case of alternating subsequences. In the case of increasing subsequences, we treat two of the three patterns for which a classical large deviations result is possible. The same rate function appears in all six cases for alternating subsequences. This rate function is in fact the rate function for the large deviations of the sum of IID symmetric Bernoulli random variables. The same rate function appears in the two cases we treat for increasing subsequences. This rate function is twice the rate function for alternating subsequences.

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1 Introduction and statement of results

The problem of analyzing the distribution of the length, L_n, of the longest increasing subsequence in a uniformly random permutation from S_n, the set of permutations of $[n] := \{1, \ldots, n\}$, has a long and distinguished history; see [1] and references therein, and see [9]. In particular, the work of Logan and Shepp [8] together with that of Vershik and Kerov [13] show that the expectation of L_n satisfies $EL_n \sim 2n^{\frac{1}{2}}$ as $n \to \infty$. This was followed over twenty years later by the profound work of Baik, Deift and Johansson [2], who proved that the distribution of L_n satisfies

$$\lim_{n\to\infty} P\left(\frac{L_n - 2n^{\frac{1}{2}}}{n^{\frac{1}{6}}} \le x\right) = F(x),$$

where F is the Tracy-Widom distribution. A large deviations result for the lower tail probabilities $P(\frac{L_n}{n^{\frac{1}{2}}} \le x)$, for $x < 2$, was given in [5], while for the upper tail probabilities $P(\frac{L_n}{n^{\frac{1}{2}}} \ge x)$, for $x > 2$, one was given in [10]. See also references therein.

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Now consider the length of the longest alternating subsequence in a uniformly random permutation from S_n. An alternating subsequence of length k in a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ is a subsequence of the form $\sigma_{i_1} > \sigma_{i_2} < \sigma_{i_3} > \sigma_{i_4} \cdots \sigma_{i_k}$ or $\sigma_{i_1} < \sigma_{i_2} > \sigma_{i_3} < \sigma_{i_4} \cdots \sigma_{i_k}$, where $1 \le i_1 < \cdots < i_k \le n$. Call the first type an initially descending alternating subsequence and call the second type an initially ascending alternating subsequence. Of course, for asymptotic results concerning the longest alternating subsequence, it doesn't matter which type one considers since the two differ from one another by at most one. Stanley derived the exact expected value and variance for initially descending alternating subsequences [12]. In particular, letting A_n denote the length of the longest alternating subsequence of either type, he showed that $EA_n \sim \frac{2}{3}n$ and that the variance is asymptotic to $\frac{8}{45}n$. It has been proven that a central limit theorem holds with a Gaussian limiting distribution [14, 12]. We are unaware of large deviations results for this permutation statistic.

In this paper we derive the large deviations for increasing subsequences and alternating subsequences in uniformly random permutations that avoid a particular pattern in S_3. We recall the definition of a pattern avoiding permutation. If $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ and $\eta = \eta_1 \cdots \eta_m \in S_m$, where $2 \le m \le n$, then we say that σ contains η as a pattern if there exists a subsequence $1 \le i_1 < i_2 < \cdots < i_m \le n$ such that for all $1 \le j, k \le m$, the inequality $\sigma_{i_j} < \sigma_{i_k}$ holds if and only if the inequality $\eta_j < \eta_k$ holds. If σ does not contain η, then we say that σ *avoids* η. We denote by $S_n^{\text{av}(\eta)}$ the set of permutations in S_n that avoid η. For any $\eta \in S_m$, we denote the uniform probability measure on $S_n^{\text{av}(\eta)}$ by $P_n^{\text{av}(\eta)}$ and denote expectations by $E_n^{\text{av}(\eta)}$.

It is well-known [3, 11] that $|S_n^{\text{av}(\eta)}| = C_n$, for every $\eta \in S_3$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the nth Catalan number. One has

$$C_n \sim \frac{4^n}{\sqrt{\pi} n^{\frac{3}{2}}}. \tag{1.1}$$

We begin with alternating subsequences. In [7] it was proven that $E_n^{\text{av}(\eta)} A_n \sim \frac{n}{2}$, that the variance of A_n under $P_n^{\text{av}}(\eta)$ is asymptotic to $\frac{1}{4}n$ and that $\frac{A_n - \frac{n}{2}}{\frac{1}{2}\sqrt{n}}$ converges in distribution to the standard Gaussian distribution, for all choices of $\eta \in S_3$. We will prove the following theorem.

Theorem 1.1. *Let $\eta \in S_3$. The longest alternating subsequence A_n satisfies*

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} \log P_n^{\text{av}(\eta)}\left(A_n \ge nx\right) = -I^{\text{alt}}(x),\ x \in [\frac{1}{2}, 1); \\ &\lim_{n\to\infty} \frac{1}{n} \log P_n^{\text{av}(\eta)}\left(A_n \le nx\right) = -I^{\text{alt}}(x),\ x \in (0, \frac{1}{2}], \end{aligned} \tag{1.2}$$

where

$$I^{\text{alt}}(x) = x \log x + (1-x)\log(1-x) + \log 2,\ x \in (0,1). \tag{1.3}$$

Remark 1.2. The expression $I^{\text{alt}}(x)$ above is in fact the relative entropy $H(\mu_x; \mu_{\frac{1}{2}})$ of μ_x with respect to $\mu_{\frac{1}{2}}$, where μ_p denotes the distribution of the Bernoulli random variable X_p satisfying $P(X_p = 1) = 1 - P(X_p = 0) = p$. From Cramèrs theorem, if $\{X_n\}_{n=1}^\infty$ are IID Bernoulli random variables with parameter $\frac{1}{2}$, and $S_n = \sum_{j=1}^n X_j$, then (1.2) also holds with $P_n^{\text{av}(\eta)}(A_n \cdots)$ replaced by $P(S_n \cdots)$. in both places. *It would be quite interesting to understand why this connection arises.*

Remark 1.3. Since $|S_n^{\text{av}(\eta)}| = C_n$, it follows from (1.1)-(1.3) that

$$\begin{aligned} &\lim_{\epsilon\to 0^+} \lim_{n\to\infty} \frac{1}{n} \log |\{\sigma \in S_n^{\text{av}(\eta)} : A_n(\sigma) \ge n(1-\epsilon)\}| = \\ &\lim_{\epsilon\to 0^+} \lim_{n\to\infty} \frac{1}{n} \log |\{\sigma \in S_n^{\text{av}(\eta)} : A_n(\sigma) \le n\epsilon\}| = \log 2. \end{aligned}$$

We now turn to increasing subsequences. In [6], the asymptotic behavior of the expectation $E_n^{\text{av}(\eta)}L_n$ and the variance $v_n(\eta)$ of the longest increasing subsequence L_n under $P_n^{\text{av}(\eta)}$ were obtained for all six permutations $\eta \in S_3$. Of course, the case $\eta = 123$ is trivial. The expectation $E_n^{\text{av}(\eta)}L_n$ is on the order n only for $\eta \in \{231, 312, 321\}$. The limiting distribution of $\frac{L_n - E_n^{\text{av}(\eta)}L_n}{v_n(\eta)}$ was calculated as well, the limit being Gaussian only for $\eta \in \{231, 312\}$. The paper culled a lot of other results in the literature in order to proceed. From the results, it is clear that a classical large deviations result is only possible for $\eta \in \{231, 312, 321\}$. We consider here $\eta \in \{231, 312\}$. In both of these cases, it was shown in [6] that $E_n^{\text{av}(\eta)}L_n = \frac{n+1}{2}$. We will prove the following theorem.

Theorem 1.4. *Let $\eta \in \{231, 312\}$. The longest increasing subsequence L_n satisfies*

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} \log P_n^{\text{av}(\eta)}\left(L_n \geq nx\right) = -I^{inc}(x),\ x \in [\frac{1}{2}, 1];\\ &\lim_{n\to\infty} \frac{1}{n} \log P_n^{\text{av}(\eta)}\left(L_n \leq nx\right) = -I^{inc}(x),\ x \in (0, \frac{1}{2}], \end{aligned} \tag{1.4}$$

where

$$\begin{aligned} &I^{inc}(x) = 2I^{\text{alt}}(x) = 2\big(x\log x + (1-x)\log(1-x) + \log 2\big),\ x \in (0,1);\\ &I^{inc}(1) = \log 4. \end{aligned} \tag{1.5}$$

Remark 1.5. The expression $I^{\text{inc}}(x)$ above is in fact the relative entropy $H(\nu_x;\nu_{\frac{1}{2}})$ of ν_x with respect to $\nu_{\frac{1}{2}}$, where ν_p denotes the distribution of one-half the sum of two independent Bernoulli random variables with parameter p. From Cramèrs theorem, if $\{Y_n\}_{n=1}^\infty$ are IID random variables with distribution $\frac{1}{2}\left(X_{\frac{1}{2}}^{(1)} + X_{\frac{1}{2}}^{(2)}\right)$, where $X_{\frac{1}{2}}^{(1)}$ and $X_{\frac{1}{2}}^{(2)}$ are IID Bernoulli random variables with parameter $\frac{1}{2}$, and $S_n = \sum_{j=1}^n Y_j$, then (1.4) also holds with $P_n^{\text{av}(\eta)}(L_n \cdots)$ replaced by $P(S_n \cdots)$. in both places.

Remark 1.6. The identity permutation in S_n is the only permutation $\sigma \in S_n$ for which $L_n(\sigma) = n$. From this fact and (1.1) alone, it follows that $\lim_{n\to\infty} \frac{1}{n} \log P_n^{\text{av}(\eta)}(L_n \geq n) = -\log 4$.

Remark 1.7. The proof of Theorem 1.4 uses generating functions. The same type of proof could be used to obtain the expectation and the variance of L_n, which is considerably simpler than the proofs of these results in [6].

The proof of Theorem 1.1 is given in section 2 and the proof of Theorem 1.4 in given in section 3.

2 Proof of Theorem 1.1

Recall that the *reverse* of a permutation $\sigma = \sigma_1 \cdots \sigma_n$ is the permutation $\sigma^{\text{rev}} := \sigma_n \cdots \sigma_1$, and the *complement* of σ is the permutation σ^{com} satisfying $\sigma_i^{\text{com}} = n+1-\sigma_i,\ i = 1, \ldots, n$. Let $\sigma^{\text{rev-com}}$ denote the permutation obtained by applying reversal and then complementation to σ (or equivalently, vice versa). Although there are six permutations η in S_3, to prove the theorem, it suffices to consider just two of them—one from $\{231, 213, 312, 132\}$ and one from $\{123, 321\}$. Indeed, for all $n \geq 3$, the operation reversal is a bijection from $S_n^{\text{av}(231)}$ to $S_n^{\text{av}(132)}$ and from $S_n^{\text{av}(123)}$ to $S_n^{\text{av}(321)}$, the operation complementation is a bijection from $S_n^{\text{av}(231)}$ to $S_n^{\text{av}(213)}$ and the operation reversal-complementation is a bijection from $S_n^{\text{av}(231)}$ to $S_n^{\text{av}(312)}$. Furthermore, $A_n(\sigma) = A_n(\sigma^{\text{com}}) = A_n(\sigma^{\text{rev}}) = A_n(\sigma^{\text{rev-com}})$, for $\sigma \in S_n$. Proposition 2.2–iii in [7] shows that the distributions of A_n under $P_n^{\text{av}(231)}$ and under $P_n^{\text{av}(321)}$ coincide. Thus, to prove the theorem, we need only consider the case $\eta = 231$.

For $n \in \mathbb{N}$ and $\sigma \in S_n$, let $A_n^{+,-}(\sigma)$ denote the longest alternating subsequence in σ that begins with an ascent and ends with a descent. An alternating subsequence that begins with an ascent and ends with of descent is of the form $\sigma_{i_1} < \sigma_{i_2} > \sigma_{i_3} < \cdots > \sigma_{i_{2k+1}}$, for $1 \le i_1 < i_2 < \cdots < i_{2k+1} \le n$, with $k \in \mathbb{N}$. If there is no such alternating subsequence in σ, then define $A_n^{+,-}(\sigma) = 1$. Note that $A_n^{+,-}(\sigma)$ takes on positive, odd integral values. It suffices to prove the theorem with $A_n^{+,-}(\sigma)$ in place of A_n since $A_n(\sigma) - A_n^{+,-}(\sigma) \in \{0, 1, 2\}$, for all $\sigma \in S_n$. For convenience in the proof, we define $A_0^{+,-} \equiv 0$.

Let

$$M_n(\lambda) = E_n^{\mathrm{av}(231)} e^{\lambda A_n^{+,-}},\ \lambda \in \mathbb{R}, n \ge 0,$$

denote the moment generating function of $A_n^{+,-}$. The main part of the proof of the theorem is the proof of the following proposition, whose proof we present at the end of this section.

Proposition 2.1.

$$\lim_{n\to\infty} \frac{1}{n} \log M_n(\lambda) = \log(e^\lambda + 1) - \log 2. \tag{2.1}$$

Let $I(x)$ denote the Legendre-Fenchel transform of the function appearing on the right hand side of (2.1); that is,

$$I(x) = \sup_{\lambda\in\mathbb{R}} \Big(\lambda x - \log(e^\lambda + 1) + \log 2\Big),\ x \in \mathbb{R}. \tag{2.2}$$

We have the following proposition.

Proposition 2.2. *The function I, defined in* (2.2)*, is given by*

$$I(x) = \begin{cases} I^{\mathrm{alt}}(x), & x \in (0,1); \\ \log 2, & x \in \{0,1\}; \\ \infty, & x \in \mathbb{R} - [0,1], \end{cases} \tag{2.3}$$

where I^{alt} is as defined in (1.3)*.*

Proposition 2.2 is well-known, as it corresponds to the case of IID symmetric Bernoulli random variables—see Remark 1.2 after Theorem 1.1; thus, we omit the proof, which is a calculus exercise.

Using Propositions 2.1 and 2.2, Theorem 1.1 follows readily from a fundamental theorem in large deviations theory, namely the Gärtner-Ellis theorem. The general version of this theorem [4] has a number of technical assumptions, which in turn require a number of technical definitions. Making certain assumptions, the version in [4] reduces to a slightly easier-to-state form that is sufficient for our needs, and which we now record.

Gärtner-Ellis theorem. *Let $\{X_n\}_{n=1}^\infty$ be a sequence of random variables. Assume that for each $\lambda \in \mathbb{R}$, the limit $\Lambda(\lambda) := \lim_{n\to\infty} \frac{1}{n} \log Ee^{\lambda X_n}$ exists as a real number. Let*

$$\Lambda^*(x) = \sup_{\lambda\in\mathbb{R}} (\lambda x - \Lambda(\lambda))$$

be the Fenchel-Legendre transform of Λ, where $\Lambda^*(x)$ is extended real-valued. Denote by $\mathcal{E}$ the set of points $y \in \mathbb{R}$ for which there exists a $\lambda \in \mathbb{R}$ such that the function $x \to \lambda x - \Lambda^*(x)$ has a unique global maximum at y. Then

i. for any closed set $C \subset \mathbb{R}$,

$$\limsup_{n\to\infty} \frac{1}{n} \log P\left(\frac{X_n}{n} \in C\right) \le - \inf_{x\in C} \Lambda^*(x); \tag{2.4}$$

ii. for any open set $G \subset \mathbb{R}$,

$$\liminf_{n\to\infty} \frac{1}{n} \log P\left(\frac{X_n}{n} \in G\right) \ge - \inf_{x\in G\cap\mathcal{E}} \Lambda^*(x). \tag{2.5}$$

We note that the set $\mathcal{E}$ is called the set of *exposed points* of Λ^*.

Proof of Theorem 1.1. We apply the Gärtner-Ellis theorem with $X_n = A_n^{+,-}$ under the probability measure $P_n^{\text{av}(\eta)}$. By Propositions 2.1 and 2.2, the function Λ is given by the right hand side of (2.1), and the function Λ^* is given by I as in (2.3). In particular then, Λ^* restricted to $(0,1)$ is given by I^{alt} as in (1.3). Thus, the set $\mathcal{E}$ is the set of $y \in \mathbb{R}$ for which there exists a $\lambda \in \mathbb{R}$ such that the extended real-valued function

$$g_\lambda(x) := \begin{cases} \lambda x - x\log x - (1-x)\log(1-x) - \log 2, & x \in (0,1); \\ \lambda x - \log 2, & x \in \{0,1\}; \\ -\infty, & x \in \mathbb{R} - [0,1], \end{cases}$$

has a unique global maximum at y. For every $\lambda \in \mathbb{R}$, one has $g''_\lambda(x) < 0$, for $x \in (0,1)$; thus, g_λ restricted to $[0,1]$ is convex. Also, $g'_\lambda(x) = \lambda + \log\frac{1-x}{x}$. Define $\lambda_y = -\log\frac{1-y}{y}$, for $y \in (0,1)$. Then g_{λ_y} has a unique maximum at y. Thus, $(0,1) \subset \mathcal{E}$.

Let $x \in [\frac{1}{2}, 1)$. Applying (2.4) with $C = [x,\infty)$ and applying (2.5) with $G = (x,\infty)$, and using the fact that I^{alt} is increasing on C and continuous at x, we obtain

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n} \log P_n^{\text{av}(\eta)} \left(A_n^{+,-} \ge nx\right) \le -I^{\text{alt}}(x); \\ &\liminf_{n\to\infty} \frac{1}{n} \log P_n^{\text{av}(\eta)} \left(A_n^{+,-} > nx\right) \ge -I^{\text{alt}}(x). \end{aligned} \tag{2.6}$$

The first line of (1.2) follows from (2.6). The second line of (1.2) follows in a similar fashion. □

It remains to prove Proposition 2.1.

Proof of Proposition 2.1. Every permutation $\sigma \in S_k^{\text{av}(231)}$ has the property that if $\sigma_j = n$, then the numbers $\{1,\dots,j-1\}$ appear in the first $j-1$ positions in σ (and then of course, the numbers $\{j,\dots,n-1\}$ appear in the last $n-j$ positions in σ.) From this fact, along with the fact that $|S_n^{\text{av}(\eta)}| = C_n$, it follows that

$$P_n^{\text{av}(231)}(\sigma_j = n) = \frac{C_{j-1}C_{n-j}}{C_n}, \text{ for } j \in [n], \tag{2.7}$$

where $C_0 = 1$. It also follows that under the conditional measure $P_n^{\text{av}(231)}|\{\sigma_j = n\}$, the permutation $\sigma_1\cdots\sigma_{j-1} \in S_{j-1}$ has the distribution $P_{j-1}^{\text{av}(231)}$, the permutation $\sigma'_{j+1}\cdots\sigma'_n$ has the distribution $P_{n-j}^{\text{av}(231)}$, where $\sigma'_k = \sigma_k - j + 1$, for $k = j+1,\dots,n$, and these two permutations are independent. From this last fact and the definition of $A_n^{+,-}$, it follows that

$$\begin{aligned} &A_n^{+,-}|\{\sigma_j = n\} \overset{\text{dist}}{=} A_{j-1}^{+,-} + 1 + A_{n-j}^{+,-},\ n \ge 3,\ j = 2,\dots,n-1; \\ &A_n^{+,-}|\{\sigma_1 = n\} \overset{\text{dist}}{=} A_n^{+,-}|\{\sigma_n = n\} \overset{\text{dist}}{=} A_{n-1}^{+,-},\ n \ge 2, \end{aligned} \tag{2.8}$$

where on the right hand side of (2.8), for any k, the random variable $A_k^{+,-}$ is considered on S_k under the measure $P_k^{\text{av}(231)}$, and where on the right hand side of the first line in (2.8), $A_{j-1}^{+,1}$ and $A_{n-j}^{+,-}$ are independent. Indeed, the first line of (2.8) follows because if $\sigma_j = n$, then a longest alternating subsequence in σ that begins with an ascent and ends with a descent is obtained by considering a longest alternating subsequence that begins with an ascent and ends with a descent from the first $j-1$ entries of σ, then using $\sigma_j = n$ for an additional ascent, then considering a longest alternating subsequence that begins with an ascent and ends with a descent from the last $n-j$ entries of σ, and concatenating these three pieces. The second line of (2.8) follows because if $\sigma_1 = n$ or $\sigma_n = n$, then this entry cannot contribute to a longest alternating subsequence that begins with an ascent and ends with a descent.

From (2.7) and (2.8), we have

$$
\begin{aligned}
&M_n(\lambda)=\sum_{j=1}^{n}P_n^{\mathrm{av}(231)}(\sigma_j=n)E_n^{\mathrm{av}(231)}(e^{\lambda A_n^{+,-}}|\sigma_j=n)=\\
&\left(\sum_{j=2}^{n-1}\frac{C_{j-1}C_{n-j}}{C_n}e^{\lambda}E_{j-1}^{\mathrm{av}(231)}e^{\lambda A_{j-1}^{+,-}}E_{n-j}^{\mathrm{av}(231)}e^{\lambda A_{n-j}^{+,-}}\right)+2\frac{C_{n-1}}{C_n}E_{n-1}^{\mathrm{av}(231)}e^{\lambda A_{n-1}^{+,-}}=\\
&e^{\lambda}\sum_{j=2}^{n-1}\frac{C_{j-1}C_{n-j}}{C_n}M_{j-1}(\lambda)M_{n-j}(\lambda)+2\frac{C_{n-1}}{C_n}M_{n-1}(\lambda),\ n\geq 3.
\end{aligned}
\tag{2.9}
$$

Multiplying the leftmost expression and the rightmost expression in (2.9) by C_nt^n, we have

$$
\begin{aligned}
&C_nM_n(\lambda)t^n=e^{\lambda}t\left(\sum_{j=2}^{n-1}C_{j-1}M_{j-1}(\lambda)C_{n-j}M_{n-j}(\lambda)\right)t^{n-1}+\\
&2tC_{n-1}M_{n-1}(\lambda)t^{n-1},\ n\geq 3.
\end{aligned}
\tag{2.10}
$$

Summing over n gives

$$
\begin{aligned}
&\sum_{n=3}^{\infty}C_nM_n(\lambda)t^n=e^{\lambda}t\sum_{n=3}^{\infty}\left(\sum_{j=2}^{n-1}C_{j-1}M_{j-1}(\lambda)C_{n-j}M_{n-j}(\lambda)\right)t^{n-1}+\\
&2t\sum_{n=3}^{\infty}C_{n-1}M_{n-1}(\lambda)t^{n-1}.
\end{aligned}
\tag{2.11}
$$

Define

$$
G_\lambda(t)=\sum_{n=0}^{\infty}C_nM_n(\lambda)t^n.
\tag{2.12}
$$

For use in some of the calculations below, note that $C_0=C_1=1$, $C_2=2$, $M_0(\lambda)=1$ and $M_1(\lambda)=M_2(\lambda)=e^{\lambda}$. We have

$$
G_\lambda^2(t)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}C_kM_k(\lambda)C_{n-k}M_{n-k}(\lambda)\right)t^n.
\tag{2.13}
$$

The double sum on the right hand side of (2.11) can be written as

$$
\begin{aligned}
&\sum_{n=3}^{\infty}\left(\sum_{j=2}^{n-1}C_{j-1}M_{j-1}(\lambda)C_{n-j}M_{n-j}(\lambda)\right)t^{n-1}=\\
&\sum_{n=3}^{\infty}\left(\sum_{k=1}^{n-2}C_kM_k(\lambda)C_{n-1-k}M_{n-1-k}(\lambda)\right)t^{n-1}=\\
&\sum_{n=3}^{\infty}\left(\sum_{k=0}^{n-1}C_kM_k(\lambda)C_{n-1-k}M_{n-1-k}(\lambda)\right)t^{n-1}-2\sum_{n=3}^{\infty}C_{n-1}M_{n-1}(\lambda)t^{n-1}=\\
&\sum_{m=2}^{\infty}\left(\sum_{k=0}^{m}C_kM_k(\lambda)C_{m-k}M_{m-k}(\lambda)\right)t^m-2\sum_{m=2}^{\infty}C_mM_m(\lambda)t^m=\\
&\left(G_\lambda^2(t)-2e^{\lambda}t-1\right)-2\left(G_\lambda(t)-e^{\lambda}t-1\right)=G_\lambda^2(t)-2G_\lambda(t)+1,
\end{aligned}
\tag{2.14}
$$

where (2.12) and (2.13) has been used for the penultimate equality. From (2.11), (2.12) and (2.14), we obtain

$$
\left(G_\lambda(t)-2e^{\lambda}t^2-e^{\lambda}t-1\right)=e^{\lambda}t\left(G_\lambda^2(t)-2G_\lambda(t)+1\right)+2t\left(G_\lambda(t)-e^{\lambda}t-1\right),
$$

or equivalently

$$e^{\lambda}tG_{\lambda}^{2}(t)+\big(2(1-e^{\lambda})t-1\big)G_{\lambda}(t)+2(e^{\lambda}-1)t+1=0.$$

Since $G_\lambda(0)=1$, the quadratic formula yields

$$G_{\lambda}(t)=\frac{1-2(1-e^{\lambda})t-\sqrt{\big(2(1-e^{\lambda})t-1\big)^{2}-4e^{\lambda}t\big(2(e^{\lambda}-1)t+1\big)}}{2e^{\lambda}t},$$

which we rewrite as

$$G_{\lambda}(t)=\frac{1-2(1-e^{\lambda})t-\sqrt{4(1-e^{2\lambda})t^{2}-4t+1}}{2e^{\lambda}t}. \tag{2.15}$$

From (2.15), it follows that the radius of convergence R_λ of the power series representing G_λ is the smaller of the absolute values of the two roots of the quadratic polynomial $4(1-e^{2\lambda})t^2-4t+1$. The roots of this polynomial are $\frac{1\pm e^{\lambda}}{2(1-e^{2\lambda})}$. Thus, we have $R_\lambda=\frac{|e^{\lambda}-1|}{2|e^{2\lambda}-1|}=\frac{1}{2(e^{\lambda}+1)}$. Consequently, from (2.12), we obtain

$$\limsup_{n\to\infty}(C_nM_n(\lambda))^{\frac{1}{n}}=2(e^{\lambda}+1). \tag{2.16}$$

From (1.1) and (2.16), we obtain

$$\limsup_{n\to\infty}\frac{1}{n}\log M_n(\lambda)=\log(e^{\lambda}+1)-\log 2. \tag{2.17}$$

In light of (2.17), to complete the proof of the proposition it suffices to show that the limit $\lim_{n\to\infty}\frac{1}{n}\log M_n(\lambda)$ exists.

From (2.12), (2.15) and (1.1), in order to show that $\lim_{n\to\infty}\frac{1}{n}\log M_n(\lambda)$ exists, it suffices to show that $\lim_{n\to\infty}a_n^{\frac{1}{n}}$ exists, where a_n is the coefficient of t^n in the power series expansion

$$\sqrt{4(1-e^{2\lambda})t^{2}-4t+1}=\sum_{n=0}^{\infty}a_nt^n. \tag{2.18}$$

Intuitively, it seems "obvious" that this limit exists, but unfortunately, we don't have a real quick proof. Recall that the two roots of $4(1-e^{2\lambda})t^2-4t+1$ are $r_1=\frac{1-e^{\lambda}}{2(1-e^{2\lambda})}>0$ and $r_2=\frac{1+e^{\lambda}}{2(1-e^{2\lambda})}$. (We suppress the dependence on λ.) We have $r_1<|r_2|$. We write

$$4(1-e^{2\lambda})t^{2}-4t+1=(1-\frac{t}{r_1})(1-\frac{t}{r_2}). \tag{2.19}$$

The Taylor series of $\sqrt{1-x}$ around $x=0$ is given by

$$\begin{aligned}&\sqrt{1-x}=\sum_{n=0}^{\infty}(-1)^n\binom{\frac12}{n}x^n=1+\sum_{n=1}^{\infty}(-1)^n\frac{\frac12(\frac12-1)\cdots(\frac12-(n-1))}{n!}x^n=\\&1-\sum_{n=1}^{\infty}\frac{(2n-2)!}{n!(n-1)!2^{2n-1}}x^n.\end{aligned} \tag{2.20}$$

Thus, from (2.19) and (2.20), we have

$$\begin{aligned}&\sqrt{4(1-e^{2\lambda})t^{2}-4t+1}=\sqrt{1-\frac{t}{r_1}}\sqrt{1-\frac{t}{r_2}}=\\&1-\sum_{n=1}^{\infty}\frac{(2n-2)!}{n!(n-1)!2^{2n-1}r_1^n}t^n-\sum_{n=1}^{\infty}\frac{(2n-2)!}{n!(n-1)2^{2n-1}!r_2^n}t^n\\&+\sum_{n=2}^{\infty}\left(\sum_{j=1}^{n}\frac{(2j-2)!}{j!(j-1)!2^{2j-1}\,r_1^j}\frac{(2(n-j)-2)!}{(n-j)!(n-j-1)!2^{2(n-j)-1}\,r_2^{n-j}}\right)t^n.\end{aligned} \tag{2.21}$$

Using Stirling's formula, one finds that the expression $\frac{(2m-2)!}{m!(m-1)!2^{2m-1}}$, for $m \in \mathbb{N}$, decays to zero on the order $m^{-\frac{3}{2}}$. In particular then, this expression is bounded and has subexponential decay. Using this with (2.18) and (2.21) and the fact that $r_1 < |r_2|$, it follows that $\lim_{n\to\infty} a_n^{\frac{1}{n}} = \frac{1}{r_1}$. This completes the proof of Proposition 2.1. □

3 Proof of Theorem 1.4

Consider $\eta \in \{231, 312\}$. For convenience, let $L_0 = 0$. Let

$$M_n(\lambda) = E_n^{\text{av}(\eta)} e^{\lambda L_n},\ \lambda \in \mathbb{R},\ n \geq 0,$$

denote the moment generating function of L_n. The main part of the proof of the theorem is the proof of the following proposition, whose proof we present at the end of this section.

Proposition 3.1. *Let $\eta \in \{231, 312\}$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \log M_n(\lambda) = 2\log(e^{\frac{\lambda}{2}} + 1) - \log 4. \tag{3.1}$$

Let $I(x)$ denote the Legendre-Fenchel transform of the function appearing on the right hand side of (3.1); that is,

$$I(x) = \sup_{\lambda \in \mathbb{R}} \Big(\lambda x - 2\log(e^{\frac{\lambda}{2}} + 1) + \log 4\Big),\ x \in \mathbb{R}. \tag{3.2}$$

The following proposition is equivalent to Proposition 2.2.

Proposition 3.2. *The function I, defined in* (3.2)*, is given by*

$$I(x) = \begin{cases} I^{\text{inc}}(x), & x \in (0,1); \\ \log 4, & x \in \{0,1\}; \\ \infty, & x \in \mathbb{R} - [0,1], \end{cases}$$

where I^{inc} is as defined in (1.5)*.*

Proof of Theorem 1.4. In light of Propositions 3.1 and 3.2, (1.4) for $x \in (0,1)$ follows from the Gärtner-Ellis theorem with the same explanation as given in the proof of Theorem 1.1. The case $x = 1$ is explained in Remark 1.6 following the statement of the theorem. □

We now turn to the proof of Proposition 3.1.

Proof of Proposition 3.1. In the first two paragraphs we begin the proof for the case $\eta = 231$. In the paragraph after that we explain why the same proof works for $\eta = 312$. After that we continue to the end of the proof for $\eta = 231$. We have

$$M_n(\lambda) = P_n^{\text{av}(231)} e^{\lambda L_n},\ \lambda \in \mathbb{R},\ n \geq 0.$$

From the discussion in the first paragraph of the proof of Proposition 2.1 and from the definition of L_n, it follows that

$$\begin{aligned} &L_n|\{\sigma_j = n\} \overset{\text{dist}}{=} L_{j-1} + L_{n-j},\ n \geq 2,\ j = 1, \ldots, n-1; \\ &L_n|\{\sigma_n = n\} \overset{\text{dist}}{=} L_{n-1} + 1,\ n \geq 2, \end{aligned} \tag{3.3}$$

where on the right hand side of (3.3), for any k, the random variable L_k is considered on S_k under the measure $P_k^{\text{av}(231)}$, and where on the right hand side of the first line in (3.3), L_{j-1} and L_{n-j} are independent. Indeed, the first line in (3.3) follows because if $\sigma_j = n$, then a longest increasing subsequence is obtained by considering a longest increasing subsequence from among the first $j-1$ entries of σ, considering a longest increasing subsequence from among the last $n-j$ entries of σ, and concatenating them. The second line of (3.3) follows because if $\sigma_n = n$, then this last entry always belongs to a longest increasing subsequence.

From (3.3) and (2.7), we have

$$\begin{aligned}
&M_n(\lambda) = \sum_{j=1}^{n} P_n^{\mathrm{av}(231)}(\sigma_j = n) E_n^{\mathrm{av}(231)}(e^{\lambda L_n}|\sigma_j = n) = \\
&\left(\sum_{j=1}^{n-1} \frac{C_{j-1}C_{n-j}}{C_n} E_{j-1}^{\mathrm{av}(231)} e^{\lambda L_{j-1}} E_{n-j}^{\mathrm{av}(231)} e^{\lambda L_{n-j}}\right) + \frac{C_{n-1}}{C_n} e^{\lambda} E_{n-1}^{\mathrm{av}(231)} e^{\lambda L_{n-1}} = \\
&\sum_{j=1}^{n-1} \frac{C_{j-1}C_{n-j}}{C_n} M_{j-1}(\lambda) M_{n-j}(\lambda) + e^{\lambda} \frac{C_{n-1}}{C_n} M_{n-1}(\lambda),\ n \ge 2.
\end{aligned} \tag{3.4}$$

In the case $\eta = 312$, the same type of reasoning as in (2.7) shows that $P_n^{\mathrm{av}(312)}(\sigma_j = 1) = \frac{C_{j-1}C_{n-j}}{C_n}$. Also, the same reasoning as in (3.3) gives

$$\begin{aligned}
&L_n|\{\sigma_j = 1\} \stackrel{\mathrm{dist}}{=} L_{j-1} + L_{n-j},\ n \ge 2,\ j = 2, \ldots, n; \\
&L_n|\{\sigma_1 = 1\} \stackrel{\mathrm{dist}}{=} L_{n-1} + 1,\ n \ge 2,
\end{aligned} \tag{3.5}$$

where on the right hand side of (3.5), for any k, the random variable L_k is considered on S_k under the measure $P_k^{\mathrm{av}(312)}$, and where on the right hand side of the first line in (3.5), L_{j-1} and L_{n-j} are independent. Using the these facts, one finds that the Laplace transform for this case also satisfies (3.4) Thus, it suffices to continue just for the case $\eta = 231$.

Multiplying the leftmost and the rightmost expressions in (3.4) by $C_n t^n$, we have

$$\begin{aligned}
&C_n M_n(\lambda) t^n = t \left(\sum_{j=1}^{n-1} C_{j-1} M_{j-1}(\lambda) C_{n-j} M_{n-j}(\lambda)\right) t^{n-1} + \\
&e^{\lambda} t C_{n-1} M_{n-1}(\lambda) t^{n-1},\ n \ge 2.
\end{aligned} \tag{3.6}$$

Summing over n gives

$$\begin{aligned}
&\sum_{n=2}^{\infty} C_n M_n(\lambda) t^n = t \sum_{n=2}^{\infty} \left(\sum_{j=1}^{n-1} C_{j-1} M_{j-1}(\lambda) C_{n-j} M_{n-j}(\lambda)\right) t^{n-1} + \\
&e^{\lambda} t \sum_{n=2}^{\infty} C_{n-1} M_{n-1}(\lambda) t^{n-1}.
\end{aligned} \tag{3.7}$$

Define

$$G_\lambda(t) = \sum_{n=0}^{\infty} C_n M_n(\lambda) t^n. \tag{3.8}$$

For use in some of the calculations below, note that $C_0 = C_1 = 1$, $M_0(\lambda) = 1$ and $M_1(\lambda) = e^{\lambda}$. We have

$$G_\lambda^2(t) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} C_k M_k(\lambda) C_{n-k} M_{n-k}(\lambda)\right) t^n. \tag{3.9}$$

The double sum on the right hand side of (3.7) can be written as

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left(\sum_{j=1}^{n-1} C_{j-1}M_{j-1}(\lambda)C_{n-j}M_{n-j}(\lambda)\right)t^{n-1} = \\
&\sum_{n=2}^{\infty}\left(\sum_{k=0}^{n-2} C_{k}M_{k}(\lambda)C_{n-1-k}M_{n-1-k}(\lambda)\right)t^{n-1} = \\
&\sum_{n=2}^{\infty}\left(\sum_{k=0}^{n-1} C_{k}M_{k}(\lambda)C_{n-1-k}M_{n-1-k}(\lambda)\right)t^{n-1} - \sum_{n=2}^{\infty} C_{n-1}M_{n-1}(\lambda)t^{n-1} = \\
&\sum_{m=1}^{\infty}\left(\sum_{k=0}^{m} C_{k}M_{k}(\lambda)C_{m-k}M_{m-k}(\lambda)\right)t^{m} - \sum_{m=1}^{\infty} C_{m}M_{m}(\lambda)t^{m} = \\
&\left(G_\lambda^2(t)-1\right)-\left(G_\lambda(t)-1\right) = G_\lambda^2(t)-G_\lambda(t),
\end{aligned} \tag{3.10}$$

where (3.8) and (3.9) has been used for the penultimate equality. From (3.7), (3.8) and (3.10) we obtain

$$\left(G_\lambda(t)-e^\lambda t-1\right) = t\left(G_\lambda^2(t)-G_\lambda(t)\right)+e^\lambda t\left(G_\lambda(t)-1\right),$$

or equivalently

$$tG_\lambda^2(t)+\left((e^\lambda-1)t-1\right)G_\lambda(t)+1=0.$$

Since $G_\lambda(0)=1$, the quadratic formula yields

$$G_\lambda(t)=\frac{1-(e^\lambda-1)t-\sqrt{\left((e^\lambda-1)t-1\right)^2-4t}}{2t},$$

which we rewrite as

$$G_\lambda(t)=\frac{1-(e^\lambda-1)t-\sqrt{(e^\lambda-1)^2t^2-2(e^\lambda+1)t+1}}{2t}. \tag{3.11}$$

From (3.11), it follows that the radius of convergence R_λ of the power series representing G_λ is the smaller of the absolute values of the two roots of the quadratic polynomial $(e^\lambda-1)^2t^2-2(e^\lambda+1)t+1$. After a bit of algebra, one finds that the two roots are $\frac{(e^{\frac{\lambda}{2}}\pm 1)^2}{(e^\lambda-1)^2}$. Thus, we have $R_\lambda=\frac{(e^{\frac{\lambda}{2}}-1)^2}{(e^\lambda-1)^2}$. Consequently, from (3.8), we obtain

$$\limsup_{n\to\infty}(C_nM_n(\lambda))^{\frac{1}{n}}=\frac{(e^\lambda-1)^2}{(e^{\frac{\lambda}{2}}-1)^2}. \tag{3.12}$$

From (1.1) and (3.12), we obtain

$$\limsup_{n\to\infty}\frac{1}{n}\log M_n(\lambda)=2\log(e^\lambda-1)-2\log(e^{\frac{\lambda}{2}}-1)-\log 4=2\log(e^{\frac{\lambda}{2}}+1)-\log 4. \tag{3.13}$$

In light of (3.13), to complete the proof of the proposition it suffices to show that the limit $\lim_{n\to\infty}\frac{1}{n}\log M_n(\lambda)$ exists. The proof of this is exactly the same as the corresponding proof in section 2. □

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