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Extended Lévy's theorem for a two-sided reflection*

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Abstract

We aim to set forth an extension of the result found in paper [6], which finds an explicit realisation of a reflecting Brownian motion with drift $-\mu$, started at x, reflecting above zero, and its local time at zero. In this paper we find a corresponding realisation for a reflecting Brownian motion with drift $-\mu$, started at x, reflected both above zero and below one, along with a corresponding expression in terms of associated local times, namely as the difference between the local time at zero and the local time at one.

Keywords: reflecting Brownian motion with drift; two-sided reflecting Brownian motion; Lévy's theorem; local time; diffusion process; normal reflection; Skorokhod map.
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1 Introduction

Lévy's classic theorem gives an explicit realisation for a standard Brownian motion reflected above zero and its local time at zero, for the full result see (1) in paper [3]. This was subsequently extended to a reflecting Brownian motion with drift in [3] and then to a reflecting Brownian motion with a non-zero initial point in [6]. We wish to extend this further to a two-sided reflecting Brownian motion and therefore we search for the explicit realisation of a process with this double reflection required. The first such result appears as the two-sided reflection of the sum of an Itô integral and an integral with respect to time, see Lemma 2 in [2]. Subsequently similar more general results begin to appear, and the result which we utilise in this paper arises in (1.8) of paper [4]. For a comprehensive history of the progression of these results for a two-sided reflection see [4, p. 171]. Result (1.8) in paper [4] (see also [5]) tells us that for the space of right-continuous functions with left limits taking values in $\mathbb{R}$, the double Skorokhod map $\Gamma_{0,a}$ on $[0,a]$ has the explicit realisation

$$\Gamma_{0,a}(\psi)(t) = \psi(t) - \big[(\psi(0)-a)^+ \wedge \inf_{u\in[0,t]} \psi(u)\big] \vee \sup_{s\in[0,t]} \big[(\psi(s)-a) \wedge \inf_{u\in[s,t]} \psi(u)\big] \tag{1.1}$$

for $t \geq 0$ and $a > 0$. From (1.1), one finds an explicit realisation for a reflecting Brownian motion $R^{-\mu,x}_{0,1}$ with drift $-\mu$, started at x, reflected both above zero and below one. We shall denote this realisation as $Z^x = (Z^x_t)_{t\geq 0}$.

When finding an explicit realisation $Z^x = (Z^x_t)_{t\geq 0}$ for the reflecting Brownian motion $R^{-\mu,x}_{0,1}$, for continuity and to allow us to draw similarities between the behaviour of the

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two processes, we wish for our Z^x to be of a similar form to the Z^x in Theorem 3.1 of [6]. From (1.1), let $\psi(t) = x - B_t^\mu$ where $B_t^\mu = B_t + \mu t$ and $B = (B_t)_{t\geq 0}$ is a standard Brownian motion started at zero, $\mu \in \mathbb{R}$ is a given and fixed constant, and take $a = 1$. Explicitly

$$\begin{aligned} Z_t^x &= x - B_t^\mu - \big[(x-1)^+ \wedge \inf_{u\in[0,t]}(x - B_u^\mu)\big] \vee \sup_{s\in[0,t]} \big[(x - B_s^\mu - 1) \wedge \inf_{u\in[s,t]}(x - B_u^\mu)\big] \quad (1.2) \\ &= -B_t^\mu - \Big(\big[0 \wedge \inf_{u\in[0,t]}(x - B_u^\mu)\big] \vee \sup_{s\in[0,t]} \big[(x - B_s^\mu - 1) \wedge \inf_{u\in[s,t]}(x - B_u^\mu)\big] - x\Big) \\ &= -B_t^\mu - \big[-x \wedge - \sup_{u\in[0,t]} B_u^\mu\big] \vee \sup_{s\in[0,t]} \big[-(B_s^\mu + 1) \wedge - \sup_{u\in[s,t]} B_u^\mu\big] \\ &= -B_t^\mu - \big[-(x \vee \sup_{u\in[0,t]} B_u^\mu)\big] \vee - \inf_{s\in[0,t]} \big[(B_s^\mu + 1) \vee \sup_{u\in[s,t]} B_u^\mu\big] \\ &= -B_t^\mu + \big(x \vee S_{0,t}^\mu\big) \wedge \inf_{s\in[0,t]} \big[(1 + B_s^\mu) \vee S_{s,t}^\mu\big] \end{aligned}$$

where $S_{s,t}^\mu \coloneqq \sup_{s\leq u\leq t} B_u^\mu$ for $0 \leq s \leq t$.

Remark 1.1. Notice the correspondence between the realisation of the Brownian motion reflecting above zero, given by

$$Y_t^x = -B_t^\mu + (x \vee S_{0,t}^\mu) \tag{1.3}$$

and the realisation of the Brownian motion reflecting between zero and one, given by

$$Z_t^x = -B_t^\mu + \big(x \vee S_{0,t}^\mu\big) \wedge \inf_{s\in[0,t]} \big[(1 + B_s^\mu) \vee S_{s,t}^\mu\big]. \tag{1.4}$$

It is also interesting to note that we can find a different expression for a reflecting Brownian motion with drift $-\mu$, started at x, reflecting both above zero and below one. This expression can be found either directly from (1.1) or via (1.2). We choose the latter method as it is easier to immediately see that the new expression does indeed satisfy our requirements. In order to find this new expression, take (1.2), multiply by -1, add 1, substitute the underlying Brownian motion process B^μ with $-B^\mu$ in order to account for the change in drift caused by multiplying by -1, and then denote the initial point of this new expression as x. This gives the following

$$\widetilde{Z}_t^x = -B_t^\mu + (x \wedge (I_{0,t}^\mu + 1)) \vee \sup_{s\in[0,t]} \big[B_s^\mu \wedge (I_{s,t}^\mu + 1)\big] \tag{1.5}$$

where $I_{s,t}^\mu \coloneqq \inf_{s\leq u\leq t} B_u^\mu$. Here we notice the correspondence with the realisation of the Brownian motion reflecting below one, given by

$$\widetilde{Y}_t^x = -B_t^\mu + (x \wedge (I_{0,t}^\mu + 1)). \tag{1.6}$$

Remark 1.2. In the case when our two-sided reflecting Brownian motion starts at the point $x = 0$, we have from (1.2) that

$$Z_t^0 = -B_t^\mu + S_{0,t}^\mu \wedge \inf_{s\in[0,t]} \big[(1 + B_s^\mu) \vee S_{s,t}^\mu\big] \tag{1.7}$$

We thus know, as can also be seen in Remark 2.2 in [4], that

$$Z_t^0 = -B_t^\mu + \inf_{s\in[0,t]} \big[((1 + B_s^\mu) \wedge S_{0,t}^\mu) \vee S_{s,t}^\mu\big] \tag{1.8}$$

due to the fact that $S_{s,t}^\mu \leq S_{0,t}^\mu$ for all $0 \leq s \leq t$. Furthermore we may also note by Theorem 6.2 in [1] that

$$Z_t^0 = \sup_{s\in[0,t]} \big[(-B_t^\mu + B_s^\mu) \wedge \inf_{u\in[s,t]}(1 - B_t^\mu + B_u^\mu)\big]. \tag{1.9}$$

This final realisation may be particularly useful in the case of the two-sided reflecting Brownian motion starting at $x = 0$ as it is the simplest form of realisation we have seen thus far.

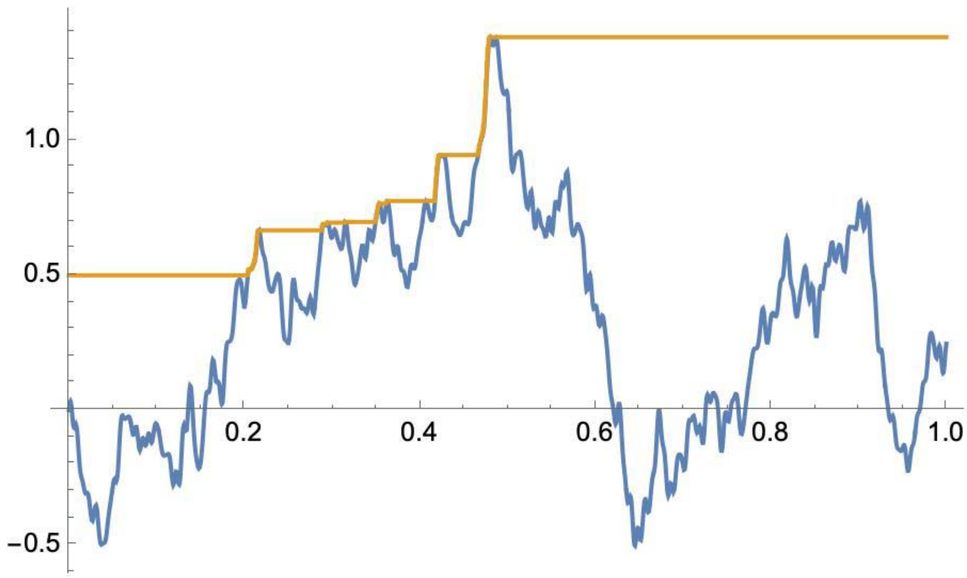

Figure 1: The blue line shows B^μ, whilst the orange line depicts $(x \vee S^\mu)$ for $x = 0.5$.

2 Extended Lévy's theorem for a two-sided reflection

Theorem 3.1 from [6] states that we have the following explicit realisation for a reflecting Brownian motion $R_{0+}^{-\mu,x}$ with drift $-\mu$, started at $x \geq 0$, and reflected above zero

$$Y^x = (x \vee S^\mu) - B^\mu = ((x \vee S_t^\mu) - B_t^\mu)_{t\geq 0}. \tag{2.1}$$

Furthermore, we have the following identity in law

$$((x \vee S^\mu) - B^\mu, (x \vee S^\mu) - x) \overset{law}{=} (R_{0+}^{-\mu,x}, \ell^0(R_{0+}^{-\mu,x})) \tag{2.2}$$

where $\ell^0(R_{0+}^{-\mu,x})$ is the local time of $R_{0+}^{-\mu,x}$ at 0. We now take a closer look at what is going on here in order to allow us to carry out the same insight in the case of the double reflection.

Obviously, $Y^x = (x \vee S^\mu) - B^\mu \geq 0$ and $(x \vee S^\mu) \geq B^\mu$ with equality when $Y^x = 0$, thus we have that the process $(x \vee S^\mu)$ increases only when Y^x, the reflecting Brownian motion,"spends time at zero", as can be seen in Figure 1. Thus it is intuitive that there may be a correspondence between $(x \vee S^\mu)$ and the local time of Y^x at zero, as was indeed shown in the aforementioned paper [6].

Now we consider similarly the case of what we will shortly show is a two-sided reflecting Brownian motion Z^x. Obviously,

$$0 \leq Z_t^x = \left(x \vee S_{0,t}^\mu\right) \wedge \inf_{s\in[0,t]} \left[(1 + B_s^\mu) \vee S_{s,t}^\mu\right] - B_t^\mu \leq 1 \tag{2.3}$$

and

$$B_t^\mu \leq Q_t^x := \left(x \vee S_{0,t}^\mu\right) \wedge \inf_{s\in[0,t]} \left[(1 + B_s^\mu) \vee S_{s,t}^\mu\right] \leq 1 + B_t^\mu \tag{2.4}$$

with equality at the lower bound when $Z_t^x = 0$ and at the upper bound when $Z_t^x = 1$. This can be visualised well with Figure 2.

Thus, Figure 2 suggests that the process $\left(\left(x \vee S_{0,t}^\mu\right) \wedge \inf_{s\in[0,t]} \left[(1 + B_s^\mu) \vee S_{s,t}^\mu\right]\right)_{t\geq 0}$ increases only when Z^x "spends time at zero", and decreases only when Z^x "spends time at one". Thus it is intuitive that there may be a correspondence between $\left(\left(x \vee S_{0,t}^\mu\right) \wedge\right.$

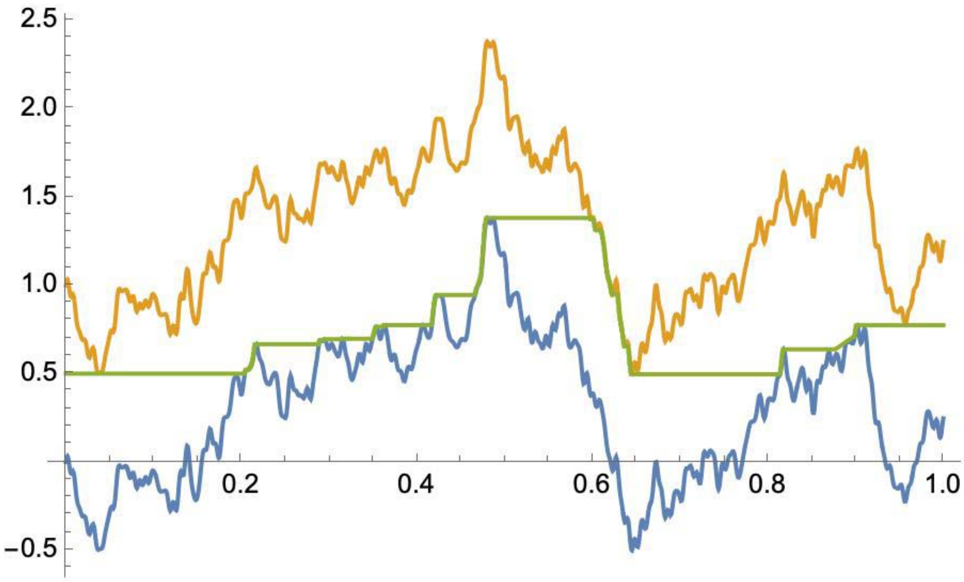

Figure 2: The blue line show the Brownian motion B^{μ}, the orange line shows $1+B^{\mu}$, whilst the green line depicts $Q_t^x = \big(x \vee S_{0,t}^{\mu}\big) \wedge \inf_{s\in[0,t]} \big[(1+B_s^{\mu}) \vee S_{s,t}^{\mu}\big]$ for $x = 0.5$.

$\inf_{s\in[0,t]} \big[(1+B_s^{\mu}) \vee S_{s,t}^{\mu}\big])_{t\geq 0}$ and the local time of Z^x at zero minus the local time of Z^x at one. This is indeed the case and is the first result that we shall prove.

We continue to use the notation as in the introduction.

Theorem 2.1 (Extended Lévy's theorem for a two-sided reflection)**.** *The following identity in law holds*

$$\Big(\big(x \vee S_{0,t}^{\mu}\big) \wedge \inf_{s\in[0,t]} \big[(1+B_s^{\mu}) \vee S_{s,t}^{\mu}\big] - B_t^{\mu}, \big(x \vee S_{0,t}^{\mu}\big) \wedge \inf_{s\in[0,t]} \big[(1+B_s^{\mu}) \vee S_{s,t}^{\mu}\big] - x\Big) \quad (2.5)$$
$$\stackrel{law}{=} (R_{0,1}^{-\mu,x}, \ell_t^0(R_{0,1}^{-\mu,x}) - \ell_t^1(R_{0,1}^{-\mu,x}))$$

where $\ell^a(R_{0,1}^{-\mu,x})$ is the local time of $R_{0,1}^{-\mu,x}$ at a as defined in (2.18) *below.*

Proof of Theorem 2.1. First we show that $R_{0,1}^{-\mu,x} \stackrel{law}{=} Z^x$. That is, we show that $Z^x = (Z_t^x)_{t\geq 0}$ is a realisation for a Brownian motion with drift $-\mu$ reflecting between 0 and 1. First recall that a Brownian motion starting at x with drift ν reflecting between 0 and 1 is a continuous strong Markov process with infinitesimal generator $\mathbb{L}^{\nu}$ defined by

$$\mathbb{L}^{\nu} f = \nu f' + \frac{1}{2} f'' \quad (2.6)$$

which acts on the domain $\mathcal{D}(\mathbb{L}^{\nu})$ given by

$$\mathcal{D}(\mathbb{L}^{\nu}) = \{f \in C_b^2((0,1)) \,|\, f'(0+) = 0, f'(1-) = 0\} \quad (2.7)$$

where $C_b^2((0,1))$ denotes the space of functions f such that $f, f', f'' \in C_b((0,1))$ for $C_b((0,1))$ the space of continuous, bounded functions on $(0,1)$, and $g(a+)$ and $g(a-)$ denote the limits of $g(y)$ as y tends to a from above and below, respectively. We know that $(\mathbb{L}^{\nu}, \mathcal{D}(\mathbb{L}^{\nu}))$ uniquely generates a family of measures $\{P_x \,|\, x \in (0,1)\}$ which define the law of our reflecting Brownian motion process.

Note that as in [6, p. 3-4], in order to show that Z_t^x is a reflecting Brownian motion of the type we want it is sufficient to show that for all $f \in \mathcal{D}(\mathbb{L}^{-\mu})$

$$f(Z_t^x) - f(Z_0^x) - \int_0^t (\mathbb{L}^{-\mu} f)(Z_s^x) ds \quad (2.8)$$

is a martingale under P_x, all $x \in (0,1)$.

Take $f \in \mathcal{D}(\mathbb{L}^{-\mu})$ and apply Itô's lemma to Z_t^x

$$\begin{aligned} f(Z_t^x) &= x + \int_0^t f'(Z_r^x)\,dZ_r^x + \frac{1}{2}\int_0^t f''(Z_r^x)\,d\langle Z^x, Z^x\rangle_r \qquad (2.9)\\ &= x + \int_0^t f'(Z_r^x)\,d((x \vee S_{0,r}^\mu) \wedge \inf_{s\in[0,r]}[(1+B_s^\mu) \vee S_{s,r}^\mu])\\ &\quad - \int_0^t f'(Z_r^x)\,dB_r^\mu + \frac{1}{2}\int_0^t f''(Z_r^x)\,dr\\ &= x - \int_0^t f'(Z_r^x)\,dB_r - \int_0^t \mu f'(Z_r^x)\,dr + \frac{1}{2}\int_0^t f''(Z_r^x)\,dr\\ &= x + \int_0^t (-\mu f' + \frac{1}{2}f'')(Z_r^x)\,dr - \int_0^t f'(Z_r^x)dB_r \end{aligned}$$

where in the third equality we use, as was suggested earlier on, that

$$Q_r^x = \left(x \vee S_{0,r}^\mu\right) \wedge \inf_{s\in[0,r]}\left[(1+B_s^\mu) \vee S_{s,r}^\mu\right] \qquad (2.10)$$

can only change in value when $Q_r^x = B_r^x$ or $Q_r^x = 1 + B_r^x$, something which can be seen by inspecting the function Q^x. Note also that $\inf_{s\in[0,r]} S_{s,r}^\mu = B_r^\mu$. Thus $d((x \vee S_{0,r}^\mu) \wedge \inf_{s\in[0,r]}[(1+B_s^\mu)\vee S_{s,r}^\mu]) = 0$ when $Z_r \neq 0$ or 1, while $f'(Z_r^x) = 0$ when $Z_r = 0$ or 1. Finally since f' is bounded this means $\int_0^t f'(Z_r^x)dB_r$ is a martingale and we have our result.

In order to proceed with the second identity in law and find a realisation for local time we recall Tanaka's formula (cf. [7, p. 222]). Given a continuous semimartigale X, for any real number a, there exists an increasing continuous process ℓ^{a+} called the right hand local time of X at a such that

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t \mathbb{1}_{X_s > a}\,dX_s + \frac{1}{2}\ell_t^{a+}(X) \qquad (2.11)$$

$$(X_t - a)^- = (X_0 - a)^- + \int_0^t -\mathbb{1}_{X_s \le a}\,dX_s + \frac{1}{2}\ell_t^{a+}(X). \qquad (2.12)$$

Using (2.11) and (2.12) respectively with our reflecting Brownian motion process Z^x in order to return expressions containing the local time of Z^x at both zero and one, we find

$$Z_t^x = x + \int_0^t \mathbb{1}_{Z_s^x > 0}\,dZ_s^x + \frac{1}{2}\ell_t^{0+}(Z^x) \qquad (2.13)$$

$$1 - Z_t^x = 1 - x - \int_0^t \mathbb{1}_{Z_s^x \le 1}\,dZ_s^x + \frac{1}{2}\ell_t^{1+}(Z^x). \qquad (2.14)$$

It is important to note that the convention used in [7] is that the integrand on the right-hand side of the above formulation of Tanaka's formula is the left-hand derivative of the left-hand side of the formula. This convention results in the local time in Tanaka's formula being the right-hand local time, defined as

$$\ell_t^{a+}(Z^x) = \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t \mathbb{1}_{\{a \le Z_s^x \le a+\varepsilon\}}\,d\langle Z^x, Z^x\rangle_s. \qquad (2.15)$$

Notice that due to our process Z^x being a reflected Brownian motion between zero and one, using the above definition of the right-hand local time would result in $\ell_t^{1+}(Z^x)$ being identically zero, a fact which can be immediately verified from (2.14). We must therefore find an expression for $1 - Z_t^x$ in terms of the left-hand local time. This can be done easily by taking the integrand on the right-hand side of Tanaka's formula to be the right-hand derivative of the left-hand side of the formula. Thus we get

$$1 - Z_t^x = 1 - x - \int_0^t \mathbb{1}_{Z_s^x < 1}\, dZ_s^x + \frac{1}{2}\ell_t^{1-}(Z^x) \tag{2.16}$$

where

$$\ell_t^{a-}(Z^x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{\{a-\varepsilon \le Z_s^x \le a\}}\, d\langle Z^x, Z^x\rangle_s. \tag{2.17}$$

Recall, again from [7], the symmetric local time ℓ_t^a of our process Z^x at fixed, real a, defined as

$$\ell_t^{a}(Z^x) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{\{a-\varepsilon \le Z_s^x \le a+\varepsilon\}}\, d\langle Z^x, Z^x\rangle_s. \tag{2.18}$$

Notice that due to the fact that $0 \le Z^x \le 1$, we have

$$\frac{1}{2}\ell_t^{0+}(Z^x) = \ell_t^0(Z^x) \quad \text{and} \quad \frac{1}{2}\ell_t^{1-}(Z^x) = \ell_t^1(Z^x). \tag{2.19}$$

Thus we may now rewrite (2.13) and (2.16) as

$$Z_t^x - x = \int_0^t \mathbb{1}_{Z_s^x > 0}\, dZ_s^x + \ell_t^0(Z^x) \tag{2.20}$$

and

$$\begin{aligned} 0 &= Z_t^x - x - \int_0^t \mathbb{1}_{Z_s^x < 1}\, dZ_s^x + \ell_t^1(Z^x) \\ &= \int_0^t 1 dZ_s^x - \int_0^t \mathbb{1}_{Z_s^x < 1}\, dZ_s^x + \ell_t^1(Z^x) \end{aligned} \tag{2.21}$$

respectively, where in the last equality we have used that

$$\int_0^t 1\, dZ_s^x = Z_t^x - x. \tag{2.22}$$

Subtracting (2.21) from (2.20), we find

$$\begin{aligned} Z_t^x - x &= \int_0^t \mathbb{1}_{Z_s^x > 0}\, dZ_s^x + \int_0^t \mathbb{1}_{Z_s^x < 1}\, dZ_s^x - \int_0^t 1\, dZ_s^x + \ell_t^0(Z^x) - \ell_t^1(Z^x) \\ &= \int_0^t (\mathbb{1}_{Z_s^x > 0} + \mathbb{1}_{Z_s^x < 1} - 1)\, dZ_s^x + \ell_t^0(Z^x) - \ell_t^1(Z^x) \\ &= \int_0^t \mathbb{1}_{0 < Z_s^x < 1}\, dZ_s^x + \ell_t^0(Z^x) - \ell_t^1(Z^x). \end{aligned} \tag{2.23}$$

We now take a closer inspection of the expression

$$\int_0^t \mathbb{1}_{0 < Z_s^x < 1}\, dZ_s^x \tag{2.24}$$

from the last line above. Recalling our definition of Z^x, we have

$$\begin{aligned} \int_0^t \mathbb{1}_{0 < Z_r^x < 1}\, dZ_r^x &= \int_0^t \mathbb{1}_{0 < Z_r^x < 1}\, d\Big[\big(x \vee S_{0,r}^\mu\big) \wedge \inf_{s \in [0,r]} \big[(1 + B_s^\mu) \vee S_{s,r}^\mu\big] - B_r^\mu\Big] \\ &= \int_0^t \mathbb{1}_{0 < Z_r^x < 1}\, d\Big[\big(x \vee S_{0,r}^\mu\big) \wedge \inf_{s \in [0,r]} \big[(1 + B_s^\mu) \vee S_{s,r}^\mu\big]\Big] - \int_0^t \mathbb{1}_{0 < Z_r^x < 1}\, dB_r^\mu \\ &= \int_0^t \mathbb{1}_{0 < Z_r^x < 1}\, dQ_r^x - \int_0^t \mathbb{1}_{0 < Z_r^x < 1}\, dB_r^\mu \end{aligned} \tag{2.25}$$

where we have recalled the definition of Q^x from (2.4). We now focus on the rather messy expression appearing above, that is

$$\int_0^t \mathbb{1}_{0<Z_r^x<1}\, dQ_r^x = \int_0^t \mathbb{1}_{0<Z_r^x<1}\, d\Big[\big(x \vee S_{0,r}^\mu\big) \wedge \inf_{s\in[0,r]} \big[(1+B_s^\mu) \vee S_{s,r}^\mu\big]\Big]. \tag{2.26}$$

We consider three distinct cases for values that Z^x can take. Firstly, the case when $Z_r^x = 0$, secondly, the case when $Z_r^x = 1$, and thirdly, the case when $0 < Z_r^x < 1$. In the first two cases, we have that

$$\mathbb{1}_{0<Z_r^x<1} = 0 \tag{2.27}$$

and therefore the integral in (2.26) is equal to zero in these cases. For the third case, we again recall, as was suggested in the introduction, and as can be seen by inspecting the function Q^x, that

$$Q_r^x = \big(x \vee S_{0,r}^\mu\big) \wedge \inf_{s\in[0,r]} \big[(1+B_s^\mu) \vee S_{s,r}^\mu\big] \tag{2.28}$$

can only change in value when $Q_r^x = B_r^x$ or $Q_r^x = 1 + B_r^x$. Thus, since $0 < Z_r^x < 1$ implies that

$$B_r^x < Q_r^x = \big(x \vee S_{0,r}^\mu\big) \wedge \inf_{s\in[0,r]} \big[(1+B_s^\mu) \vee S_{s,r}^\mu\big] < 1 + B_r^x \tag{2.29}$$

we have that Q_r^x is constant in this case, yielding $dQ_r^x = 0$, and we have shown that the integral in (2.26) is identically equal to zero in the third case as well.

Using this information, we see that equation (2.25) simplifies to

$$\int_0^t \mathbb{1}_{0<Z_s^x<1}\, dZ_s^x = -\int_0^t \mathbb{1}_{0<Z_s^x<1}\, dB_s^\mu \tag{2.30}$$

and further, equation (2.23) becomes

$$Z_t^x - x = -\int_0^t \mathbb{1}_{0<Z_s^x<1}\, dB_s^\mu + \ell_t^0(Z^x) - \ell_t^1(Z^x). \tag{2.31}$$

Now, if we can show that

$$\int_0^t \mathbb{1}_{0<Z_s^x<1}\, dB_s^\mu = B_t^\mu \tag{2.32}$$

then we have our result. In order to do this we first note that

$$\begin{aligned}\int_0^t \mathbb{1}_{0<Z_s^x<1}\, dB_s^\mu + \int_0^t \mathbb{1}_{Z_s^x=0}\, dB_s^\mu + \int_0^t \mathbb{1}_{Z_s^x=1}\, dB_s^\mu &= \int_0^t \mathbb{1}_{0\le Z_s^x\le 1}\, dB_s^\mu \\ &= \int_0^t 1\, dB_s^\mu = B_t^\mu.\end{aligned} \tag{2.33}$$

We therefore require that

$$\int_0^t \mathbb{1}_{Z_s^x=0}\, dB_s^\mu = \int_0^t \mathbb{1}_{Z_s^x=1}\, dB_s^\mu = 0. \tag{2.34}$$

Now considering $\int_0^t \mathbb{1}_{Z_s^x=0}\, dB_s^\mu$ we see that

$$\int_0^t \mathbb{1}_{Z_s^x=0}\, dB_s^\mu = \int_0^t \mathbb{1}_{Z_s^x=0}\, d(B_s + \mu s) = \int_0^t \mathbb{1}_{Z_s^x=0}\, dB_s + \mu \underbrace{\int_0^t \mathbb{1}_{Z_s^x=0}\, ds}_{=0} \tag{2.35}$$

where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion. Squaring the right-hand side of (2.35) and taking expectation, we find that

$$\mathbb{E}\bigg(\int_0^t \mathbb{1}_{Z_s^x=0}\,dB_s\bigg)^2 = \mathbb{E}\int_0^t (\mathbb{1}_{Z_s^x=0})^2\,ds = \int_0^t \mathbb{E}[\mathbb{1}_{Z_s^x=0}]\,ds = 0 \tag{2.36}$$

where in the first equality we have used the Itô isometry and in the second equality we have used Fubini's theorem and the fact that the square of the indicator function is itself. Recall, since $(\int_0^t \mathbb{1}_{Z_s^x=0} dB_s)^2 \geq 0$ then $\mathbb{E}[(\int_0^t \mathbb{1}_{Z_s^x=0} dB_s)^2] = 0$ if and only if $\int_0^t \mathbb{1}_{Z_s^x=0}\,dB_s = 0$. We therefore have that $\int_0^t \mathbb{1}_{Z_s^x=0}\,dB_s = 0$, and the same argument can be used to show $\int_0^t \mathbb{1}_{Z_s^x=1}\,dB_s = 0$, thus we have our result. □

Remark 2.2. Consider the following two versions of Tanaka's formula

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t \mathbb{1}_{X_s\geq a}\,dX_s + \frac{1}{2}\ell_t^{a-}(X) \tag{2.37}$$

$$(X_t - a)^- = (X_0 - a)^- + \int_0^t -\mathbb{1}_{X_s\leq a}\,dX_s + \frac{1}{2}\ell_t^{a+}(X). \tag{2.38}$$

Take $X_t = Z_t^x$ and $a = 0$ in (2.38)

$$(Z_t^x - 0)^- = (x - 0)^- - \int_0^t \mathbb{1}_{Z_s^x\leq 0}\,dZ_s^x + \frac{1}{2}\ell_t^{0+}(Z^x). \tag{2.39}$$

Using (2.19) and the fact that $0 \leq Z_t^x \leq 1$, we see that

$$\begin{aligned} \ell_t^0(Z^x) &= \int_0^t \mathbb{1}_{Z_s^x=0}\,dZ_s^x \\ &= \int_0^t \mathbb{1}_{Z_s^x=0}\,dQ_s^x - \int_0^t \mathbb{1}_{Z_s^x=0}\,dB_s^\mu \\ &= \int_0^t \mathbb{1}_{Z_s^x=0}\,dQ_s^x \end{aligned} \tag{2.40}$$

where in the final equality we have used (2.35) and (2.36). Similar arguments applied to (2.37) yield

$$\ell_t^1(Z^x) = -\int_0^t \mathbb{1}_{Z_s^x=1}\,dQ_s^x. \tag{2.41}$$

Note that by taking the difference between (2.40) and (2.41) one can again prove Theorem 3 by applying arguments similar to those of the original proof.

3 A coupled pair of local time realisations

We have an explicit realisation for the difference between the local times at 0 and 1, and also a realisation for each local time in integral form. However we wish to go further. We present a result for which gives a realisation of each local time in terms of the other. Recall that $R_{0,1}^{-\mu,x}$ is a Brownian motion with drift $-\mu \in \mathbb{R}$ reflected between 0 and 1 started at $x \in (0,1)$.

Theorem 3.1 (A coupled pair of local time realisations)**.** *The following identities in law hold*

$$\ell_t^0(R_{0,1}^{-\mu,x}) \stackrel{law}{=} x \vee \sup_{s\in[0,t]}(B_s^\mu + \ell_s^1(Z^x)) - x \tag{3.1}$$

$$-\ell_t^1(R_{0,1}^{-\mu,x}) \stackrel{law}{=} x \wedge \inf_{s\in[0,t]}(B_s^\mu + 1 - \ell_s^0(Z^x)) - x. \tag{3.2}$$

where $\ell^a(R_{0,1}^{-\mu,x})$ is the local time of $R_{0,1}^{-\mu,x}$ at a.

Note that this result is similar to one found in [7, p. 245], however a couple of key differences can be seen, namely that here our reflecting Brownian motion is able to have non-zero drift, and also that the Brownian motion B^μ seen here is exactly the Brownian motion used to generate our Z^x process.

Proof of Theorem 3.1. We know from Theorem 2.1 that there is no loss of generality to identify $R^{-\mu,x}_{0,1}$ with Z^x. We take (2.20) as follows

$$Z^x_t = x + \int_0^t \mathbb{1}_{Z^x_s>0}\, dZ^x_s + \ell^0_t(Z^x) \tag{3.3}$$

and we then apply Skorokhod's lemma [7, p. 239]. That is we note that from (3.3) we get

$$V^x_t \coloneqq x + \int_0^t \mathbb{1}_{Z^x_s>0}\, dZ^x_s \tag{3.4}$$

with $t \geq 0$ is a continuous process for which

$$V^x_0 = x \geq 0 \tag{3.5}$$

$$Z^x_t = V^x_t + \ell^0_t(Z^x). \tag{3.6}$$

We then see that the processes Z^x_t and $\ell^0_t(Z^x)$ satisfy

$$Z^x_t \geq 0 \tag{3.7}$$

$$\ell^0_t(Z^x) \text{ is increasing and } \ell^0_0(Z^x) = 0 \tag{3.8}$$

$$\int_0^t \mathbb{1}_{Z^x_s>0}\, d\ell^0_t(Z^x) = \int_0^t \mathbb{1}_{Z^x_s>0}\mathbb{1}_{Z^x_s=0}\, dZ^x_s = 0 \tag{3.9}$$

for $t \geq 0$, where for the first equality in (3.9) we have used the differential form of (2.40). Thus Skorokhod's Lemma tells us that we have an explicit expression for $\ell^0_t(Z^x)$ given by

$$\begin{aligned} \ell^0_t(Z^x) &= \sup_{s\in[0,t]} (-V^x_s \vee 0) \\ &= \sup_{s\in[0,t]} ((-x - \int_0^s \mathbb{1}_{Z^x_r>0}\, dZ^x_r) \vee 0) \\ &= \sup_{s\in[0,t]} (- \int_0^s \mathbb{1}_{Z^x_r>0}\, dZ^x_r \vee x) - x \\ &= x \vee \sup_{s\in[0,t]} (- \int_0^s \mathbb{1}_{Z^x_r>0}\, dZ^x_r) - x. \end{aligned} \tag{3.10}$$

Now we take a closer look at $\int_0^s \mathbb{1}_{Z^x_r>0}\, dZ^x_r$ and see that it can be written as

$$\begin{aligned} \int_0^s \mathbb{1}_{Z^x_r>0}\, dZ^x_r &= \int_0^s \mathbb{1}_{0<Z^x_r<1} + \mathbb{1}_{Z^x_r=1}\, dZ^x_r \\ &= \int_0^s \mathbb{1}_{0<Z^x_r<1}\, dZ^x_r + \int_0^s \mathbb{1}_{Z^x_r=1}\, dZ^x_r \end{aligned} \tag{3.11}$$

due to the fact that Z^x is between 0 and 1. Recall that in (2.25)–(2.36) we proved that

$$\int_0^s \mathbb{1}_{0<Z^x_r<1}\, dZ^x_r = -B^\mu_s \tag{3.12}$$

and from (2.41) we have

$$\int_0^s \mathbb{1}_{Z^x_r=1}\, dZ^x_r = -\ell^1_s(Z^x) \tag{3.13}$$

thus

$$\int_0^s \mathbb{1}_{Z_r^x>0}\,dZ_r^x = -B_s^\mu - \ell_s^1(Z^x) \tag{3.14}$$

and we can now rewrite (3.10) as

$$\ell_t^0(Z^x) = x \vee \sup_{s\in[0,t]}(B_s^\mu + \ell_s^1(Z^x)) - x \tag{3.15}$$

giving us the first of our two coupled equations in which we have written the local time at 0 of our two-sided reflecting Brownian motion process Z^x in terms of the underlying Brownian motion which can be used to generate Z^x, and the local time at 1 of Z^x. For the second of the two coupled equations we want to be able to write the local time at 1 of our two-sided reflecting Brownian motion process Z^x in terms of the underlying Brownian motion, and the local time at 0 of Z^x. It is possible to do this directly from (3.15) by exploiting the symmetries of Z^x but for a more concrete proof we will look at a similar method to the one above. We again begin with Tanaka's formula from (2.16) and (2.19), that is

$$1 - Z_t^x = 1 - x - \int_0^t \mathbb{1}_{Z_s^x<1}\,dZ_s^x + \ell_t^1(Z^x) \tag{3.16}$$

and as before we see that the conditions to apply Skorokhod's Lemma are met and thus we can state that we have an explicit expression for $\ell_t^1(Z^x)$ given by

$$\begin{aligned}
\ell_t^1(Z^x) &= \sup_{s\in[0,t]}((x-1+\int_0^s \mathbb{1}_{Z_r^x<1}\,dZ_r^x)\vee 0) && (3.17)\\
&= \sup_{s\in[0,t]}((-1+\int_0^s \mathbb{1}_{Z_r^x<1}\,dZ_r^x)\vee -x) + x\\
&= -x \vee \sup_{s\in[0,t]}(-1+\int_0^s \mathbb{1}_{Z_r^x<1}\,dZ_r^x) + x\\
&= -x \vee -\inf_{s\in[0,t]}(1-\int_0^s \mathbb{1}_{Z_r^x<1}\,dZ_r^x) + x\\
&= -(x \wedge \inf_{s\in[0,t]}(1-\int_0^s \mathbb{1}_{Z_r^x<1}\,dZ_r^x)) + x.
\end{aligned}$$

In an analogous manner to before we take a closer look at $\int_0^s \mathbb{1}_{Z_r^x<1}\,dZ_r^x$ and we see

$$\begin{aligned}
\int_0^s \mathbb{1}_{Z_r^x<1}\,dZ_r^x &= \int_0^s \mathbb{1}_{Z_r^x=0}\,dZ_r^x + \int_0^s \mathbb{1}_{0<Z_r^x<1}\,dZ_r^x && (3.18)\\
&= \ell_s^0(Z^x) - B_s^\mu
\end{aligned}$$

where in the final equality we have used (2.40) and (3.12). This yields the expression for $\ell_t^1(Z^x)$ given by

$$\ell_t^1(Z^x) = -(x \wedge \inf_{s\in[0,t]}(1-\ell_s^0(Z^x)+B_s^\mu)) + x \tag{3.19}$$

and so we have our coupled pair of equations for the local times written in terms of the other local time and the underlying Brownian motion

$$\ell_t^0(Z^x) = x \vee \sup_{s\in[0,t]}(B_s^\mu + \ell_s^1(Z^x)) - x \tag{3.20}$$

$$-\ell_t^1(Z^x) = x \wedge \inf_{s\in[0,t]}(B_s^\mu + 1 - \ell_s^0(Z^x)) - x. \tag{3.21}$$

□

Remark 3.2. Although we have only shown equality in law between our expressions on the right-hand side of (3.20) and (3.21) and the local times of the general reflecting Brownian motion $R_{0,1}^{-\mu,x}$, we have actually shown pathwise equality between these expressions and the local times of the realisation Z^x of the reflecting Brownian motion.

In Section 3, we saw that when starting with a realisation in law for a Brownian motion reflecting between 0 and 1, one can construct an expression which is equivalent in law to the difference between the local times at 0 and 1 of the reflecting Brownian motion. In this section we have started by finding realisations for each of these two local times, so now we ask, from this starting point is it possible to do the reverse and construct a new expression which is equivalent in law to the reflecting Brownian motion.

Taking inspiration from Theorem 2.1 in Section 3 we will take the difference between our two realisations of the local times of the reflecting Brownian motion process, add $x \in [0,1]$, the starting point of our reflecting Brownian motion, and then subtract $B_t^\mu = B_t + \mu t$ a Brownian motion process starting at 0 with drift $\mu \in \mathbb{R}$, with $(B_t)_{t\geq 0}$ a standard Brownian motion. We see in the above remark why we are able to do this.

This then yields the expression

$$\begin{aligned} Z_t^x &= \ell_t^0(Z^x) - \ell_t^1(Z^x) + x - B_t^\mu \\ &= x \vee \sup_{s\in[0,t]} (B_s^\mu + \ell_s^1(Z^x)) + x \wedge \inf_{s\in[0,t]} (B_s^\mu + 1 - \ell_s^0(Z^x)) - x - B_t^\mu . \end{aligned} \tag{3.22}$$

which can be used as an alternative realisation of the two-sided Brownian motion with drift $-\mu$ reflecting between 0 and 1 started at x.

Remark 3.3. Note that from (3.20) and (3.21) it is now obviously possible to find implicit expressions for $\ell_t^0(Z^x)$ and $-\ell_t^1(Z^x)$ by simply combining the two. Namely

$$\begin{aligned} \ell_t^0(Z^x) &= x \vee \sup_{s\in[0,t]} (B_s^\mu - x \wedge \inf_{r\in[0,s]} (B_r^\mu + 1 - \ell_r^0(Z^x)) + x) - x \\ &= 0 \vee \sup_{s\in[0,t]} (B_s^\mu - x \wedge \inf_{r\in[0,s]} (B_r^\mu + 1 - \ell_r^0(Z^x))) \end{aligned} \tag{3.23}$$

$$\begin{aligned} -\ell_t^1(Z^x) &= x \wedge \inf_{s\in[0,t]} (B_s^\mu + 1 - x \vee \sup_{r\in[0,s]} (B_r^\mu + \ell_r^1(Z^x)) + x) - x \\ &= 0 \wedge \inf_{s\in[0,t]} (B_s^\mu + 1 - x \vee \sup_{r\in[0,s]} (B_r^\mu + \ell_r^1(Z^x))). \end{aligned} \tag{3.24}$$

However is also interesting to note that we can obtain a much simpler implicit expression by applying Skorokhod’s Lemma either in the way we did in the proof of Theorem 5, or by applying it to a case that trivially satisfies the conditions. In particular we see that $Z_t^x = Z_t^x - \ell_t^0(Z^x) + \ell_t^0(Z^x)$, and Skorokhod’s Lemma then tells us

$$\ell_t^0(Z^x) = 0 \vee \sup_{s\in[0,t]} (\ell_s^0(Z^x) - Z_s^x). \tag{3.25}$$

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