Electronic Journal of Statistics
Vol. 17 (2023) 2120–2138
ISSN: 1935-7524
https://doi.org/10.1214/23-EJS2148

Nonregular designs from Paley's Hadamard matrices: Generalized resolution, projectivity and hidden projection property*

Guanzhou Chen

School of Statistics and Data Science, LPMC & KLMDASR
Nankai University, Tianjin, China
e-mail: guanzhou.chen@foxmail.com

Chenlu Shi

Department of Statistics
Colorado State University, Fort Collins, USA
e-mail: chenlu.shi@colostate.edu

Boxin Tang

Department of Statistics and Actuarial Science
Simon Fraser University, Burnaby, Canada
e-mail: boxint@sfu.ca

Abstract: Nonregular designs are attractive, as compared with regular designs, not just because they have flexible run sizes but also because of their performances in terms of generalized resolution, projectivity, and hidden projection property. In this paper, we conduct a comprehensive study on three classes of designs that are obtained from Paley's two constructions of Hadamard matrices. In terms of generalized resolution, we complete the study of Shi and Tang [15] on strength-two designs by adding results on strength-three designs. In terms of projectivty and hidden projection property, our results substantially expand those of Bulutoglu and Cheng [2]. For the purpose of practical applications, we conduct an extensive search of minimum G-aberration designs from those with maximum generalized resolutions and results are obtained for strength-two designs with 36, 44, 48, 52, 60, 64, 96 and 128 runs and strength-three designs with 72, 88 and 120 runs.

MSC2020 subject classifications: 62K15.

Keywords and phrases: Foldover design, minimum G-aberration, orthogonal array.

Received April 2022.

*Supported in part by the Natural Sciences and Engineering Research Council of Canada. The research started when the first author was a PhD candidate at Simon Fraser University.

1. Introduction

Two-level orthogonal arrays are a very useful class of fractional factorial designs for the planning of factorial experiments, especially for those studies that involve a large number of factors. They can be classified into regular designs and nonregular designs. Regular designs are easy to construct and have simple aliasing structures, but their run sizes are limited to powers of two. By comparison, nonregular designs allow for more flexible run sizes and also enjoy better statistical properties in terms of generalized resolution ([6]), projectivity ([1]), and hidden projection property ([25]). We refer to [27] for an excellent review on nonregular designs.

[14, 15] investigated the theoretical construction of nonregular designs with maximum generalized resolutions. Except for a special case, their results focus on orthogonal arrays of strength two. Prior to [14, 15], finding designs with maximum generalized resolution is largely computational; see, for example, [12] and [13].

Among all the factors investigated in an experiment, very often only a few of them are active. It is therefore important to examine the properties of a design when projected onto low dimensions. One way to characterize the projection properties of a design is through the concept of projectivity ([1]). A design is said to have projectivity h if its projection design onto any h factors contains all possible level combinations. For an orthogonal array of strength t, the existing results can only be used to determine whether or not it has projectivity $t + 1$ ([4, 1]).

The hidden projection property of a design provides another way of evaluating its projection designs if only the main effects and two-factor interactions are of interest ([25]). A design is said to have the hidden projection property for h factors if its projection design onto any h factors allows estimation of all main effects and all two-factor interactions. [4] showed that a strength-two orthogonal array has the hidden projection property for 4 factors if it does not have defining words of length 3 or 4. [5] further established that if a strength-three array does not have any defining word of length 4, it has the hidden projection property for 5 factors. [2] later proved that Paley designs with more than 8 runs do not have any defining words of length 3 or 4, thereby showing that Paley designs have the hidden projection property for 4 factors and their foldovers have the hidden projection property for 5 factors.

In this article, we conduct a comprehensive study on three classes of designs from Paley's Hadamard matrices in terms of generalized resolution, projectivity and hidden projection property. The three classes of designs are denoted by P_n, $\tilde{P}_{2n}$ and Q_{2n}, respectively, with their precise definitions to be given later in the paper. For now, it suffices to say that P_n is a saturated orthogonal array of strength two obtained from Paley's first construction of Hadamard matrix, $\tilde{P}_{2n}$ is the foldover of P_n, and Q_{2n} is an orthogonal array of strength two obtained by judiciously selecting n columns from Paley's second construction of Hadamard matrix of order $2n$.

[15] examined theoretical construction of designs with maximum generalized

resolutions with a focus on orthogonal arrays of strength two, and showed in particular that P_n and Q_{2n} and their subdesigns have maximum generalized resolutions. We complete their investigations by showing that $\tilde{P}_{2n}$ and many of its subdesigns, all of which are orthogonal arrays of strength three, also have maximum generalized resolutions.

More importantly, we provide a general investigation of all three classes of designs, P_n, $\tilde{P}_{2n}$ and Q_{2n} into their projectivity and hidden projection property for h factors. From [4, 5, 2], we can draw conclusions on the projectivity of P_n and Q_{2n} for $h = 3$ and of $\tilde{P}_{2n}$ for $h = 4$, and on the hidden projection property of P_n and Q_{2n} for $h = 4$ and of $\tilde{P}_{2n}$ for $h = 5$. As will be seen in Section 3, our results substantially expand these existing results of [4, 5, 2].

For practical purposes, we also study the selection problem using the minimum G-aberration criterion from the designs with maximum generalized resolutions. Besides our main focus, which is the design selection from P_n, $\tilde{P}_{2n}$ and Q_{2n}, we also consider those designs with maximum generalized resolutions obtained by [15] using tensor product construction. We tabulate our findings for strength-two designs with 36, 44, 48, 52, 60, 64, 96 and 128 runs and strength-three designs with 72, 88 and 120 runs.

The remainder of the paper is organized as follows. Section 2 of the paper introduces necessary notation and reviews some background. Section 3 studies strength-three orthogonal arrays with maximum generalized resolutions, and examines the projectivity and hidden projection property of three classes of designs P_n, $\tilde{P}_{2n}$ and Q_{2n}. Section 4 looks into the design selection problem using the minimum G-aberration criterion. The paper is concluded with some further results on the type of Hadamard matrices in Section 5. All the proofs are postponed to the Appendix.

2. Notation and background

A two-level orthogonal array of N runs, m factors and strength t, denoted by $\mathrm{OA}(N, 2^m, t)$, is an $N \times m$ matrix of ± 1 such that in any of its $N \times t$ submatrix, the 2^t possible level combinations occur equally often. Such an array can be characterized by its J-characteristics. Suppose $D = (d_{ij})$ is an $\mathrm{OA}(N, 2^m, t)$. Given a set $\mathbf{u} \subseteq \mathbb{Z}_m = \{1, \ldots, m\}$, the J-characteristic of the columns of D indexed by $\mathbf{u}$ is defined as $J_{\mathbf{u}}(D) = \sum_{i=1}^{N} \prod_{j \in \mathbf{u}} d_{ij}$. Clearly, we have $J_{\mathbf{u}}(D) = 0$ if $|\mathbf{u}| \le t$, where $|\mathbf{u}|$ is the cardinality of $\mathbf{u}$. In addition, we note that $|J_{\mathbf{u}}(D)|$ can only take values of $\{N, N-8, \ldots, N-8\lfloor N/8 \rfloor\}$ for $|\mathbf{u}| = 3, 4$ when $t = 2$, and $\{N, N-16, \ldots, N-16\lfloor N/16 \rfloor\}$ for $|\mathbf{u}| = 4$ when $t = 3$, where $\lfloor \cdot \rfloor$ is the floor function; see, for example, Lemma 3 of [17].

Let r be the smallest integer such that $\max_{|\mathbf{u}|=r} |J_{\mathbf{u}}(D)| > 0$. The generalized resolution of D is defined as $r + 1 - \max_{|\mathbf{u}|=r} |J_{\mathbf{u}}(D)|/N$ ([6]). When $N/2 < m \le N-1$, we have $r = 3$. [15] derived the following lower bound on $\max_{|\mathbf{u}|=3} |J_{\mathbf{u}}(D)|$.

Lemma 1. *Suppose D is an $OA(N, 2^m, 2)$ with $N/2 < m \le N-1$. Then* $\max_{|\mathbf{u}|=3} |J_{\mathbf{u}}(D)| \ge N - 8\lfloor (N/8)(1-\xi^{1/2}) \rfloor$, *where* $\xi = (2m-N)/((m-1)(m-2))$.

To distinguish designs with the same generalized resolution, [6] further proposed the minimum G-aberration criterion as a refinement. This criterion sequentially minimizes $F_1(N), \ldots, F_1(0), F_2(N), \ldots, F_2(0), \ldots, F_m(N), \ldots, F_m(0)$, where $F_k(l)$ is the frequency of $\mathbf{u}$'s such that $|\mathbf{u}| = k$ and $|J_{\mathbf{u}}(D)| = l$ for $k = 1, \ldots, m$ and $l = 0, \ldots, N$. For theoretical convenience, [21] introduced the criterion of minimum G_2-aberration, which aims to sequentially minimize the entries of $(A_1(D), \ldots, A_m(D))$ where $A_k(D) = \sum_{|\mathbf{u}|=k} |J_{\mathbf{u}}(D)/N|^2$.

Orthogonal arrays can be constructed from Hadamard matrices. A Hadamard matrix of order N is an $N \times N$ matrix H of ± 1 satisfying $H^T H = N I_N$, where I_N is the identity matrix of order N. Given a Hadamard matrix of order N, we can normalize one column by switching the signs of rows such that this column contains all ones, and then obtain an $\mathrm{OA}(N, 2^{N-1}, 2)$ by dropping this normalized column.

Two constructions of Hadamard matrices were proposed by [11]. Suppose s is a prime or prime power. Denote the Galois field of order s by $GF(s) = \{\alpha_1, \ldots, \alpha_s\}$ and define the function χ over $GF(s)$ such that $\chi(\alpha) = 0$ if $\alpha = 0$, $\chi(\alpha) = 1$ if $\alpha = \beta^2$ for some nonzero $\beta \in GF(s)$, and $\chi(\alpha) = -1$ otherwise. Let K be the $s \times s$ matrix with its (i, j)th entry being $\chi(\alpha_i - \alpha_j)$. Then Paley's first construction works if $s = 4l + 3$ for some integer l and leads to following Hadamard matrix of order $n = s + 1$:

$$H = \begin{bmatrix} 1 & -\mathbf{1}_s^T \\ \mathbf{1}_s & K + I_s \end{bmatrix}, \tag{1}$$

where $\mathbf{1}_s$ is a column vector of s ones. The $\mathrm{OA}(n, 2^{n-1}, 2)$ obtained by removing the first column of H in (1) is called a Paley design and is denoted by P_n hereafter. A sharp upper bound on $\max_{|\mathbf{u}|=3,4} |J_{\mathbf{u}}(P_n)|$ was established by [14].

Lemma 2. *We have* $\max_{|\mathbf{u}|=3,4} |J_{\mathbf{u}}(P_n)| \leq U_P(n) = n - 8\lceil n/8 - (n-1)^{1/2}/4 - 1/2\rceil$, *where* $\lceil \cdot \rceil$ *is the ceiling function.*

Using Lemma 2 together with Lemma 1, [15] obtained many designs with maximum generalized resolutions by dropping columns from P_n for $n = 12$, 20, 24, 28, 32, 44, 60, 72 and 80. Paley's second construction applies to the case $s = 4l + 1$ for some integer l, and yields a Hadamard matrix H of order $2n = 2s + 2$, as displayed in (2).

$$H = \begin{bmatrix} 1 & \mathbf{1}_s^T & -1 & \mathbf{1}_s^T \\ \mathbf{1}_s & K + I_s & \mathbf{1}_s & K - I_s \\ -1 & \mathbf{1}_s^T & -1 & -\mathbf{1}_s^T \\ \mathbf{1}_s & K - I_s & -\mathbf{1}_s & -K - I_s \end{bmatrix}, \qquad Q_{2n} = \begin{bmatrix} -1 & \mathbf{1}_s^T \\ \mathbf{1}_s & K - I_s \\ 1 & \mathbf{1}_s^T \\ -\mathbf{1}_s & -K - I_s \end{bmatrix}. \tag{2}$$

By multiplying the $(s+2)$th row of H in (2) by -1 and then removing the first $s+1$ columns, [15] obtained the design Q_{2n} in (2). [15] proved that Q_{2n} achieves the minimum possible $\max_{|\mathbf{u}|=3} |J_{\mathbf{u}}(D)|$ value, as given in the next lemma.

Lemma 3. *The design* Q_{2n} *in* (2) *is an* $OA(2n, 2^n, 2)$ *with* $\max_{|\mathbf{u}|=3} |J_{\mathbf{u}}(Q_{2n})| = 4$.

3. Main results

3.1. Strength-3 arrays with maximum generalized resolutions

Lemma 1 provides a lower bound on $\max_{|\mathbf{u}|=3} |J_{\mathbf{u}}(D)|$ for orthogonal arrays of strength 2. We establish a similar lower bound on $\max_{|\mathbf{u}|=4} |J_{\mathbf{u}}(D)|$ for strength-3 arrays.

Theorem 1. *Suppose D is an $OA(N, 2^m, 3)$ with $N/3 \le m \le N/2$. Then*

$$\max_{|\mathbf{u}|=4} |J_{\mathbf{u}}(D)| \ge N - 16 \left\lfloor (N/16)(1-\zeta^{1/2}) \right\rfloor,$$

where

$$\zeta = \frac{4m^3 - 3m^2N + mN^2 - 3mN + 4m - N^3/8 + 3N^2/4 - N}{m(m-1)(m-2)(m-3)}.$$

Based on Theorem 1, some designs can be shown to have maximum generalized resolutions. For a Paley design P_n, consider its foldover design

$$\tilde{P}_{2n} = \begin{bmatrix} \mathbf{1}_n & P_n \\ -\mathbf{1}_n & -P_n \end{bmatrix}.$$

Clearly, $\tilde{P}_{2n}$ is an $\mathrm{OA}(2n, 2^n, 3)$. Since $\max_{|\mathbf{u}|=4} |J_{\mathbf{u}}(\tilde{P}_{2n})| = 2\max_{|\mathbf{u}|=3,4} |J_{\mathbf{u}}(P_n)|$, a sharp upper bound on $\max_{|\mathbf{u}|=4} |J_{\mathbf{u}}(\tilde{P}_{2n})|$ follows directly from Lemma 2:

$$\max_{|\mathbf{u}|=4} |J_{\mathbf{u}}(\tilde{P}_{2n})| \le 2U_P(n) = 2n - 16\lceil n/8 - (n-1)^{1/2}/4 - 1/2 \rceil. \quad (3)$$

This shows that design $\tilde{P}_{2n}$ has a large generalized resolution as the upper bound $2U_P(n)$ on $\max_{|\mathbf{u}|=4} |J_{\mathbf{u}}(\tilde{P}_{2n})|$ is in the order of $O(n^{1/2})$. Some of the $\max_{|\mathbf{u}|=4} |J_{\mathbf{u}}(\tilde{P}_{2n})|$ values are given in Table 1 for small run sizes. Comparing the upper bound in (3) with the lower bound in Theorem 1, we deduce the next result.

Corollary 1. *Designs obtained by selecting any m columns from $\tilde{P}_{2n}$ have the maximum generalized resolutions for $2n = 24, 40, 48, 56, 64, 88, 120, 144, 160$ and $2n/3 \le m \le n$.*

We note that the special cases given by $m = n$ in Corollary 1 were previously obtained in [14].

Remark 1. *[14] found by computer search two Hadamard matrices H of order 36 with $\max_{|\mathbf{u}|=4} |J_{\mathbf{u}}(H)| = 12$. Folding over any of these two Hadamard matrices by $[H^T \ -H^T]^T$ and then selecting any m columns, we obtain $OA(72, 2^m, 3)$s with the maximum generalized resolutions for $24 \le m \le 36$ by an application of Theorem 1.*

TABLE 1
Some values of $\max_{|\mathbf{u}|=4} |J_{\mathbf{u}}(\tilde{P}_{2n})|$.

run size $2n$	24	40	48	56	64	88	96	120	136	144	160	168	208
$\max_{\lvert\mathbf{u}\rvert=4} \lvert J_{\mathbf{u}}(\tilde{P}_{2n})\rvert$	8	24	16	24	16	24	32	24	40	32	32	40	48

3.2. Projectivities of P_n, $\tilde{P}_{2n}$ and Q_{2n}

[4] pointed out that the projection of an $\mathrm{OA}(N, 2^m, t)$, say D, onto $t+1$ factors indexed by $\mathbf{u}$ has $(N - |J_{\mathbf{u}}(D)|)/2^{t+1}$ copies of the full factorial plus $|J_{\mathbf{u}}(D)|/2^t$ copies of a half replicate of the full factorial. This settles projections of P_n, $\tilde{P}_{2n}$ and Q_{2n} onto 3, 4, and 3 factors, respectively. In this subsection, we investigate the projections of these designs onto more factors. A result from [19] is useful here. We describe it next.

For any $\mathbf{s} \subseteq \mathbb{Z}_m$, let $\mathbf{r}_{\mathbf{s}}$ be an m-dimensional row vector with its jth entry being 1 if $j \in \mathbf{s}$, and -1 otherwise for $j = 1, \ldots, m$. Define a matrix $\mathbf{C}$ as

$$\mathbf{C} = \left[\mathbf{r}_{\emptyset}^T, \mathbf{r}_{\{1\}}^T, \mathbf{r}_{\{2\}}^T, \mathbf{r}_{\{1,2\}}^T, \mathbf{r}_{\{3\}}^T, \mathbf{r}_{\{1,3\}}^T, \mathbf{r}_{\{2,3\}}^T, \mathbf{r}_{\{1,2,3\}}^T, \mathbf{r}_{\{4\}}^T, \ldots, \mathbf{r}_{\{1,2,\ldots,m\}}^T\right]^T.$$

Clearly, $\mathbf{C}$ contains all possible level combinations for m factors as rows. For $\mathbf{u} \subseteq \mathbb{Z}_m$, let $\mathbf{h}_{\mathbf{u}}$ denote the Hadamard product of all the columns of $\mathbf{C}$ indexed by $\mathbf{u}$ and define

$$\mathbf{H} = \left[\mathbf{h}_{\emptyset}, \mathbf{h}_{\{1\}}, \mathbf{h}_{\{2\}}, \mathbf{h}_{\{1,2\}}, \mathbf{h}_{\{3\}}, \mathbf{h}_{\{1,3\}}, \mathbf{h}_{\{2,3\}}, \mathbf{h}_{\{1,2,3\}}, \mathbf{h}_{\{4\}}, \ldots, \mathbf{h}_{\{1,2,\ldots,m\}}\right],$$

where $\mathbf{h}_{\emptyset}$ is a column of all ones. Then the result of [19] can be stated as follows.

Lemma 4. *Suppose D is an $OA(N, 2^m, t)$. Let $N_{\mathbf{s}}$ be the frequency that $\mathbf{r}_{\mathbf{s}}$ occurs in D for $\mathbf{s} \subseteq \mathbb{Z}_m$. Then $N_{\mathbf{s}} = 2^{-m} \sum_{\mathbf{u} \subseteq \mathbb{Z}_m} h_{\mathbf{su}} J_{\mathbf{u}}(D)$, where $h_{\mathbf{su}}$ is the element on the $\mathbf{s}$th row and $\mathbf{u}$th column of $\mathbf{H}$.*

Lemma 4 reveals that any design, up to row permutations, is uniquely determined by its J-characteristics. This enables us to study the projections of a design D onto k factors through $J_{\mathbf{u}}(D)$ for $|\mathbf{u}| \leq k$.

Proposition 1. *The projection of P_n (respectively, $\tilde{P}_{2n}$) onto any 4 (respectively, 5) factors has at least $\lceil n/16 - 5U_P(n)/16 \rceil$ copies of the full factorial.*

Proposition 1 indicates that the number of full factorials contained in any four-factor projection of P_n, or five-factor projection of $\tilde{P}_{2n}$, is approximately $n/16$ for large n, since $U_P(n)$ is of order $O(n^{1/2})$. A design is said to have projectivity h if its projection onto any h factors contains at least one full factorial. Using Proposition 1, one can check that P_n (respectively, $\tilde{P}_{2n}$) has projectivity 4 (respectively, 5) when $n \geq 108$. Next, we examine the projections of Q_{2n} onto 4 and 5 factors, for which we need the following knowledge on $|J_{\mathbf{u}}(Q_{2n})|$ for $|\mathbf{u}| = 4$ and 5.

Lemma 5. *We have that $\max_{|\mathbf{u}|=4} |J_{\mathbf{u}}(Q_{2n})| \leq U_Q(2n) = 2n - 8\lceil n/4 - (n-1)^{1/2}/2 \rceil$ and that $|J_{\mathbf{u}}(Q_{2n})|$ is either 0 or 8 for $|\mathbf{u}| = 5$.*

Remark 2. *The bound $U_Q(2n)$ for Q_{2n} appears quite sharp. We have checked that the bound is attained by all $2n = 2s + 2 < 600$ with s being a prime power and all $2n = 2s + 2 < 5000$ with s being a prime. We also see that $U_Q(2n)$ is asymptotically equivalent to the bound $2U_P(n)$ for $\tilde{P}_{2n}$. This is because the inequalities (4) in the proof of Lemma 5 hold no matter whether $s \equiv 1 \pmod 4$*

or $s \equiv 3 \pmod 4$*, and are therefore an intrinsic property of the matrix* K *in* (1) *and* (2)*. We note that this property has been used to construct definitive screening designs by [26] recently.*

Lemma 5 allows us to study the projections of Q_{2n} onto 4 and 5 factors.

Proposition 2. *The projection of* Q_{2n} *onto any 4 (respectively, 5) factors has at least* $\lceil n/8 - U_Q(2n)/16 - 1\rceil$ *(respectively,* $\lceil n/16 - U_Q(2n)/8 - 6/5\rceil$*) copies of the full factorial.*

The proof of Proposition 2 is similar to that of Proposition 1 and thus omitted. It follows immediately that Q_{2n} has projectivity 4 when the run size $2n \geq 36$, and projectivity 5 when the run size $2n \geq 196$.

We now use a computer to take a closer look at the projections of P_n, $\tilde{P}_{2n}$ and Q_{2n} for small run sizes. For a design D with N runs, we denote by $f_k(l)$ the proportion of k-factor projections of D that contains l full factorials, and summarize the k-factor projection properties of D by the vector

$$\mathrm{PV}_k(D) = (f_k(0), f_k(1), \ldots, f_k(\lfloor N/2^k \rfloor)).$$

The vectors $\mathrm{PV}_4(P_n)$, $\mathrm{PV}_4(Q_{2n})$ and $\mathrm{PV}_5(Q_{2n})$ are displayed in Tables 2 and 3. The vectors $\mathrm{PV}_5(\tilde{P}_{2n})$ are omitted because we find $\mathrm{PV}_4(P_n) = \mathrm{PV}_5(\tilde{P}_{2n})$ for all $n < 108$. We conjecture this relationship holds for all n, though we cannot prove it for the moment.

Table 3 suggests that the bound $\lceil n/8 - U_Q(2n)/16 - 1\rceil$ on the number of full factorials in 4-factor projections of Q_{2n} is sharp as it is attained by all run sizes less than 196. More importantly, combining the computational results in Tables 2 and 3 and theoretical results in Propositions 1 and 2, we know exactly when designs P_n, $\tilde{P}_{2n}$ and Q_{2n} have projectivities 4 or 5. This we summarize as Theorem 2.

Theorem 2. *The design* P_n *(respectively,* $\tilde{P}_{2n}$*) has projectivity 4 (respectively, 5) when* $n \geq 68$*. The design* Q_{2n} *has projectivity 4 when* $2n \geq 36$*, and projectivity 5 when* $2n \geq 180$*.*

TABLE 2
The four-factor projections of P_n *for* $n < 108$*.*

n	$\mathrm{PV}_4(P_n) = (f_4(0), f_4(1), \ldots, f_4(\lfloor n/16 \rfloor))$
20	(100%, 0)
24	(57.1%, 42.9%)
28	(50.0%, 50.0%)
32	(39.4%, 59.1%, 1.4%)
44	(7.3%, 67.1%, 25.6%)
48	(6.1%, 51.5%, 42.4%, 0)
60	(0.4%, 24.4%, 65.8%, 9.4%)
68	(0, 10.1%, 56.7%, 33.2%, 0)
72	(0, 6.4%, 43.7%, 44.8%, 5.1%)
80	(0, 2.1%, 29.9%, 53.7%, 14.4%, 0)
84	(0, 0.9%, 18.5%, 63.9%, 16.7%, 0)
104	(0, 0.2%, 1.2%, 22.0%, 55.3%, 20.2%, 1.2%)

TABLE 3
The four- and five-factor projections of Q_{2n} for $2n < 196$.

$2n$	$\mathrm{PV}_4(Q_{2n}) = (f_4(0), f_4(1), \ldots, f_4(\lfloor n/8 \rfloor))$	$\mathrm{PV}_5(Q_{2n}) = (f_5(0), f_5(1), \ldots, f_5(\lfloor n/16 \rfloor))$
20	(100%, 0)	100%
28	(27.3%, 72.7%)	100%
36	(0, 100%, 0)	(100%, 0)
52	(0, 8.7%, 91.3%, 0)	(90.0%, 10.0%)
60	(0, 0, 55.6%, 44.4%)	(76.6%, 23.3%)
76	(0, 0, 0, 57.1%, 42.9%)	(39.1%, 57.2%, 3.7%)
84	(0, 0, 0, 23.1%, 76.9%, 0)	(22.9%, 76.3%, 7.7%)
100	(0, 0, 0, 0, 31.9%, 68.1%, 0)	(7.7%, 65.2%, 27.0%, 0)
108	(0, 0, 0, 0, 5.9%, 70.6, 23.5%)	(4.5%, 65.8%, 29.7%, 0)
124	(0, 0, 0, 0, 0, 13.6%, 45.8%, 40.7%)	(1.9%, 32.4%, 59.8%, 5.9%)
148	(0, 0, 0, 0, 0, 0, 0, 45.1%, 54.9%, 0)	(0.1%, 8.9%, 59.4%, 30.4%, 1.1%)
164	(0, 0, 0, 0, 0, 0, 0, 1.2%, 53.2%, 45.6%, 0)	(0.4%, 0.3%, 42.9%, 45.8%, 10.4%, 0)
180	(0, 0, 0, 0, 0, 0, 0, 0, 6.9%, 37.9%, 55.2%, 0)	(0, 0.1%, 22.4%, 58.5%, 18.3%, 0.6%)

3.3. Hidden projection properties of P_n, $\tilde{P}_{2n}$ and Q_{2n}

An orthogonal array is said to have the hidden projection property for h factors if in its projection onto any h factors, all the main effects and two-factor interactions are estimable under the assumption that higher-order interactions are negligible.

[2] showed that P_n does not have defining words of lengths three or four as long as $n \geq 12$ and thus has the hidden projection property for 4 factors by a result of [4]. It is also easy to deduce, according to [5], that $\tilde{P}_{2n}$ has the hidden projection property for 5 factors as long as the run size $2n$ is at least 24. In this subsection, we show that even better hidden projection properties can be achieved by P_n, $\tilde{P}_{2n}$ and also Q_{2n} for moderate n.

Lemma 6. *The design P_n (respectively, $\tilde{P}_{2n}$) has the hidden projection property for h (respectively, $h+1$) factors if $n > (h-1)(h-2)U_P(n)/2$. The design Q_{2n} has the hidden projection property for h factors if $2n > 4(h-2) + (h-2)(h-3)U_Q(2n)/2$.*

Lemma 6 guarantees that P_n (respectively, $\tilde{P}_{2n}$) has the hidden projection property for 5 (respectively, 6) factors when $n = 132$, 140, 152 and $n \geq 168$, and that Q_{2n} has the hidden projection property for 5 factors when $2n \geq 76$, and for 6 factors when $2n \geq 300$. We then proceed with a computer study of those cases not covered by Lemma 6. Combining our computational findings with Lemma 6, we obtain Theorem 3.

Theorem 3. *The design P_n (respectively, $\tilde{P}_{2n}$) has the hidden projection property for 5 (respectively, 6) factors when $n \geq 28$. The design Q_{2n} has the hidden projection property for 5 factors when $2n \geq 28$, and for 6 factors when $2n = 28$ and $2n \geq 52$.*

For a design D, let $h_{\max}(D)$ be the largest integer h such that D has the hidden projection property for h factors. We obtain the following computational results on $h_{\max}(P_n)$, $h_{\max}(\tilde{P}_{2n})$ and $h_{\max}(Q_{2n})$ as displayed in Table 4, which strengthen the general theoretical results in Theorem 3 for many cases.

Table 4
Some values of $h_{\max}(P_n)$, $h_{\max}(\tilde{P}_{2n})$ and $h_{\max}(Q_{2n})$.

n	20	24	28	32	44	48	60	68	72	80	84
$h_{\max}(P_n)$	4	4	5	6	8	7	≥ 7	≥ 7	≥ 7	≥ 6	≥ 6
$h_{\max}(\tilde{P}_{2n})$	5	5	6	7	9	8	≥ 8	≥ 8	≥ 8	≥ 7	≥ 7
$2n$	20	28	36	52	60	76	84	100	108	124	148
$h_{\max}(Q_{2n})$	4	6	5	6	7	7	≥ 8	≥ 8	≥ 8	≥ 7	≥ 7

When $n \geq 60$ for P_n, $\tilde{P}_{2n}$ and $2n \geq 84$ for Q_{2n}, we only provide a lower bound for $h_{\max}$ as the computation becomes too heavy to handle. Nonetheless, we can still see a trend that better hidden projection properties can be achieved by designs with larger run sizes. This is expected because, by Lemma 6, $h_{\max}$ should be in the order of $O(n^{1/4})$.

4. Design selection by minimum *G*-aberration

The generalized resolution, as a design selection criterion, only looks at the most severe aliasing among factorial effects. A more general design selection criterion is that of minimum G-aberration. This section is devoted to finding minimum G-aberration designs from those with maximum generalized resolutions. Our focus is on design selection from the three classes of designs P_n, $\tilde{P}_{2n}$ and Q_{2n}. Also considered are some designs by tensor product construction from [15]. In our computer search, we use J-characteristics for up to four factors, as done by most authors.

A brief review on designs with minimum G-aberration is necessary. Specifically, such $\mathrm{OA}(N, 2^m, t)$s are already available for $N = 12$, 16, 20 and $m \leq N-1$ ([18]); $N = 24$ and $m \leq 23$, $N = 28$ and $m \leq 14$, $N = 36$ and $m \leq 18$ ([13]); $N = 32$, 40 and 48 and $m \leq N/2$ ([12]). Recently, [24, 22, 23] algorithmically studied some strength-3 designs with larger run sizes. It should be noted that [13, 23, 22] have examined strength-2 designs from projections of P_{32}, strength-3 designs from projections of $\tilde{P}_{56}$ and $\tilde{P}_{64}$, respectively.

4.1. Designs from Paley's constructions

The orthogonal arrays in this subsection come from Paley's constructions of Hadamard matrices, except for those with 36 and 72 runs, which are from the two Hadamard matrices of order 36 in Remark 1.

We first consider Hadamard matrices from Paley's first construction as well as the two of order 36. Given a Hadamard matrix of order n, we first randomly select a submatrix with m columns, then obtain an $\mathrm{OA}(n, 2^{m-1}, 2)$ by normalizing and removing a randomly selected column, and an $\mathrm{OA}(2n, 2^m, 3)$ by folding over the submatrix. The procedure is repeated 200,000 times and the designs with minimum G-aberrations are selected. A complete search is done when $\binom{n}{m}$ is less than 200,000. We apply this approach to Paley's first Hadamard matrices of order 44, 60 and the two Hadamard matrices of 36. It should also be mentioned that the strength-2 designs of 44 and 60 runs obtained this way may not be subdesigns of P_{44} and P_{60}, since the normalized column need not be the first column of the Hadamard matrix. We present the search results for strength-2 orthogonal arrays of $n = 36$, 44 and 60 runs in Table 5 and strength-3 orthogonal arrays of $2n = 72$, 88 and 120 runs in Table 6. Details of all the designs in this paper are available upon request.

For these projection designs the $|J_{\mathbf{u}}|$'s can only take two values, thus the criteria of minimum G- and G_2-aberration are equivalent. Let E be the complement of an $\mathrm{OA}(n, m, 2, 2)$, say D, in an $\mathrm{OA}(n, n-1, 2, 2)$. Then the complementary design theory ([21]) states that the sequential minimization of $A_3(D)$ and $A_4(D)$ can be done by sequentially maximizing $A_3(E)$ and minimizing $A_4(E)$, where the latter is much faster when $m > n/2$. In addition, when $|J_{\mathbf{u}}(E)|$ can only be 4 or 12, we have $A_3(E) \le \binom{n-1-m}{3}(12/n)^2$ and $A_4(E) \ge \binom{n-1-m}{4}(4/n)^2$. Similar bounds can also be derived for strength-3 designs. These simple bounds enable us to identify the best projection designs when the search is incomplete. In Tables 5 and 6, we mark a value or a vector by an asterisk if it is minimized or sequentially minimized among all projections, respectively.

TABLE 5
Strength-2 designs of 36, 44 and 60 runs.

$n \times m$	(A_3, A_4)	$(F_3(12), F_4(12))$	$n \times m$	(A_3, A_4)	$(F_3(12), F_4(12))$
36×19	(26.6, 122.5)	(148, 756)	36×26	(76.5, 456.9)	(450, 2757)
36×20	(32.2, 150.4)	(184, 917)	36×27	(86.2, 536.5)	(507, 3238)
36×21	(37.7, 187.0)	(215, 1145)	36×28	(97.9, 622.7)	(582, 3745)
36×22	(44.1, 225.9)	(254, 1373)	36×29	(109.1*, 722.6)	(648*, 4347)
36×23	(50.7, 273.7)	(292, 1664)	36×30	(122.2*, 831.7)	(730*, 4995)
36×24	(59.5, 324.7)	(349, 1959)	36×31	(135.9*, 953.9)	(814*, 5725)
36×25	(67.7, 386.3)	(398, 2330)	36×32	(150.2, 1089.8)*	(901, 6539)*
44×23	(41.5, 216.9)	(407, 2174)	44×32	(120.0, 878.1)	(1195, 8786)
44×24	(47.7, 262.4)	(469, 2641)	44×33	(132.2, 999.5)	(1317, 10003)
44×25	(54.4, 311.5)	(536, 3130)	44×34	(145.2, 1131.5)	(1448, 11317)
44×26	(61.9, 365.0)	(611, 3652)	44×35	(159.0, 1277.4)	(1587, 12776)
44×27	(69.9, 430.5)	(691, 4317)	44×36	(173.7, 1437.2)	(1734, 14374)
44×28	(78.4, 501.0)	(777, 5019)	44×37	(189.1, 1611.3)	(1889, 16116)
44×29	(87.9, 582.0)	(872, 5834)	44×38	(205.5, 1800.5)	(2054, 18006)
44×30	(97.9, 670.5)	(973, 6715)	44×39	(222.8, 2006.2)	(2227, 20063)
44×31	(108.6, 767.8)	(1081, 7680)	44×40	(240.9, 2229.1)*	(2409, 22291)*
60×31	(77.2, 554.0)	(1610, 11647)	60×44	(231.3, 2382.9)	(4849, 50049)
60×32	(85.2, 631.4)	(1775, 13262)	60×45	(248.0, 2615.0)	(5201, 54923)
60×33	(94.1, 718.0)	(1964, 15078)	60×46	(265.4, 2863.7)	(5566, 60143)
60×34	(103.3, 813.9)	(2156, 17093)	60×47	(283.6, 3130.1)	(5948, 65739)
60×35	(113.0, 920.2)	(2360, 19336)	60×48	(302.7, 3414.5)	(6351, 71709)
60×36	(123.7, 1033.2)	(2586, 21697)	60×49	(322.5, 3717.9)	(6768, 78081)
60×37	(134.8, 1158.5)	(2820, 24327)	60×50	(343.2, 4041.0)	(7202, 84867)
60×38	(146.4, 1295.4)	(3062, 27206)	60×51	(364.9, 4384.8)	(7659, 92085)
60×39	(158.8, 1444.4)	(3323, 30343)	60×52	(387.3, 4750.1)	(8130, 99755)
60×40	(171.8, 1603.3)	(3597, 33670)	60×53	(410.7, 5137.6)	(8623, 107891)
60×41	(185.7, 1777.6)	(3889, 37337)	60×54	(435.0, 5548.5)	(9133, 116520)
60×42	(200.1, 1964.4)	(4193, 41258)	60×55	(460.2, 5983.5)*	(9663, 125654)*
60×43	(215.3, 2166.2)	(4514, 45497)	60×56	(486.3, 6443.7)*	(10212, 135318)*

Next we study designs from Q_{2n}'s with run sizes 52, 60 and 76 and search for those with minimum G-aberration. Although the minimum G_2-aberration cri-

Table 6
Strength-3 designs of 72, 88 and 120 runs.

$2n \times m$	A_4	$F_4(24)$	$2n \times m$	A_4	$F_4(24)$	$2n \times m$	A_4	$F_4(24)$
72×9	2.8	13	72×17	70.2	413	72×25	381.9	2285
72×10	5.1	25	72×18	90.7	536	72×26	451.7	2705
72×11	8.5	45	72×19	115.5	685	72×27	530.8	3181
72×12	13.4	74	72×20	144.9	861	72×28	619.6	3714
72×13	20.0	113	72×21	179.4	1068	72×29	719.2	4313
72×14	28.5	163	72×22	219.8	1311	72×30	830.2*	4980*
72×15	39.4	228	72×23	266.7	1593	72×31	953.4*	5720*
72×16	53.2	311	72×24	320.4	1916	72×32	1089.7*	6538*
88×9	2.0	14	88×20	115.5	1142	88×31	765.7	7648
88×10	3.9	33	88×21	143.1	1416	88×32	875.6	8748
88×11	6.8	61	88×22	175.6	1741	88×33	996.8	9961
88×12	10.6	98	88×23	213.1	2116	88×34	1129.9	11293
88×13	15.7	148	88×24	256.5	2552	88×35	1276.0	12754
88×14	22.4	214	88×25	305.8	3044	88×36	1436.0	14357
88×15	31.2	302	88×26	362.0	3607	88×37	1610.4	16102
88×16	42.1	409	88×27	425.3	4239	88×38	1800.1*	18000*
88×17	55.3	539	88×28	497.1	4959	88×39	2006.0*	20060*
88×18	72.0	707	88×29	577.2	5762	88×40	2229.0*	22290*
88×19	91.8	904	88×30	666.4	6654			
120×9	1.6	29	120×25	218.6	4567	120×41	1774.2	37241
120×10	2.9	55	120×26	259.1	5418	120×42	1961.4	41172
120×11	4.7	92	120×27	304.3	6365	120×43	2162.8	45403
120×12	7.5	149	120×28	355.6	7443	120×44	2379.6	49957
120×13	11.2	227	120×29	413.0	8646	120×45	2612.0	54839
120×14	15.9	323	120×30	477.3	9998	120×46	2861.3	60075
120×15	22.0	449	120×31	548.3	11488	120×47	3127.9	65677
120×16	29.9	614	120×32	627.3	13148	120×48	3412.5	71654
120×17	39.5	813	120×33	714.3	14975	120×49	3716.1	78030
120×18	51.4	1062	120×34	810.4	16996	120×50	4039.6	84825
120×19	65.5	1359	120×35	915.3	19197	120×51	4383.6	92051
120×20	82.2	1707	120×36	1030.3	21613	120×52	4749.1	99727
120×21	101.9	2118	120×37	1155.6	24245	120×53	5137.0	107874
120×22	125.3	2611	120×38	1292.0	27111	120×54	5548.1*	116508*
120×23	152.2	3175	120×39	1439.9	30216	120×55	5983.4*	125650*
120×24	183.3	3826	120×40	1600.6	33592	120×56	6443.7*	135317*

terion and complementary design theory cannot be applied to find such designs because $|J_{\mathbf{u}}(Q_{2n})|$ takes three values for $|\mathbf{u}| = 4$, we can still use the minimum G_e-aberration to accelerate the search as suggested by [8]. For a design D with N runs, the criterion of minimum G_e-aberration sequentially minimizes $A_{1,e}(D), \ldots, A_{m,e}(D)$ where $A_{k,e}(D) = \sum_{|\mathbf{u}|=k} |J_{\mathbf{u}}(D)/N|^e$ for some $e > 0$. It can be shown that for $\mathrm{OA}(2n, 2^m, 2)$s studied here, the minimum G- and G_e-aberration criteria are equivalent if we take $e > \log \binom{m}{4} / \{\log(20) - \log(12)\}$. For each $2n \times m$, a complete search is done if $\binom{n}{m} < 200{,}000$ otherwise a total of 200,000 random subdesigns from Q_{2n} are compared then the best one is selected. The results are displayed in Table 7. We mark a value or a vector by an asterisk if it is minimized or sequentially minimized among all projections, respectively.

TABLE 7
Strength-2 designs of 52, 60 and 76 runs.

$2n \times m$	A_4	$F_4(20,12)$	$2n \times m$	A_4	$F_4(20,12)$	$2n \times m$	A_4	$F_4(20,12)$
52×4	0.01*	$(0,0)^*$	52×12	17.84	$(30,225)$	52×20	178.07	$(420,1896)$
52×5	0.08*	$(0,1)^*$	52×13	25.77	$(50,305)$	52×21	220.08*	$(520,2341)^*$
52×6	0.37	$(0,6)$	52×14	36.46	$(74,423)$	52×22	269.08*	$(636,2862)^*$
52×7	1.01	$(0,17)$	52×15	49.97	$(110,555)$	52×23	325.77*	$(770,3465)^*$
52×8	2.02	$(0,34)$	52×16	66.15	$(149,723)$	52×24	390.92*	$(924,4158)^*$
52×9	3.96	$(5,53)$	52×17	86.89	$(201,935)$	52×25	465.38*	$(1100,4950)^*$
52×10	6.92	$(9,93)$	52×18	112.02	$(260,1204)$			
52×11	11.89	$(20,150)$	52×19	142.32	$(334,1520)$			
60×5	0.02*	$(0,0)^*$	60×14	31.15	$(97,460)$	60×23	283.91	$(980,3938)$
60×6	0.28	$(0,6)$	60×15	41.80	$(135,600)$	60×24	340.88	$(1179,4722)$
60×7	0.80	$(0,18)$	60×16	57.44	$(188,824)$	60×25	405.91*	$(1405,5620)^*$
60×8	1.63	$(0,37)$	60×17	75.68	$(251,1078)$	60×26	479.85*	$(1661,6644)^*$
60×9	3.23	$(5,60)$	60×18	97.19	$(328,1367)$	60×27	563.33*	$(1950,7800)^*$
60×10	5.91	$(14,98)$	60×19	123.50	$(420,1729)$	60×28	657.22*	$(2275,9100)^*$
60×11	9.68	$(26,153)$	60×20	154.69	$(529,2158)$	60×29	762.38*	$(2639,10556)^*$
60×12	14.86	$(43,227)$	60×21	191.29	$(656,2664)$			
60×13	22.24	$(66,338)$	60×22	234.18	$(805,3257)$			
76×6	0.04*	$(0,0)^*$	76×17	58.80	$(514,814)$	76×28	522.71	$(4666,7030)$
76×7	0.47	$(2,11)$	76×18	75.62	$(660,1050)$	76×29	606.27	$(5417,8138)$
76×8	1.06	$(5,24)$	76×19	96.90	$(853,1329)$	76×30	699.85	$(6253,9396)$
76×9	2.32	$(17,38)$	76×20	122.14	$(1079,1669)$	76×31	803.95	$(7186,10787)$
76×10	4.11	$(33,60)$	76×21	150.89	$(1336,2053)$	76×32	918.94	$(8216,12324)$
76×11	7.23	$(60,105)$	76×22	185.18	$(1641,2519)$	76×33	1045.96	$(9352,14028)$
76×12	11.43	$(92,178)$	76×23	224.66	$(1999,3034)$	76×34	1185.53*	$(10600,15900)^*$
76×13	16.78	$(142,242)$	76×24	270.32	$(2405,3655)$	76×35	1338.53*	$(11968,17952)^*$
76×14	24.14	$(205,349)$	76×25	322.13	$(2868,4351)$	76×36	1505.84*	$(13464,20196)^*$
76×15	33.32	$(285,478)$	76×26	381.22	$(3399,5137)$	76×37	1688.37*	$(15096,22644)^*$
76×16	44.71	$(385,635)$	76×27	447.44	$(3996,6009)$			

4.2. Designs from the tensor product method

Besides designs from Paley's constructions, [15] constructed some strength-2 orthogonal arrays with maximum generalized resolutions by the tensor product $D = H_{n_1} \otimes B$ for $n_1 = 2$ and 4, where $B = (b_1, \ldots, b_{m_2})$ is an $\mathrm{OA}(n_2, 2^{m_2}, 2)$,

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \text{and} \quad H_4 = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

We provide some theoretical results to select such designs by the minimum G-aberration criterion. For convenience, we use again the equivalence of minimum G- and G_e-aberrations for large e and present our results in terms of the latter.

Proposition 3. *Suppose $D = H_{n_1} \otimes B$ for $n_1 = 2$ or 4.*

(i) We have $A_{3,e}(D) = \gamma_1 A_{3,e}(B)$ and $A_{4,e}(D) = \gamma_2 A_{4,e}(B) + \gamma_3$, where γ_1, γ_2 and γ_3 are positive constants depending on H_{n_1} and e.

(ii) Let $g(k) = \sum_{i<j} |J(b_i, b_j, b_k)/n_2|^e$ for $k = 1, \ldots, m_2$ and suppose $g(k_0) = \max_{1 \le k \le m_2} g(k)$. Then designs obtained by successively removing columns of $H_{n_1} \otimes b_{k_0}$ from D have minimum $A_{3,e}$ values among all projections of D.

With a sufficiently large e, part (i) of Proposition 3 implies that the G-aberration property of D is determined by that of B, and that it is preferable to use a B with minimum G-aberration. This is feasible as catalogues of designs with minimum or small G-aberration for small run sizes are readily available in [18] and [13]. After that, we apply part (ii) of Proposition 3 to delete columns from $H_{n_1} \otimes B$ to cover all cases. Following this procedure, we obtain the designs of 48, 64, 96 and 128 runs displayed in Tables 8 and 9. We note that when $m \leq 56$ for 64-run designs, it is better to take $A = H_4$ and B as 16-run minimum G-aberration designs in [18] than to take $A = H_2$ and B as the 32-run designs in [13].

5. Further results

The three- and four-column J-characteristics of a design, as we have seen, play a crucial role in its generalized resolutions and projection properties. [14] showed that these J-characteristics bear a close relationship to the *type* of Hadamard matrices. We conclude the paper with more results on the type of certain Hadamard matrices.

The concept of type was introduced by [10] and further studied in [9]. Let H be a Hadamard matrix of order N. By permutation and negation of rows and columns, any four columns of H that can be transformed into the following form

$$\begin{bmatrix} \mathbf{1}_a & \mathbf{1}_a & \mathbf{1}_a & \mathbf{1}_a \\ \mathbf{1}_b & \mathbf{1}_b & \mathbf{1}_b & -\mathbf{1}_b \\ \mathbf{1}_b & \mathbf{1}_b & -\mathbf{1}_b & \mathbf{1}_b \\ \mathbf{1}_a & \mathbf{1}_a & -\mathbf{1}_a & -\mathbf{1}_a \\ \mathbf{1}_b & -\mathbf{1}_b & \mathbf{1}_b & \mathbf{1}_b \\ \mathbf{1}_a & -\mathbf{1}_a & \mathbf{1}_a & -\mathbf{1}_a \\ \mathbf{1}_a & -\mathbf{1}_a & -\mathbf{1}_a & \mathbf{1}_a \\ \mathbf{1}_b & -\mathbf{1}_b & -\mathbf{1}_b & -\mathbf{1}_b \end{bmatrix}$$

where $a + b = N/4$ and $0 \leq b \leq \lfloor N/8 \rfloor$, is said to be of type b. A Hadamard matrix is of type b if it has a set of four columns of type b but has no set of four columns of type less than b. [14] established a connection between the type of H and the $\mathrm{OA}(N, 2^{N-1}, 2)$ derived from H, which can be rephrased as the following lemma.

Table 8

Strength-2 designs of 48 runs.

$N \times m$	A_3	$F_3(8)$	$N \times m$	A_3	$F_3(8)$	$N \times m$	A_3	$F_3(8)$
48×25	42.2	1520	48×32	99.6	3584	48×39	197.3	7104
48×26	48.9	1760	48×33	112.0	4032	48×40	213.3	7680
48×27	55.6	2000	48×34	124.4	4480	48×41	233.3	8400
48×28	62.2	2240	48×35	136.9	4928	48×42	253.3	9120
48×29	71.6	2576	48×36	149.3	5376	48×43	273.3	9840
48×30	80.9	2912	48×37	165.3	5952	48×44	293.3	10560
48×31	90.2	3248	48×38	181.3	6528			

Table 9
Strength-2 designs of 64, 96 and 128 runs.

$N \times m$	A_3	$F_3(16)$	$N \times m$	A_3	$F_3(16)$	$N \times m$	A_3	$F_3(16)$
64×33	16.0	256	64×43	178.0	2848	64×53	376.0	6016
64×34	32.0	512	64×44	192.0	3072	64×54	400.0	6400
64×35	48.0	768	64×45	208.0	3328	64×55	424.0	6784
64×36	64.0	1024	64×46	224.0	3584	64×56	448.0	7168
64×37	92.0	1472	64×47	240.0	3840	64×57	477.9	7646
64×38	104.0	1664	64×48	256.0	4096	64×58	504.0	8064
64×39	116.0	1856	64×49	280.0	4480	64×59	532.0	8512
64×40	128.0	2048	64×50	304.0	4864	64×60	560.0	8960
64×41	150.0	2400	64×51	328.0	5248	64×61	590.0	9440
64×42	164.0	2624	64×52	352.0	5632	64×62	620.0	9920
96×49	124.0	4464	96×64	398.2	14336	96×79	821.3	29568
96×50	136.0	4896	96×65	427.1	15376	96×80	853.3	30720
96×51	148.0	5328	96×66	450.7	16224	96×81	904.0	32544
96×52	160.0	5760	96×67	474.2	17072	96×82	940.4	33856
96×53	191.1	6880	96×68	497.8	17920	96×83	976.9	35168
96×54	208.0	7488	96×69	522.7	18816	96×84	1013.3	36480
96×55	224.9	8096	96×70	547.6	19712	96×85	1053.3	37920
96×56	241.8	8704	96×71	572.4	20608	96×86	1093.3	39360
96×57	261.3	9408	96×72	597.3	21504	96×87	1133.3	40800
96×58	280.9	10112	96×73	634.7	22848	96×88	1173.3	42240
96×59	300.4	10816	96×74	664.9	23936	96×89	1217.3	43824
96×60	320.0	11520	96×75	695.1	25024	96×90	1261.3	45408
96×61	342.2	12320	96×76	725.3	26112	96×91	1305.3	46992
96×62	360.9	12992	96×77	757.3	27264	96×92	1349.3	48576
96×63	379.6	13664	96×78	789.3	28416			
128×65	310.5	19872	128×85	749.2	47952	128×105	1476.2	94480
128×66	327.0	20928	128×86	778.5	49824	128×106	1521.5	97376
128×67	343.5	21984	128×87	807.8	51696	128×107	1566.8	100272
128×68	360.0	23040	128×88	837.0	53568	128×108	1612.0	103168
128×69	379.8	24304	128×89	869.2	55632	128×109	1660.8	106288
128×70	398.5	25504	128×90	901.5	57696	128×110	1709.5	109408
128×71	417.2	26704	128×91	933.8	59760	128×111	1758.2	112528
128×72	436.0	27904	128×92	966.0	61824	128×112	1807.0	115648
128×73	457.0	29248	128×93	1001.5	64096	128×113	1859.2	118992
128×74	478.0	30592	128×94	1037.0	66368	128×114	1911.5	122336
128×75	499.0	31936	128×95	1072.5	68640	128×115	1963.8	125680
128×76	520.0	33280	128×96	1108.0	70912	128×116	2016.0	129024
128×77	544.5	34848	128×97	1147.5	73440	128×117	2072.0	132608
128×78	568.0	36352	128×98	1186.0	75904	128×118	2128.0	136192
128×79	591.5	37856	128×99	1224.5	78368	128×119	2184.0	139776
128×80	615.0	39360	128×100	1263.0	80832	128×120	2240.0	143360
128×81	642.0	41088	128×101	1304.8	83504	128×121	2300.0	147200
128×82	668.0	42752	128×102	1346.5	86176	128×122	2360.0	151040
128×83	694.0	44416	128×103	1388.2	88848	128×123	2420.0	154880
128×84	720.0	46080	128×104	1430.0	91520	128×124	2480.0	158720

Lemma 7. *A Hadamard matrix H has type b if and only if* $\max_{|\mathbf{u}|=4} |J_{\mathbf{u}}(H)| = N - 8b$.

Lemma 7 is useful for finding the type of a Hadamard matrix; it can also be taken as a definition of type for anyone who finds the original definition cumbersome.

Proposition 4. *Let H_1 and H_2 be any two Hadamard matrices of orders N_1 and N_2, respectively. Then $H_1 \otimes H_2$ has type 0.*

The special case that H_1 is of order 2 was considered by [14]. Proposition 4 shows that a tensor product inevitably introduces defining words of lengths 4, and thus cannot be used to construct designs with the attractive properties as described in Section 3.

Proposition 5. *Hadamard matrices from Paley's second construction are of type 1.*

Proposition 2 implies that appending more columns to Q_{2n} will lead to severe aliasing among certain three or four columns. As a result, we cannot obtain designs with large generalized resolutions or good projection properties from them.

Propositions 4 and 5 are worth documenting even though they are somewhat negative. They convey a message that we should look elsewhere if we want to find Hadamard matrices of large types.

Appendix: Proofs

Proof of Theorem 1. [3] showed that for $N/3 \le m \le N/2$, any $\mathrm{OA}(N, 2^m, 3)$ can be written as $D = [V^T \ -V^T]^T$ where $V = [v_1, \ldots, v_m]$ is an $(N/2) \times m$ matrix of ± 1 with orthogonal columns. Clearly, we have $\max_{|\mathbf{u}|=4} |J_{\mathbf{u}}(D)| = 2\max_{|\mathbf{u}|=4} |J_{\mathbf{u}}(V)|$. The rest of the proof is similar to that for Theorem 1 in [15]. Let $n' = N/2$ and $m' = n' - m$. Then there exist real vectors $w_1, \ldots, w_{m'}$ such that $(n')^{-1/2}[v_1, \ldots, v_m, w_1, \ldots, w_{m'}]$ form an orthonormal basis for the n'-dimensional Euclidean space. We first consider the scenario $m' \ge 4$. Note that

$$
\begin{aligned}
&\sum_{distinct\, i_1,i_2,i_3,i_4} J(v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4})^2 \\
&= \sum_{distinct\, i_1,i_2,i_3} \left\{ (n')^2 - \sum_{i_4=1}^{m'} J(v_{i_1}, v_{i_2}, v_{i_3}, w_{i_4})^2 \right\} \\
&= m(m-1)(m-2)(n')^2 - \sum_{i_1 \ne i_2} \sum_{i_4=1}^{m'} \left\{ (n')^2 - \sum_{i_3 \ne i_4} J(v_{i_1}, v_{i_2}, w_{i_3}, w_{i_4})^2 \right\} \\
&= \{m(m-1)(m-2) - m(m-1)m'\}(n')^2 \\
&\quad + \sum_{i_1=1}^{m} \sum_{i_3 \ne i_4} \left\{ (n')^2 - \sum_{i_2 \ne i_3, i_4} J(v_{i_1}, w_{i_2}, w_{i_3}, w_{i_4})^2 \right\} \\
&= \{m(m-1)(m-2) - m(m-1)m' + mm'(m'-1) - m'(m'-1)(m'-2)\}(n')^2 \\
&\quad + \sum_{distinct\, i_1,i_2,i_3,i_4} J(w_{i_1}, w_{i_2}, w_{i_3}, w_{i_4})^2,
\end{aligned}
$$

where, for example, we use $J(v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4})$ to denote the J-characteristics of columns v_{i_1}, v_{i_2}, v_{i_3} and v_{i_4}. Thus we have that $\sum_{distinct\, i_1,i_2,i_3,i_4} J(v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4})^2 \geq \{m(m-1)(m-2) - m(m-1)m' + mm'(m'-1) - m'(m'-1)(m'-2)\}(n')^2$. It can be easily verified that the equality holds for $m' \leq 3$. Therefore, $\max_{|\mathbf{u}|=4} J_{\mathbf{u}}^2(V) \geq \{m(m-1)(m-2) - m(m-1)m' + mm'(m'-1) - m'(m'-1)(m'-2)\}(n')^2/\{m(m-1)(m-2)(m-3)\}$. Note that $n' - \max_{|\mathbf{u}|=4} |J_{\mathbf{u}}(V)|$ must be a multiple of 8 ([14]). The result follows by some tedious algebra. □

Proof of Proposition 1. Let D_0 be the projected design of P_n onto certain 4 factors. By Lemma 4, for any $\mathbf{s} \subseteq \mathbb{Z}_4$, the frequency of $\mathbf{r_s}$ occurs in D_0 is given by $N_{\mathbf{s}} = 2^{-4}\{n + \sum_{\emptyset \neq \mathbf{u} \subseteq \mathbb{Z}_4} h_{\mathbf{su}} J_{\mathbf{u}}(D_0)\}$. Recall that $J_{\mathbf{u}}(D_0) = 0$ for $|\mathbf{u}| = 1, 2$ and that $|J_{\mathbf{u}}(D_0)| \leq U_P(n)$ for $|\mathbf{u}| = 3, 4$. Then we have $N_{\mathbf{s}} \geq 2^{-4}\{n - 5U_P(n)\}$ since $h_{\mathbf{su}} = \pm 1$. The result on P_n follows by the fact that $N_{\mathbf{s}}$ must be an integer and that $\mathbf{s}$ is arbitrary. The proof for $\tilde{P}_{2n}$ can be done similarly by noting that $J_{\mathbf{u}}(\tilde{P}_{2n}) = 0$ for $|\mathbf{u}| \leq 3$ and $|\mathbf{u}| = 5$ and that $|J_{\mathbf{u}}(\tilde{P}_{2n})| \leq 2U_P(n)$ for $|\mathbf{u}| = 4$. □

Proof of Lemma 5. The arguments are similar to the proofs for Theorem 2.1 of [2] and Theorem 5 of [15]. For simplicity, we outline the proof for $|\mathbf{u}| = 4$ and omit that for $|\mathbf{u}| = 5$. Let's write $Q_{2n} = [q_0, q_1, \ldots, q_s]$, where $s = n - 1$. Then for any distinct integers $i_1, i_2, i_3, i_4 \in \{1, \ldots, s\}$, by some simple algebra we have $J(q_0, q_{i_1}, q_{i_2}, q_{i_3}) = 2\sum_{y \in GF(s) \setminus \{\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}\}} \chi((y - \alpha_{i_1})(y - \alpha_{i_2})(y - \alpha_{i_3}))$ and $J(q_{i_1}, q_{i_2}, q_{i_3}, q_{i_4}) = 2\sum_{y \in GF(s) \setminus \{\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3}, \alpha_{i_4}\}} \chi((y - \alpha_{i_1})(y - \alpha_{i_2})(y - \alpha_{i_3})(y - \alpha_{i_4})) + 2$. Let $N(s, k)$ be the number of solutions $(z, y) \in GF(s) \times GF(s)$ of $z^2 = \prod_{j=1}^{k}(y - \alpha_{i_j})$. Then we have $J(q_0, q_{i_1}, q_{i_2}, q_{i_3}) = 2N(s, 3) - 2s$ and $J(q_{i_1}, q_{i_2}, q_{i_3}, q_{i_4}) = 2N(s, 4) + 2 - 2s$. By a result of [7] quoted by [16], we know that

$$|N(s,3) - s| \leq 2s^{1/2} \quad \text{and} \quad |N(s,4) - s + 1| \leq 2s^{1/2}, \tag{4}$$

from which it follows that $\max_{|\mathbf{u}|=4} |J_{\mathbf{u}}(Q_{2n})| \leq 4s^{1/2}$. The upper bound on $\max_{|\mathbf{u}|=4} |J_{\mathbf{u}}(Q_{2n})|$ follows by noting that $(2n - |J_{\mathbf{u}}(Q_{2n})|)/8$ must be an integer. □

Proof of Lemma 6. Suppose X is a subdesign of P_n for h factors. Then the model matrix M for all the main effects and two-factor interactions of these h factors can be written as $M = [\mathbf{1}_n \ X \ Y]$, where Y is an $n \times \{h(h-1)/2\}$ matrix consisting of all the pairwise Hadamard products of columns of X. It can then be checked that in each row of the information matrix $M^T M$, there are at most $(h-1)(h-2)/2$ nonzero off-diagonal elements whose absolute values are all bounded above by $\max_{|\mathbf{u}|=3,4} |J_{\mathbf{u}}(P_n)| \leq U_P(n)$. A square matrix $Z = (z_{ij})$ is said to be strictly diagonally dominant if $|z_{ii}| > \sum_{j \neq i} |z_{ij}|$ for all i; by Levy-Desplanques theorem, such a matrix must be nonsingular. Therefore, if $n > (h-1)(h-2)U_P(n)/2$, $M^T M$ is strictly diagonally dominant and thus nonsingular. This completes the proof for P_n. The proofs for $\tilde{P}_{2n}$ and Q_{2n} are similar and thus omitted. □

Proof of Proposition 3. With a slight abuse of notation, write $H_{n_1} = [h_1, \dots, h_{n_1}]$. Then invoking Lemma 2 of [20], we have

$$\begin{aligned} A_{3,e}(D) &= \sum_{distinct\ \{(i_1,i_2),(j_1,j_2),(k_1,k_2)\}} |J(h_{i_1} \otimes b_{i_2}, h_{j_1} \otimes b_{j_2}, h_{k_1} \otimes b_{k_2})/(n_1 n_2)|^e \\ &= \sum_{1 \le i_1, j_1, k_1 \le n_1} \sum_{i_2 < j_2 < k_2} |J(h_{i_1}, h_{j_1}, h_{k_1})/n_1|^e |J(b_{i_2}, b_{j_2}, b_{k_2})/n_2|^e, \end{aligned}$$

since $J(b_{i_2}, b_{j_2}, b_{k_2}) = 0$ as long as i_2, j_2 and k_2 have common elements. Therefore $A_{3,e}(D) = \gamma_1 A_{3,e}(B)$ with $\gamma_1 = \sum_{1 \le i_1, j_1, k_1 \le n_1} |J(h_{i_1}, h_{j_1}, h_{k_1})/n_1|^e$. The proof for the result on $A_{4,e}(D)$ is similar. Part (ii) can be done by observing that at each time a column of $H \otimes b_{k_0}$ is removed, $A_{3,e}$ decreases by the same and also the maximum possible amount. □

Proof of Proposition 4. Let $h_1^{(1)}, h_2^{(1)}$ be two columns of H_1 and $h_1^{(2)}, h_2^{(2)}$ be two columns of H_2. Then we have that $J(h_1^{(1)} \otimes h_1^{(2)}, h_1^{(1)} \otimes h_2^{(2)}, h_2^{(1)} \otimes h_1^{(2)}, h_2^{(1)} \otimes h_2^{(2)}) = J(h_1^{(1)}, h_1^{(1)}, h_2^{(1)}, h_2^{(1)}) J(h_1^{(2)}, h_2^{(2)}, h_1^{(2)}, h_2^{(2)}) = N_1 N_2$. Proposition 4 now follows from Lemma 7. □

Proof of Proposition 5. Write H in (2) as

$$H = \begin{bmatrix} F & G \\ G & -F \end{bmatrix}, \qquad \text{where} \quad F = \begin{bmatrix} 1 & \mathbf{1}_s^T \\ \mathbf{1}_s & K + I_s \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} -1 & \mathbf{1}_s^T \\ \mathbf{1}_s & K - I_s \end{bmatrix}.$$

For $1 \le i < j \le n$, let f_i and f_j (respectively, g_i and g_j) be the ith and jth column of F (respectively, G). Then the J-characteristic of the following four columns of H

$$\begin{bmatrix} f_i & f_j & g_i & g_j \\ g_i & g_j & -f_i & -f_j \end{bmatrix}$$

is $2J(f_i, f_j, g_i, g_j) = 2J(f_i g_i, f_j g_j)$. Note that the column $f_i g_i$ is all ones except for the ith entry, which is -1. One can easily see that $2J(f_i g_i, f_j g_j) = 2\{(n-2) - 2\} = 2n - 8$. On the other hand, since $2n$ is not a multiple of 8, $\max_{|\mathbf{u}|=4} |J_{\mathbf{u}}(H)|$ can be at most $2n - 8$ ([4]). Therefore, we have $\max_{|\mathbf{u}|=4} |J_{\mathbf{u}}(H)| = 2n - 8$ and result follows by Lemma 7. □

References

[1] Box, G. and Tyssedal, J. (1996). Projective properties of certain orthogonal arrays. *Biometrika* **83** 950–955. MR1440059

[2] Bulutoglu, D. A. and Cheng, C.-S. (2003). Hidden projection properties of some nonregular fractional factorial designs and their applications. *Ann. Statist.* **31** 1012–1026. MR1994740

[3] Butler, N. A. (2007). Results for two-level fractional factorial designs of resolution IV or more. *J. Statist. Plann. Inference* **137** 317–323. MR2292860

[4] Cheng, C.-S. (1995). Some projection properties of orthogonal arrays. *Ann. Statist.* **23** 1223–1233. MR1353503

[5] Cheng, C.-S. (1998). Some hidden projection properties of orthogonal arrays with strength three. *Biometrika* **85** 491–495. MR1649130

[6] Deng, L.-Y. and Tang, B. (1999). Generalized resolution and minimum aberration criteria for Plackett-Burman and other nonregular factorial designs. *Statist. Sinica* **9** 1071–1082. MR1744824

[7] Hasse, H. (1936). Zur Theorie der abstrakten elliptischen Funktionenkörper. I-III. *J. Reine Angew. Math.* **175** 55–62, 69–88, 193–208. MR1581496

[8] Ingram, D. and Tang, B. (2005). Minimum G aberration design construction and design tables for 24 runs. *J. Qual. Technol.* **37** 101–114.

[9] Kharaghani, H. and Tayfeh-Rezaie, B. (2013). Hadamard matrices of order 32. *J. Combin. Des.* **21** 212–221. MR3037086

[10] Kimura, H. (1994). Classification of Hadamard matrices of order 28. *Discrete Math.* **133** 171–180. MR1298972

[11] Paley, R. E. A. C. (1933). On orthogonal matrices. *J. Math. Phys.* **12** 311–320.

[12] Schoen, E. D. and Mee, R. W. (2012). Two-level designs of strength 3 and up to 48 runs. *J. R. Stat. Soc. Ser. C. Appl. Stat.* **61** 163–174. MR2877591

[13] Schoen, E. D., Vo-Thanh, N. and Goos, P. (2017). Two-level orthogonal screening designs with 24, 28, 32, and 36 runs. *J. Amer. Statist. Assoc.* **112** 1354–1369. MR3735383

[14] Shi, C. and Tang, B. (2018). Designs from good Hadamard matrices. *Bernoulli* **24** 661–671. MR3706772

[15] Shi, C. and Tang, B. (2021). On construction of nonregular two-level factorial designs with maximum generalized resolution. *Statist. Sinica* **33** . MR4575316

[16] Stark, H. M. (1973). On the Riemann hypothesis in hyperelliptic function fields. In *Analytic Number Theory (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972)* 285–302. MR0332793

[17] Stufken, J. and Tang, B. (2007). Complete enumeration of two-level orthogonal arrays of strength d with $d + 2$ constraints. *Ann. Statist.* **35** 793–814. MR2336869

[18] Sun, D. X., Li, W. and Ye, K. Q. (2008). Algorithmic construction of catalogs of non-isomorphic two-level orthogonal designs for economic run sizes. *Statist. Appl.* **6** 141–155.

[19] Tang, B. (2001). Theory of J-characteristics for fractional factorial designs and projection justification of minimum G_2-aberration. *Biometrika* **88** 401–407. MR1844840

[20] Tang, B. (2006). Orthogonal arrays robust to nonnegligible two-factor interactions. *Biometrika* **93** 137–146. MR2277746

[21] Tang, B. and Deng, L.-Y. (1999). Minimum G_2-aberration for nonregular fractional factorial designs. *Ann. Statist.* **27** 1914–1926. MR1765622

[22] Vazquez, A. R., Goos, P. and Schoen, E. D. (2019). Constructing two-level designs by concatenation of strength-3 orthogonal arrays. *Tech-*

nometrics **61** 219–232. MR3957143

[23] Vazquez, A. R., Schoen, E. D. and Goos, P. (2021). Two-level orthogonal screening designs with 80, 96, and 112 runs, and up to 29 factors. *J. Qual. Technol.* MR3735383

[24] Vazquez, A. R. and Xu, H. (2019). Construction of two-level nonregular designs of strength three with large run sizes. *Technometrics* **61** 341–353. MR3987624

[25] Wang, J. C. and Wu, C. F. J. (1995). A hidden projection property of Plackett-Burman and related designs. *Statist. Sinica* **5** 235–250. MR1329295

[26] Wang, Y., Liu, S. and Lin, D. K. J. (2022). On definitive screening designs using Paley's conference matrices. *Statist. Probab. Lett.* **181** Paper No. 109267, 6. MR4329125

[27] Xu, H., Phoa, F. K. H. and Wong, W. K. (2009). Recent developments in nonregular fractional factorial designs. *Stat. Surv.* **3** 18–46. MR2520978