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Scaling limits of directed polymers in spatial-correlated environment[*][†]

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Abstract

We consider a directed polymer model in dimension $1+1$, where the random walk is attracted to stable law and the environment is independent in time variable and correlated in space variable. We obtain the scaling limit in the intermediate disorder regime for partition function, and show that the rescaled point-to-point partition function of directed polymers converges in the space of continuous functions to the solution of a stochastic heat equation driven by time-white spatial-colored noise. The scaling limit of the polymer transition probability is also established in the path space. The proof of the tightness is based on the gradient estimates for symmetric random walks in the domain of normal attraction of α-stable law which are established in this paper.

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1 introduction

The directed polymer model in random environment was originally introduced in [31]. It was formulated as the polymer measure in [10, 33]. The directed polymer is described by a random probability distribution on the path space $(\mathbb{Z}^d)^{\mathbb{Z}_+}$ of random walks on the d-dimensional lattice. For fixed environment $\omega = \{\omega(i,x), (i,x) \in \mathbb{Z}_+ \times \mathbb{Z}^d\}$ which is a family of real valued, non-constant, and identically distributed random variables on a probability space $(\Omega, \mathcal{G}, \mathbf{P})$, any $n \geq 1$, the polymer measure is the probability measure on the path space $((\mathbb{Z}^d)^{\mathbb{Z}_+}, \mathcal{F}, \mathbb{P})$ defined by

$$\mathbb{P}^{\omega}_{n,\beta}(S) := \frac{1}{Z_n(\beta,\omega)} e^{\beta \sum_{i=1}^{n} \omega(i,S_i)} \mathbb{P}(S), \tag{1.1}$$

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where $\beta > 0$ is the inverse temperature, $S = \{S_n, n \geq 0\}$ is a random walk starting from origin in $\mathbb{Z}^d$, $Z_n(\beta, \omega)$ is the point-to-line partition function defined by

$$Z_n(\beta;\omega) := \mathbb{E}\left(e^{\beta \sum_{i=1}^n \omega(i,S_i)}\right). \tag{1.2}$$

Here, we denote by $\mathbb{E}$ and $\mathbf{E}$ the expectation with respect to $\mathbb{P}$ and $\mathbf{P}$. The quantity $p_n(\beta) := \frac{1}{n}\log Z_n(\beta;\omega)$ is called the free energy. Let $Z_{n,x}(\beta;\omega)$ be the point-to-point partition function

$$Z_{n,x}(\beta;\omega) := \mathbb{E}\left(e^{\beta \sum_{i=1}^n \omega(i,S_i)} I_{\{S_n = x\}}\right). \tag{1.3}$$

The distribution density for the polymer endpoint is thus

$$\mathbb{P}^\omega_{n,\beta}(S_n = x) := \frac{Z_{n,x}(\beta;\omega)}{Z_n(\beta;\omega)}. \tag{1.4}$$

Assume that for β sufficiently small,

$$\lambda(\beta) := \log \mathbf{E} e^{\beta\omega(i,x)} < \infty. \tag{1.5}$$

The normalized partition function is defined by

$$W_n := Z_n(\beta;\omega)\exp\{-n\lambda(\beta)\},\ n \geq 1. \tag{1.6}$$

It is known that if the environment variables are independent, then

$$p(\beta) := \lim_{n\to\infty} \frac{1}{n} \log Z_n(\beta;\omega), \quad W_\infty = \lim_{n\to\infty} W_n \tag{1.7}$$

exist $\mathbf{P}$-$a.s.$ and either the limit W_∞ is $\mathbf{P}$-$a.s.$ positive, or it is $\mathbf{P}$-$a.s.$ zero (cf. Theorem 2.1 and Theorem 3.1 in [15]). The polymer is in weak disorder regime if W_∞ is $\mathbf{P}$-$a.s.$ positive, and in strong disorder regime if W_∞ is $\mathbf{P}$-$a.s.$ zero. When $d = 1$, all $\beta > 0$ are in the strong disorder regime.

For the $d = 1$ case, Alberts, Khanin and Quastel [1] introduced a new disorder regime for directed polymers by scaling the inverse temperature β with the length of the polymer n. They showed that when the environment consists of i.i.d. random variables and $\{S_n, n \geq 0\}$ is the simple symmetric random walk, the following convergences hold: the rescaled partition function

$$Z_n(n^{-1/4}\beta;\omega)e^{-n\lambda(n^{-1/4}\beta)} \xrightarrow{(d)} \mathcal{Z}_{\sqrt{2}\beta}$$

and the rescaled point-to-point partition function

$$\frac{1}{2}\sqrt{n} Z_{nt,\sqrt{n}x}(n^{-1/4}\beta;\omega)e^{-n\lambda(n^{-1/4}\beta)} \xrightarrow{(d)} \mathcal{Z}_{\sqrt{2}\beta}(t,x) \quad \text{in } C((0,1]\times\mathbb{R}),$$

where $\{Z_{t,x};\ t \in \mathbb{R}_+, x \in \mathbb{R}\}$ is a linear interpolation of $\{Z_{m,k};\ m \in \mathbb{Z}_+, k \in \mathbb{Z}\}$, $\mathcal{Z}_{\sqrt{2}\beta} = \int \mathcal{Z}_{\sqrt{2}\beta}(1,x)dx$ and $\mathcal{Z}(t,x) := \mathcal{Z}_{\sqrt{2}\beta}(t,x)$ is the mild solution of the stochastic heat equation

$$\begin{cases} \partial_t \mathcal{Z} = \frac{1}{2}\Delta\mathcal{Z} + \sqrt{2}\beta\mathcal{Z}\dot{W}, \\ \mathcal{Z}(0,x) = \delta_0(x). \end{cases} \tag{1.8}$$

Here, $\xrightarrow{(d)}$ denotes the convergence in law. The result illustrates an intermediate disorder regime between weak and strong disorder regime.

Caravenna, Sun and Zygouras [14] extended the invariant principle of multi-linear polynomials of independent random variables in [39], and provided a unified framework to study the continuum and weak disorder scaling limits of statistical mechanics systems that are disorder relevant. In particular, a directed polymer with random walks attracted to stable laws (long-range directed polymer) was also studied in [14] and the convergence in the sense of the finite dimensional distributions was obtained. The intermediate disorder regime of the directed polymer with a heavy-tail disorder was studied in [3], [4] and [23]. Joseph [34] considered a model of discrete space-time stochastic heat equations, and showed that an appropriate scaling limit of the model with Lipschitz continuous initial data can get the following stochastic partial differential equation

$$\partial_t \mathcal{Z} = -\nu(-\Delta)^{\alpha/2}\mathcal{Z} + \sigma(\mathcal{Z})\dot{W}. \tag{1.9}$$

Rang [43] first considered time independent and space correlated environment. Furthermore, see [20] for multiple non-intersecting random walks, [21] for the Brownian directed polymer in Poissonian environment, [45] for the polymer given by the occupation field of a Poisson system of independent random walks, and the references therein.

Caravenna, Sun and Zygouras ([14], P.25) expected that the convergence in the sense of the finite dimensional distributions can be upgraded to convergence in the space of continuous functions equipped with uniform topology for the long-range directed polymer. In this paper, we study the problem for general long-range directed polymers in an environment which is independent in time variable and correlated in space variable. That is, the random walk $\{S_n, n \geq 0\}$ is in the domain of normal attraction of a stable law of index $\alpha \in (0, 2]$ with period q, and the environment $\omega = \{\omega(i, x), (i, x) \in \mathbb{Z}_+ \times \mathbb{Z}^d\}$ is an autoregressive integrated moving average model (cf. [27] [28] [43]):

$$\omega(i,x) = \sum_{y\in\mathbb{Z}} a_y \xi(i, x+y), \;\; a_y \sim c_r|y|^{-r}, \; c_r > 0, \; \frac{1}{2} < r < 1,$$

where $\{\xi(i,x) : i \in \mathbb{Z}_+, x \in \mathbb{Z}\}$ is a family of i.i.d. centered variables with an exponential moment. We show that when $\alpha \in (2r-1, 2]$, the rescaled partition function converges in distribution, and when $\alpha \in (1, 2]$, the rescaled point-to-point partition function converges weakly in path space to the solution of a stochastic heat equation driven by time-white spatial-colored noise, i.e.,

$$\frac{1}{q} n^{1/\alpha} Z_{nt, n^{1/\alpha}x}(\beta_n; \omega) e^{-n\lambda(\beta_n)} \xrightarrow{(d)} \mathcal{Z}_\beta(t,x) \quad \text{in } C((0,1]\times\mathbb{R}), \tag{1.10}$$

where $\beta_n = \beta n^{-\frac{1}{2}-\frac{1}{2\alpha}+\frac{r}{\alpha}}$, $\mathcal{Z}(t,x) := \mathcal{Z}_\beta(t,x)$ is the mild solution of the following fractional stochastic heat equation with initial $\mathcal{Z}(0,x) = \delta_0(x)$:

$$\partial_t \mathcal{Z} = -\nu(-\Delta)^{\alpha/2}\mathcal{Z} + \beta\mathcal{Z}\dot{\mathcal{W}}, \tag{1.11}$$

where $\dot{\mathcal{W}}$ is a time-white spatial-colored noise. The scaling limit of the polymer transition probability is also established in this paper.

In particular, when environment variables are independent, $\dot{\mathcal{W}}$ is a time-space white noise, our result on the rescaled point-to-point partition function upgrades Theorem 3.8 in [14] to the convergence in law in $C((0,1]\times\mathbb{R})$ equipped with locally uniform topology. The result is exactly what Caravenna, Sun and Zygouras [14] expected.

Our approach is based on Lindeberg's argument and the hypercontractive technique in multilinear polynomials (cf. [39]), and a gradient estimate for symmetric random walks in the domain of normal attraction of α-stable law. Precisely, we use Lindeberg's argument and the hypercontractive technique to study convergence of finite dimensional distributions. This strategy converts the environment variables to Gaussian ones. Mossel, O'Donnell and Oleszkiewicz [39] established an invariance principle which gives an error bound of the distributions of two random multilinear polynomial, when a multilinear polynomial of a sequence of independent random variables is replaced by independent Gaussian random variables with the same mean and the variance. The result was extended in [14]. Although the environment random variables in this paper are correlated in space variable, they still have the hypercontractivity since the multilinear polynomials in our model can be expanded into multilinear polynomials of independent random variables. This observation allows us to use Lindeberg's argument and the hypercontractive technique to our case. We use the characteristic function approach to obtain a gradient estimate for symmetric random walk in the domain of normal attraction of α-stable law, and apply the gradient estimate to establish the tightness for the rescaled point-to-point partition function. Hardy-Littewood's inequality, Minkowski's integral inequality and the gradient estimate for symmetric random walks play important role in the proof of tightness.

The rest of the paper is organized as follows. In Section 2, we state the main results and give some notation. The proofs of the main results are presented in sections 3–5. In Section 3, we first study the scaling limit of a modified point-to-line partition function and then show Theorem 2.1. In Section 4, we prove Theorem 2.2. A sketch proof of Theorem 2.3 is given in Section 5. In Appendix A, we recall briefly the elementary theory of time-white spatial-colored noise and stochastic integral with respect to a time-white spatial-colored noise. In appendix B, we present a gradient estimate for symmetric random walk in the domain of normal attraction of α-stable law. In appendix C, we give a proof existence and uniqueness of the mild solution to the fractional stochastic heat equation with the δ_0 initial data. In appendix D, we give some moment estimates for an autoregressive integrated moving average model.

Since the completion of this paper, there has been a recent work [44] on the scaling limit of a long-range directed polymer in a random environment that is correlated in time and independent in space, which complements the result of this paper.

2 Main results

In this section, we first introduce the model and some conditions in this paper, then state the main results.

(A.1). *Let $\{S_n, n \geq 0\}$ be a symmetric random walk with period q starting from the origin on $\mathbb{Z}$ and in the domain of normal attraction of a stable law of index $\alpha \in (0,2]$, i.e.,*

$$\frac{S_n}{n^{1/\alpha}} \xrightarrow{(d)} Y \text{ as } n \to \infty,$$

where the random variable Y has characteristic function

$$\mathbb{E}\left(e^{\iota u Y}\right) = e^{-\nu|u|^\alpha},$$

for some $\nu > 0$ and $\iota = \sqrt{-1}$. We assume that there exists a function $h(u)$ such that $h(u) \to 0$ as $|u| \to 0$ and the characteristic function ϕ of S_1 satisfies

$$\phi(u) = 1 - \nu|u|^\alpha + |u|^\alpha h(u) \text{ as } u \to 0. \tag{2.1}$$

□

Remark 2.1. In this paper, we only deal with a symmetric random walk, in the normal domain of attraction to a stable law. In this case, the characteristic function has a simple approximation which can simplify the proofs of the asymptotics and the gradient estimate. For the general case in the domain of attraction of a stable law, some properties of slowly varying functions should be required (cf. [9]).

By the inversion formula, it is known that Y has a bounded and differentiable density function $g(x)$. Define

$$g(t,x) := \frac{1}{t^{1/\alpha}} g\left(\frac{x}{t^{1/\alpha}}\right), \ \ t > 0, \ x \in \mathbb{R}.$$

Let $\mathbb{P}(S_1 \in q\mathbb{Z} + \ell) = 1$ for some $\ell \in \{0, 1, \cdots, q-1\}$. Since S_1 is symmetric, for the case $\ell \neq 0$, if $q \neq 1$, then q is an even number and $\ell = q/2$. Define

$$p(n,k) := \mathbb{P}(S_n = k), \qquad n \geq 0, \ k \in \mathbb{Z}.$$

Let $F(x) := \mathbb{P}(S_1 \leq x)$ be the distribution function of S_1. Then in the $\alpha = 2$ case,

$$\mathbb{E}(S_1) = 0, \ \sigma^2 = \mathbb{E}(S_1^2) = 2\nu < \infty, \ Y \sim N(0, \sigma^2);$$

in the $\alpha \in (0,2)$ case, there exist constant $c > 0$ and function $\beta(x)$ (cf. Theorem 2.6.7 in [32]) such that

$$1 - F(x) = \frac{c + \beta(x)}{x^\alpha}, \qquad \text{for all } x > 0, \text{ and } \beta(x) \to 0 \text{ as } |x| \to \infty. \tag{2.2}$$

By Lemma 3.1 in [5], if $\beta(x)x^{-\alpha}$ is decreasing in x on $[x_0, \infty)$ for some $x_0 \geq 0$, then (2.1) is valid.

(A.2). *Let the environment $\omega = \{\omega(i,x), (i,x) \in \mathbb{Z}_+ \times \mathbb{Z}\}$ have the following form as in [43]:*

$$\begin{cases} \omega(i,x) = \sum_{y \in \mathbb{Z}} a_y \xi(i, x+y) \text{ if } r \in (\frac{1}{2}, 1), \\ \\ \omega(i,x) = \xi(i,x) \text{ if } r = 1, \end{cases}$$

where $a_y \geq 0$, $a_y \sim c_r |y|^{-r}$, $c_r > 0$, and $\{\xi(i,x); i \in \mathbb{Z}_+, x \in \mathbb{Z}\}$ is a family of independent and identically distributed random variables with $\mathbf{E}(\xi(i,x)) = 0, \mathbf{E}(|\xi(i,x)|^2) = 1$. We assume that $\xi(i,x)$ satisfies

$$\mathbf{E} e^{\beta |\xi(i,x)|} < \infty \tag{2.3}$$

for β sufficiently small which implies (1.5). *For convenience, in the $r \in (\frac{1}{2}, 1)$ case, we take c_r such that $\lambda_r := \frac{4c_r^2 \Gamma(2r-1)\Gamma(1-r)}{\Gamma(r)} = H(2H-1)$, where $H = \frac{3}{2} - r$. Set*

$$\gamma(z) = \mathbf{E}(\omega(1,x)\omega(1,x-z)) = \sum_{y \in \mathbb{Z}} a_y a_{y-z}. \tag{2.4}$$

□

Remark 2.2. An example of the environment satisfying (A.2) is a version of autoregressive integrated moving average model $ARIMA\,(0, 1-r, 0)$ (cf. [27] [28]).

Let $r \in (\frac{1}{2}, 1)$ and take $c = \sqrt{\frac{H(2H-1)\Gamma(r)\Gamma(1-r)}{\Gamma(2r-1)}}$. Set

$$a_0 = c, \ \ a_k = a_{-k} = \frac{c}{2} \frac{\Gamma(k+1-r)}{\Gamma(k+1)\Gamma(1-r)} \geq 0, \ \ k \geq 1.$$

Then (see Theorem 1 in [27]),

$$\gamma(k) = \frac{c^2(-1)^k \Gamma(2r-1)}{\Gamma(k+r)\Gamma(-k+r)} = \frac{c^2 \Gamma(r) \prod_{j=0}^{k-1}(-j+k-r)\Gamma(2r-1)}{\Gamma(k+r)} \geq 0,$$

By Stirling formula, $a_k \sim \frac{c}{2} \frac{k^{-r}}{\Gamma(1-r)}$,

$$\gamma(k) \sim \frac{c^2\Gamma(2r-1)}{\Gamma(r)\Gamma(1-r)}k^{1-2r} = H(2H-1)k^{2H-2}.$$

□

By Remark 2.2, under the condition (A.2), we have that for $r \in (\frac{1}{2}, 1)$,

$$\gamma([z]) \sim K(z) \text{ as } |z| \to \infty \tag{2.5}$$

where $[z]$ denotes the integer part of z, and the function K is defined by

$$K(z) = \begin{cases} H(2H-1)|z|^{2H-2} & \text{if } r \in (\frac{1}{2}, 1), \\ \delta_0(z) & \text{if } r = 1. \end{cases} \tag{2.6}$$

The third assumption in this paper is the following (A.3).

(A.3). $\frac{1}{2} < r < \frac{1}{2}(1+\alpha)$.

Remark 2.3. In this paper, some basic estimates are based on the inequality that the norm $\|\cdot\|_{\mathcal{L}^k_K}$ in $\mathcal{L}^k_K$ can be controlled with the L^{2r}-norm $\|\cdot\|_{2r}$ in (see (3.16)). When $0 < r \le 1/2$, the inequality does not hold. The condition $\alpha > 2r - 1$ should be optimal.

□

2.1 Stochastic heat equation

Consider the following fractional stochastic heat equation:

$$\partial_t \mathcal{Z} = -\nu(-\Delta)^{\alpha/2}\mathcal{Z} + \sigma\mathcal{Z}\dot{\mathcal{W}}, \tag{2.7}$$

where $\sigma > 0$ is a constant, and $\dot{\mathcal{W}}$ is a Gaussian noise with the covariance

$$\mathrm{Cov}(\mathcal{W}(\varphi), \mathcal{W}(\psi)) = \int_0^1 \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(s,x)K(x-y)\psi(s,y)dsdxdy \tag{2.8}$$

and $K(x)$ is defined as (2.6). The scaling limit of the polymer transition probability is also established in this paper.

We rewrite it in Duhamel form

$$\mathcal{Z}(t,x) = \int_{\mathbb{R}} g(t,x-y)\mathcal{Z}(0,y)dy + \sigma\int_0^t \int_{\mathbb{R}} g(t-s,x-y)\mathcal{Z}(s,y)\mathcal{W}(dsdy). \tag{2.9}$$

A mild solution of (2.7) is a progressively measurable process $\mathcal{Z}(t,x)$ such that

$$\int_0^t \int_{\mathbb{R}} g(t-s,x-y)\mathbf{E}(|\mathcal{Z}(s,y)|^2)dyds < \infty \ \text{ for } t > 0,\ x \in \mathbb{R}, \tag{2.10}$$

and (2.9) holds.

We use the notion $\mathcal{Z}_0(\cdot) = \mathcal{Z}(0,\cdot)$ to denote the initial data of the fractional stochastic heat equation (2.7). For the bounded initial data $\mathcal{Z}_0$ case, the existence and uniqueness of a mild solution of (2.7) can be founded in [25]. Foondun Joseph and Li [24] studies the approximation problem of a class of SPDEs including (2.7) by systems of interacting stochastic differential equations. The following proposition includes the initial data $\mathcal{Z}_0 = \delta_0$ case. For convenience, we will give a proof of the proposition in Appendix C.

Proposition 2.1. Let $\mathcal{Z}_0$ be a $\mathcal{F}_0$-measurable initial data. If $\mathcal{Z}_0 = \delta_0$ or bounded, then there exists a unique mild solution to equation (2.7) with initial data $\mathcal{Z}_0$. The solution can be written by

$$\begin{aligned} \mathcal{Z}(t,x) = \int_{\mathbb{R}} g(t,x-y)\mathcal{Z}_0(y)dy + \sum_{k=1}^{\infty} \sigma^k \int_{\Delta_k(t)} \int_{\mathbb{R}^{k+1}} \\ g(t-t_k, x-x_k)\prod_{i=1}^k g(t_i - t_{i-1}, x_i - x_{i-1})\mathcal{Z}_0(x_0)dx_0\mathcal{W}(dt_idx_i), \end{aligned} \tag{2.11}$$

where $\Delta_k(t) = \{0 \le t_1 < \cdots < t_k \le t\}$, $t_0 = 0$. Furthermore, The series in (2.11) converges in L^2. In particular, when $\mathcal{Z}_0 = \delta_0$, there exists a positive constant C such that for any $t \in [0,T]$,

$$\mathbf{E}(|\mathcal{Z}(t,x)|^2) \le Cg^2(t,x). \tag{2.12}$$

When $\mathcal{Z}_0$ is bounded, $\sup_{t\in[0,1],x\in\mathbb{R}} \mathbf{E}(|\mathcal{Z}(t,x)|^2) < \infty$.

Denote by

$$\sigma(\beta,q) = \begin{cases} \sqrt{q}\beta & \text{if } r = 1, \\ \\ \beta & \text{if } \frac{1}{2} < r < 1. \end{cases} \tag{2.13}$$

Remark 2.4. The quantity $\sigma(\beta,q)$ is the diffusion coefficient of the scaling limit (see the following main results). An interesting phenomenon is that the $\sigma(\beta,q)$ is independent of q in space-correlated environment case. The phenomenon is due to the difference in the variance of sum of random variables in the two environments (see Lemma 3.5).

2.2 Main results

Theorem 2.1. Let $Z_n(\beta;\omega)$ be the partition function which is defined by (1.2), i.e.,

$$Z_n(\beta;\omega) := \mathbb{E}\left(e^{\beta\sum_{i=1}^n \omega(i,S_i)}\right).$$

Assume that (A.1), (A.2) and (A.3) hold. Set $\beta_n = \beta n^{-\frac{1}{2}-\frac{1}{2\alpha}+\frac{r}{\alpha}}$. Then we have the following convergences for the rescaled point-to-line partition function

$$Z_n(\beta_n;\omega)e^{-n\lambda(\beta_n)} \xrightarrow{(d)} \mathcal{Z}_{\sigma(\beta,q)}(1,*), \tag{2.14}$$

and

$$\lim_{n\to\infty} \mathbf{E}\left(\left(Z_n(\beta_n;\omega)e^{-n\lambda(\beta_n)}\right)^2\right) = \mathbf{E}\left(\left(\mathcal{Z}_{\sigma(\beta,q)}(1,*)\right)^2\right), \tag{2.15}$$

where $\mathcal{Z}_{\sigma(\beta,q)}(1,*) = \int \mathcal{Z}_{\sigma(\beta,q)}(1,x)dx$, and $\mathcal{Z}_{\sigma(\beta,q)}(t,x)$ is the mild solution of (2.7) with $\sigma = \sigma(\beta,q)$ and initial data $\mathcal{Z}_0 = \delta_0$.

Remark 2.5. Caravenna,Sun and Zygouras [12] proposed a new point of view to interpret disorder relevance for disordered systems. The viewpoint focuses on the existence of a non-trivial, random continuum limit when disorder scales to zero in a particular way as a function of the lattice spacing. Theorem 2.1 proves that for a class of directed polymer models where the increments of the walk lie in the domain of attraction of an α-stable law, and the environment is a r-fractional autoregressive moving average model, with $r \in (1/2,1]$ and $1/2 < r < (1+\alpha)/2$, the rescaled partition function has a non-trivial, random limit when the disorder scales to zero in the speed $n^{-\frac{1}{2}-\frac{1}{2\alpha}+\frac{r}{\alpha}}$. Furthermore, the following theorem establishes the convergence of the rescaled point-to-point partition function in path space. The results also show that there is a transition between weak and strong disorder at $\beta = 0$ for the class of directed polymer models.

Theorem 2.2. Let $Z_{n,x}(\beta;\omega)$ be the point-to-point partition function which is defined by (1.3), i.e.,

$$Z_{n,x}(\beta;\omega) := \mathbb{E}\left(e^{\beta\sum_{i=1}^n \omega(i,S_i)} I_{\{S_n=x\}}\right).$$

Let $\alpha \in (1,2]$, (A.1) and (A.2) hold. Then we have the following convergences for the rescaled point-to-point partition function

$$\left(\frac{1}{q}n^{1/\alpha} Z_{nt,n^{1/\alpha}x}(\beta_n;\omega)e^{-n\lambda(\beta_n)}\right)_{t\in(0,1],x\in\mathbb{R}} \xrightarrow{(d)} \left(\mathcal{Z}_{\sigma(\beta,q)}(t,x)\right)_{t\in(0,1],x\in\mathbb{R}} \tag{2.16}$$

with respect to the locally uniform topology on $C((0,1]\times\mathbb{R})$, and for any $t\in(0,1], x\in\mathbb{R}$,

$$\lim_{n\to\infty}\mathbf{E}\left(\left(n^{1/\alpha}Z_{nt,n^{1/\alpha}x}(\beta_n;\omega)e^{-n\lambda(\beta_n)}/q\right)^2\right)=\mathbf{E}\left(\left(\mathcal{Z}_{\sigma(\beta,q)}(t,x)\right)^2\right). \tag{2.17}$$

Theorem 2.3. Let $\alpha\in(1,2]$, (A.1) and (A.2) hold. Then we have the following convergences for the rescaled polymer transition probabilities

$$\begin{aligned}&\left(\frac{1}{q}n^{1/\alpha}\mathbf{P}^{\omega}_{n,\beta_n}(S_{nt}=n^{1/\alpha}x|S_{ns}=n^{1/\alpha}y)\right)_{(s,y;t,x)\in\mathfrak{D}}\\&\xrightarrow{(d)}\left(\frac{\mathcal{Z}_{\sigma(\beta,q)}(s,y;t,x)\int\mathcal{Z}_{\sigma(\beta,q)}(t,x;1,\lambda)\,d\lambda}{\int\mathcal{Z}_{\sigma(\beta,q)}(s,y;1,\lambda)d\lambda}\right)_{(s,y;t,x)\in\mathfrak{D}}\end{aligned} \tag{2.18}$$

with respect to the locally uniform topology on $C(\mathfrak{D})$, where $\mathfrak{D}=\{(s,y;t,x);\ 0\le s<t\le 1, x,y\in\mathbb{R}\}$, $\mathcal{Z}_{\sigma(\beta,q)}(s,y;t,x)$ is the mild solution of the stochastic heat equation

$$\partial_t\mathcal{Z}=-\nu(-\Delta_x))^{\alpha/2}\mathcal{Z}+\sigma(\beta,q)\mathcal{Z}\dot{\mathcal{W}},\quad \mathcal{Z}(s,y;s,x)=\delta_0(x-y). \tag{2.19}$$

Remark 2.6. The field $(t,x)\mapsto Z_{nt,n^{1/\alpha}x}(\beta_n;\omega)$ is defined exactly on the points where $(nt,n^{1/\alpha}x)$ takes values in $\{(i,k);\ i\in\mathbb{Z}_+,k\in q\mathbb{Z}+i\ell\}$, but we can use a linear interpolation scheme to extend it to the whole space (see [1]). The linear interpolation scheme is defined concretely in Section 4. We also extend the field $(s,y;t,x)\mapsto\mathbf{P}^{\omega}_{n,\beta_n}(S_{nt}=n^{1/\alpha}x|S_{ns}=n^{1/\alpha}y)$ to the domain $\{(s,y;t,x);\ 0\le s<t\le 1,x,y\in\mathbb{R}\}$.

Remark 2.7. For the $r=1$ case, Theorem 2.2 upgrades Theorem 3.8 in [14] to the convergence in law in $C((0,1]\times\mathbb{R})$ which is expected in [14]. Hardy-Littewood's inequality, Minkowski's integral inequality and the gradient estimate for symmetric random walks play important role in the proof of tightness.

Remark 2.8. In this paper, we only consider the auto-regressive environment with exponential moments. Our study depends on the auto-regressive representation of the environment. It is expected that some results of the directed polymer with heavy-tailed disorder (cf. [3], [4], [23]) can be extended to correlated environments with heavy tails.

Remark 2.9. Berger and Lacoin (Theorem 2.4 and Theorem 2.7 in [3]) considered the joint convergence of a modified partition function together with environment. For the auto-regressive environment, we can also study the joint convergence of the modified partition function and the environment.

3 The scaling limit of the point-to-line partition function

In this section, we give the proof of Theorem 2.1. We first study the scaling limit of a modified point-to-line partition function defined by

$$\mathfrak{Z}_n(\beta;\omega)=\mathbb{E}\left(\prod_{i=1}^{n}(1+\beta\omega(i,S_i))\right), \tag{3.1}$$

The modified point-to-line partition function $\mathfrak{Z}_n$ is an approximation of the point-to-line partition function and it is more convenient to study the convergence of the modified partition function than that of the partition function. On the other hand, we can write that

$$Z_n(\beta;\omega)e^{-n\lambda(\beta)}=\mathbb{E}\left(\prod_{i=1}^{n}(1+\beta\widetilde{\omega}(i,S_i))\right),$$

where

$$\widetilde{\omega}(i,x)=\frac{e^{\beta\omega(i,x)-\lambda(\beta)}-1}{\beta}.$$

Therefore, by estimating the error between two environment variables $\widetilde{\omega}$ and ω, we can obtain the convergence of the partition function from that of the modified partition function. In the first subsection of this section, we study the convergence of the modified partition function. This is the crucial part. In the second subsection of this section, we estimate the error between two environment variables $\widetilde{\omega}$ and ω, and prove Theorem 2.1.

Note that $\mathfrak{Z}_n(\beta;\omega)$ can be approximated by a multilinear polynomials of $\omega(i,x)$, $(i,x) \in \mathbb{Z}_+ \times \mathbb{Z}$. We use Lindeberg's argument to replace the environment variables $\omega(i,x)$, $(i,x) \in \mathbb{Z}_+ \times \mathbb{Z}$ by some Gaussian variables $\mu(i,x)$, $(i,x) \in \mathbb{Z}_+ \times \mathbb{Z}$, and apply the hypercontractive technique in multilinear polynomials (cf. [39]) to control the error of $\mathfrak{Z}_n(\beta;\omega)$ and $\mathfrak{Z}_n(\beta;\mu)$. This strategy converts the environment variables to Gaussian ones. Then we can use techniques of weighted U-statistics for Gaussian variables to show that $\mathfrak{Z}_n(\beta;\mu)$ converges in law to the Wiener chaos of $\mathcal{Z}_{\sigma(\beta,q)}(1,*)$. Theorem 2.1 will be obtained by estimating L^2-error between $Z_n(\beta_n;\omega)e^{-n\lambda(\beta_n)}$ and the modified point-to-line partition function $\mathfrak{Z}_n(\beta_n;\omega)$.

The following weak convergence result will be applied repeatedly (cf. Chapter 1, Theorem 4.2. in [8]).

Lemma 3.1. Let Y_k^n, Y_k, Y^n, Y be real-valued random variables and assume that for each fixed n the Y_k^n and Y^n are defined on a common probability space. If $Y_k^n \to Y^n$ in probability uniformly in n as $k \to \infty$, $Y_k^n \to Y_k$ in distribution as $n \to \infty$, and $Y_k \to Y$ in distribution as $k \to \infty$, then $Y^n \to Y$ in distribution as $n \to \infty$.

We also use the following Beta integral formula

$$\int_{t_i \geq 0, t_1+\cdots+t_n \leq 1} \left(1 - \sum_{i=1}^n t_i\right)^{\beta-1} \prod_{j=1}^n t_j^{\alpha_j - 1} dt_j = \frac{\Gamma(\beta)\prod_{j=1}^n \Gamma(\alpha_j)}{\Gamma(\alpha_1 + \cdots + \alpha_n + \beta)}, \tag{3.2}$$

where $\beta > 0$, $\alpha_j > 0$, $j = 1, \cdots, n$.

3.1 The scaling limit of the modified point-to-line partition function

In this subsection, we show the following scaling limit theorem for the modified point-to-line partition function $\mathfrak{Z}_n(\beta;\omega)$. This is the main step of the proof of Theorem 2.1.

Theorem 3.1. Assume that (A.1), (A.2) and (A.3) hold. Set $\beta_n = \beta n^{-\frac{1}{2} - \frac{1}{2\alpha} + \frac{r}{\alpha}}$. Then we have the following convergences for the rescaled point-to-line partition function.

$$\mathfrak{Z}_n(\beta_n;\omega) \xrightarrow{(d)} \mathcal{Z}_{\sigma(\beta,q)}(1,*), \tag{3.3}$$

and

$$\lim_{n\to\infty} \mathbf{E}\left((\mathfrak{Z}_n(\beta_n;\omega))^2 \right) = \mathbf{E}\left(\left(\mathcal{Z}_{\sigma(\beta,q)}(1,*)\right)^2 \right).$$

The proof of Theorem 3.1 proceeds in three steps.

Step 1. We expand the modified point-to-line partition function $\mathfrak{Z}_n(\beta;\omega)$ into a discrete chaos expansion, and give some estimates for the coefficients of the chaos expansion. The main estimates in this step are Lemma 3.2 and Lemma 3.3.

Let us first introduce some notations. Set $\Delta_k(t) = \{0 \leq t_1 < \cdots < t_k \leq t\}$,

$$\begin{cases} \mathbb{T} := \left\{(i,x) \in \mathbb{Z}^2;\ i \in \mathbb{Z}_+, x \in q\mathbb{Z} + i\ell\right\}, \\ \mathbb{D}_n := \left\{ \left(\dfrac{i}{n}, \dfrac{x}{n^{\frac{1}{\alpha}}}\right); (i,x) \in \mathbb{T}, 1 \leq i \leq n \right\}, \\ \Delta\mathbb{D}_n^k := \left\{(\mathbf{t},\mathbf{x}) = ((t_1,x_1),\cdots,(t_k,x_k)) \in \mathbb{D}_n^k; 0 \leq t_1 < \cdots < t_k \leq 1\right\}. \end{cases} \tag{3.4}$$

Define

$$p_n^k(\mathbf{t},\mathbf{x}) = P(S_{nt_1} = n^{1/\alpha}x_1, \cdots, S_{nt_k} = n^{1/\alpha}x_k), \quad (\mathbf{t},\mathbf{x}) \in \Delta\mathbb{D}_n^k, \tag{3.5}$$

where $x_0 = 0$ and

$$\omega_n^k(\mathbf{t},\mathbf{x}) = \prod_{i=1}^k \omega\left(nt_i, n^{\frac{1}{\alpha}} x_i\right), \quad (\mathbf{t},\mathbf{x}) \in \mathbb{D}_n^k.$$

We extend $p_n^k(\mathbf{t},\mathbf{x})$ from $\Delta\mathbb{D}_n^k$ to $\Delta_k(1)\times\mathbb{R}^k$ by defining

$$p_n^k(\mathbf{t},\mathbf{x}) := p_n^k(\mathbf{s},\mathbf{y}) \text{ for all } (\mathbf{t},\mathbf{x}) \in \mathcal{C}_n^k(\mathbf{s},\mathbf{y}) := \prod_{i=1}^k \mathcal{C}_n(s_i, y_i),\ (\mathbf{s},\mathbf{y}) \in \Delta\mathbb{D}_n^k,$$

where

$$\mathcal{C}_n(t,x) := \left(t - \frac{1}{n}, t\right] \times \left(x - \frac{q}{n^{\frac{1}{\alpha}}}, x\right]. \tag{3.6}$$

Then, for each $\mathbf{t} \in \Delta_k(1)$,

$$\widehat{p}_n^k(\mathbf{t},\mathbf{x}) := \left(q^{-1} n^{\frac{1}{\alpha}}\right)^k p_n^k(\mathbf{t},\mathbf{x}) \tag{3.7}$$

is a probability density on $\mathbb{R}^k$. Similarly, $\omega_n^k(\mathbf{t},\mathbf{x})$ can be extended to the whole space $[0,1]^k \times \mathbb{R}^k$ by setting

$$\omega_n^k(\mathbf{t},\mathbf{x}) = \omega_n^k(\mathbf{s},\mathbf{y}) \text{ for all } (\mathbf{t},\mathbf{x}) \in \mathcal{C}_n^k(\mathbf{s},\mathbf{y}),\ (\mathbf{s},\mathbf{y}) \in \mathbb{D}_n^k. \tag{3.8}$$

We abbreviate p_n^1 to p_n, $\widehat{p}_n^1$ to $\widehat{p}_n$, and ω_n^1 to ω_n.

Now, by expanding the product $\prod_{i=1}^n (1+\beta_n\omega(i,S_i))$ along each path of the random walk, we obtain

$$\prod_{i=1}^n (1+\beta_n\omega(i,S_i)) = 1 + \sum_{k=1}^n \beta_n^k \sum_{(\mathbf{t},\mathbf{x})\in\Delta\mathbb{D}_n^k} \prod_{i=1}^k \omega\left(nt_i, n^{\frac{1}{\alpha}} x_i\right).$$

Then by the Markov property for random walk, we can give a series expansion of $\mathfrak{Z}_n(\beta_n;\omega)$:

$$\mathfrak{Z}_n(\beta_n;\omega) = \mathbb{E}\left(\prod_{i=1}^n (1+\beta_n\omega(i,S_i))\right) = 1 + \sum_{k=1}^n \beta_n^k \sum_{(\mathbf{t},\mathbf{x})\in\Delta\mathbb{D}_n^k} p_n^k(\mathbf{t},\mathbf{x})\omega_n^k(\mathbf{t},\mathbf{x}). \tag{3.9}$$

Define

$$\psi_n^k(\mathbf{t},\mathbf{x}) := \mathrm{Sym}\{\widehat{p}_n^k(\mathbf{t},\mathbf{x}) I_{\Delta_k(1)\times\mathbb{R}^k}(\mathbf{t},\mathbf{x})\}, \quad (\mathbf{t},\mathbf{x}) \in [0,1]^k \times \mathbb{R}^k, \tag{3.10}$$

where the symmetrization of a function f on $[0,1]^k \times \mathbb{R}^k$ is defined by

$$\mathrm{Sym}\{f\}(\mathbf{t},\mathbf{x}) = \frac{1}{k!} \sum_{\pi\in\mathbf{S}_k} f(\pi\mathbf{t}, \pi\mathbf{x})$$

and $\mathbf{S}_k$ is the group of permutations on $\{1,2,\cdots,k\}$.

Note that when $t_i = t_j$ for some $i \neq j$, $\psi_n^k(\mathbf{t},\mathbf{x}) = 0$. We have that $\psi_n^k(\mathbf{t},\mathbf{x}) = 0$ for any $k \geq n+1$. Therefore, we can write

$$\mathfrak{Z}_n(\beta_n;\omega) = 1 + \sum_{k=1}^\infty (\beta\sqrt{q})^k \theta_n^k \sum_{(\mathbf{t},\mathbf{x})\in\mathbb{D}_n^k} \psi_n^k(\mathbf{t},\mathbf{x})\omega_n^k(\mathbf{t},\mathbf{x}), \tag{3.11}$$

where

$$\theta_n = q^{\frac{1}{2}} n^{-\frac{1}{2} - \frac{3}{2\alpha} + \frac{r}{\alpha}}. \tag{3.12}$$

Since the volume of each cell $\mathcal{C}_n^k(\mathbf{s},\mathbf{y})$ equals $qn^{-(\alpha+1)/\alpha}$, we have

$$\mathfrak{Z}_n(\beta_n;\omega)=1+\sum_{k=1}^{\infty}(\beta\sqrt{q})^k\theta_n^k q^{-k}n^{k(\alpha+1)/\alpha}\int_{[0,1]^k}\int_{\mathbb{R}^k}\psi_n^k(\mathbf{t},\mathbf{x})\omega_n^k(\mathbf{t},\mathbf{x})\,d\mathbf{t}d\mathbf{x}. \tag{3.13}$$

In order to study the convergence of $\mathfrak{Z}_n(\beta_n;\omega)$, we need to estimate some moments and covariance of $\omega_n^k(\mathbf{t},\mathbf{x})$. For any $k\geq 1$, we define the rescaled covariance of ω_n^k by

$$\begin{aligned}\gamma_n^k(\mathbf{x}-\mathbf{y})=&n^{k(2r-1)/\alpha}\mathbf{E}\left(\omega_n^k(\mathbf{t},\mathbf{x})\omega_n^k(\mathbf{t},\mathbf{y})\right)\\ =&n^{k(2r-1)/\alpha}\prod_{i=1}^k\gamma\left(n^{1/\alpha}x_i-n^{1/\alpha}y_i\right),\quad (\mathbf{t},\mathbf{x}),(\mathbf{t},\mathbf{y})\in\mathbb{D}_n^k,\end{aligned} \tag{3.14}$$

and extend it to the whole space $\mathbb{R}^k\times\mathbb{R}^k$ by defining

$$\gamma_n^k(\mathbf{x}'-\mathbf{y}')=\gamma_n^k(\mathbf{x}-\mathbf{y}),\ (\mathbf{t},\mathbf{x}')\in\mathcal{C}_n^k(\mathbf{t},\mathbf{x}),(\mathbf{t},\mathbf{y}')\in\mathcal{C}_n^k(\mathbf{t},\mathbf{y}),$$

for any $(\mathbf{t},\mathbf{x}),(\mathbf{t},\mathbf{y})\in\mathbb{D}_n^k$. We abbreviate γ_n^1 to γ_n.

By the definition of $K(z)$ in (2.6), in the $r\in(\frac{1}{2},1)$ case, $K(z)=H(2H-1)|z|^{2H-2}$ for $z\in\mathbb{R}$, and by (2.5), $\gamma([z])\sim K(z)$ as $|z|\to\infty$, i.e., $\lim_{|z|\to\infty}\frac{\gamma([z])}{K(z)}=1$, and so, there exist positive constants C_1 and L such that $0\leq\gamma([z])\leq C_1K(z)$ for $|z|\geq L$. Set $C_2=\sup_{|z|\leq L}\gamma([z])$ and $C_3=\inf_{|z|\leq L}K(z)>0$. Then

$$0\leq\gamma([z])\leq C_4K(z)\text{ for }z\in\mathbb{R},$$

where $C_4=\max\left\{\frac{C_2}{C_3},C_1\right\}$. Therefore, for any function $\varphi\in\mathcal{L}_K^k([0,1]^k\times\mathbb{R}^k)$,

$$\begin{aligned}&\left|\int_{[0,1]^k}\int_{\mathbb{R}^{2k}}\varphi(\mathbf{t},\mathbf{x})\gamma_n(\mathbf{x}-\mathbf{y})\varphi(\mathbf{t},\mathbf{y})d\mathbf{t}d\mathbf{x}d\mathbf{y}\right|\\ \leq&\int_{[0,1]^k}\int_{\mathbb{R}^{2k}}|\varphi(\mathbf{t},\mathbf{x})|\gamma_n(\mathbf{x}-\mathbf{y})|\varphi(\mathbf{t},\mathbf{y})|d\mathbf{t}d\mathbf{x}d\mathbf{y}\\ =&\sum_{(\mathbf{t},\mathbf{x})\in\mathbb{D}_n^k,(\mathbf{t},\mathbf{y})\in\mathbb{D}_n^k}\int_{\mathcal{C}_n^k(\mathbf{t},\mathbf{x})\times\mathcal{C}_n^k(\mathbf{t},\mathbf{y})}|\varphi(\mathbf{t}',\mathbf{x}')|n^{\frac{k(2r-1)}{\alpha}}\gamma\left(n^{\frac{1}{\alpha}}(\mathbf{x}'-\mathbf{y}')\right)|\varphi(\mathbf{t}',\mathbf{y}')|d\mathbf{t}'d\mathbf{x}'d\mathbf{y}'\\ \leq&C_4^k\sum_{(\mathbf{t},\mathbf{x})\in\mathbb{D}_n^k,(\mathbf{t},\mathbf{y})\in\mathbb{D}_n^k}\int_{\mathcal{C}_n^k(\mathbf{t},\mathbf{x})\times\mathcal{C}_n^k(\mathbf{t},\mathbf{y})}|\varphi(\mathbf{t}',\mathbf{x}')|\,n^{\frac{k(2r-1)}{\alpha}}K\left(n^{\frac{1}{\alpha}}(\mathbf{x}'-\mathbf{y}')\right)|\varphi(\mathbf{t}',\mathbf{y}')|\,d\mathbf{t}'d\mathbf{x}'d\mathbf{y}'\\ =&C_4^k\||\varphi|\|_{\mathcal{L}_K^k}^2,\end{aligned}$$

where

$$\|\varphi\|_{\mathcal{L}_K^k}^2:=\int_{[0,1]^k}\int_{\mathbb{R}^{2k}}\varphi(\mathbf{t},\mathbf{x})\prod_{i=1}^kK(x_i-y_i)\varphi(\mathbf{t},\mathbf{y})d\mathbf{t}d\mathbf{x}d\mathbf{y}.$$

Therefore, we have

$$\left|\int_{[0,1]^k}\int_{\mathbb{R}^{2k}}\varphi(\mathbf{t},\mathbf{x})\gamma_n(\mathbf{x}-\mathbf{y})\varphi(\mathbf{t},\mathbf{y})d\mathbf{t}d\mathbf{x}d\mathbf{y}\right|\leq C\||\varphi|\|_{\mathcal{L}_K^k}^2. \tag{3.15}$$

For the $r=1$ case, (3.15) is obvious. For the norm $\|\cdot\|_{\mathcal{L}_K^k}$, if

$$\sup_{t\in[0,1]^k}\int_{\mathbb{R}^k}|\varphi(\mathbf{t},\mathbf{x})|d\mathbf{x}\leq A<\infty,$$

then by Hardy-Littewood's inequality (cf. Theorem 1 in [46], P.119) and Hölder's inequality, there exists a positive constant A_H such that

$$\begin{aligned}|||\varphi|||^2_{\mathcal{L}^k_K} &\le A_H \int_{[0,1]^k}\left(\int_{\mathbb{R}^k}|\varphi(\mathbf{t},\mathbf{x})|^{\frac{2}{3-2r}}d\mathbf{x}\right)^{3-2r}d\mathbf{t}\\ &\le A_H A^{2-2r}\int_{[0,1]^k}\int_{\mathbb{R}^k}|\varphi(\mathbf{t},\mathbf{x})|^{2r}d\mathbf{x}d\mathbf{t}.\end{aligned} \tag{3.16}$$

The following two lemmas give some estimates of ψ_n^k.

Lemma 3.2. Assume that (A.1), (A.2) and (A.3) hold. Then there exists a positive constant C such that for any $n \ge 1$, $k \ge 1$,

$$\|g_k\|^2_{\mathcal{L}^k_K} \le \frac{C^k\Gamma^{k+1}\left(1+\frac{1-2r}{\alpha}\right)}{\Gamma\left((k+1)\left(1+\frac{1-2r}{\alpha}\right)\right)} \tag{3.17}$$

and

$$k!\|\psi_n^k\|^2_{\mathcal{L}^k_K} \le \frac{C^k\Gamma^{k+1}\left(1+\frac{1-2r}{\alpha}\right)}{\Gamma\left((k+1)\left(1+\frac{1-2r}{\alpha}\right)\right)}, \tag{3.18}$$

where

$$g_k(\mathbf{t},\mathbf{x}) := \prod_{i=1}^k g(t_i-t_{i-1},x_i-x_{i-1}), \quad (\mathbf{t},\mathbf{x})\in\Delta_k(1)\times\mathbb{R}^k. \tag{3.19}$$

In particular, by $\alpha > 2r-1$, (3.17) and (3.18) are summable, and so,

$$\sum_{k=1}^\infty(\beta\sqrt{q})^{2k}\|g_k\|^2_{\mathcal{L}^k_K} < \infty, \tag{3.20}$$

and

$$\lim_{l\to\infty}\limsup_{n\to\infty}\sum_{k\ge l}(\beta\sqrt{q})^{2k}k!\|\psi_n^k\|^2_{\mathcal{L}^k_K} = 0. \tag{3.21}$$

Proof. The proofs of (3.17) and (3.18) are similar. Next, we prove (3.18). Noting that for any $t\in(0,1]$

$$\int_{\mathbb{R}} q^{-1}n^{\frac{1}{\alpha}}p_n^1(t,x)dx = \sum_{i\in\mathbb{Z}}P(S_{nt}=i)=1.$$

By Hardy-Littewood's inequality and Hölder's inequality,

$$\begin{aligned}&k!\|\psi_n^k\|^2_{\mathcal{L}^k_K}\\ =&\left(q^{-1}n^{\frac{1}{\alpha}}\right)^{2k}\int_{\Delta_k(1)}\int_{\mathbb{R}^{2k}}p_n^k(\mathbf{t},\mathbf{x})p_n^k(\mathbf{t},\mathbf{y})\prod_{i=1}^k K(x_i-y_i)dt_idx_idy_i\\ \le& C^k\int_{\Delta_k(1)}\prod_{i=1}^k\int_{\mathbb{R}}\left(\left(q^{-1}n^{\frac{1}{\alpha}}\right)p(n(t_i-t_{i-1}),n^{1/\alpha}x_i)\right)^{2r}dx_idt_i.\end{aligned} \tag{3.22}$$

By the Gnedenko local limit theorem (see Lemma B.1 in Appendix for a proof),

$$\sup_{(n,k)\in\mathbb{T}}\left|\frac{n^{1/\alpha}}{q}p(n,k)-g(k/n^{1/\alpha})\right| \to 0 \text{ as } n\to\infty. \tag{3.23}$$

Since the density g is bounded, there exists $A\in(0,+\infty)$ such that

$$p(n,k)\le qAn^{-1/\alpha}, \qquad \text{for all } n\in\mathbb{Z}_+, k\in\mathbb{Z}. \tag{3.24}$$

Therefore,

$$
\begin{aligned}
k!\|\psi_n^k\|^2_{\mathcal{L}^k_K} \leq& C^K \int_{\Delta_k(1)} \prod_{i=1}^k \int_{\mathbb{R}} \left(\left(q^{-1} n^{\frac{1}{\alpha}} \right) p(n(t_i - t_{i-1}), n^{1/\alpha} x_i) \right)^{2r} dx_i dt_i \\
\leq& C^k A^{k(2r-1)} \int_{\Delta_k(1)} \prod_{i=1}^k \int_{\mathbb{R}} \left(q^{-1} n^{\frac{1}{\alpha}} \right) p(n(t_i - t_{i-1}), n^{1/\alpha} x_i)(t_i - t_{i-1})^{\frac{1-2r}{\alpha}} dx_i dt_i \\
\leq& C^k A^{k(2r-1)} \int_{\Delta_k(1)} \prod_{i=1}^k (t_i - t_{i-1})^{\frac{1-2r}{\alpha}} dt_i \\
\leq& \frac{C^k A^{k(2r-1)} \Gamma^k \left(1 + \frac{1-2r}{\alpha}\right)}{\Gamma\left(k\left(1 + \frac{1-2r}{\alpha}\right) + 1\right)},
\end{aligned}
\tag{3.25}
$$

and so, (3.18) is valid. □

Lemma 3.3. Assume that (A.1), (A.2) and (A.3) hold. Then for every $k \geq 1$,

$$\lim_{n\to\infty} \||\psi_n^k - G_k|\|^2_{\mathcal{L}^k_K} = 0, \tag{3.26}$$

where

$$G_k(\mathbf{t}, \mathbf{x}) := \mathrm{Sym}\{g_k(\mathbf{t}, \mathbf{x})\}, \quad (\mathbf{t}, \mathbf{x}) \in [0,1]^k \times \mathbb{R}^k. \tag{3.27}$$

Proof. By the local limit theorem (see Lemma B.1 in Appendix),

$$\lim_{n\to\infty} \psi_n^k(\mathbf{t}, \mathbf{x}) = G_k(\mathbf{t}, \mathbf{x}) \text{ for any } (\mathbf{t}, \mathbf{x}) \in \Delta_k(1) \times \mathbb{R}^k.$$

For any $\delta > 0$ and $M \geq 1$, we can write

$$\||\psi_n^k - G_k|\|^2_{\mathcal{L}^k_K} \leq 2\||\psi_n^k - G_k| I_{D_1}\|^2_{\mathcal{L}^k_K} + 4\|\psi_n^k I_{D_1^c}\|^2_{\mathcal{L}^k_K} + 4\|G_k I_{D_1^c}\|^2_{\mathcal{L}^k_K},$$

where

$$D_1 := \cap_{i=1}^k \{t_i - t_{i-1} > \delta, |x_i| < M, |y_i| < M\}.$$

Since $\psi_n^k - G_k$ is bounded and $\prod_{i=1}^k K(x_i - y_i)$ is integrable on D_1, by the dominated convergence theorem,

$$\||\psi_n^k - G_k| I_{D_1}\|^2_{\mathcal{L}^k_K} \to 0 \text{ as } n \to \infty.$$

Note that $D_1^c = D_2 \cup D_3$, where

$$D_2 := \cup_{i=1}^k \{t_i - t_{i-1} \leq \delta\}, \quad D_3 := \cup_{i=1}^k \{|x_i| \geq M\} \cup \left(\cup_{i=1}^k \{|y_i| \geq M\}\right).$$

By the proof of Lemma 3.2,

$$k!\|\psi_n^k I_{D_2}\|^2_{\mathcal{L}^k_K} \leq C^k \underset{\substack{s_i \geq 0, i=1,\cdots,k, s_1+\cdots+s_k \leq 1 \\ \cup_{i=1}^k \{s_i \leq \delta\}}}{\int \cdots \int} \prod_{i=1}^k s_i^{\frac{1-2r}{\alpha}} ds_i \to 0 \text{ uniformly in } n \text{ as } \delta \to 0$$

and

$$k!\|\psi_n^k I_{D_3}\|^2_{\mathcal{L}^k_K} \leq C^k \mathbb{P}\left(\max_{1\leq i\leq n} |S_i| \geq n^{\frac{1}{\alpha}} M\right) \frac{\Gamma^k \left(1 + \frac{1-2r}{\alpha}\right)}{\Gamma\left(k\left(1 + \frac{1-2r}{\alpha}\right) + 1\right)}.$$

By the Montgomery-Smith inequality (cf. [38]),

$$\mathbb{P}\left(\max_{1\leq i\leq n} |S_i| \geq n^{\frac{1}{\alpha}} M\right) \leq 9\mathbb{P}\left(|S_n| \geq n^{\frac{1}{\alpha}} M/30\right) \to 9\mathbb{P}\left(|Y| \geq M/30\right)$$

uniformly in M as $n \to \infty$, where Y is the symmetric α-stable variable. Noting $P(|Y| \geq M) \to 0$ as $M \to \infty$, for any $\epsilon > 0$, for small $\delta > 0$ enough and $M \geq 1$ large enough, we have that

$$\limsup_{n\to\infty} \|\psi_n^k I_{D_1^c}\|^2_{\mathcal{L}^k_K} \leq \epsilon.$$

Similarly, for any $\epsilon > 0$, for small $\delta > 0$ enough and $M \geq 1$ large enough,

$$\limsup_{n\to\infty} \|G_k I_{D_1^c}\|^2_{\mathcal{L}^k_K} \leq \epsilon.$$

Thus, (3.26) is valid. □

Step 2. We define a discrete Gaussian chaos $\mathfrak{Z}_n(\beta_n;\mu)$ which has the same coefficients as the chaos expansion of $\mathfrak{Z}_n(\beta;\omega)$. We use Lindeberg's argument and the hypercontractive technique to show that the two chaos have the same asymptotic distribution. Then we study the convergence of the Gaussian chaos $\mathfrak{Z}_n(\beta_n;\mu)$ using techniques of weighted U-statistics for Gaussian variables. This step includes Lemma 3.4, Lemma 3.5 and Lemma 3.6.

Let $\{\eta(i,x),(i,x) \in \mathbb{Z}_+ \times \mathbb{Z}\}$ be a family of i.i.d. standard Gaussian random variables, and independent of $\{\xi(i,x) \in \mathbb{Z}_+ \times \mathbb{Z}\}$. Set

$$\mu(i,x) = \sum_{-\infty}^{+\infty} a_y \eta(i, x+y). \tag{3.28}$$

Then $\{\eta(i,x),(i,x) \in \mathbb{Z}_+ \times \mathbb{Z}\}$ and $\{\omega(i,x),(i,x) \in \mathbb{Z}_+ \times \mathbb{Z}\}$ have the same correlation structure. Define

$$\mu_n^k(\mathbf{t},\mathbf{x}) = \prod_{i=1}^k \mu\left(nt_i, n^{\frac{1}{\alpha}} x_i\right), \quad (\mathbf{t},\mathbf{x}) \in \mathbb{D}_n^k. \tag{3.29}$$

We also extend μ_n^k to the whole space $[0,1]^k \times \mathbb{R}^k$ by defining

$$\mu_n^k(\mathbf{t},\mathbf{x}) = \mu_n^k(\mathbf{s},\mathbf{y}) \text{ for all } (\mathbf{t},\mathbf{x}) \in \mathcal{C}_n^k(\mathbf{s},\mathbf{y}),\ (\mathbf{s},\mathbf{y}) \in \mathbb{D}_n^k.$$

Define

$$\mathfrak{Z}_n(\beta_n;\mu) = 1 + \sum_{k=1}^{\infty} (\beta\sqrt{q})^k \theta_n^k \sum_{(\mathbf{t},\mathbf{x})\in\mathbb{D}_n^k} \psi_n^k(\mathbf{t},\mathbf{x}) \mu_n^k(\mathbf{t},\mathbf{x}). \tag{3.30}$$

Let $\mathfrak{Z}_n^{\leq l}(\beta_n;\omega)$ be the sum of the first $l+1$ terms in $\mathfrak{Z}_n(\beta_n;\omega)$, i.e.,

$$\mathfrak{Z}_n^{\leq l}(\beta_n;\omega) := 1 + \sum_{k=1}^{l} (\beta\sqrt{q})^k \theta_n^k \sum_{(\mathbf{t},\mathbf{x})\in\mathbb{D}_n^k} \psi_n^k(\mathbf{t},\mathbf{x}) \omega_n^k(\mathbf{t},\mathbf{x}),$$

and let $\mathfrak{Z}_n^{\leq l}(\beta_n;\mu)$ be defined in same way.

We first use Lindeberg's argument and the hypercontractive technique in multilinear polynomials (cf. [39]) to prove that both $\mathfrak{Z}_n^{\leq l}(\beta_n;\omega)$ and $\mathfrak{Z}_n^{\leq l}(\beta_n;\mu)$ have the same limiting distribution as $n \to \infty$. Let us first recall the conception of hypercontractivity (cf. [39]).

Let $1 \leq p \leq q < \infty$, $\tau \in (0,1)$. A variable X is said to be (p,q,τ)- hypercontractive if

$$\|a + \tau X\|_q \leq \|a + X\|_p \text{ for all } a \in \mathbb{R}.$$

It is known that if $E(X) = 0$ and $E(|X|^q) < \infty$ where $q > 2$, then X is $(2,q,\tau)$- hypercontractive with $\tau = \frac{\|X\|_2}{2(q-1)^{\frac{1}{2}}\|X\|_q}$.

Generally, let n be a positive integer and let $\mathcal{X}_i$ be a collection of orthonormal real random variables, one of which is the constant 1, $i = 1, \cdots, n$. We call $\mathcal{X} = \{\mathcal{X}_1, \cdots, \mathcal{X}_n\}$

an orthonormal ensemble. $\mathcal{X}$ is said to be independent if $\mathcal{X}_1, \cdots, \mathcal{X}_n$ are independent families of random variables. A multi-index $\boldsymbol{\sigma}$ is a sequence $(\sigma_1, \cdots, \sigma_n)$ in $\mathbb{Z}_+^n$. The degree of $\boldsymbol{\sigma}$, denoted $|\boldsymbol{\sigma}|$, is the number of elements in $\{1 \leq i \leq n; \sigma_i > 0\}$. Let $\{x_{i,j}\}_{1\leq i\leq n, j\geq 0}$ be a doubly-indexed set of real constants. We write $x_{\boldsymbol{\sigma}} := \Pi_{i=1}^n x_{i,\sigma_i}$. A multilinear polynomial over $\mathcal{X}$ is defined by

$$Q(x) := \sum_{\boldsymbol{\sigma}} c_{\boldsymbol{\sigma}} x_{\boldsymbol{\sigma}}.$$

For $0 < \tau < 1$, define

$$(T_\tau Q)(x) := \sum_{\boldsymbol{\sigma}} \tau^{|\boldsymbol{\sigma}|} c_{\boldsymbol{\sigma}} x_{\boldsymbol{\sigma}}.$$

For $1 \leq p < q < \infty$ and $0 < \tau < 1$, we say that $\mathcal{X}$ is (p, q, τ)-hypercontractive if

$$\| T_\tau Q \|_q \leq \| Q \|_p$$

for every multilinear polynomial Q over $\mathcal{X}$.

Lemma 3.4. Assume that (A.1), (A.2) and (A.3) hold. Then for each $l \geq 1$, both $\mathfrak{Z}_n^{\leq l}(\beta_n; \omega)$ and $\mathfrak{Z}_n^{\leq l}(\beta_n; \mu)$ have the same limiting distributions as $n \to \infty$, that is, for any $t \in \mathbb{R}$,

$$\lim_{n\to\infty} \left| \mathbf{E}\left(\exp\left\{it\mathfrak{Z}_n^{\leq l}(\beta_n;\omega)\right\}\right) - \mathbf{E}\left(\exp\left\{it\mathfrak{Z}_n^{\leq l}(\beta_n;\mu)\right\}\right) \right| = 0.$$

Equivalently, for any bounded continuous function $f \in C_b^{(3)}(\mathbb{R})$,

$$\lim_{n\to\infty} \left| \mathbf{E}\left(f\left(\mathfrak{Z}_n^{\leq l}(\beta_n;\omega)\right)\right) - \mathbf{E}\left(f\left(\mathfrak{Z}_n^{\leq l}(\beta_n;\mu)\right)\right) \right| = 0.$$

Proof. Since the $\omega(i, x)$ and $\mu(i, x)$, $i \in \mathbb{Z}_+$, $x \in \mathbb{Z}$, are correlated in space variable, some invariance principles in [14] and [39] can not be applied directly to our model. But by time-independence, the multilinear polynomials in our model can be expanded into multilinear polynomials of independent random variables. This observation allows us to apply Lindeberg's argument and the hypercontractive technique to our case. Next, we use Lindeberg's argument and the hypercontractive technique to show the lemma.

Recall that $\mathbb{T} := \left\{(i, x) \in \mathbb{Z}^2;\ i \in \mathbb{Z}_+, x \in q\mathbb{Z} + i\ell\right\}$. We choose a sequence of finite subsets

$$\mathbb{T}_M := \{(i, x_{i,m}) \in \mathbb{T}; 1 \leq i \leq M, 1 \leq m \leq M\}, \quad M \geq 1$$

in $\mathbb{T}$ such that $\mathbb{T}_M \subset \mathbb{T}_{M'}$ for $M' \geq M$, and $\mathbb{T}_M \uparrow \mathbb{T}$ as $M \uparrow \infty$. In fact, we can take $x_{i,m} = q(m - [M/2]) + i\ell$, $i = 1, \cdots, M$, $m = 1, \cdots, M$. For each $l \geq 1$ and $M \geq 1$, define

$$\begin{aligned}
&\mathfrak{Z}_{n,M}^{\leq l}(\beta_n;\omega) \\
&:= 1 + \sum_{k=1}^{l} (\beta\sqrt{q})^k \theta_n^k \sum_{(\mathbf{t},\mathbf{x})\in\mathbb{D}_{n,M}^k} \psi_n^k(\mathbf{t},\mathbf{x})\omega_n^k(\mathbf{t},\mathbf{x}) \\
&= 1 + \sum_{k=1}^{l} (\beta\sqrt{q})^k \theta_n^k (q^{-1}n^{1+1/\alpha})^k \int_{[0,1]^k}\int_{\mathbb{R}^k} I_{\mathcal{C}_{n,M}^k}(\mathbf{s},\mathbf{y})\psi_n^k(\mathbf{s},\mathbf{y})\omega_n^k(\mathbf{s},\mathbf{y})d\mathbf{s}d\mathbf{y},
\end{aligned}$$

where I_C is the indicator function of C,

$$\mathbb{D}_{n,M} := \left\{\left(\frac{i}{n}, \frac{x}{n^{\frac{1}{\alpha}}}\right); (i,x) \in \mathbb{T}_M\right\}, \quad \mathcal{C}_{n,M}^k = \cup_{(\mathbf{t},\mathbf{x})\in\mathbb{D}_{n,M}^k} \mathcal{C}_n^k(\mathbf{t},\mathbf{x}), \quad \mathcal{C}_n^k(\mathbf{t},\mathbf{x}) = \prod_{i=1}^k \mathcal{C}_n(t_i, x_i),$$

and $\mathcal{C}_n(t, x)$ is defined in (3.6).

For given $f:\mathbb{R}\to\mathbb{R}$ with

$$C_f := \max\{\|f'\|_\infty, \|f''\|_\infty, \|f'''\|_\infty\} < \infty,$$

we have

$$\begin{aligned}
&\left|\mathbf{E}\left(f\left(\mathfrak{Z}_n^{\le l}(\beta_n;\omega)\right) - f\left(\mathfrak{Z}_n^{\le l}(\beta_n;\mu)\right)\right)\right| \\
\le&\left|\mathbf{E}\left(f\left(\mathfrak{Z}_n^{\le l}(\beta_n;\omega)\right) - f\left(\mathfrak{Z}_{n,M}^{\le l}(\beta_n;\omega)\right)\right)\right| + \left|\mathbf{E}\left(f\left(\mathfrak{Z}_n^{\le l}(\beta_n;\mu)\right) - f\left(\mathfrak{Z}_{n,M}^{\le l}(\beta_n;\mu)\right)\right)\right| \\
&+\left|\mathbf{E}\left(f\left(\mathfrak{Z}_{n,M}^{\le l}(\beta_n;\omega)\right) - f\left(\mathfrak{Z}_{n,M}^{\le l}(\beta_n;\mu)\right)\right)\right|. \qquad (3.31)
\end{aligned}$$

Let us first estimate the third term on the right side. Set

$$\omega_i := (1, \omega(i, x_{i,1}), \cdots, \omega(i, x_{i,M})), \quad \mu_i := (1, \mu(i, x_{i,1}), \cdots, \mu(i, x_{i,M})), \quad 1 \le i \le M.$$

Define the intermediate sequences between $(\mu_1, \cdots, \mu_M)$ and $(\omega_1, \cdots, \omega_M)$ as follows:

$$\mathbf{X}^{(j)} = (\mathbf{X}_1^{(j)}, \cdots, \mathbf{X}_M^{(j)}) := (\omega_1, \cdots, \omega_j, \mu_{j+1}, \cdots, \mu_M),\ j = 0, 1, \cdots, M.$$

Then the components $\mathbf{X}_k^{(j)} = (X_{k,0}^{(j)}, X_{k,1}^{(j)}, \cdots, X_{k,M}^{(j)})$, $k = 1, \cdots, M$, $j = 0, 1, \cdots, M$ have the following forms:

$$X_{k,0}^{(j)} = 1,\ X_{k,i}^{(j)} = \omega(k, x_{k,i}),\ 1 \le k \le j; \quad X_{k,i}^{(j)} = \mu(k, x_{k,i}),\ j+1 \le k \le M,\ i = 1, \cdots, M.$$

For each $\sigma = (\sigma_1, \cdots, \sigma_M) \in \{0, 1, \cdots, M\}^M$, let $1 \le i_1 < i_2, \cdots < i_{|\sigma|} \le M$ be the integer such that $\sigma_{i_k} \ne 0$, $k = 1, \cdots, |\sigma|$. Denote by $\mathbf{m}_\sigma = (i_1, \cdots, i_{|\sigma|})$ and $\mathbf{x}_\sigma = (x_{i_1,\sigma_{i_1}}, \cdots, x_{i_{|\sigma|},\sigma_{i_{|\sigma|}}})$. Then we can write $\mathfrak{Z}_{n,M}^{\le l}(\beta_n;\mu)$ as a multilinear polynomial of $\mathbf{X}^{(0)}$:

$$\begin{aligned}
\mathfrak{Z}_{n,M}^{\le l}(\beta_n;\mu) =& 1 + \sum_{k=1}^{l} (\beta\sqrt{q})^k \theta_n^k \sum_{(\mathbf{i},\mathbf{x})\in\mathbb{T}_M^k} \psi_n^k\left(\frac{\mathbf{i}}{n}, \frac{\mathbf{x}}{n^{1/\alpha}}\right) \mu_n^k\left(\frac{\mathbf{i}}{n}, \frac{\mathbf{x}}{n^{1/\alpha}}\right) \\
=& 1 + \sum_{k=1}^{l} \sum_{\substack{\sigma=(\sigma_1,\cdots,\sigma_M)\in\{0,1,\cdots,M\}^M, \\ |\sigma|=k}} (\beta\sqrt{q}\theta_n)^{|\sigma|} \psi_n^{|\sigma|}\left(\frac{\mathbf{m}_\sigma}{n}, \frac{\mathbf{x}_\sigma}{n^{1/\alpha}}\right) \prod_{i=1}^{M} X_{i,\sigma_i}^{(0)} \\
=& \sum_{\substack{\sigma=(\sigma_1,\cdots,\sigma_M)\in\{0,1,\cdots,M\}^M, \\ |\sigma|\le l}} c_\sigma \prod_{k=1}^{M} X_{k,\sigma_k}^{(0)} := Q(\mathbf{X}^{(0)}),
\end{aligned}$$

where

$$c_\sigma = (\beta\sqrt{q}\theta_n)^{|\sigma|} \psi_n^{|\sigma|}\left(\frac{\mathbf{m}_\sigma}{n}, \frac{\mathbf{x}_\sigma}{n^{1/\alpha}}\right).$$

We define $Q(\mathbf{X}^{(j)})$ to be the multilinear polynomial of $\mathbf{X}^{(j)}$ by substituting $\mathbf{X}^{(0)}$ with $\mathbf{X}^{(j)}$ in $Q(\mathbf{X}^{(0)})$. Then for each $j = 1, \cdots, M$, we can write

$$\begin{aligned}
Q(\mathbf{X}^{(j)}) =& \sum_{\substack{\sigma=(\sigma_1,\cdots,\sigma_M)\in\{0,1,\cdots,M\}^M \\ |\sigma|\le l}} c_\sigma \prod_{k=1}^{M} X_{k,\sigma_k}^{(j)} \\
=& \sum_{\sigma:|\sigma|\le l, \sigma_j=0} c_\sigma \prod_{k\ne j, 1\le k\le M} X_{k,\sigma_k}^{(j)} + \sum_{\sigma:|\sigma|\le l,\sigma_j>0} c_\sigma \omega(j, x_{j,\sigma_j}) \prod_{k\ne j,1\le k\le M} X_{k,\sigma_k}^{(j)} \\
=:& \widetilde{Q}_j + \mathbf{R}_j.
\end{aligned}$$

Similarly, we write $Q(\mathbf{X}^{(j-1)}) = \widetilde{Q}_j + \mathbf{S}_j$, where

$$\mathbf{S}_j = \sum_{\sigma:|\sigma|\le l,\sigma_j>0} c_\sigma \mu(j, x_{j,\sigma_j}) \prod_{k\ne j,1\le k\le M} X_{k,\sigma_k}^{(j)}.$$

Then

$$f\left(\mathfrak{Z}^{\leq l}_{n,M}(\beta_n;\omega)\right) - f\left(\mathfrak{Z}^{\leq l}_{n,M}(\beta_n;\mu)\right) = \sum_{j=1}^{M}\left(f\left(Q(\mathbf{X}^{(j)})\right) - f\left(Q(\mathbf{X}^{(j-1)})\right)\right)$$
$$= \sum_{j=1}^{M}\left(f\left(\widetilde{Q}_j + \mathbf{R}_j\right) - f\left(\widetilde{Q}_j + \mathbf{S}_j\right)\right). \tag{3.32}$$

Using the Taylor expansion:

$$|f(x+y) - (f(x) + f'(x)y + \frac{1}{2}f''(x)y^2)| \leq \frac{C_f}{6}|y|^3,$$

in particular, we have

$$\left|\mathbf{E}\left(f\left(\widetilde{Q}_j + \mathbf{R}_j\right) - \left(f(\widetilde{Q}_j) + f'(\widetilde{Q}_j)\mathbf{R}_j + \frac{1}{2}f''(\widetilde{Q}_j)\mathbf{R}_j^2\right)\right)\right| \leq \frac{C_f}{6}\mathbf{E}(|\mathbf{R}_j|^3)$$

and

$$\left|\mathbf{E}\left(f\left(\widetilde{Q}_j + \mathbf{S}_j\right) - \left(f(\widetilde{Q}_j) + f'(\widetilde{Q}_j)\mathbf{S}_j + \frac{1}{2}f''(\widetilde{Q}_j)\mathbf{S}_j^2\right)\right)\right| \leq \frac{C_f}{6}\mathbf{E}(|\mathbf{S}_j|^3).$$

Note that ω_j and $\prod_{k\neq j, 1\leq k\leq M} X^{(j)}_{k,\sigma_k}$ are independent, and μ_j and $\prod_{k\neq j, 1\leq k\leq M} X^{(j)}_{k,\sigma_k}$ are also independent. Then

$$\mathbf{E}\left(f'(\widetilde{Q}_j)\mathbf{R}_j\right) = \sum_{\sigma:|\sigma|\leq l, \sigma_j>0} c_\sigma \mathbf{E}\left(f'(\widetilde{Q}_j)\prod_{k\neq j, 1\leq k\leq M} X^{(j)}_{k,\sigma_k}\right)\mathbf{E}\left(\omega(j, x_{j,\sigma_j})\right)$$
$$= \sum_{\sigma:|\sigma|\leq l, \sigma_j>0} c_\sigma \mathbf{E}\left(f'(\widetilde{Q}_j)\prod_{k\neq j, 1\leq k\leq M} X^{(j)}_{k,\sigma_k}\right)\mathbf{E}\left(\mu(j, x_{j,\sigma_j})\right) = \mathbf{E}\left(f'(\widetilde{Q}_j)\mathbf{S}_j\right),$$

and

$$\mathbf{E}\left(f^{(2)}(\widetilde{Q}_j)\mathbf{R}_j^2\right)$$
$$= \sum_{\substack{\sigma^1_j>0, \sigma^2_j>0,\\ |\sigma^1|\leq l, |\sigma^2|\leq l}} c_{\boldsymbol{\sigma}^1} c_{\boldsymbol{\sigma}^2} \mathbf{E}\left(f^{(2)}(\widetilde{Q}_j)\omega(j, x_{j,\sigma^1_j})\omega(j, x_{j,\sigma^2_j})\left(\prod_{k\neq j, 1\leq k\leq M} X^{(j)}_{k,\sigma^1_k}\right)\left(\prod_{k\neq j, 1\leq k\leq M} X^{(j)}_{k,\sigma^2_k}\right)\right)$$
$$= \sum_{\substack{\sigma^1_j>0, \sigma^2_j>0,\\ |\sigma^1|\leq l, |\sigma^2|\leq l}} c_{\boldsymbol{\sigma}^1} c_{\boldsymbol{\sigma}^2} \mathbf{E}\left(f^{(2)}(\widetilde{Q}_j)\left(\prod_{k\neq j, 1\leq k\leq M} X^{(j)}_{k,\sigma^1_k}\right)\left(\prod_{k\neq j, 1\leq k\leq M} X^{(j)}_{k,\sigma^2_k}\right)\right)$$
$$\times \mathbf{E}\left(\omega(j, x_{j,\sigma^1_j})\omega(j, x_{j,\sigma^2_j})\right)$$
$$= \sum_{\substack{\sigma^1_j>0, \sigma^2_j>0,\\ |\sigma^1|\leq l, |\sigma^2|\leq l}} c_{\boldsymbol{\sigma}^1} c_{\boldsymbol{\sigma}^2} \mathbf{E}\left(f^{(2)}(\widetilde{Q}_j)\left(\prod_{k\neq j, 1\leq k\leq M} X^{(j)}_{k,\sigma^1_k}\right)\left(\prod_{k\neq j, 1\leq k\leq M} X^{(j)}_{k,\sigma^2_k}\right)\right)$$
$$\times \mathbf{E}\left(\mu(j, x_{j,\sigma^1_j})\mu(j, x_{j,\sigma^2_j})\right)$$
$$= \mathbf{E}\left(f^{(2)}(\widetilde{Q}_j)\mathbf{S}_j^2\right).$$

Therefore, we have that

$$\left|\mathbf{E}\left(f\left(\widetilde{Q}_j + \mathbf{R}_j\right) - f\left(\widetilde{Q}_j + \mathbf{S}_j\right)\right)\right| \leq \frac{C_f}{6}\left(\mathbf{E}(|\mathbf{R}_j|^3) + \mathbf{E}(|\mathbf{S}_j|^3)\right). \tag{3.33}$$

Next, we estimate $\mathbf{E}(|\mathbf{R}_j|^3)$ and $\mathbf{E}(|\mathbf{S}_j|^3)$. For each $L \geq 1$, define

$$\omega_L(i,x) := \sum_{|y|\leq L} a_y \xi(i, x+y),$$
$$\mu_L(i,x) := \sum_{|y|\leq L} a_y \mu(i, x+y), \quad x \in \{x_{i,1}, \cdots, x_{i,M}\},\ 1 \leq i \leq M.$$

Set

$$\omega_i^L := (\omega_L(i,x_{i,1}), \cdots, \omega_L(i,x_{i,M})), \quad \mu_i^L := (\mu_L(i,x_{i,1}), \cdots, \mu_L(i,x_{i,M})), \quad 1 \leq i \leq M,$$

and substituting ω and μ with ω_L and μ_L, we can define $\mathbf{X}^{(j,L)}$, $\mathbf{X}_k^{(j,L)}$, $X_{k,i}^{(j,L)}$, $k = 1, \cdots, M$, $i = 1, \cdots, M$, $j = 0, 1, \cdots, M$. Denote by

$$\mathbf{R}_{j,L} = \sum_{\sigma:|\sigma|\leq l, \sigma_j > 0} c_\sigma \omega_L(j, x_{j,\sigma_j}) \prod_{k\neq j, 1\leq k\leq M} X_{k,\sigma_k}^{(j,L)},$$
$$\mathbf{S}_{j,L} = \sum_{\sigma:|\sigma|\leq l, \sigma_j > 0} c_\sigma \mu_L(j, x_{j,\sigma_j}) \prod_{k\neq j, 1\leq k\leq M} X_{k,\sigma_k}^{(j,L)}.$$

Then $\mathbf{R}_{j,L}$ and $\mathbf{S}_{j,L}$ can be expanded into two multilinear polynomials with degree $\leq l$ over $\mathcal{X}^{L+\widetilde{M}} = \{\mathcal{X}_1^{L+\widetilde{M}}, \cdots, \mathcal{X}_M^{L+\widetilde{M}}\}$, where $\widetilde{M} = \max\{|x_{i,1}|, \cdots, |x_{i,M}|\} + 1$,

$$\mathcal{X}_i^{L+\widetilde{M}} = \left\{1,\ \xi(i,x),\ \eta(i,x);\ |x| \leq 2(L+\widetilde{M})\right\}, \quad i = 1, \cdots, M.$$

Since $\mathcal{X}^{L+\widetilde{M}}$ is $(2,3,\tau)$-hypercontractive independent ensemble with

$$\tau = \min\left\{\frac{1}{2^{3/2}\|\xi\|_3}, \frac{1}{2^{3/2}\|\eta\|_3}\right\},$$

by Proposition 3.12 in [39], we have that

$$\mathbf{E}(|\mathbf{R}_{j,L}|^3) + \mathbf{E}(|\mathbf{S}_{j,L}|^3) \leq \tau^{-3l}\left((\mathbf{E}(|\mathbf{R}_{j,L}|^2))^{3/2} + (\mathbf{E}(|\mathbf{S}_{j,L}|^2))^{3/2}\right).$$

Letting $L \to \infty$, we obtain

$$\mathbf{E}(|\mathbf{R}_j|^3) + \mathbf{E}(|\mathbf{S}_j|^3) \leq \tau^{-3l}\left((\mathbf{E}(|\mathbf{R}_j|^2))^{3/2} + (\mathbf{E}(|\mathbf{S}_j|^2))^{3/2}\right).$$

Therefore

$$\left|\mathbf{E}\left(f\left(\widetilde{Q}_j + \mathbf{R}_j\right) - f\left(\widetilde{Q}_j + \mathbf{S}_j\right)\right)\right| \leq \frac{C_f}{6}\tau^{-3l}\left((\mathbf{E}(|\mathbf{R}_j|^2))^{3/2} + (\mathbf{E}(|\mathbf{S}_j|^2))^{3/2}\right). \quad (3.34)$$

For $(\mathbf{t},\mathbf{x}) \in \mathbb{D}_{n,M}^k$, $\frac{j}{n} \in \mathbf{t}$ means that $\frac{j}{n}$ is a component of $\mathbf{t}$. Note that $\mathbf{X}_j^{(j)}$ and $\mathbf{X}_j^{(j-1)}$ are independent of $\mathbf{X}_1^{(j)}, \cdots, \mathbf{X}_{j-1}^{(j)}, \mathbf{X}_{j+1}^{(j)}, \cdots, \mathbf{X}_M^{(j)}$, and have the same mean and variance. Therefore, by (3.15) and noting that $\psi_n^k(\mathbf{t},\mathbf{x})$ is a probability density,

$$\begin{aligned}
&\mathbf{E}(|\mathbf{R}_j|^2) + \mathbf{E}(|\mathbf{S}_j|^2) \\
=&2\mathbf{E}\left(\left(\sum_{k=1}^{l}(\beta\sqrt{q})^k\theta_n^k \sum_{(\mathbf{t},\mathbf{x})\in\mathbb{D}_{n,M}^k \text{ and } \frac{j}{n}\in\mathbf{t}} \psi_n^k(\mathbf{t},\mathbf{x})\left(\prod_{i=1}^k \mathbf{X}_{nt_i, n^{1/\alpha}x_i}^{(j)}\right)\right)^2\right) \\
=&2\sum_{k=1}^{l}(\beta\sqrt{q})^{2k}k!k\int_{[0,1]^k}\int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\psi_n^k(\mathbf{t},\mathbf{x})I_{\mathcal{C}_{n,M}^k}(\mathbf{t},\mathbf{x})\gamma_n(\mathbf{x}-\mathbf{y})\,\psi_n^k(\mathbf{t},\mathbf{y})I_{\mathcal{C}_{n,M}^k}(\mathbf{t},\mathbf{y})I_j^n(\mathbf{t})d\mathbf{t}d\mathbf{x}d\mathbf{y},
\end{aligned}$$

where $\mathcal{C}_{n,M}^k = \cup_{(\mathbf{t},\mathbf{x})\in\mathbb{D}_{n,M}^k}\mathcal{C}_n^k(\mathbf{t},\mathbf{x})$,

$$I_j^n(\mathbf{t}) = \sum_{i=1}^k I_{(\frac{j-1}{n},\frac{j}{n}]}(t_i), \quad \mathbf{t} = (t_1,\cdots,t_k),$$

and I_C is the indicator function of C. Then by (3.15),

$$\begin{aligned}&\sum_{k=1}^l(\beta\sqrt{q})^{2k}k!k\int_{[0,1]^k}\int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\psi_n^k(\mathbf{t},\mathbf{x})I_{\mathcal{C}_{n,M}^k}(\mathbf{t},\mathbf{x})\gamma_n(\mathbf{x}-\mathbf{y})\,\psi_n^k(\mathbf{t},\mathbf{y})I_{\mathcal{C}_{n,M}^k}(\mathbf{t},\mathbf{y})I_j^n(\mathbf{t})d\mathbf{t}d\mathbf{x}d\mathbf{y}\\ \le& C_1(l)\sum_{k=1}^l\left\|I_j^nI_{\mathcal{C}_{n,M}^k}\psi_n^k\right\|_{\mathcal{L}_K^k}^2,\end{aligned}$$

where $C_1(l)$ are positive constants independent of n and M. By (3.18),

$$\sum_{k=1}^l\left\|\psi_n^k\right\|_{\mathcal{L}_K^k}^2 \le C_2(l),$$

where $C_2(l)$ is a positive constant independent of n and M. Therefore,

$$\sup_{1\le j\le M}\left\{\mathbf{E}(|\mathbf{R}_j|^2)+\mathbf{E}(|\mathbf{S}_j|^2)\right\} \le C_1(l)\sum_{k=1}^l\left\|\psi_n^k\right\|_{\mathcal{L}_K^k}^2 \le C_1(l)C_2(l). \tag{3.35}$$

From (3.26), we have that, for each $1\le k\le l$,

$$\lim_{n\to\infty}\left\||\psi_n^k - G_k|\right\|_{\mathcal{L}_K^k}^2 = 0.$$

From (3.17), and for each $t>0$, $I_j^n(t)\to 0$ uniformly in j as $n\to\infty$,

$$\left\|G_kI_j^nI_{\mathcal{C}_{n,M}^k}\right\|_{\mathcal{L}_K^k}^2 \to 0$$

uniformly in j, M as $n\to\infty$. Then

$$\left\|I_j^nI_{\mathcal{C}_{n,M}^k}\psi_n^k\right\|_{\mathcal{L}_K^k}^2 \le 2\left\||\psi_n^k-G_k|\right\|_{\mathcal{L}_K^k}^2 + 2\left\|G_kI_j^nI_{\mathcal{C}_{n,M}^k}\right\|_{\mathcal{L}_K^k}^2 \to 0 \tag{3.36}$$

uniformly in j, M as $n\to\infty$, where C is a positive constant independent of n, M. Note that

$$\sum_{j=1}^M I_j^n(\mathbf{t}) = \sum_{i=1}^k I_{(0,\frac{M}{n}]}(t_i).$$

We have

$$\begin{aligned}&\sum_{j=1}^M\left\|I_j^nI_{\mathcal{C}_{n,M}^k}\psi_n^k\right\|_{\mathcal{L}_K^k}^2\\ =&\int_{[0,1]^k}\int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\psi_n^k(\mathbf{t},\mathbf{x})I_{\mathcal{C}_{n,M}^k}(\mathbf{t},\mathbf{x})\gamma_n(\mathbf{x}-\mathbf{y})\,\psi_n^k(\mathbf{t},\mathbf{y})I_{\mathcal{C}_{n,M}^k}(\mathbf{t},\mathbf{y})\sum_{j=1}^M I_j^n(\mathbf{t})d\mathbf{t}d\mathbf{x}d\mathbf{y}\\ =&\sum_{i=1}^k\int_{[0,1]^k}\int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\psi_n^k(\mathbf{t},\mathbf{x})I_{\mathcal{C}_{n,M}^k}(\mathbf{t},\mathbf{x})\gamma_n(\mathbf{x}-\mathbf{y})\,\psi_n^k(\mathbf{t},\mathbf{y})I_{\mathcal{C}_{n,M}^k}(\mathbf{t},\mathbf{y})I_{(0,\frac{M}{n}]}(t_i)d\mathbf{t}d\mathbf{x}d\mathbf{y}\\ \le& k\left\|\psi_n^k\right\|_{\mathcal{L}_K^k}^2 \le l\left\|\psi_n^k\right\|_{\mathcal{L}_K^k}^2.\end{aligned}$$

Now, by (3.32) and (3.34), there exists a positive constant $C_3(l)$ independent of n and M such that

$$\begin{aligned}&\left|\mathbf{E}\left(f\left(\mathfrak{Z}_{n,M}^{\le l}(\beta_n;\omega)\right)-f\left(\mathfrak{Z}_{n,M}^{\le l}(\beta_n;\mu)\right)\right)\right|\\ \le&\frac{C_3(l)}{3}C_f\tau^{-3l}\left(\sum_{k=1}^{l}\sum_{j=1}^{M}\left\|I_j^n I_{\mathcal{C}_{n,M}^k}\psi_n^k\right\|_{\mathcal{L}_K^k}^2\right)\sup_{1\le j\le M}\left(\sum_{k=1}^{l}\left\|I_j^n I_{\mathcal{C}_{n,M}^k}\psi_n^k\right\|_{\mathcal{L}_K^k}^2\right)^{1/2}\\ \le&\frac{C_3(l)l}{3}2C_f\tau^{-3l}\sum_{k=1}^{l}\left\|\psi_n^k\right\|_{\mathcal{L}_K^k}^2\left(\left\||\psi_n^k-G_k|\right\|_{\mathcal{L}_K^k}^2+\left\|G_k I_j^n I_{\mathcal{C}_{n,M}^k}\right\|_{\mathcal{L}_K^k}^2\right)\to 0,\end{aligned}$$

uniformly in M as $n\to\infty$.

For any $\epsilon>0$, choose $n_\epsilon\ge 1$ such that for all $n\ge n_\epsilon$, $M\ge 1$,

$$\left|\mathbf{E}\left(f\left(\mathfrak{Z}_{n,M}^{\le l}(\beta_n;\omega)\right)-f\left(\mathfrak{Z}_{n,M}^{\le l}(\beta_n;\mu)\right)\right)\right|\le\epsilon.$$

Since for each $n\ge n_\epsilon$ fixed, as $M\to\infty$, $\mathcal{C}_n^k\setminus\mathcal{C}_{n,M}^k\to\emptyset$ where $\mathcal{C}_n^k=\cup_{(\mathbf{t},\mathbf{x})\in\mathbb{D}_n^k}\mathcal{C}_n^k(\mathbf{t},\mathbf{x})$, we have that as $M\to\infty$,

$$\mathbf{E}\left(\left(\mathfrak{Z}_n^{\le l}(\beta_n;\omega)-\mathfrak{Z}_{n,M}^{\le l}(\beta_n;\omega)\right)^2\right)\le C_l\sum_{k=1}^{l}(\beta\sqrt{q})^{2k}k!k\left\|(I_{\mathcal{C}_n^k}-I_{\mathcal{C}_{n,M}^k})\psi_n^k\right\|_{\mathcal{L}_K^k}^2\to 0,$$

Therefore, as $M\to\infty$,

$$\mathfrak{Z}_n^{\le l}(\beta_n;\omega)-\mathfrak{Z}_{n,M}^{\le l}(\beta_n;\omega)\xrightarrow{\mathbf{P}}0,\quad \mathfrak{Z}_n^{\le l}(\beta_n;\mu)-\mathfrak{Z}_{n,M}^{\le l}(\beta_n;\mu)\xrightarrow{\mathbf{P}}0,$$

and so, for $f\in C_b(\mathbb{R})$,

$$\left|\mathbf{E}\left(f\left(\mathfrak{Z}_n^{\le l}(\beta_n;\omega)\right)-f\left(\mathfrak{Z}_{n,M}^{\le l}(\beta_n;\omega)\right)\right)\right|\xrightarrow{M\to\infty}0,$$

$$\left|\mathbf{E}\left(f\left(\mathfrak{Z}_n^{\le l}(\beta_n;\mu)\right)-f\left(\mathfrak{Z}_{n,M}^{\le l}(\beta_n;\mu)\right)\right)\right|\xrightarrow{M\to\infty}0.$$

Now, in (3.31), letting $M\to\infty$, we get that for all $n\ge n_\epsilon$,

$$\left|\mathbf{E}\left(f\left(\mathfrak{Z}_n^{\le l}(\beta_n;\omega)\right)-f\left(\mathfrak{Z}_n^{\le l}(\beta_n;\mu)\right)\right)\right|\le\epsilon.$$

Finally, letting first $n\to\infty$, then $\epsilon\to 0$, we obtain that

$$\lim_{n\to\infty}\left|\mathbf{E}\left(f\left(\mathfrak{Z}_n^{\le l}(\beta_n;\omega)\right)-f\left(\mathfrak{Z}_n^{\le l}(\beta_n;\mu)\right)\right)\right|=0.\qquad\square$$

Next, we study the convergence of $\mathfrak{Z}_n^{\le l}(\beta_n;\mu)$.

Lemma 3.5. Assume that (A.1), (A.2) and (A.3) hold. Let $k\ge 1$ and $\varphi\in L^{2r}([0,1]^k\times\mathbb{R}^k)$. Assume that

$$\sup_{t\in[0,1]^k}\int_{\mathbb{R}^k}|\varphi(\mathbf{t},\mathbf{x})|d\mathbf{x}<\infty. \tag{3.37}$$

Set

$$\overline{\varphi}(\mathbf{s},\mathbf{y})=(q^{-1}n^{1+1/\alpha})^k\sum_{(\mathbf{t},\mathbf{x})\in\mathbb{D}_n^k}I_{\mathcal{C}_n^k(\mathbf{t},\mathbf{x})}(\mathbf{s},\mathbf{y})\left(\int_{\mathcal{C}_n^k(\mathbf{t},\mathbf{x})}\varphi(\mathbf{s},\mathbf{y})d\mathbf{s}d\mathbf{y}\right),$$

and define the weighted U-statistics via

$$\mathcal{S}_k^n(\varphi)=\theta_n^k\sum_{(\mathbf{t},\mathbf{x})\in\mathbb{D}_n^k}\overline{\varphi}(\mathbf{t},\mathbf{x})\mu_n^k\left(\mathbf{t},\mathbf{x}\right),$$

where $\theta_n = q^{\frac{1}{2}} n^{-\frac{1}{2}-\frac{3}{2\alpha}+\frac{r}{\alpha}}$ and $\mu_n^k(\mathbf{t},\mathbf{x})$ is defined by (3.29). Then, as $n\to\infty$,

$$\mathcal{S}_k^n(\varphi) \xrightarrow{(d)} \begin{cases} \int_{[0,1]^k}\int_{\mathbb{R}^k} \varphi(\mathbf{t},\mathbf{x})\mathcal{W}^{\otimes k}(d\mathbf{t}\,d\mathbf{x}) & \text{if } r=1, \\ \frac{1}{q^{k/2}}\int_{[0,1]^k}\int_{\mathbb{R}^k} \varphi(\mathbf{t},\mathbf{x})\mathcal{W}^{\otimes k}(d\mathbf{t}\,d\mathbf{x}) & \text{if } r<1. \end{cases} \tag{3.38}$$

Proof. By the definition of $\overline{\varphi}$, we have

$$\|\overline{\varphi}-\varphi\|_{L^{2r}}^{2r} := \int_{[0,1]}\int_{\mathbb{R}^k} \left|\overline{\varphi}(\mathbf{t},\mathbf{x})-\varphi(\mathbf{t},\mathbf{x})\right|^{2r} d\mathbf{t}d\mathbf{x} \to 0 \text{ as } n\to\infty. \tag{3.39}$$

When $k=1$, $\mathcal{S}_1^n(\varphi)$ is a Gaussian random variable with zero mean. We first consider $\varphi(t,x) = I_{(t_0,t_1]\times(x_0,x_1]}(t,x)$, where $0\le t_0<t_1\le 1$, $-\infty<x_0<x_1<\infty$, then

$$\begin{aligned} &\mathbf{E}\left(\left|\mathcal{S}_1^n(\varphi) - \theta_n \sum_{(t,x)\in\mathbb{D}_n} \varphi(t,x)\mu_n(t,x)\right|^2\right) \\ &\le \sum_{(t,x)\in\mathbb{D}_n,(t,y)\in\mathbb{D}_n} \int_{\mathcal{C}_n(t,x)\times\mathcal{C}_n(t,y)} |\overline{\varphi}(t',x')-\varphi(t',x')|\gamma_n(x'-y')|\overline{\varphi}(t',y')-\varphi(t',y')|dt'dx'dy' \\ &= \int_0^1\int_{\mathbb{R}^2} |\overline{\varphi}(t,x)-\varphi(t,x)|\gamma_n(x-y)|\overline{\varphi}(t,y)-\varphi(t,y)|dtdxdy. \end{aligned}$$

Therefore, by (3.15), (3.16), and (3.39), we have

$$\lim_{n\to\infty}\mathbf{E}\left(\left|\mathcal{S}_1^n(\varphi) - \theta_n \sum_{(t,x)\in\mathbb{D}_n} \varphi(t,x)\mu_n(t,x)\right|^2\right) = 0.$$

Next, let us estimate $\mathbf{E}\left(\left(\sum_{(t,x)\in\mathbb{D}_n}\varphi(t,x)\mu_n(t,x)\right)^2\right)$.

$$\begin{aligned} &\mathbf{E}\left(\left(\sum_{(t,x)\in\mathbb{D}_n}\varphi(t,x)\mu_n(t,x)\right)^2\right) \\ &= \sum_{nt_0<i\le nt_1}\ \sum_{n^{1/\alpha}x_0<qk+i\ell,qj+i\ell\le n^{1/\alpha}x_1} \gamma(qk+i\ell-(qj+i\ell)) + O(n) \\ &= \sum_{nt_0<i\le nt_1}\ \sum_{0\le l\le n^{1/\alpha}(x_1-x_0)/q} \gamma(ql) \sum_{\substack{|k-j|=l \\ (n^{1/\alpha}x_0-i\ell)/q<k,j\le(n^{1/\alpha}x_1-i\ell)/q}} 1 + O(n). \end{aligned}$$

Noting that

$$\sum_{\substack{|k-j|=0 \\ (n^{1/\alpha}x_0-i\ell)/q<k,j\le(n^{1/\alpha}x_1-i\ell)/q}} 1 = n^{1/\alpha}(x_1-x_0),$$

and for $l\neq 0$,

$$\sum_{\substack{|k-j|=l \\ (n^{1/\alpha}x_0-i\ell)/q<k,j\le(n^{1/\alpha}x_1-i\ell)/q}} 1 = 2n^{1/\alpha}(x_1-x_0),$$

we have

$$\begin{aligned}&\mathbf{E}\left(\left(\sum_{(t,x)\in\mathbb{D}_n}\varphi(t,x)\mu_n(t,x)\right)^2\right)\\=&\frac{1}{q}n(t_1-t_0)\bigg(n^{1/\alpha}(x_1-x_0)\gamma(0)+2n^{1/\alpha}(x_1-x_0)\sum_{l=1}^{n^{1/\alpha}(x_1-x_0)/q}\gamma(ql)\\&-2q\sum_{l=1}^{n^{1/\alpha}(x_1-x_0)/q}l\gamma(ql)\bigg)+O(n).\end{aligned}$$

Thus, if $r=1$, then

$$\lim_{n\to\infty}\theta_n^2\mathbf{E}\left(\left(\sum_{(t,x)\in\mathbb{D}_n}\varphi(t,x)\mu_n(t,x)\right)^2\right)=(t_1-t_0)(x_1-x_0)$$

If $r<1$, then

$$\begin{aligned}&\lim_{n\to\infty}\theta_n^2\mathbf{E}\left(\left(\sum_{(t,x)\in\mathbb{D}_n}\varphi(t,x)\mu_n(t,x)\right)^2\right)\\=&\lim_{n\to\infty}n^{\frac{2r-2}{\alpha}}(t_1-t_0)\bigg(2(x_1-x_0)\lambda_r q^{1-2r}\sum_{l=1}^{n^{1/\alpha}(x_1-x_0)/q}l^{1-2r}\\&-2qn^{-\frac{1}{\alpha}}\sum_{l=1}^{n^{1/\alpha}(x_1-x_0)/q}\lambda_r q^{1-2r}l^{2-2r}\bigg)\\=&\frac{1}{q}\lim_{n\to\infty}n^{\frac{2r-2}{\alpha}}(t_1-t_0)\bigg(2(x_1-x_0)\lambda_r(2-2r)^{-1}\left(n^{1/\alpha}(x_1-x_0)\right)^{2-2r}\\&-2n^{-\frac{1}{\alpha}}\lambda_r(3-2r)^{-1}\left(n^{1/\alpha}(x_1-x_0)\right)^{3-2r}\bigg)\\=&\frac{1}{q}(t_1-t_0)(x_1-x_0)^{3-2r}=\frac{1}{q}\int_{[0,1]}\int_{\mathbb{R}^2}\varphi(t,x)K(x-y)\varphi(t,y)dtdxdy,\end{aligned}$$

where $\lambda_r=(1-r)(3-2r)$. Thus, for $\varphi(t,x)=I_{(t_0,t_1]\times(x_0,x_1]}(t,x)$,

$$\mathcal{S}_1^n(\varphi)\xrightarrow{(d)}\begin{cases}\int_{[0,1]}\int_{\mathbb{R}}\varphi(t,x)\mathcal{W}(dt\,dx) & \text{if } r=1,\\ \frac{1}{\sqrt{q}}\int_{[0,1]}\int_{\mathbb{R}}\varphi(t,x)\mathcal{W}(dt\,dx) & \text{if } r<1.\end{cases}$$

which implies that this also holds for any simple function.

Let us now complete the proof in the case $k=1$. We only consider the $r<1$ case. For any $\varphi\in L^{2r}([0,1]\times\mathbb{R})$ with

$$\sup_{t\in[0,1]}\int_{\mathbb{R}}|\varphi(t,x)|dx<\infty,$$

choose a sequence of simple functions $\varphi^{(m)}$ such that

$$|\varphi^{(m)}(t,x)|\le|\varphi(t,x)| \text{ for all } (t,x)\in[0,1]\times\mathbb{R} \text{ and } \|\varphi^{(m)}-\varphi\|_{L^{2r}}\to 0 \text{ as } m\to\infty.$$

By (3.16), (3.39), we have

$$\lim_{n\to\infty}\mathbf{E}\left(\left(\mathcal{S}_1^n(\varphi)\right)^2\right)=\frac{1}{q}\int_{[0,1]}\int_{\mathbb{R}^2}\varphi(t,x)K\left(x-y\right)\varphi(t,y)dtdxdy,$$

and so

$$S_1^n(\varphi)\overset{(d)}{\to}\frac{1}{\sqrt{q}}I_1^{\mathcal{W}}(\varphi):=\frac{1}{\sqrt{q}}\int_0^1\int_{\mathbb{R}}\varphi(t,x)\mathcal{W}(dtdx).$$

Therefore, we complete the proof of (3.38) in $k=1$ case.

By the Cramér-Wold method, for any $m\geq 1$, $\varphi_1,\cdots,\varphi_m\in L^{2r}([0,1]\times\mathbb{R})$, if

$$\sup_{1\leq i\leq m}\sup_{t\in[0,1]}\int_{\mathbb{R}}|\varphi_i(t,x)|dx<\infty,$$

then we have the joint convergence

$$\left(\mathcal{S}_1^n(\varphi_1),\cdots,\mathcal{S}_1^n(\varphi_m)\right)\overset{(d)}{\longrightarrow}\left(I_1^{\mathcal{W}}(\varphi_1),\cdots,I_1^{\mathcal{W}}(\varphi_m)\right).$$

Next, let us extend the conclusion to $k\geq 2$. We first consider functions of the form

$$\varphi(\mathbf{t},\mathbf{x})=\varphi_1(t_1,x_1)\cdots\varphi_k(t_k,x_k),\tag{3.40}$$

where $\varphi_1,\cdots,\varphi_k\in L^{2r}([0,1]\times\mathbb{R})$ satisfies

$$\sup_{1\leq i\leq k}\sup_{t\in[0,1]}\int_{\mathbb{R}}|\varphi_i(t,x)|dx<\infty.$$

If $\varphi_i(t,x)\varphi_j(t,y)=0$, $t\in[0,1]$, $x,y\in\mathbb{R}$, $1\leq i<j\leq k$, then for such functions φ, by Lemma A.2, as $n\to\infty$,

$$\mathcal{S}_k^n(\varphi)=\prod_{j=1}^k\mathcal{S}_1^n(\varphi_j)\overset{(d)}{\longrightarrow}\frac{1}{q^{k/2}}\prod_{j=1}^k\int_0^1\int_{\mathbb{R}}\varphi_j(t,x)\mathcal{W}(dt\,dx)=\frac{1}{q^{k/2}}\int_{[0,1]^k}\int_{\mathbb{R}^k}\varphi(\mathbf{t},\mathbf{x})\mathcal{W}(d\mathbf{t}\,d\mathbf{x}).$$

For general $\varphi\in L^{2r}([0,1]^k\times\mathbb{R}^k)$ satisfying (3.40), for each $m\geq 2$, we define

$$\varphi_{j,i}^{(m)}(t,x)=\varphi_j(t,x)I_{\left[\frac{i-1}{m},\frac{i}{m}\right)}(t),\quad i=1,\cdots,m,\ j=1,\cdots,k,$$

and

$$\varphi^{(m)}(\mathbf{t},\mathbf{x})=\sum_{1\leq i_1,\cdots,i_k\leq m,i_j\neq i_l\ \ for\ \ i\neq l}\prod_{j=1}^k\varphi_{j,i_j}^{(m)}(t_j,x_j)=\varphi(\mathbf{t},\mathbf{x})I_{B_m}.$$

where

$$B_m=\bigcup_{1\leq i_1,\cdots,i_k\leq m,i_j\neq i_l\ \ for\ \ j\neq l}\left[\frac{i_1-1}{m},\frac{i_1}{m}\right)\times\cdots\times\left[\frac{i_k-1}{m},\frac{i_k}{m}\right).$$

Then for each $1\leq i_1,\cdots,i_k\leq m$ with $i_j\neq i_l$ for any $j\neq l$, $\varphi_{1,i_1}^{(m)}(t_1,x_1),\ \cdots,\ \varphi_{k,i_k}^{(m)}(t_k,x_k)\in L^{2r}([0,1]\times\mathbb{R})$ satisfy $\varphi_{l,i_l}^{(m)}(t,x)\varphi_{j,i_j}^{(m)}(t,y)=0$, $t\in[0,1]$, $x,y\in\mathbb{R}$, $1\leq l<j\leq k$, and

$$|\varphi^{(m)}(\mathbf{t},\mathbf{x})|\leq|\varphi(\mathbf{t},\mathbf{x})|\text{ for all }(\mathbf{t},\mathbf{x})\in[0,1]^k\times\mathbb{R}^k.$$

By the Lebesgue measure $|B_m^c|\to 0$ as $m\to\infty$, we have

$$\|\varphi^{(m)}-\varphi\|_{L^{2r}}\leq\|\varphi I_{B_m^c}\|_{L^{2r}}\to 0\text{ as }m\to\infty.$$

Since for each $m \geq 2$, as $n \to \infty$,

$$S_k^n(\varphi^{(m)}) = \sum_{1\leq i_1,\cdots,i_k\leq m, i_j\neq i_l \ for \ j\neq l} S_k^n\left(\prod_{j=1}^k \varphi_{j,i_j}^{(m)}\right)$$
$$\xrightarrow{(d)} \sum_{1\leq i_1,\cdots,i_k\leq m, i_j\neq i_l \ for \ j\neq l} I_k^{\mathcal{W}}\left(\prod_{j=1}^k \varphi_{j,i_j}^{(m)}\right) = I_k^{\mathcal{W}}\left(\varphi^{(m)}\right).$$

By (3.15), (3.16) and (3.39), there exists a positive constant A only depending on φ such that for any $n \geq 1$,

$$\mathbf{E}\left(\left(S_k^n(\varphi) - S_k^n(\varphi^{(m)})\right)^2\right) \leq A\|\varphi^{(m)} - \varphi\|_{L^{2r}} \leq A\|\varphi I_{B_m^c}\|_{L^{2r}},$$

and

$$\mathbf{E}\left(\left(I_k^{\mathcal{W}}(\varphi) - I_k^{\mathcal{W}}\left(\varphi^{(m)}\right)\right)^2\right) \leq A\|\varphi^{(m)} - \varphi\|_{L^{2r}} \leq A\|\varphi I_{B_m^c}\|_{L^{2r}}.$$

Therefore, $S_k^n(\varphi) \xrightarrow{(d)} I_k^{\mathcal{W}}(\varphi)$.

Finally, for any $\phi \in L^{2r}([0,1]^k \times \mathbb{R}^k)$ satisfying (3.37), choose a sequence of functions $\phi^{(m)}(\mathbf{t},\mathbf{x}) = \sum_{j=1}^{l_m} \phi_{1,j}^{(m)}(t_1,x_1)\cdots\phi_{k,j}^{(m)}(t_k,x_k)$, such that

$$|\phi^{(m)}(\mathbf{t},\mathbf{x})| \leq |\phi(\mathbf{t},\mathbf{x})| \text{ for all } (\mathbf{t},\mathbf{x}) \in [0,1]^k \times \mathbb{R}^k \text{ and } \|\phi^{(m)} - \phi\|_{L^{2r}} \to 0 \text{ as } m \to \infty.$$

Then by (3.16), (3.39), we accomplish the proof. □

Remark 3.1. It is obvious that the operators $\varphi \to \mathcal{S}_k^n(\varphi)$, $k \geq 1$, have a natural symmetrizing property. Thus $\mathcal{S}_k^n(\varphi) = \mathcal{S}_k^n(\mathrm{Sym}\{\varphi\})$.

Remark 3.2. Note that ψ_k^n is already constant on $\mathcal{C}_n^k(\mathbf{t},\mathbf{x})$, $(\mathbf{t},\mathbf{x}) \in \Delta\mathbb{D}_n^k$, so that $\overline{\psi_n^k} = \psi_n^k$. Thus, by the definition of $\mathcal{S}_k^n$,

$$\mathcal{S}_k^n(\psi_n^k) = \theta_n^k \sum_{(\mathbf{t},\mathbf{x})\in\mathbb{D}_n^k} \psi_n^k(\mathbf{t},\mathbf{x})\mu_n^k(\mathbf{t},\mathbf{x}).$$

By the definition of the $\mathcal{Z}_{\sigma(\beta,q)}(1,*)$ (see Theorem 2.1), we have that

$$\mathcal{Z}_{\sigma(\beta,q)}(1,*) = 1 + \sum_{k=1}^{\infty}(\sigma(\beta,q))^k \int_{\Delta_k(1)}\int_{\mathbb{R}^k}\prod_{i=1}^k g(t_i - t_{i-1}, x_i - x_{i-1})\mathcal{W}(dt_i dx_i),$$

where $x_0 = 0$, $t_0 = 0$ and $\Delta_k(1) = \{0 \leq t_1 < \cdots < t_k \leq 1\}$. Define

$$\mathcal{Z}_{\sigma(\beta,q)}^{\leq l}(1,*) = 1 + \sum_{k=1}^{l}(\sigma(\beta,q))^k \int_{\Delta_k(1)}\int_{\mathbb{R}^k}\prod_{i=1}^k g(t_i - t_{i-1}, x_i - x_{i-1})\mathcal{W}(dt_i dx_i).$$

Lemma 3.6. Assume that (A.1), (A.2) and (A.3) hold. Then for each $l \geq 1$, $\mathfrak{Z}_n^{\leq l}(\beta_n;\omega)$ converges in distribution to $\mathcal{Z}_{\sigma(\beta,q)}^{\leq l}(1,*)$.

Proof. Since $g(x)$ is the density function of the symmetric stable distribution on $\mathbb{R}$, $\alpha \in (2r-1,2]$, it is known that $g(t,x)$ is continuous on $(0,1] \times \mathbb{R}$ and

$$\int_{\mathbb{R}} g(t,x)dx = 1,$$

and

$$\int_0^1 \int_{\mathbb{R}} |g(t,x)|^{2r} dxdt = \int_0^1 \int_{\mathbb{R}} t^{(1-2r)/\alpha} |g(x)|^{2r} dxdt = \frac{\alpha}{\alpha+1-2r} \int_{\mathbb{R}} |g(x)|^{2r} dx < \infty.$$

Set

$$g_n^k(\mathbf{s},\mathbf{y}) = (q^{-1} n^{1+1/\alpha})^k \sum_{(\mathbf{t},\mathbf{x}) \in \mathbb{D}_n^k} I_{\mathcal{C}_n^k(\mathbf{t},\mathbf{x})}(\mathbf{s},\mathbf{y}) \left(\int_{\mathcal{C}_n^k(\mathbf{t},\mathbf{x})} g_k(\mathbf{s},\mathbf{y}) d\mathbf{s} d\mathbf{y} \right).$$

Define

$$\mathfrak{Z}_n^{\leq l}(g) := 1 + \sum_{k=1}^{l} (\beta\sqrt{q})^k \mathcal{S}_k^n(g_k).$$

Therefore, Lemma 3.5 yields

$$\mathfrak{Z}_n^{\leq l}(g) \overset{(d)}{\to} \mathcal{Z}_{\sigma(\beta,q)}^{\leq l}(1,*) \text{ as } n \to \infty.$$

We write

$$\begin{aligned} &\mathbf{E}\left(\left(\mathfrak{Z}_n^{\leq l}(g) - \mathfrak{Z}_n^{\leq l}(\beta_n;\omega) \right)^2 \right) \\ =& \sum_{k=1}^{l} \beta^{2k} k! \int_{[0,1]^k} \int_{\mathbb{R}^{2k}} (\psi_n^k(\mathbf{t},\mathbf{x}) - g_n^k(\mathbf{t},\mathbf{x})) \gamma_n^k(\mathbf{x}-\mathbf{y}) (\psi_n^k(\mathbf{t},\mathbf{y}) - g_n^k(\mathbf{t},\mathbf{y})) d\mathbf{t} d\mathbf{x} d\mathbf{y}. \end{aligned}$$

By (3.15), we have

$$\begin{aligned} &\int_{[0,1]^k} \int_{\mathbb{R}^{2k}} (\psi_n^k(\mathbf{t},\mathbf{x}) - g_n^k(\mathbf{t},\mathbf{x})) \gamma_n^k(\mathbf{x}-\mathbf{y}) (\psi_n^k(\mathbf{t},\mathbf{y}) - g_n^k(\mathbf{t},\mathbf{y})) d\mathbf{t} d\mathbf{x} d\mathbf{y} \\ &\leq C^k |||\psi_n^k(\mathbf{t},\mathbf{x}) - g_n^k|||^2_{\mathcal{L}_K^k} \\ &\leq C^k \left(|||\psi_n^k(\mathbf{t},\mathbf{x}) - G_k|||^2_{\mathcal{L}_K^k} + |||g_n^k - G_k|||^2_{\mathcal{L}_K^k} \right). \end{aligned}$$

Therefore, by Lemma 3.3,

$$\mathbf{E}\left(\left(\mathfrak{Z}_n^{\leq l}(g) - \mathfrak{Z}_n^{\leq l}(\beta_n;\omega) \right)^2 \right) \to 0 \text{ as } n \to \infty.$$

and so, $\mathfrak{Z}_n^{\leq l}(\beta_n;\omega) \overset{(d)}{\to} \mathcal{Z}_{\sigma(\beta,q)}^{\leq l}(1,*)$. □

Step 3. Combining the conclusions in previous two steps, we show Theorem 3.1 via Lemma 3.1.

Proof of Theorem 3.1. Define

$$Y_l^n = \mathfrak{Z}_n^{\leq l}(\beta_n;\omega) := 1 + \sum_{k=1}^{l} (\beta\sqrt{q})^k \theta_n^k \sum_{(\mathbf{t},\mathbf{x}) \in \mathbb{D}_n^k} \psi_n^k(\mathbf{t},\mathbf{x}) \omega_n^k(\mathbf{t},\mathbf{x}),$$

$$Y_l = \mathcal{Z}_{\sigma(\beta,q)}^{\leq l}(1,*) = 1 + \sum_{k=1}^{l} (\sigma(\beta,q))^k \int_{\Delta_k(1)} \int_{\mathbb{R}^k} \prod_{i=1}^{k} g(t_i - t_{i-1}, x_i - x_{i-1}) \mathcal{W}(dt_i dx_i),$$

and

$$Y^n = \mathfrak{Z}_n(\beta_n;\omega), \qquad Y = \mathcal{Z}_{\sigma(\beta,q)}(1,*).$$

Then

$$\mathbf{E}\left((Y_l - Y)^2 \right) = \sum_{k=l+1}^{\infty} (\sigma(\beta),q)^{2k} \|g_k\|_{\mathcal{L}^k}$$

and

$$\mathbf{E}\left((Y_l^n - Y^n)^2\right) = \sum_{k=l+1}^{\infty} (\sqrt{q}\beta)^{2k} \int_{\Delta_k(1)} \int_{\mathbb{R}^{2k}} \psi_n^k(\mathbf{t},\mathbf{x})\gamma_n^k\,(\mathbf{x}-\mathbf{y})\,\psi_n^k(\mathbf{t},\mathbf{y}) d\mathbf{t} d\mathbf{x} d\mathbf{y}.$$

By the conditions (3.15), we have

$$\int_{\Delta_k(1)} \int_{\mathbb{R}^{2k}} \psi_n^k(\mathbf{t},\mathbf{x})\gamma_n^k\,(\mathbf{x}-\mathbf{y})\,\psi_n^k(\mathbf{t},\mathbf{y}) d\mathbf{t} d\mathbf{x} d\mathbf{y} \le C^k \|\psi_n^k(\mathbf{t},\mathbf{x})\|^2_{\mathcal{L}_K^k}.$$

Therefore, by Lemma 3.2, $Y_l^n \to Y^n$ in probability uniformly in n as $l \to \infty$, and $Y_l \to Y$ in probability as $l \to \infty$. By Lemma 3.4 and Lemma 3.6, for each $l \ge 1$, $Y_l^n \to Y_l$ in distribution as $n \to \infty$. Therefore, by Lemma 3.1, $Y^n \to Y$ in distribution as $n \to \infty$. □

3.2 Proof of Theorem 2.1

In this subsection, we show Theorem 2.1 by estimating the L^2-error between $Z_n(\beta_n;\omega)e^{-n\lambda(\beta_n)}$ and the modified point-to-line partition function $\mathfrak{Z}_n(\beta_n;\omega)$.

Denote by

$$\widetilde{\omega}(i,x) := \frac{e^{\beta_n \omega(i,x) - \lambda(\beta_n)} - 1}{\beta_n},$$

and set

$$\vartheta(i,x) := \widetilde{\omega}(i,x) - \omega(i,x),$$

where $\lambda(\beta) = \log \mathbf{E} e^{\beta\omega(i,x)}$ and $\beta_n = \beta n^{-\frac{1}{2}-\frac{1}{2\alpha}+\frac{r}{\alpha}}$. Then $\mathbf{E}(\widetilde{\omega}(i,x)) = 0$, and we can write

$$\begin{aligned} Z_n(\beta_n;\omega)e^{-n\lambda(\beta_n)} =& \mathbb{E}\left(\prod_{i=1}^n (1+\beta_n\widetilde{\omega}(i,S_i))\right) \\ =& 1 + \sum_{k=1}^{\infty} (\beta\sqrt{q})^k \theta_n^k \sum_{(\mathbf{t},\mathbf{x})\in\mathbb{D}_n^k} \psi_n^k(\mathbf{t},\mathbf{x})\widetilde{\omega}_n^k(\mathbf{t},\mathbf{x}). \end{aligned}$$

The following lemma gives an error estimate between the two environments.

Lemma 3.7. Assume that (A.1), (A.2) and (A.3) hold. Then we can choose a positive integer $M_0 \ge 1$ such that for any $M \ge M_0$,

$$\begin{cases} M\left(-\frac{1}{2}-\frac{1}{2\alpha}+\frac{r}{\alpha}\right) + \frac{2r-1}{\alpha} < 0, \\ |\mathbf{E}(\widetilde{\omega}(i,x)\widetilde{\omega}(i,y))| \le C_M \gamma(x-y) + O(\beta_n^M), \\ |\mathbf{E}(\vartheta(i,x)\vartheta(i,y))| \le C_M \gamma(x-y) + O(\beta_n^M), \end{cases} \tag{3.41}$$

where $O(\beta_n)$ is independent of (i,x,y), and C_M is a positive constant independent of n.

Proof. We only prove the $r \in (1/2, 1)$ case. The $r = 1$ case is similar. Since $\frac{2r-1}{\alpha} < 1$, we can choose $M_0 \ge 1$ such that for all $M \ge M_0$,

$$M\left(-\frac{1}{2}-\frac{1}{2\alpha}+\frac{r}{\alpha}\right) + \frac{2r-1}{\alpha} < 0.$$

For any integer $M \ge M_0$, by a Taylor expansion, we have

$$\begin{aligned} \mathbf{E}\left(e^{\beta_n(\omega(i,x)+\omega(i,y))}\right) =& 1 + \sum_{k=1}^{M+1} \frac{\beta_n^k}{k!} \mathbf{E}((\omega(i,x)+\omega(i,y))^k) + O(\beta_n^{M+2}) \\ =& 1 + \sum_{k=1}^{M+1} \sum_{l=1}^{k} \frac{\beta_n^k}{k!} C_k^l \mathbf{E}\left(\omega^l(i,x)\omega^{k-l}(i,y)\right) + O(\beta_n^{M+2}), \end{aligned}$$

and

$$\begin{aligned}
&\mathbf{E}\left(e^{\beta_n\omega(i,x)}\right)\mathbf{E}\left(e^{\beta_n\omega(i,y)}\right)\\
=&\left(1+\sum_{k=1}^{M+1}\frac{\beta_n^k}{k!}\mathbf{E}(\omega^k(i,x))+O(\beta_n^{M+2})\right)\left(1+\sum_{j=1}^{M+1}\frac{\beta_n^j}{j!}\mathbf{E}(\omega^j(i,y)+O(\beta_n^{M+2})\right)\\
=&1+\sum_{k=1}^{M+1}\sum_{l=1}^{k}\frac{\beta_n^k}{k!}C_k^l\mathbf{E}\left(\omega^l(i,x)\right)\mathbf{E}\left(\omega^{k-l}(i,y)\right)+O(\beta_n^{M+2}).
\end{aligned}$$

By Lemma D.1, for any $k,j\geq 1$, there is a positive constant $C_{k,j}$ such that

$$\left|\mathbf{E}\left(\omega^l(i,x)\omega^{k-l}(i,y)\right)-\mathbf{E}\left(\omega^l(i,x)\right)\mathbf{E}\left(\omega^{k-l}(i,y)\right)\right|\leq C_{k,j}\gamma(x-y).$$

Therefore, there is a positive constant $\widetilde{C}_M$ such that

$$\begin{aligned}
|\mathbf{E}(\widetilde{\omega}(i,x)\widetilde{\omega}(i,y))|=&\frac{e^{-2\lambda(\beta_n)}}{\beta_n^2}\left|\mathbf{E}\left(e^{\beta_n(\omega(i,x)+\omega(i,y))}\right)-\mathbf{E}\left(e^{\beta_n\omega(i,x)}\right)\mathbf{E}\left(e^{\beta_n\omega(i,y)}\right)\right|\\
\leq&\frac{e^{-2\lambda(\beta_n)}}{\beta_n^2}\left(\widetilde{C}_M\beta_n^2\gamma(x-y)+O(\beta_n^{M+2})\right)\\
\leq&\widehat{C}_M\gamma(x-y)+O(\beta_n^M).
\end{aligned}$$

Finally, we show the third estimate in (3.41). It is obvious that

$$\begin{aligned}
&\mathbf{E}(\vartheta(i,x)\vartheta(i,y))\\
=&\mathbf{E}\left((\widetilde{\omega}(i,x)-\omega(i,x))(\widetilde{\omega}(i,y)-\omega(i,y))\right)\\
=&\mathbf{E}(\widetilde{\omega}(i,x)\widetilde{\omega}(i,y))-\mathbf{E}(\widetilde{\omega}(i,x)\omega(i,y))-\mathbf{E}(\omega(i,x)\widetilde{\omega}(i,y))+\gamma(x-y).
\end{aligned}$$

Then, we can write

$$\mathbf{E}(\widetilde{\omega}(i,x)\omega(i,y))=\mathbf{E}\left(\frac{e^{\beta_n\omega(i,x)-\lambda(\beta_n)}-1}{\beta_n}\cdot\omega(i,y)\right)=\frac{e^{-\lambda(\beta_n)}}{\beta_n}\mathbf{E}\left(\omega(i,y)e^{\beta_n\omega(i,x)}\right).$$

Using again a Taylor expansion, we have

$$\begin{aligned}
|\mathbf{E}(\widetilde{\omega}(i,x)\omega(i,y))|=&\frac{e^{-\lambda(\beta_n)}}{\beta_n}\left|\left(\sum_{k=1}^{M+1}\frac{\beta_n^k}{k!}\mathbf{E}(\omega(i,y)\omega^k(i,x))+O(\beta_n^{M+2})\right)\right|\\
\leq&C_M\gamma(x-y)+O(\beta_n^{M+1}).
\end{aligned}$$

Therefore, the third estimate in (3.41) holds. □

Proof of Theorem 2.1. Denote by

$$\left(Z_n(\beta_n;\omega)e^{-n\lambda(\beta_n)}\right)^{>l}:=\sum_{k=l+1}^{\infty}(\beta\sqrt{q})^k\theta_n^k\sum_{(\mathbf{t},\mathbf{x})\in\mathbb{D}_n^k}\psi_n^k(\mathbf{t},\mathbf{x})\widetilde{\omega}_n^k(\mathbf{t},\mathbf{x}). \tag{3.42}$$

$$\left(Z_n(\beta_n;\omega)e^{-n\lambda(\beta_n)}\right)^{\leq l}:=1+\sum_{k=1}^{l}(\beta\sqrt{q})^k\theta_n^k\sum_{(\mathbf{t},\mathbf{x})\in\mathbb{D}_n^k}\psi_n^k(\mathbf{t},\mathbf{x})\widetilde{\omega}_n^k(\mathbf{t},\mathbf{x}). \tag{3.43}$$

Let γ_n^k and $\widehat{p}_n^k$ be defined by (3.7) and (3.14), i.e.,

$$\widehat{p}_n^k(\mathbf{t},\mathbf{x}):=\left(q^{-1}n^{\frac{1}{\alpha}}\right)^k p_n^k(\mathbf{t},\mathbf{x}),\ \gamma_n^k(\mathbf{x}-\mathbf{y})=n^{k(2r-1)/\alpha}\mathbf{E}\left(\omega_n^k(\mathbf{t},\mathbf{x})\omega_n^k(\mathbf{t},\mathbf{y})\right).$$

Recall $\beta_n = \beta n^{-\frac{1}{2}-\frac{1}{2\alpha}+\frac{r}{\alpha}}$. By Lemma 3.7, there exist positive integer $M \geq 1$ and positive constant C_M such that (3.41) holds. In particular, $\beta_n^M n^{\frac{2r-1}{\alpha}} \leq 1$. Then by (3.41), we have

$$\begin{aligned}
&\mathbf{E}\left(\left(Z_n(\beta_n;\omega)e^{-n\lambda(\beta_n)}\right)^{>l}\right)^2 \\
\leq& \sum_{k=l+1}^{\infty} A^k \int_{\Delta_k(1)} \int_{\mathbb{R}^{2k}} \widehat{p}_n^k(\mathbf{t},\mathbf{x})\widehat{p}_n^k(\mathbf{t},\mathbf{y}) \prod_{i=1}^{k} \Big(\gamma_n\big(x_i-y_i\big) + O(n^{\frac{2r-1}{\alpha}}\beta_n^M)\Big) dt_i dx_i dy_i \\
=& \sum_{k=l+1}^{\infty} A^k \int_{\Delta_k(1)} \prod_{i=1}^{k} \Big(\int_{\mathbb{R}^2} \widehat{p}_n^1(t_i-t_{i-1},x_i)\gamma_n\big(x_i-y_i\big)\widehat{p}_n^1(t_i-t_{i-1},y_i) dx_i dy_i + O(1)\Big) dt_i,
\end{aligned}$$

where A is a positive constant. Note that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \widehat{p}_n^1(t_i-t_{i-1},x_i)\gamma_n\big(x_i-y_i\big)\widehat{p}_n^1(t_i-t_{i-1},y_i) dx_i dy_i \\
\leq& C \int_{\mathbb{R}^2} \widehat{p}_n^1(t_i-t_{i-1},x_i) K\big(x_i-y_i\big)\widehat{p}_n^1(t_i-t_{i-1},y_i) dx_i dy_i \\
\leq& C_1 \int_{\mathbb{R}} |\widehat{p}_n^1(t_i-t_{i-1},x_i)|^{2r} dx_i \\
\leq& C_2 (t_i-t_{i-1})^{\frac{1-2r}{\alpha}},
\end{aligned}$$

where C, C_1, C_2 are positive constants. Noting that $(t_i-t_{i-1})^{\frac{1-2r}{\alpha}} \geq 1$, we have

$$\begin{aligned}
&\mathbf{E}\left(\left(Z_n(\beta_n;\omega)e^{-n\lambda(\beta_n)}\right)^{>l}\right)^2 \qquad (3.44)\\
\leq& \sum_{k=l+1}^{\infty} A^k C_2^k \int_{\Delta_k(1)} \prod_{i=1}^{k} \Big((t_i-t_{i-1})^{\frac{1-2r}{\alpha}} + O(1)\Big) dt_i \\
\leq& \sum_{k=l+1}^{\infty} C_3^k \int_{\Delta_k(1)} \prod_{i=1}^{k} (t_i-t_{i-1})^{\frac{1-2r}{\alpha}} dt_i \\
\leq& \sum_{k=l+1}^{\infty} \frac{C_3^k \Gamma^k\left(1+\frac{1-2r}{\alpha}\right)}{\Gamma\left(k\left(1+\frac{1-2r}{\alpha}\right)+1\right)} \longrightarrow 0 \qquad (3.45)
\end{aligned}$$

uniformly in n as $l \to \infty$, where C_2, C_3 are positive constants.

Define

$$\widetilde{\mathcal{S}}_k^n = \theta_n^k \sum_{(\mathbf{t},\mathbf{x})\in\mathbb{D}_n^k} \psi_n^k(\mathbf{t},\mathbf{x})\widetilde{\omega}_n^k(\mathbf{t},\mathbf{x}),$$

and

$$\mathcal{S}_k^n = \theta_n^k \sum_{(\mathbf{t},\mathbf{x})\in\mathbb{D}_n^k} \psi_n^k(\mathbf{t},\mathbf{x})\omega_n^k(\mathbf{t},\mathbf{x}).$$

Next, we prove that for any $k \geq 1$,

$$\lim_{n\to\infty} \mathbf{E}\left(\left(\widetilde{\mathcal{S}}_k^n - \mathcal{S}_k^n\right)^2\right) = 0. \qquad (3.46)$$

For $k \geq 2$, $1 \leq l \leq k$, set

$$\mathcal{S}_{k,l}^n = \theta_n^k \sum_{(\mathbf{t},\mathbf{x})\in\mathbb{D}_n^k} \psi_n^k(\mathbf{t},\mathbf{x}) \prod_{i=1}^{l-1} \widetilde{\omega}_n(t_i,x_i)\,\vartheta_n(t_l,x_l) \prod_{j=l+1}^{k} \omega_n(t_j,x_j).$$

Then

$$\left(\mathbf{E}\left(\left(\widetilde{\mathcal{S}}_k^n-\mathcal{S}_k^n\right)^2\right)\right)^{\frac{1}{2}}\leq\sum_{l=1}^k\left(\mathbf{E}\left(\left(\mathcal{S}_{k,l}^n\right)^2\right)\right)^{\frac{1}{2}}.$$

For each $1\leq l\leq k$, by (3.41) and the assumption (A.3), we have

$$\begin{aligned}
&\mathbf{E}\left(\left(\mathcal{S}_{k,l}^n\right)^2\right)\\
\leq& Cn^{\frac{k(2r-1)}{\alpha}}\int_{\Delta_k(1)}\int_{\mathbb{R}^{2k}}\widehat{p}_n^k(\mathbf{t},\mathbf{x})\widehat{p}_n^k(\mathbf{t},\mathbf{y})\prod_{i=1}^{l-1}\left(n^{-\frac{2r-1}{\alpha}}\gamma_n(x_i-y_i)+O(\beta_n^M)\right)\\
&\times\left(n^{-\frac{2r-1}{\alpha}}\gamma_n(x_l-y_l)+O(\beta_n^M)\right)\prod_{j=l+1}^k n^{-\frac{2r-1}{\alpha}}\gamma_n(x_j-y_j)d\mathbf{t}d\mathbf{x}d\mathbf{y}\\
\leq& C_1\int_{\Delta_k(1)}\int_{\mathbb{R}^{2k}}\widehat{p}_n^k(\mathbf{t},\mathbf{x})\widehat{p}_n^k(\mathbf{t},\mathbf{y})\prod_{i=1}^{l}\left(K(x_i-y_i)+O(n^{\frac{2r-1}{\alpha}}\beta_n^M)\right)\prod_{j=l+1}^k K(x_j-y_j)d\mathbf{t}d\mathbf{x}d\mathbf{y}\\
\leq& C_3\int_{\Delta_k(1)}\prod_{i=1}^k(t_i-t_{i-1})^{\frac{1-2r}{\alpha}}dt_i\longrightarrow 0 \text{ as } n\to\infty.
\end{aligned}$$

Therefore, (3.46) holds, and so

$$\lim_{n\to\infty}\mathbf{E}\left(\left(\left(Z_n(\beta_n;\omega)e^{-n\lambda(\beta_n)}\right)^{\leq l}\right)-\mathfrak{Z}_n(\beta_n;\omega)^{\leq l}\right)^2=0. \tag{3.47}$$

Combining (3.44) and (3.47), we get

$$\lim_{n\to\infty}\mathbf{E}\left(\left(Z_n(\beta_n;\omega)e^{-n\lambda(\beta_n)}\right)-\mathfrak{Z}_n(\beta_n;\omega)\right)^2=0.$$

Thus, the conclusion of Theorem 2.1 holds. □

4 The scaling limit of the point-to-point partition function

In this section, we give the proof of Theorem 2.2. The convergence of the finite dimensional distributions in Theorem 2.2 is similar to the proof of Theorem 2.1. An important estimate in the tightness is the following gradient estimate for symmetric random walk in the domain of normal attraction of α-stable law.

Lemma 4.1. Let $\{S_n,n\geq 0\}$ be a symmetric random walk starting from the origin on $\mathbb{Z}$ and in the domain of normal attraction of a stable law of index $\alpha\in(0,2]$. Assume that the characteristic function $\varphi(u)$ of S_1 satisfies (2.1). Then there exists a constant such that for any $n\geq 1$, $m\in\mathbb{Z}$, $y\in\mathbb{Z}$,

$$\sup_{x\in\mathbb{Z}}|p(n,x+y)-p(n,x)|\leq\frac{C|y|}{n^{2/\alpha}}, \tag{4.1}$$

$$\sup_{x\in\mathbb{Z}}|p(n+m,x)-p(n,x)|\leq\frac{Cm}{n^{1+1/\alpha}}. \tag{4.2}$$

One can see [37] for a proof of the $\alpha=2$ case. A proof for general case is given in Appendix B. The proof also gives the local limit theorem for symmetric random walk in the domain of normal attraction of α-stable law.

We consider the modified point-to-point partition function defined by

$$\mathfrak{Z}_{n,x}(\beta;\omega)=\mathbb{E}\left(\prod_{i=1}^n\left(1+\beta\omega(i,S_i)\right)I_{\{S_n=x\}}\right).$$

For each $n \geq 1$, we first define a stochastic process $z_n(t,x)$ which is right continuous in time and space,

$$z_n(t,x) = \frac{1}{q} n^{1/\alpha} \mathfrak{Z}_{nt, n^{1/\alpha}x}(\beta_n; \omega) \text{ for any } (t,x) \in \mathbb{D}_n, \tag{4.3}$$

and takes a constant value in interior of each cell $\mathcal{C}_n(t,x)$, $(t,x) \in \mathbb{D}_n$.

For each $n \geq 1$, we also define a linear interpolation process $Z_n(t,x)$ which is a continuous process in time and space, and

$$Z_n(t,x) = z_n(t,x) \text{ for any } (t,x) \in \mathbb{D}_n. \tag{4.4}$$

Set $\mathcal{R}_n = \{\mathcal{C}_n(t,x);\ (t,x) \in \mathbb{D}_n\}$. For each $n \geq 1$, we define a linear interpolation process that extends Z_n to a continuous process. The definition is as follows (see [1]): if (t,x) is a corner point of the left-hand side of a rectangle in $\mathcal{R}_n$, define $Z_n(t,x) = z_n(t,x)$; then for space–time points (t,x) on the left edges of rectangles in $\mathcal{R}_n$, define $Z_n(t,x)$ by linear interpolation of the values on the corners that the edge connects, and finally for (t,x) on the interior points of rectangles, define $Z_n(t,x)$ by linear interpolation of the values at the four boundary corners.

Theorem 4.1. Let $\alpha \in (1,2]$, (A.1) and (A.2) hold. Then

$$(z_n(t,x))_{t\in(0,1],x\in\mathbb{R}} \xrightarrow{(d)} (\mathcal{Z}_{\sigma(\beta,q)}(t,x))_{t\in(0,1],x\in\mathbb{R}}, \tag{4.5}$$

with respect to the Skorohod topology on $D((0,1]\times\mathbb{R})$, and

$$(Z_n(t,x))_{t\in(0,1],x\in\mathbb{R}} \xrightarrow{(d)} (\mathcal{Z}_{\sigma(\beta,q)}(t,x))_{t\in(0,1],x\in\mathbb{R}}, \tag{4.6}$$

with respect to the locally uniform topology on $C((0,1]\times\mathbb{R})$, where $\mathcal{Z}_{\sigma(\beta,q)}(t,x)$ is the mild solution of (2.7) with $\sigma = \sigma(\beta,q)$, and initial data $\mathcal{Z}_0(x) = \delta_0(x)$. Furthermore,

$$\lim_{n\to\infty} \mathbf{E}\left((z_n(t,x))^2 \right) = \lim_{n\to\infty} \mathbf{E}\left((Z_n(t,x))^2 \right) = \mathbf{E}\left(\left(\mathcal{Z}_{\sigma(\beta,q)}(t,x)\right)^2 \right). \tag{4.7}$$

The convergence of the finite dimensional distributions is the same as the proof of Theorem 3.1. The proof of the tightness is based on Minkowski's integral inequality, and the gradient estimates for symmetric random walks.

4.1 Convergence of finite dimensional distributions

For each $t \in (0,1]$, set

$$\Delta\mathbb{D}_n^k(t) = \left\{ (\mathbf{t},\mathbf{x}) = ((t_1,x_1),\cdots,(t_k,x_k)) \in \mathbb{D}_n^k; 0 \leq t_1 < \cdots < t_k \leq t \right\}.$$

For each $(t,x) \in \mathbb{D}_n$, let $p_{n,t,x}^k(\mathbf{t},\mathbf{x})$ be the joint probability of $(S_{nt_1} = n^{1/\alpha}x_1, \cdots, S_{nt_k} = n^{1/\alpha}x_k)$ under the condition $S_{nt} = n^{1/\alpha}x$, i.e.,

$$\begin{aligned} &p_{n,t,x}^k(\mathbf{t},\mathbf{x}) \\ &= \mathbb{P}(S_{nt_1} = n^{1/\alpha}x_1, \cdots, S_{nt_k} = n^{1/\alpha}x_k | S_{nt} = n^{1/\alpha}x) \\ &= \frac{p(n(t-t_k), n^{1/\alpha}(x-x_k))}{p(nt, n^{1/\alpha}x)} \prod_{j=1}^k p(n(t_j - t_{j-1}), n^{1/\alpha}(x_j - x_{j-1})), \quad (\mathbf{t},\mathbf{x}) \in \Delta\mathbb{D}_n^k, \end{aligned}$$

where $(t_0,x_0) = (0,0)$. We extend $p_{n,t,x}^k(\mathbf{t},\mathbf{x})$ from $\Delta\mathbb{D}_n^k(t)$ to $\Delta_k(t)\times\mathbb{R}^k$ in the same way as $p_n^k(\mathbf{t},\mathbf{x})$. Set

$$\psi_{n,t,x}^k(\mathbf{t},\mathbf{x}) := \left(q^{-1}n^{1/\alpha}\right)^k \mathrm{Sym}\{p_{n,t,x}^k(\mathbf{t},\mathbf{x}) I_{\Delta_k(t)\times\mathbb{R}^k}(\mathbf{t},\mathbf{x})\}, \quad (\mathbf{t},\mathbf{x}) \in [0,t]^k \times \mathbb{R}^k.$$

Then for any $(t,x)\in\mathbb{D}_n$,

$$\mathfrak{Z}_{nt,n^{1/\alpha}x}(\beta_n;\omega)=p(nt,n^{1/\alpha}x)\left(1+\sum_{k=1}^{n}\beta_n^k\sum_{(\mathbf{t},\mathbf{x})\in\Delta\mathbb{D}_n^k(t)}\psi_{n,t,x}^k(\mathbf{t},\mathbf{x})\omega_n^k(\mathbf{t},\mathbf{x})\right),$$

where $\beta_n=\beta n^{-\frac{1}{2}-\frac{1}{2\alpha}+\frac{r}{\alpha}}$, $\omega_n^k(\mathbf{t},\mathbf{x})=\prod_{i=1}^k\omega\left(nt_i,n^{\frac{1}{\alpha}}x_i\right)$,

Therefore, using the same approach as proof of Theorem 3.1, we can obtain that for any finite points $(t_1,x_1),\cdots,(t_m,x_m)\in(0,\infty)\times\mathbb{R}$, as $n\to\infty$,

$$(z_n(t_1,x_1),\cdots,z_n(t_m,x_m))\xrightarrow{(d)}\left(\mathcal{Z}_{\sigma(\beta,q)}(t_1,x_1),\cdots,\mathcal{Z}_{\sigma(\beta,q)}(t_m,x_m)\right),$$

and so

$$(Z_n(t_1,x_1),\cdots,Z_n(t_m,x_m))\xrightarrow{(d)}\left(\mathcal{Z}_{\sigma(\beta,q)}(t_1,x_1),\cdots,\mathcal{Z}_{\sigma(\beta,q)}(t_m,x_m)\right).$$

□

4.2 Tightness

By the symmetry and Markov property of $\{S_n,n\ge 0\}$, for any $n\ge 1$, $k\le n$, $x\in q\mathbb{Z}+n\ell$,

$$\begin{aligned}
&\mathbb{E}\left(\prod_{i=1}^{k}(1+\beta\omega(i,S_i))\,I_{\{S_n=x\}}\right)-\mathbb{E}\left(\prod_{i=1}^{k-1}(1+\beta\omega(i,S_i))\,I_{\{S_n=x\}}\right)\\
&=\sum_{y\in\mathbb{Z}}\mathbb{E}\left(\prod_{i=1}^{k-1}(1+\beta\omega(i,S_i))\,I_{\{S_{k-1}=y\}}\right)\mathbb{E}\left(\beta\omega(k,y+S_1)p(n-k,x-(y+S_1))\right)\\
&=\sum_{y\in q\mathbb{Z}+(k-1)\ell}\mathfrak{Z}_{k-1,y}(\beta,\omega)\sum_{z\in q\mathbb{Z}+\ell}p(1,z)\beta\omega(k,y+z)p(n-k,x-y-z)\\
&=\sum_{y\in q\mathbb{Z}+(k-1)\ell}\mathfrak{Z}_{k-1,y}(\beta,\omega)\sum_{z\in q\mathbb{Z}+k\ell}p(1,z-y)\beta\omega(k,z)p(n-k,x-z)\\
&=\sum_{z\in q\mathbb{Z}+k\ell}\overline{\mathfrak{Z}}_{k-1,z}(\beta,\omega)\beta\omega(k,z)p(n-k,z-x),
\end{aligned}\tag{4.8}$$

where

$$\overline{\mathfrak{Z}}_{k-1,z}(\beta;\omega)=\sum_{y\in q\mathbb{Z}+(k-1)\ell}p(1,z-y)\mathfrak{Z}_{k-1,y}(\beta;\omega).$$

Suming (4.8) from 1 to n, we have

$$\mathfrak{Z}_{n,x}(\beta;\omega)-p(n,x)=\beta\sum_{i=1}^{n}\sum_{y\in q\mathbb{Z}+i\ell}p(n-i,x-y)\overline{\mathfrak{Z}}_{i-1,y}(\beta;\omega)\omega(i,y).\tag{4.9}$$

We define the rescaled transition probability $\overline{p}_n(t,x)$ which is right continuous in time and space,

$$\overline{p}_n(t,x)=\frac{n^{1/\alpha}}{q}p(nt,n^{1/\alpha}x)\ \text{ for any }(t,x)\in\mathbb{D}_n,\tag{4.10}$$

and takes a constant value in interior of each cell $\mathcal{C}_n(t,x)$, $(t,x)\in\mathbb{D}_n$. Similarly, let $\overline{z}_n(t,x)$ be a right continuous piecewise constant extension of $\overline{\mathfrak{Z}}$:

$$\overline{z}_n(t,x):=\frac{n^{1/\alpha}}{q}\overline{\mathfrak{Z}}_{nt,n^{1/\alpha}x}(\beta_n;\omega),\quad(t,x)\in\mathbb{D}_n.$$

We also extend $\omega_n(t,x) := \omega(nt, n^{\frac{1}{\alpha}}x)$ to a right continuous piecewise constant function. Then for any $(t,x) \in \mathbb{D}_n$,

$$\begin{aligned}\overline{z}_n(t,x) &= \sum_{y\in n^{-1/\alpha}\mathbb{Z}} \overline{p}_n\left(1/n, x-y\right) z_n((t-1/n)^+, y) q n^{-1/\alpha} \\ &= \int \overline{p}_n\left(1/n, x-y\right) z_n((t-1/n)^+, y) dy,\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}z_n(t,x) =& \overline{p}_n(t,x) + \beta q n^{-1-\frac{1}{\alpha}} \sum_{s\in[0,t]\cap n^{-1}\mathbb{Z}} \sum_{y\in n^{-\frac{1}{\alpha}}\mathbb{Z}} \overline{p}_n(t-s,x-y)\overline{z}_n(s,y)\overline{\omega}_n(s,y) \\ =& \overline{p}_n(t,x) + \beta \int_0^t \int_{\mathbb{R}} \overline{p}_n(t-s,x-y)\overline{z}_n(s,y)\overline{\omega}_n(s,y) ds dy,\end{aligned} \tag{4.12}$$

where $\overline{\omega}_n(s,y) := n^{\frac{1}{2}-\frac{1}{2\alpha}+\frac{r}{\alpha}}\omega_n(s,y)$.

Next, we show the tightness of $z_n(t,x)$. Let us first prove the following a priori estimate.

Lemma 4.2. Let $\alpha \in (1,2]$, (A.1) and (A.2) hold. Then there exists a positive constant C_m such that for any $n \geq 1$, $t \in (0,1]$, $x \in \mathbb{R}$,

$$\left\|z_n^2(t,x)\right\|_m \leq C_m \overline{p}_n(t,x)/t^{1/\alpha}, \tag{4.13}$$

and

$$\left\|\overline{z}_n^2(t,x)\right\|_m \leq C_m \overline{p}_n(t,x)/t^{1/\alpha}. \tag{4.14}$$

where $\|Z\|_m = \left(\mathbf{E}\left(|Z|^m\right)\right)^{1/m}$

Proof. By the condition (2.3) and the definition of $z_n(t,x)$,

$$\mathbf{E}\left(z_n^{2m}(t,x)\right) < \infty \;\text{ and }\; \mathbf{E}\left(\overline{z}_n^{2m}(t,x)\right) < \infty \text{ for any } m\geq 1, n\geq 1, t\in(0,1], x\in\mathbb{R}.$$

We only need to consider $(t,x) \in \mathbb{D}_n$.

Let us first consider the $r \in (1/2, 1)$ case. We write

$$z_n(t,x) = \overline{p}_n(t,x) + \beta q n^{-1-\frac{1}{\alpha}} \sum_{s\in[0,t]\cap n^{-1}\mathbb{Z}} X_s,$$

where

$$X_s := \sum_{y\in n^{-\frac{1}{\alpha}}\mathbb{Z}} \overline{p}_n(t-s,x-y)\overline{z}_n(s,y)\overline{\omega}_n(s,y), \quad s \in [0,t]\cap n^{-1}\mathbb{Z}.$$

Note that the $\overline{z}_n(s,\cdot)$ terms are independent of $\mathcal{G}_s := \sigma(\omega_n(s,y), y \in n^{-\frac{1}{\alpha}}\mathbb{Z})$. It is known that $\{X_s, s\in[0,t]\cap n^{-1}\mathbb{Z}\}$ is a sequence of martingale differences. Then by discrete Burkholder's inequality (cf. [26], Theorem 2.10), and Minkowski's integral inequality, we have that for any $m \geq 1$,

$$\begin{aligned}&\left(\mathbf{E}\left(n^{-1-\frac{1}{\alpha}}\left|\sum_{s\in[0,t]\cap n^{-1}\mathbb{Z}} X_s\right|^{2m}\right)\right)^{1/m} \\ \leq& 72 m n^{-2-\frac{2}{\alpha}}\left(\mathbf{E}\left(\left|\sum_{s\in[0,t]\cap n^{-1}\mathbb{Z}} X_s^2\right|^m\right)\right)^{1/m} \\ \leq& 72 m n^{-2-\frac{2}{\alpha}} \sum_{s\in[0,t]\cap n^{-1}\mathbb{Z}} \|X_s^2\|_m.\end{aligned}$$

For $s \in [0,t] \cap n^{-1}\mathbb{Z}$, we have

$$\begin{aligned}
&\mathbf{E}(|X_s|^{2m}) \\
&= \sum_{y_1 \in n^{-\frac{1}{\alpha}}\mathbb{Z}} \cdots \sum_{y_{2m} \in n^{-\frac{1}{\alpha}}\mathbb{Z}} \Big(\prod_{j=1}^{2m} \overline{p}_n(t-s, x-y_j)\Big) \mathbf{E}\Big(\prod_{j=1}^{2m} \overline{z}_n(s,y_j)\Big) \mathbf{E}\Big(\prod_{j=1}^{2m} \overline{\omega}_n(s,y_j)\Big) \\
&\le \sum_{y_1 \in n^{-\frac{1}{\alpha}}\mathbb{Z}} \cdots \sum_{y_{2m} \in n^{-\frac{1}{\alpha}}\mathbb{Z}} \Big(\prod_{j=1}^{2m} \overline{p}_n(t-s, x-y_j)\Big) \Big(\prod_{j=1}^{2m} \|\overline{z}_n(s,y_j)\|_{2m}\Big) \mathbf{E}\Big(\prod_{j=1}^{2m} \overline{\omega}_n(s,y_j)\Big).
\end{aligned}$$

Next, we first compute $\mathbf{E}\left(\prod_{j=1}^{2m} \omega(i,x_j)\right)$. For each $\boldsymbol{u} := (u_1, u_2, \cdots, u_{2m}) \in \mathbb{Z}^{2m}$, there exist integer numbers $1 \le k \le 2m$ and $l_1, \cdots, l_k \ge 1$, and a k-division $A_1, \cdots, A_k$ of $\{1, 2, \cdots, 2m\}$ such that $|A_h| = l_h$, $1 \le h \le k$ and the mapping $\{1, 2, \cdots, 2m\} \ni j \to u_j \in \mathbb{Z}$ is different constant on each A_h, $1 \le h \le k$. Let $\mathcal{U}_k$ denote the set of all such k-divisions. For each such division $(A_1, \cdots, A_k)$, we write $A_h = \{v_{h,1}, \cdots, v_{h,l_h}\}$ and set

$$\mathbb{Z}^{2m}_{A_1,\cdots,A_k} = \{\boldsymbol{u} \in \mathbb{Z}^{2m};\ j \to u_j \text{ is different constant on each } A_h, 1 \le h \le k\}.$$

Then we can write

$$\begin{aligned}
\mathbf{E}\left(\prod_{j=1}^{2m} \omega(i,x_j)\right) &= \sum_{\boldsymbol{u} \in \mathbb{Z}^{2m}} \prod_{j=1}^{2m} a_{u_j - x_j} \mathbf{E}\left(\prod_{j=1}^{2m} \xi(1,u_j)\right) \\
&= \sum_{k=1}^{m} \sum_{\substack{(A_1,\cdots,A_k) \in \mathcal{U}_k, \\ l_1 \ge 2, \cdots, l_k \ge 2}} \sum_{\boldsymbol{u} \in \mathbb{Z}^{2m}_{A_1,\cdots,A_k}} \prod_{j=1}^{2m} a_{u_j - x_j} \mathbf{E}\left(\prod_{j=1}^{2m} \xi(1,u_j)\right),
\end{aligned}$$

where the last equality is due to $\mathbf{E}\left(\prod_{j=1}^{2m} \xi(1,u_j)\right) = 0$ if $l_h = 1$ for some $1 \le h \le k$.

For each $1 \le k \le m$, any $(A_1, \cdots, A_k) \in \mathcal{U}_k$ with $l_1 \ge 2, \cdots, l_k \ge 2$, we have

$$\begin{aligned}
&\left|\sum_{\boldsymbol{u} \in \mathbb{Z}^{2m}_{A_1,\cdots,A_k}} \prod_{j=1}^{2m} a_{u_j - x_j} \mathbf{E}\left(\prod_{j=1}^{2m} \xi(1,u_j)\right)\right| \\
&\le \sum_{u_1 \in \mathbb{Z}} \cdots \sum_{u_k \in \mathbb{Z}} \prod_{h=1}^{k} \left(\prod_{j=1}^{l_h} a_{u_h - x_{v_{h,j}}}\right) \left|\prod_{h=1}^{k} \mathbf{E}\left(\xi^{l_h}(1,1)\right)\right| \\
&\le C_m \sum_{u_1 \in \mathbb{Z}} \cdots \sum_{u_k \in \mathbb{Z}} \prod_{h=1}^{k} \left(\prod_{j=1}^{l_h} a_{u_h - x_{v_{h,j}}}\right),
\end{aligned}$$

where $C_m = \sup_{\substack{2 \le l_s \le 2m, 1 \le h \le k \le m, \\ l_1 + \cdots + l_k = 2m}} \left|\prod_{h=1}^{k} \mathbf{E}\left(\xi^{l_h}(1,1)\right)\right|$. Therefore

$$\mathbf{E}\left(\prod_{j=1}^{2m} \omega(i,x_j)\right) \le C_m \sum_{k=1}^{m} \sum_{\substack{(A_1,\cdots,A_k) \in \mathcal{U}_k, \\ l_1 \ge 2, \cdots, l_k \ge 2}} \sum_{u_1 \in \mathbb{Z}} \cdots \sum_{u_k \in \mathbb{Z}} \prod_{h=1}^{k} \left(\prod_{j=1}^{l_h} a_{u_h - x_{v_{h,j}}}\right).$$

Now, let us return the estimate of $\mathbf{E}(|X_s|^{2m})$. Then by the above inequality, we have that

$$\begin{aligned}
&\mathbf{E}(|X_s|^{2m})\\
\le& C_m \sum_{k=1}^{m} \sum_{\substack{(A_1,\cdots,A_k)\in\mathcal{U}_k,\\ l_1\ge 2,\cdots,l_k\ge 2}} \sum_{y_1\in n^{-\frac{1}{\alpha}}\mathbb{Z}} \cdots \sum_{y_{2m}\in n^{-\frac{1}{\alpha}}\mathbb{Z}} \sum_{u_1\in\mathbb{Z}} \cdots \sum_{u_k\in\mathbb{Z}}\\
&\times \prod_{h=1}^{k} \left(\prod_{j=1}^{l_h} n^{\frac{1}{2}-\frac{1}{2\alpha}+\frac{r}{\alpha}} a_{u_h - n^{\frac{1}{\alpha}} y_{v_{h,j}}} \|\overline{z}_n(s,y_{v_{h,j}})\|_{2m} \overline{p}_n(t-s, x-y_{v_{h,j}}) \right)\\
\le& C_m \sum_{k=1}^{m} \sum_{\substack{(A_1,\cdots,A_k)\in\mathcal{U}_k,\\ l_1\ge 2,\cdots,l_k\ge 2}} \sum_{u_1\in\mathbb{Z}} \cdots \sum_{u_k\in\mathbb{Z}} \prod_{h=1}^{k} \Bigg(\sum_{y_{v_{h,1}}\in n^{-\frac{1}{\alpha}}\mathbb{Z}} \cdots \sum_{y_{v_{h,l_h}}\in n^{-\frac{1}{\alpha}}\mathbb{Z}}\\
&\times \prod_{j=1}^{l_h} n^{\frac{1}{2}-\frac{1}{2\alpha}+\frac{r}{\alpha}} a_{u_h - n^{\frac{1}{\alpha}} y_{v_{h,j}}} \|\overline{z}_n(s,y_{v_{h,j}})\|_{2m} \overline{p}_n(t-s, x-y_{v_{h,j}}) \Bigg)\\
=& C_m \sum_{k=1}^{m} \sum_{\substack{(A_1,\cdots,A_k)\in\mathcal{U}_k,\\ l_1\ge 2,\cdots,l_k\ge 2}} \sum_{u_1\in\mathbb{Z}} \cdots \sum_{u_k\in\mathbb{Z}} \prod_{h=1}^{k}\\
&\times \left(\sum_{y_h\in n^{-\frac{1}{\alpha}}\mathbb{Z}} n^{\frac{1}{2}-\frac{1}{2\alpha}+\frac{r}{\alpha}} a_{u_h - n^{\frac{1}{\alpha}} y_h} \|\overline{z}_n(s,y_h)\|_{2m} \overline{p}_n(t-s, x-y_h) \right)^{l_h}.
\end{aligned}$$

Note that we can write

$$\begin{aligned}
&\left(\sum_{y_h\in n^{-\frac{1}{\alpha}}\mathbb{Z}} n^{\frac{1}{2}-\frac{1}{2\alpha}+\frac{r}{\alpha}} a_{u_h - n^{\frac{1}{\alpha}} y_h} \|\overline{z}_n(s,y_h)\|_{2m} \overline{p}_n(t-s, x-y_h) \right)^{l_h}\\
=& \Bigg(\sum_{y_h\in n^{-\frac{1}{\alpha}}\mathbb{Z}} \sum_{y'_h\in n^{-\frac{1}{\alpha}}\mathbb{Z}} n^{1-\frac{1}{\alpha}+\frac{2r}{\alpha}} a_{u_h - n^{\frac{1}{\alpha}} y_h} a_{u_h - n^{\frac{1}{\alpha}} y'_h} \|\overline{z}_n(s,y_h)\|_{2m} \|\overline{z}_n(s,y'_h)\|_{2m}\\
&\times \overline{p}_n(t-s, x-y_h) \overline{p}_n(t-s, x-y'_h) \Bigg)^{l_h/2},
\end{aligned}$$

and that $l_h/2 \ge 1$ for all $1\le h\le m$. Using the Minkowski inequality: $(\sum_{u\in\mathbb{Z}} |x_u|^p)^{1/p} \le \sum_{u\in\mathbb{Z}} |x_u|$ for $p\ge 1$, we have that

$$\begin{aligned}
&\sum_{u_1\in\mathbb{Z}} \cdots \sum_{u_k\in\mathbb{Z}} \prod_{h=1}^{k} \left(\sum_{y_h\in n^{-\frac{1}{\alpha}}\mathbb{Z}} n^{\frac{1}{2}-\frac{1}{2\alpha}+\frac{r}{\alpha}} a_{u_h - n^{\frac{1}{\alpha}} y_h} \|\overline{z}_n(s,y_h)\|_{2m} \overline{p}_n(t-s, x-y_h) \right)^{l_h}\\
\le& \prod_{h=1}^{k} \Bigg(\sum_{y_h\in n^{-\frac{1}{\alpha}}\mathbb{Z}} \sum_{y'_h\in n^{-\frac{1}{\alpha}}\mathbb{Z}} \sum_{u_h\in\mathbb{Z}} n^{1-\frac{1}{\alpha}+\frac{2r}{\alpha}} a_{u_h - n^{\frac{1}{\alpha}} y_h} a_{u_h - n^{\frac{1}{\alpha}} y'_h} \|\overline{z}_n(s,y_h)\|_{2m} \|\overline{z}_n(s,y'_h)\|_{2m}\\
&\times \overline{p}_n(t-s, x-y_h) \overline{p}_n(t-s, x-y'_h) \Bigg)^{l_h/2}\\
=& \left(\sum_{y\in n^{-\frac{1}{\alpha}}\mathbb{Z}} \sum_{y'\in n^{-\frac{1}{\alpha}}\mathbb{Z}} n\gamma_n(y-y') \|\overline{z}_n(s,y)\|_{2m} \|\overline{z}_n(s,y')\|_{2m} \overline{p}_n(t-s, x-y) \overline{p}_n(t-s, x-y') \right)^{m}
\end{aligned}$$

We can write

$$\begin{aligned}&\sum_{y\in n^{-\frac{1}{\alpha}}\mathbb{Z}}\sum_{y'\in n^{-\frac{1}{\alpha}}\mathbb{Z}} n\gamma_n(y-y')\|\overline{z}_n(s,y)\|_{2m}\|\overline{z}_n(s,y')\|_{2m}\overline{p}_n(t-s,x-y)\overline{p}_n(t-s,x-y')\\ =&q^{-2}n^{1+\frac{2}{\alpha}}\int_{\mathbb{R}}\int_{\mathbb{R}}\overline{p}_n(t-s,x-y)\overline{p}_n(t-s,x-y')\|\overline{z}^2_n(s,y)\|^{\frac{1}{2}}_m\|\overline{z}^2_n(s,y')\|^{\frac{1}{2}}_m\gamma_n(y-y')dydy'.\end{aligned}$$

Thus, there exists a positive constant $\widehat{A}_m$ such that for any $n\geq 1$, $t\in(0,1]$, $x\in\mathbb{R}$,

$$\begin{aligned}&(\mathbf{E}(|X_s|^{2m}))^{1/m}\\ \leq&\widehat{A}_m n^{1+\frac{2}{\alpha}}\int_{\mathbb{R}}\int_{\mathbb{R}}\overline{p}_n(t-s,x-y)\overline{p}_n(t-s,x-y')\|\overline{z}^2_n(s,y)\|^{\frac{1}{2}}_m\|\overline{z}^2_n(s,y')\|^{\frac{1}{2}}_m\gamma_n(y-y')dydy',\end{aligned}$$

and

$$\begin{aligned}&\left(\mathbf{E}\left(n^{-1-\frac{1}{\alpha}}\left|\sum_{s\in[0,t]\cap n^{-1}\mathbb{Z}}X_s\right|^{2m}\right)\right)^{1/m}\\ \leq&72mn^{-2-\frac{2}{\alpha}}\sum_{s\in[0,t]\cap n^{-1}\mathbb{Z}}\|X_s^2\|_m\\ \leq&72m\widehat{A}_m\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}\overline{p}_n(t-s,x-y)\overline{p}_n(t-s,x-y')\|\overline{z}^2_n(s,y)\|^{\frac{1}{2}}_m\|\overline{z}^2_n(s,y')\|^{\frac{1}{2}}_m\gamma_n(y-y')dydy'ds.\end{aligned}$$

Therefore, there exists a positive constant C_m such that for any $n\geq 1$, $t\in(0,1]$, $s\in[0,t]$, $x\in\mathbb{R}$,

$$\begin{aligned}\left\|z^2_n(t,x)\right\|_m\leq&C_m(\overline{p}_n(t,x))^2+C_m\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}\overline{p}_n(t-s,x-y)\overline{p}_n(t-s,x-y')\\ &\|\overline{z}^2_n(s,y)\|^{\frac{1}{2}}_m\|\overline{z}^2_n(s,y')\|^{\frac{1}{2}}_m\gamma_n(y-y')dydy'ds.\end{aligned}\tag{4.15}$$

By the proof of (3.15), there exists a positive constant C_2 such that

$$0\leq\gamma([z])\leq C_2K(z)\text{ for }z\in\mathbb{R}.$$

Set $\varphi_{t,x}(s,y)=\overline{p}_n(t-s,x-y)\|\overline{z}^2_n(s,y)\|^{\frac{1}{2}}_m$, $0\leq s\leq t, x\in\mathbb{R}$, then

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}\overline{p}_n(t-s,x-y)\overline{p}_n(t-s,x-y')\|\overline{z}^2_n(s,y)\|^{\frac{1}{2}}_m\|\overline{z}^2_n(s,y')\|^{\frac{1}{2}}_m\gamma_n(y-y')dydy'ds\\ \leq&\sum_{\substack{(s,y),(s,z)\in\mathbb{D}_n\\ s\leq t}}\int_{\mathcal{C}_n(s,y)\times\mathcal{C}_n(s,z)}\varphi_{t,x}(s',y')n^{\frac{2r-1}{\alpha}}\gamma\left(n^{\frac{1}{\alpha}}(y'-z')\right)\varphi_{t,x}(s',z')ds'dy'dz'\\ \leq&C_2\sum_{\substack{(s,y),(s,z)\in\mathbb{D}_n\\ s\leq t}}\int_{\mathcal{C}_n(s,y)\times\mathcal{C}_n(s,z)}\varphi_{t,x}(s',y')n^{\frac{2r-1}{\alpha}}K\left(n^{\frac{1}{\alpha}}(y'-z')\right)\varphi_{t,x}(s',z')ds'dy'dz'\\ =&C_2\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}\overline{p}_n(t-s,x-y)\overline{p}_n(t-s,x-y')K(y-y')\|\overline{z}^2_n(s,y)\|^{\frac{1}{2}}_m\|\overline{z}^2_n(s,y')\|^{\frac{1}{2}}_m dydy'ds.\end{aligned}$$

Therefore, by Hardy-Littewood's inequality and Hölder's inequality we have

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}\overline{p}_n(t-s,x-y)\overline{p}_n(t-s,x-y')\|\overline{z}^2_n(s,y)\|^{\frac{1}{2}}_m\|\overline{z}^2_n(s,y')\|^{\frac{1}{2}}_m\gamma_n(y-y')dydy'ds\\ \leq&C_3\int_0^t\int_{\mathbb{R}}|\overline{p}_n(t-s,x-y)|^{2r}\|\overline{z}^2_n(s,y)\|_m dyds\\ \leq&C_4\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}|\overline{p}_n(t-s,x-y)|^{2r}\overline{p}_n(1/n,y-z)\left\|z^2_n((s-1/n)^+,z)\right\|_m dzdyds,\end{aligned}$$

where C_3 and C_4 are universal constants independent of n and t. Therefore, there exists a positive constants C_m such that for any $n \geq 1$, $t \in (0,1]$, $x \in \mathbb{R}$,

$$\begin{aligned}
&\left\| z_n^2(t,x) \right\|_m \\
&\leq C_m(\overline{p}_n(t,x))^2 \\
&\quad + C_m \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} |\overline{p}_n(t-s,x-y)|^{2r} \overline{p}_n(1/n,y-z) \left\| z_n^2((s-1/n)^+,z) \right\|_m dzdyds. \quad (4.16)
\end{aligned}$$

Iterating the inequality, we can obtain

$$\left\| z_n^2(t,x) \right\|_m \leq C_m(\overline{p}_n(t,x))^2 + \sum_{k=1}^{\lfloor nt \rfloor} C_m^k \mathbf{I}_{n,k}(t,x),$$

where

$$\begin{aligned}
&\mathbf{I}_{n,k}(t,x) \\
&= \int_{\Delta_k(t,1/n)} \int_{\mathbb{R}^{2k}} |\overline{p}_n(t-t_1,x-x_1)|^{2r} \overline{p}_n(1/n,x_1-x_2) \\
&\quad \times \left(\prod_{j=2}^k |\overline{p}_n(t_{j-1}-t_j-1/n, x_{2(j-1)}-x_{2j-1})|^{2r} \overline{p}_n(1/n,x_{2j-1}-x_{2j}) \right) \\
&\quad \times p_n^2(t_k-1/n,x_{2k}) dx_1 \cdots dx_{2k} dt_1 \cdots dt_k,
\end{aligned}$$

and $x_0 = x$, $\Delta_k(t,1/n) = \{1/n \leq t_j \leq t_{j-1} - 1/n, j = 2, \cdots, k, t_1 \leq t_0 = t\}$.

By (3.23) and (3.24), there exists a positive constant C such that

$$\begin{aligned}
&|\overline{p}_n(t-t_1,x-x_1)|^{2r} \overline{p}_n(1/n,x_1-x_2) \left(\prod_{j=2}^k |\overline{p}_n(t_{j-1}-t_j-1/n, x_{2(j-1)}-x_{2j-1})|^{2r} \right. \\
&\left. \times \overline{p}_n(1/n,x_{2j-1}-x_{2j}) \right) p_n^2(t_k-1/n,x_{2k}) \\
\leq& \frac{C^k}{(t_k-1/n)^{1/\alpha}(t-t_1)^{(2r-1)/\alpha}} \prod_{i=2}^k \frac{1}{(t_{i-1}-t_i-1/n)^{(2r-1)/\alpha}} \overline{p}_n(t-t_1,x-x_1)\overline{p}_n(1/n,x_1-x_2) \\
&\times \left(\prod_{j=2}^k \overline{p}_n(t_{j-1}-t_j-1/n, x_{2(j-1)}-x_{2j-1}) \overline{p}_n(1/n,x_{2j-1}-x_{2j}) \right) \overline{p}_n(t_k-1/n,x_{2k}).
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{\mathbb{R}^{2k}} \overline{p}_n(t-t_1,x-x_1)\overline{p}_n(1/n,x_1-x_2) \left(\prod_{j=2}^k \overline{p}_n(t_{j-1}-t_j-1/n, x_{2(j-1)}-x_{2j-1}) \right. \\
&\quad \left. \times \overline{p}_n(1/n,x_{2j-1}-x_{2j}) \right) \overline{p}_n(t_k-1/n,x_{2k}) dx_1 \cdots dx_{2k} = \overline{p}_n(t,x),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Delta_k(t,1/n)} \frac{1}{(t_k-1/n)^{1/\alpha}(t-t_1)^{(2r-1)/\alpha}} \prod_{i=2}^k \frac{1}{(t_{i-1}-t_i-1/n)^{(2r-1)/\alpha}} dt_1 \cdots dt_k \\
&\leq \left(t - \frac{k}{n} \right)^{((k-1)(2r-1)+1)/\alpha} \frac{\Gamma\left(1-\frac{1}{\alpha}\right)\Gamma^k\left(1+\frac{1-2r}{\alpha}\right)}{\Gamma\left(k\left(1+\frac{1-2r}{\alpha}\right)+1-\frac{1}{\alpha}\right)},
\end{aligned}$$

we have

$$\mathbf{I}_{n,k}(t,x) \leq \frac{\Gamma\left(1-\frac{1}{\alpha}\right)\Gamma^k\left(1+\frac{1-2r}{\alpha}\right)}{\Gamma\left(k\left(1+\frac{1-2r}{\alpha}\right)+1-\frac{1}{\alpha}\right)} C_1^k \overline{p}_n(t,x).$$

Therefore, there exists a positive constants C_m such that for any $n \geq 1$, $t \in (0,1]$, $x \in \mathbb{R}$,

$$\left\|z_n^2(t,x)\right\|_m \leq C_m(\overline{p}_n(t,x))^2 + C_m\overline{p}_n(t,x)$$

(4.13) is valid. Finally, by (4.11) and Jensen's inequality, we obtain (4.14).

Next, we consider the $r=1$ case. For each $n \geq 1$, $(t,x) \in \mathbb{D}_n$, and any positive integer $M \geq 2$, define

$$X_{(s,y)} := \overline{p}_n(t-s,x-y)\overline{z}_n(s,y)\overline{\omega}_n(s,y), \quad s \in [0,t] \cap n^{-1}\mathbb{Z}, y \in n^{-\frac{1}{\alpha}}\mathbb{Z} \cap [-M,M],$$

Note that the $\overline{z}_n(s,\cdot)$ terms are independent of $\omega_n(s,y), y \in n^{-\frac{1}{\alpha}}\mathbb{Z}$, and for each $s \in [0, t \cap n^{-1}\mathbb{Z}]$, $\omega_n(s,y), y \in n^{-\frac{1}{\alpha}}\mathbb{Z}$ are also independent. Then, applying discrete Burkholder's inequality (cf. Theorem 2.10 in [26]) and Minkowski's integral inequality in the order of temporal and spatial variables, we have that for any $m \geq 1$,

$$\begin{aligned}
&\left(\mathbf{E}\left(\left|n^{-1-\frac{1}{\alpha}} \sum_{s\in[0,t]\cap n^{-1}\mathbb{Z}} \sum_{y\in n^{-\frac{1}{\alpha}}\mathbb{Z}\cap[-M,M]} X_{(s,y)}\right|^{2m}\right)\right)^{1/m} \\
\leq& 72mn^{-2-\frac{2}{\alpha}} \sum_{s\in[0,t]\cap n^{-1}\mathbb{Z}} \left(\mathbf{E}\left(\left(\sum_{y\in n^{-\frac{1}{\alpha}}\mathbb{Z}\cap[-M,M]} X_{(s,y)}\right)^{2m}\right)\right)^{1/m} \\
\leq& (72m)^2 n^{-2-\frac{2}{\alpha}} \sum_{s\in[0,t]\cap n^{-1}\mathbb{Z}} \sum_{y\in n^{-\frac{1}{\alpha}}\mathbb{Z}\cap[-M,M]} \|X^2_{(s,y)}\|_m.
\end{aligned}$$

Letting $M \to \infty$, we obtain

$$\begin{aligned}
&\left(\mathbf{E}\left(\left|n^{-1-\frac{1}{\alpha}} \sum_{s\in[0,t]\cap n^{-1}\mathbb{Z}} \sum_{y\in n^{-\frac{1}{\alpha}}\mathbb{Z}} X_{(s,y)}\right|^{2m}\right)\right)^{1/m} \\
\leq& (72m)^2 n^{-2-\frac{2}{\alpha}} \sum_{s\in[0,t]\cap n^{-1}\mathbb{Z}} \sum_{y\in n^{-\frac{1}{\alpha}}\mathbb{Z}} \|X^2_{(s,y)}\|_m \\
\leq& (72m)^2 \int_0^t \int_{\mathbb{R}} \overline{p}_n^2(t-s,x-y)\|\overline{z}_n^2(s,y)\|_m dyds.
\end{aligned}$$

Therefore, there exists a positive constants C_m such that for any $n \geq 1$, $t \in (0,1]$, $x \in \mathbb{R}$,

$$\left\|z_n^2(t,x)\right\|_m \leq C_m(\overline{p}_n(t,x))^2 + C_m\int_0^t \int_{\mathbb{R}} |\overline{p}_n(t-s,x-y)|^2 \left\|z_n^2((s-1/n)^+,z)\right\|_m dyds.$$

Finally, using the above inequality, we can obtained (4.13) and (4.14) via the same as the proof of the $r \in (1/2,1)$ case. □

Next, let us estimate the modulus of continuity of z_n.

Lemma 4.3. Let $\alpha \in (1,2]$, (A.1) and (A.2) hold. Then there exist constants $\kappa_1 > 0$, $\kappa_2 > 0$ and a positive function $(0,1/4] \ni \epsilon \to C_\epsilon$ such that for any $n \in \mathbb{Z}_+$ with $n \geq 1+1/\epsilon$, $t \in [2\epsilon,1]$ $x \in \mathbb{R}$, $h \geq 0$, $\delta \geq 0$,

$$\left(\mathbf{E}\left(\left(z_n(t+h,x+\delta)-z_n(t,x)\right)^{2m}\right)\right)^{1/m} \leq C_\epsilon\left(h_n^{\kappa_1}+\delta_n^{\kappa_2}\right), \tag{4.17}$$

where $h_n = h \vee \frac{1}{n}$ and $\delta_n = \delta \vee \frac{1}{n^{1/\alpha}}$.

Proof. Without of loss generality, we assume $\epsilon = \frac{i_\epsilon}{n}$ for some integer $i_\epsilon < n/4$, and $(t,x),(t+h,x+\delta) \in \mathbb{D}_n$.

$$z_n(\epsilon,x) = \overline{p}_n(\epsilon,x) + \beta \int_0^\epsilon \int_{\mathbb{R}} \overline{p}_n(\epsilon - s, y-x)\overline{z}_n(s,y)\omega_n(s,y)dsdy.$$

For any $0 < 2\epsilon < t$, we can rewrite

$$\begin{aligned} z_n(t,x) &= \int_{\mathbb{R}} \overline{p}_n(t-\epsilon, y-x) z_n(\epsilon,y)dy + \beta \int_\epsilon^t \int_{\mathbb{R}} \overline{p}_n(t-s,y-x)\overline{z}_n(s,y)\omega_n(s,y)dsdy \\ &=: A_{n,\epsilon}(t,x) + \beta U_{n,\epsilon}(t,x). \end{aligned}$$

Then for $h > 0$ and $\delta > 0$,

$$\begin{aligned} &\left\|(z_n(t+h,x+\delta) - z_n(t,x))^2\right\|_m \\ \le &\left\|(A_{n,\epsilon}(t+h,x+\delta) - A_{n,\epsilon}(t,x))^2\right\|_m + \beta^2 \left\|(U_{n,\epsilon}(t+h,x+\delta) - U_{n,\epsilon}(t,x))^2\right\|_m . \end{aligned}$$

By Hölder's inequality, we have that for $2\epsilon \le t \le 1$,

$$\begin{aligned} &\left\|(A_{n,\epsilon}(t+h,x+\delta) - A_{n,\epsilon}(t,x))^2\right\|_m \\ \le &\left(\int_{\mathbb{R}} \left|\overline{p}_n(t+h-\epsilon,y+\delta) - \overline{p}_n(t-\epsilon,y)\right|^2 dy\right) \left(\mathbf{E}\left(\left(\int_{\mathbb{R}} z_n^2(\epsilon,y)dy\right)^m\right)\right)^{1/m}. \end{aligned}$$

By Minkowski's integral inequality, (4.14) and (3.24),

$$\begin{aligned} \left(\mathbf{E}\left(\left(\int_{\mathbb{R}} z_n^2(\epsilon,y)dy\right)^m\right)\right)^{1/m} &\le \int_{\mathbb{R}} \left(\mathbf{E}\left(z_n^{2m}((\epsilon,y))\right)\right)^{1/m} dy \\ &\le \frac{C}{\epsilon^{1/\alpha}} \int_{\mathbb{R}} \overline{p}_n(\epsilon,y)dy = \frac{C}{\epsilon^{1/\alpha}}. \end{aligned}$$

By the gradient estimate (4.1) and (4.2), we have

$$\begin{aligned} &\int_{\mathbb{R}} \left|\overline{p}_n(t+h-\epsilon,y+\delta) - \overline{p}_n(t-\epsilon,y)\right|^2 dy \\ \le &2\int_{\mathbb{R}} \left|\overline{p}_n(t+h-\epsilon,y) - \overline{p}_n(t-\epsilon,y)\right|^2 dy + 2\int_{\mathbb{R}} \left|\overline{p}_n(t+h-\epsilon,y-\delta) - \overline{p}_n(t+h-\epsilon,y)\right|^2 dy \\ \le &C\int_{\mathbb{R}} \frac{n^{\frac{1}{\alpha}} n h_n}{(n\epsilon)^{1+\frac{1}{\alpha}}} \left(\overline{p}_n(t+h-\epsilon,y) + \overline{p}_n(t-\epsilon,y)\right) dy \\ &+ C\int_{\mathbb{R}} \frac{n^{\frac{1}{\alpha}} n\delta_n}{(n\epsilon)^{\frac{2}{\alpha}}} \left(\overline{p}_n(t+h-\epsilon,y-\delta) + \overline{p}_n(t+h-\epsilon,y)\right) dy \\ \le &\frac{Ch_n}{\epsilon^{1+\frac{1}{\alpha}}} + \frac{C\delta_n}{\epsilon^{\frac{2}{\alpha}}}. \end{aligned}$$

Next, we estimate the term U. By Minkowski's inequality,

$$\begin{aligned} &\left\|(U_{n,\epsilon}(t+h,x+\delta) - U_{n,\epsilon}(t,x))^2\right\|_m \\ \le &\left\|\left(\int_\epsilon^t \int_{\mathbb{R}} \left(\overline{p}_n(t+h-s,y-x-\delta) - \overline{p}_n(t-s,y-x-\delta)\right) \overline{z}_n(s,y)\omega_n(s,y)dsdy\right)^2\right\|_m \\ &+ \left\|\left(\int_t^{t+h} \int_{\mathbb{R}} \overline{p}_n(t+h-s,y-x-\delta)\overline{z}_n(s,y)\omega_n(s,y)dsdy\right)^2\right\|_m \\ &+ \left\|\left(\int_\epsilon^t \int_{\mathbb{R}} \left(\overline{p}_n(t-s,y-x-\delta) - \overline{p}_n(t-s,y-x)\right) \overline{z}_n(s,y)\omega_n(s,y)dsdy\right)^2\right\|_m \\ =: &Q_1 + Q_2 + Q_3. \end{aligned}$$

Using the same way in the proof of (4.16), by discrete Burkholder's inequality, Hölder's inequality and Minkowski's integral inequality, we can obtain that

$$Q_1 \leq \int_\epsilon^t \int_{\mathbb{R}} |\overline{p}_n(t+h-s, y-x-\delta) - \overline{p}_n(t-s, y-x-\delta)|^{2r} \left\|\overline{z}_n^2(s,y)\right\|_m dyds.$$

By (4.14) and (3.24), for any $s \geq 2\epsilon$, $\left\|\overline{z}_n^2(s,y)\right\|_m \leq \frac{C}{\epsilon^{2/\alpha}}$. Thus,

$$Q_1 \leq \frac{C}{\epsilon^{2/\alpha}} \int_0^{t-\epsilon} \int_0^\infty |\overline{p}_n(s+h,y) - \overline{p}_n(s,y)|^{2r} dyds$$

If $r < \frac{2\alpha+1}{2(\alpha+1)}$, i.e., $(1-2r)(1+1/\alpha)+1>0$, then by (4.2) and (3.24),

$$\begin{aligned}
&\int_0^{t-\epsilon} \int_0^\infty |\overline{p}_n(s+h,y) - \overline{p}_n(s,y)|^{2r} dyds \\
\leq &\int_0^{t-\epsilon} \int_0^\infty \frac{Ch_n^{2r-1}}{s^{(2r-1)(1+1/\alpha)}} \left(\overline{p}_n(s+h,y) + \overline{p}_n(s,y)\right) dyds \leq C_1 h_n^{2r-1}.
\end{aligned}$$

If $r > \frac{2\alpha+1}{2(\alpha+1)}$, choose $\epsilon > 0$ such that $\eta := (2(\alpha+1)r - (2\alpha+1))(1/\alpha+\epsilon) < 2r-1$, then $(1+\eta-2r)(1+1/\alpha) - \eta/\alpha + 1 > 0$ and

$$\begin{aligned}
&\int_0^{t-\epsilon} \int_0^\infty |\overline{p}_n(s+h,y) - \overline{p}_n(s,y)|^{2r} dyds \\
\leq &\int_0^{t-\epsilon} \int_0^\infty \frac{Ch_n^{2r-1-\eta}}{s^{(2r-1-\eta)(1+1/\alpha)+\eta/\alpha}} \left(\overline{p}_n(s+h,y) + \overline{p}_n(s,y)\right) dyds \leq C_1 h_n^{2r-1-\eta}.
\end{aligned}$$

If $r = \frac{2\alpha+1}{2(\alpha+1)}$, then for any $0 < \eta < 2r-1$,

$$\begin{aligned}
&\int_0^{t-\epsilon} \int_0^\infty |\overline{p}_n(s+h,y) - \overline{p}_n(s,y)|^{2r} dyds \\
\leq &\int_0^{t-\epsilon} \int_0^\infty \frac{Ch_n^{2r-1-\eta}}{s^{(2r-1-\eta)(1+1/\alpha)+\eta/\alpha}} \left(\overline{p}_n(s+h,y) + \overline{p}_n(s,y)\right) dyds \leq C_1 h_n^{2r-1-\eta}.
\end{aligned}$$

We also have in the same way

$$\begin{aligned}
Q_2 \leq &\frac{C}{\epsilon^{2/\alpha}} \int_t^{t+h} \int_{\mathbb{R}} |\overline{p}_n(t+h-s, y-x-\delta)|^{2r} dyds \\
\leq &\frac{C_1}{\epsilon^{2/\alpha}} \int_t^{t+h} \int_{\mathbb{R}} (t+h-s)^{-(2r-1)/\alpha} \overline{p}_n(t+h-s, y-x-\delta) dyds = \frac{C_1}{\epsilon^{2/\alpha}} h^{(1+\alpha-2r)/\alpha}.
\end{aligned}$$

If $r < \frac{2+\alpha}{4}$, then $\frac{2(1-2r)}{\alpha} + 1 > 0$, by the gradient estimate (4.1)

$$Q_3 \leq \frac{C}{\epsilon^{2/\alpha}} \int_\epsilon^t \int_{\mathbb{R}} |\overline{p}_n(t-s, y-x-\delta) - \overline{p}_n(t-s, y-x)|^{2r} dyds \leq \frac{2C_1 \delta_n^{2r-1}}{\epsilon^{1+4r/\alpha}}.$$

If $r = \frac{2+\alpha}{4}$, then for any $0 < \eta < 2r-1$,

$$\int_\epsilon^t \int_{\mathbb{R}} |\overline{p}_n(t-s, y-x-\delta) - \overline{p}_n(t-s, y-x)|^{2r} dyds \leq C_1 \delta_n^{2r-1-\eta}.$$

If $r > \frac{2+\alpha}{4}$, choose $\epsilon > 0$ such that $\eta := (4r - (2+\alpha))(1+\epsilon) < 2r-1$, then $2(1+\eta-2r)\alpha - \eta/\alpha + 1 > 0$ and

$$\int_\epsilon^t \int_{\mathbb{R}} |\overline{p}_n(t-s, y-x-\delta) - \overline{p}_n(t-s, y-x)|^{2r} dyds \leq C_1 \delta_n^{2r-1-\eta}.$$

Thus, (4.17) holds. □

We will use the following lemma to show the tightness.

Lemma 4.4 ([35]). Let $Y_n(t,x)$ be a sequence of stochastic processes on $[0,1]\times[0,1]$. Denote by

$$w_\delta(Y^n) = \sup_{\substack{(t,x),\,(s,y)\in[0,1]^2,\\ |t-s|+|x-y|<\delta}} |Y^n(t,x)-Y^n(s,y)|.$$

Suppose there exist positive constants $\kappa>2$, λ, C and a sequence $\delta_n\downarrow 0$, such that for all large enough n, for all $(t,x),(s,y)\in[0,1]^2$ and $|t-s|+|x-y|>\delta_n$,

$$\mathbf{E}\left(|Y^n(t,x)-Y^n(s,y)|^\lambda\right) \le C\left(|t-s|^\kappa+|x-y|^\kappa\right),$$

and for all $\epsilon,\rho>0$, for all large n,

$$\mathbf{P}\left(w_{\delta_n}(Y^n)>\epsilon\right)<\rho.$$

Then for all ϵ, $\rho>0$, there is a $0<\delta<1$ such that for all large n,

$$\mathbf{P}\left(w_{\delta}(Y^n)>\epsilon\right)<\rho.$$

□

The proof of theorem 4.1. We only need to show the tightness. By Lemma 4.3, for any $\epsilon\in(0,1/4)$, $M\in(0,\infty)$, there exist strictly positive constants $m\ge 1$, $\kappa>4, C(\epsilon,M)$ such that for any $n\ge 1+1/\epsilon$, for all $u,v\in[\epsilon,1]$, $|x|\vee|y|\le M$ with $|u-v|+|x-y|\ge 1/n^{1/\alpha}$,

$$\mathbf{E}\left(|z_n(u,x)-z_n(v,y)|^{2m}\right) \le C(\epsilon,M)\left(|u-v|^\kappa+|x-y|^\kappa\right). \tag{4.18}$$

Set

$$w_\delta(z_n) := \sup_{\substack{(t,x),\,(s,y)\in[0,1]\times[-M,M],\\ |t-s|+|x-y|<\delta}} |z_n(t,x)-z_n(s,y)|.$$

Noting that the number of points in $\mathbb{D}_n\cap([0,1]\times[-M,M])$ is less than $2(M+2)n\times n^{1/\alpha}$, we have

$$\begin{aligned}\mathbf{P}\left(w_{1/n^{1/\alpha}}(z_n)>\epsilon\right) &\le \sum_{(t,x)\in\mathbb{D}_n\cap([0,1]\times[-M,M])} \mathbf{P}\left(|z_n(t,x)-z_n(t-,x-)|>\epsilon\right)\\ &\le C_1 n^{1+\frac{1}{\alpha}} \sup_{(t,x)\in\mathbb{D}_n\cap([0,1]\times[-M,M])} \frac{\mathbf{E}\left(|z_n(t,x)-z_n(t-,x-)|^{2m}\right)}{\epsilon^{2m}}\\ &\le \frac{C_2 n^{1+\frac{1}{\alpha}}}{\epsilon^{2m}}\frac{1}{n^{4/\alpha}} \to 0 \text{ as } n\to\infty,\end{aligned}$$

where C_1 and C_2 are universal constants independent of n and ϵ. Therefore, by Lemma 4.4 and Theorem 15.5 in [8], $\{z_n, n\ge 1\}$ is tight in $D([\epsilon,1]\times[-M,M])$, and so $\{Z_n, n\ge 1\}$ is tight in $C([\epsilon,1]\times[-M,M])$. By arbitrariness of ϵ and M, we obtain the tightness of $\{z_n, n\ge 1\}$ in $D((0,1]\times\mathbb{R})$, and the tightness of $\{Z_n, n\ge 1\}$ in $C((0,1]\times\mathbb{R})$. □

4.3 The proof of Theorem 2.2

Proof of Theorem 2.2. We only show the $r\in(1/2,1)$ case. By Lemma 3.7, the convergence of the finite dimensional distributions can be obtained via the same as the proof of Theorem 2.1.

Next, we give a sketch of the tightness. Define the right continuous processes in time and space:

$$\widetilde{z}_n(t,x) := n^{\frac{1}{\alpha}}\mathfrak{z}_{nt,n^{1/\alpha}x}(\beta_n;\widetilde{\omega})$$

and

$$\overline{\widetilde{z}}_n(t,x) := n^{\frac{1}{\alpha}} \overline{\mathfrak{Z}}_{nt,n^{1/\alpha}x}(\beta_n;\widetilde{\omega}) = p\left(1, n^{1/\alpha}(x-y)\right) \widetilde{z}_n((t-1/n)^+, y).$$

Then

$$\widetilde{z}_n(t,x) = \overline{p}_n(t,x) + \beta_n \int_0^t \int_{\mathbb{R}} \overline{p}_n(t-s,x-y) \overline{\widetilde{z}}_n(s,y) \widetilde{\omega}_n(s,y) dsdy. \tag{4.19}$$

Set $\widetilde{\gamma}_n(x-y) = n^{\frac{2r-1}{\alpha}} \mathbf{E}(\widetilde{\omega}_n(nt, n^{1/\alpha}x)\widetilde{\omega}_n(nt, n^{1/\alpha}y))$. By Lemma 3.7, Hardy-Littewood's inequality and Hölder's inequality, we have

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{p}_n(t-s,x-y_1)\overline{p}_n(t-s,x-y_2)\widetilde{\gamma}_n(y_1-y_2)\overline{\widetilde{z}}_n(s,y_1)\overline{\widetilde{z}}_n(s,y_2)dy_1dy_2ds \\
\le& C_1 \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{p}_n(t-s,x-y_1)\overline{p}_n(t-s,x-y_2)\left(K(y_1-y_2) + O(n^{\frac{2r-1}{\alpha}}\beta_n^2)\right) \\
&\quad \overline{\widetilde{z}}_n(s,y_1)\overline{\widetilde{z}}_n(s,y_2)dy_1dy_2ds \\
\le& C_2 \int_0^t \int_{\mathbb{R}} |\overline{p}_n(t-s,x-y)|^{2r} \overline{\widetilde{z}}^2_n(s,y)dyds + \int_0^t \int_{\mathbb{R}} |\overline{p}_n(t-s,x-y)|^2 \overline{\widetilde{z}}^2_n(s,y)dyds \\
\le& C_3 \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \left(|\overline{p}_n(t-s,x-y)|^{2r} + |\overline{p}_n(t-s,x-y)|^2\right) \overline{p}_n(1/n,y-z)\widetilde{z}^2_n(s,z)dzdyds,
\end{aligned}$$

where C_1, C_2 and C_3 are universal constants independent of n and t. Therefore, using the same approach as the proof of (4.16), there exists a positive constants C_m such that for any $n \ge 1$, $t \in (0,1]$, $x \in \mathbb{R}$,

$$\begin{aligned}
&\left\|\widetilde{z}^2_n(t,x)\right\|_m \\
&\le C_m(\overline{p}_n(t,x))^2 + C_m \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \left(|\overline{p}_n(t-s,x-y)|^{2r} + |\overline{p}_n(t-s,x-y)|^2\right) \\
&\qquad \overline{p}_n(1/n,y-z) \left\|\widetilde{z}^2_n(s,z)\right\|_m dzdyds.
\end{aligned} \tag{4.20}$$

Iterating the inequality, there exists a positive constants C_m such that for any $n \ge 1$, $t \in (0,1]$, $x \in \mathbb{R}$,

$$\left\|z^2_n(t,x)\right\|_m \le C_m(\overline{p}_n(t,x))^2 + C_m \sum_{k=1}^{\lfloor nt \rfloor} C_m^k \mathbf{I}_{n,k}(t,x),$$

where

$$\begin{aligned}
&\mathbf{I}_{n,k}(t,x) \\
\le& \overline{p}_n(t,x) \int_{\Delta_k(t,1/n)} \frac{1}{(t_k - 1/n)^{1/\alpha}(t-t_1)^{1/\alpha}} \prod_{i=2}^k \frac{1}{(t_{i-1}-t_i-1/n)^{1/\alpha}} dt_1 \cdots dt_k \\
\le& \frac{\Gamma\left(1-\frac{1}{\alpha}\right)\Gamma^k\left(1-\frac{1}{\alpha}\right)}{\Gamma\left(k\left(1-\frac{1}{\alpha}\right)+1-\frac{1}{\alpha}\right)} C_m^k \overline{p}_n(t,x),
\end{aligned}$$

and $x_0 = x$, $\Delta_k(t,1/n) = \{1/n \le t_j \le t_{j-1} - 1/n, j = 2, \cdots, k, t_1 \le t_0 = t\}$. Therefore, there exists a positive constants C_m such that for any $n \ge 1$, $t \in (0,1]$, $x \in \mathbb{R}$,

$$\left\|\widetilde{z}^2_n(t,x)\right\|_m \le C_m(\overline{p}_n(t,x))^2 + C_m\overline{p}_n(t,x), \quad \left\|\overline{\widetilde{z}}^2_n(t,x)\right\|_m \le C_m(\overline{p}_n(t,x))^2 + C_m\overline{p}_n(t,x). \tag{4.21}$$

Similarly, there exist constant $\kappa_1 > 0$, $\kappa_2 > 0$ and a positive function $(0,1/4] \ni \epsilon \to C_\epsilon$ such that for any $n \in \mathbb{Z}_+$ with $n \ge 1 + 1/\epsilon$, $t \in [2\epsilon, 1]$, $x \in \mathbb{R}$, $h > 0$, $\delta > 0$,

$$\left(\mathbf{E}\left((\widetilde{z}_n(t+h,x+\delta) - \widetilde{z}_n(t,x))^{2m}\right)\right)^{1/m} \le C_\epsilon \left(h_n^{\kappa_1} + \delta_n^{\kappa_2}\right). \tag{4.22}$$

We complete the proof of Theorem 2.2. □

5 The scaling limit of the polymer transition probability

In this section, we give the proof of Theorem 2.3. This follows the same scheme as before. For $0 \le m < k \le n$ and $x, y \in \mathbb{Z}$, we define the four-parameter field $Z^\omega(m, y; k, x; \beta)$ by

$$Z^\omega(m,y;k,x;\beta) = \mathbb{P}\left(\exp\left\{\beta \sum_{i=m+1}^{k} \omega(i,S_i)\right\}\mathbf{1}\{S_k = x\}\Big| S_m = y\right). \tag{5.1}$$

Then the polymer transition probabilities are

$$\mathbf{P}^\omega_{n,\beta}(S_k = x | S_m = y) = \frac{Z^\omega(m,y;k,x;\beta) Z^\omega(k,x;n,*;\beta)}{Z^\omega(m,y;n,*;\beta)},$$

where

$$Z^\omega(k,x;n,*;\beta) = \sum_{z\in\mathbb{Z}} Z^\omega(k,x;n,z;\beta).$$

We consider the modified partition function

$$\mathfrak{Z}^\omega(m,y;k,x;\beta) = \mathbf{P}\left(\prod_{i=m+1}^{k}\big(1+\beta\omega(i,S_i)\big)\mathbf{1}\{S_k = x\}\Big| S_m = y\right). \tag{5.2}$$

Then

$$e^{-n(t-s)\lambda(\beta_n)} Z^\omega\big(ns, n^{1/\alpha}y; nt, n^{1/\alpha}x; \beta_n\big) = \mathfrak{Z}^{\widetilde{\omega}}\big(ns, n^{1/\alpha}y; nt, n^{1/\alpha}x; \beta_n\big),$$

where $\widetilde{\omega}(i,x) = \omega(i,x) + \vartheta(i,x)$ and (3.41) holds. Thus, using the same approach as the proof of Theorem 2.2, we only need to show the following result.

Theorem 5.1. Let $\alpha \in (1,2]$, (A.1) and (A.2) hold. Then

$$\begin{aligned}&\left(\frac{1}{q} n^{1/\alpha}\mathfrak{Z}^\omega\big(ns, n^{1/\alpha}y; nt, n^{1/\alpha}x; \beta_n\big)\right)_{(s,y;t,x)\in\mathfrak{D}} \\ &\qquad \xrightarrow{(d)} \left(\frac{\mathcal{Z}_{\sigma(\beta,q)}(s,y;t,x)\int \mathcal{Z}_{\sigma(\beta,q)}(t,x;1,\lambda)\,d\lambda}{\int \mathcal{Z}_{\sigma(\beta,q)}(s,y;1,\lambda)d\lambda}\right)_{(s,y;t,x)\in\mathfrak{D}}\end{aligned} \tag{5.3}$$

with respect to the locally uniform topology on $C(\mathfrak{D})$, where $\mathfrak{D} = \{(s,y;t,x);\ 0 \le s < t \le 1, x, y \in \mathbb{R}\}$.

Proof. Let $p^k_{n,t,x}(\mathbf{t},\mathbf{x})$ be the joint probability of $(S_{nt_1} = n^{1/\alpha}x_1, \cdots, S_{nt_k} = n^{1/\alpha}x_k)$ under the condition $S_{ns} = n^{1/\alpha}y$, $S_{nt} = n^{1/\alpha}x$, where $s \le t_1 < \cdots < t_k \le t$, i.e.,

$$\begin{aligned}p^k_{n,s,y;t,x}(\mathbf{t},\mathbf{x}) &= \mathbb{P}(S_{nt_1} = n^{1/\alpha}x_1, \cdots, S_{nt_k} = n^{1/\alpha}x_k | S_{ns} = n^{1/\alpha y}, S_{nt} = n^{1/\alpha}x) \\ &= \frac{p(n(t-t_k), n^{1/\alpha}(x-x_k))}{p(n(t-s), n^{1/\alpha}(x-y))}\prod_{j=1}^{k} p(n(t_j - t_{j-1}), n^{1/\alpha}(x_j - x_{j-1})),\end{aligned}$$

where $(t_0, x_0) = (s, y)$.

These kernels are space–time shifts of the kernels $p^k_{n,t,x}$. By shift invariance of the random walk and the environment, we have

$$\frac{n^{1/\alpha}}{q}\mathfrak{Z}^\omega\big(ns, n^{1/\alpha}y; nt, n^{1/\alpha}x; \beta_n\big) \overset{(d)}{=} \frac{n^{1/\alpha}}{q}\mathfrak{Z}^\omega\big(0,0; n(t-s), n^{1/\alpha}(x-y); \beta_n\big).$$

For a finite collection of space–time points $(s_i, y_i; t_i, x_i)$, the joint convergence of

$$\frac{n^{1/\alpha}}{q}\mathfrak{Z}^\omega\big(ns_i, n^{1/\alpha}y_i; nt_i, n^{1/\alpha}x_i; \beta_n\big)$$

follows from the above subsection approach.

Note that the law of the environment field is invariant under a similar time reversal. More precisely, define a field ω_n by $\omega_n(i,x) = \omega(n-i,x)$. Then it is clear that

$$\big(1+\beta\omega(n-m,y)\big)\mathfrak{Z}^{\omega}(m,y;k,x;\beta) = \big(1+\beta\omega(n-k,x)\big)\mathfrak{Z}^{\omega_n}(n-k,x;n-m,y;\beta).$$

Following the explanation in [1] and the reversibility of the random walk, the tightness of the field $(t,x)\to n^{1/\alpha}\mathfrak{Z}_{nt,n^{1/\alpha}x}(\beta_n;\omega)$ is sufficient to prove tightness of the field

$$(s,y;t,x)\mapsto n^{1/\alpha}\mathfrak{Z}^{\omega}\big(ns,n^{1/\alpha}y;nt,n^{1/\alpha}x;\beta_n\big).$$

Indeed, the tightness of $n^{1/\alpha}\mathfrak{Z}_{nt,n^{1/\alpha}x}(\beta_n;\omega)$ implies tightness of $n^{1/\alpha}\mathfrak{Z}^{\omega}\big(ns,n^{1/\alpha}y;nt, n^{1/\alpha}x;\beta_n\big)$ in the forward (t,x) variables, and tightness in the (s,y) variables follows from the reversibility of the random walk and the invariance of the law of the environment under a similar time reversal. □

A Stochastic integral with respect to a time-white spatial-colored noise

In this section, we briefly introduce the time-white spatial-colored noise and stochastic integral with respect to a time-white spatial-colored noise (cf. [1] [22] [29] [30] [40] [43] [47]).

Let $K(x) = H(2H-1)|x|^{2H-2}$, $\frac{1}{2}<H\le 1$. A time-white spatial-colored noise $\mathcal{W}$ with the kernel K is a mean zero Gaussian process $\{\mathcal{W}(\varphi),\varphi\in C_c^\infty([0,1]\times\mathbb{R})\}$ defined on some probability space $(\Omega_{\mathcal{W}},\mathcal{F}_{\mathcal{W}},\mathbf{E})$ with covariance

$$\mathrm{Cov}(\mathcal{W}(\varphi),\mathcal{W}(\psi)) = \int_{[0,1]}\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(s,x)K(x-y)\psi(s,y)dsdxdy, \tag{A.1}$$

where $C_c^\infty([0,1]\times\mathbb{R})$ is the space of smooth functions with compact support on $[0,1]\times\mathbb{R}$. Set $\mathcal{W}_{tx} = \mathcal{W}([0,t]\times[0,x])$, and let $\dot{\mathcal{W}}$ denote the derivative $\frac{\partial^2\mathcal{W}_{tx}}{\partial t\partial x}$ in the sense of Schwartz distribution, that is,

$$\dot{\mathcal{W}}(\varphi) = \int_{[0,1]}\int_{\mathbb{R}}\mathcal{W}_{tx}\frac{\partial^2\varphi(t,x)}{\partial t\partial x}dtdx,\quad \varphi\in C_c^\infty([0,1]\times\mathbb{R}).$$

Then $\dot{\mathcal{W}}(\varphi) = \mathcal{W}(\varphi)$. Therefore, we also use $\dot{\mathcal{W}}$ to denote this time-white spatial-colored noise.

A.1 Itô stochastic integral

Next, let us define the stochastic integral with respect to $\mathcal{W}$. Consider the Hilbert space:

$$\mathcal{L}_K = \left\{f;[0,1]\times\mathbb{R}\to\mathbb{R};\,\|\,f\,\|_{\mathcal{L}_K}^2 := \int_{[0,1]}\int_{\mathbb{R}}f(t,x)K(x-y)f(t,y)dtdxdy<\infty\right\},$$

Remark A.1. By Hardy-Littlewood's inequality (Theorem 1 in [46], P.119), for some positive constant A_H,

$$\int_{[0,1]}\int_{\mathbb{R}}\int_{\mathbb{R}}f(s,u)K(u-v)f(s,v)dsdudv \le A_H\int_{[0,1]}\left(\int_{\mathbb{R}}|f(s,u)|^{\frac{1}{H}}du\right)^{2H}ds. \tag{A.2}$$

For any $f\in\mathcal{L}_K$, choose $f_n\in C_c^\infty([0,1]\times\mathbb{R})$ such that

$$\|\,f_n-f\,\|_{\mathcal{L}_K}^2\to 0 \text{ as } n\to\infty,$$

and define

$$\int_{[0,1]} \int_{\mathbb{R}} f(t,x)\mathcal{W}(dtdx) = \lim_{n\to\infty} \mathcal{W}(f_n) \text{ in } L^2(\Omega_{\mathcal{W}}, \mathcal{F}_{\mathcal{W}}, \mathbf{P}).$$

We now define the stochastic integral with respect to $\mathcal{W}$. For each $t \geq 0$, define $\mathcal{F}_t$ to be the σ-field generated by

$$\left\{ \int_{[0,1]} \int_{\mathbb{R}} I_{[0,t]}(s)\varphi(x)\mathcal{W}(dsdx);\ \varphi \text{ is smooth function on } \mathbb{R} \text{ with compact support} \right\}.$$

Define

$$\mathcal{S} := \left\{ f(t,x) = \sum_{i=1}^n X_i I_{(a_i,b_i]}(t)\varphi_i(x);\ \begin{array}{l} 0 < a_1 < b_1 < \cdots < a_n < b_n < \infty, \\ X_i \in \mathcal{F}_{a_i}, \varphi_i \text{ is smooth function on } \mathbb{R} \text{ with} \\ \text{compact support}, i = 1, \cdots, n, n \geq 1 \end{array} \right\}.$$

For $f \in \mathcal{S}$, we define the stochastic integral as

$$\mathcal{W}(f) := \int_{[0,1]} \int_{\mathbb{R}} f(t,x)\mathcal{W}(dtdx) = \sum_{i=1}^n X_i \int_{[0,1]} \int_{\mathbb{R}} I_{(a_i,b_i]}(t)\varphi(x)\mathcal{W}(dtdx). \tag{A.3}$$

Let $\mathcal{P}_s$ denote the σ-fields generated by $\mathcal{S}$ and set

$$\mathcal{L}_K(\Omega, \mathcal{P}_s) = \left\{ f; [0,1] \times \mathbb{R} \times \Omega \to \mathbb{R};\ \ \mathcal{P}_s - \text{measurable and } \mathbf{E}(\| f \|^2_{\mathcal{L}_K}) < \infty \right\}.$$

Then $\mathcal{S}$ is dense in $\mathcal{L}_K(\Omega, \mathcal{P}_s)$. Consequently, if $f \in \mathcal{L}_K(\Omega, \mathcal{P}_s)$, then there exist $f_n \in \mathcal{S}$, $n \geq 1$ such that

$$\lim_{n\to\infty} \mathbf{E}\left(\|f_n - f\|^2_{\mathcal{L}_K} \right) = 0.$$

Thus, there is a limit $I := \lim_{n\to\infty} \int_{\mathbb{R}_+} \int_{\mathbb{R}} f_n(t,x)\mathcal{W}(dtdx)$ in $\mathcal{L}_K(\Omega, \mathcal{P}_s)$ which we call the stochastic integral

$$\mathcal{W}(f) := \int_{\mathbb{R}_+} \int_{\mathbb{R}} f(t,x)\mathcal{W}(dtdx). \tag{A.4}$$

Then from (A.1),

$$\mathrm{Cov}(\mathcal{W}(f_1), \mathcal{W}(f_2)) = \mathbf{E}\left(\langle f_1, f_2 \rangle_{\mathcal{L}_K} \right) \quad \text{for any } f_1, f_2 \in \mathcal{L}_K(\Omega, \mathcal{P}_s), \tag{A.5}$$

where

$$\langle f_1, f_2 \rangle_{\mathcal{L}_K} = \int_{[0,1]} \int_{\mathbb{R}} f_1(t,x)K(x-y)f_2(t,y)dtdxdy.$$

Lemma A.1 (Burkholder's inequality)**.** *For any $p \geq 2$, there exists a positive constant C_p such that*

$$\mathbf{E}\left(\left| \int_{\mathbb{R}_+} \int_{\mathbb{R}} f(t,x)\mathcal{W}(dtdx) \right|^p \right) \leq C_p \mathbf{E}\left(\left| \| f \|^2_{\mathcal{L}_K} \right|^{p/2} \right). \tag{A.6}$$

Proof. Since

$$M_t := \int_0^t \int_{\mathbb{R}} f(s,x)\mathcal{W}(dsdx) = \int_{\mathbb{R}_+} \int_{\mathbb{R}} I_{[0,t]}(s)f(s,x)\mathcal{W}(dsdx), \quad t \geq 0$$

is a martingle and $[M]_t = \int_0^t \int_{\mathbb{R}} f(s,x)K(x-y)f(s,y)dsdxdy$, by Burkholder's inequality for martingale, there exists a positive constant C_p such that

$$\mathbf{E}\left(\sup_{t\geq 0} |M_t|^p \right) \leq C_p \mathbf{E}\left(|[M]_\infty|^{p/2} \right).$$

Thus, (A.6) holds. ☐

A.2 Multiple stochastic integral

For $k \in \mathbb{Z}_+$, define the following Hilbert space:

$$\mathcal{L}_K^k = \Big\{ f; [0,1]^k \times \mathbb{R}^k \to \mathbb{R}; \quad \text{symmetric function, and} \\ \| f \|_{\mathcal{L}_K^k}^2 := \int_{[0,1]^k} \int_{\mathbb{R}^{2k}} f(\mathbf{t}, \mathbf{x}) \prod_{i=1}^k K(x_i - y_i) f(\mathbf{t}, \mathbf{y}) d\mathbf{t} d\mathbf{x} d\mathbf{y} < \infty \Big\},$$

where $\mathbf{t} = (t_1, t_2, \cdots, t_k)$, $\mathbf{x} = (x_1, x_2, \cdots, x_k)$, $\mathbf{y} = (y_1, y_2, \cdots, y_k)$. Let f be a function on $[0,1]^k \times \mathbb{R}^k$. The symmetrization of f is defined by

$$\mathrm{Sym}(f)(\mathbf{t}, \mathbf{x}) = \frac{1}{k!} \sum_{\pi \in \mathbf{S}_k} f(\pi \mathbf{t}, \pi \mathbf{x})$$

where $\mathbf{S}_k$ is the group of permutations on $\{1, 2, \cdots, k\}$.

For $f_1, f_2, \cdots, f_k \in \mathcal{L}_K$, $f_1 \otimes f_2 \otimes \cdots \otimes f_k = f_1(t_1, x_1) f_2(t_2, x_2) \cdots f_k(t_k, x_x)$ denotes the tensor product of $f_1, f_2, \cdots, f_k$. When $f_j = f$ for all $j = 1, \cdots, k$, abbreviate $f_1 \otimes f_2 \otimes \cdots \otimes f_k$ to $f^{\otimes k}$. We also denote by

$$\mathcal{W}^{\otimes k}(d\mathbf{t} d\mathbf{x}) = \mathcal{W}(dt_1 dx_1) \cdots \mathcal{W}(dt_k dx_k).$$

For $f \in \mathcal{L}_K$ with $\|f\|_{\mathcal{L}_K} = 1$, define the multiple stochastic integral of $f^{\otimes k}$ with respect to $\mathcal{W}$ by

$$I_k^{\mathcal{W}}(f) := \int_{[0,1]^k} \int_{\mathbb{R}^k} f^{\otimes k}(\mathbf{t}, \mathbf{x}) \mathcal{W}^{\otimes k}(d\mathbf{t} d\mathbf{x}) := \mathrm{H}_k(\mathcal{W}(f)),$$

where H_k is the Hermite polynomial of degree k, i.e., $\mathrm{H}_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}$. For $f_1, f_2, \cdots, f_k \in \mathcal{L}_K$, using the polarization identity (cf. (2.12) in [30]):

$$\mathrm{Sym}(f_1 \otimes f_2 \otimes \cdots \otimes f_k) = \frac{1}{2^n n!} \sum_{\boldsymbol{\epsilon} \in \{-1,1\}^k} \epsilon_1 \cdots \epsilon_k (\epsilon_1 f_1 + \cdots + \epsilon_k f_k)^{\otimes k},$$

the multiple stochastic integral of $\mathrm{Sym}(f_1 \otimes f_2 \otimes \cdots \otimes f_k)$ is defined by

$$I_n^{\mathcal{W}}\left(\mathrm{Sym}(f_1 \otimes f_2 \otimes \cdots \otimes f_k)\right) = \frac{1}{2^k k!} \sum_{\boldsymbol{\epsilon} \in \{-1,1\}^k} \epsilon_1 \cdots \epsilon_k I_k^{\mathcal{W}}\left((\epsilon_1 f_1 + \cdots + \epsilon_k f_k)^{\otimes k}\right).$$

Then we can extend to symmetric functions f in $\mathcal{L}_K^k$ by the density argument:

$$I_k^{\mathcal{W}}(f) := \int_{[0,1]^k} \int_{\mathbb{R}^k} f(\mathbf{t}, \mathbf{x}) \mathcal{W}^{\otimes k}(d\mathbf{t} d\mathbf{x}).$$

The multiple stochastic integral has the following property:

$$\mathrm{Cov}(I_j^{\mathcal{W}}(f), I_k^{\mathcal{W}}(g)) = \begin{cases} k! \langle f, g \rangle_{\mathcal{L}_K^k} & \text{if } j = k, \quad f, g \in \mathcal{L}_K^k \\ 0 & \text{if } j \neq k. \end{cases} \tag{A.7}$$

For general functions f with $\|f\|_{\mathcal{L}_K^k}^2 < \infty$, define

$$I_k^{\mathcal{W}}(f) := I_k^{\mathcal{W}}(\mathrm{Sym} f).$$

Remark A.2. Let $f : \Delta_k \times \mathbb{R}^k \to \mathbb{R}$, where $\Delta_k = \{0 = t_0 < t_1 < t_2 < \cdots < t_k \leq 1\}$, satisfy

$$\| f \|_{\mathcal{L}_K^k}^2 := \int_{\Delta_k} \int_{\mathbb{R}^{2k}} f(\mathbf{t}, \mathbf{x}) \prod_{i=1}^k K(x_i - y_i) f(\mathbf{t}, \mathbf{y}) d\mathbf{t} d\mathbf{x} d\mathbf{y} < \infty.$$

We extend f to a function on $[0,1]^k \times \mathbb{R}^k$ by defining $f(t,x)=0$ for $(\mathbf{t},\mathbf{x}) \notin \Delta_k$. Then

$$I_k^{\mathcal{W}}(f) = I_k^{\mathcal{W}}(\mathrm{Sym}f) = \int_{\Delta_k}\int_{\mathbb{R}^k} f(\mathbf{t},\mathbf{x})\mathcal{W}^{\otimes k}(d\mathbf{t}d\mathbf{x}).$$

and

$$\mathbf{E}\left(\int_{\Delta_k}\int_{\mathbb{R}^k} f(\mathbf{t},\mathbf{x})\mathcal{W}(d\mathbf{t}d\mathbf{x})\right)^2 = \|f\|^2_{\mathcal{L}_K^k} = k!\|\mathrm{Sym}f\|^2_{\mathcal{L}_K^k}. \tag{A.8}$$

Lemma A.2. Let $f_k(t,x) \in \mathcal{L}_K$, $k=1,\cdots,n$ be a collection of orthonormal functions in $\mathcal{L}_K$, $n \geq 2$, i.e., for any $k \neq j$, $\langle f_k, f_j\rangle_{\mathcal{L}_K} = 0$. Then

$$I_n^{\mathcal{W}}(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = \prod_{k=1}^n I_1^{\mathcal{W}}(f_k). \tag{A.9}$$

Proof. Firstly, by the orthonormality condition, we have that $I_1^{\mathcal{W}}(f_1),\cdots,I_1^{\mathcal{W}}(f_n)$ are independent normal random variables, $\mathbf{E}(I_1^{\mathcal{W}}(f_k))=0$, $\mathbf{E}((I_1^{\mathcal{W}}(f_k))^2) = \|f_k\|^2_{\mathcal{L}_K}$, $1 \leq k \leq n$, and

$$\|\epsilon_1 f_1 + \cdots + \epsilon_n f_n\|_{\mathcal{L}_K} = \left(\sum_{k=1}^n \|f_k\|^2_{\mathcal{L}_K}\right)^{1/2} := A_f, \quad \text{for all } \boldsymbol{\epsilon} \in \{-1,1\}^n$$

Then, by the definition of multiple stochastic integral, we have

$$\begin{aligned}
&I_n^{\mathcal{W}}(f_1 \otimes f_2 \otimes \cdots \otimes f_k)\\
=&\frac{(A_f)^n}{2^n n!}\sum_{\boldsymbol{\epsilon}\in\{-1,1\}^n} \epsilon_1\cdots\epsilon_n I_n^{\mathcal{W}}\left((\epsilon_1 f_1 + \cdots + \epsilon_n f_n)^{\otimes n}\right)\\
=&\frac{(A_f)^n}{2^n n!}\sum_{\boldsymbol{\epsilon}\in\{-1,1\}^n} \epsilon_1\cdots\epsilon_n H_n\left(I_1^{\mathcal{W}}\left(\frac{\epsilon_1 f_1 + \cdots + \epsilon_n f_n}{A_f}\right)\right)\\
=&\frac{(A_f)^n}{2^n n!}\sum_{\boldsymbol{\epsilon}\in\{-1,1\}^n} \epsilon_1\cdots\epsilon_n H_n\left(\frac{1}{A_f}\sum_{k=1}^n \epsilon_k I_1^{\mathcal{W}}(f_k)\right).
\end{aligned}$$

It is known that (cf. Theorem A.1 in [30])

$$H_n(x) = \sum_{j=0}^{[n/2]} \frac{n!(-1)^j x^{n-2j}}{2^j j!(n-2j)!},\ x \in \mathbb{R},\ n \geq 0.$$

Then

$$\begin{aligned}
&(A_f)^n \sum_{\boldsymbol{\epsilon}\in\{-1,1\}^n} \epsilon_1\cdots\epsilon_n H_n\left(\frac{1}{A_f}\sum_{k=1}^n \epsilon_k I_1^{\mathcal{W}}(f_k)\right)\\
=&\sum_{\boldsymbol{\epsilon}\in\{-1,1\}^n} \epsilon_1\cdots\epsilon_n\left(\sum_{k=1}^n \epsilon_k I_1^{\mathcal{W}}(f_k)\right)^n\\
&+\sum_{j=1}^{[n/2]}\frac{n!(-1)^j (A_f)^{2j}}{2^j j!(n-2j)!}\sum_{\boldsymbol{\epsilon}\in\{-1,1\}^n} \epsilon_1\cdots\epsilon_n\left(\sum_{k=1}^n \epsilon_k I_1^{\mathcal{W}}(f_k)\right)^{n-2j}.
\end{aligned}$$

For any $j \geq 1$, we consider the following $n(n-2j)$-order multilinear polynomial

$$f(x_1,\cdots,x_n) = \sum_{\boldsymbol{\epsilon}\in\{-1,1\}^n} \epsilon_1\cdots\epsilon_n\left(\sum_{k=1}^n \epsilon_k x_k\right)^{n-2j}.$$

If $f(x_1,\cdots,x_n)\not\equiv 0$, then the zeros of multilinear polynomial f have at most $n(n-2j)$. It is clear that $f(0,\cdots,0)=0$. For any $1\le k\le n$, set

$$B_{n,k}=\{(x_1,\cdots,x_n)\in\{0,1\}^n;\ x_1+\cdots+x_n=k\}.$$

Then when n is a even, for any odd $1\le k\le n$, and $(x_1,\cdots,x_n)\in B_{n,k}$, $f(x_1,\cdots,x_n)=0$, and so, the zeros of polynomial f have

$$1+\sum_{k=0}^{n/2-1}\frac{n!}{(2k+1)!(n-(2k+1))!}>n(n-2).$$

In fact, $n=2$, 4 or 6, the above inequality is obvious. When $n\ge 8$,

$$1+\sum_{k=0}^{n/2-1}\frac{n!}{(2k+1)!(n-(2k+1))!}\ge 1+2n+\frac{n(n-1)(n-2)}{6}\times 2>n(n-2).$$

Similarly, when n is a odd, for any even $1\le k\le n$, and $(x_1,\cdots,x_n)\in B_{n,k}$, $f(x_1,\cdots,x_n)=0$. In this case, the zeros of polynomial f have

$$1+\sum_{k=1}^{[n/2]}\frac{n!}{(2k)!(n-2k)!}>n(n-2).$$

Therefore, $f(x_1,\cdots,x_n)\equiv 0$. In particular,

$$\sum_{j=1}^{[n/2]}\frac{n!(-1)^j(A_f)^{2j}}{2^j j!(n-2j)!}\sum_{\epsilon\in\{-1,1\}^n}\epsilon_1\cdots\epsilon_n\left(\sum_{k=1}^{n}\epsilon_k I_1^{\mathcal{W}}(f_k)\right)^{n-2j}=0.$$

Thus

$$\begin{aligned}I_n^{\mathcal{W}}(f_1\otimes\cdots\otimes f_n)=&\frac{(A_f)^n}{2^n n!}\sum_{\epsilon\in\{-1,1\}^n}\epsilon_1\cdots\epsilon_n H_n\left(\frac{1}{A_f}\sum_{k=1}^{n}\epsilon_k I_1^{\mathcal{W}}(f_k)\right)\\ =&\frac{1}{2^n n!}\sum_{\epsilon\in\{-1,1\}^n}\epsilon_1\cdots\epsilon_n\left(\sum_{k=1}^{n}\epsilon_k I_1^{\mathcal{W}}(f_k)\right)^n=\prod_{k=1}^{n}I_1^{\mathcal{W}}(f_k).\qquad\square\end{aligned}$$

B Proof of Lemma 4.1 and the local central limit

Proof of Lemma 4.1. Without loss of generality, we assume $q=1$ by the transformation $S_1'=\frac{S_1-\ell}{q}$. Choose $\delta\in(0,1/4)$ such that for $|u|\le\delta$,

$$|\phi(u)-1|\le\frac{1}{2}$$

and

$$\log\phi(u)=-\nu|u|^\alpha+|u|^\alpha h(u)\le-\frac{1}{2}\nu|u|^\alpha.$$

Therefore, for all $|u|\le\delta n^{1/\alpha}$,

$$\left(\phi\left(un^{-1/\alpha}\right)\right)^n=e^{-\nu|u|^\alpha+|u|^\alpha h(un^{-1/\alpha})}\le e^{-\frac{1}{2}\nu|u|^\alpha}.\tag{B.1}$$

Noting $q=1$, by Theorem 1.4.2 in [32], there exists $\beta>0$ such that

$$|\phi(u)|\le e^{-\beta}\text{ for }\delta\le|u|\le\pi.$$

Thus, by the inversion formula, we can write

$$\begin{aligned} p(n,x) &= \frac{1}{2\pi n^{1/\alpha}} \int_{[-n^{1/\alpha}\pi, n^{1/\alpha}\pi]} \left(\phi\left(sn^{-1/\alpha}\right)\right)^n e^{-\iota x s n^{-1/\alpha}} ds \\ &= \ell(n,x) + \frac{1}{2\pi n^{1/\alpha}} \int_{|s|\le \delta n^{1/\alpha}} e^{-\nu|s|^\alpha + |s|^\alpha h(sn^{-1/\alpha})} e^{-\iota x s n^{-1/\alpha}} ds, \end{aligned}$$

where

$$\ell(n.x) = \frac{1}{2\pi n^{1/\alpha}} \int_{\delta n^{1/\alpha} < |s| \le \pi n^{1/\alpha}} \left(\phi\left(sn^{-1/\alpha}\right)\right)^n e^{-\iota x s n^{-1/\alpha}} ds.$$

Note that $|e^{\iota y} - 1| \le |y|$ for $y \in \mathbb{R}$. For any $x, y \in \mathbb{R}$,

$$\begin{aligned} &\frac{1}{2\pi n^{1/\alpha}} \int_{|s|\le \delta n^{1/\alpha}} e^{-\nu|s|^\alpha + |s|^\alpha h(sn^{-1/\alpha})} \left| e^{-\iota(x+y)s/n^{1/\alpha}} - e^{-\iota x s/n^{1/\alpha}} \right| ds \\ &\le \frac{1}{2\pi n^{1/\alpha}} \int_{|s|\le \delta n^{1/\alpha}} \frac{|s||y|}{n^{1/\alpha}} e^{-\frac{\nu}{2}|s|^\alpha} ds \\ &\le \frac{|y|}{2\pi n^{2/\alpha}} \int_{\mathbb{R}} e^{-\frac{\nu}{2}|s|^\alpha} |s| ds \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{2\pi n^{1/\alpha}} \int_{\delta n^{1/\alpha} < |s| \le \pi n^{1/\alpha}} \left(\phi\left(sn^{-1/\alpha}\right)\right)^n \left| e^{-\iota(x+y)sn^{-1/\alpha}} - e^{-\iota x s n^{-1/\alpha}} \right| ds \\ &\le \frac{|y|}{2\pi n^{2/\alpha}} e^{-\beta n} \int_{\delta n^{1/\alpha} < |s| \le \pi n^{1/\alpha}} |s| ds \\ &= \frac{|y|}{2\pi n^{1/\alpha}} e^{-\beta n} \pi n^{2/\alpha}. \end{aligned}$$

Therefore, (4.1) holds.

Next, let us show (4.2). we write

$$\begin{aligned} p(n+1,x) &= \frac{1}{2\pi n^{1/\alpha}} \int_{[-n^{1/\alpha}\pi, n^{1/\alpha}\pi]} \left(\phi\left(sn^{-1/\alpha}\right)\right)^{n+1} e^{-\iota x s n^{-1/\alpha}} ds \\ &= \widehat{\ell}(n,x) + \frac{1}{2\pi n^{1/\alpha}} \int_{|s|\le \delta n^{1/\alpha}} e^{\frac{n+1}{n}(-\nu|s|^\alpha + |s|^\alpha h(sn^{-1/\alpha}))} e^{-\iota x s/n^{1/\alpha}} ds, \end{aligned}$$

where

$$\widehat{\ell}(n,x) = \frac{1}{2\pi n^{1/\alpha}} \int_{\delta n^{1/\alpha} < |s| \le \pi n^{1/\alpha}} \left(\phi\left(sn^{-1/\alpha}\right)\right)^{n+1} e^{-\iota x s n^{-1/\alpha}} ds.$$

Noting that for any $\alpha > 0$, $\int_0^\infty e^{-s^\alpha} ds < \infty$, we have that

$$\sup_{x\in\mathbb{Z}} \max\{\widehat{\ell}(n,x), \ell(n,x)\} = O(e^{-\beta n}),$$

and

$$\begin{aligned} &|p(n,x) - p(n+1,x)| \\ &\le O(e^{-\beta n}) + \frac{1}{2\pi n^{1/\alpha}} \int_{|s|\le \delta n^{1/\alpha}} e^{-\frac{1}{2}\nu|s|^\alpha} \left| e^{\frac{1}{n}(-\nu|s|^\alpha + |s|^\alpha h(sn^{-1/\alpha}))} - 1 \right| ds \\ &\le O(e^{-\beta n}) + \frac{1}{2\pi n^{1+1/\alpha}} O(1) \int_{\mathbb{R}} e^{-\frac{1}{2}\nu|s|^\alpha} |s|^\alpha ds \\ &= O\left(\frac{1}{n^{1+1/\alpha}}\right). \end{aligned}$$

Therefore, (4.2) is valid. □

The same as the proof of Lemma 4.1 is also to give the following the Gnedenko local limit theorem (cf. Theorem 4.2.1 in [32]).

Lemma B.1 (Local limit theorem)**.** Let $\{S_n, n\geq 0\}$ be a symmetric random walk starting from the origin on $\mathbb{Z}$ and in the domain of normal attraction of a stable law of index $\alpha\in(0,2]$. Assume that the characteristic function $\phi(u)$ of S_1 satisfies (2.1). Then

$$\lim_{n\to\infty}\sup_{k\in\mathbb{T}} n^{1/\alpha}\left|\frac{1}{q}p(n,k)-g(n,k)\right|=0. \tag{B.2}$$

Proof. Without loss of generality, we assume $q=1$. Then choose $\delta\in(0,1/4)$ such that for $|u|\leq\delta$,

$$|\phi(u)-1|\leq\frac{1}{2},\ \ |\psi(u)-1|\leq\frac{1}{2}$$

where $\psi(u)=e^{-\nu|u|^\alpha}$, and

$$\log\phi(u)=-\nu|u|^\alpha+|u|^\alpha h(u)\leq-\frac{1}{2}\nu|u|^\alpha.$$

Choose $\beta>0$ such that

$$|\phi(u)|\leq e^{-\beta}\text{ and }|\psi(u)|\leq e^{-\beta}\text{ for }\delta\leq|u|\leq\pi.$$

Thus, by the inversion formula, we can write

$$p(n,k)=O(e^{-n\beta})+\frac{1}{2\pi n^{1/\alpha}}\int_{|s|\leq\delta n^{1/\alpha}}e^{-\nu|s|^\alpha+|s|^\alpha h(sn^{-1/\alpha})}e^{-\iota n^{-1/\alpha}ks}ds,$$

and

$$\begin{aligned}g(n,k)&=\frac{1}{2\pi n^{1/\alpha}}\int_{\mathbb{R}}\left(\psi\left(sn^{-1/\alpha}\right)\right)^n e^{-\iota n^{-1/\alpha}ks}ds\\&=O(e^{-nt\beta})+\frac{1}{2\pi n^{1/\alpha}}\int_{|s|\leq\delta n^{1/\alpha}}e^{-\nu|s|^\alpha}e^{-\iota n^{-1/\alpha}ks}ds.\end{aligned}$$

Therefore

$$\begin{aligned}&\sup_{k\in\mathbb{Z}} n^{1/\alpha}\left|p(n,k)-g(n,k)\right|\\&\leq O(e^{-nt\beta})+\frac{1}{2\pi}\int_{|s|\leq\delta n^{1/\alpha}}e^{-\nu|s|^\alpha}\left|e^{|s|^\alpha h(sn^{-1/\alpha})}-1\right|ds,\end{aligned}$$

which yields (B.2) by the dominated convergence. □

C Proof of Proposition 2.1

Proof of Proposition 2.1. We only give the proof for the case $\frac{1}{2}<r<1$. Without loss of generality we can assume the initial data is non-random because we could always take the conditional expectation given $\mathcal{F}_0$.

Let us first give a priori estimate. By (2.9), we have

$$\begin{aligned}&\mathbf{E}(|\mathcal{Z}(t,x)|^2)\\=&\left(\int_{\mathbb{R}}g(t,x-y)\mathcal{Z}_0(y)dy\right)^2\\&+\sigma^2\int_0^t\int_{\mathbb{R}^2}g(t-s,x-y)K(y-z)g(t-s,x-z)\mathbf{E}(\mathcal{Z}(s,y)\mathcal{Z}(s,z))dydzds.\end{aligned}$$

By Hölder inequality,

$$\mathbf{E}(\mathcal{Z}(s,y)\mathcal{Z}(s,z))\leq\mathbf{E}(|\mathcal{Z}(s,y)|^2)^{\frac{1}{2}}\mathbf{E}(|\mathcal{Z}(s,z)|^2)^{\frac{1}{2}}.$$

Thus, by Hardy-Littewood's inequality and Hölder's inequality, there exists a positive constant C such that

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}^2} g(t-s,x-y)K(y-z)g(t-s,x-z)\mathbf{E}(\mathcal{Z}(s,y)\mathcal{Z}(s,z))dydzds\\ \leq& C\int_0^t\int_{\mathbb{R}} g^{2r}(t-s,x-y)\mathbf{E}(|\mathcal{Z}(s,y)|^2)dyds.\end{aligned}$$

Then, there exists a positive constant C such that

$$\begin{aligned}\mathbf{E}(|\mathcal{Z}(t,x)|^2) \leq &\left(\int_{\mathbb{R}} g(t,x-y)\mathcal{Z}_0(y)dy\right)^2\\ &+ C\int_0^t\int_{\mathbb{R}} g^{2r}(t-s,x-y)\mathbf{E}(|\mathcal{Z}(s,y)|^2)dyds.\end{aligned} \tag{C.1}$$

Iterating the inequality, we obtain that

$$\mathbf{E}(|\mathcal{Z}(t,x)|^2) \leq \sum_{n=0}^{\infty} C^n \mathbf{E}(I_n(t,x)), \tag{C.2}$$

where

$$\begin{aligned}I_n(t,x) = &\int_{\Delta_n(t)}\int_{\mathbb{R}^n}\prod_{i=0}^{n-1} g^{2r}(t_i-t_{i+1},x_i-x_{i+1})\\ &\times\left(\int_{\mathbb{R}} g(t_n,x_n-y)\mathcal{Z}_0(y)dy\right)^2\prod_{i=1}^{n} dx_i dt_i.\end{aligned} \tag{C.3}$$

Noting that there exists a positive C_1 such that for any $n \geq 1$,

$$\prod_{i=0}^{n-1} g^{2r}(t_i-t_{i+1},x_i-x_{i+1}) \leq C_1^n \prod_{i=0}^{n-1}\left(\frac{1}{(t_i-t_{i+1})^{\frac{2r-1}{\alpha}}} g(t_i-t_{i+1},x_i-x_{i+1})\right),$$

we have

$$\begin{aligned}&\mathbb{E}(I_n(t,x))\\ \leq&\frac{C_1^n\Gamma^n\left(1-\frac{2r-1}{\alpha}\right)}{\Gamma\left(n\left(1-\frac{2r-1}{\alpha}\right)\right)} t^{\frac{(n-1)(\alpha+1-2r)}{\alpha}}\\ &\int_0^t (t-s)^{\frac{1-2r}{\alpha}}\int_{\mathbb{R}} g(t-s,x-z)\mathbb{E}\left(\left(\int_{\mathbb{R}} g(s,z-y)\mathcal{Z}_0(y)dy\right)^2\right)dzds.\end{aligned}$$

Therefore, there exists positive constant C' such that for any $t \in [0,T]$, $x \in \mathbb{R}$,

$$\begin{aligned}&\mathbf{E}(|\mathcal{Z}(t,x)|^2)\\ &\leq \sum_{n=0}^{\infty} C^n \mathbf{E}(I_n(t,x))\\ &\leq C'\mathbb{E}\left(\left(\int_{\mathbb{R}} g(t,x-y)\mathcal{Z}_0(y)dy\right)^2\right)\\ &\quad + C'\int_0^t (t-s)^{\frac{1-2r}{\alpha}}\int_{\mathbb{R}} g(t-s,x-z)\mathbb{E}\left(\left(\int_{\mathbb{R}} g(s,z-y)\mathcal{Z}_0(y)dy\right)^2\right)dzds.\end{aligned} \tag{C.4}$$

In particular, if $\mathcal{Z}_0 = \delta_0$, then for some constant $C \in (0,\infty)$,

$$\mathbf{E}\left(\int_{\mathbb{R}} g(s,z-y)\mathcal{Z}_0(y)dy\right)^2 = g^2(s,z)), \tag{C.5}$$

and

$$\int_0^t (t-s)^{\frac{1-2r}{\alpha}} \int_{\mathbb{R}} g(t-s,x-z) \left(\int_{\mathbb{R}} g(s,z-y)\mathcal{Z}_0(y)dy \right)^2 dzds \leq Cg(t,x). \tag{C.6}$$

Therefore, for some constant $C \in (0,\infty)$,

$$\mathbf{E}(|\mathcal{Z}(t,x)|^2) \leq Cg^2(t,x). \tag{C.7}$$

If $\mathcal{Z}_0$ is bounded, then

$$\sup_{t\in[0,1],x\in\mathbb{R}} \mathbf{E}(|\mathcal{Z}(t,x)|^2) < \infty. \tag{C.8}$$

Next, we prove the existence and the uniqueness. Let first show the uniqueness. Let $\mathcal{Z}_i(t,x)$, $i=1,2$, be the two mild solutions of (2.7) with initial data $\mathcal{Z}(0,\cdot)=\mathcal{Z}_0$. Set $\bar{\mathcal{Z}}(t,x) = \mathcal{Z}_1(t,x) - \mathcal{Z}_2(t,x)$. Then

$$\mathbf{E}(|\bar{\mathcal{Z}}(t,x)|^2) \leq C \int_0^t \int_{\mathbb{R}} g^{2r}(t-s,x-y)E(|\bar{\mathcal{Z}}(s,y)|^2)dyds.$$

Iterating the inequality, we obtain that for any $N \geq 1$,

$$\begin{aligned}\mathbf{E}(|\bar{\mathcal{Z}}(t,x)|^2) \leq & \frac{C_1^{N+1}\Gamma^{N+1}\left(1-\frac{2r-1}{\alpha}\right)}{\Gamma\left((N+1)\left(1-\frac{2r-1}{\alpha}\right)\right)} t^{\frac{N(\alpha+1-2r)}{\alpha}} \\ & \times \int_0^t (t-s)^{\frac{1-2r}{\alpha}} \int_{\mathbb{R}} g(t-s,x-z)\mathbf{E}(\bar{\mathcal{Z}}^2(s,z))dzds.\end{aligned} \tag{C.9}$$

Therefore, let $N\to\infty$, we obtain $\mathbf{E}(|\bar{\mathcal{Z}}(t,x)|^2) = 0$ for any $t\in[0,T]$ and $x\in\mathbb{R}$. The uniqueness is proved.

We use the Picard iteration to prove the existence. Let $\mathcal{Z}^0(t,x)=0$, for $n\geq 0$ define

$$\mathcal{Z}^{n+1}(t,x) = \int_{\mathbb{R}} g(t,x-y)\mathcal{Z}_0(y)dy + \sigma \int_0^t \int_{\mathbb{R}} g(t-s,x-y)\mathcal{Z}^n(s,y)\mathcal{W}(dsdy). \tag{C.10}$$

and

$$\bar{\mathcal{Z}}^n(t,x) = \mathcal{Z}^{n+1}(t,x) - \mathcal{Z}^n(t,x).$$

Then these processes are progressively measurable by construction, and

$$\bar{\mathcal{Z}}^{n+1}(t,x) = \sigma \int_0^t \int_{\mathbb{R}} g(t-s,x-y)\bar{\mathcal{Z}}^n(s,y)\mathcal{W}(dsdy).$$

Therefore, there exists positive constant C such that for any $t\in[0,T]$,

$$\mathbf{E}(|\bar{\mathcal{Z}}^{n+1}(t,x)|^2) \leq C \int_0^t \int_{\mathbb{R}} g^{2r}(t-s,x-y)E(|\bar{\mathcal{Z}}^n(s,y)|^2)dyds.$$

For $\mathcal{Z}_0=\delta_0$ case, set

$$f^n(t) = \sup_{x\in\mathbb{R},s\in[0,t]} s^{(2r-1)/\alpha}\mathbf{E}(|\bar{\mathcal{Z}}^n(s,x)|^2).$$

Then

$$\begin{aligned}f^1(t) = & \sup_{x\in\mathbb{R},s\in[0,t]} s^{(2r-1)/\alpha}\mathbf{E}\left(\left(\int_{\mathbb{R}} g(s,x-y)\mathcal{Z}^1(y)dy\right)^2\right) \\ \leq & C \sup_{x\in\mathbb{R},s\in[0,t]} s^{(2r-1)/\alpha}\int_0^s (s-u)^{(1-2r)/\alpha}u^{-1/\alpha}ds < \infty\end{aligned}$$

and

$$f^{n+1}(t) \le C_1 \int_0^t \frac{f^n(s)}{(t-s)^{(2r-1)/\alpha}} ds,$$

where C, C_1 are positive constants. Iterating the inequality, we have

$$f^{n+1}(t) \le C \int_0^t \int_0^s \frac{f^{n-1}(u)}{((t-s)(s-u))^{(2r-1)/\alpha}} duds \le C' \int_0^t f^{n-1}(u) du,$$

where C, C' are positive constants. Therefore, $f^n(t) \le \frac{(C't)^{n/2}}{(n/2)!}$, and

$$\mathcal{Z}(t,x) = \sum_{k=0}^{\infty} \bar{\mathcal{Z}}^n(t,x)$$

is progressively measurable. By

$$\sup_{x\in\mathbb{R}, s\in[0,t]} s^{(2r-1)/\alpha} \mathbf{E}(|\bar{\mathcal{Z}}(s,x)|^2) < \infty,$$

we have

$$\int_0^t \int_{\mathbb{R}} g(t-s, x-y) E(|\mathcal{Z}^n(s,y)|^2) dyds < \infty.$$

Thus, $\mathcal{Z}(t,x)$ solves the equation (2.7) with initial data $\mathcal{Z}(0,\cdot) = \mathcal{Z}_0$.

For $\mathcal{Z}_0$ is bounded case, set

$$f^n(t) = \sup_{x\in\mathbb{R}, s\in[0,t]} \mathbf{E}(|\bar{\mathcal{Z}}^n(s,x)|^2).$$

Then $f^1(t) < \infty$, and $f^{n+1}(t) \le C \int_0^t f^n(s) ds$. Therefore, $f^n(t) \le \frac{(C't)^n}{n!}$, and so, $\mathcal{Z}(t,x) = \sum_{k=0}^{\infty} \bar{\mathcal{Z}}^n(t,x)$ is progressively measurable and solves the equation (2.7) with initial data $\mathcal{Z}_0$. □

D Some moment estimates for an autoregressive integrated moving average model

In this section, we give some moment estimates for an autoregressive integrated moving average model. These moment estimates play an important role in Lemma 3.7.

Lemma D.1. Let $\frac{1}{2} < r < 1$ and the environment $\omega = \{\omega(i,x), (i,x) \in \mathbb{Z}_+ \times \mathbb{Z}\}$ satisfy (A.2). Then for any $k, j \ge 1$, there is positive constant $C_{k,j}$ such that

$$\left|\mathbf{E}(\omega^k(i,x)\omega^j(i,y)) - \mathbf{E}(\omega^k(i,x))\mathbf{E}(\omega^j(i,y))\right| \le C_{k,j}\gamma(x-y). \tag{D.1}$$

Proof. Let us recall the definition of ω as follows

$$\omega(i,x) = \sum_{y\in\mathbb{Z}} a_y \xi(i, x+y),$$

where $a_y \ge 0$, $a_y \sim c_r|y|^{-r}$, $c_r > 0$, and $\{\xi(i,x);\ i \in \mathbb{Z}_+, x \in \mathbb{Z}\}$ is a family of independent and identically distributed random variables with $\mathbf{E}(\xi(i,x)) = 0$, $\mathbf{E}(|\xi(i,x)|^2) = 1$, and $\mathbf{E}e^{\beta|\xi(i,x)|} < \infty$ for β sufficiently small. For $N, M \ge 1$, set

$$\mathbb{Z}_0^{M,N} := \{(u_1, \cdots, u_{M+N}) \in \mathbb{Z}^{M+N};\ \{u_m, 1 \le m \le M\} \cap \{u_{M+n}, 1 \le n \le N\} = \emptyset\},$$

and $\mathbb{Z}_1^{M,N} := \mathbb{Z}^{M+N} \setminus \mathbb{Z}_0^{M,N}$. Note that for $\boldsymbol{u} := (u_1, u_2, \cdots, u_k, u_{k+1}, \cdots, u_{k+j}) \in \mathbb{Z}_0^{k,j}$, $\prod_{m=1}^k \xi(i, x+u_m)$ is independent of $\prod_{n=1}^j \xi(i, x+u_{k+n})$. We can write that

$$\begin{aligned}
&\mathbf{E}\left(\omega^k(i,x)\omega^j(i,y)\right) \\
=&\mathbf{E}\left(\left(\sum_{u\in\mathbb{Z}} a_u \xi(1, x+u)\right)^k \left(\sum_{v\in\mathbb{Z}} a_v \xi(1, y+v)\right)^j\right) \\
=&\mathbf{E}\left(\left(\sum_{u\in\mathbb{Z}} a_u \xi(1, x+u)\right)^k \left(\sum_{v\in\mathbb{Z}} a_{x-y+v} \xi(1, x+v)\right)^j\right) \\
=&\sum_{\boldsymbol{u}\in\mathbb{Z}_0^{k,j}} \mathbf{E}\left(\prod_{m=1}^k \left(a_{u_m}\xi(1,x+u_m)\right)\right)\mathbf{E}\left(\prod_{n=1}^j \left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right) \\
&+ \sum_{\boldsymbol{u}\in\mathbb{Z}_1^{k,j}} \mathbf{E}\left(\prod_{m=1}^k \left(a_{u_m}\xi(1,x+u_m)\right)\prod_{n=1}^j \left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right).
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{\boldsymbol{u}\in\mathbb{Z}_0^{k,j}} \mathbf{E}\left(\prod_{m=1}^k \left(a_{u_m}\xi(1,x+u_m)\right)\right)\mathbf{E}\left(\prod_{n=1}^j \left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right) \\
=&\sum_{\boldsymbol{u}\in\mathbb{Z}^{k+j}} \mathbf{E}\left(\prod_{m=1}^k \left(a_{u_m}\xi(1,x+u_m)\right)\right)\mathbf{E}\left(\prod_{n=1}^j \left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right) \\
&- \sum_{\boldsymbol{u}\in\mathbb{Z}_1^{k,j}} \mathbf{E}\left(\prod_{m=1}^k \left(a_{u_m}\xi(1,x+u_m)\right)\right)\mathbf{E}\left(\prod_{n=1}^j \left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right) \\
=&\mathbf{E}(\omega^k(i,x))\mathbf{E}(\omega^j(i,y)) \\
&- \sum_{\boldsymbol{u}\in\mathbb{Z}_1^{k,j}} \mathbf{E}\left(\prod_{m=1}^k \left(a_{u_m}\xi(1,x+u_m)\right)\right)\mathbf{E}\left(\prod_{n=1}^j \left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right).
\end{aligned}$$

Next, let us estimate

$$\sum_{\boldsymbol{u}\in\mathbb{Z}_1^{k,j}} \mathbf{E}\left(\prod_{m=1}^k \left(a_{u_m}\xi(1,x+u_m)\right)\right)\mathbf{E}\left(\prod_{n=1}^j \left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right)$$

and

$$\sum_{\boldsymbol{u}\in\mathbb{Z}_1^{k,j}} \mathbf{E}\left(\prod_{m=1}^k \left(a_{u_m}\xi(1,x+u_m)\right)\prod_{n=1}^j \left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right).$$

For $\boldsymbol{u} \in \mathbb{Z}_1^{k,j}$, there exist $1 \le m_0 \le k$, $1 \le n_0 \le j$ such that $u_{m_0} = u_{k+n_0}$. For each such couple (u_{m_0}, u_{k+n_0}) fixed, let $N(\boldsymbol{u})$ denote the number of components in $\boldsymbol{u}$ that are not equal to u_{m_0}, and define

$$\mathbb{Z}_{1,l}^{k,j} = \mathbb{Z}_1^{k,j} \cap \{\boldsymbol{u};\ N(\boldsymbol{u}) = l\},\ l = 0, 1, \cdots, k+j-2.$$

Then

$$\sum_{\boldsymbol{u}\in\mathbb{Z}_1^{k,j},u_{m_0}=u_{k+n_0}}\mathbf{E}\left(\prod_{m=1}^{k}\left(a_{u_m}\xi(1,x+u_m)\right)\right)\mathbf{E}\left(\prod_{n=1}^{j}\left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right)$$
$$=\sum_{l=0}^{k+j-2}\sum_{\boldsymbol{u}\in\mathbb{Z}_{1,l}^{k,j}}\mathbf{E}\left(\prod_{m=1}^{k}\left(a_{u_m}\xi(1,x+u_m)\right)\right)\mathbf{E}\left(\prod_{n=1}^{j}\left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right).$$

Next, we show the following claim:

Claim A. For any $k\geq 1, j\geq 1$, there exists a positive constant $C_{k,j}$ such that for any $0\leq l\leq k+j-2$, any couple (u_{m_0},u_{k+n_0}) with $u_{m_0}=u_{k+n_0}$ for some $1\leq m_0\leq k$ and $1\leq n_0\leq j$,

$$\left|\sum_{\boldsymbol{u}\in\mathbb{Z}_{1,l}^{k,j}}\mathbf{E}\left(\prod_{m=1}^{k}\left(a_{u_m}\xi(1,x+u_m)\right)\right)\mathbf{E}\left(\prod_{n=1}^{j}\left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right)\right|\leq C_{k,j}\gamma(x-y). \tag{D.2}$$

When $l=0$,

$$\left|\sum_{\boldsymbol{u}\in\mathbb{Z}_{1,0}^{k,j}}\mathbf{E}\left(\prod_{m=1}^{k}\left(a_{u_m}\xi(1,x+u_m)\right)\right)\mathbf{E}\left(\prod_{n=1}^{j}\left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right)\right|$$
$$=\left|\sum_{u_{m_0}\in\mathbb{Z}}a_{u_{m_0}}^{k}a_{x-y+u_{m_0}}^{j}\mathbf{E}(\xi^k(1,x+u_{m_0}))\mathbf{E}(\xi^j(1,x+u_{m_0}))\right|\leq C\gamma(x-y),$$

where $C=L^{k+j-2}\mathbf{E}(\xi^k(1,1))\mathbf{E}(\xi^j(1,1))$, and $L=\sup_{x\in\mathbb{Z}}a_x$. That is, the Claim A holds for $l=0$.

If the Claim A holds for any $0\leq l\leq l_0\leq k+j-2$, i,e, for any $k\geq 1$ and $j\geq 1$, there exists a positive constant $C_{k,j}$ such that for all $0\leq l\leq l_0$, all couple (u_{m_0},u_{k+n_0}) with $u_{m_0}=u_{k+n_0}$ for some $1\leq m_0\leq k$ and $1\leq n_0\leq j$,

$$\left|\sum_{\boldsymbol{u}\in\mathbb{Z}_{1,l}^{k,j}}\mathbf{E}\left(\prod_{m=1}^{k}\left(a_{u_m}\xi(1,x+u_m)\right)\right)\mathbf{E}\left(\prod_{n=1}^{j}\left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right)\right|\leq C_{k,j}\gamma(x-y),$$

then for $l=l_0+1$, for any $\boldsymbol{u}\in\mathbb{Z}_{1,l}^{k,j}$, there exist $l_1\geq 1$, $1\leq m_s\leq k$, $s=1,\cdots,l_1$, or $1\leq m_s\leq j$, $s=1,\cdots,l_1$ such that

$$\mathbf{E}\left(\prod_{m=1}^{k}\left(a_{u_m}\xi(1,x+u_m)\right)\right)\mathbf{E}\left(\prod_{n=1}^{j}\left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right)$$
$$=a_{u_{m_1}}^{l_1}\mathbf{E}\left(\xi^{l_1}(1,1)\right)\mathbf{E}\left(\prod_{\substack{m\neq m_s\\ s=1,\cdots,l_1}}\left(a_{u_m}\xi(1,x+u_m)\right)\right)\mathbf{E}\left(\prod_{n=1}^{j}\left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right),$$

or

$$\mathbf{E}\left(\prod_{m=1}^{k}\left(a_{u_m}\xi(1,x+u_m)\right)\right)\mathbf{E}\left(\prod_{n=1}^{j}\left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right)$$
$$=a_{u_{m_1}}^{l_1}\mathbf{E}\left(\xi^{l_1}(1,1)\right)\mathbf{E}\left(\prod_{m=1}^{k}\left(a_{u_m}\xi(1,x+u_m)\right)\right)\mathbf{E}\left(\prod_{\substack{n\neq m_s\\ s=1,\cdots,l_1}}\left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right).$$

Therefore, noting $\mathbf{E}\left(\xi(1,1)\right)=0$, we have that

$$
\begin{aligned}
&\left|\sum_{\boldsymbol{u}\in\mathbb{Z}_{1,l}^{k,j}}\mathbf{E}\left(\prod_{m=1}^{k}\left(a_{u_m}\xi(1,x+u_m)\right)\right)\mathbf{E}\left(\prod_{n=1}^{j}\left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right)\right|\\
\le& C\sum_{l_1=2}^{\min\{k-1,l\}}\left|\sum_{\boldsymbol{u}\in\mathbb{Z}_{1,l-l_1}^{k-l_1,j}}\mathbf{E}\left(\prod_{m=1}^{k-l_1}\left(a_{u_m}\xi(1,x+u_m)\right)\right)\mathbf{E}\left(\prod_{n=1}^{j}\left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right)\right|\\
&+C\sum_{l_1=2}^{\min\{j-1,l\}}\left|\sum_{\boldsymbol{u}\in\mathbb{Z}_{1,l-l_1}^{k-l_1,j}}\mathbf{E}\left(\prod_{m=1}^{k}\left(a_{u_m}\xi(1,x+u_m)\right)\right)\mathbf{E}\left(\prod_{n=1}^{j-l_1}\left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right)\right|\\
\le& 2CC_{k,j}\gamma(x-y),
\end{aligned}
$$

where $C:=\sup_{2\le l_1\le k+j}\sum_{u\in\mathbb{Z}}a_u^{l_1}|\mathbf{E}\left(\xi^{l_1}(1,1)\right)|$, and the last inequality is due $l-l_1\le l_0$. That is, when $l=l_0+1$, the Claim A also holds. Thus, the Claim A is true, and so,

$$
\begin{aligned}
&\left|\sum_{\boldsymbol{u}\in\mathbb{Z}_{1}^{k,j}}\mathbf{E}\left(\prod_{m=1}^{k}\left(a_{u_m}\xi(1,x+u_m)\right)\right)\mathbf{E}\left(\prod_{n=1}^{j}\left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right)\right|\\
\le& kj\max_{\substack{1\le m_0\le k\\1\le n_0\le j}}\left|\sum_{\boldsymbol{u}\in\mathbb{Z}_{1}^{k,j},u_{m_0}=u_{k+n_0}}\mathbf{E}\left(\prod_{m=1}^{k}\left(a_{u_m}\xi(1,x+u_m)\right)\right)\mathbf{E}\left(\prod_{n=1}^{j}\left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right)\right|\\
=& kj(k+j-2)C_{k,j}\gamma(x-y).
\end{aligned}
$$

Similarly, we can also obtain that for some constant $C_{k,j}$,

$$
\left|\sum_{\boldsymbol{u}\in\mathbb{Z}_{1}^{k,j}}\mathbf{E}\left(\prod_{m=1}^{k}\left(a_{u_m}\xi(1,x+u_m)\right)\prod_{n=1}^{j}\left(a_{x-y+u_{k+n}}\xi(1,x+u_{k+n})\right)\right)\right|\le C_{k,j}\gamma(x-y).
$$

Now, we complete the proof of (D.1). □

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