Electronic Journal of Probability

Electron. J. Probab. **28** (2023), article no. 64, 1–81.
ISSN: 1083-6489 https://doi.org/10.1214/23-EJP954

Harnack inequalities and Gaussian estimates for random walks on metric measure spaces*

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Abstract

We characterize Gaussian estimates for transition probability of a discrete time Markov chain in terms of geometric properties of the underlying state space. In particular, we show that the following are equivalent: (1) Two sided Gaussian bounds on heat kernel (2) A scale invariant Parabolic Harnack inequality (3) Volume doubling property and a scale invariant Poincaré inequality. The underlying state space is a metric measure space, a setting that includes both manifolds and graphs as special cases. An important feature of our work is that our techniques are robust to small perturbations of the underlying space and the Markov kernel. In particular, we show the stability of the above properties under quasi-isometries. We discuss various applications and examples.

Keywords: random walks; Gaussian estimates; parabolic Harnack inequality.
MSC2020 subject classifications: 60J05; 58J65; 60J35; 58J35.
Submitted to EJP on February 17, 2021, final version accepted on May 2, 2023.

1 Introduction

The goal of this work is to characterize Gaussian estimates for Markov chains and parabolic Harnack inequality for a corresponding discrete time version of heat equation by two geometric properties on the state space: Large scale volume doubling property and Poincaré inequality. Our main result is that equivalence between the following properties:

(1) Two sided Gaussian bounds on heat kernel.

(2) Parabolic Harnack inequality.

(3) Volume doubling property and a scale invariant Poincaré inequality.

*Both the authors were partially supported by NSF grants DMS 1004771 and DMS 1404435. The first author was supported in part by NSERC (Canada) and the Canada research chairs program.
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The above result is due to Grigor'yan [30] and Saloff-Coste [67] for diffusion on Riemannian manifolds (or more generally, local Dirichlet spaces due to Sturm [74]) and extended to random walks on graphs by Delmotte [25]. Different versions of this result was shown for diffusions, random walks on graphs, and jump processes admitting for more general space-time scaling (space need not scale like square of time). A (partial) list of works in this direction are [11, 4, 5, 15, 35, 36, 75]. We emphasize that we only consider the standard space time scaling (quadratic in distance) and leave more general cases for future work (see Problem 10.10).

The hardest and most useful implication in the equivalence above is that the conjunction of the volume doubling property and Poincaré inequality implies the two sided Gaussian estimates and parabolic Harnack inequality. Also, an important consequence of this characterization is the stability of Gaussian estimates and parabolic Harnack inequality under quasi-isometric transformation of the underlying space. This result extends Moser's stability result of Harnack inequalities on $\mathbb{R}^n$ [58, 59]. Moser's work also provides an alternate approach to regularity of weak solutions to elliptic equations previously obtained by de Giorgi and Nash [22, 62]. Examples of spaces satisfying Gaussian heat kernel bounds include manifolds with non-negative Ricci curvature [16, 55, 13], convex domains and complement of convex domains [39], subelliptic operators on groups of polynomial growth [41, 77], weighted Euclidean spaces [34]. We refer to [71, Section 3.3] for a more extensive list of examples and further details. Recently, this characterization of parabolic Harnack inequality led to a similar characterization and stability of the elliptic Harnack inequality [11, 9, 6].

The goal of this work is to obtain the equivalence for random walks on metric spaces. A motivation comes from the work of Hebisch and Saloff-Coste [41] on random walks on groups. By the main results of [41], we know that many natural translation-invariant Markov chains on groups (discrete and continuous groups) of polynomial volume growth satisfy two-sided Gaussian estimates. However the arguments in [41] for proving Gaussian lower bounds are specific to the case of translation-invariant Markov chains as the authors of [41] note "We want to emphasize that a number of key points of the argument presented below are specific to the case of translation invariant Markov chains". To this end they conjecture "We have no doubt that, if G has polynomial volume growth a corresponding Gaussian lower bound holds for (non transition-invariant) Markov chains as well. However, we have not been able to prove this result. We hope to come back to this question in the future." [41, Remark 2]. Our work validates their conjecture.

1.1 Main results

Next, we state a version of our main result in a restricted setting. Recall that a weighted Riemannian manifold (M, g, μ) is a Riemannian manifold (M, g) equipped with a measure μ such that $\mu(dy) = \sigma(y)\nu(dy)$, where ν is the Riemannian measure and $\sigma \in \mathcal{C}^\infty(M)$ is the *weight function*. We denote integer intervals by $[\![a, b]\!] := \{i \in \mathbb{Z} : a \le i \le b\}$.

Theorem 1.1. *Let (M, g, μ) be a complete non-compact, weighted Riemannian manifold such that there exists $K \ge 0$ such that $\mathrm{Ric} \ge -Kg$. Furthermore there exists $C_1 \ge 1$ such that the weight function σ satisfies $C_1^{-1} \le \sigma(x)/\sigma(y) \le C_1$ for all $x, y \in M$ with $d(x, y) \le 1$. Consider a Markov chain on M with a symmetric kernel p_k with respect to μ. Further we assume that there exists $C_0 > 1, h > 0, h' \ge h$ such that*

$$C_0^{-1} \frac{\mathbf{1}_{B(x,h)}(y)}{V(x,h)} \le p_1(x,y) \le C_0 \frac{\mathbf{1}_{B(x,h')}(y)}{V(x,h')} \tag{1.1}$$

for all $x \in M$ and for μ-almost every $y \in M$. Then the following properties are equivalent:

(a) *The parabolic Harnack inequality: there exists* $\eta \in (0,1)$, $C_H > 1, R_H > 0$ *such that for all balls* $B(x,r)$, $x \in M$, $r > R_H$ *and for all non-negative functions* $u : \mathbb{N} \times M \to \mathbb{R}$ *that satisfies* $\partial_k u + \Delta u_k \equiv 0$ *in* $[\![0, \lfloor 4\eta^2 r^2 \rfloor]\!] \times B(x,r)$*, we have*

$$\sup_{Q_\ominus} u \le C_H \inf_{Q_\oplus} u$$

where

$$Q_\ominus = [\![\lceil (\eta^2/2) r^2 \rceil, \lfloor \eta^2 r^2 \rfloor]\!] \times B(x, (\eta/2) r),$$
$$Q_\oplus = [\![\lceil 2\eta^2 r^2 \rceil, \lfloor 4\eta^2 r^2 \rfloor]\!] \times B(x, (\eta/2) r)$$

(b) *Two sided Gaussian bounds on the heat kernel: there exists* $C_1, C_2 > 0$ *such that for all* $x, y \in M$ *and for all* $n \in \mathbb{N}^*$ *satisfying* $n \ge 2$*, we have*

$$p_n(x,y) \le \frac{C_1}{V(x,\sqrt{n})} \exp\left(-\frac{d(x,y)^2}{C_2 n}\right) \tag{1.2}$$

Further there exists $c_1, c_2, c_3 > 0$ *such that for all* $x, y \in M$ *satisfying* $d(x,y) \le c_3 n$ *and for all* $n \in \mathbb{N}^*$ *satisfying* $n \ge 2$

$$p_n(x,y) \ge \frac{c_1}{V(x,\sqrt{n})} \exp\left(-\frac{d(x,y)^2}{c_2 n}\right) \tag{1.3}$$

(c) *The conjunction of*

- *The volume doubling property: there exists* $C_D > 0$ *such that for all* $x \in M$*, for all* $r > 0$ *we have*
$$V(x, 2r) \le C_D V(x,r)$$
- *The Poincaré inequality: there exists* $C_P > 0$*,* $\kappa \ge 1$ *such that for any ball* $B = B(x,r)$*,* $x \in M$*,* $r > 1$ *and for all* $f \in L^2(M,\mu)$*, we have*

$$\int_B |f - f_B|^2 \, d\mu \le C_P r^2 \int_{\kappa B} \left(\frac{1}{V(y,1)} \int_{B(y,1)} |f(z) - f(y)|^2 \, \mu(dz) \right) \mu(dy), \tag{1.4}$$

where $\kappa B = B(x, \kappa r)$*,* $f_B = \frac{1}{\mu(B)} \int_B f \, d\mu$*.*

The role of Theorem 1.1 is only to illustrate our main result introducing additional definitions. We provide an unified approach to study random walks on both discrete and continuous spaces. We prove Theorem 1.1 as a corollary of a general result that also gives an alternate proof of Delmotte's theorem [25, Theroem 1.7] for random walks on graphs. We also provide a characterization of Gaussian upper bounds on heat kernel in Theorem 7.18.

Given the previous works on characterization of parabolic Harnack inequality and Gaussian bounds [30, 67, 74, 25, 42] our results should not be surprising. However we encounter new difficulties that had to be resolved here and which were not present in previous works. The Sobolev inequalities in the previous settings are of the form

$$\|f\|_2^{2\delta/(\delta-2)} \le \frac{Cr^2}{V_\mu(x,r)^{2/\delta}} \left(\mathcal{E}(f,f) + r^{-2} \|f\|_2^2 \right) \tag{1.5}$$

for all 'nice' functions f supported in $B(x,r)$. However for discrete time Markov chains, the Dirichlet form satisfies the inequality $\mathcal{E}(f,f) = \langle (I-P)f, f \rangle \le 2 \|f\|_2^2$. This along with (1.5) implies that $L^2(B(x,r)) \subseteq L^{2\delta/(\delta-2)}(B(x,r))$ for all balls $B(x,r)$ which can happen only if the space is discrete. Hence for discrete time Markov chains on Riemannian manifolds the Sobolev inequality (1.5) cannot possibly be true. We prove and rely on a weaker form of the Sobolev inequality (1.5) which seems to be too weak to run Moser's iteration directly to prove parabolic Harnack inequality (See Theorem 5.1). Instead we use Moser's iteration to prove a version of the mean value inequality which in turn gives Gaussian upper bounds. We adapt a method of [42] which uses elliptic Harnack inequality and Gaussian upper bounds to prove Gaussian lower bounds (See Section 8). Another difficulty that is new to our setting is explained in the beginning of Section 7.3.

1.2 Outline for the paper

This paper is organized as follows. In Section 2, we present the setting of quasi-geodesic spaces satisfying certain doubling hypotheses, study its basic properties and develop techniques that would let us compare discrete and continuous spaces. In Section 3, we introduce Poincaré inequalities and discuss various examples. We study the relationship between our new Poincaré inequalities on metric measure spaces with previous versions on manifolds on graphs. We study stability properties of Poincaré inequalities. In Section 4, we introduce hypotheses on Markov chain, Dirichlet forms and recall their basic properties. In Section 5, we introduce and prove a Sobolev inequality under the assumptions of large scale volume doubling and Poincaré inequality. In Section 6, we use Sobolev inequality and Poincaré inequality to run the Moser iteration argument to prove elliptic Harnack inequality.

Section 7 is devoted to the proof of Gaussian upper bounds using Sobolev inequality. In addition, we show that Sobolev inequality is equivalent to the conjunction of Gaussian upper bounds on the heat kernel and large scale volume doubling property. In Section 8 we prove Gaussian lower bounds using elliptic Harnack inequality and Gaussian upper bounds. This completes the proof that large scale volume doubling property and Poincaré inequality implies two sided Gaussian bound on the heat kernel. In Section 9, we prove parabolic Harnack inequality using Gaussian bounds. Moreover, we prove large scale volume doubling property and Poincaré inequality using parabolic Harnack inequality, and thereby completing the proof of the characterization parabolic Harnack inequality and Gaussian bounds. In Section 10, we mention some applications of Gaussian estimates and describe some examples. Many of the proofs that rely on small modifications to known methods have been omitted but an interested reader can find more complete proofs on arXiv [61].

2 Metric geometry

Let (M, d, μ) be a locally compact metric measure space where μ is a Radon measure with full support. Let $\mathcal{B}(M)$ denote the Borel σ-algebra on (M, d). Let $B(x, r) := \{y \in M : d(x, y) \le r\}$ denote the closed ball in M for metric d with center x and radius $r > 0$. Let $V(x, r) := \mu(B(x, r))$ denote the volume of the closed ball centered at x of radius r. Since M is a Radon measure with full support, we have that $V(x, r)$ is finite and positive for all $x \in M$ and for all $r > 0$. In this section, we introduce some assumptions on the metric d and measure μ and study some consequences of those assumptions.

2.1 Quasi-geodesic spaces

The main assumption on the metric d of the metric measure space (M, d, μ) is that of quasi-geodesicity. In Riemannian geometry, the distance between two points of a manifold is defined as the infimum of lengths of curves joining them. Such a relation between distance and length of curves is observed more generally in length spaces.

Definition 2.1. *Let (M, d) be a metric space. For $x, y \in M$, a* path *from x to y is a continuous map $\gamma : [0, 1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$. We define the* length

$L(\gamma) \in [0, \infty]$ *of a path* γ *is the supremum*

$$L(\gamma) = \sup_{P[0,1]} \sum_i d(\gamma(t_{i-1}), \gamma(t_i)).$$

taken over all partition $0 = t_0 < t_1 < \ldots < t_k = 1$ *of* $[0, 1]$.

The length of a path is a non-negative real number or $+\infty$.

Definition 2.2. *The* inner metric *or* length metric *associated with metric space* (M, d) *is the function* $d_i : M \times M \to [0, \infty]$ *defined by*

$$d_i(x, y) = \inf_\gamma L(\gamma)$$

where the infimum is taken over all paths γ *from* x *to* y. (M, d) *is called a* length space *if* $d_i = d$. *A metric for which* $d = d_i$ *is called an intrinsic metric.*

One of the goals of this work is to provide an unified approach to the study of random walks in continuous spaces like Riemannian manifolds and discrete spaces like graphs. We would like to consider spaces more general than length spaces to include disconnected metric spaces like graphs. Quasi-geodesic spaces provides a natural setting to include both length spaces and graphs as special cases. Quasi-geodesic spaces are equipped with a weak notion of geodesics called chains. We recall the following definition of chain and various notions of geodesicity as presented by Tessera in [76].

Definition 2.3. *Consider a metric space* (M, d) *and* $b > 0$. *For* $x, y \in M$, *a* b-chain *between from* x *to* y, *is a sequence* $\gamma : x = x_0, x_2, \ldots, x_m = y$ *in* M *such that for every* $0 \le i < m$, $d(x_i, x_{i+1}) \le b$. *We define the* length $l(\gamma)$ *of a* b*-chain* $\gamma : x_0, x_1, \ldots, x_m$ *by setting*

$$l(\gamma) = \sum_{i=0}^{m-1} d(x_i, x_{i+1}).$$

Define a new distance function $d_b : M \times M \to [0, \infty]$ *as*

$$d_b(x, y) = \inf_\gamma l(\gamma) \tag{2.1}$$

where γ *runs over every* b*-chain from* x *to* y. *We say a metric space* (M, d) *is*

- b*-geodesic if* $d(x, y) = d_b(x, y)$ *for all* $x, y \in M$.
- *quasi-*b*-geodesic if there exists* $C > 0$ *such that* $d_b(x, y) \le C d(x, y)$ *for all* $x, y \in M$.
- *quasi-geodesic if there exists* $b > 0$ *such that* (M, d) *is quasi-*b*-geodesic.*

The following lemma guarantees that quasi-geodesic spaces are endowed with sufficiently short chains at many length scales. The proof follows easily from the definition of quasi-geodesic space.

Lemma 2.4 (Chain lemma)**.** *Let* (M, d) *be a quasi-*b*-geodesic space for some* $b > 0$. *Then there exists* $C_1 \ge 1$ *such that for all* $b_1 \ge b$ *and for all* $x, y \in M$, *there exists a* b_1*-chain* $x = x_0, x_1, \ldots, x_m = y$ *with* $m \le \left\lceil \frac{C_1 d(x,y)}{b_1} \right\rceil$.

2.2 Doubling hypotheses

The main assumption that we recall below on the Radon measure μ is the doubling property. For a metric measure space (M, d, μ), we denote volume of balls by $V(x, r) = \mu(B(x, r))$.

Definition 2.5. *We define the following doubling hypothesis:*

$(VD)_{\mathrm{loc}}$ *We say a space* (M, d, μ) *satisfies the local volume doubling property* $(VD)_{\mathrm{loc}}$*, if for all* $r > 0$*, there exists* C_r *such that*

$$V(x, 2r) \le C_r V(x, r) \qquad (VD)_{\mathrm{loc}}$$

for all $x \in M$.

$(VD)_\infty$ *We say a space* (M, d, μ) *satisfies the large scale doubling property* $(VD)_\infty$*, if there exists positive reals* C_{r_0}, r_0 *such that*

$$V(x, 2r) \le C_{r_0} V(x, r) \qquad (VD)_\infty$$

for all $x \in M$ *and* $r \ge r_0$.

(VD) *We say a space* (M, d, μ) *satisfies the global volume doubling property* (VD)*, if there exists a constant* $C_D > 0$ *such that*

$$V(x, 2r) \le C_D V(x, r) \qquad (VD)$$

for all $x \in M$ *and* $r > 0$.

We collect some basic properties of spaces satisfying the above doubling hypothesis $(VD)_{\mathrm{loc}}$ and $(VD)_\infty$.

Lemma 2.6 ([21, Lemma 2.1])**.** *If* (M, d, μ) *satisfies* $(VD)_{\mathrm{loc}}$*, then for all* $r_1, r_2 > 0$*, there exists* C_{r_1,r_2} *such that*

$$V(x, r_2) \le C_{r_1,r_2} V(x, r_1) \tag{2.2}$$

for all $x \in M$*. In particular, for all* $x, y \in M$*, such that* $d(x, y) \le R$*, we have*

$$V(x, r) \le C_{r,R+r} V(y, r)$$

The large scale doubling property $(VD)_\infty$ along with $(VD)_{\mathrm{loc}}$ implies a polynomial volume growth upper bound.

Lemma 2.7. *Let* (M, d, μ) *be a metric measure space satisfying* $(VD)_{\mathrm{loc}}$ *and* $(VD)_\infty$*. Then for all* $b > 0$*, there exists* $C_b > 0$ *such that*

$$V(x, 2r) \le C_b V(x, r) \tag{2.3}$$

for all $x \in M$ *and* $r \ge b$*. Moreover this* C_b *satisfies*

$$\frac{V(x, r)}{V(x, s)} \le C_b \left(\frac{r}{s}\right)^\delta \tag{2.4}$$

for all $x \in M$*, for all* $b \le s < r$ *and for all* $\delta \ge \log_2 C_b$*. Furthermore*

$$\frac{V(x, r)}{V(y, s)} \le C_b^2 \left(\frac{r}{s}\right)^\delta \tag{2.5}$$

holds for all $b \le s \le r$*, for all* $x \in M$*, for all* $y \in B(x, r)$ *and for all* $\delta \ge \log_2 C_b$.

The equation (2.4) implies a polynomial upper bound on the volume growth. In quasi-geodesic spaces, we can reverse the inequality (2.4) and obtain a polynomial lower bound for all radii small enough compared to the diameter. The property stated in following lemma is often called the reverse volume doubling property which follows from [44, Exercise 13.1].

Lemma 2.8 ([44, Exercise 13.1])**.** *Let* (M, d, μ) *be a quasi-*b*-geodesic space with the measure* μ *satisfying* $(VD)_{\mathrm{loc}}$ *and* $(VD)_\infty$*. Then there exists* $c, \gamma > 0$ *such that*

$$\frac{V(x, r)}{V(x, s)} \ge c \left(\frac{r}{s}\right)^\gamma \tag{2.6}$$

for all $x \in M$ *and for all* $b \le s \le r \le \mathrm{diam}(M)$*, where* $\mathrm{diam}(M) = \sup\{d(x, y) : x, y \in M\}$ *denotes the diameter of* (M, d, μ).

2.3 Quasi-isometry

One of the goals of this work is to develop arguments which are robust to small perturbations in the geometry of the underlying space; for example addition of few edges in a graph or small changes in the metric of a Riemannian manifold. We study properties that depends mainly on the large scale geometry of the underlying space. In this spirit, the concept of quasi-isometry was introduced by Kanai in [48] and in the more restricted setting of groups by Gromov in [37]. Informally, two metric spaces are quasi-isometric if they have the same large scale geometry. Here is a precise definition:

Definition 2.9. *A map $\phi : (M_1, d_1) \to (M_2, d_2)$, between metric spaces is called a quasi-isometry if the following conditions are satisfied:*

(i) There exists $a \geq 1$ and $b \geq 0$ such that

$$a^{-1}d_1(x_1, x_2) - b \leq d_2(\phi(x_1), \phi(x_2)) \leq a d_1(x_1, x_2) + b$$

for all $x_1, x_2 \in M_1$.

(ii) There exists $\epsilon > 0$, such that for all $y \in M_2$ there exists $x \in M_1$ such that $d_2(\phi(x), y) < \epsilon$.

We say metric spaces (M_1, d_1) and (M_2, d_2) are quasi-isometric if there exists a quasi-isometry $\phi : (M_1, d_1) \to (M_2, d_2)$.

We remark that a quasi-isometry is not necessarily a continuous, injective or surjective map. However, we can construct a *quasi-inverse* $\phi^- : (M_2, d_2) \to (M_1, d_1)$ as $\phi^-(y) = x$ where $x \in M_1$ is chosen so that $d_2(\phi(x), y) < \epsilon$ where ϵ is given by the above definition.

We now describe some well-known examples of quasi-isometry. The space $\mathbb{R}^d$ with Euclidean metric and $\mathbb{Z}^d$ with standard graph metric are quasi-isometric. Consider a finitely generated group Γ with a finite system of generator A. For an element $\gamma \neq 1$, let $|\gamma|_A$ denote the smallest positive integer k such that a product of k elements of $A \cup A^{-1}$, and put $|1|_A = 0$. This $|\cdot|_A$ is called the *word norm* of Γ and defines a *word metric* $d_A(\gamma_1, \gamma_2) = \left|\gamma_1^{-1}\gamma_2\right|_A$. In other words, d_A is the graph metric in the Cayley graph of Γ corresponding to the symmetric generating set $A \cup A^{-1}$. Assume two finite generating sets A and B of a group Γ which induces metric d_A and d_B respectively. Then (Γ, d_A) and (Γ, d_B) are quasi-isometric (See [65, Proposition 1.15]). Therefore every finitely generated group defines a unique word metric space up to quasi-isometry and we may often view a finitely generated group up as a metric space without explicitly specifying the generating set. A large class of examples of quasi-isometry is given by the Švarc-Milnor theorem. [65, Theorem 1.18].

Theorem 2.10 (Švarc-Milnor theorem)**.** *Suppose that (M, d) is a length space and Γ is a finitely generated group equipped with a word metric acting properly and cocompactly by isometries on M. Then Γ is quasi-isometric to (M, d). The map $\gamma \mapsto \gamma.x_0$ is a quasi-isometry for each fixed base point $x_0 \in M$.*

Note that the quasi-isometry between $\mathbb{Z}^d$ and $\mathbb{R}^d$ is a special case of Theorem 2.10. We will give a general construction of *net* which approximates a quasi-geodesic space using a graph with combinatorial metric in next subsection.

The notion of quasi-isometry was extended to metric measure spaces by Couhlon and Saloff-Coste in [21] which they called "isometry at infinity". Let (M_i, d_i, μ_i), $i = 1, 2$ be two metric measure spaces. Define

$$V_i(y, r) = \mu_i\left(\{z \in M_i : d_i(y, z) \leq r\}\right).$$

Definition 2.11. *A map $\phi : (M_1, d_1, \mu_1) \to (M_2, d_2, \mu_2)$, between metric measure spaces is called a* quasi-isometry *if the following conditions are satisfied:*

(i) $\phi : (M_1, d_1) \to (M_2, d_2)$ *is a quasi-isometry between metric spaces;*

(ii) *There exists a constant* $C > 0$ *such that*

$$C^{-1}V_1(x,1) \le V_2(\phi(x),1) \le CV_1(x,1)$$

for all $x \in M_1$*.*

We say metric measure spaces (M_1, d_1, μ_1) *and* (M_2, d_2, μ_2) *are quasi-isometric if there exists a quasi-isometry* $\phi : (M_1, d_1, \mu_1) \to (M_2, d_2, \mu_2)$*.*

The arguments in this work implies that the long term behavior of natural random walks depends mainly on the large scale geometry of the quasi-geodesic space. Other important examples of properties invariant under quasi-isometries are large scale doubling and Poincaré inequality. (See Proposition 2.14 and Proposition 3.16). We conclude this subsection by proving that the large scale doubling property is preserved by quasi-isometries for metric measure spaces satisfying $(VD)_{\rm loc}$. It is due to Couhlon and Saloff-Coste in [21]. We need the following definition:

Definition 2.12. *Let* (M, d) *be a metric space with* $X \subseteq M$ *and let* $R > 0$*. Then a subset* Y *of* X *is* R-separated *if* $d(y_1, y_2) > R$ *whenever* y_1 *and* y_2 *are distinct points of* Y*, and a* R*-separated subset* Y *of* X *is called maximal if it is maximal among all* R*-separated subsets of* X *with respect to the partial order of inclusion.*

The existence of maximal R-separated subsets follows from Zorn's lemma.

The following lemma compares volume of balls between quasi-isometric metric measure spaces.

Lemma 2.13 ([21, Proposition 2.2])**.** *Let* $\Phi : (M_1, d_1, \mu_1)$ *and* (M_2, d_2, μ_2) *be a quasi-isometry between metric measure spaces satisfying* $(VD)_{\rm loc}$*. Then for all* $h > 0$*, there exists* $C_h > 0$ *such that*

$$C_h^{-1}V_1(x, C_h^{-1}r) \le V_2(\Phi(x), r) \le C_hV_1(x, C_hr)$$

for all $x \in M_1$ *and for all* $r \ge h$*.*

Proof. We denote balls and volumes by B_i and V_i respectively for $i = 1, 2$. Let $R \ge h$ such that $aR - b = R' > 0$ where a, b is from Definition 2.9. Let Y be a maximal R-separated subset of $B(x, r)$. Thus $B(x, r) \subseteq \cup_{y\in Y} B(y, R)$. Hence

$$V_1(x, r) \le \sum_{y\in Y} V_1(y, R) \tag{2.7}$$

By Lemma 2.6 and Definition 2.11, we have

$$V_1(y, R) \le C_{1,R}V_1(y, 1) \le C_{1,R}CV_2(\Phi(y), 1). \tag{2.8}$$

for all $y \in Y$. The balls $\{B(y, R/2)\}_{y\in Y}$ are pairwise disjoint and hence the balls $B(\Phi(x_i), R'/2)$ are pairwise disjoint. By Lemma 2.6

$$V_2(\Phi(x_i), h) \le C_{h,R'}V_2(\Phi(x_i), R'/2) \tag{2.9}$$

Combining 2.7, 2.8 and 2.9

$$\begin{aligned} V_1(x, r) &\le \sum_{y\in Y} C_{1,R}CC_{1,R'}V_2(\Phi(y), R'/2) \\ &\le C_{1,R}CC_{1,R'}V_2(\Phi(x), ar + b + R'/2) \end{aligned} \tag{2.10}$$

The last step follows from the definition of quasi-isometry, triangle inequality and that $B(\Phi(x_i), R'/2)$ are pairwise disjoint. We can choose C_2 large enough so that, $ar + b +$

$R'/2 \le C_2 r$ for all $r \in [h, \infty)$. Hence by Lemma 2.7, we have the desired lower bound on V_2 for all $r \ge R$ and by Lemma 2.6 for all $r \ge h$. Similar argument applied to quasi-inverse Φ^{-1} yields

$$V_2(\Phi(x), r) \le C V_1(\Phi^{-1} \circ \Phi(x), Cr).$$

The conclusion follows from the fact that $d_1(\Phi^{-1} \circ \Phi(x), x)$ is bounded uniformly for all $x \in M_1$. □

For metric measure spaces satisfying $(VD)_{\mathrm{loc}}$, the condition $(VD)_\infty$ is preserved by quasi-isometries. This is the content of the following lemma.

Proposition 2.14 ([21, Proposition 2.3])**.** *Let (M_1, d_1, μ_1) and (M_2, d_2, μ_2) be quasi-isometric spaces satisfying $(VD)_{\mathrm{loc}}$. Then (M_1, d_1, μ_1) satisfies $(VD)_\infty$ if and only if (M_2, d_2, μ_2) satisfies $(VD)_\infty$.*

Proof. Let $\Phi : M_2 \to M_1$ be a quasi-isometry. Using Lemma 2.13, there exists $C > 0$ such that

$$C^{-1} V_2(x, C^{-1} r) \le V_1(\Phi(x), r) \le C V_2(x, Cr)$$

for all $x \in M_2$ and $r \ge 1$. Hence by (2.4), we have

$$\frac{V_2(x, 2r)}{V_2(x, r)} \le C^2 \frac{V_1(\Phi(x), 2Cr)}{V_1(\Phi(x), C^{-1} r)} \le C^2 C_D (2C^2)^\delta$$

for all $r \ge \max(C, 1)$. □

2.4 Approximating quasi-geodesic spaces by graphs

One might think of $\mathbb{Z}^d$ as a graph approximation or discretization of $\mathbb{R}^d$. More generally, we can approximate quasi-geodesic spaces by graphs. By [76, Proposition 6.2], a metric space is quasi-isometric to a graph if and only if it is quasi-geodesic. Therefore quasi-geodesic spaces form a natural class of metric spaces that can be roughly approximated by graphs.

We begin by recalling some standard definitions and notation from graph theory. We restrict ourselves to simple graphs. A *graph* G is a pair $G = (V, E)$ where V is a set (finite or infinite) called the *vertices* of G and E is a subset of $\mathcal{P}_2(V)$ (*i.e.*,two-element subsets of V) called the *edges* of G. A graph (V, E) is *countable* (resp. *infinite*) if V is a countable (resp. infinite) set. We say that p is a neighbor of q (or in short $p \sim q$), if $\{x, y\} \in E$. The *degree* of p is the number of neighbors of p, that is $\deg(p) = |\{q \in V : p \sim q\}|$. A graph (V, E) is said to have *bounded degree* if $\sup_{v \in V} \deg(v) < \infty$.

A finite sequence $(p_0, p_1, \ldots, p_l)$ of points in V is called a *path* from p_0 to p_l of *length* l, if each p_k is a neighbor of p_{k-1}. A graph $G = (V, E)$ is said to be *connected* if for all $p, q \in V$, there exists a path from p to q. For points $p, q \in V$ of a graph $G = (V, E)$, let $d_G(p, q)$ denote the minimum of the lengths of paths from p to q with $d_G(p, q) = +\infty$ if there exists no path from p to q. This makes (V, d_G) an extended metric space. The graph (V, E) is connected if and only if (V, d_G) is a metric space. The extended metric d_G is called *graph metric* or *combinatorial metric* of G. Notice that we can recover a graph (V, E) from its (extended) metric space structure (V, d_G) and vice-versa.

Using the above identification, we view a connected graph as a metric space. We would like to view a connected graph as a metric measure space. This motivates the definition of weighted graph. A *weight* $m : V \to (0, \infty)$ on a graph (V, E) is a positive function on V. With a slight abuse of notation, m induces a measure on V (also denoted by m) as

$$m(A) = \sum_{v \in A} m(v)$$

for each $A \subseteq V$. A *weighted graph* is a graph (V,E) endowed with a weight m. By the above, we will identify a weighted graph $G=(V,E)$ with weight m as a (possibly extended) metric measure space (V, d_G, m).

The definition of ϵ-net is due to Kanai in the setting of Riemannian manifolds (See [48]) and was extended in [21] for weighted Riemannian manifolds.

Definition 2.15. *A ϵ-net of a metric measure space (M,d,μ) is a weighted graph $G=(V,E)$ with weight m described as follows: The vertices V is a maximal ϵ-separated subset of M. The edges E are defined by $\{x,y\} \in E$ if and only if $0 < d(x,y) \le 3\epsilon$. The weight m is defined as $m(x) = \mu\left(B(x,\epsilon)\right)$. Let d_G denote the graph metric of G. We often alternatively view the ϵ-net as (extended) metric measure space (V, d_G, m) defined by the corresponding weighted graph.*

The above definition does not guarantee ϵ-net to be a connected graph. However it is connected and countable in many situations as described in the lemma below. We collect the basic properties of nets in Proposition 2.16 which builds on the ideas of [48], [50] and [21].

Proposition 2.16. *Let (M,d,μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\rm loc}$ and let $\epsilon \ge b$. Let $G=(X,E)$ be an ϵ-net of (M,d,μ) with weight m and let (X,d_G,m) denote the corresponding extended metric measure space. Then we have the following:*

(a) The collection of balls $I = \{B(x,\epsilon/2) : x \in X\}$ is pairwise disjoint and the collection $J=\{B(x,\epsilon): x\in X\}$ covers M where $B(.,.)$ denotes closed metric ball in (M,d).

(b) Bounded degree property: The graph (X,E) is of bounded degree, that is

$$\sup_{p\in X} \deg(p) < \infty.$$

(c) (X, d_G, m) satisfies $(VD)_{\rm loc}$.

(d) There exists $A>0$ such that

$$\frac{1}{3\epsilon} d(x,y) \le d_G(x,y) \le A d(x,y) + A \tag{2.11}$$

for all $x,y \in X$. Therefore G is a connected graph and (X,d_G,m) is a metric measure space.

(e) The metric measure spaces (M,d,μ) and (X,d_G,m) are quasi-isometric.

(f) X is a countable set. Moreover if $\mathrm{diameter}(M,d)=\infty$, then X is an infinite set.

(g) If (M,d,μ) satisfies $(VD)_\infty$, then so does (X,d_G,m).

(h) Finite overlap property: Define

$$N_p(\delta) = |\{x \in X : d(x,p) \le \delta\}|.$$

for each $\delta>0$ and $p\in M$. Then $\sup_{p\in M} N_p(\delta) < \infty$.

The bounded degree property is true for all weighted graphs (X,d,m) satisfying $(VD)_{\rm loc}$ as shown below.

Lemma 2.17. *Let (X,d,m) be a metric measure space satisfying $(VD)_{\rm loc}$ that corresponds to a weighted graph $G=(X,E)$ with weight m. Then G is of bounded degree and*

$$\sup_{x,y\in X: x\sim y} \frac{m(y)}{m(x)} = C_m < \infty \tag{2.12}$$

Proof. By $(VD)_{\rm loc}$, there exists $C>0$ such that

$$m(y) \le V(x,1) \le CV(x,1/2) = Cm(x)$$

for all $x,y\in X$ with $x\sim y$. The above inequality shows that $C_m \le C$ and $\sup_{x\in X}\deg(x) \le C^2$ □

3 Poincaré inequalities

Poincaré inequalities and its many variants are functional inequalities that have been extensively studied. Many results in classical theory of Sobolev spaces, Hölder regularity estimates for solutions of elliptic and parabolic partial differential equations, properties of harmonic functions, Harnack inequalities can be generalized to spaces satisfying volume doubling and a Poincaré inequality. See the introduction in [40] for a survey and references.

Definition 3.1. *We say that a complete weighted Riemannian manifold (M,g) with measure μ satisfies a Poincaré inequality $(P)_{Rm}$ if there exists $C_1>0$, $C_2\geq 1$ such that for all $f\in C^\infty(M)$, for all $x\in M$ and for all $r>0$,*

$$\int_{B(x,r)} \left|f(y)-f_{B(x,r)}\right|^2 \mu(dy) \leq C_1 r^2 \int_{B(x,C_2 r)} \left|\operatorname{grad} f(y)\right|^2 \mu(dy) \qquad (P)_{Rm}$$

where $f_{B(x,r)}$ denote the μ-average of f in $B(x,r)$

$$f_B = \frac{1}{V(x,r)} \int_{B(x,r)} f(y)\mu(dy),$$

and $|\operatorname{grad} f|$ denotes the Riemannian length of the gradient vector.

The above inequality is sometimes called a weak, local, scale-invariant or L^2 Poincaré inequality but we will refrain from using such adjectives. The word *local* means that we are interested in average and integrals *around some point* x. This is in contrast with *global* Poincaré inequality in which average and integrals are over the whole space M. The Poincaré inequality is *scale-invariant* or *uniform* to emphasize the fact the the constants C_1 and C_2 is independent of x or r. For $1\leq p<\infty$, we might replace $(P)_{Rm}$ with the L^p *Poincaré inequality*

$$\int_{B(x,r)} \left|f(y)-f_{B(x,r)}\right|^p \mu(dy) \leq C_1 r^p \int_{B(x,C_2 r)} \left|\operatorname{grad} f(y)\right|^p \mu(dy).$$

instead of L^2 version presented above. For spaces satisfying global doubling property, one can always take $C_2=1$ in $(P)_{Rm}$. This is due to D. Jerison by a Whitney decomposition argument [45] (see also [70, Section 5.3.2]). The Poincaré inequality with $C_2=1$ is called *strong* Poincaré inequality as opposed to the *weak* inequality $(P)_{Rm}$.

3.1 Gradient and Poincaré inequality at a given scale

To generalize the Poincaré inequality $(P)_{Rm}$ to metric measure spaces, we must find a suitable definition for "length of gradient". We will consider a class of random walks that spreads over different distances. Therefore we define length of gradient over different scales for a metric measure space. We use the following definition due to [76] for length of gradient at a scale h for a function $f: M\to\mathbb{R}$ with $f\in L^\infty(M,\mu)$ (denoted by $|\nabla f|_h$).

Definition 3.2. *Let (M,d,μ) be a metric measure space and let $f\in L^\infty_{\mathrm{loc}}(M,\mu)$. Then length of gradient at a scale h for a function f is defined as*

$$|\nabla f|_h(x) = \left(\frac{1}{V(x,h)}\int_{B(x,h)} |f(y)-f(x)|^2 \mu(dy)\right)^{1/2}. \qquad (3.1)$$

for all $x\in M$.

Remark 3.3. Our definition of $|\nabla f|_h$ coincides with $|\nabla f|_{h,2}$ in the notation of Tessera [76].

Now that we are armed with length of gradient, we define the corresponding Poincaré inequality.

Definition 3.4. *We say that a metric measure space (M, d, μ) satisfies a Poincaré inequality at scale h, if there exists $C_1 > 0$, $C_2 \geq 1, r_0 > 0$ such that for all $f \in L^\infty_{\mathrm{loc}}(M, \mu)$, for all $x \in M$ and for all $r \geq r_0$.*

$$\int_{B(x,r)} \left|f(y) - f_{B(x,r)}\right|^2 \mu(dy) \leq C_1 r^2 \int_{B(x,C_2 r)} \left(|\nabla f|_h(y)\right)^2 \mu(dy) \qquad (P)_h$$

where $f_{B(x,r)}$ denote the μ-average of f in $B(x,r)$

$$f_B = \frac{1}{V(x,r)} \int_{B(x,r)} f(y)\mu(dy).$$

We will denote the above inequality by $P_h(r_0, C_1, C_2)$ or simply (P_h).

The rest of the section is devoted to the study of various properties and examples of the above Poincaré inequality $(P)_h$. In particular, we will show that for a weighted Riemannian manifold the Poincaré inequality at scale h $(P)_h$, generalizes the Poincaré inequality $(P)_{Rm}$ under some mild hypothesis. One of the main results that we will see in this section is that Poincaré inequality $(P)_h$ is preserved under quasi-isometries.

The following simple fact will be frequently used in rest of this section. Let (M, d, μ) be a metric measure space and let $A \subset M$ with $0 < \mu(A) < \infty$. Then for every function $f \in L^\infty(A)$

$$\inf_{\alpha \in \mathbb{R}} \int_A |f(y) - \alpha|^2 \mu(dy) = \int_A |f(y) - f_A|^2 \mu(dy) \tag{3.2}$$

where f_A is the μ-average of f in A,

$$f_A = \frac{1}{\mu(A)} \int_A f \, d\mu.$$

In other words, mean minimizes squared error.

A Poincaré inequality at scale h implies a Poincaré inequality at all larger scales h' with $h' \geq h$.

Lemma 3.5. *Let (M, d, μ) be a metric measure space satisfying $(VD)_{\mathrm{loc}}$ and Poincaré inequality $(P)_h$ at scale h. Then for all $h' \geq h$, (M, d, μ) satisfies $(P)_{h'}$.*

Proof. Assume $P_h(r_0, C_1, C_2)$. Then for all functions $f \in L^\infty_{\mathrm{loc}}$ and for all balls $B(x,r)$ with $r \geq r_0$ and $x \in M$, we have

$$\begin{aligned}
&\int_{B(x,r)} \left|f - f_{B(x,r}\right|^2 d\mu \\
&\leq C_1 r^2 \int_{B(x,C_2 r)} \left|\nabla_h f\right|^2 d\mu \\
&= C_1 r^2 \int_{B(x,C_2 r)} \int_{B(x,C_2 r + h')} |f(y) - f(z)|^2 \frac{\mathbf{1}_{d(x,y)\leq h}}{V(y,h)} \, dz \, dy \\
&\leq C_{h,h'} C_1 r^2 \int_{B(x,C_2 r)} \int_{B(x,C_2 r + h')} |f(y) - f(z)|^2 \frac{\mathbf{1}_{d(x,y)\leq h'}}{V(y,h')} \, dz \, dy
\end{aligned}$$

which is $P_{h'}(r_0, C_1 C_{h,h'}, C_2)$. In the last line above, we used Lemma 2.6. □

Remark 3.6. A question now arises: At what scales h does a Poincaré inequality $(P)_h$ hold? We have a satisfactory answer for length spaces and graphs. If a graph satisfies Poincaré inequality at some scale, it satisfies Poincaré inequality at all scales $h \geq 1$ (See

Corollary 3.15). Moreover, a graph does not satisfy Poincaré inequality for scales smaller than 1 because the gradient at scales smaller than 1 is identically zero. If a length space satisfies Poincaré inequality at some scale, then it satisfies Poincaré inequality at all positive scales (See Corollary 3.17). We will see in Proposition 9.9 that if $(P)_h$ is satisfied at for some $h > 0$ then $(P)_h$ is true for all $h > b$. We analyze an example which is neither a graph nor a length space (See Example 3.22) to show that $h > b$ is the best possible bound.

We now show that the constant r_0 in $P_h(r_0, C_1, C_2)$ is flexible.

Lemma 3.7. *Assume the Poincaré inequality $P_h(r_0, C_1, C_2)$ holds for a metric measure space (M, d, μ). Then for every $r_1 > 0$ and there exists constants C_1', C_2' such that the Poincaré inequality $P_h(r_1, C_1', C_2')$ holds.*

Proof. The non-trivial case to check is $r_1 < r_0$. Assume $B(x, r)$ with $r_1 \le r < r_0$. Then for all functions $f \in L^\infty_{\mathrm{loc}}(M)$, by (3.2) we have

$$\begin{aligned}\int_{B(x,r)} \left|f - f_{B(x,r)}\right|^2 d\mu &\le \int_{B(x,r)} \left|f - f_{B(x,r_0)}\right|^2 d\mu \\ &\le \int_{B(x,r_0)} \left|f - f_{B(x,r_0)}\right|^2 d\mu.\end{aligned}$$

Combining the above inequality with $P_h(r_0, C_1, C_2)$ yields

$$\int_{B(x,r)} \left|f(y) - f_{B(x,r)}\right|^2 dy \le C_1 r_0^2 \int_{B(x,C_2 r_0)} |\nabla f|_h^2 \, d\mu.$$

Hence we can choose $C_1' = C_1(r_0/r_1)^2$ and $C_2' = C_2(r_0/r_1)$. □

3.2 Robustness under quasi-isometry

Since quasi-isometry between metric measure spaces satisfying $(VD)_{\mathrm{loc}}$ is an equivalence relation, we may expect that a quasi-isometry preserves certain invariants of such spaces. For instance, we saw in Proposition 2.14 that quasi-isometry preserves the large scale doubling property. In this section, we shall see that quasi-isometry preserves Poincaré inequality $(P)_h$. The approach for proving robustness of functional inequalities can traced back to the seminal works of Kanai [48, 49, 50] and further developments by Couhlon and Saloff-Coste [21].

The idea is to show that a functional inequality on the metric measure space is equivalent to a similar functional inequality on its net. Since quasi-isometry is an equivalence relation, it suffices to show that the functional inequality on graphs is preserved under quasi-isometries. To compare functional inequalities back and forth between a metric measure space and its net, we need to be able to transfer functions on metric measure space to functions on its net and vice-versa. We start by recalling those tools.

As before, let (M, d, μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\mathrm{loc}}$ and let (X, d_G, m) be its ϵ-net for some fixed $\epsilon \ge b$. By Proposition 2.16, we have that (X, d_G, m) is a metric measure space satisfying $(VD)_{\mathrm{loc}}$. Moreover the graph corresponding to (X, d_G, m) is connected, countable with bounded degree. Let $D_X = \sup_{x \in X} \deg(x) < \infty$ be the maximum degree. We will denote closed balls in (M, d, μ) and (X, d_G, μ) by B and B_G respectively. Similarly, we denote their corresponding volumes by V and V_G respectively.

Given a function $g \in L^\infty_{\mathrm{loc}}(M, \mu)$, we a define a function $\tilde{g} : X \to \mathbb{R}$ on its net as

$$\tilde{g}(x) = \frac{1}{V(x,\epsilon)} \int_{B(x,\epsilon)} g \, d\mu. \tag{3.3}$$

for all $x \in M$. Conversely, given a function $f : X \to \mathbb{R}$ on the net, we define $\hat{f} : M \to \mathbb{R}$ as

$$\hat{f} = \sum_{x \in X} f(x)\theta_x \tag{3.4}$$

where $\theta_x : M \to \mathbb{R}$ is defined by

$$\theta_x(p) = \frac{\mathbf{1}_{B(x,\epsilon)}(p)}{\sum_{y \in X} \mathbf{1}_{B(y,\epsilon)}(p)}. \tag{3.5}$$

The sum in (3.4) and denominator of (3.5) is a finite sum due to the finite overlap property of Proposition 2.16(h). Moreover, there exists a constant $c_X > 0$ such that $\{\theta_x\}_{x \in X}$ is a partition of unity ($\sum_{x \in X} \theta_x \equiv 1$) satisfying

$$c_X \mathbf{1}_{B(x,\epsilon)} \le \theta_x \le \mathbf{1}_{B(x,\epsilon)} \tag{3.6}$$

for all $x \in X$. The above properties of the partition of unity θ_x can be verified using Proposition 2.16.

We will now compare norms and gradients for the transfer of functions between metric measure space and its net. For a metric measure space (M, d, μ) and for all $f \in L^\infty_{\text{loc}}(A)$ where $A \subset M$, we denote by

$$\|f\|_{p,A} = \left(\int_A |f|^p \, d\mu \right)^{1/p}.$$

We adapt the same notation for its net by considering it as a metric measure space.

Definition 3.8. *For a function $f : X \to \mathbb{R}$ on a graph (X, E), we define the* discrete gradient *of f at x as*

$$\delta f(x) = \left(\sum_{y \sim x} |f(y) - f(x)|^2 \right)^{1/2}.$$

This definition of discrete gradient was used to define Poincaré inequality for graphs in [21]. We now show that our definition of $|\nabla f|_1$ is comparable to δf.

Lemma 3.9. *Let (X, d_G, m) be a weighted graph satisfying $(VD)_{\text{loc}}$. Then there exists $C > 0$ such that*

$$C^{-1} |\nabla f|_1(x) \le \delta f(x) \le C |\nabla f|_1(x)$$

for all functions $f : X \to \mathbb{R}$ and for all $x \in X$.

Proof. We write the gradient as

$$(|\nabla f|_1(x))^2 = \frac{1}{m(x) + \sum_{y \in X : y \sim x} m(y)} \sum_{y \in X : y \sim x} |f(y) - f(x)|^2 m(y).$$

The conclusion immediately follows from Lemma 2.17. □

Remark 3.10. Therefore our Poincaré inequality $(P)_1$ generalizes the Poincaré inequality for graphs considered by Delmotte [23, 25]. Using the above lemma, our definition of $(P)_1$ for graphs is equivalent to the L^2 version of (P) for graphs in [21].

The next lemma compares gradient of a function on net and with its metric measure space version.

Lemma 3.11. *Let (M,d,μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\mathrm{loc}}$ and let (X,d,m) be its ϵ-net for some $\epsilon \ge b$. For all $h>0$, there exists positive reals C, C' such that for all $x \in M$, for all $r \ge 1$, and for all functions $f : X \to \mathbb{R}$, we have*

$$\left\| \left| \nabla \hat{f} \right|_h \right\|_{2,B(x,r)}^2 \le C \left\| \delta f \right\|_{2,B_G(\bar{x},C'r)}^2$$

where $\bar{x} \in X$ is such that $d(x,\bar{x}) \le \epsilon$ and $\hat{f} : M \to \mathbb{R}$ is defined as in (3.4).

Proof. Using Lemma 2.6, Proposition 2.16 (a) and (2.11), there exists $C_1 > 0$ such that

$$\begin{aligned} &\int_{B(x,r)} \int_M \left| \hat{f}(y) - \hat{f}(z) \right|^2 \frac{1_{d(y,z)\le h}}{V(y,h)}\, dz\, dy \\ &\le \sum_{s\in B_G(\bar{x},C_1 r)} \int_{B(s,\epsilon)} \int_M \left| \hat{f}(y) - \hat{f}(z) \right|^2 \frac{1_{d(y,z)\le h}}{V(y,h)}\, dz\, dy \end{aligned} \tag{3.7}$$

for all $x \in M$ and $r \ge 1$. For all $s \in X$, $y \in B(s,\epsilon)$ and $z \in B(y,h)$, we have

$$\begin{aligned} \hat{f}(y) - \hat{f}(z) &= \sum_{t\in X} f(t)(\theta_t(y) - \theta_t(z)) = \sum_{t\in X} (f(t) - f(s))(\theta_t(y) - \theta_t(z)) \\ &= \sum_{t\in X, d(s,t)\le 2\epsilon+h} (f(t) - f(s))(\theta_t(y) - \theta_t(z)) \end{aligned}$$

For the last line, if $d(s,t) > 2\epsilon + h$, then by triangle inequality $d(t,y) > h + \epsilon$, $d(t,z) > \epsilon$ and therefore $\theta_t(y) = \theta_t(z) = 0$ whenever $d(s,t) > 2\epsilon + h$. Let $D_X < \infty$ be the maximum degree of the net from Proposition 2.16(b) and $n_0 = A(h + 2\epsilon) + A + h$ where A is from (2.11). Since $|B_G(s,n_0)| \le 2N^{n_0}$, we have

$$\left| \hat{f}(p_1) - \hat{f}(p_2) \right| \le 2 \sum_{t\in B_G(s,n_0)} |f(t) - f(s)| \le 4D_X^{n_0} \sup_{t\in B_G(s,n_0)} |f(t) - f(s)| \tag{3.8}$$

Let $p_0, p_1, \ldots, p_{d_G(s,t)}$ be a path from s to t. For all $t \in B_G(s,n_0)$, by Cauchy-Schwarz inequality we have

$$|f(t) - f(s)|^2 \le \left(\sum_{i=0}^{d_G(t,s)-1} (f(p_i) - f(p_{i+1})) \right)^2 \le n_0 \sum_{p\in B_G(s,n_0)} |\delta f(p)|^2. \tag{3.9}$$

Combining (3.7), (3.8) and (3.9)

$$\begin{aligned} \left\| \left| \nabla \hat{f} \right|_h \right\|_{2,B(x,r)}^2 &\le \sum_{s\in B_G(\bar{x},C_1 r)} 4N^{2n_0} n_0 \sum_{t\in B_G(s,n_0)} |\delta f(t)|^2 m(s) \\ &\le \sum_{s\in B_G(\bar{x},C_1 r)} C_m^{n_0} 4N^{2n_0} n_0 \sum_{t\in B_G(s,n_0)} |\delta f(t)|^2 m(t) \\ &\le 8C_m^{n_0} D_X^{3n_0} n_0 \sum_{s\in B_G(\bar{x},C_3 r)} |\delta f(t)|^2 m(t) \end{aligned}$$

for all $x \in M$ and all $r \ge 1$. The second line follows from Proposition 2.16 and the last line from $|B(t,n_0)| \le 2D_X^{n_0}$. ☐

The following proposition shows that Poincaré inequalities can be transferred between a metric measure space and its net.

Proposition 3.12. *Let (M,d,μ) be a b-quasi-geodesic space satisfying $(VD)_{\mathrm{loc}}$ and let (X,d,m) be its ϵ-net for some $\epsilon \ge b$. Then for all $h \ge 5\epsilon$, (X,d_G,m) satisfies $(P)_1$ if and only if (M,d,μ) satisfies $(P)_h$.*

Remark 3.13. In general, we do not know if the inequality $h \geq 5\epsilon$ in the above statement is required. We believe that $h > b$ is sufficient but we are unable to prove this.

Proof of Proposition 3.12. Suppose (X, d_G, m) satisfies $P_1(r_0, C_1', C_2')$. Let $g \in L^\infty_{\text{loc}}$ and let $\tilde{g} : X \to \mathbb{R}$ be defined as (3.3). Let $x \in M$ and $r \geq r_0$ be arbitrary. Let $\bar{x} \in X$ be such that $d(x, \bar{x}) \leq \epsilon$. There exists $C_1 > 0$ such that, we have

$$\begin{aligned}
&\int_{B(x,r)} \left|g(y) - g_{B(x,r)}\right|^2 dy \\
&\leq \int_{B(x,r)} |g(y) - \alpha|^2 dy \leq \sum_{p \in B_G(\bar{x}, C_1 r)} \int_{B(p,\epsilon)} |g(y) - \alpha|^2 dy \\
&\leq 2 \sum_{p \in B_G(\bar{x}, C_1 r)} \left(\int_{B(p,\epsilon)} |g(y) - \tilde{g}(p)|^2 dy + m(p)|\tilde{g}(p) - \alpha|^2 \right)
\end{aligned} \tag{3.10}$$

for all $\alpha \in \mathbb{R}$ and all functions g. The second line above follows from (3.2), Proposition 2.16 (a) and (2.11). The last line follows from $(a+b)^2 \leq 2(a^2 + b^2)$. The first term above is bounded using Jensen's inequality as

$$\int_{B(p,\epsilon)} |g(y) - \tilde{g}(p)|^2 dy \leq \frac{1}{V(p,\epsilon)} \int_{B(p,\epsilon)} \int_{B(p,\epsilon)} |g(y) - g(z)|^2 \, dz \, dy$$

Hence by Lemma 2.6, we have

$$\begin{aligned}
I_1 &= \sum_{p \in B_G(\bar{x}, C_1 r)} \int_{B(p,\epsilon)} |g(y) - \tilde{g}(p)|^2 dy \\
&\leq C_{\epsilon, 6\epsilon} \sum_{p \in B_G(\bar{x}, C_1 r)} \int_{B(p,\epsilon)} \int_{B(p,\epsilon)} |g(y) - g(z)|^2 \frac{1_{d(y,z) \leq 2\epsilon}}{V(y, 5\epsilon)} \, dz \, dy \\
&\leq C_2 \left\| |\nabla g|_{5\epsilon} \right\|^2_{2, B(x, C_3 r)}
\end{aligned} \tag{3.11}$$

for some C_2, C_3 large enough. We used Lemma 2.6 and triangle inequality in second line above and Proposition 2.16(h) and (2.11) in the last line. Choose $\alpha = \tilde{g}_{B_G(\bar{x}, C_1 r)}$ in (3.10), so as to apply $P_1(r_0, C_1', C_2')$ on (X, d, m) to bound the second term in (3.10) as

$$I_2 = \sum_{p \in B_G(\bar{x}, C_1 r)} m(p) |\tilde{g}(p) - \alpha|^2 \leq C_4 r^2 \left\| \delta \tilde{g} \right\|^2_{2, B_G(\bar{x}, C_5 r)} \tag{3.12}$$

For all $p, q \in X$ satisfying $p \sim q$, by Jensen's inequality and triangle inequality we have

$$\begin{aligned}
|\tilde{g}(p) - \tilde{g}(q)|^2 &\leq \frac{1}{m(p) m(q)} \int_{B(p,\epsilon)} \int_{B(q,\epsilon)} |g(y) - g(z)|^2 \, dz \, dy \\
&\leq \frac{1}{m(p) m(q)} \int_{B(p,\epsilon)} \int_{B(q,\epsilon)} |g(y) - g(z)|^2 1_{d(y,z) \leq 5\epsilon} \, dz \, dy
\end{aligned}$$

Hence for all $p \in X$,

$$\begin{aligned}
&|\delta \tilde{g}(p)|^2 m(p) \\
&\leq \sum_{q \in X, q \sim p} \frac{1}{V(q,\epsilon)} \int_{B(p,4\epsilon)} \int_{B(p,4\epsilon)} |g(y) - g(z)|^2 1_{d(y,z) \leq 5\epsilon} \, dz \, dy \\
&\leq C_{\epsilon, 9\epsilon} \sum_{q \in X, q \sim p} \int_{B(p,4\epsilon)} \int_{B(y,4\epsilon)} |g(y) - g(z)|^2 \frac{1_{d(y,z) \leq 5\epsilon}}{V(y, 5\epsilon)} \, dz \, dy \\
&\leq C_{\epsilon, 9\epsilon} D_X \int_{B(p,4\epsilon)} \int_{B(p,4\epsilon)} |g(y) - g(z)|^2 \frac{1_{d(y,z) \leq 5\epsilon}}{V(y, 5\epsilon)} \, dz \, dy
\end{aligned} \tag{3.13}$$

The third line follows from Lemma 2.6 and the last line from bounded degree property of Proposition 2.16(b). Combining (3.12), (3.13) along with (2.11) and Proposition 2.16(h) gives

$$I_2 \le C_6 r^2 \left\| |\nabla g|_{5\epsilon} \right\|^2_{2,B(x,C_7 r)} . \tag{3.14}$$

Combining (3.10), (3.11) and (3.14) yields Poincaré inequality $(P)_{5\epsilon}$ for (M,d,μ). By Lemma 3.5, we get $(P)_h$ for all $h \ge 5\epsilon$.

Conversely, suppose that (M,d,μ) satisfies $P_h(r_1, C_3', C_4')$ for some $h \ge 5\epsilon$. Let $f : X \to \mathbb{R}$ be an arbitrary function and define $\hat{f} : M \to \mathbb{R}$ as in (3.4). Denote $B_G(p,r)$ be an arbitrary ball in (X,d,m) where $r \ge r_1$. Then using (3.2), $(VD)_{\rm loc}$ and the inequality $(a+b)^2 \le 2(a^2+b^2)$ we have

$$\begin{aligned} &\sum_{q\in B_G(p,r)} \left|f(q) - f_{B_G(x,r)}\right|^2 m(q) \qquad (3.15)\\ &\le \sum_{q\in B_G(p,n)} |f(q)-\alpha|^2 m(q) \le C_{\epsilon/2} \sum_{q\in B_G(x,n)} \int_{B(q,\epsilon/2)} |f(q)-\alpha|^2 \, d\mu \\ &\le 2C_{\epsilon/2} \sum_{q\in B_G(p,n)} \int_{B(q,\epsilon/2)} \left(\left|f(q)-\hat{f}(y)\right|^2 + \left|\hat{f}(y)-\alpha\right|^2 \right) dy \end{aligned}$$

for all $\alpha \in \mathbb{R}$. Using Proposition 2.16(a) and (2.11),there exists positive reals C_8, C_{11}, C_{12} such that for all $r \ge \min(1, r_1/C_8)$ and all functions f, we have

$$\begin{aligned} J_2 = \sum_{q\in B_G(p,r)} \int_{B(q,\epsilon/2)} \left|\hat{f}(y)-\alpha\right|^2 dy &\le \int_{B(p,C_8 r)} \left|\hat{f}(y)-\alpha\right|^2 dy \\ &\le C_9 r^2 \left\| \left|\nabla \hat{f}\right|_h \right\|^2_{2,B(p,C_{10} r)} \le C_{11} r^2 \left\|\delta f\right\|^2_{2,B_G(p,C_{12} r)} . \end{aligned} \tag{3.16}$$

In the second step above, we fix $\alpha = \hat{f}_{B(p,C_2 r)}$ and apply Poincaré inequality $(P)_h$ and in the last line we apply Lemma 3.11. Let $q \in X$ and $y \in B(q,\epsilon/2)$. Since $\hat{f}(y) = \sum_{t\in X: d_G(t,q)\le 1} \theta_t(y) f(t)$, we have

$$\begin{aligned} \left|f(q)-\hat{f}(y)\right| &= \left| \sum_{t\in X: d(t,q)\le 1} \theta_t(y)(f(q)-f(t)) \right| \le \sum_{t\in X: d(t,q)\le 1} |(f(q)-f(t))| \\ &\le \delta f(q) \sqrt{D_X}. \end{aligned}$$

The last line follows from Cauchy-Schwarz inequality and maximum degree D_X from Proposition 2.16(b). Using this estimate, we have

$$\begin{aligned} J_1 &= \sum_{q\in B_G(p,r)} \int_{B(q,\epsilon/2)} \left|f(y)-\hat{f}(y)\right|^2 dy \le D_X C_{\epsilon/2} \sum_{y\in B_G(p,r)} |\delta f(q)|^2 m(q) \\ &\le D_X C_{\epsilon/2} \left\|\delta f\right\|^2_{2,B_G(p,r)} . \end{aligned} \tag{3.17}$$

Thus $(P)_1$ for (X,d,m) follows from (3.15), (3.16) and (3.17) along with Lemma 3.9. □

We now show that Poincaré inequality $(P)_1$ is preserved under quasi-isometry for graphs.

Let (X,d,m) be a weighted graph. Then for the closed balls in the graph, we have $B(x,r) = B(x,\lfloor r \rfloor)$. Hence by Lemmas 3.7 and 3.9, we have the following equivalent definition of $(P)_1$: A weighted graph (X,d,m) satisfies $(P)_1$, if there exists $C_1 > 0$, $C_2 \ge 1$ such that for all $f : X \to \mathbb{R}$, for all $x \in X$ and for all $n \in \mathbb{N}^*$.

$$\sum_{y\in B(x,n)} \left|f(y) - f_{B(x,n)}\right|^2 \mu(dy) \le C_1 n^2 \sum_{B(x,C_2 n)} |\delta f(y)|^2 m(y) \tag{3.18}$$

where $f_{B(x,n)}$ is the average of f in $B(x,n)$ with respect to measure m. We will use the alternate definition for the proposition below.

Proposition 3.14 ([21], Proposition 4.2)**.** *Consider weighted graphs (X_1, d_1, m_1) and (X_2, d_2, m_2) that satisfy $(VD)_{\rm loc}$ and are quasi-isometric. Then (X_1, d_1, m_1) satisfies $(P)_1$ if and only if (X_2, d_2, m_2) satisfies $(P)_1$.*

Corollary 3.15. *Let (X, d, m) be a weighted graph satisfying $(VD)_{\rm loc}$ and let $h \geq 1$. Then (X, d, m) satisfies $(P)_1$ if and only if (X, d, m) satisfies $(P)_h$.*

Proof. By Lemma 3.5, $(P)_1$ implies $(P)_h$.

Conversely, assume (X, d, m) satisfies $(P)_h$. Fix $k = \lfloor h \rfloor$. Since $|\nabla f|_h = |\nabla f|_k$ for all functions $f : X \to \mathbb{R}$, (X, d, m) satisfies $(P)_k$. k*-fuzz* of a weighted graph is defined as the weighted graph (X, d_k, m) where the edges are defined by $d_k(x, y) = 1$ if and only if $1 \leq d(x, y) \leq k$ for $x, y \in X$. It is straightforward to verify that the k-fuzz (X, d_k, m) satisfies $(VD)_{\rm loc}$ and is quasi-isometric to (X, d, m). Since (X, d, m) satisfies $(P)_k$, the k-fuzz (X, d_k, m) satisfies $(P)_1$. Hence by Proposition 3.14, (X, d, m) satisfies $(P)_1$. □

As outlined at the start, the robustness of Poincaré inequality on graphs in Proposition 3.14 can be transferred to arbitrary quasi-geodesic spaces using Proposition 3.12.

Proposition 3.16. *For $i = 1, 2$, let (M_i, d_i, μ_i) be quasi-b_i-geodesic spaces satisfying $(VD)_{\rm loc}$. Assume (M_1, d_1, μ_1) and (M_2, d_2, μ_2) are quasi-isometric. Let $h_1 \geq 5b_1$ and for all $h_2 \geq 5b_2$. Then (M_1, d_1, μ_1) satisfies $(P)_{h_1}$ if and only if (M_2, d_2, μ_2) satisfies $(P)_{h_2}$.*

Proof. It is a direct consequence of Propositions 3.12 and 3.14. □

The above Proposition along with the fact that length space is b-geodesic for all $b > 0$ gives the following.

Corollary 3.17. *Let (M, d, μ) be a length space satisfying $(VD)_{\rm loc}$. Then for every $h_1, h_2 > 0$, (M, d, μ) satisfies $(P)_{h_1}$ if and only if (M, d, μ) satisfies $(P)_{h_2}$.*

3.3 Poincaré inequalities in Riemannian manifolds

In this subsection, we see the relationship between various Poincaré inequalities on a weighted Riemannian manifold. We start by introducing some Poincaré inequalities from [21].

Definition 3.18. *We say that a complete weighted Riemannian manifold (M, g) with measure μ satisfies $(P)_\infty$ if there exists $r_0 > 0$, $C_1 > 0$, $C_2 \geq 1$ such that for all $f \in C^\infty(M)$, for all $x \in M$ and for all $r \geq r_0$, we have*

$$\int_{B(x,r)} \left|f(y) - f_{B(x,r)}\right|^2 \mu(dy) \leq C_{r_0} r^2 \int_{B(x,C_2 r)} \left|\operatorname{grad} f(y)\right|^2 \mu(dy) \qquad (P)_\infty$$

where $f_{B(x,r)}$ denote the average of f in $B(x,r)$ with respect to μ. We say that a complete weighted Riemannian manifold (M, g) with measure μ satisfies $(P)_{\rm loc}$ if there exists $C_1 > 0$, $C_2 \geq 1$ such that for all $f \in C^\infty(M)$, for all $x \in M$ and for all $r \geq 0$, we have

$$\int_{B(x,r)} \left|f(y) - f_{B(x,r)}\right|^2 \mu(dy) \leq C_r \int_{B(x,C_2 r)} \left|\operatorname{grad} f(y)\right|^2 \mu(dy) \qquad (P)_{\rm loc}$$

where $f_{B(x,r)}$ denote the average of f in $B(x,r)$ with respect to μ.

It is clear that $(P)_{Rm}$ implies $(P)_\infty$ and $(P)_{\rm loc}$. The inequality $(P)_{\rm loc}$ is a weak assumption. For instance, manifolds with a lower bound on Ricci curvature satisfy $(P)_{\rm loc}$. Inequality $(P)_\infty$ is a large scale version of $(P)_{Rm}$.

Proposition 3.19 ([21, Proposition 6.10]). *Let (M, g, μ) be a weighted Riemannian manifold satisfying $(VD)_{\mathrm{loc}}$ and $(P)_{\mathrm{loc}}$ and let (X, d, m) be its ϵ-net for some $\epsilon > 0$. Then (M, g) with measure μ satisfies $(P)_\infty$ if and only if (X, d, m) satisfies $(P)_1$.*

Propositions 3.19 and 3.12 along with Corollary 3.17 gives the following

Proposition 3.20. *Let (M, g, μ) be a weighted Riemannian manifold with Riemannian distance d. Denote by (X, d_G, m) be an ϵ-net of (M, d, μ) for some $\epsilon > 0$. Assume (M, d, μ) satisfies $(VD)_{\mathrm{loc}}$ and $(P)_{\mathrm{loc}}$. Then the following are equivalent:*

(a) (M, d, μ) *satisfies* $(P)_\infty$.
(b) (M, d, μ) *satisfies* $(P)_h$ *for some* $h > 0$.
(c) (M, d, μ) *satisfies* $(P)_h$ *for all* $h > 0$.
(d) (X, d_G, m) *satisfies* $(P)_1$.
(e) (X, d_G, m) *satisfies* $(P)_h$ *for some* $h \geq 1$.

3.4 Poincaré inequality: examples and Non-examples

A large class of examples for $(P)_h$ can be obtained from Proposition 3.16 and 3.20. For instance, Buser proved $(P)_{Rm}$ for Riemannian manifolds with non-negative Ricci curvature. Therefore by Proposition 3.20, Riemannian manifolds with non-negative Ricci curvature satisfy $(P)_h$ for all positive scales h. The following example is from [34].

Example 3.21. [Euclidean space with radial weights] Consider $\mathbb{R}^n$ with standard Riemannian metric g, Euclidean distance d and measure $d\mu_\alpha(x) = (1+|x|^2)^{\alpha/2}\, dx$. It is easy to verify that $(\mathbb{R}^n, d, \mu_\alpha)$ satisfies $(VD)_{\mathrm{loc}}$ and $(P)_{\mathrm{loc}}$. Moreover $(\mathbb{R}^n, d, \mu_\alpha)$ satisfies $(VD)_\infty$ if and only if $\alpha > -n$. If $n \geq 2$, then $(\mathbb{R}^n, d, \mu_\alpha)$ satisfies $(P)_\infty$ and therefore $(P)_h$ for all values of $\alpha \in \mathbb{R}$ and $h > 0$ (See Remark 3.13 in [34]). However, $(\mathbb{R}, d, \mu_\alpha)$ does not satisfy $(P)_\infty$ for $\alpha \geq 1$. It can be easily seen using the test function $f_\alpha(x) = \int_0^x (1+t^2)^{-\alpha/2}\, dt$. By [34, Theorem 7.1(i)], $(\mathbb{R}, d, \mu_\alpha)$ satisfies $(P)_\infty$ if $-1 < \alpha < 1$. Due to an unpublished result of Grigor'yan and Saloff-Coste, $(\mathbb{R}, d, \mu_\alpha)$ satisfies $(P)_\infty$ if and only if $\alpha < 1$.

Example 3.22. We describe an example of quasi-geodesic space which is neither a graph, nor a length space. Consider the 'Broken line' $BL \subset \mathbb{R}$

$$BL = \bigcup_{n \in \mathbb{Z}} [n - 1/4, n + 1/4]$$

It is quasi-b-geodesic if and only if $b \geq 1/2$. We equip it with the Euclidean distance d and restriction of Lebesgue measure μ on BL. $(P)_h$ is not true for (BL, d, μ) if $h \leq 1/2$. It can be seen using the test function $f : BL \to \mathbb{R}$ defined by $f(x) = (-1)^{\lfloor x+1/4 \rfloor}$. However, it can be shown that for (BL, d, μ) satisfies $(P)_h$ for all $h > 1/2$.

Example 3.23 (Hyperbolic space). Consider the Hyperbolic n-space $\mathbb{H}^n$ equipped with Riemannian distance d_H and Riemannian measure μ. $(\mathbb{H}^n, d_H, \mu)$ satisfies $(VD)_{\mathrm{loc}}$ and $(P)_{\mathrm{loc}}$. However $(\mathbb{H}^n, d, \mu)$ does not satisfy $(VD)_{\mathrm{loc}}$ because the volume of balls grows exponentially. Further $(\mathbb{H}^n, d_H, \mu)$ does not satisfy the Poincaré inequality $(P)_\infty$.

Another example in a similar spirit is the infinite d-regular tree $\mathbb{T}_d$ equipped with graph distance metric and counting measure. It is easy to very that if $d \geq 3$, $\mathbb{T}_d$ does not satisfy $(VD)_\infty$ and does not satisfy $(P)_h$ for all $h > 0$.

Examples 3.21 and 3.23 illustrates all the four possibilities that can occur with properties $(VD)_\infty$ and $(P)_\infty$. It is summarized in the table below.

4 Markov kernel, semigroup and Dirichlet forms

In this section, we consider Markov chains on metric measure space (M, d, μ). Let $\mathcal{B}$ denote the Borel σ-algebra on (M, d). Our work concerns long term behavior of a natural

Table 1: Examples of spaces in relation to the properties $(VD)_\infty$ and $(P)_\infty$.

$(VD)_\infty$	$(P)_\infty$	Examples
True	True	$(\mathbb{R}^n, d, \mu_\alpha)$ with $n \geq 2$ and $\alpha > -n$ or $n = 1$ and $\alpha \in (-1, 1)$
True	False	$(\mathbb{R}, d, \mu_\alpha)$ with $\alpha \geq 1$
False	True	$(\mathbb{R}^n, d, \mu_\alpha)$ with $\alpha \leq -n$
False	False	$(\mathbb{H}^n, d_H, \mu)$

family of Markov chains on the state space M. We will recall some standard definitions and facts about discrete time Markov chains.

A *Markov transition function* is a map $\mathcal{P} : M \times \mathcal{B} : [0, \infty)$ such that $x \mapsto (x, A)$ is $\mathcal{B}$-measurable function on M for all $A \in \mathcal{B}$ and $A \mapsto \mathcal{P}(x, A)$ is a probability measure on $(M, \mathcal{B})$ for all $x \in M$. A Markov transition function $\mathcal{P}$ on $(M, \mathcal{B})$ is *μ-symmetric* if

$$\int_M \int_M u_1(x) u_2(y) \mathcal{P}(x, dy) \mu(dx) = \int_M \int_M u_1(x) u_2(y) \mathcal{P}(x, dy) \mu(dx) \tag{4.1}$$

for all measurable functions $u_1, u_2 : M \to [0, \infty)$.

Remark 4.1. For the rest of this work, we assume that the our Markov transition function is μ-symmetric with respect to some measure μ.

Associated with a μ-symmetric Markov transition function $\mathcal{P}$ is a *Markov operator* P, which is a linear operator defined by

$$Pf(x) = \int_M f(y) \mathcal{P}(x, dy) \tag{4.2}$$

on the set of bounded measurable functions. The operator P extends as a contraction operator on $L^p(M) = L^p(M, \mu)$ for all $p \in [1, \infty]$. With a slight abuse of notation, we denote this extension again by $P : L^p(M) \to L^p(M)$ for each $1 \leq p \leq \infty$. Moreover P is positivity preserving, *i.e.* if $f \geq 0$ then $Pf \geq 0$.

The n-th iteration P^n of the operator P is just the operator associated with kernel $\mathcal{P}^n$ defined inductively by

$$\mathcal{P}^n(x, A) := \int_M \mathcal{P}^{n-1}(z, A) \mathcal{P}(x, dz)$$

for all $x \in M$, for all measurable sets $A \in \mathcal{B}$ and $\mathcal{P}^1 := \mathcal{P}$. We now have the *Markov semigroup* of linear operators $(P^n)_{n \in \mathbb{N}_0}$ where P^0 is the identity operator on $L^2(M)$. The Chapman-Kolmogorov equation is given by

$$\mathcal{P}^{m+n}(x, A) = \int_M \mathcal{P}^n(z, A) \mathcal{P}^m(x, dz) \tag{4.3}$$

for all $A \in \mathcal{B}$ and for all $m, n \in \mathbb{N}^*$. By Fubini's theorem, (4.3) implies the semigroup property $P^{m+n} f = P^m P^n f$ for all $m, n \in \mathbb{N}_0$ and $f \in L^1(M)$.

The operator $\Delta := I - P$ is the *Laplacian* which generates the *Dirichlet form*

$$\mathcal{E}(f, f) = \langle f, \Delta f \rangle_{L^2(M)} = \frac{1}{2} \int_M \int_M (f(x) - f(y))^2 \, \mathcal{P}(x, dy) \, \mu(dx).$$

on $L^2(M)$ with full domain $\mathcal{D}(\mathcal{E}) = L^2(M)$.

For every Markov transition function $\mathcal{P}$ on $(M, \mathcal{B})$ there exists a *Markov chain* $(X_n, \mathbb{P}_x)_{n \in \mathbb{N}_0, x \in M}$ on some path space $(\Omega, \mathcal{F})$ such that

$$\mathcal{P}(x, A) = \mathbb{P}_x(X_1 \in A).$$

(one can always choose the canonical path space $\Omega = M^{\otimes \mathbb{N}_0}, \mathcal{F} = \mathcal{B}^{\otimes \mathbb{N}_0}$ and $X_n(\omega) = \omega_n$ for $\omega = (\omega_0, \omega_1, \ldots)$.) The transition function $\mathcal{P}^n$ is then given by $\mathcal{P}^n(x, A) = \mathbb{P}_x(X_n \in A)$ and the operator P^n by $P^n f(x) = \mathbb{E}_x f(X_n)$. The μ-symmetry of $\mathcal{P}$ is equivalent to the μ-*reversibility* of the Markov chain:

$$\mathbb{P}_\mu(X_0 \in A, X_1 \in B) = \mathbb{P}_\mu(X_1 \in A, X_0 \in B)$$

where $\mathbb{P}_\mu$ is a measure (not necessarily a probability measure) defined by $\mathbb{P}_\mu(\cdot) := \int_M \mathbb{P}_x(\cdot)\mu(dx)$.

If $\mathcal{P}(x, \cdot) \ll \mu$ for all $x \in M$, we denote its kernel by $p : M \times M \to [0, \infty)$, that is

$$\mathcal{P}(x, A) = \int_A p(x, y)\mu(dy)$$

for all $x \in M$ and for all $A \in \mathcal{B}$. The kernel p is called a *Markov kernel* with respect to μ. The kernel $p(x, \cdot)$ is the Radon-Nikodym derivative of $\mathcal{P}(x, \cdot)$ with respect to μ, that is $\mathcal{P}(x, A) = \int_A p(x, y)\mu(dy)$ for all $x \in M$ and all $A \in \mathcal{B}$. The μ-symmetry of $\mathcal{P}$ implies symmetry of kernel, that is $p(x, y) = p(y, x)$ for all $\mu \times \mu$ almost every $(x, y) \in M \times M$. By definition, we have $p(x, \cdot) \in L^1(M, \mu)$ for all $x \in M$. However, we assume that $p(x, \cdot) \in L^\infty(M, \mu)$ for all $x \in M$. Under the assumption $p(x, \cdot) \in L^1 \cap L^\infty$, we define iteratively

$$p_{k+1}(x, y) := [P p_k(x, \cdot)](y) = \int_M p_k(x, z) p_1(y, z)\mu(dz) \tag{4.4}$$

where $p_1 := p$ and $k \in \mathbb{N}^*$. The function p_k for $k \in \mathbb{N}^*$ is called the *heat kernel*. The following basic properties of heat kernel defined in (4.4) are easy to verify.

Lemma 4.2. *Let (M, d, μ) be a metric measure space and let $\mathcal{P}$ be a μ-symmetric Markov transition function satisfying $\mathcal{P}(x, \cdot) \ll \mu$ for all $x \in M$. Let $p_1(x, \cdot) = \frac{d\mathcal{P}(x, \cdot)}{d\mu}$ denote the corresponding Markov kernel. Assume further that $p_1(x, \cdot) \in L^\infty(M, \mu)$ for all $x \in M$. The the kernel p_k defined in* (4.4) *satisfies*

- *(a)* $p_k(x, \cdot) = \frac{d\mathcal{P}^k(x, \cdot)}{d\mu}$ *for all $k \in \mathbb{N}^*$. That is $\mathcal{P}^k(x, A) = \int_A p_k(x, z)\mu(dz)$ for all $x \in M$, for all $k \in \mathbb{N}^*$ and for all $A \in \mathcal{B}$.*
- *(b)* $p_k(x, y) = p_k(y, x) \in [0, \infty)$ *for all $x, y \in M$ and for all $k \geq 2$.*
- *(c)* $p_{k+l}(x, y) = P^k(p_l(x, \cdot))(y)$ *for all $x, y \in M$ and for all $k, l \in \mathbb{N}^*$.*

The following natural example falls under our framework.

Example 4.3. Let (M, d, μ) satisfy $(VD)_{\text{loc}}$ and let $h > 0$. Consider the natural *ball walk* with Markov kernel k with respect to μ defined as $k(x, y) = \frac{\mathbf{1}_{B(x,h)}(y)}{V(x,h)}$. The corresponding Markov transition function $\mathcal{K}$ is not necessarily μ-symmetric because $k(x, y) \neq k(y, x)$ in general. Consider the measure $\mu' \ll \mu$ with $\frac{d\mu'}{d\mu}(x) = V(x, h)$. The Markov kernel of $\mathcal{K}$ with respect to μ' is $p(x, y) = \frac{\mathbf{1}_{B(x,h)}(y)}{V(x,h)V(y,h)}$. Hence $\mathcal{K}$ is μ'-symmetric. Such ball walks on compact Riemannian manifolds were studied in [53].

A Markov chain $(X_n, \mathbb{P}_x)_{n \in \mathbb{N}_0, x \in M}$ is said to be *lazy* if $\inf_{x \in M} \mathbb{P}_x(X_1 = x) > 0$.

Example 4.4. Consider a metric measure space (M, d, μ) with a μ-symmetric Markov transition function $\mathcal{P}$. Define the Markov transition function

$$\mathcal{P}_L(x, A) := \frac{1}{2}(\mathcal{P}(x, A) + \delta_x(A))$$

where $\delta_x(A) = \mathbf{1}_A(x)$ denotes the Dirac measure at x. Note that $\mathcal{P}_L$ μ-symmetric and corresponds to a lazy Markov chain. Assume $\mathcal{P}$ has a kernel p with respect to μ. Then

$\mathcal{P}_L$ has a kernel with respect to μ if and only if $\delta_x \ll \mu$ for all $x \in M$. If P is the Markov operator corresponding to $\mathcal{P}$, then $P_L = (I + P)/2$ is the Markov operator corresponding to $\mathcal{P}_L$, where I is the identity operator on $L^p(M)$. Hence the corresponding Laplacian operators Δ and Δ_L are related by $\Delta_L = \Delta/2$.

Some basic properties of a symmetric Markov kernel are listed without proof in the lemmas below.

Lemma 4.5 (Folklore)**.** *Let $\mathcal{P}$ denote a μ-symmetric Markov transition function over a metric measure space (M, d, μ) and let P be the corresponding Markov operator. Then P is a contraction on all $L^p(M, \mu)$, that is*

$$\|Pf\|_p \le \|f\|_p \tag{4.5}$$

for all $p \in [1, \infty]$ and for all $f \in L^p(M)$. A consequence of (4.5) *is the inequality*

$$\mathcal{E}(f, f) = \langle (I - P)f, f\rangle \le 2\|f\|_2^2 \tag{4.6}$$

for all $f \in L^2(M)$. Moreover P is self-adjoint on $L^2(M)$, that is

$$\langle f, Pg\rangle = \langle Pf, g\rangle \tag{4.7}$$

for all $f, g \in L^2(M, \mu)$ where $\langle f_1, f_2\rangle = \int_M f_1 f_2 \, d\mu$ denotes the inner product on $L^2(M, \mu)$.

Lemma 4.6 (Folklore)**.** *Let $\mathcal{P}$ denote a μ-symmetric Markov transition function over a metric measure space (M, d, μ) and let p be the corresponding Markov kernel. Then for all $x \in M$, the function*

$$n \mapsto p_{2n}(x, x) \tag{4.8}$$

is non-increasing. Moreover we have

$$p_{2n}(x, y) \le p_{2n}(x, x)^{1/2} p_{2n}(y, y)^{1/2} \tag{4.9}$$

for all $x, y \in M$ and for all $n \in \mathbb{N}^*$.

4.1 Assumptions on the Markov chain

We introduce the main assumptions on the Markov chain in the following definition.

Definition 4.7. *For $h > 0$, a Markov transition function $\mathcal{P}$ on $(M, \mathcal{B})$ is said to be* (h, h')-compatible *with (M, d, μ) if*

(a) $\mathcal{P}$ is μ-symmetric.

(b) There exists a kernel p_1 such that $\mathcal{P}(x, A) = \int_A p_1(x, y)\mu(dy)$ for all $x \in M$ and for all $A \in \mathcal{B}$. By (a), we have $p_1(x, y) = p_1(y, x)$ for all $\mu \times \mu$-almost every $(x, y) \in M \times M$.

(c) There exists reals $c_1, C_1 > 0$ and $h' \ge h$ such that

$$\frac{c_1}{V(x, h)} \mathbf{1}_{B(x,h)}(y) \le p_1(x, y) \le \frac{C_1}{V(x, h')} \mathbf{1}_{B(x,h')}(y) \tag{4.10}$$

for all $x \in M$ and for μ-almost every $y \in M$.

(d) There exists $\alpha > 0$ such that

$$p_2(x, y) \ge \alpha p_1(x, y) \tag{4.11}$$

for all $x \in M$ and for μ-almost every $y \in M$, where p_2 is defined by (4.4)*.*

*The corresponding Markov kernel p_1 is said to be (h,h')-*compatible *with (M,d,μ). If a Markov transition function $\mathcal{P}$ satisfies (a),(b),(c) above we say that $\mathcal{P}$ (respectively p_1) is* weakly (h,h')-compatible *with (M,d,μ).*

Similarly, we say the corresponding Markov operator P is (weakly) (h,h')-compatible with (M,d,μ) if the Markov transition function $\mathcal{P}$ is (weakly) (h,h')-compatible with (M,d,μ).

We record some important consequences of Condition (d) in Definition 4.7.

Lemma 4.8. *Let (M,d,μ) be a metric measure space and let P be Markov operator that is (h,h')-compatible with (M,d,μ). Then the corresponding kernel p_k satisfies*

$$p_{k+1}(x,y) \geq \alpha p_k(x,y) \tag{4.12}$$

for all $x,y \in M$ and for all $k \geq 2$ where α is same as in (4.11)*. Moreover the operator $(P-(\alpha/2)I)^2$ is positivity preserving, that is if $f: M \to \mathbb{R}$ satisfies $f \geq 0$, then $(P-(\alpha/2)I)^2 f \geq 0$.*

Proof. Since P^k is a Markov operator, by (4.11) and Lemma 4.2(c) we have

$$p_{k+2}(x,y) - \alpha p_{k+1}(x,y) = P^k\left[p_2(x,\cdot) - \alpha p_1(x,\cdot)\right](y) \geq 0$$

for all $k \in \mathbb{N}^*$ and for all $x,y \in M$. This proves (4.12).

By (4.11) and $f \geq 0$, we have

$$\begin{aligned}(P-(\alpha/2)I)^2 f(x) &= (P^2-\alpha P)f(x) + (\alpha/2)^2 f(x)\\ &\geq (P^2-\alpha P)f(x) = \int_M f(y)(p_2(x,y)-\alpha p_1(x,y))\,dy \geq 0\end{aligned}$$

for all $x \in M$. □

The following lemma shows that a large enough convolution power of a weakly compatible kernel is compatible under some mild conditions.

Lemma 4.9. *Let (M,d,μ) be a quasi-b-geodesic space satisfying $(VD)_{\mathrm{loc}}$ and let p_1 be a Markov kernel weakly (h,h')-compatible with (M,d,μ) for some $h > b$. Then there exists $k \in \mathbb{N}^*$ for which p_l is (h,lh')-compatible with (M,d,μ) for all $l \in \mathbb{N}^*$ such that $l \geq k$.*

Proof. Properties (a) and (b) of Definition 4.7 follows directly from the weak compatibility of p_1. It only remains to check properties (c) and (d). Assume that p_1 satisfies (4.10). Let $x,y \in M$ with $d(x,y) \leq h'$. By Lemma 2.4, there exists even number $k \in \mathbb{N}^*$ such that for all $l \geq k \geq 2$, there exists a b-chain $x_0,x_1,\ldots,x_l$ with $x_0 = x$, $x_l = y$. Define $h_1 = (h-b)/2$. By Chapman-Kolmogorov equation

$$\begin{aligned}&p_l(x,y)\\ &\geq \int_{B(x_{l-1},h_1)}\cdots\int_{B(x_1,h_1)} p(x,y_1)p(y_1,y_2)\ldots p(y_{l-1},y)\,dy_1dy_2\ldots dy_{l-1}\\ &\geq \int_{B(x_{l-1},h_1)}\cdots\int_{B(x_1,h_1)} \frac{c_1^{l-1}}{V(x,h)V(y_1,h)\ldots V(y_{l-1},h)}\,dy_1dy_2\ldots dy_{l-1}\\ &\geq \int_{B(x_{l-1},h_1)}\cdots\int_{B(x_1,h_1)} \frac{c_1^{l-1}C_{h,2h}^{2-l}}{V(x,h)V(x_1,h)\ldots V(x_{l-1},h)}\,dy_1dy_2\ldots dy_{l-1}\\ &\geq \frac{c_1^{l-1}C_{h,2h}^{2-l}}{V(x,h)}\end{aligned} \tag{4.13}$$

The third line above follows weakly (h, h')-compatible condition (4.10) and the fourth line follows from Lemma 2.6. Combining with the fact that p is weakly (h, h')-compatible along with Lemma 2.6 gives the following lower bound: For all $l \geq k$ and $l \in \mathbb{N}^*$, there exists $c_{1,l} > 0$ such that

$$\min(p_l(x,y), p_{l+1}(x,y)) \geq \frac{c_{1,l}}{V(x,'h)} \mathbf{1}_{B(x,h')}(y) \tag{4.14}$$

for all $x, y \in M$. Hence by (4.14) and (4.10) we get $p_{l+1} \geq \alpha_l p_1$ for some $\alpha_l > 0$. Since P is positivity preserving, we have

$$p_{2l}(x,y) = \left(P^{l-1} p_{l+1}(x,.)\right)(y) \geq \alpha_l \left(P^{l-1} p_1(x,.)\right)(y) = \alpha_l p_l(x,y)$$

which is condition (d) of Definition 4.7. Note that (4.14) implies that p_l satisfies the lower bound in condition (c) of Definition 4.7.

Now we turn to the corresponding upper bound for p_l. Since P is a contraction on L^∞, there exists $C_1 > 0$ such that $p_m(x,y) \leq C_1/V(x,h)$ for all $x, y \in M$ and all $m \in \mathbb{N}^*$. By triangle inequality $p_m(x,y) = 0$ if $d(x,y) > mh'$ for all $m \in \mathbb{N}^*$ and for all $x, y \in M$. Hence by Lemma 2.6 we have the desired conclusion. □

Remark 4.10. We now justify the condition $h > b$ in the above lemma. It is to avoid pathological examples of the following kind: Consider a ball walk of Example 4.3 with $h \leq 1/2$ on Broken line space (BL, d, μ) from Example 3.22. It is easy to check that such a random walk never leaves a connected component. Similarly, the ball walk of Example 4.3 with $h < 1$ on a graph always stays at one point.

4.2 Gaussian estimates

The main property of a Markov kernel that we are interested in are Gaussian estimates for its iterated kernel p_n.

Definition 4.11. *A μ-symmetric Markov kernel p on (M, d, μ) is said to satisfy Gaussian upper bound (GUE) if there exists $C_1, C_2 > 0$ such that*

$$p_n(x,y) \leq \frac{C_1}{V(x,\sqrt{n})} \exp\left(-d(x,y)^2/C_2 n\right) \tag{GUE}$$

for all $x, y \in M$ and for all $n \in \mathbb{N}^*$ satisfying $n \geq 2$.

Analogously, a μ-symmetric Markov kernel p on a metric measure space (M, d, μ) is said to satisfy Gaussian lower bound (GLE) if there exists $c_1, c_2, c_3 > 0$ such that

$$p_n(x,y) \geq \frac{c_1}{V(x,\sqrt{n})} \exp\left(-d(x,y)^2/c_2 n\right) \tag{GLE}$$

for all $x, y \in M$ satisfying $d(x,y) \leq c_3 n$ and for all $n \in \mathbb{N}^*$ satisfying $n \geq 2$.

A μ-symmetric Markov kernel p on a metric measure space (M, d, μ) is said to satisfy two sided Gaussian bound (GE) if it satisfies (GUE) and (GLE).

The condition $d(x,y) \leq c_3 n$ in (GLE) is needed because $p_n(x,y)$ vanishes for compatible kernels if $d(x,y) \geq cn$ for some constant $c > 0$. In many situations, the above Gaussian estimates are equivalent to the following (a priori weaker) estimates which are easier to prove. We require the estimates in Definition 4.11 to hold only for large enough n in the definition below.

Definition 4.12. *A μ-symmetric Markov kernel p on (M, d, μ) is said to satisfy Gaussian upper bound $(GUE)_\infty$ if there exists $C_1, C_2, n_0 > 0$ such that*

$$p_n(x,y) \leq \frac{C_1}{V(x,\sqrt{n})} \exp\left(-d(x,y)^2/C_2 n\right) \tag{$(GUE)_\infty$}$$

for all $x, y \in M$ and for all $n \in \mathbb{N}^*$ such that $n \geq n_0$.

The conditions $(GLE)_\infty$ and $(GE)_\infty$ are defined analogously.

Under mild conditions, we show that $(GE)_\infty$ implies (GE).

Lemma 4.13. *Let (M, d, μ) be a quasi-b-geodesic space satisfying $(VD)_{\mathrm{loc}}$ and let p_1 be a Markov kernel weakly (h, h')-compatible with (M, d, μ) for some $h > b$. The following hold:*

(a) If p_1 satisfies $(GUE)_\infty$, then p_1 satisfies (GUE).

(b) If p_1 satisfies $(GLE)_\infty$, then p_1 satisfies (GLE).

(c) If p_1 satisfies $(GE)_\infty$, then p_1 satisfies (GE).

Proof. Note that p satisfies (4.10).

(a) The Gaussian upper estimate for p_n where $n \geq n_0$ follows from $(GUE)_\infty$. If $n < n_0$, we simply use that P is a contraction in L^∞ along with (4.10) to obtain

$$\begin{aligned} p_n(x,y) &\leq \frac{C_1 \mathbf{1}_{B(x,n_0h')}(y)}{V(x,h')} \\ &\leq \frac{C_2}{V(x,\sqrt{n})} \exp\left(-\frac{d(x,y)^2}{C_2 n}\right) \end{aligned}$$

for all $x, y \in M$ and for all $n < n_0$. The first line above follows from triangle inequality, $\|P\|_{L^\infty \to L^\infty} = 1$ and (4.10). The second line follows from Lemma 2.6.

(b) The Gaussian lower bounds for p_n where $n \geq n_0$ follows from $(GLE)_\infty$. Let $h_1 = \min(h/2, h-b)$. Using ideas similar to the proof of Lemma 4.9 (see (4.13)), there exists $c_2, c_3, c_4 > 0$ such that

$$\begin{aligned} &p_n(x,y) \\ &\geq \int_{B(x,h_1)} \cdots \int_{B(x,h_1)} p(x,y_1)p(y_1,y_2)\ldots p(y_{n-1},y)\, dy_1 dy_2 \ldots dy_{n-1} \\ &\geq \frac{c_2 c_3^n \mathbf{1}_{B(x,b)}(y)}{V(x,h)} \geq \frac{c_4}{V(x,\sqrt{n})} \exp\left(-\frac{d(x,y)^2}{c_4 n}\right) \end{aligned}$$

for all $n < n_0$ and for all $x, y \in M$ such that $d(x,y) \leq (b/n_0)n$.

(c) It is a direct consequence of (a) and (b). □

We state the following elementary lemma without proof.

Lemma 4.14. *Let (M, d, μ) be a quasi-b-geodesic space satisfying $(VD)_{\mathrm{loc}}$ and let p be a Markov kernel weakly (h, h')-compatible with (M, d, μ) for some $h > b$. For some $k \in \mathbb{N}^*$, if p_k satisfies $(GE)_\infty$ then p satisfies (GE).*

We describe two examples that does not fall under the framework given by Definition 4.7 but nevertheless the methods developed in this work still applies.

Example 4.15 (Random walk with jumps supported in an annulus)**.** Consider a measured, complete, *length* space (M, d, μ) satisfying $(VD)_{\mathrm{loc}}$ and $\mathrm{diam}(M) = +\infty$. The P be a μ-symmetric Markov operator whose kernel $p(x,y)$ satisfies the following estimate: there exists $C_1 > 0$ and $h > 0$ such that

$$C_1^{-1} \frac{\mathbf{1}_{B(x,2h)\setminus B(x,h)}(y)}{V(x,h)} \leq p(x,y) \leq C_1 \frac{\mathbf{1}_{B(x,2h)\setminus B(x,h)}(y)}{V(x,h)} \tag{4.15}$$

for all $x \in M$ and for μ-almost every $y \in M$.

In this case, it is easy to verify that the density p_2 is $(h/5, 4h)$-compatible with (M, d, μ). Note that for all $x \in M$, there exists $z \in M$ such that $d(x, z) = 3h/2$. Note that by Lemma 2.6 and (4.15), there exists $C_2 > 0$ such that for all $x, y \in M$ with $d(x, y) \le h/5$

$$p_2(x,y) \ge \int_{B(z,h/4)} p_1(x,w)p_1(y,w)\,\mu(dw) \ge \frac{C_2^{-1}}{V(x,h/5)}$$

and for all $x, y \in M$ with $d(x, y) \le 4h$ we have

$$p_2(x,y) \le \int_{B(x,2h)} p_1(x,w)p_1(y,w)\,\mu(dw) \le \frac{C_2}{V(x,4h)}.$$

Therefore p_2 is $(h/5, 4h)$ compatible with (M, d, μ).

For example, it is clear that the application to Liouville property will not be affected if we replace the operator P by P^2. If the underlying space satisfies volume doubling and Poincaré inequality we can use our main results to obtain Gaussian estimates $(GE)_\infty$ provided (M, d, μ) satisfies $(VD)_\infty$ and $(P)_h$. To prove the above statement, we simply note by Theorem 1.1 that p_2 satisfies (GE) and by a similar argument p_3 satifies (GE).

Example 4.16. We describe another example similar to Example 4.15. Consider $\mathbb{R}^n$ equipped with Euclidean distance d and Lebesgue measure μ. Let e denote an arbitrary unit vector in $\mathbb{R}^n$. Consider the μ-symmetric random walk with the kernel

$$p(x,y) = \frac{\mathbf{1}_{B(x+2e,1)\cup B(x-2e,1)}(y)}{2V(x,1)}.$$

Although p is not compatible with $(\mathbb{R}^n, d, \mu)$, similar to Example 4.15 one can check that $(\mathbb{R}^n, d, \mu)$ satisfies that p_2 and p_3 are $(1/3, 9)$-compatible with $(\mathbb{R}^n, d, \mu)$ and that the kernel p_k satisfies $(GE)_\infty$.

4.3 Comparison of Dirichlet forms

Let (M, d, μ) be a metric measure space with a μ-symmetric Markov operator P and corresponding kernel p. Recall that we defined the Dirichlet form $\mathcal{E}(f, g) = \langle f, \Delta g\rangle$ for $f, g \in L^2(M)$. We define another Dirichlet form $\mathcal{E}_*$ which is the Dirichlet form corresponding to the Markov operator P^2, that is

$$\mathcal{E}_*(f,g) = \langle f, (I-P^2)g\rangle = \|f\|_2^2 - \|Pf\|_2^2\,.$$

for all $f, g \in L^2(M)$.

Remark 4.17. Functional inequalities involving the Dirichlet form (for instance Nash, Sobolev, log Sobolev, Poincaré inequalities) can be transferred to an inequality concerning the Markov semigroup, which in turn sheds light on asymptotic behavior of Markov chains. For a continuous time Markov semigroup $(P_t)_{t\ge 0}$ a crucial identity to carry out this is $\frac{d\|P_t f\|_2^2}{dt} = -2\mathcal{E}(P_t f, P_t f)$ (for instance [3, Theorems 4.2.5 and 6.3.1]) By the above definition, we have a similar identity for discrete time Markov semigroup:

$$\partial_k \left\|P^k f\right\|_2^2 := \left\|P^{k+1} f\right\|_2^2 - \left\|P^k f\right\|_2^2 = -\mathcal{E}_*(P^k f, P^k f).$$

for all $f \in L^2(M)$. This is the main reason why we sometimes prefer $\mathcal{E}_*$ instead of $\mathcal{E}$.

The above remark motivates us to compare the Dirichlet forms $\mathcal{E}$ and $\mathcal{E}_*$.

Lemma 4.18. *Consider a μ-symmetric Markov chain on (M, d, μ) with Markov operator P and Dirichlet forms $\mathcal{E}$ and $\mathcal{E}_*$ defined as above. We have the following:*

(a) $\mathcal{E}_*(f, f) \le 2\mathcal{E}(f, f)$ *for all* $f \in L^2(M)$.

(b) *Assume further that P has a strongly (h,h')-compatible kernel p with respect to (M,d,μ). Then there exists a constant $C>0$ such that $\mathcal{E}(f,f)\le C\mathcal{E}_*(f,f)$ for all $f\in L^2(M)$.*

Proof. (a) Note that

$$\langle Pf,f\rangle \le \frac{1}{2}\left(\langle Pf,Pf\rangle+\langle f,f\rangle\right)=\frac{1}{2}\left(\langle P^2f,f\rangle+\langle f,f\rangle\right)$$

Hence

$$\mathcal{E}(f,f)=\langle f,f\rangle_M-\langle Pf,f\rangle_M\ge \langle f,f\rangle-\frac{1}{2}\left(\langle P^2f,f\rangle+\langle f,f\rangle\right)=\frac{1}{2}\mathcal{E}_*(f,f)$$

(b) The conclusion follows from Property (d) of Definition 4.7 by observing that

$$\mathcal{E}(f,f)=\frac{1}{2}\int_M\int_M (f(x)-f(y))^2 p(x,y)\,dx\,dy \tag{4.16}$$

$$\mathcal{E}_*(f,f)=\frac{1}{2}\int_M\int_M (f(x)-f(y))^2 p_2(x,y)\,dx\,dy \tag{4.17}$$

□

Remark 4.19. The inequality $\mathcal{E}(f,f)\le C\mathcal{E}_*(f,f)$ is not true in general. Consider nearest neighbor (simple) random walk on a finite bipartite graph. Let f be a function on the graph that assigns $+1$ to one partition and -1 to other. It is easy to check that $Pf=-f$ and therefore $2\|f\|_2^2=\mathcal{E}(f,f)\le C\mathcal{E}_*(f,f)=0$ fails.

4.4 Markov chains killed on exiting a ball

To obtain lower bounds on the heat kernel, we consider the corresponding Markov process killed on exiting a ball B (See Section 8). Moreover functional inequalities like Nash and Sobolev inequalities that we will encounter are local to balls. Motivated by these considerations, we introduce Markov chains killed on exiting a ball and their corresponding Markov operator and kernel. Let $(X_n)_{n\in\mathbb{N}}$ be a Markov chain on (M,d,μ) driven by a μ-symmetric Markov operator P with kernel p_1 with respect to μ. The corresponding Markov chain $(X_n^B)_{n\in\mathbb{N}}$ that is killed on exiting a ball B has state space $B\cup\{\partial_B\}$ where ∂_B is the absorbing *cemetery state*. The Markov chain $(X_n^B)_{n\in\mathbb{N}}$ killed on exiting B is defined as

$$X_n^B=\begin{cases}X_n & \text{if } n<\zeta\\ \partial_B & \text{if } n\ge\zeta\end{cases}$$

where ζ is the *lifetime* of the process defined by

$$\zeta=\min\{k\ :\ X_k\notin B\}.$$

For the killed Markov chain, we consider functions $f:B\cup\partial_{\mathcal{B}}\to\mathbb{R}$ with the 'Dirichlet' boundary condition $f(\partial_{\mathcal{B}})=0$. Therefore, we can define corresponding quantities like Markov kernel and Markov operator just by restriction to B. Define the *restricted kernel* $p_B:B\times B\to\mathbb{R}$, as a restriction of p_1 on $B\times B$. We endow B with the measure μ_B which is the restriction of μ to all Borel subsets of B. We denote by $L^2(B)=L^2(B,\mu_B)$. We define the Markov operator P_B with kernel p_B with respect to μ_B as

$$P_Bf(x):=\int_B f(y)p_1(x,y)\,\mu(dy)=\int_B p_B(x,y)f(y)\mu_B(dy). \tag{4.18}$$

Define the corresponding Dirichlet forms

$$\mathcal{E}^B(f,f):=\langle f,(I-P_B)f\rangle_{L^2(B)},\qquad \mathcal{E}_*^B(f,f):=\langle f,(I-P_B^2)f\rangle_{L^2(B)} \tag{4.19}$$

for all $f \in L^2(B)$. Similar to (4.4), we define the kernel $p_k^B(x,y)$ iteratively as

$$p_{k+1}^B(x,y) := \left[P_B p_k^B(x,\cdot)\right](y) = \int_B p_k^B(x,z) p_1^B(y,z)\,\mu(dz) \tag{4.20}$$

for all $k \in \mathbb{N}^*$ and for all $x, y \in B$. It is easy to check that the proof of Lemma 4.2 (b),(c) applies to the kernel p_B. As before, the function $(x,y) \mapsto p_k^B(x,y)$ is well-defined for all $k \geq 2$. Further $p_B(x,\cdot) \in L^1(B)$ for all $x \in M$. It is easy to see that

$$p_k^B(x,y) \leq p_k(x,y) \tag{4.21}$$

for all $x, y \in M$ and for all $k \geq 2$.

The operator P_B is positivity preserving, that is $f \geq 0$ implies $P_B f \geq 0$. However unlike P, the operator P_B is not necessarily conservative, that is $P_B \mathbf{1} \neq \mathbf{1}$ in general. Analogous to (4.5), we have that P_B is a contraction on all $L^p(B)$ for all $1 \leq p \leq +\infty$. We also define the corresponding 'Dirichlet Laplacian' $\Delta_{P_B} := I - P_B$.

The following comparison of Dirichlet forms is well-known.

Lemma 4.20. *Let $f \in L^2(B)$ and let $\tilde{f} \in L^2(M)$ denote an extension of f defined by*

$$\tilde{f} = \begin{cases} f & \text{in } B \\ 0 & \text{in } B^c. \end{cases} \tag{4.22}$$

Then

(a) $\mathcal{E}^B(f,f) = \mathcal{E}(\tilde{f},\tilde{f})$.
(b) $\mathcal{E}_*^B(f,f) \geq \mathcal{E}_*(\tilde{f},\tilde{f})$.

We warn the reader of the following abuse of notation. We may consider a function $f \in L^2(B)$ as a function in $L^2(M)$ using the extension given by (4.22). Alternatively we may consider a function $f \in L^2(M)$ as a function in $L^2(B)$ by the restriction $f|_B$.

5 Sobolev-type inequalities

We recall the difficulty arising due to Sobolev inequalities mentioned in the introduction. The Sobolev inequalities in the previous works [67, 23, 25, 74] are of the form

$$\|f\|_{2\delta/(\delta-2)}^2 \leq \frac{Cr^2}{V_\mu(x,r)^{2/\delta}} \left(\mathcal{E}(f,f) + r^{-2}\|f\|_2^2\right) \tag{5.1}$$

for all 'nice' functions f supported in $B(x,r)$. However (5.1) along with (4.6) implies that $L^2(B(x,r)) \subseteq L^{2\delta/(\delta-2)}(B(x,r))$ for all balls $B(x,r)$ which can happen only if the space is discrete. Hence for discrete time Markov chains on continuous spaces the Sobolev inequality (5.1) fails to hold. In this section, we prove a weaker form of the above Sobolev inequality (see (5.2)) and study its properties. In the next two sections, we will use the Sobolev inequality (5.2) to run the Moser's iterative method and obtain elliptic Harnack inequality and Gaussian upper bounds.

We adapt the approach of [67] to obtain a Sobolev inequality using $(VD)_\infty$ and $(P)_\infty$. The main result of this section is the following Sobolev inequality.

Theorem 5.1. *Let (M,d,μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\mathrm{loc}}$, $(VD)_\infty$ and Poincaré inequality at scale h $(P)_h$. Suppose that a Markov operator P has a kernel p that is (h,h')-compatible with respect to μ. Let P_B and $\mathcal{E}^B$ denote the corresponding Markov operator and Dirichlet form restricted to a ball $B \subset M$. Then there exists $\delta > 2$ and $C_S > 0$ such that for all $r > 0$, for all $x \in M$, and for all $f \in L^2(B)$, we have*

$$\|P_B f\|_{2\delta/(\delta-2)}^2 \leq \frac{C_S r^2}{V(x,r)^{2/\delta}} \left(\mathcal{E}^B(f,f) + r^{-2}\|f\|_2^2\right) \tag{5.2}$$

where $B = B(x,r)$.

Remark 5.2. Since P_B is a contraction, note that (5.1) implies (5.2). Since we rely on the weaker Sobolev inequality (5.2), our methods give an unified approach to Gaussian bounds for graphs and continuous spaces. However we will encounter new difficulties due to (5.2).

Let $s > 0$ and $f \in L^1_{\mathrm{loc}}(M,\mu)$. We define f_s as

$$f_s(x) := f_{B(x,s)} = \frac{1}{V(x,s)} \int_{B(x,s)} f(x)\, \mu(dx). \tag{5.3}$$

5.1 Pseudo-Poincaré and Nash inequalities

As in [67, Lemma 2.4], we need a pseudo-Poincaré inequality.

Lemma 5.3 (Pseudo-Poincaré inequality)**.** *Under the hypotheses of Theorem 5.1, there exists* $C_0 > 0$ *and* $s_0 > 0$ *such that*

$$\|f - f_s\|_2^2 \le C_0 s^2 \mathcal{E}(f,f) \tag{5.4}$$

for all $f \in L^2(M)$ *and for all* $s \ge s_0$.

We omit the proof as it follows from Poincaré inequality using the same argument in [67, Lemma 2.4]. The following lemma is a consequence of doubling hypothesis.

Lemma 5.4. *Let* (M,d,μ) *be a measure space satisfying* $(VD)_{\mathrm{loc}}$ *and* $(VD)_\infty$*. Then for all* $b > 0$*, there exists* $C_b > 0$*,* $\delta > 2$ *such that*

$$\|f_s\|_2^2 \le \frac{C_b}{V(x,r)} \left(\frac{r}{s}\right)^\delta \|f\|_1^2 \tag{5.5}$$

for all $f \in L^1(M)$ *is supported in* $B = B(x,r)$ *and for all* $b \le s \le r$

Proof. By Hölder inequality, we have

$$\|f_s\|_2^2 \le \|f_s\|_\infty \|f_s\|_1 . \tag{5.6}$$

Since f is supported in $B(x,r)$ and $s \le r$ we have

$$\|f_s\|_\infty \le \|f\|_1 \sup_{y \in B(x,r+s)} \frac{1}{V(y,s)} \le \frac{\|f\|_1}{V(x,r)} \sup_{y \in B(x,r+s)} \frac{V(y,3r)}{V(y,s)}.$$

By (2.4), there exists $C_1 > 0$ and $\delta > 2$ such that

$$\|f_s\|_\infty \le \frac{C_1}{V(x,r)} \left(\frac{r}{s}\right)^\delta \|f\|_1 \tag{5.7}$$

for all $b \le s \le r$ and for all $f \in L^1$ supported in $B(x,r)$.

Further there exists $C_2 > 0$ such that

$$\begin{aligned} \|f_s\|_1 &= \int_{B(x,r+s)} |f_s(y)| \mu(dy) \le \int_{B(x,r+s)} \frac{1}{V(y,s)} \int_{B(y,s)} |f(z)|\, \mu(dz)\, \mu(dy) \\ &\le \int_{B(x,r+s)} |f(z)| \int_{B(z,s)} \frac{1}{V(y,s)}\, \mu(dy)\, \mu(dz) \\ &\le C_2 \int_{B(x,r+s)} |f(z)| \int_{B(z,s)} \frac{1}{V(z,s)}\, \mu(dy)\, \mu(dz) = C_2 \|f\|_1 \end{aligned} \tag{5.8}$$

for all $b \le s \le r$ and for all $f \in L^1$ supported in $B(x,r)$. The second line follows from Fubini's theorem and (5.8) above follows from (2.5). The desired conclusion (5.5) follows from (5.6), (5.7) and (5.8). □

Next, we show a Nash inequality using the pseudo-Poincaré inequality and doubling hypotheses by adapting the approach of [67, Theorem 2.1].

Proposition 5.5. *Let (M, d, μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\mathrm{loc}}$, $(VD)_\infty$ and Poincaré inequality at scale h $(P)_h$. Suppose that a Markov operator P has a kernel p that is (h, h')-compatible with respect to μ. Let $\mathcal{E}$ denote the Dirichlet form corresponding to P. Then there exists $\delta > 2$, $C_N > 0$ such that*

$$\|Pf\|_2^{2+(4/\delta)} \le \frac{C_N r^2}{V(x,r)^{2/\delta}} \left(\mathcal{E}(f,f) + r^{-2} \|f\|_2^2\right) \|f\|_1^{4/\delta} \tag{5.9}$$

for all $r > 0$, for all $x \in M$, and for all $f \in L^2(M)$ with f supported in $B(x,r)$.

Proof. We start with an observation that (5.9) follows directly for small values of r. Let $r_0 > 0$ be an arbitrary constant. If $r \le r_0$, by (4.10) and (2.2), there exists $C_1, C_2 > 0$ such that for all functions $f \in L^1(M)$ supported in $B(x,r)$, we have

$$\begin{aligned}\|Pf\|_\infty &\le \|f\|_1 \sup_{y \in B(x, r+h')} \frac{C_1}{V(y,h')} \le \|f\|_1 \sup_{y \in B(x, r_0+h')} \frac{C_1}{V(y,h')} \\ &\le \frac{C_1}{V(x,r_0)} \|f\|_1 \sup_{y \in B(x,r_0+h')} \frac{V(y, 2r_0+h')}{V(y,h')} \le \frac{C_2}{V(x,r)} \|f\|_1 . \end{aligned} \tag{5.10}$$

By Hölder inequality along with (5.10) and (4.5), we have $C_3 > 0$ such that

$$\|Pf\|_2 \le \|Pf\|_\infty^{1/2} \|Pf\|_1^{1/2} \le \frac{C_3}{V(x,r)^{1/2}} \|f\|_1 \tag{5.11}$$

for all function $f \in L^2(M)$ supported in $B(x,r)$ with $r \le r_0$. By (5.11) and (4.5) and by the choice $C_N \ge C_3^{4/\delta}$, it suffices to show (5.9) for the case $r > r_0$.

Note that

$$\|Pf\|_2 \le \|Pf - (Pf)_s\|_2 + \|(Pf)_s\|_2 . \tag{5.12}$$

We use pseudo-Poincaré inequality (Lemma 5.3) to bound the first term and use the (h,h')-compatibility of P along with doubling hypotheses to bound the second term. To obtain (5.9), we minimize the bound on right hand side of (5.12) by varying s.

By Lemma 5.4, there exists $C_0 \ge 1$ and $r_0 > 0$ such that

$$\|Pf - (Pf)_s\|_2 \le C_0 s \sqrt{\mathcal{E}(Pf, Pf)} \tag{5.13}$$

for all $f \in L^2(M)$ and for all $s \ge r_0$.

By (5.5) and (4.5), there exists $C_4 > 0$ and $\delta > 2$ such that

$$\|(Pf)_s\|_2 \le \frac{C_4}{V(x,r)^{1/2}} \left(\frac{r}{s}\right)^{\delta/2} \|f\|_1 \tag{5.14}$$

for all $f \in L^2(M)$ supported in $B(x,r)$ and for all $r_0 \le s \le r$. Combining (5.12), (5.13), (5.14), we obtain

$$\|Pf\|_2 \le C_0 s \left(\sqrt{\mathcal{E}(Pf,Pf)} + r^{-1} \|Pf\|_2\right) + \frac{C_4}{V(x,r)^{1/2}} \left(\frac{r}{s}\right)^{\delta/2} \|f\|_1 \tag{5.15}$$

for all $f \in L^2(M)$ supported in $B(x,r)$ and for all $s \ge r_0$ and for all $r \ge r_0$. The desired result follows from minimizing the right hand side by choosing s. □

Before we proceed, we restate the above Nash inequality for functions defined on balls.

Corollary 5.6. *Let (M,d,μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\rm loc}$, $(VD)_\infty$ and Poincaré inequality at scale h $(P)_h$. Suppose that a Markov operator P has a kernel p that is (h,h')-compatible with respect to μ. Let P_B and $\mathcal{E}^B$ denote the corresponding Markov operator and Dirichlet form restricted to a ball $B \subset M$. Then there exists $\delta > 2$, $C_N > 0$ such that*

$$\|P_B f\|_2^{2+(4/\delta)} \le \frac{C_N r^2}{V(x,r)^{2/\delta}} \left(\mathcal{E}^B(f,f) + r^{-2}\|f\|_2^2\right) \|f\|_1^{4/\delta} \tag{5.16}$$

for all $r > 0$, for all $x \in M$, and for all $f \in L^2(M)$ with f supported in $B(x,r)$.

Proof. We define $\tilde f \in L^2(M)$ as in (4.22). Since $P\tilde f = P_B f$ on B, we have $\|P_B f\|_2 \le \left\|P\tilde f\right\|_2$. Combining this observation along with $\|f\|_p = \left\|\tilde f\right\|_p$, Lemma 4.20(a) and Proposition (5.5) yields (5.16). □

Remark 5.7. It is easy to prove Nash inequality (5.16) using Sobolev inequality (5.2) just by an application of Hölder inequality

$$\|P_B f\|_2 \le \|P_B f\|_{2\delta/(\delta-2)}^{\delta/(\delta+2)} \|P_B f\|_1^{2/(\delta+2)} \le \|P_B f\|_{2\delta/(\delta-2)}^{\delta/(\delta+2)} \|f\|_1^{2/(\delta+2)}$$

along with the fact that P_B is a contraction on $L^1(B)$. However proving (5.2) using (5.16) is harder. There is a direct and elementary approach using slicing of functions developed in [2]. Their approach was used by Delmotte in the setting of graphs [23, Theorem 4.4] to prove a Sobolev inequality. However those slicing techniques not so seem to apply directly for proving (5.2), since the (sub-Markov) operator P_B does not commute with the slicing maps $f \mapsto (f-s)_+ \wedge t$. It is an interesting open problem to make this approach work for our Sobolev-type inequalities.

5.2 Ultracontractivity estimate on balls

In light of the above remark, we adapt a different approach based on Hardy-Littlewood-Sobolev theory for discrete time Markov semigroups as developed in [20, Theorems 5 and 6] (see also [14] for related work). Our approach is to obtain an upper bound for $\|P_B^k\|_{1\to\infty}$ using (5.16) which in turn is used to prove the Sobolev inequality (5.2).

Lemma 5.8. *Let (M,d,μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\rm loc}$, $(VD)_\infty$. Suppose that a Markov operator P has a kernel p that is (h,h')-compatible with respect to μ. Let P_B and $\mathcal{E}^B$ denote the corresponding Markov operator and Dirichlet form restricted to a ball $B \subset M$. Further assume that the operators P_B satisfy the Nash inequality* (5.16) *with constant $\delta > 2$. There exists $C_u > 0$ such that*

$$\|P_B^k\|_{1\to\infty} \le \frac{C_u(1+r^2)^{\delta/2}}{V(x,r)} \frac{(1+r^{-2})^{k-1}}{k^{\delta/2}} \tag{5.17}$$

for all $x \in M$, for all $r > 0$ and for all $k \in \mathbb{N}^*$ where $B = B(x,r)$.

Remark 5.9. If two side Gaussian estimate (GE) holds for p_k and if we choose $r \asymp \sqrt{k}$, then the upper bound (5.17) is sharp up to a constant factor.

Proof of Lemma 5.8. Let $x \in M$, $r > 0$ and $B = B(x,r)$. Our first step is an upper bound for $\|P_B^k\|_{1\to 2}$. Let $f \in L^1(B)$ be an arbitrary function with $\|f\|_1 = 1$. The constants in this proof do not depend on the choice of $x \in M$, $k \in \mathbb{N}^*$, $r > 0$ or $f \in L^1(B)$.

Then by Hölder inequality,

$$\|P_B f\|_2^2 \le \|P_B f\|_1 \|P_B f\|_\infty \le \|f\|_1 \|P_B f\|_\infty = \|P_B f\|_\infty .$$

By (5.10), there exists $C_1 > 0$ such that

$$\|P_B f\|_2^2 \le \|P_B f\|_\infty \le \frac{C_1(1+r^2)^{\delta/2}}{V(x,r)}. \tag{5.18}$$

By (5.16), along with Lemma 4.20 and Lemma 4.18(b), there exists $C_N > 0$ such that

$$\|P_B g\|_2^{2+(4/\delta)} \le \frac{C_N r^2}{V(x,r)^{2/\delta}} \left(\mathcal{E}_*^B(g,g) + r^{-2}\|g\|_2^2\right) \|g\|_1^{4/\delta} \tag{5.19}$$

for all $r > 0$, for all $x \in M$, and for all $g \in L^2(B)$ where $B = B(x,r)$. Define

$$v_k := (1+r^{-2})^{-(k-1)} \left\|P_B^k f\right\|_2^2$$

for all $k \in \mathbb{N}^*$. Substituting $g = P_B^k f$ in (5.19) and using the fact that $\left\|P_B^k f\right\|_1 \le \|f\|_1 = 1$ and $\mathcal{E}_*^B(P_B^k f, P_B^k f) = \left\|P_B^k f\right\|_2^2 - \left\|P_B^{k+1} f\right\|_2^2$, we obtain the following difference inequality for v_k:

$$v_{k+1}^{1+(2/\delta)} \le \frac{C_N(1+r^2)}{V(x,r)^{2/\delta}}(v_k - v_{k+1}) \tag{5.20}$$

for all $k \in \mathbb{N}^*$. The desired estimate follows from solving the difference inequality given by (5.20). □

We are ready to prove the Sobolev inequality (5.2) using the ultracontractivity estimate (5.17) above.

For an operator T, we define the operator $(I-T)^{1/2}$ as

$$(I-T)^{1/2} = \sum_{k=0}^{\infty} a_k T^k$$

where a_k is defined by the Taylor series $(1-x)^\alpha = \sum_{k=0}^\infty a_k x^k$ for $x \in (-1,1)$. By a classical estimate on coefficient of Taylor series, there exists $C_a > 0$ such that

$$\frac{C_a^{-1}}{(k+1)^{1/2}} \le a_k \le \frac{C_a}{(k+1)^{1/2}} \tag{5.21}$$

for all $k \in \mathbb{N}_{\ge 0}$.

5.3 Local Sobolev inequality

We use the ultracontractivity estimate (5.17) to obtain Sobolev inequality (5.2).

Proposition 5.10. *Let (M,d,μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\mathrm{loc}}$, $(VD)_\infty$. Suppose that a Markov operator P has a kernel p that is (h,h')-compatible with respect to μ. Let P_B and $\mathcal{E}^B$ denote the corresponding Markov operator and Dirichlet form restricted to a ball $B \subset M$. Assume that there exists $C_u > 0$ such that*

$$\left\|P_B^k\right\|_{1\to\infty} \le \frac{C_u(1+r^2)^{\delta/2}}{V(x,r)} \frac{(1+r^{-2})^{k-1}}{k^{\delta/2}} \tag{5.22}$$

for all $x \in M$, for all $r > 0$ and for all $k \in \mathbb{N}^*$ where $B = B(x,r)$. Then we have the Sobolev inequality (5.2).

Proof. As in the proof of Nash inequality (5.9), we start by considering the case $r \le 1$. By (5.10), there exists $C_2 > 0$ such that

$$\|P_B\|_{1\to\infty} \le \frac{C_1}{V(x,r)} \tag{5.23}$$

for all balls $B = B(x,r)$ with $r \le 1$. Since P_B is a contraction on all $L^p(B)$, we have

$$\|P_B\|_{2(\delta-1)/(\delta-2)\to 2(\delta-1)/(\delta-2)} \le 1. \tag{5.24}$$

Applying Riesz-Thorin interpolation between (5.23) and (5.24) yields

$$\|P_B\|_{2\to 2\delta/(\delta-2)} \le \left(\frac{C_1}{V(x,r)}\right)^{1/\delta}$$

for all balls $B = B(x,r)$ with $r \le 1$. By choosing $C_S \ge C_1^{2/\delta}$, we have (5.2) for all balls $B(x,r)$ with $r \le 1$.

Next we consider the case $r > 1$. Since

$$\mathcal{E}^B(f,f) + r^{-2}\|f\|_2^2 = \left\|\left((1+r^{-2})I - P_B\right)^{1/2} f\right\|^2,$$

it suffices to show that there exists $C_2 > 0$ such that

$$\left\|P_B\left(I-(1+r^{-2})^{-1}P_B\right)^{-1/2}\right\|_{2\to 2\delta/(\delta-2)} \le C_2\frac{(1+r^2)^{1/2}}{V(x,r)^{1/\delta}} \tag{5.25}$$

for all balls $B = B(x,r)$ with $r > 1$. To see this, note that $C_S = \max(C_1^{2/\delta}, 2C_2^2)$ satisfies (5.2). Define

$$E(B) := \frac{(1+r^2)}{\mu(B)}, \qquad T_B := P_B\left(I-(1+r^{-2})^{-1}P_B\right)^{-1/2}. \tag{5.26}$$

Let $p \in [1,\delta)$ and $q \in [\delta/(\delta-1),\infty)$ satisfy

$$p^{-1} = q^{-1} + \delta^{-1}. \tag{5.27}$$

For all $p \in [1,\delta)$ and $q \in [\delta/(\delta-1),\infty)$ satisfying (5.27), we show that the operator T_B is of weak-type (p,q). An application of Marcinkiewicz interpolation then yields (5.2). Recall that $T_B = \sum_{k=1}^{\infty} a_{k-1}(1+r^{-2})^{-(k-1)}P_B^k$. For $N \in \mathbb{N}^*$, we define operators

$$R_{B,N} := \sum_{k=1}^{N} a_{k-1}(1+r^{-2})^{-(k-1)}P_B^k, \qquad S_{B,N} := T_B - R_{B,N}.$$

By (5.22) and Riesz-Thorin interpolation, we obtain

$$\left\|P_B^k\right\|_{p\to\infty} \le C_u^{1/p}E(B)^{\delta/(2p)}\frac{(1+r^{-2})^{(k-1)/p}}{k^{\delta/(2p)}} \tag{5.28}$$

for all balls B, for all $k \in \mathbb{N}^*$ and for all $1 \le p < \infty$. For each $p \in [1,\delta)$, there exists $C_3 > 0$ such that

$$\begin{aligned}\|S_{B,N}\|_{p\to\infty} &\le \sum_{k=N+1}^{\infty} a_{k-1}(1+r^{-2})^{-(k-1)}\left\|P_B^k\right\|_{p\to\infty}\\ &\le C_u^{1/p}E(B)^{\delta/(2p)}C_a\sum_{k=N+1}^{\infty} k^{-1/2}k^{-\delta/(2p)}\\ &\le C_3E(B)^{\delta/(2p)}N^{-\delta/(2q)}\end{aligned} \tag{5.29}$$

for all balls B, where q is given by (5.27). In (5.29) C_3 depends only on p,q,δ but not on $B = B(x,r)$. In the second line above we use (5.28) and (5.21) and we used (5.27) and

$p \in [1, \delta)$ in the last line. By the same argument as above and increasing $C_3 = C_3(p)$ if necessary, we may assume that

$$\|T_B\|_{p\to\infty} \le C_3 E(B)^{\delta/(2p)} \tag{5.30}$$

for all balls B.

Let $g \in L^p(B)$ satisfy $\|g\|_p = 1$. For $\lambda > 0$, let $N_0 = N_0(\lambda, B)$ denote the smallest positive integer such that $C_3 E(B)^{\delta/(2p)} N_0^{-\delta/(2q)} \le \lambda/2$. By union bound, for each $p \in [1, \delta)$ and q given by (5.27), there exists $C_4, C_5 > 0$ such that

$$\begin{aligned}\mu_B\{x \in B \,:\, |T_B g(x)| > \lambda\} &\le \mu_B\{x \in B \,:\, |R_{B,N_0} g(x)| > \lambda/2\} \\ &\quad + \mu_B\{x \in B \,:\, |S_{B,N_0} g(x)| > \lambda/2\} \\ &\le \mu_B\{x \in B \,:\, |R_{B,N_0} g(x)| > \lambda/2\} \\ &\le (2/\lambda)^p \|R_{B,N_0} g\|_p^p \\ &\le C_u^p (2/\lambda)^p \left(\sum_{k=1}^{N_0} k^{-1/2}\right)^p \le C_4 (2C_u)^p \lambda^{-p} N_0^{p/2} \\ &\le C_5 E(B)^{q/2} \lambda^{-q} \end{aligned} \tag{5.31}$$

for all balls $B = B(x, r)$. In the second step above we used the definition of N_0. The third step follows from Chebyshev inequality, the fourth step follows from (5.21) and $\|P_B^k\|_{p\to p} \le 1$. The last step (5.31) follows from (5.21), (5.27), (5.30) and the definition of N_0. By Marcinkiewicz interpolation theorem and the estimates given by (5.31), there exists $C_6 > 0$ such that

$$\|T_B\|_{2\to 2\delta/(\delta-2)} \le C_6 \sqrt{E(B)}$$

for all balls $B = B(x, r)$. This is precisely (5.25) which we intended to prove. ☐

We record two important consequences of Proposition 5.10 first of which is the proof of Theorem 5.1

Proof of Theorem 5.1. Theorem 5.1 follows from Corollary 5.6, Lemma 5.8 and Proposition 5.10. ☐

The next corollary shows that Sobolev inequality is necessarily true under doubling hypothesis and Gaussian upper bounds (GUE).

Corollary 5.11. *Let (M, d, μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\text{loc}}$, $(VD)_\infty$. Suppose that a Markov operator P has a kernel p that is (h, h') compatible with respect to μ. Further assume that iterated kernel p_k that satisfies (GUE). Let P_B and $\mathcal{E}^B$ denote the corresponding Markov operator and Dirichlet form restricted to a ball $B \subset M$. Then the Sobolev inequality* (5.2) *holds.*

Proof. By Proposition 5.10 it suffices to show the ultracontractivity estimate (5.22) on $\|P_B^k\|_{1\to\infty}$. By (GUE), there exists $C_1 > 0$ such that

$$\|P_B^k\|_{1\to\infty} \le \sup_{y\in B, z\in B} p_k(y, z) \le \sup_{y\in B(x,r)} \frac{C_1}{V(y, \sqrt{k})} \tag{5.32}$$

for all balls $B = B(x, r)$ and for all $k \in \mathbb{N}^*$. By (2.4), there exists $\delta > 2$ and $C_2 > 0$ such that

$$\sup_{y\in B(x,r)} \frac{1}{V(x, \sqrt{k})} \le \frac{1}{V(x, r)} \sup_{y\in B(x,r)} \frac{V(x, 2(r \vee \sqrt{k}))}{V(y, \sqrt{k})} \le \frac{1}{V(x, r)} C_2 \left(\frac{2(r \vee \sqrt{k})}{\sqrt{k}}\right)^\delta \tag{5.33}$$

for all balls $B(x, r)$ and for all $k \in \mathbb{N}^*$. The desired estimate (5.22) follows from (5.32) and (5.33). ☐

5.4 Sobolev inequality implies large scale doubling property

Next, we show that Sobolev inequality implies $(VD)_\infty$ under natural hypotheses. More precisely

Proposition 5.12. *Let (M,d,μ) be a metric measure space satisfying $(VD)_{\rm loc}$. Let P be (h,h') compatible Markov operator in a metric measure space (M,d,μ) satisfying Sobolev inequality* (5.2)*. Then (M,d,μ) satisfies the large scale doubling property $(VD)_\infty$.*

We need the following volume comparison lemma.

Lemma 5.13. *Let (M,d,μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\rm loc}$ and let $h' \geq b > 0$. Then there exists $C_0>0$ such that*

$$V(x,r+h') \leq C_0 V(x,r) \tag{5.34}$$

for all $x \in M$ and for all $r \geq 3h'$.

Proof. Let Y be a maximal h'-separated subset of $B(x,r)$ where $x\in M$ and $r \geq 3h'$. The collection of balls $\{B(y,h'/2)\ :\ y \in Y\}$ are disjoint and hence

$$V(x,r) \geq \sum_{y \in Y \cap B(x,r-h')} V(y,h'/2). \tag{5.35}$$

However since $B(x,r) \subseteq \cup_{y\in Y} B(y,h')$ and $r \geq 3h'$, we have

$$\emptyset \neq B(x,r-2h') \subseteq \cup_{y \in Y\cap B(x,r-h')} B(y,h'), \tag{5.36}$$

By quasi-b-geodesicity and $b \leq h'$, there exists $C_1>0$ such that for all $z \in B(x,r+h')$, there exists a b-chain $x_0,x_1,\ldots,x_m$ b-chain from x to z such that

$$x_i \in B(x,r-2h') \text{ and } d(x_i,z) \leq C_1 h'. \tag{5.37}$$

Combining (5.36) and (5.37), we obtain

$$B(x,r+h') \subseteq \cup_{y\in Y\cap B(x,r-h')} B(y,(C_1+1)h'). \tag{5.38}$$

Combining (5.38),Lemma 2.6 and (5.35), we obtain

$$\begin{aligned} V(x,r+h') &\leq \sum_{y\in Y\cap B(x,r-h')} V(y,(C_1+1)h') \\ &\leq C_{h'/2,(C_1+1)h'} \sum_{y\in Y\cap B(x,r-h')} V(y,h'/2) \leq C_{h'/2,(C_1+1)h'} V(x,r). \end{aligned}$$ ☐

Proof of Proposition 5.12. . We adapt the argument of [18, Proposition 2.1]. However unlike in [18, Proposition 2.1], we do consider volumes of arbitrarily small balls.

Let $x \in M$ and $r \geq 3h'$ be arbitrary. For $s>0$, define the 'tent function'

$$f_s(y) = \max(s - d(x,y), 0).$$

By (h,h') compatibility of P, we have $P_{B(x,r)} f_{3h'} \geq h' \mathbf{1}_{B(x,h')}$. Therefore by applying (5.2), we have

$$(h')^2 V(x,h')^{(\delta-2)/\delta} \leq \frac{C_S r^2}{V(x,r)^{2/\delta}} \left((h')^2 V(x,4h') + r^{-2}(3h')^2 V(x,3h') \right)$$

for all $r \geq 3h'$ and for all $x \in M$. Combined with Lemma 2.6, there exists $C_1>0$ such that

$$\frac{V(x,r)}{V(x,h')} \leq C_1 r^\delta \tag{5.39}$$

for all $r \geq 3h'$ and for all $x \in M$.

Let $3h' \leq s \leq r$. Then by (h, h') compatibility of P, we have $P_{B(x,r)} f_s \geq (s/6) \mathbf{1}_{B(x,s/2)}$. Hence by Sobolev inequality (5.2), (4.10) and Lemma 4.20(a), we obtain

$$(s/6)^2 V(x, s/2)^{(\delta-2)/\delta} \leq \frac{C_S r^2}{V(x,r)^{2/\delta}} \left((h')^2 V(x, s+h) + r^{-2} s^2 V(x,s)\right)$$

Combined with Lemma 5.13, there exists $C_2 > 0$ such that

$$V(x,s) \geq \left(\frac{s^\delta V(x,r)}{C_2 r^\delta}\right)^{2/\delta} V(x, s/2)^{(\delta-2)/\delta} \tag{5.40}$$

for all $x \in M$ and for all $3h' \leq s \leq r$. We replace s by $s/2$ in (5.40) and iterate to obtain

$$V(x,s) \geq 4^{-\sum_{j=0}^{i-1} j(\delta-2)^j/\delta^j} \left(\frac{s^\delta V(x,r)}{C_2 r^\delta}\right)^{(2/\delta)\sum_{j=0}^{i-1}(\delta-2)^j/\delta^j} V(x, s/2^i)^{(\delta-2)^i/\delta^i} \tag{5.41}$$

for all $3h' \leq s/2^{i-1} \leq s \leq r$. In particular if we choose $i = \lceil \log_2(s/3h') \rceil$, we have $(3h')/2 \leq s/2^i \leq 3h'$. Hence by (5.41) and (5.39),we have

$$V(x,s) \geq 4^{-\sum_{j=0}^{\infty} j(\delta-2)^j/\delta^j} \left(\frac{s^\delta V(x,r)}{C_2 r^\delta}\right)^{(2/\delta)\sum_{j=0}^{i-1}(\delta-2)^j/\delta^j} \left(\frac{V(x,r)}{C_1 r^\delta}\right)^{(\delta-2)^i/\delta^i} \tag{5.42}$$

for all $x \in M$ and for all $3h' \leq s \leq r$, where $i = \lceil \log_2(s/3h') \rceil$. By (5.42), there exists $C_3 > 0$ such that

$$\frac{V(x,r)}{V(x,s)} \leq C_3 (r/s)^\delta s^{(\delta-2)^i/\delta^{i-1}} \tag{5.43}$$

for all $x \in M$ and for all $3h' \leq s \leq r$, where $i = \lceil \log_2(s/3h') \rceil$. Since the map $s \mapsto \exp\left(\delta((\delta-2)/\delta)^{\lceil \log_2(s/3h') \rceil} \ln s\right)$ is bounded in $[3h', \infty)$, by (5.43) there exists $C_4 > 0$ such that

$$\frac{V(x,r)}{V(x,s)} \leq C_4 \left(\frac{r}{s}\right)^\delta$$

for all $x \in M$ and for all $3h' \leq s \leq r$. The above equation clearly implies $(VD)_\infty$. □

6 Elliptic Harnack inequality

In this section, we prove elliptic Harnack inequality for non-negative harmonic functions. As before, we consider a metric measure space (M, d, μ) and a Markov operator P that is (h, h')-compatible with (M, d, μ). Recall that the operator $\Delta := I - P$ is the Laplacian corresponding to P.

6.1 Harmonic functions

Definition 6.1. *Let P be a Markov operator on (M, d, μ). A function $f : U \to \mathbb{R}$ is P-harmonic in $B(x,r)$ if*

$$\Delta f(y) = f(y) - Pf(y) = 0$$

for all $y \in B(x,r)$.

Similarly, we say $f : M \to R$ is P-subharmonic (resp. P-superharmonic) in $B(x,r)$ if

$$\Delta f(y) \leq 0 \text{ (resp. } \geq 0)$$

for all $y \in B(x,r)$.

We say a function $f : M \to \mathbb{R}$ is P-harmonic (resp. subharmonic, superharmonic) if $\Delta f \equiv 0$ (resp. $\Delta f \leq 0$, $\Delta f \geq 0$).

Remark 6.2.

(a) Consider a Markov operator P that is (h,h')-compatible with (M,d,μ). By (4.10), $Pf(y)$ depends only on f in $B(y,h')$. Therefore the property that $f: M \to \mathbb{R}$ is P-harmonic in $B(x,r)$ depends only on the values of f in $B(x,r+h')$. Hence in this case it suffices to have $B(x,r+h') \subseteq \mathrm{Domain}(f)$.

(b) We use the term harmonic instead of P-harmonic if the Markov operator P is clear from the context. Same holds for superharmonic or subharmonic functions.

The main result of the section is the following elliptic Harnack inequality.

Theorem 6.3 (Elliptic Harnack inequality)**.** *Let (M,d,μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\mathrm{loc}}$, $(VD)_\infty$ and Poincaré inequality at scale h $(P)_h$. Suppose that a Markov operator P has a kernel p that is (h,h')-compatible with respect to μ for some $h > b$. Then there exists $c > 0, r_0 > 0, C_E > 0$ such that for all $x \in M$, for all $r \geq r_0$ and for all non-negative functions $u : B(x,r) \to \mathbb{R}_{\geq 0}$ that are P-harmonic in $B(x,r)$ the following Harnack inequality holds:*

$$\sup_{x\in B(x,cr)} u \leq C_E \inf_{x\in B(x,cr)} u. \tag{6.1}$$

In (6.1), the sup and inf must be understood as essential sup and essential inf with respect to μ.

We follow Moser's iteration method [58] to prove the elliptic Harnack inequality. Our approach is an adaptation of Delmotte's approach except that we have to rely on a weaker version of Sobolev inequality and a modified version of John-Nirenberg inequality. Moser's iteration relies on estimating the quantities

$$\phi(u,p,B') := \left(\frac{1}{\mu(B')}\int_{B'} |u|^p \, d\mu\right)^{1/p} \tag{6.2}$$

for different balls $B' \subset B$ and for different values of $p \in \mathbb{R}\setminus\{0\}$. By Jensen's inequality, $p \mapsto \phi(u,p,B')$ is non-decreasing function. The function ϕ satisfies $\lim_{p\to-\infty}\phi(u,p,cB) = \inf_{cB} u$ and $\lim_{p\to+\infty}\phi(u,p,cB) = \sup_{cB} u$ [47, Lemma 14.1.4]. To obtain (6.1), Moser's iterative method relies on establishing bounds of the form $\phi(u,p_1,B') \leq C_{p_1,p_2}\phi(u,p_2,B'')$ for different values of $p_1,p_2 \in \mathbb{R}\setminus\{0\}$ satisfying $p_1 < p_2$. Sobolev inequality and Poincaré inequality are crucial ingredients to run this iterative procedure. For a function f that is defined on a ball B, we denote the mean integral by

$$f_B = \fint_B f\, d\mu = \frac{1}{\mu(B)}\int_B f\, d\mu.$$

We start with a local version of the above elliptic Harnack inequality.

Lemma 6.4. *Let (M,d,μ) be a quasi-b-geodesic space satisfying $(VD)_{\mathrm{loc}}$ and let P be a weakly (h,h')-compatible Markov operator with (M,d,μ) for some $h > b$. There exists $C > 0$ and $r_0 > 0$ such that*

$$u(y) \leq Cu(z) \tag{6.3}$$

for all $x \in M$, for all $r \geq r_0$, for all $y,z \in B(x,r/2)$ satisfying $d(y,z) \leq h'$ and for all non-negative functions $u : B(x,r+h') \to \mathbb{R}$ harmonic in $B(x,r)$.

Proof. There exists $c_1 > 0$ and $l \in \mathbb{N}^*$ such that

$$p_l(z,w) = p_l(w,z) \geq \frac{c_1 \mathbf{1}_{B(z,2h')}(w)}{V(w,h')} \tag{6.4}$$

for all $y, w \in M$. The proof of (6.4) is analogous to that of (4.13). Therefore by (6.4), $(VD)_{\text{loc}}$ weak (h, h')-compatibility of p_1 and triangle inequality, there exists $c_2 > 0$ such that

$$p_l(z,w) \geq \frac{c_1 \mathbf{1}_{B(z,2h')}(w)}{V(w,h')} \geq \frac{c_1 \mathbf{1}_{B(y,h')}(w)}{V(w,h')} \geq c_2 p_1(w,y) = c_2 p_1(y,w) \tag{6.5}$$

for all $y, z, w \in M$ satisfying $d(y,z) \leq h'$.

Choose r_0 large enough so that $r/2 + lh' \leq r + h'$ for all $r \geq r_0$. Note that for every harmonic function $u : B(x, r + h') \to \mathbb{R}$ in $B(x,r)$ with $r \geq r_0$ and for all $z \in B(x, r/2)$, we have

$$u(z) = P^l u(z) = \int_{B(z,lh')} p_l(z,w) u(w) \mu(dw) \tag{6.6}$$

By (6.6) and (6.5), we obtain

$$u(z) = \int_{B(z,lh')} p_l(z,w) u(w) \mu(dw) \geq c_2 \int_{B(y,h')} p_1(y,w) u(w) \mu(dw) = c_2 u(y) \tag{6.7}$$

for all non-negative harmonic functions u in $B(x,r)$ for all $x \in M$, for all $z, y \in B(x, r/2)$ with $r \geq r_0$. The choice $C = c_2^{-1}$ satisfies (6.3). □

6.2 John-Nirenberg inequality

Moser [58], used John-Nirenberg inequality to obtain an estimate of the form $\phi(u, -q, B') \leq C' \phi(u, q, B')$ for some $q, C' > 0$. An alternative approach is to use an abstract lemma of Bombieri and Guisti was later proposed by Moser [70, Section 2.2.3].

John-Nirenberg inequality is an estimate on distribution of functions of bounded mean oscillation which were introduced in [46]. A locally integrable function $f : B \to \mathbb{R}$ define is of bounded mean oscillation (BMO) if

$$\|f\|_{BMO(B)} := \sup_{B' \subset B} \frac{1}{\mu(B')} \int_{B'} |f - f_{B'}| \, d\mu < \infty.$$

John-Nirenberg inequality states that functions of bounded mean oscillation have an exponentially decaying distribution function.

In [1, Theorem 5.2] a version of John-Nirenberg inequality is shown for spaces satisfying the doubling hypothesis (VD). However for us, the metric measure space (M, d, μ) only satisfies $(VD)_{\text{loc}}$ and $(VD)_\infty$. Since we do not have doubling hypothesis on arbitrarily small balls, we introduce a modified version of BMO seminorm (BMO seminorm at scale h) defined as

$$\|f\|_{BMO(B(x_0,r_0)),h} = \sup_{B(y,r) \subseteq B(x_0,r_0), r \geq h} \frac{1}{V(y,r)} \int_{B(y,r)} \left| f - f_{B(y,r)} \right| d\mu. \tag{6.8}$$

Our proof is motivated by the presentation in [1]. We start by recalling the Vitali covering lemma.

Lemma 6.5 (Vitali covering lemma)**.** *[44, Theorem 1.2] Let $\mathcal{F}$ be a family of balls with positive and uniformly bounded radii in a metric space (M, d). Then there exists a disjoint subfamily $\mathcal{G} \subseteq \mathcal{F}$ such that*

$$\bigcup_{B \in \mathcal{F}} B \subseteq \bigcup_{B \in \mathcal{G}} 5B.$$

In fact, every ball $B \in \mathcal{F}$ meets a ball $B' \in \mathcal{G}$ with radius at least half that of B and therefore satisfies $B \subseteq 5B'$.

A crucial ingredient in the proof of John-Nirenberg inequality is the following version of Calderón-Zygmund decomposition lemma. Since we replaced (VD) by weaker assumptions $(VD)_{\rm loc}$ and $(VD)_\infty$, we need some other method to control the behavior of a BMO function at small length scales. This is why we assume a local Harnack inequality (by Lemma 6.4 the local Harnack inequality holds for harmonic functions).

Lemma 6.6 (Calderón-Zygmund decomposition lemma)**.** *Let (M,d,μ) be a metric measure space satisfying $(VD)_{\rm loc}$ and $(VD)_\infty$. Let f be a non-negative locally integrable function on $B(x_0,11r_0)$ for some $r_0 \ge r_1 \ge h > 0$. Further we assume that there exists $C_1 \ge 1$ such that f satisfies the local Harnack inequality*

$$f(y) \le C_1 f(z) \tag{6.9}$$

for all $y,z \in B(x_0, r_0+h)$ satisfying $d(y,z) \le h$. Further, assume that

$$\lambda_0 \ge \frac{1}{V(x_0,r)} \int_{B(x_0,11r_0)} f \, d\mu \tag{6.10}$$

Then there exists countable (possibly finite) family of disjoint balls $\mathcal{F} = \{B_i\}$ of disjoint balls centered in $B(x_0,r)$ and satisfying $5B_i \subseteq B(x_0,11r_0)$ for all $B_i \in \mathcal{F}_0$ so that

- *(i)* $f(x) \le C_1\lambda_0$ *for all* $x \in B(x_0,r_0) \setminus \left(\bigcup_{B_i\in\mathcal{F}} 5B_i\right)$.
- *(ii)* $\lambda_0 < \fint_{B_i} f\,d\mu \le C_2\lambda_0$ *for all* $B_i \in \mathcal{F}_0$.
- *(iii)* $C_2^{-1}\lambda_0 < \fint_{5B_i} f\,d\mu \le \lambda_0$ *for all* $B_i \in \mathcal{F}_0$.

The family of balls $\mathcal{F}_0$ satisfying the above conditions are called Calderón-Zygmund balls at level λ_0. Moreover if $\lambda_0 \le \lambda_1 \le \ldots \le \lambda_N$, then the family Calderón-Zygmund balls $\mathcal{F}_n$ corresponding to different levels λ_n may be chosen in such a way that every $B_i(\lambda_{n+1}) \in \mathcal{F}_{n+1}$ is contained in some $5B_j(\lambda_n)$ where $B_j(\lambda_n) \in \mathcal{F}_n$.

Proof. We denote $B(x_0,r_0)$ as B_0. Define a maximal function

$$M_{B_0}f(x) = M_{B(x_0,r_0)}f(x) = \sup_{\substack{B(y,r)\subset B(x_0,r_0+h):\\ y\in B(x_0,r_0), r\ge h, B(y,r)\ni x}} \fint_{B(y,r)} f\,d\mu$$

for all $x \in B(x_0,r)$. We define

$$E_\lambda = \left\{x \in B(x_0,r_0) \;:\; M_{B(x_0,r_0)}f(x) > \lambda\right\}.$$

First consider λ_N. By definition for every $x \in E_{\lambda_N}$, there exists a ball $B_x = B(y_x,r_x)$ satisfying $y_x \in B_0$, $x \in B_x$, $B_x \subseteq B(x_0,r_0+h)$, $r_x \ge h$ and

$$\lambda_0 \le \lambda_1 \le \ldots \le \lambda_N < \fint_{B_x} f\,d\mu. \tag{6.11}$$

Let $k = k_x \in \mathbb{N}^*$ be such that $5^{k-1}r_x \le 2r_0 \le 5^k r_x$. Then $B_0 \subseteq 5^k B_x \subseteq 11B_0$. Combining this with (6.10), we have

$$\fint_{5^kB_x} f\,d\mu \le \frac{1}{\mu(B_0)}\int_{11B_0} f\,d\mu \le \lambda_0 \le \lambda_N.$$

However since $\fint_{B_x} f\,d\mu > \lambda_N$, there exist smallest $n_x \ge 1$ such that

$$\fint_{5^{n_x}B_x} f\,d\mu \le \lambda_N \tag{6.12}$$

and

$$\fint_{5^j B_x} f\,d\mu > \lambda_N \tag{6.13}$$

for all $j = 0, 1, \ldots, n_x - 1$. The balls $5^{n_x-1}B_x$ forms a covering of E_{λ_N}. Therefore by Vitali covering lemma (Lemma 6.5), we pick a family $\mathcal{F}_N$ of pairwise disjoint balls $B_i = 5^{n_{x_i}-1}B_{x_i}$ satisfying $E_{\lambda_N} \subseteq \bigcup_{B\in\mathcal{F}_N} 5B$. We now check the construction above satisfies the desired properties. By (6.12), (6.13) and (2.4), there exists $C_h > 0, \delta > 0$ such that

$$\lambda_N < \fint_{5^{n_x-1}B_x} f\,d\mu \le C_h 5^\delta \fint_{5^{n_x}B_x} f\,d\mu \le C_h 5^\delta \lambda_N.$$

Choosing $C_2 = 5^\delta C_h$, we obtain properties (ii) and (iii) of Calderón-Zygmund decomposition.

It remains to verify (i). If $x \in B_0 \setminus \left(\bigcup_{B_i\in\mathcal{F}} 5B_i\right) \subseteq B_0 \setminus E_{\lambda_N}$, we have $M_{B_0}f(x) \le \lambda_N$. Therefore by (6.9), we have

$$\lambda_N \ge M_{B_0}f(x) \ge \fint_{B(x,h)} f\,d\mu \ge C_1^{-1}f(x).$$

This give property (i). We have now constructed the desired decomposition at level λ_N. Next we consider λ_{N-1}. The rest of the properties follow from the argument in [1, Lemma 3.8].

□

Remark 6.7. In the above proof, we use (6.9) to obtain property (i) of the Calderón-Zygmund decomposition. Typically property (i) is proved using Lebesgue differentiation theorem. However the proof of Lebesgue differentiation theorem typically follows from (VD) (See [1] and [44, Theorem 1.8]). Even though (VD) is not necessary for Lebesgue differentiation theorem, it is not clear whether Lebesgue differentiation theorem holds in our setting.

Next, we prove the John-Nirenberg inequality for spaces satisfying $(VD)_{\mathrm{loc}}$ and $(VD)_\infty$.

Proposition 6.8 (John-Nirenberg inequality)**.** *Let (M, d, μ) be a metric measure space satisfying $(VD)_{\mathrm{loc}}$ and $(VD)_\infty$. Let f be a non-negative locally integrable function on $B(x_0, 11r_0)$ for some $r_0 \ge h > 0$. Further we assume that there exists $C_1 \ge 1$ such that f satisfies the local Harnack inequality*

$$f(y) \le C_1 f(z) \tag{6.14}$$

for all $y, z \in B(x_0, r_0 + h)$ satisfying $d(y, z) \le h$. Then there exists $C_2 > 0$ such that

$$\mu\left(\{x \in B_0 \,:\, |f - f_{B_0}|\} > \lambda\right) \le C_2\mu(B_0)\exp(-\lambda/(C_2\,\|f\|_{\mathrm{BMO}(11B_0),h})) \tag{6.15}$$

for all $\lambda > 0$. The constant C_2 depends only on C_1, h and constants associated with doubling hypotheses $(VD)_{\mathrm{loc}}$ and $(VD)_\infty$.

Proof. The proof of this version of John-Nirenberg inequality follows from the same argument given in [1, Theorem 5.2] except that we use the version of Calderón-Zygmund decomposition given in Lemma 6.6. □

Following [23, Corollaire, p. 25], we obtain:

Corollary 6.9. *Let (M, d, μ) be a metric measure space satisfying $(VD)_{\mathrm{loc}}$ and $(VD)_\infty$. Let f be a non-negative locally integrable function on $B(x_0, 11r_0)$ for some $r_0 \ge h' > 0$. Further we assume that there exists $C_1 \ge 1$ such that f satisfies the local Harnack inequality*

$$f(y) \le C_1 f(z) \tag{6.16}$$

for all $y, z \in B(x_0, r_0 + h')$ *satisfying* $d(y,z) \le h$. *Then there exists* $c_0, C_0 > 0$ *such that*

$$\int_{B_0} e^{\left(c_0 f(y)/\|f\|_{\mathrm{BMO}(11B_0),h'}\right)}\,dy \int_{B_0} e^{\left(-c_0 f(y)/\|f\|_{\mathrm{BMO}(11B_0),h'}\right)}\,dy \le C_0^2 \mu(B_0)^2 \tag{6.17}$$

where $B_0 = B(x_0, r_0)$. *The constants* c_0, C_0 *depends only on* C_1, h' *and constants associated with doubling hypotheses* $(VD)_{\mathrm{loc}}$ *and* $(VD)_\infty$.

6.3 Discrete calculus

Before we dive into computations, we introduce simplifying notations and collect basic rules that mimics calculus rules in a discrete setting. Let f be a function on $\mathbb{N} \times M$ or on M. Depending on context, we may abbreviate $f(k,x)$ to $f_k(x), f_k$ or even f.

1. 'Gradient'
$$\nabla_{xy} f := f(y) - f(x) \tag{6.18}$$
and the 'time derivative'
$$\partial_k f(x) := f(k+1,x) - f(k,x). \tag{6.19}$$
2. Differentiation of product
$$\nabla_{xy}(fg) = (\nabla_{xy} f) g(y) + (\nabla_{xy} g) f(x). \tag{6.20}$$
3. Differentiation of square
$$\nabla_{xy} f^2 = 2(\nabla_{xy} f) f(x) + (\nabla_{xy} f)^2. \tag{6.21}$$
4. The same formulas for the 'time derivatives':
$$\partial_k(fg) = (\partial_k f) g_{k+1} + (\partial_k g) f_k \tag{6.22}$$
and
$$\partial_k(f^2) = 2(\partial_k f) f_k + (\partial_k f)^2. \tag{6.23}$$
5. Let $\Delta = I - P$ denote the Laplacian corresponding to a μ-symmetric Markov operator P with kernel p_1. Then
$$\Delta f(x) := (I-P) f(x) = \int_M p_1(x,y) \nabla_{yx} f\,dy.$$
6. Integration by parts: If $f, g \in L^2(M,\mu)$, then
$$\int_M \Delta f(x) g(x) dx = \frac{1}{2} \int_M \int_M (\nabla_{xy} f)(\nabla_{xy} g) p_1(x,y)\,dy\,dx. \tag{6.24}$$
7. Consider a μ-symmetric Markov operator with kernel p_1. We define $|\nabla f|$ corresponding to the Markov operator P as
$$|\nabla_P f|^2(x) := \int_M (\nabla_{xy} f)^2 p_1(x,y)\,dy. \tag{6.25}$$

We caution the reader to be aware of different uses of the symbol ∇ in (3.1), (6.18) and (6.25) with slight change in subscript. The subscript could be a positive real number, a pair of points or a Markov operator. We hope the different notations of ∇ would be clear from the context.

6.4 Logarithm of a harmonic function

If u is a positive harmonic function, then we show that $\log u$ has bounded BMO semi-norm. This combined with John-Nirenberg inequality yields $\phi(u, -q, c_1 B) \le C' \phi(u, q, c_1 B)$ for some $q, C' > 0$ and $c_1 \in (0, 1)$.

Lemma 6.10. *Let (M, d, μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\mathrm{loc}}$, $(VD)_\infty$ and Poincaré inequality at scale h $(P)_h$. Suppose that a Markov operator P has a kernel p that is (h, h')-compatible with respect to μ for some $h > b$. Let u be a positive P-harmonic function on $B = B(x, r)$. Let η be a non-negative function on B satisfying $\mathrm{supp}(\eta) \subset B(x, (r/2) - h')$. There exists $C_0 > 0$ and $r_0 > 2h'$ satisfies*

$$\int_{B/2}\int_{B/2}\left(\ln\frac{u(y)}{u(z)}\right)^2 \eta(z)^2 p_1(y,z)\,dy\,dz \le C_0 \int_{B/2}\int_{B/2}(\nabla_{yz}\eta)^2\, p_1(y,z)\,dy\,dz \tag{6.26}$$

for all balls B, for all functions u, η satisfying the above requirements.

Proof. Define $\psi := \eta^2/u$. By product rule (6.20)

$$\nabla_{yz}\psi = \nabla_{yz}(1/u)\eta(z)^2 + (1/u(y))\nabla_{yz}(u^2). \tag{6.27}$$

Using integration by parts (6.24) along with $\mathrm{supp}(\eta) \subset B(x, (r/2) - h')$, we deduce

$$\int_{B/2}\int_{B/2} p_1(y,z)(\nabla_{yz}\psi)(\nabla_{yz}u)\,dy\,dz = 0. \tag{6.28}$$

Combining (6.27), (6.28), we have

$$\begin{aligned} &-\int_{B/2}\int_{B/2} p_1(y,z)(\nabla_{yz}u)\left(\nabla_{yz}\frac{1}{u}\right)\eta(z)^2\,dy\,dz \\ &\le \int_{B/2}\int_{B/2} p_1(y,z)|\nabla_{yz}u|\left|\nabla_{yz}\eta^2\right|\frac{1}{u(y)}\,dy\,dz. \end{aligned} \tag{6.29}$$

By Lemma 6.4, u satisfies the local Harnack inequality on $B/2$ for large enough balls B. Hence there exists $c_1, C_1 > 0$ and $r_0 > 2h'$ such that

$$-(\nabla_{yz}u)\left(\nabla_{yz}\frac{1}{u}\right) = \frac{(u(y)-u(z))^2}{u(y)u(z)} \ge c_1\left(\ln\frac{u(y)}{u(z)}\right)^2 \tag{6.30}$$

$$|\nabla_{yz}u|/u(y) \le C_1\left|\ln\frac{u(y)}{u(z)}\right| \tag{6.31}$$

for all positive P-harmonic functions u on $B = B(x, r)$, for all $y, z \in B/2$ with $d(y, z) \le h'$ and $r > r_0$. Combining (6.29), (6.30) and (6.31), we obtain

$$\begin{aligned} &\int_{B/2}\int_{B/2} p_1(y,z)\left(\ln\frac{u(y)}{u(z)}\right)^2\eta(z)^2\,dy\,dz \\ &\le \frac{C_1}{c_1}\int_{B/2}\int_{B/2} p_1(y,z)|\nabla_{yz}\eta|(\eta(y)+\eta(z))\left|\ln\frac{u(y)}{u(z)}\right|\,dy\,dz \end{aligned} \tag{6.32}$$

Since $p_1(y, z) = p_1(z, y)$ for $\mu \times \mu$-almost every $(y, z) \in M \times M$, we have

$$\begin{aligned} &\int_{B/2}\int_{B/2} p_1(y,z)|\nabla_{yz}\eta|\eta(y)\left|\ln\frac{u(y)}{u(z)}\right|\,dy\,dz \\ &= \int_{B/2}\int_{B/2} p_1(y,z)|\nabla_{yz}\eta|\eta(z)\left|\ln\frac{u(y)}{u(z)}\right|\,dy\,dz \end{aligned} \tag{6.33}$$

By (6.32) and (6.33)

$$\begin{aligned}&\int_{B/2}\int_{B/2} p_1(y,z)\left(\ln\frac{u(y)}{u(z)}\right)^2\eta(z)^2\,dy\,dz\\&\le \frac{2C_1}{c_1}\int_{B/2}\int_{B/2} p_1(y,z)|\nabla_{yz}\eta|\eta(z)\left|\ln\frac{u(y)}{u(z)}\right|\,dy\,dz \qquad (6.34)\end{aligned}$$

By Hölder inequality

$$\begin{aligned}&\left(\int_{B/2}\int_{B/2} p_1(y,z)|\nabla_{yz}\eta|\eta(z)\left|\ln\frac{u(y)}{u(z)}\right|\,dy\,dz\right)^2\\&\le \int_{B/2}\int_{B/2} p_1(y,z)|\nabla_{yz}\eta|^2\,dy\,dz\cdot\int_{B/2}\int_{B/2} p_1(y,z)\left(\ln\frac{u(x)}{u(y)}\right)^2\eta(z)^2\,dy\,dz. \qquad (6.35)\end{aligned}$$

Combining (6.34) and (6.35), we obtain (6.26) with $C_0=4C_1^2/c_1^2$. □

In the next proposition, we show that logarithm of a harmonic function has bounded mean oscillation. Then using John-Nirenberg inequality we prove a weak form of elliptic Harnack inequality.

Proposition 6.11. *Under the assumptions of Theorem 6.3, there exists $q>0$, $c_0\in(0,1)$ and $C_0,r_0>0$ such that*

$$\phi(u,-q,c_0B)\le C_0\phi(u,q,c_0B) \qquad (6.36)$$

for all P-harmonic functions u on $B=B(x,r)$ with $r\ge r_0$ and for all $x\in M$.

Proof. The proof follows from the same argument as [23, Théorème 3.3] where the use of [23, Lemme 3.2] is replaced Lemma 6.10. □

6.5 Mean value inequality for subharmonic functions

For the rest of the section, we will rely on $(VD)_\infty$, $(VD)_{\mathrm{loc}}$ and the obolev inequality (5.2) to prove Theorem 6.3. We obtain various inequalities on subharmonic functions. The following elementary property of subharmonic and superharmonic functions and follows immediately from Jensen's inequality.

Lemma 6.12. *Let P be a Markov operator.*

(a) If f is a non-negative function that is P-subharmonic in $B(x,r)$, then f^p is P-subharmonic in $B(x,r)$ for all $p\in[1,\infty)$.

(b) If f is a positive function that is P-superharmonic in $B(x,r)$, then f^p is P-subharmonic in $B(x,r)$ for all $p<0$.

Moser's iteration relies on repeated application of the following Lemma.

Lemma 6.13. *Let (M,d,μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\mathrm{loc}}$ and $(VD)_\infty$. Suppose that a Markov operator P has a kernel p that is (h,h')-compatible with respect to μ for some $h>b$. Further assume that P satisfies the Sobolev inequality* (5.2). *There exists $C_0>0$ such that*

$$\phi(u,2(1+2/\delta),B(x,(1-\sigma)r-h')\le C_0\sigma^{-\delta/(\delta+2)}\phi(u,2,B(x,r+h')) \qquad (6.37)$$

for all $x\in M$, for all $r\ge 3h'$, for all $\sigma\in(0,1/2)$ and for all functions u that are non-negative and P-subharmonic on $B(x,r)$.

Proof. The proof of this result follows from a similar argument as that of [23, Proposition 5.2] where we use the version of Sobolev ineqaulity given by (5.2) instead. Although the Sobolev inequality takes a different form compared to [23], we use the subharmonicity of u to compare $P_B u$ with u (away from boundary of u). □

The following reverse Poincaré inequality.

Lemma 6.14 (Reverse Poincaré inequality)**.** *Let (M, d, μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\mathrm{loc}}$ and $(VD)_\infty$. Suppose that a Markov operator P has a kernel p that is weakly (h, h')-compatible with respect to μ for some $h > b$. For all $\Omega > 1$, there exists $C = C(\Omega)$ such that for all P-harmonic functions u, for all $x \in M$ and for all $r > 3h'/(\Omega - 1)$*

$$\int_{B(x,r)} |\nabla_P u|^2 \, d\mu \le C r^{-2} \int_{B(x,\Omega r)} u^2 \, d\mu. \tag{6.38}$$

In particular, there exists $C_R = C(2)$ such that such that for all P-harmonic functions u, for all $x \in M$ and for all $r > 3h'$

$$\int_{B(x,r)} |\nabla_P u|^2 \, d\mu \le C_R r^{-2} \int_{B(x,2r)} u^2 \, d\mu. \tag{6.39}$$

Proof. Define

$$\psi(y) := \max\left(0, \min\left(1, \frac{\Omega r - h' - d(x,y)}{(\Omega - 1)r - 2h'}\right)\right). \tag{6.40}$$

Note that $\psi \equiv 0$ in $B(x, \Omega r - h')^c$ and $\psi \equiv 1$ in $B(x, r + h')$. Since $\Delta u = (I - P)u = 0$, for all $r > 3h'/(\Omega - 1)$ and for all $x \in M$ we have

$$\begin{aligned}
0 &= -\int_M \psi^2(y) u(y) \Delta u(y) \, dy = -\int_{B(x,\Omega r - h')} \psi^2(y) u(y) \Delta u(y) \, dy \\
&= -\frac{1}{2} \int_{B(x,\Omega r)} \int_{B(x,\Omega r)} p_1(y,z) \left(\nabla_{yz}(\psi^2 u)\right) (\nabla_{yz} u) \, dy \, dz \\
&= -\frac{1}{2} \int_{B(x,\Omega r)} \int_{B(x,\Omega r)} p_1(y,z) \psi^2(y) \left(\nabla_{yz} u\right)^2 \, dy \, dz \\
&\quad - \frac{1}{2} \int_{B(x,\Omega r)} \int_{B(x,\Omega r)} p_1(y,z) u(z) \left(\nabla_{yz} \psi^2\right) (\nabla_{yz} u) \, dy \, dz.
\end{aligned} \tag{6.41}$$

The above steps follows from integration by parts (6.24) and product rule (6.20). We use the inequality $ab \le a^2/4 + b^2$ to obtain

$$\begin{aligned}
\left|u(z) \left(\nabla_{yz} \psi^2\right) (\nabla_{yz} u)\right| &= |(\psi(y) + \psi(z)) u(z) (\nabla_{yz} \psi)(\nabla_{yz} u)| \\
&\le \frac{1}{4} (\psi^2(y) + \psi^2(z)) \left(\nabla_{yz} u\right)^2 + 2u^2(z) \left(\nabla_{yz} \psi\right)^2.
\end{aligned} \tag{6.42}$$

Since $p_1(y,z) = p_1(z,y)$ for $\mu \times \mu$-almost every (y,z), we have

$$\int_{B_1} \int_{B_1} p_1(y,z) \psi^2(y) \left(\nabla_{yz} u\right)^2 \, dy \, dz = \int_{B_1} \int_{B_1} p_1(y,z) \psi^2(z) \left(\nabla_{yz} u\right)^2 \, dy \, dz \tag{6.43}$$

where $B_1 := B(x, \Omega r)$. Combining (6.41), (6.42) and (6.43)

$$\int_{B_1} \int_{B_1} p_1(y,z) \psi^2(y) \left(\nabla_{yz} u\right)^2 \, dy \, dz \le 4 \int_{B_1} \int_{B_1} p_1(y,z) u^2(z) \left(\nabla_{yz} \psi\right)^2 \, dy \, dz. \tag{6.44}$$

The inequality $(a + b)^2 \le 2(a^2 + b^2)$ along with product rule (6.20) implies

$$\begin{aligned}
\int_{B_1} \int_{B_1} p_1(y,z) \left(\nabla_{yz}(\psi u)\right)^2 \, dy \, dz &\le 2 \int_{B_1} \int_{B_1} p_1(y,z) \psi^2(y) \left(\nabla_{yz} u\right)^2 \, dy \, dz \\
&\quad + 2 \int_{B_1} \int_{B_1} p_1(y,z) u^2(z) \left(\nabla_{yz} \psi\right)^2 \, dy \, dz.
\end{aligned} \tag{6.45}$$

Combining (6.44) and (6.45), we obtain

$$\int_{B_1}\int_{B_1} p_1(y,z)\left(\nabla_{yz}(\psi u)\right)^2\,dy\,dz \le 10\int_{B_1}\int_{B_1} p_1(y,z)u^2(z)\left(\nabla_{yz}\psi\right)^2\,dy\,dz. \tag{6.46}$$

By (6.40) and (4.10), there exists $C_1>0$ such that

$$\left(\nabla_{yz}\psi\right)^2 p_1(y,z) \le (3h')^2(\Omega-1)^{-2}r^{-2}p_1(y,z)$$

for all $y\in M$, for μ-almost every $z\in M$ and for all $r>3h'/(\Omega-1)$. Therefore, we have

$$\int_{B_1}\int_{B_1} p_1(y,z)\left(\nabla_{yz}(\psi u)\right)^2\,dy\,dz \le (3h')^2(\Omega-1)^{-2}r^{-2}\int_{B_1} u^2(z)\,dz. \tag{6.47}$$

for all P-harmonic functions u, for all $r>3h'/(\Omega-1)$ and for all $x\in M$. Since $\psi\equiv 1$ in $B(x,r+h')$ the desired inequality (6.38) follows from (6.47). □

The next lemma is a L^2-mean value inequality for positive P-subharmonic functions.

Lemma 6.15. *Let (M,d,μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\rm loc}$ and $(VD)_\infty$. Suppose that a Markov operator P has a kernel p that is (h,h')-compatible with respect to μ for some $h>b$. Further assume that P satisfies the Sobolev inequality* (5.2)*. There exists $C_1>0$ and $r_1>0$ such that*

$$\phi(u,\infty,B(x,r/6)) \le C\phi(u,2,B(x,r+h')) \tag{6.48}$$

for all $x\in M$, for all $r\ge r_1$ and for all functions u that are non-negative and P-subharmonic on $B(x,r)$.

Proof. Define a sequence of radii iteratively by $r(1)=r+h'$,

$$r(i+1) = (r(i)-h')\left(1-\frac{1}{3^{i+1}}\right)-h'$$

for $i=1,2,\ldots,\lceil\log r\rceil$. By the above definition, there exists $r_0>0$ such that

$$r(\lceil\log r\rceil+2)-h' \ge r\left(1-\sum_{j=1}^{i}3^{-(i+1)}\right)-4h'(\log r+3) \ge r/2 \ge 3h' \tag{6.49}$$

for all $r\ge r_0$. We define the balls $B_i=B(x,r(i))$ for $i\in\mathbb{N}^*$ and exponents $p_i=(1+2/\delta)^i$ for $i\in\mathbb{N}_{\ge 0}$. By Lemma 6.12 u^{p_i} is P-subharmonic for all $i\in\mathbb{N}_{\ge 0}$. By applying Lemma 6.13 to the function $u^{p_{i-1}}$ that is P-subharmonic in B_i, we obtain

$$\phi(u,2p_i,B_{i+1}) \le C_0^{1/p_{i-1}}3^{-(i+1)/p_i}\phi(u,2p_{i-1},B_i) \tag{6.50}$$

for $i=1,2,\ldots,\lceil\log r\rceil$ and $r\ge r_0$. Combining the estimates in (6.50), there exists $C_2>0$ such that

$$\phi(u,2p_{\lceil\log r\rceil},B_{\lceil\log r\rceil+1}) \le C_2\phi(u,2,B(x,r+h')) \tag{6.51}$$

for all $x\in M$, for all $r\ge r_0$ and for all non-negative subharmonic u in $B(x,r)$. There exists $C_3,C_4>0$ such that

$$\begin{aligned}
\sup_{B(x,r/2)} u^{2p_{\lceil\log r\rceil}} &\le \sup_{B(x,r/2)} P(u^{2p_{\lceil\log r\rceil}})\\
&\le \sup_{y\in B_{\lceil\log r\rceil+1}}\frac{C_3}{V(y,h')}\int_{B_{\lceil\log r\rceil+1}} u^{2p_{\lceil\log r\rceil}}\,d\mu\\
&\le \frac{C_3}{\mu(B_{\lceil\log r\rceil+1})}\left(\sup_{y\in B(x,r)}\frac{V(y,2r)}{V(y,h')}\right)\int_{B_{\lceil\log r\rceil+1}} u^{2p_{\lceil\log r\rceil}}\,d\mu\\
&\le C_4r^\delta \fint_{B_{\lceil\log r\rceil+1}} u^{2p_{\lceil\log r\rceil}}\,d\mu
\end{aligned} \tag{6.52}$$

The first line above follows from Lemma 6.12, the second line follows from (4.10) and (6.49), the third line follows from (6.49) and the last line from (2.4) and (6.49). Combining (6.51) and (6.52), we obtain (6.48). □

The next lemma is analogous to Lemma 6.13 and will be used for an iteration procedure.

Lemma 6.16. *Let (M, d, μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\mathrm{loc}}$ and $(VD)_\infty$. Suppose that a Markov operator P has a kernel p that is (h, h')-compatible with (M, d, μ) for some $h > b$. Further assume that P satisfies the Sobolev inequality (5.2). There exists $C_0 > 0, r_0 > 0$ such that*

$$\begin{aligned}\int_{B(x,r/2)} \int_{B(x,r/2)} \psi(y)^2 |\nabla_{yz}(u^p)| p_1(y,z)\, dy\, dz \\ \le C_0 \left(\frac{2p}{2p-1}\right)^2 \int_{B(x,r/2)} \int_{B(x,r/2)} u(y)^{2p} |\nabla_{yz}\psi| p_1(y,z)\, dy\, dz\end{aligned} \tag{6.53}$$

for all $x \in M$, for all $r \ge r_0$, for all $p \in (0,1] \setminus \{1/2\}$, for all positive functions u that are P-harmonic on $B(x,r)$ and for all $\psi \ge 0$ with $\operatorname{supp}(\psi) \subseteq B(x, r/2 - h')$.

Proof. Let $\eta := u^{2p-1}\psi$, where $\psi \ge 0$ satisfies $\operatorname{supp}(\psi) \subseteq B(x, r/2 - h')$ and $u > 0$ is P-harmonic in $B(x,r)$. By product rule (6.20)

$$\nabla_{yz}\eta = \left(\nabla_{yz}(u^{2p-1})\right)\psi(y)^2 + u(z)^{2p-1}\left(\nabla_{yz}\psi^2\right).$$

By integration by parts (6.24), we obtain

$$\begin{aligned}&\int_B \int_B p_1(y,z)(\nabla_{yz}u)\left(\nabla_{yz}(u^{2p-1})\right)\psi(y)^2\, dy\, dz \\ &= -\int_B \int_B p_1(y,z)\left(\nabla_{yz}u\right) u(z)^{2p-1}\left(\nabla_{yz}(\psi^2)\right)\, dy\, dz\end{aligned} \tag{6.54}$$

where $B := B(x, r/2)$. There exists $C_1 > 0$ such that

$$|2p-1| \left(\nabla_{yz}(u^p)\right)^2 \le p^2 (\nabla_{yz}u)(\nabla_{yz}(u^{2p-1})) \tag{6.55}$$

$$|\nabla_{yz}u| u(z)^{p-1} \le C_1 p^{-1} |\nabla_{yz}(u^p)|. \tag{6.56}$$

for all $p \in (0,1]$, for all $y, z \in M$ with $d(y,z) \le h'$ and for all positive u. The estimate (6.55) is elementary and is a version of Stroock-Varopoulos inequality. The proof of (6.55) is essentially contained in [60, Lemma 2.4]. The estimate (6.56) follows from mean value theorem and the local Harnack inequality given by Lemma 6.4. Combining (6.54), (6.55) and (6.56), we have

$$\begin{aligned}&C_1^{-1}\frac{|2p-1|}{p}\int_B \int_B p_1(y,z)\psi(y)^2 |\nabla_{yz}(u^p)|^2\, dy\, dz \\ &\le \int_B \int_B p_1(y,z) u(z)^p |\nabla_{yz}\psi| |\psi(y) + \psi(z)| |\nabla_{yz}(u^p)|\, dy\, dz \\ &\le \left(\int_B \int_B p_1(y,z) u(y)^{2p} |\nabla_{yz}\psi|^2\, dy\, dz\right)^{1/2} \\ &\quad \times \left(\int_B \int_B p_1(y,z) 2(\psi(y)^2 + \psi(z)^2) |\nabla_{yz}(u^p)|^2\, dy\, dz\right)^{1/2}.\end{aligned} \tag{6.57}$$

We use Cauchy-Schwarz inequality and $(a+b)^2 \le 2(a^2+b^2)$ in the last step. By the $\mu \times \mu$-almost everywhere symmetry of p_1, we have

$$\int_B \int_B p_1(y,z)\psi(z)^2 |\nabla_{yz}(u^p)|^2\, dy\, dz = \int_B \int_B p_1(y,z)\psi(y)^2 |\nabla_{yz}(u^p)|^2\, dy\, dz. \tag{6.58}$$

Combining (6.57) and (6.58) yields (6.53). □

We do another iteration procedure between the exponents q and 2 using Lemma 6.16.

Lemma 6.17. *Let (M,d,μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\mathrm{loc}}$ and $(VD)_\infty$. Suppose that a Markov operator P has a kernel p that is (h,h')-compatible to (M,d,μ) for some $h>b$. Further assume that P Sobolev inequality* (5.2)*. For any fixed $q>0$, there exists $C_1>0, c_1\in(0,1/2)$ and $r_1>0$ such that*

$$\phi(u,2,B(x,c_1r)) \le C_1\phi(u,q,B(x,r/2)) \tag{6.59}$$

for all $x\in M$, for all $r\ge r_1$ and for all functions u that are non-negative and P-subharmonic on $B(x,r)$.

Proof. If $q\ge 2$, then (6.59) follows from Jensen's inequality. Hence it suffices to consider $q\in(0,2)$.

Define $\theta := \delta/(\delta-2)$. We slightly decrease q if necessary so that $q\theta^k \neq 1/2$ for all $i\in\mathbb{N}$. Define $k\in\mathbb{N}^*$ as the integer that satisfies $q\theta^{k-1} < 2 \le q\theta^k$. Define $c_1 := 4^{-k}$ and iteratively define

$$s_i := 2s_{i-1} + 2h'$$

for $i=1,\dots,k$, where $s_0 := c_1 r$. Fix $r_0>0$ such that $s_k \le r/2-h'$ for all $r\ge r_0$ where k and s_k are defined as above.

Define $q_i := q\theta^i/2$, $B_i = B(x,s_{k-i})$ for $i=0,1,\dots,k$. Define the functions

$$\psi_i(y) = \max\left(0, \min\left(1, \frac{2s_{k-i-1}+h'-d(x,y)}{s_{k-i-1}}\right)\right)$$

for $i=0,1,\dots,k-1$. Note that $\psi_i\equiv 1$ in $B(x,s_{k-i-1}+h')$ and $\psi\equiv 0$ in $B(x,s_{k-i}-h')^c$.

By Sobolev inequality (5.2) there exists $C_2>0$ such that

$$\begin{aligned}\left(\int_{B_i}\left(P_{B_i}(\psi_i u^{q_i})(y)\right)^{2\theta} dy\right)^{1/\theta} \le{}& \frac{C_2 s_{k-i}^2}{\mu(B_i)^{2/\delta}}\int_{B_i}\int_{B_i} p_1(y,z)|\nabla_{yz}(\psi_i u^{q_i})|^2\,dy\,dz\\ &+\frac{C_2}{\mu(B_i)^{2/\delta}}\int_{B_i}\psi_i(y)^2 u(y)^{2q_i}\,dy\end{aligned} \tag{6.60}$$

for all $i=0,1,\dots,k-1$. By (4.10) and Lemma 6.4 there exists $C_3>0$ such that

$$P_{B_i}(\psi u^{q_i})(y) = \int_{B(y,h')} u^{q_i}(z)p_1(y,z)\,dz \ge C_3^{-q_i}u^{q_i}(y)$$

for all $y\in B_{i+1}$. Therefore

$$\left(\int_{B_{i+1}} u(y)^{2q_{i+1}}\,dy\right)^{1/\theta} \le \left(C_3^{q_i}\int_{B_i}\left(P_{B_{i+1}}(\psi u^{q_i})(y)\right)^{2\theta}\,dy\right)^{1/\theta} \tag{6.61}$$

for $x\in B_{i+1}$. There exists $C_4,C_5,C_6>0$ such that

$$\begin{aligned}&\int_{B_i}\int_{B_i} p_1(y,z)|\nabla_{yz}(\psi u^{q_i})|^2\,dy\,dz\\ &\le 2\int_{B_i}\int_{B_i} p_1(y,z)\psi(y)^2|\nabla_{yz}(u^{q_i})|^2\,dy\,dz + 2\int_{B_i}\int_{B_i} p_1(y,z)|\nabla_{yz}\psi|^2 u(z)^{2q_i}\,dy\,dz\\ &\le C_4\left[\left(\frac{2q_i}{2q_i-1}\right)^2+1\right]\int_{B_i}\int_{B_i} p_1(y,z)|\nabla_{yz}\psi_i|^2 u(y)^{2q_i}\,dy\,dz\\ &\le \frac{C_5}{s_{k-i-1}^2}\left[\left(\frac{2q_i}{2q_i-1}\right)^2+1\right]\int_{B_i} u(z)^{2q_i}\,dz\\ &\le \frac{C_6}{s_{k-i-1}^2}\int_{B_i} u(z)^{2q_i}\,dz.\end{aligned} \tag{6.62}$$

In the first step above, we used product rule (6.20) and the inequality $(a+b)^2 \le 2(a^2+b^2)$. In the second step we use Lemma 6.16 and in the third step we use (4.10). In the last step, we simply bound $2q_i/|2q_i-1|$ by $\max_{0\le i\le k} 2p_i/|2p_i-1| < \infty$.

Combining (6.60), (6.61), (6.62) along with $s_{k-i}/s_{k-i-1} \le 4^k$ yields

$$\left(\int_{B_{i+1}} u(y)^{2q_{i+1}}\,dy\right)^{1/\theta} \le \frac{C_7}{\mu(B_i)^{2/\delta}} \int_{B_i} u(y)^{2q_i}\,dy$$

for some $C_7 > 0$. Combined with $r \ge r_0$ and (2.4), we deduce

$$\phi(u, 2q_{i+1}, B_{i+1}) \le C_8 \phi(u, 2q_i, B_i) \tag{6.63}$$

for $i = 0, 1, \dots, k-1$, for all $x \in M$, for all $r \ge r_0$ and for all P-harmonic $u > 0$. The estimates (6.63) along with Jensen's inequality implies (6.59) with $C_1 = C_8^k$ and $c_1 = 4^{-k}$. □

We are now ready to prove elliptic Harnack inequality.

Proof of Theorem 6.3. It suffices to consider the case $u > 0$ because we can replace $u \ge 0$ by $u + \epsilon$ and let $\epsilon \downarrow 0$.

Note that we have Sobolev inequality (5.2) by Theorem 5.1. There exists $r_0 > 0$ $C_i > 0, c_i \in (0,1)$ for $1 \le i \le 5$ such that for all $x \in M$ and for all $r \ge r_0$ and for all positive functions u that are P-harmonic on $B := B(x,r)$

$$\begin{aligned}\phi(u, \infty, c_1 B) &\le C_1 \phi(u, c_2, B)\\ &\le C_2 \phi(u, q, c_3 B)\\ &\le C_3 \phi(u, -q, c_4, B)\\ &\le C_4 \phi(u, -\infty, c_5 B).\end{aligned}$$

The first line above follows from Lemma 6.15, the second line above follows from Lemma 6.17 and the third line follows from Proposition 6.11. The last line follows from applying Lemma 6.15 to the function $u^{-q/2}$ which is subharmonic by Lemma 6.12(b). Choosing $c = \min(c_1, c_5)$ yields the elliptic Harnack inequality. □

The constant $c \in (0,1)$ in (6.1) is flexible. More precisely, we can slightly improve the conclusion of Theorem 6.3 for b-geodesic spaces by an easy chaining.

Corollary 6.18 (Elliptic Harnack inequality)**.** *Let (M,d,μ) be a b-geodesic space satisfying $(VD)_{\mathrm{loc}}$, $(VD)_\infty$ and Poincaré inequality $(P)_h$ at scale h. Suppose that a Markov operator P has a kernel p that is (h,h')-compatible with (M,d,μ) for some $h > b$. Then for all $c \in (0,1)$, there exists $r_0 > 0, C_E > 0$ such that for all $x \in M$, for all $r \ge r_0$ and for all non-negative functions $u : B(x,r) \to \mathbb{R}_{\ge 0}$ that are P-harmonic in $B(x,r)$ the following Harnack inequality holds:*

$$\sup_{x \in B(x,cr)} u \le C_E \inf_{x \in B(x,cr)} u. \tag{6.64}$$

The above corollary is a consequence of Theorem 6.3 applied repeatedly to a sequence of points in an approximate geodesic. We do not use the above corollary. The proof of Corollary 6.18 is left to the reader.

6.6 Applications of elliptic Harnack inequality

We present two immediate and well-known applications of elliptic Harnack inequality.

Proposition 6.19 (Liouville property). *Let (M,d,μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\mathrm{loc}}$, $(VD)_\infty$ and Poincaré inequality $(P)_h$ at scale h. Suppose that a Markov operator P has a kernel p that is (h,h')-compatible with (M,d,μ) for some $h>b$. Then all non-negative P-harmonic functions are constant (strong Liouville property). Therefore all bounded harmonic functions are constant (weak Liouville property).*

Proof. Let u be a non-negative harmonic function. Then $v = u - \inf u$ is a non-negative harmonic function with $\inf v = 0$. By elliptic Harnack inequality, there exists $c \in (0,1)$ and $C > 1$ such that $\sup_{B(x,cr)} v \le C \inf_{B(x,cr)} v$ for all large enough r. Letting $r \to \infty$, we have $\sup_M v \le 0$ which implies $v \equiv 0$. This proves strong Liouville property. The weak Liouville property follows from the observation that for any bounded harmonic function h, the function $h - \inf h$ is a non-negative harmonic function. □

The following Hölder regularity-type estimate is a direct consequence of elliptic Harnack inequality. Our argument is an adaptation of Moser's argument [58, Section 5] which uses an oscillation inequality.

Proposition 6.20. *There exists $c \in (0,1)$, $\alpha > 0$, $C > 0$ and $r_1 > 0$ such that*

$$|u(y) - u(z)| \le C \left(\frac{\max(d(y,z),1)}{r} \right)^{\alpha} \sup_{B(x,r)} u \tag{6.65}$$

for all $y,z \in B(x,cr)$, for all $x \in M$, for all $r \ge r_1$ and for all non-negative functions $u : M \to \mathbb{R}$ that is P-harmonic on $B(x,r)$ with $B(x,r) \ne M$.

Proof. The proof follows from using Moser's argument [58, Section 5] by iterating the oscillation lemma at all large enough scales. □

Note that above result does not give Hölder continuity for harmonic functions which is in contrast to [58, Section 5]. However we will see that Proposition 6.20 is useful. In particular, we use Proposition 6.20 to prove Gaussian lower bounds in Section 8.

7 Gaussian upper bounds

The goal of this section is to prove the following Gaussian upper bounds using Sobolev inequality. The results of this section rely only on $(VD)_{\mathrm{loc}}$, $(VD)_\infty$ and the Sobolev inequality (5.2). We do not assume the Poincaré inequality $(P)_h$ to show Gaussian upper bounds. More precisely, we show

Proposition 7.1. *Let (M,d,μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\mathrm{loc}}$ and $(VD)_\infty$. Suppose that a Markov operator P has a kernel p that is (h,h')-compatible with (M,d,μ) for some $h>b$. Further assume that P satisfies the Sobolev inequality* (5.2)*. There exists $C>0$ such that*

$$p_n(x,y) \le \frac{C}{V(x,\sqrt{n})} \exp\left(\frac{-d(x,y)^2}{Cn} \right) \tag{7.1}$$

for all $x \in M$ and for all $n \in \mathbb{N}_{\ge 2}$.

The first step is to obtain the following on-diagonal upper bound.

Proposition 7.2. *Let (M,d,μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\mathrm{loc}}$ and $(VD)_\infty$. Suppose that a Markov operator P has a kernel p that is (h,h')-compatible with (M,d,μ) for some $h>b$. Further assume that P satisfies the Sobolev inequality* (5.2)*. There exists $C_0>0$ such that*

$$p_n(x,x) \le \frac{C}{V(x,\sqrt{n})} \tag{7.2}$$

for all $x \in M$ and for all $n \in \mathbb{N}_{\ge 2}$.

A crucial ingredient in the proof of Proposition 7.2 is a L^1 to L^∞ mean value inequality for the solutions of a heat equation. We again rely on Moser's iterative method and the calculations are similar but more involved than those encountered in Section 6.5 for harmonic functions. The lazy walk defined in Example 4.4 will play an important role in this section. Recall that for a Markov operator P, the corresponding 'lazy' versions of Markov operator and Laplacian are given by

$$P_L = (I+P)/2 \text{ and } \Delta_L = \Delta/2 = (I-P)/2. \tag{7.3}$$

For $a, b \in \mathbb{N}$, we denote the integer intervals by

$$[\![a,b]\!] := \{i \in \mathbb{N} \,:\, a \le i \le b\}.$$

The following definition is analogous to Definition 6.1. Caloric functions are solutions to heat equation.

Definition 7.3. *Let P be a Markov operator on (M,d,μ) and let $a,b \in \mathbb{N}$. A function $u : \mathbb{N} \times M \to \mathbb{R}$ is P-caloric (respectively P_L-caloric) in $[\![a,b]\!] \times B(x,r)$ if*

$$\partial_k u(y) + \Delta u_k(y) = 0 \qquad (\textit{respectively } \partial_k u(y) + \Delta_L u_k(y) = 0)$$

for all $k \in [\![a,b]\!]$ and for all $y \in B(x,r)$.

Similarly, we say a function $u : \mathbb{N} \times R \to R$ is P-subcaloric (resp. P-supercaloric) in $[\![a,b]\!] \times B(x,r)$ if

$$\partial_k u(y) + \Delta u_k(y) \le 0 \qquad (\textit{respectively } \ge 0)$$

for all $k \in [\![a,b]\!]$ and for all $y \in B(x,r)$. Analogously, we define P_L-subcaloric and P_L-supercaloric functions simply by replacing Δ with Δ_L in the equation above.

Remark 7.4.

(a) We can restate the above definitions using $\partial_k u + \Delta u_k = u_{k+1} - P u_k$ and $\partial_k u + \Delta_L u_k = u_{k+1} - P_L u_k$.

(b) Consider a Markov operator P that is (h,h')-compatible with (M,d,μ). Similar to Remark 6.2(a), the property that $u : \mathbb{N} \times M \to \mathbb{R}$ is P-caloric (or P_L-caloric) in $[\![a,b]\!] \times B(x,r)$ depends only on the value of u in $[\![a,b+1]\!] \times B(x,r+h')$. Therefore it suffices if the function u has a domain that satisfies $[\![a,b+1]\!] \times B(x,r+h') \subseteq \mathrm{Domain}(u)$.

Although our eventual goal is to prove parabolic Harnack inequality for P-caloric functions, the Moser's iteration procedure is applied to P_L-caloric functions. The laziness is introduced to handle certain technical difficulties that arise due to discreteness of time. Another method to avoid these technical difficulties that arise due to discreteness of time is to carry out Moser's iteration method for solutions of the continuous time heat equation $\frac{\partial u}{\partial t} + \Delta u = 0$ (See [25, Section 2] for this method on graphs).

In continuous time case the product rule of differentiation implies $\frac{\partial (u^2)}{\partial t} = 2u \frac{\partial u}{\partial u}$; however for discrete time the analogous formula is $\partial_k(u^2) = 2u_k \partial u_k + (\partial_k u)^2$. The 'error term' $(\partial_k u)^2$ due to discreteness of time is a source of difficulty in the proofs of Caccioppoli inequality and an integral maximum principle for P-caloric and P-subcaloric functions. However as we shall see, this 'error term' can be handled using a Cauchy-Schwarz inequality for P_L-caloric and P_L-subcaloric functions (See Remark 7.9). As a result, we will primarily be concerned with P_L-caloric and P_L-subcaloric functions for now. The assumption (d) in Definition 4.7 will allow to compare the random walks driven by P and P_L.

The following lemma and its proof is analogous to that of Lemma 6.12.

Lemma 7.5. *Let P be a Markov operator. Assume that the function $u : \mathbb{N} \times M \to \mathbb{R}_{\geq 0}$ is P-subcaloric in $[\![a,b]\!] \times B(x,r)$ for some $x \in M$, $r > 0$ and $a, b \in \mathbb{N}$. Then u^p is a P-subcaloric in $[\![a,b]\!] \times B(x,r)$ for all $p \geq 1$.*

Proof. Note that

$$u_{k+1}^p(y) = (\partial_k u + u_k)^p(y) \leq (-\Delta u_k + u_k)^p = (Pu_k(y))^p \leq P(u_k^p)(y)$$

for all $(k,y) \in [\![a,b]\!] \times B(x,r)$. The first inequality above follows from the fact that $u \geq 0$ is P-subcaloric in $[\![a,b]\!] \times B(x,r)$ and the second follows from Jensen's inequality. □

For a function $f : \mathbb{N} \times M \to \mathbb{R}$ and a Markov operator P on M, we denote the function $Pf : \mathbb{N} \times M \to \mathbb{R}$

$$Pf(k,x) := (Pf(k,\cdot))(x) = (Pf_k)(x)$$

for all $k \in \mathbb{N}$ and for all $x \in M$. We require the following property of subcaloric functions.

Lemma 7.6. *Let (M,d,μ) be a metric measure space and let P be a Markov operator that is (h,h')-compatible to (M,d,μ). If $u : \mathbb{N} \times M \to \mathbb{R}$ is P_L-subcaloric in $[\![a,b]\!] \times B(x,r)$, then Pu is P_L-subcaloric in $[\![a,b]\!] \times B(x, r-h')$) for all $x \in M$ and for all $r > h'$.*

Proof. If $(k,y) \in [\![a,b]\!] \times B(x, r-h')$ and $u : \mathbb{N} \times M \to \mathbb{R}$ is P_L-subcaloric in $[\![a,b]\!] \times B(x,r)$, then

$$[(Pu)_{k+1} - P_L(Pu)_k](y) = P\left(u_{k+1} - P_L u_k\right)(y) \leq 0.$$

In the above equality, we used that P and P_L commute. The inequality follows from (4.10) and the fact that any Markov operator is positivity preserving. □

7.1 Mean value inequality for subcaloric functions

We will prove the following mean value inequality in a weak form. The inequality bounds from above a weak version of L^∞ norm on a space-time cylinder by a weak version of L^1 norm. Our version of the mean value inequality in Lemma 7.7 is weaker than the one known for graphs [18, Theorem 4.1] mainly because we rely on a weaker Sobolev-type inequality (5.2). Although the mean value inequality is weaker, we will obtain on-diagonal upper bounds using Lemma 7.7. Using an integral maximum principle argument, we will obtain Gaussian upper bounds in Section 7.

Lemma 7.7. *Under the assumptions of Proposition 7.2, there exists $C_1 > 0, n_1 > 0$ such that*

$$\inf_{k \in [\![0,n]\!]} \sup_{y \in B(x,\sqrt{n}/2)} P^{2\lceil \log \sqrt{n} \rceil + 2} u_k(y) \leq \frac{C_1}{V(x,\sqrt{n})} \sup_{k \in [\![0,n]\!]} \int_{B(x,\sqrt{n}+h')} u_k \, d\mu \tag{7.4}$$

for all $n \in \mathbb{N}^*$ satisfying $n > n_1$, for all $x \in M$, for all non-negative functions $u : \mathbb{N} \times M \to \mathbb{R}$ that is P_L-subcaloric in $[\![0,n]\!] \times B(x,\sqrt{n})$.

The proof of Lemma 7.7 relies on Moser's iteration procedure. Couhlon and Grigor'yan [18, Section 4] obtained a similar (stronger) mean value inequality in the graph setting using an iteration procedure. However they relied on a Faber-Krahn inequality that is equivalent to the Sobolev inequality (5.1) and therefore does not hold for discrete time Markov chains on continuous spaces.

In this section, we carry out Moser's iteration procedure for subcaloric functions relying on the weaker[1] Sobolev inequality (5.2). To prove the elementary iterative step of iteration, we need the following discrete Caccioppoli inequality. The proof is an adaptation [18, Proposition 4.5]. The next two Lemmas together may be regarded as the parabolic version of Lemma 6.13.

[1] 'weaker' compared to Sobolev inequalities in [67, 68, 74, 23, 25, 42].

Lemma 7.8 (Caccioppoli inequality)**.** *Under the assumptions on Proposition 7.2, we have*

$$\int_M \partial_k(u^2)\psi^2\,d\mu + \frac{1}{8}\mathcal{E}(u_k\psi, u_k\psi) \le \frac{17}{8}\int_M\int_M |\nabla_{yz}\psi|^2 u_k^2(y)p_1(y,z)\,dy\,dz \tag{7.5}$$

for all $x \in M$, for all $r > 0$, for all non-negative functions $\psi : M \to \mathbb{R}_{\ge 0}$ satisfying $\mathrm{supp}(\psi) \subseteq B(x,r)$, for all $a, b \in \mathbb{N}$, for all $k \in [\![a,b]\!]$ and for all non-negative functions $u : \mathbb{N} \times M \to \mathbb{R}_{\ge 0}$ such that u is P_L-subcaloric in $[\![a,b]\!] \times B(x,r)$.

Proof. Fix $x \in M$, $r > h'$ and define $B := B(x, r+h')$. Let $u : \mathbb{N} \times M \to \mathbb{R}_{\ge 0}$ be such that u is P_L-subcaloric in $[\![a,b]\!] \times B(x,r)$. We start with the elementary inequality

$$\partial_k(u^2)(y) \le -u_k(y)\Delta u_k(y) + \frac{1}{4}\left(\Delta u_k(y)\right)^2 \tag{7.6}$$

for all $(k,y) \in [\![a,b]\!] \times B(x,r)$, as we now show. Since u is P_L-subcaloric in $[\![a,b]\!] \times B(x,r)$, we have $u_{k+1}(y) \le P_L u_k(y)$ for all $(k,y) \in [\![a,b]\!] \times B(x,r)$. Combined with the fact that u is non-negative, we have $u_{k+1}^2(y) \le (P_L u_k(y))^2$ for all $(k,y) \in [\![a,b]\!] \times B(x,r)$ which can be rearranged into (7.6).

Let $(k,y) \in [\![a,b]\!] \times B(x,r)$. Recall that $B = B(x, r+h')$. Using (7.6), integration by parts (6.24) and $\mathrm{supp}(\psi) \subseteq B(x,r)$, we have

$$\begin{aligned}\int_B \psi^2 \partial_k(u^2)\,d\mu \le -\frac{1}{2}\int_B\int_B (\nabla_{yz}u_k)\left(\nabla_{yz}(u_k\psi^2)\right)p_1(y,z)\,dy\,dz \\ + \frac{1}{4}\int_B (\Delta u_k(y))^2\psi^2(y)\,dy.\end{aligned} \tag{7.7}$$

The second term in (7.7) can be handled using Cauchy-Schwarz inequality as

$$\begin{aligned}(\Delta u_k(y))^2 &= \left(-\int_M (\nabla_{yz}u_k)p_1(y,z)\,dz\right)^2 \\ &\le \left(\int_M p_1(y,z)\,dz\right)\left(\int_M (\nabla_{yz}u_k)^2 p_1(y,z)\,dz\right) \\ &= \int_M (\nabla_{yz}u_k)^2 p_1(y,z)\,dz.\end{aligned} \tag{7.8}$$

For the first term in (7.7), we use product rule (6.20)

$$\nabla_{yz}(u_k\psi^2) = u_k(z)\nabla_{yz}\psi^2 + \psi^2(y)\nabla_{yz}u_k. \tag{7.9}$$

Combining (7.7), (7.8) and (7.9), we have

$$\begin{aligned}&\int_B \psi^2(y)\partial_k(u^2)(y)\,dy + \frac{1}{4}\int_B\int_B (\nabla_{yz}u_k)^2\,\psi^2(y)p_1(y,z)\,dy\,dz \\ &\le -\frac{1}{2}\int_B\int_B u_k(z)\left(\nabla_{yz}\psi^2\right)(\nabla_{yz}u_k)\,p_1(y,z)\,dy\,dz.\end{aligned} \tag{7.10}$$

The right side of (7.10) can be bounded using $t_1t_2 \le t_1^2/8 + 2t_2^2$ as

$$\begin{aligned}\left|-u_k(z)\left(\nabla_{yz}\psi^2\right)(\nabla_{yz}u_k)\right| &\le u_k(z)\psi(y)|(\nabla_{yz}\psi)(\nabla_{yz}u_k)| \\ &\quad + u_k(z)\psi(z)|(\nabla_{yz}\psi)(\nabla_{yz}u_k)| \\ &\le \frac{1}{8}(\psi^2(y)+\psi^2(z))(\nabla_{yz}u_k)^2 \\ &\quad + 4u_k^2(z)|\nabla_{yz}\psi|^2.\end{aligned} \tag{7.11}$$

Using $p_1(y,z) = p_1(z,y)$ for $\mu \times \mu$-almost every (y,z), we obtain

$$\int_B \int_B \psi^2(y) \left(\nabla_{yz} u_k\right)^2 p_1(y,z)\, dy\, dz = \int_B \int_B \psi^2(z) \left(\nabla_{yz} u_k\right)^2 p_1(y,z)\, dy\, dz. \tag{7.12}$$

Combining (7.10), (7.11) and (7.12), we deduce

$$\begin{aligned} &\int_B \psi^2(y) \partial_k(u^2)(y)\, dy + \frac{1}{8} \int_B \int_B \left(\nabla_{yz} u_k\right)^2 \psi^2(y) p_1(y,z)\, dy\, dz \\ &\le 2 \int_B \int_B u_k^2(z) \left(\nabla_{yz} \psi\right)^2 p_1(y,z)\, dy\, dz. \end{aligned} \tag{7.13}$$

Since $\operatorname{supp}(\psi) \subseteq B(x_0, r - h')$, using integration by parts (6.24) we have

$$\mathcal{E}(\psi u_k, \psi u_k) = \frac{1}{2} \int_B \int_B \left|\nabla_{yz}(u_k \psi)\right|^2 p_1(y,z)\, dy\, dz. \tag{7.14}$$

Using product rule (6.20) and the inequality $(t_1 + t_2)^2 \le 2(t_1^2 + t_2^2)$, we obtain

$$\begin{aligned} \left|\nabla_{yz}(u_k \psi)\right|^2 &= \left|\psi(y)(\nabla_{yz} u_k) + u_k(z)(\nabla_{yz} \psi)\right|^2 \\ &\le 2 \left(\psi^2(y)(\nabla_{yz} u_k)^2 + u_k^2(z)(\nabla_{yz} \psi)^2\right). \end{aligned} \tag{7.15}$$

Combining (7.13), (7.14), (7.15) and $\mu \times \mu$-almost everywhere symmetry of p_1 yields (7.5). ☐

Remark 7.9. Recall the product rule of differentiation $\partial_k(u^2) = 2u_k \partial u_k + (\partial_k u)^2$ gives rise to the 'error term' $(\partial_k u)^2$ which occurs due to discreteness of time. This error term occurs in (7.7) and is controlled using Cauchy-Schwarz inequality in (7.8). However the estimate given by (7.8) is sufficient to prove Caccioppoli inequality only in the presence of some laziness. A similar difficulty arises in the proof of discrete integral maximum principle and is the reason behind considering the operator P_L as opposed to P in this section.

Next, we prove the elementary iterative step of Moser's iteration in parabolic setting. The proof relies on Caccioppoli inequality (7.5) and Sobolev inequality (5.2). Let μ_c denote the counting measure on $\mathbb{N}$ and let (M, d, μ) be a metric measure space. We denote the product measure on $\mathbb{N} \times M$ by $\tilde{\mu} := \mu_c \times \mu$. Similar to (6.2), we define

$$\tilde{\phi}(u, p, Q) := \left(\frac{1}{\tilde{\mu}(Q)} \int_Q u^p \, d\tilde{\mu} \right)^{1/p} \tag{7.16}$$

for all $p > 0$, for all $Q \subset \mathbb{N} \times M$ and for all functions $u : \mathbb{N} \times M \to \mathbb{R}_{\ge 0}$.

Lemma 7.10. *Under the assumptions of Proposition 7.2, for all $K_1 \ge 1$, there exists $C_1 > 0, r_1 > 0$ (depending on K_1) such that*

$$\begin{aligned} &\tilde{\phi}(Pu, 2 + (4/\delta), [\![\lceil (1-\sigma^2)a_0 + \sigma^2 a_1 \rceil, a_1]\!] \times B(x, (1-\sigma)r - h')) \\ &\quad \le C_1 \sigma^{-1} \tilde{\phi}(u, 2, [\![a_0, a_1]\!] \times B(x, r + h')) \end{aligned} \tag{7.17}$$

for all $\sigma \in (0, 1/2)$, for all $x \in M$, for all $r \ge r_1$, for all $a_0, a_1 \in \mathbb{N}$ satisfying $K_1^{-1} r^2 \le a_2 - a_1 \le K_1 r^2$ and for all non-negative functions $u : \mathbb{N} \times M \to \mathbb{R}_{\ge 0}$ such that u is P_L-subcaloric in $[\![a_0, a_1]\!] \times B(x, r)$.

Proof. Let $x \in M$, $\sigma \in (0, 1/2)$ and let $r > r_1 \ge 4h'$, where r_1 will be determined later. Let u be a non-negative function that is P_L-subcaloric in $[\![a_0, a_1]\!] \times B(x, r)$.

We start by defining appropriate cut-off functions in space and time. Define $B := B(x, r+h')$ and $\psi : M \to \mathbb{R}_{\geq 0}$ as

$$\psi_\sigma(y) := \max\left(0, \min\left(1, \frac{r - d(x,y)}{\sigma r}\right)\right).$$

Note that $\operatorname{supp}(\psi_\sigma) \subseteq B(x,r)$ and $\psi \equiv 1$ on $B(x,(1-\sigma)r)$. Define $a_\sigma := \lceil (1-\sigma^2)a_0 + \sigma^2 a_1 \rceil$ and $\chi : \mathbb{N} \to \mathbb{R}$ as

$$\chi_\sigma(k) = \begin{cases} 1 & \text{if } k \geq a_\sigma \\ 0 & \text{if } k \leq a_0 \\ \frac{k-a_0}{a_\sigma - a_0} & \text{otherwise.} \end{cases}$$

Since u is non-negative and P_L-subcaloric in $[\![a_0, a_1]\!] \times B(x,r)$, by Caccioppoli inequality (Lemma 7.8) and product rule (6.22), we obtain

$$\begin{aligned} &\int_B \left(\partial_k(\chi_\sigma u)^2\right)\psi_\sigma^2\, d\mu + \frac{\chi_\sigma^2(k+1)}{8}\mathcal{E}^B(\psi_\sigma u_k, \psi_\sigma u_k) \\ &\leq \frac{17}{8}\chi_\sigma^2(k+1)\int_B\int_B |\nabla_{yz}\psi_\sigma|^2 u_k^2(y) p_1(y,z)\, dy\, dz + \partial_k \chi_\sigma^2 \int_B u_k^2 \psi_\sigma^2\, d\mu \end{aligned} \tag{7.18}$$

for all $k \in [a, b)$. Since p_1 is (h, h')-compatible with (M, d, μ), we have

$$|\nabla_{yz}\psi|^2 p_1(y,z) \leq \frac{(h')^2}{(\sigma r)^2} p_1(y,z). \tag{7.19}$$

We use product rule (6.22), triangle inequality, $\chi_\sigma \leq 1$ and $a_\sigma - a_0 \geq \sigma^2(a_1 - a_0) \geq \sigma^2 K_1^{-1} r^2$ to deduce

$$\left|\partial_k \chi_\sigma^2\right| \leq (\chi_\sigma(k+1) + \chi_\sigma(k))|\partial_k\chi_\sigma| \leq 2|\partial_k \chi_\sigma| \leq \frac{2}{(a_\sigma - a_0)} \leq \frac{2K_1}{\sigma^2 r^2} \tag{7.20}$$

Combining (7.18), (7.19) and (7.20), there exists $C_2 > 0$ such that

$$\int_B \psi_\sigma^2\left(\partial_k(\chi_\sigma u)^2\right)\, d\mu + \frac{\chi_\sigma^2(k+1)}{8}\mathcal{E}^B(\psi_\sigma u_k, \psi_\sigma u_k) \leq \frac{C_2}{\sigma^2 r^2}\int_B u_k^2\, d\mu \tag{7.21}$$

for all $k \in [\![a_0, a_1]\!]$. In (7.21), C_2 depends only on K_1 and h'.

Adding (7.21), from $k = a_0$ to $k \in [\![a_0, a_1]\!]$, yields

$$\sup_{k \in [a_\sigma, a_1]} \int_B \psi_\sigma^2 u_k^2\, d\mu \leq \frac{C_2}{\sigma^2 r^2}\sum_{k=a_0}^{a_1}\int_B u_k^2\, d\mu \tag{7.22}$$

$$\sum_{k=a_\sigma}^{a_1}\mathcal{E}(\psi_\sigma u_k, \psi_\sigma u_k) \leq \frac{8C_2}{\sigma^2 r^2}\sum_{k=a_0}^{a_1}\int_B u_k^2\, d\mu. \tag{7.23}$$

Define $w_k := P_B(\psi_\sigma u_k)$. Since $\psi \equiv 1$ on $B(x,(1-\sigma)r)$, by (4.10) $w_k = P_B(\psi_\sigma u_k) = Pu_k$ on $B(x,(1-\sigma)r - h')$. Combined with Hölder inequality, we have

$$\int_{B(x,(1-\sigma)r-h')} (Pu_k)^{2(1+2/\delta)}\, d\mu \leq \left(\int_B w_k^2\, d\mu\right)^{2/\delta}\left(\int_B w_k^{2\delta/(\delta-2)}\, d\mu\right)^{(\delta-2)/\delta}. \tag{7.24}$$

Since P_B is a contraction in $L^2(B)$, we have

$$\int_B w_k^2\, d\mu \leq \int_B \psi_\sigma^2 u_k^2\, d\mu. \tag{7.25}$$

By Sobolev inequality (5.2), Lemma 4.20(a) and (4.10), we obtain

$$\left(\int_B w_k^{2\delta/(\delta-2)}\,d\mu\right)^{(\delta-2)/\delta} \le \frac{C_S r^2}{V(x_0,r)^{2/\delta}}\left(\mathcal{E}(\psi_\sigma u_k,\psi_\sigma u_k)+r^{-2}\int_B \psi_\sigma^2 u_k^2\,d\mu\right) \tag{7.26}$$

By (7.22), (7.23), (7.24), (7.25), (7.26) and $a_1-a_0 \le K_1 r^2$, there exists $C_3>0$ such that

$$\sum_{k=a_\sigma}^{a_1}\int_{B(x,(1-\sigma)r-h')}(Pu_k)^{2(1+2/\delta)}\,d\mu \le \frac{C_4 r^2}{V(x,r)^{2/\delta}}\left((r\sigma)^{-2}\sum_{k=a_0}^{a_1}\int_B u_k^2\,d\mu\right)^{1+2/\delta}. \tag{7.27}$$

We choose $r_1 \ge 4h'$ so that $a_\sigma \le a_{1/2} \le (a_0+a_1)/2$ for all $a_0,a_1\in\mathbb{N}$ so that $a_1-a_0 \ge K_1^{-1}r_1^2$. Since $r\ge 4h'$ and $\sigma<1/2$, we have $(1-\sigma)r-h' \ge (r/2)-h' \ge r/4$. Hence by (2.4), $K_1^{-1}r^2 \le a_1-a_0 \le K_1r^2$ along with (7.27), we have (7.17). □

Proof of Lemma 7.7. We carry out Moser's iteration in two stages. In the first stage of the iteration, we obtain a L^1 to L^2 mean value inequality and in the second stage we show a L^2 to L^∞ mean value inequality. Combining the two stages yields the desired L^1 to L^∞ mean value inequality. The proof relies on repeated application of the elementary iterative step given by Lemma 7.10.

Let $r_1(0):=\sqrt{n}+h'$, $a_1(0):=0$, $N:=\lceil \log\sqrt{n}\rceil$ and $\theta:=1+(2/\delta)$. For the first stage of iteration, we iteratively define the quantities

$$\begin{aligned} r_1(i+1) &:= (r_1(i)-h')\left(1-\frac{4^{-1}}{3^{N+1-i}}\right)-h' \\ a_1(i+1) &:= \left\lceil \left(1-\frac{4^{-2}}{9^{N_r+1-i}}\right)a_1(i)+\frac{4^{-2}}{9^{N_r+1-i}}n\right\rceil \end{aligned}$$

for $i=0,1,\dots,N$. We define a non-increasing sequence of space-time cylinders

$$Q_i(i) = [\![a_1(i),n]\!]\times B(x,r_i),\qquad \text{for } i=0,1,\dots,N+1.$$

The following estimates are straightforward from definitions of r_1 and a_1: There exists $n_0>0$ such that for all $n\ge n_0$, we have

$$\begin{aligned} r_1(N+1) &\ge \sqrt{n}\left(1-4^{-1}\sum_{j=1}^{N+1}3^{-j}\right)-2(\log\sqrt{n}+3+h') \\ &\ge (7/8)\sqrt{n}-2(\log\sqrt{n}+3+h') \ge (6/7)\sqrt{n}, \end{aligned} \tag{7.28}$$

$$\begin{aligned} n-a_1(N+1) &\ge n\left(1-4^{-2}\sum_{j=1}^{N+1}9^{-j}\right)-2(N+1) \\ &\ge (31/32)n-2(\log\sqrt{n}+2) \ge (15/16)n. \end{aligned} \tag{7.29}$$

Let $u:\mathbb{N}\times M\to\mathbb{R}_{\ge 0}$ be an arbitrary non-negative function that is P_L-subcaloric in $[\![0,n]\!]\times B(x,\sqrt{n})$ where $n\ge n_1$. By Lemma 7.6 P^iu is P_L-subcaloric in $[\![0,n]\!]\times B(x,\sqrt{n}-ih')$ and therefore P_L-subcaloric in $[\![a_1(i),n]\!]\times B(x,r_1(i)-h')$ for all $i=0,1,\dots,N+1$. Hence by applying Lemma 7.10 for the function P^iu which is P_L-subcaloric on $[\![a_1(i),n]\!]\times B(x,r_1(i)-h')$ with $\sigma=4^{-1}3^{-(N+1-i)}$, we have $C_2>0$ such that

$$\tilde{\phi}(P^{i+1}u,2\theta,Q_{i+1}) \le C_2 3^{N+1-i}\tilde{\phi}(P^iu,2,Q_i) \tag{7.30}$$

for all $i=0,1,\dots,N$. We may choose $K_1=8$ in the application of Lemma 7.10 above due to (7.28) and (7.29).

By Hölder inequality along with (7.28), (7.29) and (2.4), there exists $C_3 > 0$ such that

$$\tilde{\phi}(P^{i+1}u, 2, Q_1(i+1)) \le C_3 \tilde{\phi}(P^{i+1}u, 1, Q_1(i+1))^{\alpha} \tilde{\phi}(P^{i+1}u, 2\theta, Q_1(i+1))^{\beta} \tag{7.31}$$

for all $i = 1, 2, \ldots, N$, where $\alpha = 1 - \beta = 2/(\delta + 4)$. By (4.10), $u \ge 0$, (7.28), (7.29) and (2.4), there exists $C_4 > 0$ such that

$$\tilde{\phi}(P^i u, 1, Q_1(i)) \le C_4 \tilde{\phi}(u, 1, Q_1(0)) \tag{7.32}$$

for all $i = 0, 1, \ldots, N+1$. Combining (7.30), (7.31), (7.32), there exists $C_5 > 0$ such that

$$\tilde{\phi}(P^{i+1}u, 2, Q_1(i+1)) \le C_5 3^{\beta(N+1-i)} \tilde{\phi}(u, 1, Q_1(0))^{\alpha} \tilde{\phi}(P^i u, 2, Q_1(i))^{\beta} \tag{7.33}$$

for $i = 1, \ldots, N$. By iterating (7.33), we obtain

$$\tilde{\phi}(P^{N+1}u, 2, Q_1(N+1)) \le C_5^{\sum_{i=0}^{\infty} \beta^i} 3^{\sum_{i=1}^{\infty} i\beta^i} \tilde{\phi}(u, 1, Q_1(0))^{(1-\beta^N)} \tilde{\phi}(Pu, 2, Q_1(1))^{\beta^N}. \tag{7.34}$$

Since $u \ge 0$, by Hölder inequality, (4.10) and (2.4), there exists $C_6, C_7 > 0$ such that

$$\begin{aligned} \int_{B(x,r_1(1))} (Pu_i)^2 \, d\mu &\le \left(\sup_{B(x,r_1(1))} Pu_i \right) \int_{B(x,r_1(1))} Pu_i \, d\mu \\ &\le \left(\int_{B(x,\sqrt{n}+h')} u_i \, d\mu \right)^2 \sup_{y \in B(x,\sqrt{n})} \frac{C_6}{V(y,h')} \\ &\le \frac{C_7 n^{\delta/2}}{V(x,\sqrt{n})} \left(\sup_{i \in [\![0,n]\!]} \int_{B(x,\sqrt{n}+h')} u_i \, d\mu \right)^2 \end{aligned} \tag{7.35}$$

for all $i \in [\![0,n]\!]$. Combining (7.34), (7.35) along with (2.4) yields

$$\tilde{\phi}(P^{N+1}u, 2, Q_1(N+1)) \le \frac{C_8}{V(x,\sqrt{n})} \sup_{k \in [\![0,n]\!]} \int_{B(x,\sqrt{n}+h')} u_k \, d\mu \tag{7.36}$$

for some $C_8 > 0$. The inequality (7.36) is a L^1 to L^2 mean value inequality and this concludes the first part of iteration.

For the second part, we define $v = P^{N+1}u$, $a_2(0) = a_1(N+1)$ and $r_2(0) = r_1(N+1)$. As before, we iteratively define

$$\begin{aligned} r_2(i+1) &:= (r_2(i) - h') \left(1 - \frac{4^{-1}}{3^{i+1}}\right) - h', \\ a_2(i+1) &:= \left\lceil \left(1 - \frac{4^{-2}}{9^{i+1}}\right) a_2(i) + \frac{4^{-2}}{9^{i+1}} n \right\rceil \end{aligned}$$

for $i = 1, 2, \ldots, N+1$. As before, define a non-increasing sequence of space-time cylinders by $Q_2(i) := [\![a_2(i), n]\!] \times B(x, r_2(i))$ for $i = 0, 1, \ldots, n$. Note that $Q_2(0) = Q_1(N+1)$.

Similar to (7.28) and (7.29), there exists $n_1 \ge n_0$ such that for all $n \ge n_1$,

$$r_2(i) \ge r_2(N+1) \ge \sqrt{n}/2 \tag{7.37}$$

$$n - a_2(i) \ge n - a_2(N+1) \ge n/2 \tag{7.38}$$

for all $i = 0, 1, \ldots, N+1$. By Jensen's inequality, we have

$$(P^{i+1}v)^{\theta^{i+1}} \le \left(P \left[(P^i v)^{\theta^i} \right] \right)^{\theta} \tag{7.39}$$

for all $i \in \mathbb{N}$. By Lemma 7.6 and Lemma 7.5, the function $(P^i v)^{\theta^i}$ is P_L-subcaloric in $[\![a_2(i), n]\!] \times B(x, r_2(i) - h)$ for all $i = 0, 1, \ldots, N+1$. Therefore by Lemma 7.10 for the function $(P^i v)^{\theta^i}$ and (7.39), there exists $C_9 > 0$ such that

$$\tilde{\phi}(P^{i+1}v, 2\theta^{i+1}, Q_2(i+1)) \le C_9^{\theta^{-i}} 3^{(i+1)\theta^{-i}} \tilde{\phi}(P^i v, 2\theta^i, Q_2(i)) \tag{7.40}$$

for $i = 0, 1, \ldots, N-1$. Iterating the inequalities (7.40), there exists $C_{10} > 0$ such that

$$\tilde{\phi}(P^N v, 2\theta^N, Q_2(N)) \le C_{10}\tilde{\phi}(v, 2, Q_2(0)) = C_{10}\tilde{\phi}(v, 2, Q_1(N+1)). \tag{7.41}$$

There exists $C_{11}, C_{12}, C_{13} > 0$ such that, for all $k \in N$

$$\begin{aligned} \sup_{y \in B(x, r_2(N+1))} P^{N+1} v_k(y) &\le C_{11} \fint_{B(y,h')} P^N v_k \, d\mu \\ &\le C_{11} \left(\fint_{B(y,h')} (P^N v_k)^{2\theta^N} \, d\mu \right)^{1/(2\theta^N)} \\ &\le C_{12} n^{\delta/(4\theta^N)} \left(\fint_{B(x, r_2(N))} (P^N v_k)^{2\theta^N} \, d\mu \right)^{1/(2\theta^N)} \\ &\le C_{13} \left(\fint_{B(x, r_2(N))} (P^N v_k)^{2\theta^N} \, d\mu \right)^{1/(2\theta^N)} \end{aligned} \tag{7.42}$$

The first line above follows from (4.10), the second line follows from Jensen's inequality, the third line follows from (2.4) and the last line follows from the fact that $n \mapsto n^{\delta/(4\theta^{\log n})}$ is bounded in $[2, \infty)$. By (7.41), (7.42) and $v = P^{N+1}u$, we have a L^2 to L^∞ mean value inequality

$$\inf_{k \in [\![0,n]\!]} \sup_{B(x, r_2(N+1))} P^{2N+2} u \le C_{10} C_{13} \tilde{\phi}(P^{N+1}u, 2, Q_1(N+1)). \tag{7.43}$$

Combining (7.36) and (7.43), we have the desired inequality (7.4). □

7.2 On-diagonal upper bound

The following lemma provides a useful example of P_L-caloric function.

Lemma 7.11. *Let (M, d, μ) be a metric measure space. Let P be Markov operator equipped with kernel $(p_k)_{k \in \mathbb{N}}$ that is (h, h')-compatible with (M, d, μ). Define for all $k \in \mathbb{N}$, the function $h_k : M \times M \to \mathbb{R}$ by*

$$h_k(x, y) := \left(P_L^k p_2(x, \cdot) \right)(y) = 2^{-n} \sum_{i=0}^{n} \binom{n}{i} p_{i+2}(x, y) \tag{7.44}$$

where $P_L = (I + P)/2$ as before. Then for all $x \in M$, the function

$$(k, y) \mapsto h_k(x, y)$$

is P_L-caloric in $\mathbb{N} \times M$.

Proof. The second equality in (7.44) is a consequence of binomial theorem and Lemma 4.2(c). Note that

$$P_L(h_k(x, \cdot))(y) = P_L(P_L^k p_2(x, \cdot))(y) = P_L^{k+1} \left(p_2(x, \cdot) \right)(y) = h_{k+1}(x, y).$$

Therefore $(k, y) \mapsto h_k(x, y)$ is P_L-caloric in $\mathbb{N} \times M$ for all $x \in M$. □

We are ready to prove Proposition 7.2 using the mean value inequality (7.4).

Proof of Proposition 7.2. Let $h_k(x,y)$ be defined as (7.44). Choose $n_1 \in \mathbb{N}$ such that

$$2\lceil \log \sqrt{n} \rceil + 4 \leq n \tag{7.45}$$

for all $n \geq n_1$. By Lemma 7.7, Lemma 7.11 and $\int_M h_k(x,y)\,dy = 1$, there exists $n_2 \geq n_1$ and $C_1 > 0$ such that the P_L-caloric function $(k,y) \mapsto h_k(x,y)$ satisfies the mean value inequality

$$\inf_{k \in [\![0,n]\!]} P^{2\lceil \log \sqrt{n} \rceil + 2} h_k(x,x) \leq \inf_{k \in [\![0,n]\!]} \sup_{y \in B(x,\sqrt{n}/2)} P^{2\lceil \log \sqrt{n} \rceil + 2} h_k(x,y) \leq \frac{C_1}{V(x,\sqrt{n})} \tag{7.46}$$

for all $x \in M$ and for all $n \in \mathbb{N}$ satisfying $n \geq n_2$.

By (4.11), we have $p_2(x,\cdot) - \alpha p_1(x,\cdot) \geq 0$ μ-almost everywhere for each $x \in M$. By (4.12) of Lemma 4.8 and Lemma 4.6, we have

$$p_k(x,x) \leq \alpha^{-1} p_{2\lceil k/2 \rceil}(x,x) \leq \alpha^{-1} p_{2\lceil k/2 \rceil}(x,x) \geq \alpha^{-1} p_{2n}(x,x) \tag{7.47}$$

for all $x \in M$ and for all $2 \leq k \leq 2n$. By (7.47) and (7.45),

$$P^{2\lceil \log \sqrt{n} \rceil + 2} h_k(x,x) \geq \alpha^{-1} p_{2n}(x,x) \tag{7.48}$$

for all $x \in M$, for all $k \in [\![0,n]\!]$ and for all $n \geq n_2$. Combining (7.48), (4.12), (7.44) and (7.46), there exists $C_2 > 0$ such that

$$p_n(x,x) \leq \frac{C_2}{V(x,\sqrt{n})} \tag{7.49}$$

for all $n \geq 2n_2$. Since P is a contraction in L^∞ by (4.10), Lemma 4.2(c) and (2.4), there exists $C_3, C_4 > 0$ and $\delta > 2$ such that

$$p_n(x,x) \leq \frac{C_3}{V(x,h')} \leq \frac{C_4 n^{\delta/2}}{V(x,\sqrt{n})} \tag{7.50}$$

for all $x \in M$ and for all $n \in \mathbb{N}$ with $n \geq 2$. Combining (7.49) and (7.50) gives the diagonal bound (7.2). □

7.3 Discrete integral maximum principle

We use Discrete integral maximum principle and diagonal upper bound to obtain Gaussian upper bounds. This approach is detailed in [19] for graphs. A crucial assumption in [19] is the laziness assumption for the corresponding Markov chain $(X_n)_{n \in N}$ given by $\inf_{x \in M} \mathbb{P}_x(X_1 = x) > 0$. As explained in [19, Section 3] this laziness assumption is not too restrictive for graphs because under natural conditions the iterated operator P^2 corresponds to a lazy Markov chain. However this fails to be true for continuous spaces.

Since the laziness assumption is unavoidable for discrete integral maximum principle, we consider the Markov operator $P_L = (I+P)/2$ instead of P. Using discrete integral maximum principle corresponding to P_L and diagonal estimate on p_k, we obtain off-diagonal estimates on h_k defined in (7.44). We rely on careful comparison between off-diagonal estimates of h_k and the Markov kernel p_k. The comparison arguments are new but elementary and involves Stirling's approximation. Our comparison arguments rely crucially on the compatibility assumption (4.11). Similar comparison arguments for off-diagonal estimates was carried out in [25, Section 3.2] to compare Markov chains on graphs with its corresponding continuous time version.

The main technical tool to prove Gaussian upper bounds is the following discrete integral maximum principle. The proposition adapted from [19, Proposition 2.1] and the proof follows from essentially the same argument.

Proposition 7.12 (Discrete integral maximum principle)**.** *Suppose that P is a Markov operator that is (h,h')-compatible with a metric measure space (M,d,μ). Let f be a strictly positive continuous function on $[\![0,n]\!]\times M$ such that,*

$$\partial_k f(x) + \frac{|\nabla_P f_{k+1}|^2}{4f_{k+1}}(x) \le 0. \tag{7.51}$$

for all $x\in M$ and $k\in[\![0,n-1]\!]$ where $|\nabla_P f|$ is as defined in (6.25)*. Let $u:\mathbb{N}\times M$ bounded function that is P_L-caloric on $[\![0,n-1]\!]\times M$ satisfying $\operatorname{supp}(u_0)\subset B(w,R)$ for some $w\in M, R\in(0,\infty)$. Then the function*

$$k\mapsto J_k = J_k(u) := \int_M u_k^2 f_k \, d\mu$$

is non-increasing in $[\![0,n]\!]$.

The following lemma essentially follow from [19, Proposition 2.5]. Lemma 7.13 provides a weight function f that will be used in the application of discrete integral maximum principle.

Lemma 7.13. *Let (M,d,μ) be a metric measure space and let P be a Markov operator that is (h,h')-compatible with (M,d,μ). Let $\sigma: M\to\mathbb{R}$ be a 1-Lipschitz function such that $\inf\sigma\ge h'$. There exists a positive number D_1 such that for all $D\ge D_1$, the weight function*

$$f_k(x) = f_k^D(x) := \exp\left(-\frac{\sigma^2(x)}{D(n+1-k)}\right) \tag{7.52}$$

satisfies

$$\partial_k f(x) + \frac{|\nabla_P f_{k+1}|^2}{4f_{k+1}}(x) \le 0.$$

for all $x\in M$, for all $n\in\mathbb{N}^*$ and $k\in[\![0,n-1]\!]$.

Next, we need the following estimate on h_k defined in (7.44). The proof uses the diagonal estimate in Proposition 7.2.

Lemma 7.14. *Under the assumptions of Proposition 7.1, there exists $C_0>0$ such that*

$$\int_M h_n^2(x,y)dy \le \frac{C_0}{V(x,\sqrt{n+2})} \tag{7.53}$$

for all $n\in\mathbb{N}$ and for all $x\in M$ where h is as defined in (7.44)*.*

Proof. By (7.44) of Lemma 7.11, Lemma 4.2(c) and Vandermonde's convolution formula, we have

$$\begin{aligned}\int_M (h_n(x,y))^2\,dy &= 4^{-n}\int_M\left(\sum_{i=0}^n\binom{n}{i}p_{i+2}(x,y)\right)^2 dy\\ &= 4^{-n}\sum_{i=0}^{2n}\binom{2n}{i}p_{i+4}(x,x)\end{aligned} \tag{7.54}$$

for all $x\in M$. By Proposition 7.2, there exists $C_1>0$ such that

$$p_k(x,x)\le\frac{C_1}{V(x,\sqrt{k})}$$

for all $k \geq 2$ and for all $x \in M$. Combined with (7.54) and (2.4), we obtain $C_2 > 0, \delta > 2$ such that

$$\int_M (h_n(x,y))^2 \, dy \leq 4^{-n} \sum_{i=0}^{2n} \binom{2n}{i} \frac{C_1}{V(x, \sqrt{i+4})} \leq \frac{C_2}{V(x, \sqrt{2n+4})} 4^{-n} \sum_{i=0}^{2n} \binom{2n}{i} \left(\frac{2n+4}{i+4} \right)^{\delta/2} \tag{7.55}$$

for all $n \in \mathbb{N}$ and all $x \in M$. By the above inequality, we have

$$4^{-n} \sum_{i=0}^{2n} \binom{2n}{i} \left(\frac{2n+4}{i+4} \right)^{\delta/2} \leq 4^{-n} \sum_{i=0}^{2n} \binom{2n}{i} \left(\frac{2n+4}{i+4} \right)^{\kappa} \leq 4^{2\kappa} \kappa! \sum_{i=0}^{2n} \binom{2n+\kappa}{i+\kappa} 2^{-(2n+\kappa)} \leq 4^{2\kappa} \kappa! \tag{7.56}$$

where $\kappa := \lceil \delta/2 \rceil \in \mathbb{N}^*$. Combining (7.55), (7.56) along with (2.4) implies (7.53). □

Our next result involves repeated application of the discrete integral maximum principle.

Lemma 7.15. *Let (M, d, μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\mathrm{loc}}$ and $(VD)_\infty$. Suppose that a Markov operator P has a kernel p that is (h, h')-compatible with (M, d, μ) for some $h > b$. Further assume that P satisfies the Sobolev inequality (5.2). Define*

$$E_D(k, x) := \int_M h_k^2(x, z) \exp\left(\frac{d_1^2(x, z)}{Dk} \right) dz \tag{7.57}$$

for all $k \in \mathbb{N}^*$ and $x \in M$, where $d_1(x, z) := \max(d(x, z), h')$ and h_k is defined by (7.44). There exists $C, D > 0$ such that

$$E_D(k, x) \leq \frac{C}{V(x, \sqrt{k})} \tag{7.58}$$

for all $x \in M$ and for all $k \in \mathbb{N}^*$.

The proof follows from a modification of the argument in [19, Proposition 5.4] using the diagonal upper bound in Lemma 7.14. We omit the details. We use Lemma 7.15 to prove a Gaussian upper bound for h_k.

Lemma 7.16. *Under the assumptions of Proposition 7.1, there exists positive reals C_0, D_0 such that*

$$h_{2k}(x, y) \leq \frac{C_0}{V(x, \sqrt{k})} \exp\left(-\frac{d^2(x, y)}{D_0 k} \right) \tag{7.59}$$

for all $x, y \in M$ and for all $k \in \mathbb{N}^*$.

Proof. By triangle inequality and the inequality $(a+b)^2 \leq 2(a^2 + b^2)$, we have

$$d_1(x, y)^2 \leq 2(d_1(x, z)^2 + d_1(y, z)^2) \tag{7.60}$$

for all $x, y, z \in M$, where $d_1(x, y) := \max(d(x, y), h')$ as before. By (4.12), (7.60) and

Cauchy-Schwarz inequality we have

$$\begin{aligned} h_{2k}(x,y) &= \sum_{i=0}^{2k} \binom{2k}{i} \left(\frac{1}{2}\right)^{2k} p_{i+2}(x,y) \\ &\le \alpha^{-2} \sum_{i=0}^{2k} \binom{2k}{i} \left(\frac{1}{2}\right)^{2k} p_{i+4}(x,y) = \alpha^{-2} \int_M h_k(x,z) h_k(y,z)\, dz \\ &\le \alpha^{-2} \int_M h_k(x,z) h_k(z,y) e^{d_1(x,z)^2/2Dk} e^{d_1(z,y)^2/2Dk} e^{-d_1(x,y)^2/4Dk}\, dz \\ &\le \alpha^{-2} \sqrt{E_D(k,x) E_D(k,y)} e^{-d_1^2(x,y)/4Dk} \\ &\le \alpha^{-2} \sqrt{E_D(k,x) E_D(k,y)} e^{-d(x,y)^2/4Dk} \end{aligned} \tag{7.61}$$

for all $x, y \in M$, for all $k \in \mathbb{N}^*$ and for all $D > 0$, where $\alpha > 0$ is from (4.11). The equality in the second line above follows from a calculation analogous to (7.54).

The bound (7.61) and Lemma 7.15 implies that there exists $C_1, D_1 > 0$ such that

$$h_{2k}(x,y) \le \frac{C_1}{\left(V(x,\sqrt{k}) V(y,\sqrt{k})\right)^{1/2}} \exp\left(-\frac{d^2(x,y)}{D_1 k}\right) \tag{7.62}$$

for all $x, y \in M$ and for all $k \in \mathbb{N}^*$. However by (2.4), there exists $C_2, C_3, C_4 > 0, \delta > 0$ such that

$$\begin{aligned} \frac{V(x,\sqrt{k})}{V(y,\sqrt{k})} &\le \frac{V(y,\sqrt{k} + d(x,y))}{V(y,\sqrt{k})} \le C_2 \left(1 + \frac{d(x,y)}{\sqrt{k}}\right)^{\delta} \\ &\le C_3 \left(1 + \frac{d^2(x,y)}{k}\right)^{\delta/2} \le C_4 \exp\left(\frac{d^2(x,y)}{2 D_1 k}\right) \end{aligned} \tag{7.63}$$

for all $x, y \in M$. Combining (7.62) and (7.63) yields the desired Gaussian upper bound (7.59). □

7.4 Comparison with lazy random walks

We want to convert the Gaussian bounds on h_k given by Lemma 7.16 to Gaussian bounds on p_k. To accomplish this we need the following elementary polynomial identities.

Lemma 7.17. *For all $\beta > 0$ and for all $n \in \mathbb{N}^*$, we have the following polynomial identities*

$$\begin{aligned} z^n &= \sum_{k \in [\![1,n]\!], k \text{ odd}} \binom{n}{k} \beta^{n-k} (z-\beta)^{k-1} z \\ &\quad + \sum_{k \in [\![1,n-1]\!], k \text{ odd}} \binom{n-1}{k} \beta^{n-1-k} (z-\beta)^{k-1} (z^2 - 2\beta z), \end{aligned} \tag{7.64}$$

$$\begin{aligned} \left(\frac{1+z}{2}\right)^n &= \frac{1}{2^n} + \sum_{k \in [\![1,n]\!], k \text{ odd}} \binom{n}{k} \left(\frac{1+\beta}{2}\right)^{n-k} \left(\frac{1}{2}\right)^k (z-\beta)^{k-1} z \\ &\quad + \sum_{k \in [\![1,n-1]\!], k \text{ odd}} s_{n,k} \left(\frac{1+\beta}{2}\right)^{n-1-k} \left(\frac{1}{2}\right)^{k+1} (z-\beta)^{k-1} (z^2 - 2\beta z) \end{aligned} \tag{7.65}$$

where $(z-\beta)^0 = 1$ and

$$s_{n,k} = (1+\beta)^{-(n-1-k)} \sum_{i=k+1}^{n} \binom{n}{i} \binom{i-1}{k} \beta^{i-1-k} \ge \binom{n-1}{k}.$$

Proof. Note that

$$z^n = z\left(\frac{z^n-(2\beta-z)^n}{2(z-\beta)}\right)+(z^2-2\beta z)\left(\frac{z^{n-1}-(2\beta-z)^{n-1}}{2(z-\beta)}\right) \tag{7.66}$$

for all $z \neq \beta$. To obtain (7.64), we expand $z^n, z^{n-1}, (2\beta-z)^n, (2\beta-z)^{n-1}$ in (7.66) using binomial expansion and the substitution

$$z = \beta + (z-\beta) \text{ and } 2\beta - z = \beta - (z-\beta).$$

To show (7.65), we use binomial expansion on $(1+z)^n$ and then use (7.64) to obtain

$$\begin{aligned}(1+z)^n &= 1+\sum_{i=1}^n \binom{n}{i} z^i \\ &= 1+\sum_{i=1}^n \sum_{k\in[\![1,i]\!],k \text{ odd}} \binom{n}{i}\binom{i}{k}\beta^{i-k}(z-\beta)^{k-1}z \\ &\quad+\sum_{i=1}^n \sum_{k\in[\![1,i-1]\!],k \text{ odd}} \binom{n}{i}\binom{i-1}{k}\beta^{i-1-k}(z-\beta)^{k-1}(z^2-2\beta z).\end{aligned} \tag{7.67}$$

The coefficient of $(z-\beta)^{k-1}z$ in (7.67) is

$$\sum_{i=k}^n \binom{n}{i}\binom{i}{k}\beta^{i-k} = \binom{n}{k}\sum_{i=k}^n \binom{n-k}{i-k}\beta^{i-k} = \binom{n}{k}(1+\beta)^{n-k}.$$

Similarly, the coefficient of $(z-\beta)^{k-1}(z^2-2\beta z)$ in (7.67) is

$$\begin{aligned}\sum_{i=k+1}^n \binom{n}{i}\binom{i-1}{k}\beta^{i-1-k} &= \binom{n-1}{k}\sum_{i=k+1}^n \frac{n}{i}\binom{n-1-k}{i-1-k}\beta^{i-1-k} \\ &\geq \binom{n-1}{k}\sum_{i=k+1}^n \binom{n-1-k}{i-1-k}\beta^{i-1-k} \\ &= \binom{n-1}{k}(1+\beta)^{n-1-k}.\end{aligned}$$

This gives (7.65) with $s_{n,k} \geq \binom{n-1}{k}$. ☐

We are now prepared to prove Gaussian upper bounds for p_k.

Proof of Proposition 7.1. By Lemma 4.8 there exists $\beta > 0$ such that $u_k, v_k : M \times M \to \mathbb{R}$ satisfy

$$u_k(x,y) := \left[(P-\beta I)^k p_2(x,\cdot)\right](y) \geq 0, \tag{7.68}$$

$$v_k(x,y) := \left[(P-\beta I)^k (p_3(x,.)-2\beta p_2(x,\cdot))\right](y) \geq 0 \tag{7.69}$$

for all $x, y \in M$ and for all even non-negative integers k. For instance $\beta = \alpha/2$ where α is given by (4.11) would satisfy the above requirements.

Using Lemma 4.2(c) and (7.64) of Lemma 7.17, we have

$$\begin{aligned}p_{n+1}(x,y) &= \left[P^n p_1(x,\cdot)\right](y) \\ &= \sum_{k\in[\![1,n]\!],k \text{ odd}} \binom{n}{k}\beta^{n-k}u_{k-1}(x,y) \\ &\quad+\sum_{k\in[\![1,n-1]\!],k \text{ odd}} \binom{n-1}{k}\beta^{n-1-k}v_{k-1}(x,y)\end{aligned} \tag{7.70}$$

for all $n \in \mathbb{N}^*$ and for all $x, y \in M$. By (7.68), (7.69), Lemma 4.8 and Lemma 7.17, we have

$$\begin{aligned} h_{2n}(x,y) &= \left[((I+P)/2)^{2n} p_2(x,\cdot) \right](y) \\ &\geq \alpha \sum_{k \in [\![1,2n]\!], k \text{ odd}} \binom{2n}{k} \left(\frac{1+\beta}{2}\right)^{2n-k} \left(\frac{1}{2}\right)^k u_{k-1}(x,y) \\ &\quad + \alpha \sum_{1 \leq k \leq 2n-1, k \text{ odd}} s_{2n,k} \left(\frac{1+\beta}{2}\right)^{2n-1-k} \left(\frac{1}{2}\right)^{k+1} v_{k-1}(x,y) \end{aligned} \tag{7.71}$$

for all $x, y \in M$. Define the ratio of coefficients in (7.70) and (7.71) as

$$a_{k,n} = \frac{\binom{2n}{k}\left(\frac{1+\beta}{2}\right)^{2n-k}\left(\frac{1}{2}\right)^k}{\binom{n}{k}\beta^{n-k}} \text{ and } b_{l,n} = \frac{\binom{2n-1}{l}\left(\frac{1+\beta}{2}\right)^{2n-1-l}\left(\frac{1}{2}\right)^{l+1}}{\binom{n-1}{l}\beta^{n-1-l}} \tag{7.72}$$

for each $k \in [\![1,n]\!]$ and for each $l \in [\![1,l-1]\!]$. If $k \in [\![1,n-1]\!]$, then

$$\frac{a_{k+1,n}}{a_{k,n}} = \frac{\beta}{1+\beta}\frac{2n-k}{n-k}.$$

Therefore $a_{k+1,n} \geq a_{k,n}$ if and only if $k \geq n(1-\beta)$. Thus $a_{k,n}$ reaches minimum for $k = \lceil n(1-\beta) \rceil$. By Stirling's approximation there exists constant $C_1 > 0$ such that for all $r \in \mathbb{N}^*$,

$$C_1^{-1} r^{r+(1/2)} e^{-r} \leq r! \leq C_1 r^{r+(1/2)} e^{-r}.$$

We use the Stirling's approximation to estimate $a_{k,n}$ at $k = n(1-\beta)+\epsilon$ where $\epsilon = \lceil n(1-\beta) \rceil - n(1-\beta) \in [0,1)$. There exists $c_1 > 0$ such that

$$\begin{aligned} \min_{k \in [\![1,n]\!]} a_{k,n} &\geq a_{\lceil n(1-\beta) \rceil, n} \\ &\geq \frac{C_1^{-4}(2n)^{2n+(1/2)}e^{-2n}(\beta n-\epsilon)^{\beta n+(1/2)-\epsilon}e^{-\beta n+\epsilon}\,(1+\beta)^{(1+\beta)n-\epsilon}}{2^{2n}n^{n+(1/2)}e^{-n}(n(1+\beta)-\epsilon)^{n(1+\beta)+(1/2)-\epsilon}e^{-n(1+\beta)+\epsilon}\beta^{\beta n-\epsilon}} \\ &\geq c_1 \end{aligned}$$

for all $n \in \mathbb{N}^*$ satisfying $n \geq 2/\beta$. Therefore there exists $c_2 > 0$ such that

$$a_{k,n} \geq c_2 \tag{7.73}$$

for all $n \in \mathbb{N}^*$ and for all $k \in [\![1,k]\!]$. Similarly,

$$b_{l,n} = \frac{1}{2} a_{l+1,n} \geq \frac{1}{2} c_2 \tag{7.74}$$

for all $n \in \mathbb{N}^*$ and for all $l \in [\![1,n-1]\!]$. Combining (7.68), (7.69), (7.70), (7.71), (7.72), (7.73) and (7.74), there exists $c_3 > 0$ such that

$$h_{2n}(x,y) \geq c_3 p_{n+1}(x,y) \tag{7.75}$$

for all $n \in \mathbb{N}^*$, and for all $x, y \in M$. Combining (7.75) along with Lemma 7.16 yields the Gaussian upper bound (7.1). □

We have shown the following equivalence

Theorem 7.18. *Let (M,d,μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\mathrm{loc}}$. Suppose that a Markov operator P has a kernel p that is (h,h')-compatible with (M,d,μ) for some $h > b$. Then the following are equivalent:*

(i) Sobolev inequality (5.2)*.*

(ii) Large scale volume doubling property $(VD)_\infty$ *and Gaussian upper bounds* (GUE)*.*

Proof. By Corollary 5.11, (ii) implies (i).

Next, we assume the Sobolev inequality (5.2). By Proposition 5.12 we have $(VD)_\infty$. In addition, by Proposition 7.1 we have (GUE). This proves (i) implies (ii). □

8 Gaussian lower bounds

In this section, we use elliptic Harnack inequality and Gaussian upper bounds to establish Gaussian lower bounds. The proofs in this section is adapted from [42]. In [42], Hebisch and Saloff-Coste provide an alternate approach to prove parabolic Harnack inequality using elliptic Harnack inequality and Gaussian upper bounds. This method avoids relying on the full strength of Moser's iteration method in parabolic setting.

The main result of this section is the following Gaussian lower bound.

Proposition 8.1. *Let* (M, d, μ) *be a quasi-*b*-geodesic metric measure space satisfying* $(VD)_{\mathrm{loc}}$*,* $(VD)_\infty$*,* $\mathrm{diam}(M) = \infty$ *and Poincaré inequality at scale* h $(P)_h$*. Suppose that a Markov operator* P *has a kernel* p *that is* (h, h')*-compatible with respect to* μ *for some* $h > b$*. Then the corresponding kernel* p_k *satisfies Gaussian lower bounds* (GLE)*.*

Note that under the assumptions of Proposition 8.1, we have Gaussian upper bounds (GUE). This is a direct consequence of Theorem 5.1 and Proposition 7.1.

We focus on the case $\mathrm{diam}(M) = \infty$ just for simplicity. In fact, we expect these methods to work when $\mathrm{diam}(M) < \infty$. However when the space has finite diameter, it is important to find optimal constants (or close to optimal) for various functional inequalities. To compute these optimal constants, one has to exploit the specific structure of the Markov chain under consideration. We plan to address the finite diameter case in a sequel.

The first step is to obtain lower bounds on $p_k(x, x)$. It is well-known that Gaussian upper bounds implies a matching diagonal lower bounds. The follows from [42, Proof of (3.1)].

Lemma 8.2. *Under the assumption of Proposition 8.1, there exists* $c_0 > 0$ *such that*

$$p_n(x, x) \ge \frac{c_0}{V(x, \sqrt{n})}$$

for all $x \in M$ *and for all* $n \in \mathbb{N}$ *satisfying* $n \ge 2$*.*

The following lemma is a discrete time analog of [42, Lemma 3.7], where we transfer the on-diagonal lower bound given by Lemma 8.2 to on-diagonal lower bound for the 'Dirichlet kernel' p_k^B on a ball B defined in (4.20). We omit its proof as it follows from the same argument as in [42].

Lemma 8.3. *Under the assumptions of Proposition 8.1, there exists* $c > 0$ *and* $A > \max(1, h')$ *such that*

$$p_n^{B(x,r)}(x, x) \ge \frac{c}{V(x, \sqrt{n})}$$

for all $x \in M$*, for all* $n \in \mathbb{N}^*$ *with* $n \ge 2$ *and for all* $r \ge A\sqrt{n}$

Our next result is a bound on the spectrum of P_B or alternatively on the Dirichlet Laplacian Δ_{P_B}. The following Proposition is a discrete time analog of [42, Theorem 2.5]. However unlike [42], we cannot apply the stronger Sobolev inequality (5.1). Nevertheless, the weaker variant (5.2) is sufficient for that argument.

Proposition 8.4. *Let* (M, d, μ) *be a quasi-*b*-geodesic metric measure space satisfying* $(VD)_{\mathrm{loc}}$*,* $(VD)_\infty$ *and Poincaré inequality at scale* h $(P)_h$*. Suppose that a Markov* *operator P has a kernel p that is (h, h')-compatible with respect to μ for some $h > b$. Then there exists positive reals a, ϵ_0 such that*

$$\left\| P_{B(x,r)} \right\|_{2\to 2} := \sup_{f \in L^2(B(x,r)), \|f\|_2 = 1} \left\| P_{B(x,r)} f \right\|_2 \leq 1 - \frac{a}{r^2} \tag{8.1}$$

for all $x \in M$ and for all $r \in \mathbb{R}$ satisfying $r \geq h'$ and $r \leq \epsilon_0 \operatorname{diam}(M)$.

Remark 8.5.

(a) A simple consequence of Proposition 8.4 is that there exists $a, \epsilon_0 > 0$ such that

$$\operatorname{Spectrum}(P_B) \subseteq \left[-\left(1 - ar^{-2}\right), 1 - ar^{-2} \right], \quad \operatorname{Spectrum}(\Delta_{P_B}) \subseteq \left[ar^{-2}, 2 - ar^{-2} \right]$$

for all $x \in M$ and for all r satisfying $r \geq h'$ and $r \leq \epsilon_0 \operatorname{diam}(M)$.

(b) If $\operatorname{diam}(M) = \infty$, then for all balls $B = B(x, r)$ with $r \in (0, \infty)$, we have

$$\|P_B\|_{2\to 2} < 1.$$

The case $r \geq h'$ is clear from Proposition 8.4. The case $r < h'$ follows from $\|P_B\|_{2\to 2} \leq \left\| P_{B(x,h')} \right\|_{2\to 2}$.

(c) Note that if $\operatorname{diam}(M) < \infty$, then the conclusion Proposition 8.4 is vacuously true as one can choose $\epsilon_0 = h'/(2 \operatorname{diam}(M))$. However if $h' \ll \operatorname{diam}(M)$ and if we have good control of the constants in various functional inequalities, we can prove useful estimates which in turn yields applications to estimates on mixing times. We will extend the techniques developed here to finite diameter spaces elsewhere.

(d) Note that the condition $r \leq \epsilon_0 \operatorname{diam}(M)$ is necessary. Too see this consider the case when $\operatorname{diam}(M) < \infty$ and $B(x, r) = M$. It is clear that (8.1) fails to be true because $P_{B(x,r)} \mathbf{1} = \mathbf{1}$.

8.1 Near diagonal lower bound

As in [42, Proposition 3.5], the following near diagonal estimate is an important step in obtaining Gaussian lower bounds.

Proposition 8.6 (Near diagonal lower bound)**.** *Under the assumptions of Proposition 8.1, there exists positive reals ϵ_1, c_1 such that p_k satisfies the lower bound*

$$\inf_{y \in B(x, \epsilon_1 \sqrt{k})} p_k(x, y) \geq \frac{c_1}{V(x, \sqrt{k})} \tag{8.2}$$

for all $x \in M$ and for all $k \in \mathbb{N}^*$ satisfying $k \geq 2$.

From the above near diagonal lower bound, we will see that the Gaussian lower bound follows by a well-established 'chaining argument'.

The idea behind the proof of Proposition 8.6 is to convert the elliptic Hölder-like regularity estimate (Proposition 6.20) into a parabolic Hölder-like regularity estimate for the function $(k, y) \mapsto p_k^B(x, y)$ as follows:

Lemma 8.7. *Under the assumptions of Proposition 8.1, for all $\sigma > 0$ and all $A \geq 1$, there exists three positive reals $C_{\sigma,A}$, $\epsilon_0 \leq A$ and $N_0 \geq 2$ such that*

$$\left| p_k^B(x, y) - p_k^B(x, x) \right| \leq \left[\sigma + C_{\sigma,A} \left(\frac{d(x, y) \vee 1}{\sqrt{k}} \right)^{\alpha} \right] \frac{1}{V(x, \sqrt{k})} \tag{8.3}$$

for all $x \in M$, $k \in \mathbb{N}^*$ with $k \geq N_0$ and for all $y \in B(x, \epsilon_0 \sqrt{k})$, where $B = B(x, A\sqrt{k})$ and α is the exponent in (6.65).

The proof of Lemma 8.7 is long and involves many technical estimates. We will need some upper bounds on $p_k^B(y,z)$ and its 'time derivative'

$$\partial_k p^B(y,z) := p_{k+1}(y,z) - p_k(y,z)$$

for all $y, z \in B$.

Lemma 8.8. *Under the assumptions of Proposition 8.1, the following estimates hold:*

(i) There exists $C_1, D_1 > 0$ such that

$$p_j^{B(x,A\sqrt{k})}(y,z) \le \frac{C_1}{V(y,\sqrt{j})} \exp\left(-\frac{d(y,z)^2}{D_1 j}\right) \tag{8.4}$$

for all $x \in M$, for all $k \in \mathbb{N}^*$, for all $j \ge 2$, for all $A \ge 1$ and for all $y, z \in B(x, A\sqrt{k})$.

(ii) There exists $C_2, \delta > 0$ such that

$$\left|\partial_k p^{B(x,A\sqrt{k})}(y,z)\right| \le \frac{C_2 A^\delta}{kV(x,\sqrt{k})} \tag{8.5}$$

for all $x \in M$, for all $k \in \mathbb{N}_{\ge 2}$, for all $A \ge 1$ and for all $y, z \in B(x, A\sqrt{k})$.

(iii) For all $A > 1 \vee h'$, there exists $\epsilon, a_1 > 0$, such that for all $\theta \in (0,1)$, there exists C_θ such that,

$$p_j^{B(x,A\sqrt{k})}(y,z) \le \frac{C_\theta A^\delta}{V(x,\sqrt{k})} \left(1 - \frac{a_1}{A^2 k}\right)^j \tag{8.6}$$

for all $x \in M$, for all $k \in \mathbb{N}^*$, for all $j \in \mathbb{N}$ satisfying $j \ge \max(2, \theta k)$ and for all $y, z \in B(x, A\sqrt{k})$.

The above Lemma follows from the argument in [42, Proof of Lemma 3.9].

Remark 8.9. The constants C_1, C_1, C_2, C_θ and a_1 in Lemma 8.8 do not depend on A, x and k.

Lemma 8.7 now follows from the argument in [42, Section 3.4], where the use of [42, Lemma 3.9] is replaced with Lemma 8.8 instead. Next, we prove the near diagonal lower bound using Lemmas 8.7 and 8.3.

Proof of Proposition 8.6. By Lemma 8.3, there exists $A \ge 1 \vee h'$ and $c > 0$ such that

$$p_k^{B(x,A\sqrt{k})}(x,x) \ge \frac{c}{V(x,\sqrt{k})} \tag{8.7}$$

for all $x \in M$ and for all $k \in \mathbb{N}^*$ with $k \ge 2$. By Lemma 8.7, there exists $C_1 > 1, N_1 \ge 2$, $\epsilon \in (0,1), \alpha > 0$ such that

$$\left|p_k^B(x,y) - p_k^B(x,x)\right| \le \left[\frac{c}{3} + C_1 \left(\frac{d(x,y) \vee 1}{\sqrt{k}}\right)^\alpha\right] \frac{1}{V(x,\sqrt{k})} \tag{8.8}$$

for all $x \in M$, for all $k \in \mathbb{N}^*$ with $k \ge N_0$, for all $y \in B(x, \epsilon\sqrt{k})$ where $B = B(x, A\sqrt{k})$. Next, we choose $\epsilon_1 \in (0,\epsilon)$ and $N_1 \ge N_0$ such that for all $k \ge N_1$, we have

$$C_1 \left(\frac{\epsilon_1\sqrt{k} \vee 1}{\sqrt{k}}\right)^\alpha \le C_1 \max(\epsilon^\alpha, N_0^{-\alpha/2}) \le \frac{c}{3}.$$

By the above choice of ϵ_1, N_1 along with (8.7), (8.8) and the triangle inequality, we have

$$\inf_{y \in B(x,\epsilon_1\sqrt{k})} p_k^{B(x,A\sqrt{k})}(x,y) \ge \frac{c}{3V(x,\sqrt{k})}$$

for all $x \in M$ and for all $k \in \mathbb{N}^*$ with $k \geq N_1$. Since $p_k^B \leq p_k$, the above equation yields the desired near diagonal lower bound (8.2) for all $k \geq N_1$.

If $k \in [\![2, N_1]\!]$, then we reduce ϵ if necessary so that $\epsilon \leq h/\sqrt{N_1}$. Hence $d(x,y) \leq \epsilon\sqrt{k}$ and $k \leq N_1$ implies $d(x,y) \leq h$. Therefore by (4.12) of Lemma 4.8 and (4.10), we obtain (8.2) for all $k \in [\![2, N_1]\!]$. □

8.2 Off-diagonal lower bounds

The near diagonal lower bound of Proposition 8.6 can be easily upgraded to full Gaussian lower bounds (GLE) by a well-known chaining argument (See [41, Theorem 5.1], [25, Theorem 3.8]). For general quasi-geodesic spaces, we rely on the chain lemma (Lemma 2.4). We now prove the main result of this section, i.e. Gaussian lower bound.

Proof of Proposition 8.1. By Lemma 2.4 there exists $C_1 > 1$ such that for all $b_1 \geq b$ and for all $x, y \in M$, there exists a b_1-chain $x = x_0, x_1, \ldots, x_m = y$ with

$$m \leq \left\lceil \frac{C_1 d(x,y)}{b_1} \right\rceil. \tag{8.9}$$

By Proposition 8.6, there exists $\epsilon > 0, c_1 > 0$ such that

$$\inf_{y \in B(x,\epsilon\sqrt{k})} p_k(x,y) \geq \frac{c_1}{V(x,\sqrt{k})} \tag{8.10}$$

for all $x \in M$ and for all $k \geq 2$. If

$$s := \frac{C_1 \epsilon^2 k}{C_2 d(x,y)} \geq b, \tag{8.11}$$

then there exists a s-chain $x = x_0, x_1, \ldots, x_m = y$ between x and y with

$$m := \left\lceil \frac{C_2 d(x,y)^2}{\epsilon^2 k} \right\rceil. \tag{8.12}$$

However (8.11) holds whenever $d(x,y) \leq c_3 k$ and $c_3 \leq C_1\epsilon^2/C_2 b$. If $C_2 \geq 1$ and $d(x,y) \geq \epsilon\sqrt{k}$, we have

$$m := \left\lceil \frac{C_2 d(x,y)^2}{\epsilon^2 k} \right\rceil \leq \frac{2C_2 d(x,y)^2}{\epsilon^2 k}. \tag{8.13}$$

If $d(x,y) \leq c_3 k$ and $c_3 \leq \epsilon/\sqrt{(2C_2)}$, we have

$$\frac{k}{m} \geq \frac{\epsilon^2 k^2}{C_2 d(x,y)^2} \geq \frac{\epsilon^2}{C_2 c_3^2} \geq 2. \tag{8.14}$$

We fix $c_3 = \min\left(\epsilon/\sqrt{(2C_2)}, C_1\epsilon^2/C_2 b\right)$, so that (8.11), (8.12) and (8.14) are satisfied. We will fix $C_2 \geq 1$ later.

We will require

$$d(x_i, x_{i+1}) \leq s = \frac{C_1\epsilon^2 k}{C_2 d(x,y)} \leq \frac{\epsilon}{3}\sqrt{\frac{k}{2m}} \leq \frac{\epsilon}{3}\sqrt{\left\lfloor \frac{k}{m} \right\rfloor} \tag{8.15}$$

for all $i = 0, 1, \ldots, m-1$ and for all $k \geq m$. We fix $C_2 := 36C_1^2 \geq 1$, so that by (8.13) we deduce

$$s = \frac{C_1\epsilon^2 k}{C_2 d(x,y)} \leq \frac{\epsilon}{3}\left(\frac{\epsilon^2 k^2}{4C_2 d(x,y)^2}\right)^{1/2} \leq \frac{\epsilon}{3}\sqrt{\frac{k}{2m}} \leq \frac{\epsilon}{3}\sqrt{\left\lfloor \frac{k}{m} \right\rfloor} \tag{8.16}$$

for all $x, y \in M$ and $k \in \mathbb{N}^*$ such that $d(x,y) \ge \epsilon\sqrt{k}$ and $k/m \ge 2$, where s, m is as defined in (8.11) and (8.12). Define $k_0, \ldots, k_{m-1}$ such that

$$k_i := \left\lfloor \frac{k}{m} \right\rfloor \text{ or } \left\lfloor \frac{k}{m} \right\rfloor + 1$$

satisfying $\sum_{i=0}^{m-1} k_i = k$. Consider the s-chain $x = x_0, \ldots, x_m = y$ between x and y where s, m are given by (8.11), (8.12). By (8.16) and definition of k_i, for all $w_i \in B(x_i, (\epsilon/3)\sqrt{\lfloor k/m \rfloor})$, for $i = 0, 1, \ldots, m-1$ we have

$$d(w_i, w_{i+1}) \le \epsilon\sqrt{\lfloor k/m \rfloor} \le \epsilon\sqrt{k_i}.$$

Therefore by (8.10), (8.14) and (2.5), there exists $c_4, c_5 \in (0,1)$ such that for all for $i = 0, 1, \ldots, m-1$, $w_i \in B(x_i, (\epsilon/3)\sqrt{\lfloor k/m \rfloor})$, we have

$$p_{k_i}(w_i, w_{i+1}) \ge \frac{c_1}{V(w_i, \sqrt{k_i})} \ge \frac{c_4}{V(w_i, \sqrt{\lfloor k/m \rfloor})} \ge \frac{c_5}{V(x_i, \sqrt{\lfloor k/m \rfloor})} \tag{8.17}$$

for all $x, y \in M$, $k \ge 2$ satisfying $d(x,y) \ge \epsilon\sqrt{k}$ and $d(x,y) \le c_3 k$.

Define $B_i = B(x_i, (\epsilon/3)\sqrt{\lfloor k/m \rfloor})$. By Chapman-Kolmogorov equation and (8.17), for all $x, y \in M$, $k \ge 2$ satisfying $d(x,y) \ge \epsilon\sqrt{k}$ and $d(x,y) \le c_3 k$, we obtain

$$\begin{aligned} & p_k(x,y) \\ &= \int_M \cdots \int_M p(x_0, w_1) p(w_1, w_2) \ldots p(w_{m-2}, w_{m-1}) p(w_{m-1}, y)\, dw_1 \ldots dw_{m-1} \\ &\ge \int_{B_{m-1}} \cdots \int_{B_1} p(x_0, w_1) p(w_1, w_2) \ldots p(w_{m-2}, w_{m-1}) p(w_{m-1}, y)\, dw_1 \ldots dw_{m-1} \\ &\ge \frac{c_5^{m-1}}{V(x, \sqrt{k})} \prod_{i=1}^{m-1} \frac{V(x_i, (\epsilon/3)\sqrt{\lfloor k/m \rfloor})}{V(x_i, \sqrt{\lfloor k/m \rfloor})} \end{aligned} \tag{8.18}$$

By (2.4), (8.13), (8.14) and (8.18), there exists $c_6, c_7 \in (0,1)$ such that

$$\begin{aligned} p_k(x,y) &\ge \frac{c_6^m}{V(x, \sqrt{k})} \ge \exp\left(\frac{2C_2 d(x,y)^2 \log c_6}{\epsilon^2 k} \right) \frac{1}{V(x, \sqrt{k})} \\ &\ge \frac{1}{V(x, \sqrt{k})} \exp\left(-\frac{d(x,y)^2}{c_7 k} \right) \end{aligned} \tag{8.19}$$

for all $x, y \in M$, $k \ge 2$ satisfying $d(x,y) \ge \epsilon\sqrt{k}$ and $d(x,y) \le c_3 k$. This yields (GLE) for the case $d(x,y) \ge \epsilon\sqrt{k}$.

The case $d(x,y) \le \epsilon\sqrt{k}$ follows from (8.10). This completes the proof of (GLE). □

9 Parabolic Harnack inequality

In this section, we use the two sided Gaussian estimates on the heat kernel to prove parabolic Harnack inequality. Moreover, we show the necessity of Poincaré inequality and large scale volume doubling using parabolic Harnack inequality.

Based on ideas of Nash [62], Fabes and Stroock [29] gave a proof of parabolic Harnack inequality using Gaussian bounds on the heat kernel for uniformly elliptic operators on $\mathbb{R}^n$. This idea of using Gaussian estimates on the heat kernel to prove parabolic Harnack inequality was extended in various settings [72, 66, 25, 8]. Delmotte [25] introduced a discrete version of balayage formula to prove parabolic Harnack inequality on graphs. We use a direct adaptation of Delmotte's method to prove parabolic Harnack inequality.

Recall that we defined caloric function as solutions to the discrete time heat equation $\partial_k u + \Delta u_k = 0$ in Definition 7.3. We introduce the parabolic Harnack inequality for non-negative caloric functions.

Definition 9.1. *Let (M,d,μ) be a metric measure space and let P be a Markov operator on (M,d,μ). Let $0<\zeta<1$ and $0<\theta_1<\theta_2<\theta_3<\theta_4$. We that a μ-symmetric Markov operator P (or equivalently its heat kernel p_k) on (M,d,μ) satisfies the discrete-time parabolic Harnack inequality*

$$H(\zeta,\theta_1,\theta_2,\theta_3,\theta_4)$$

if there exists positive reals C,R such that for all $x\in M, r\in\mathbb{R}, a\in\mathbb{N}$ with $r>R$ and every non-negative P-caloric function $u:\mathbb{N}\times M\to\mathbb{R}_{\geq 0}$ on

$$Q=[\![a,a+\lfloor\theta_4 r^2\rfloor]\!]\times B(x,r),$$

we have

$$\sup_{Q_\ominus} u\leq C\inf_{Q_\oplus} u,$$

where

$$Q_\ominus:=[\![a+\lceil\theta_1 r^2\rceil,a+\lfloor\theta_2 r^2\rfloor]\!]\times B(x,\zeta r),$$
$$Q_\oplus:=[\![a+\lceil\theta_3 r^2\rceil,a+\lfloor\theta_4 r^2\rfloor]\!]\times B(x,\zeta r).$$

Remark 9.2.

(i) The exact values of the constants $\zeta\in(0,1)$ and $\theta_1,\theta_2,\theta_2,\theta_4$ are unimportant. For example, for graphs and length spaces if the parabolic Harnack inequality is satisfied for one set of constants, then it is satisfied for every other set of constants. The argument in [8, Proposition 5.2(iv)] can be adapted for graphs and length spaces in the above discrete-time setting.

(ii) It suffices to consider the case $a=0$ in the definition above by simply by shifting the function in the time component.

(iii) Analogous to Remark 7.4(b), if P is (h,h')-compatible with (M,d,μ) we may only require the function u to be defined on a smaller domain.

9.1 Gaussian estimates implies parabolic Harnack inequality

In this subsection, we prove the following parabolic Harnack inequality using two sided Gaussian bounds.

Proposition 9.3. *Let (M,d,μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\text{loc}}$. Suppose that a Markov operator P has a kernel p_k that is weakly (h,h')-compatible with respect to μ for some $h>b$. Moreover, suppose that p_k satisfies two sided Gaussian estimate (GE). Then there exists $\eta\in(0,1)$ such that P satisfies the parabolic Harnack inequality $H(\eta/2,\eta^2/2,\eta^2,2\eta^2,4\eta^2)$.*

First we start by verifying that Gaussian lower bound implies large scale volume doubling property.

Lemma 9.4. *Let (M,d,μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\text{loc}}$. Suppose that a Markov operator P has a kernel p_k that satisfies (GLE). Then (M,d,μ) satisfies $(VD)_\infty$.*

Proof. By (GLE) there exists $c_1,c_2,c_3>0$ such that

$$p_n(x,y)\geq\frac{c_1}{V(x,\sqrt{n})}\exp\left(-d(x,y)^2/c_2 n\right)$$

for all $x,y\in M$ satisfying $d(x,y)\leq c_3 n$ and for all $n\in\mathbb{N}^*$. Therefore there exists $N_1\geq 1$ such that $4\sqrt{n}\leq c_3 n$ for all $n\geq N_1$. By the Gaussian lower bound above

$$1=\int_M p_n(x,y)\,dy\geq\int_{B(x,4\sqrt{n})} p_n(x,y)\,dy\geq\frac{V(x,4\sqrt{n})}{V(x,\sqrt{n})}c_1\exp(-4/c_2)$$

for all $x \in M$ and for all $n \geq N_1$. Therefore there exists $R := N_1^2$ such that for all $x \in M$ and for all $r \geq R$, we have

$$V(x,r) \geq V(x,\lfloor r \rfloor) \geq c_1 \exp(-4/c_2) V(x, 4\lfloor r \rfloor) \geq c_1 \exp(-4/c_2) V(x, 2r). \qquad \square$$

We show the following near diagonal lower bounds as a consequence of two sided Gaussian bound (GE).

Lemma 9.5. *Under the assumptions of Proposition 9.3, there exists $c_1 > 0$, $\eta \in (0,1)$ and $R_0 > 0$ such that for all $x \in M$, for all $r \geq R_0$, for all $y, z \in B(x, \eta r)$, for all $k \in \mathbb{N}^*$ satisfying $(\eta r)^2 \leq k \leq (2\eta r)^2$, we have*

$$p_k^{B(x,r)}(y,z) \geq \frac{c_1}{V(x,\sqrt{k})}. \tag{9.1}$$

Proof. We abbreviate $B(x,r)$ by B. We denote the exit time from ball B by

$$\tau := \min\{n \;:\; X_n \notin B\}$$

where $(X_n)_{n\in\mathbb{N}}$ is the Markov chain on M corresponding to the kernel p_k.

By strong Markov property and μ-symmetry, the Dirichlet kernel p_k^B can be expressed in terms of p_k as

$$p_k^B(y,z) = p_k(y,z) - \mathbb{E}_y\left[p_{k-\tau}(z, X_\tau)\mathbf{1}_{[\![1,k-1]\!]}(\tau)\right] \tag{9.2}$$

for all $n \geq 2$ and for all $x \in M$, where $\mathbb{E}_y$ denotes that the Markov chain starts at $X_0 = y$. We choose $R_0 > (1-\eta)^{-1}h'$, so that by (4.10)

$$\mathbb{E}_y\left[p_{k-\tau}(z, X_\tau)\mathbf{1}_{[\![1,k-1]\!]}(\tau)\right] = \mathbb{E}_y\left[p_{k-\tau}(z, X_\tau)\mathbf{1}_{[\![2,k-2]\!]}(\tau)\right]$$

for all $y, z \in B(x, \eta r)$, for all $k \geq 2$, for all $x \in M$ and for all $r \in \mathbb{R}$ with $r \geq R_0$. Combining this with (9.2) and $X_\tau \notin B$, we have

$$p_k^B(y,z) \geq p_k(y,z) - \sup_{l\in[\![2,k]\!]} \sup_{w \notin B(x,r)} p_l(z,w) \tag{9.3}$$

for all $y, z \in B(x, \eta r)$, for all $k \geq 2$, for all $x \in M$ and for all $r \in \mathbb{R}$ with $r \geq (1-\eta)^{-1}h'$.

Note that by Lemma 9.4 we have $(VD)_\infty$. Therefore by (GLE), (2.4) and $k \geq (\eta r)^2$, there exists $c_2, c_3 > 0$ and $R_1 > 0$ such that

$$p_k(y,z) \geq \frac{c_2}{V(y,\sqrt{k})} \exp\left(-\frac{(2\eta r)^2}{c_2(\eta r)^2}\right) \geq \frac{c_3}{V(x,\sqrt{k})} \tag{9.4}$$

for all $x \in M$, for all $r \geq R_1$, for all $\eta \in (0,1)$, for all $y, z \in B(x, \eta r)$ and for all $k \in \mathbb{N}^*$ satisfying $(\eta r)^2 \leq k$.

For the second term in (9.3) by (GUE), there exists $C_1 > 0$ such that

$$p_l(z,w) \leq \frac{C_1}{V(z,\sqrt{l})} \exp\left(-\frac{d(z,w)^2}{C_1 l}\right) \leq \frac{C_1}{V(z,\sqrt{l})} \exp\left(-\frac{(1-\eta)^2 r^2}{C_1 l}\right)$$

for all $l \in \mathbb{N}^*$ with $l \geq 2$, for all $x \in M$, for all $r > 0$, for all $\eta \in (0,1)$, for all $z \in B(x,\eta r)$ and for all $w \notin B(x,r)$. Combined this with (2.4) and $k \leq (2\eta r)^2$, there exists $C_2, C_3, C_4, \delta > 0$ such that for all $\eta \in (0,1/2)$, for all $x \in M$, for all $z \in B(x,\eta r)$, for all $k \in \mathbb{N}^*$ satisfying $(\eta r)^2 \leq k \leq (2\eta r)^2$, for all $l \in [\![2,k]\!]$ and for all $w \notin B(x,r)$, we have

$$\begin{aligned} p_l(z,w) &\leq \frac{C_2}{V(z,\sqrt{k})} \left(\frac{k}{l}\right)^\delta \exp\left(-\frac{(1-\eta)^2 r^2}{C_1 l}\right) \\ &\leq \frac{C_3 \eta^{2\delta}}{V(x,\sqrt{k})} \left(\frac{r^2}{l}\right)^\delta \exp\left(-\frac{r^2}{4C_1 l}\right) \\ &< \frac{C_4 \eta^{2\delta}}{V(x,\sqrt{k})}. \end{aligned} \tag{9.5}$$

The second line above follows from $\eta < 1/2$ and (2.4) and the last line follows from the fact the function $t \mapsto t^\delta \exp(-t/4C_1)$ is bounded in $(0,\infty)$. Combining (9.3), (9.4) and (9.5), there exists $c_1 > 0$ and $R_0 > 0$ such that p_k^B satisfies (9.1). □

The following lemma provides a discrete time version of Balayage decomposition for the heat equation.

Lemma 9.6. *Let (M,d,μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\mathrm{loc}}$. Suppose that a Markov operator P has a kernel p_k that is weakly (h,h')-compatible with respect to μ for some $h > b$. Then for all $x \in M$, for all $r > h'$, for all r_1 such that $0 < r_1 < r_1 + h' < r$, for all $a, b \in \mathbb{N}$, for all non-negative function $u : N \times M \to \mathbb{R}_{\geq 0}$ that is P-caloric in $[\![a,b]\!] \times B(x,r)$, there exists a non-negative function $v : N \times M \to \mathbb{R}$ (depending on u) such that $\mathrm{supp}(v) \subseteq [\![a+1,b]\!] \times (B(x,r_1+h') \setminus B(x,r_1))$ and for all $y \in B(x,r_1)$ and for all $k \in [\![a,b+1]\!]$, we have*

$$u(k,y) = \int_{B(x,r_1+h')} p^B_{k-a}(y,z)u(a,z)\,dz + \sum_{l=a+1}^{k-1} \int_{B(x,r_1+h')} p^B_{k-l}(y,w)v(l,w)\,dw, \tag{9.6}$$

where $B = B(x,r)$.

Proof. Denote by $B_1 = B(x,r_1+h')$ and $B = B(x,r)$. Define

$$v_1(k,y) = u(k,y) - \int_{B(x,r_1+h')} p^B_{k-a}(y,z)u(a,z)\,dz$$

for all $(k,y) \in [\![a+1,b+1]\!] \times B(x,r_1+h')$. Note that

$$(k,y) \mapsto \int_{B_1} p^B_{k-a}(y,z)u(a,z)\,dz$$

is P-caloric in $[\![a+1,b]\!] \times B(x,r_1)$. Since $u \geq 0$, by (4.10) we have $v_1(a+1,y) = 0$ for all $y \in B(x,r_1)$ and by maximum principle $v_1 \geq 0$ in $[\![a+1,b+1]\!] \times B(x,r_1)$.

Next, we construct $v : \mathbb{N} \times M \to \mathbb{R}$ iteratively. We assume that $\mathrm{supp}(v) \subseteq [\![a+1,b]\!] \times (B(x,r_1+h') \setminus B(x,r_1))$. Define $v(a+1,y) = v_1(a+1,y)$ for all $y \in B(x,r_1+h') \setminus B(x,r)$.

Since v_1 is a difference of two P-caloric functions, we have v_1 is P-caloric in $[\![a+1,b]\!] \times B(x,r_1)$. We repeat this construction iteratively by defining

$$v_{i+1}(k,y) = v_i(k,y) - \int_{B(x,r_1+h')} p^B_{k-a-i}(y,z)v_i(a+i,z)\,dz \tag{9.7}$$

for all $(k,y) \in [\![a+i+1,b+1]\!] \times B(x,r_1+h')$ and

$$v(a+i+1,w) = v_{i+1}(a+i+1,w)$$

for all $w \in B(x,r_1+h') \setminus B(x,r_1)$ and $i = 0,1,\ldots,b-a-1$. By the same argument as above, v_i is non-negative and caloric in $[\![a+i,b]\!] \times B(x,r_1)$ for all $i = 0,1,\ldots,b-a+1$. Further

$$u_i(a+i,z) = 0 \tag{9.8}$$

for all z in $B(x,r_1)$ and $i = 1,2,\ldots,b-a$. Combining (9.7), (9.8) and gives (9.6). □

We are now ready to prove the parabolic Harnack inequality.

Proof of Proposition 9.3. Let $\eta \in (0,1)$ be as given by Lemma 9.5. Note that for all $r > 12h'/\eta$, we have $\eta r - h' > 2\eta r/3 > \eta/2$. Moreover for all $r > 12h'/\eta$, for all $y \in B(x,\eta r/2)$ and for all $z \in B(x,\eta r) \setminus B(x,\eta r - h')$ we have $d(y,z) > 2h'$. Let $R_1 :=$

$1 + \max(R_0, 12h'/\eta, 10/\eta)$ where R_0 is the constant from Lemma 9.5. By the above remarks, (4.10) and Lemma 9.6, for all $x \in M$, for all $r \geq R_1$, for all non-negative function u that is P-caloric in $[\![0, \lfloor 4\eta^2 r^2 \rfloor]\!] \times B$ where $B = B(x,r)$, there exists a non-negative function v supported in $B(x,\eta r) \setminus B(x, \eta r - h')$ such that

$$u(k,y) = \int_{B(x,\eta r)} p_k^B(y,z) u(a,z)\, dz + \sum_{l=1}^{k-2} \int_{B(x,\eta r)} p_{k-l}^B(y,w) v(l,w)\, dw \tag{9.9}$$

for all $(k,y) \in [\![1, \lfloor 4\eta^2 r^2 \rfloor + 1]\!] \times B(x, \eta r/2)$.

For some fixed $x \in M$ and $r > R_1$, we define

$$Q_\ominus := [\![\lceil \eta^2 r^2/2 \rceil, \lfloor \eta^2 r^2 \rfloor]\!] \times B(x,\eta r/2), Q_\oplus := [\![\lceil 2\eta^2 r^2 \rceil, \lfloor 4\eta^2 r^2 \rfloor]\!] \times B(x,\eta r/2) \tag{9.10}$$

and $Q := [\![0, \lfloor 4\eta^2 r^2 \rfloor]\!] \times B(x, \eta r)$.

By Lemma 9.4 we have $(VD)_\infty$. Therefore by Lemma 9.5 and (2.4) there exists $c_1, c_2 > 0$ such that for all $x \in M$, for all $r \geq R_1$, for all $y \in B(x,\eta r/2)$, for all $z \in B(x, \eta r)$, we have

$$\inf_{(k,y)\in Q_\oplus} p_k^B(y,z) \geq \inf_{k \in [\![\lceil 2\eta^2 r^2 \rceil, \lfloor 4\eta^2 r^2 \rfloor]\!]} \frac{c_1}{V(x,\sqrt{k})} \geq \frac{c_1}{V(x, 2\eta r)} \geq \frac{c_2}{V(x,\eta r)}. \tag{9.11}$$

Similarly by Lemma 9.5 for all $x \in M$, for all $r \geq R_1$, for all $y \in B(x, \eta r/2)$, for all $z \in B(x,\eta r) \setminus B(x, \eta r - h')$, for all $l \in [\![1, \lfloor \eta^2 r^2 \rfloor - 2]\!]$ we have

$$\inf_{(k,y)\in Q_\oplus} p_{k-l}^B(y,z) \geq \inf_{k \in [\![\lceil 2\eta^2 r^2 \rceil, \lfloor 4\eta^2 r^2 \rfloor]\!]} \frac{c_1}{V(x,\sqrt{(k-l)})} \geq \frac{c_2}{V(x,\eta r)}. \tag{9.12}$$

For upper bounds in $Q_\oplus$ we simply use (GUE) as follows. By (GUE) and (2.5), there exists $C_1, C_2 > 0$ such that for all $x \in M$, for all $r \geq R_1$, for all $y \in B(x, \eta r/2)$, for all $z \in B(x,\eta r)$ we have

$$\sup_{(k,y)\in Q_\ominus} p_k^B(y,z) \leq \sup_{(k,y)\in Q_\ominus} p_k(y,z) \leq \sup_{(k,y)\in Q_\ominus} \frac{C_1}{V(y,\sqrt{k})} \leq \frac{C_2}{V(x,\eta r)}. \tag{9.13}$$

Similarly by (GUE) and (2.4), there exists $C_3, C_4, C_5, \delta > 0$ such that for all $x \in M$, for all $r \geq R_1$, for all $y \in B(x,\eta r/2)$, for all $z \in B(x,\eta r) \setminus B(x, \eta r - h')$, for all $(k,y) \in Q_\ominus$ and for all $l \in [\![1, k-2]\!]$ we have

$$\begin{aligned} p_{k-l}^B(y,z) \leq p_{k-l}(y,z) &\leq \frac{C_3}{V(y,\sqrt{(k-l)})} \exp\left(-\frac{d(y,z)^2}{C_3(k-l)}\right) \\ &\leq \frac{C_4}{V(y,\eta r)} \left(\frac{\eta^2 r^2}{(k-l)}\right)^{\delta/2} \exp\left(-\frac{\eta^2 r^2}{36 C_3 (k-l)}\right) \\ &\leq \frac{C_5}{V(x,\eta r)}. \end{aligned} \tag{9.14}$$

The last line follows from the fact that the function $t \mapsto t^{\delta/2} \exp(-t/(36C_3))$ is bounded in $(0,\infty)$ along with (2.5).

Combining the inequalities (9.11), (9.12), (9.13) and (9.14) along with the balayage formula (9.9) for all $x \in M$, for all $r \geq R_1$, for all non-negative function u that is P-caloric in $[\![0, \lfloor 4\eta^2 r^2 \rfloor]\!] \times B(x,r)$, we have

$$\sup_{(k,y)\in Q_\ominus} u(k,y) \leq c_2^{-1} \max(C_2, C_5) \inf_{(k,y)\in Q_\oplus} u(k,y)$$

where $Q_\ominus, Q_\oplus$ are as defined in (9.10). Note that by Remark 9.2(ii), we have the desired Harnack inequality. □

9.2 Necessity of Poincaré inequality and large scale volume doubling

In the previous sections, we have obtain two-sided Gaussian bounds on the heat kernel and parabolic Harnack inequality assuming large scale volume doubling and a Poincaré inequality. Now we show that large scale volume doubling and Poincaré inequality are necessary to have two-sided Gaussian bounds on the heat kernel and parabolic Harnack inequality. The was first proved by Saloff-Coste in [67, Theorem 3.1] using an argument due to Kusuoka and Stroock [52].Delmotte [25] followed the same strategy in discrete-time setting for random walk on graphs. The following proposition follows from the argument in [67, 25].

Proposition 9.7. *Let (M, d, μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\text{loc}}$. Suppose that a Markov operator P has a kernel p_k that is weakly (h, h')-compatible with respect to μ for some $h > b$ and there exists $\eta \in (0, 1)$ such that P satisfies the parabolic Harnack inequality $H(\eta/2, \eta^2/2, \eta^2, 2\eta^2, 4\eta^2)$. Then (M, d, μ) satisfies $(VD)_\infty$ and $(P)_{h'}$.*

We now have all the ingredients to prove our main result in a slightly weaker form.

Proposition 9.8. *Let (M, d, μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\text{loc}}$ and $\operatorname{diam}(M) = +\infty$. Suppose that a Markov operator P has a kernel p that is (h, h')-compatible with (M, d, μ) with either $h = h' > b$ or $h' > h \geq 5b$. Then the following are equivalent:*

- *(i) Parabolic Harnack inequality: there exists $\eta \in (0, 1)$ such that P satisfies the parabolic Harnack inequality $H(\eta/2, \eta^2/2, \eta^2, 2\eta^2, 4\eta^2)$.*
- *(ii) Gaussian bounds on the heat kernel: the heat kernel p_k satisfies (GE).*
- *(iii) The conjunction of large scale volume doubling property $(VD)_\infty$ and Poincaré inequality $(P)_h$.*

Proof. The implication "(iii) implies (ii)" follows from Theorem 5.1, Proposition 7.1 and Proposition 8.1. (ii) implies (i) follows from Proposition 9.3. (i) implies (iii) follows from Proposition 9.7 and Corollary 3.17. □

Next, we answer the question raised in Remark 3.6.

Proposition 9.9. *Let (M, d, μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\text{loc}}$, $(VD)_\infty$, $(P)_{h'}$ for some $h' > b$ and $\operatorname{diam}(M) = +\infty$. Then (M, d, μ) satisfies $(P)_h$ for all $h > b$.*

Proof. By Lemma 3.5 it suffices to consider the case $b < h < h'$. Consider the Markov chain with density

$$p(x, y) = \frac{\mathbf{1}_{B(x,h)}(y)}{Q(x)Q(y)\sqrt{V(x, h)V(y, h)}}$$

that is symmetric with respect to the measure $\mu'(dx) = Q(x)\mu(dx)$, where

$$Q(x) = \int_M \frac{\mathbf{1}_{B(x,h)}(y)}{\sqrt{V(x, h)V(y, h)}}\, \mu(dy).$$

By $(VD)_{\text{loc}}$, there exists $C_1 > 0$ such that

$$C_1^{-1} \leq Q(x) \leq C_1 \tag{9.15}$$

for all $x \in M$. Therefore the space (M, d, μ') satisfies $(VD)_{\text{loc}}$, $(VD)_\infty$, $(P)_{h'}$ for some $h' > b$. Moreover by (9.15), p is weakly (h, h)-compatible with (M, d, μ'). By the same argument as Lemma 4.9, there exists $l \in N^*$ such that p_l is (h', lh) compatible with

(M, d, μ'). Therefore by Proposition 9.8 and Lemma 4.14 the kernel p_k satisfies (GE). The Poincaré inequality $(P)_h$ for (M, d, μ') then follows from Propositions 9.3 and 9.7. An easy comparison argument using (9.15) gives $(P)_h$ for (M, d, μ). □

The following the main result of our work.

Theorem 9.10. *Let (M, d, μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\text{loc}}$ and $\text{diam}(M) = +\infty$. Suppose that a Markov operator P has a kernel p that is (h, h')-compatible with (M, d, μ), where $h' \geq h > b$. Then the following are equivalent:*

(i) Parabolic Harnack inequality: there exists $\eta \in (0, 1)$ such that P satisfies the parabolic Harnack inequality $H(\eta/2, \eta^2/2, \eta^2, 2\eta^2, 4\eta^2)$.

(ii) Gaussian bounds on the heat kernel: the heat kernel p_k satisfies (GE).

(iii) The conjunction of large scale volume doubling property $(VD)_\infty$ and Poincaré inequality $(P)_h$.

Proof. Combining Propositions 9.8 and 9.9 yields the desired result. □

As announced in the introduction, we will show Theorem 1.1 and [25, Theorem 1.7] are covered by our results. Since [25, Theorem 1.7] is a special case of Theorem 9.10, it remains to verify Theorem 1.1.

Proof of Theorem 1.1. We need only to check the implication (c) implies (b) as the other implications follow as in Theorem 9.10. Although p_1 is only weakly (h, h')-compatible to (M, d, μ), by Lemma 4.9, Theorem 9.10 and Lemma 4.14, we have that p_k satisfies (GE). □

10 Applications and examples

Perhaps the most important application of the characterization of parabolic Harnack inequality and Gaussian bounds on the heat kernel is the stability under quasi-isometries.

Theorem 10.1. *Let (M_i, d_i, μ_i) be a quasi-b_i-geodesic metric measure spaces satisfying $(VD)_{\text{loc}}$ and $\text{diam}(M_i) = +\infty$, for $i = 1, 2$. Moreover we assume that $(M_1, d_1, \mu_1$ and (M_2, d_2, μ_2) are quasi-isometric metric measure spaces. Suppose that a Markov operator P_i has a kernel that is (h_i, h_i')-compatible with (M_i, d_i, μ_i) with $h_i' \geq h_i > b_i$ for $i = 1, 2$. Then*

(i) The kernel corresponding to P_1 satisfies (GE) if and only if the kernel corresponding to P_2 satisfies (GE).

(ii) The operator P_1 satisfies the Harnack inequality $H(\eta/2, \eta^2/2, \eta^2, 2\eta^2, 4\eta^2)$ for some $\eta \in (0, 1)$ if and only if P_2 satisfies $H(\zeta/2, \zeta^2/2, \zeta^2, 2\zeta^2, 4\zeta^2)$ for some $\zeta \in (0, 1)$.

Proof. The is a direct consequence of Theorem 9.10 along with stability of $(VD)_\infty$ given by Proposition 2.14, stability of $(P)_h$ given by Proposition 3.16, Proposition 9.9 and Lemma 3.5. □

Recall that we proved an elliptic Hölder regularity estimate for P-harmonic functions in Proposition 6.20 and we used the regularity in the proof of Gaussian lower bounds (Lemma 8.7). There is an analogous parabolic Hölder regularity estimate which follows from parabolic Harnack inequality. The proof is similar, for example the proof given in [70, Theorem 5.4.7] can be adapted for the present setting. Such parabolic Hölder continuity estimates were first obtained by Nash [62].

Proposition 10.2. *Let (M, d, μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\text{loc}}$ and $\operatorname{diam}(M) = +\infty$. Suppose that a Markov operator P has a kernel p that is weakly (h, h')-compatible with (M, d, μ) and satisfies parabolic Harnack inequality $H(\eta/2, \eta^2/2, \eta^2, 2\eta^2, 4\eta^2)$ for some $\eta \in (0, 1)$. Then there exists $C > 0$, $R > 0$ and $\alpha > 0$ such that for all $x \in M$, for all $r > R$ and for any non-negative function $u : \mathbb{N} \times M \to \mathbb{R}$ that is P-caloric in $[\![0, \lfloor 4\eta^2 r^2 \rfloor]\!] \times B(x, r) = Q$, we have the regularity estimate*

$$\sup_{(k_1,x_1),(k_2,x_2) \in [\![\lceil 2\eta^2 r^2 \rceil, \lfloor 4\eta^2 r^2 \rfloor]\!] \times B(x,r)} \frac{|u(k_1, x_1) - u(k_2, x_2)|}{(\max(1, |k_1 - k_2| + d(x_1, x_2))^{\alpha}} \leq \frac{C}{r^{\alpha}} \sup_Q u.$$

Note that we do not obtain continuity, because we do not have Hólder continuity estimate at arbitrarily small distances. Another application of elliptic Harnack inequality is Liouville property for harmonic functions that was shown in Proposition 6.19.

Next, we turn attention to application of two sides Gaussian estimates (GE). Of course, the estimates given by (GE) has enough information to determine whether or not the the random walk is transient. The estimate given by [25, Proposition 4.3] can be easily generalized to metric measure spaces in which case we obtain

Proposition 10.3. *Let (M, d, μ) be a quasi-b-geodesic metric measure space satisfying $(VD)_{\text{loc}}$ and $\operatorname{diam}(M) = +\infty$. Consider a μ-symmetric Markov operator P that is (h, h')-compatible with (M, d, μ) for some $h > b$ and whose kernel p_k satisfies (GE). Then the random walk corresponding to P is transient if and only if*

$$\sum_{n=1}^{\infty} \frac{n}{V(x, n)} < +\infty \tag{10.1}$$

for some $x \in M$.

It is easy to see that the convergence of the series in (10.1) does not depend on the choice of $x \in M$. Unless the space is discrete, we do not have a 'Green's function' as the Green operator $\Delta^{-1} = \sum_{i=0}^{\infty} P^i$ does not have a kernel as there is 'delta mass' singularity at the starting point. However, we may consider the off-diagonal part of the Green operator given by the "Green's function" $G(x, y) = \sum_{i=1}^{\infty} p_i(x, y)$. The estimate given by [25, Proposition 4.3] can be again generalized as follows.

Proposition 10.4. *Under the assumptions of Proposition 10.3, there exists $C > 0$ such that*

$$C \sum_{n=\lceil d(x,y) \rceil}^{\infty} \frac{n}{V(x, n)} \leq G(x, y) := \sum_{i=1}^{\infty} p_i(x, y) \leq C \sum_{n=\lceil d(x,y) \rceil}^{\infty} \frac{n}{V(x, n)} \tag{10.2}$$

for some $x \in M$ and for all $y \in M$ with $d(x, y) > h'$.

As noted in [41, Theorem 9.1], the Gaussian estimate is sufficient to prove law of iterated logarithm in a weak form. The proof in [41] can be generalized for metric measure spaces.

Proposition 10.5. *Under the assumptions of Proposition 10.3, there exist $C > 0$ such that for all starting points $X_0 \in M$*

$$C^{-1} \leq \limsup \frac{d(X_0, X_n)}{(n \log \log n)^{1/2}} \leq C$$

almost surely, where $(X_k)_{k \in N}$ is the Markov chain corresponding to P.

We refer the reader to [41, Section 9] for other probabilistic applications in similar spirit.

If the space has finite diameter the techniques developed here can be used to prove upper and lower bounds on *mixing times*. In this case μ is a finite measure on M and can be normalized if necessary to be the stationary probability measure. Roughly speaking, in this case for (h,h)-compatible Markov operator on a space with diameter D, it takes $(D/h)^2$ steps of the Markov chain to get close to the stationary distribution μ. The Poincaré inequality and Gaussian upper bounds can be used to obtain upper bounds on mixing time as outlined in [28, Lemma 2.1 and Remark 1 after Lemma 2.2]. For lower bounds on the mixing time one would need Gaussian lower bounds. We plan to address these questions in a sequel and obtain results complementary to those in [53]. We refer the reader to [26, 27] for other recent works in this direction.

10.1 Harmonic functions with polynomial volume growth

In [17], Colding and Minicozzi proved that the space of harmonic functions with polynomial volume growth with fixed rate on a manifold satisfying volume doubling and Poincaré inequality is finite dimensional. As a corollary, they prove a conjecture of S. T. Yau on manifolds that asserts the above property for Riemannian manifolds with non-negative Ricci curvature. A recent surprising application of this result is an alternate proof of Gromov's theorem on groups of polynomial volume growth due to Kleiner [51].

Definition 10.6. *For a metric measure space (M,d,μ) and a μ-symmetric Markov operator P on M, we define the space of P-harmonic functions with growth rate d as the vector space $\mathcal{H}_d(M,P)$ consisting of all P-harmonic functions u such that there exists $C>0, p\in M$ (depending on u) such that $|u(x)| \le C(1+d(x,p)^\gamma)$ for all $x \in M$.*

We have the following theorem that would extend the result of Colding and Minicozzi for random walks on metric measure spaces.

Theorem 10.7. *Let (M,d,μ) be a quasi-geodesic metric measure spaces satisfying $\operatorname{diam}(M) = +\infty$, volume doubling hypotheses $(VD)_{\text{loc}}$, $(VD)_\infty$ and Poincaré inequality $(P)_h$. Let P be a Markov operator that is (h,h')-compatible with (M,d,μ). Then the space of P-harmonic functions $\mathcal{H}_d(M,P)$ with a fixed growth rate d is finite dimensional for any $d \ge 0$.*

The proof of Colding and Minicozzi's theorem in [17] relies on three ingredients: volume doubling hypotheses (VD), a Poincaré inequality and a reverse Poincaré inequality for harmonic functions. We have all the three ingredients (with some changes) as we showed the reverse Poincaré inequality in Lemma 6.14. A caveat is that we have to rely on weaker versions of all the three ingredients but nevertheless we will see that Theorem 10.7 can be proved using the techniques introduced of [17]. T. Delmotte adapted an alternate approach due to P. Li [54] to prove a similar statement for random walks on graphs satisfying doubling and Poincaré inequality [24].

The following below is a slightly weaker version of [17, Proposition 2.5].

Proposition 10.8. *Let (M,d,μ) be a quasi-geodesic metric measure spaces satisfying $\operatorname{diam}(M) = +\infty$, volume doubling hypotheses $(VD)_{\text{loc}}$, $(VD)_\infty$ and Poincaré inequality $(P)_h$ and let P be a Markov operator that is (h,h')-compatible with (M,d,μ). There exists $\epsilon \in (0,1)$ such that for all $p \in M$, for all $k \ge 1$ satisfying $r \ge k/\epsilon$ and for all functions $f_1, f_2, \ldots, f_n \in L^\infty_{\text{loc}}(M)$ satisfying*

$$\int_{B(p,r)} f_i^2 \, d\mu = V(p,r) \tag{10.3}$$

for all $i = 1,2,\ldots,n$;

$$\left| \int_{B(p,r)} f_i f_j \, d\mu \right| < \frac{V(p,r)}{2} \tag{10.4}$$

for all $1 \le i < j \le n$*; and*

$$\int_{B(p,2r)} \left(f_i^2 + (2r)^2 |\nabla_P f_i|^2\right) d\mu \le k^2 V(p,r) \tag{10.5}$$

for all $i = 1, 2, \ldots, n$*, we have* $n \le \mathcal{N}$*, where* $\mathcal{N}$ *depends on* k *but does not depend on* $r \ge k/\epsilon$ *or* $p \in M$*.*

Proof. The proof follows from modifying the proof of [17, Proposition 2.5] by using our modified versions of volume doubling, Poincaré and reverse Poincaré inequalities. □

The following proposition and its proof is a slight modification of a result due to Colding and Minicozzi [17, Proposition 4.16].

Proposition 10.9. *Consider a metric measure space* (M, d, μ) *satisfying the hypotheses* $(VD)_{\mathrm{loc}}$*,* $(VD)_\infty$ *and* $\mathrm{diam}(M) = +\infty$*. Let* P *be a Markov operator that is* (h, h')*-compatible with* (M, d, μ) *for some* $0 < h \le h'$*. Suppose that* $u_1, u_2, \ldots, u_{2k} \in \mathcal{H}_d(M, P)$ *are linearly independent. There exists* $\delta > 0$*,* $p \in M$ *such that for all* $d > 0$*,* $\Omega > 1$ *and* $m_0 > 0$*, there exists* $m \ge m_0$*,* $l \ge \frac{k}{2}\Omega^{-4d-\delta}$*, and functions* $v_1, \ldots, v_l$ *in the linear span of* u_i *such that*

$$2\Omega^{4d+2\delta} V(p, \Omega^m) = 2\Omega^{4d+2\delta} \int_{B(p,\Omega^m)} v_i^2 \, d\mu \ge \int_{B(p,\Omega^{m+1})} v_i^2 \, d\mu \tag{10.6}$$

and

$$\int_{B(p,\Omega^m)} v_i v_j \, d\mu = \delta_{i,j} V(p, \Omega^m). \tag{10.7}$$

In Proposition 10.9, we may choose δ as the constant in Lemma 2.7. We are now ready to prove Theorem 10.7.

Proof of Theorem 10.7. Fix $\Omega > \max(4, 3h')$, $d > 0$ and $p \in M$. Let $A \ge 1$, $C_A > 0$ and $C_D > 0$ and $\delta = \log_2 C_D$ be as given in Lemma 2.7. Let C_R be as given by Lemma 6.14 and set $k^2 = (8C_R + 2)\Omega^{4d+2\delta}$. Let $\epsilon \in (0, 1)$ be given by Proposition 10.8. We choose $m_0 \in \mathbb{N}^*$ such that $\Omega^{m_0} \ge k/\epsilon$. Let $\dim \mathcal{H}_d(M, P) \ge \mathcal{N}_0 := 4\Omega^{4d+2\delta}\mathcal{N}$ where $\mathcal{N}$ is given by Proposition 10.8 where k is as defined above.

Suppose that $u_1, u_2, \ldots, u_{\mathcal{N}_0} \in \mathcal{H}_d(M, P)$ be linearly independent. Then by Proposition 10.9 and reverse Poincaré inequality (Lemma 6.14) there exists $C_R > 0$ and $m > m_0$ such that for all $f \in L^\infty_{\mathrm{loc}}(M, \mu)$, we have harmonic functions $v_1, v_2, \ldots, v_l$ satisfying

$$l \ge \frac{1}{4}\mathcal{N}_0 \Omega^{-4d-2\delta} = \mathcal{N}, \tag{10.8}$$

$$\int_{B(p,\Omega^m)} v_i v_j \, d\mu = V(p, \Omega^m)\delta_{i,j}, \tag{10.9}$$

$$\int_{B(p,\Omega^{m+1})} v_i^2 \, d\mu \le 2C_R \Omega^{4d+2\delta} V(p, \Omega^m), \tag{10.10}$$

and

$$\int_{B(p,2\Omega^m)} |\nabla_P v_i|^2 \, d\mu \le C_R \Omega^{-2m} \int_{B(p,4\Omega^m)} v_i^2 \, d\mu \le 2\Omega^{4d+2\delta-2m} V(p, \Omega^m). \tag{10.11}$$

Note that (10.9), (10.10), (10.11) and $\Omega > 4$ implies that $v_1, v_2, \ldots, v_l$ satisfy (10.3), (10.3)

$$\int_{B(p,2\Omega^m)} v_i^2 + (2\Omega^m)^2 |\nabla_P v_i|^2 \, d\mu \le (8C_R + 2)\Omega^{4d+2\delta} V(p, r) \tag{10.12}$$

for all $i = 1, \ldots, l$. Note that (10.8), (10.9), (10.12) along with Proposition 10.8 implies the desired contradiction. Therefore $\dim \mathcal{H}_d(M, P) < \mathcal{N}_0 < \infty$. □

10.2 Directions for future work

We end with a direction for future work. One of the features of our work is that it provides an unified approach to Gaussian estimates for discrete time Markov chains on both discrete and continuous spaces. Recently, there has been considerable interest in analysis and probability on fractals and fractal-like manifolds and graphs. For many natural family of fractals the heat kernel satisfies sub-Gaussian estimates of the form

$$p_t(x,y) \asymp \frac{C_1}{V(x,t^{1/\beta})} \exp\left(-C_2\left(\frac{d(x,y)^\beta}{t}\right)^{1/(\beta-1)}\right)$$

for all $t>0$ and for all $x,y\in M$ and $\beta>1$ is a parameter (See [10, Theorem 1.5(e)] for an early example). Here $\asymp$ means that both inequalities $\le$ and $\ge$ hold with different values of constants C_1, C_2. Similar to the characterizations of Gaussian estimates in [30, 67, 74, 25, 40] there exists various characterizations for sub-Gaussian estimates both in the setting of diffusions on local Dirichlet spaces [5] and for discrete time Markov chains on graphs [4, 7, 35, 36]. As in the case of Gaussian estimates, it is desirable to obtain characterizations of sub-Gaussian estimates that are stable under quasi-isometries. This was achieved using a condition called cutoff-Sobolev inequality first introduced by Barlow and Bass [4] (See also [5]). Our work naturally raises an analogous question for sub-Gaussian estimates on Markov chains with more general space-time scaling.

Problem 10.10. Characterize sub-Gaussian estimates for discrete time Markov chains on quasi-geodesic metric measure spaces using geometric conditions that are stable with respect to quasi-isometries.

Another direction for future work is to clarify the applications to mixing times in the finite diameter case as mentioned in Remark 8.5(b).

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