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Stochastic primitive equations with horizontal viscosity and diffusivity*

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Abstract

We establish the existence and uniqueness of pathwise strong solutions to the stochastic 3D primitive equations with only horizontal viscosity and diffusivity driven by transport noise on a cylindrical domain $M = (-h, 0) \times G$, $G \subset \mathbb{R}^2$ bounded and smooth, with the physical Dirichlet boundary conditions on the lateral part of the boundary. Compared to the deterministic case where the uniqueness of z-weak solutions holds in L^2, more regular initial data are necessary to establish uniqueness in the anisotropic space $H^1_z L^2_{xy}$ so that the existence of local pathwise solutions can be deduced from the Gyöngy-Krylov theorem. Global existence is established using the logarithmic Sobolev embedding, the stochastic Gronwall lemma and an iterated stopping time argument.

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1 Introduction and main results

The 3D primitive equations, one of the fundamental models for geophysical flows, describe oceanic and atmospheric dynamics. They are derived from the compressible Navier-Stokes equations assuming hydrostatic balance and the Boussinesq approximation. The subject of this work is the initial value problem for the primitive equations with horizontal viscosity and diffusivity and the physical lateral Dirichlet boundary conditions.

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The study of problems with partial viscosity and diffusivity is motivated by the fact that in many geophysical models, the horizontal viscosity and diffusivity is considered dominant and the vertical one is neglected (see e.g. [40]). Models with partial viscosity and diffusivity are also interesting from the analytical point of view since they combine features of both parabolic diffusion equations in horizontal directions (represented by the term $-\Delta_H$) and hyperbolic transport equations in the vertical direction (represented by the nonlinear term $w\partial_z v$), see (1.1) below. Roughly speaking, one thus expects that regularity is preserved in the vertical direction while it is smoothed in horizontal directions. Following this intuition, we may identify classes of initial data for which the problem is locally or even globally well-posed.

1.1 Primitive equations with horizontal viscosity and diffusivity

Let $t > 0$ and let $G \subset \mathbb{R}^2$ be a bounded and smooth domain. We consider the primitive equations on a cylindrical domain M of depth $h > 0$, defined by $M = G \times (-h, 0)$. The boundary ∂M decomposes into a lateral, upper and bottom parts as

$$\Gamma_l := \partial G \times (-h, 0), \quad \Gamma_u := \overline{G} \times \{0\}, \quad \Gamma_b := \overline{G} \times \{-h\}.$$

The primitive equations describe the velocity $u = (v, w) \colon M \to \mathbb{R}^3$, the temperature $T : M \to \mathbb{R}$ and the pressure $p \colon M \to \mathbb{R}$ of a fluid, where $v = (v_1, v_2)$ denotes the horizontal components and w stands for the vertical component of velocity. The primitive equations with horizontal viscosity and diffusivity are

$$\begin{aligned} \partial_t v - \nu_v \Delta_H v + k \times v + \frac{1}{\rho_0} \nabla_H p & \\ + v \cdot \nabla_H v + w \partial_z v &= f_v + \sigma_1(v, \nabla_H v, T, \nabla_H T) \dot{W}_1, \\ \partial_t T - \nu_T \Delta_H T + v \cdot \nabla_H T + w \partial_z T &= f_T + \sigma_2(v, \nabla_H v, T, \nabla_H T) \dot{W}_2, \\ \partial_z p &= -\rho g, \\ \operatorname{div}_H v + \partial_z w &= 0, \\ \rho &= \rho_0(1 - \beta_T(T - T_r)), \end{aligned} \tag{1.1}$$

in $M \times (0, t)$ with

$$w = 0 \quad \text{on } \Gamma_u \cup \Gamma_b \times (0, t). \tag{1.2}$$

For the prognostic variables v, T, we have the initial conditions

$$v(t = 0) = v_0, \qquad T(t = 0) = T_0, \tag{1.3}$$

and the boundary conditions

$$v = 0 \quad \text{and} \quad \partial_{n_G} T = 0 \quad \text{on } \Gamma_l \times (0, t), \tag{1.4}$$

where n_G is the outer normal to G on Γ_l (since M is cylindrical n_G does not depend on the vertical coordinate). The first boundary condition in (1.4) is a lateral no-slip boundary condition for v, the second one is a Neumann-type condition for T. The condition (1.2) on w is considered to be part of the system (1.1) since w is a diagnostic variable, see (2.8).

In what follows, we will denote the variables of the horizontal domain by $(x, y) \in G$ and the vertical coordinate by $z \in (-h, 0)$. We define $\nabla_H = (\partial_x, \partial_y)^T$, $\operatorname{div}_H = \nabla_H{}^*$ and $\Delta_H = \partial_x^2 + \partial_y^2$ to be the horizontal gradient, divergence and Laplacian, respectively. Also let $v \cdot \nabla_H = v_1 \partial_x + v_2 \partial_y$ and let $k \times v = k_0(-v_2, v_1)$ be the Coriolis force. The terms $\sigma_i \dot{W}_i$ model the stochastic forces. The constants $\nu_v, \nu_T > 0$ are the horizontal viscosity and horizontal diffusivity, $k_0 \geq 0$ is the Coriolis parameter, $\rho_0, \beta_T, g > 0$ denote the reference density, the expansion coefficient and the gravity, respectively. Note that for the primitive equations the nonlinear term $w\partial_z v$ is of worse order compared to the nonlinearity of the Navier-Stokes equations since $w = w(v)$ given by (2.8) also involves a first order derivative. However, the pressure p in (1.1) is essentially two-dimensional up to a linear shift, see (2.16) below.

The general anisotropic primitive equations with full viscosity and diffusivity are obtained from (1.1) if one replaces the term $\nu_v \Delta_H$ by $\nu_v \Delta_H + \nu_{v,z}\partial_{zz}$, where $\nu_{v,z} > 0$ is the vertical viscosity, in the equation for v and similarly in the equation for T and provides additional appropriate boundary conditions in (1.2).

1.2 Previous results

Deterministic results. The mathematical analysis of the initial value problem for primitive equations with full viscosity and diffusivity was started by Lions, Temam and Wang [42, 43, 44] and launched significant activity in the field. In comparison to the 3D Navier-Stokes equations, the primitive equations are globally well-posed for initial data in $H^1(M)$ by a breakthrough result of Cao and Titi [9], see also Kobelkov [37]. The more realistic case of Dirichlet boundary conditions on the velocity on the bottom and lateral part of the boundary, also called physical boundary conditions in this context, in non-cylindrical domains were handled by Kukavica and Ziane [39]. These results were refined to global well-posedness for initial data with $v_0, \partial_z v_0 \in L^2$, see [35], or $v_0 \in L^1(-h, 0; L^\infty(G))$, see [21].

The primitive equations with only horizontal viscosity and diffusivity are of particular interest in the field of numerical weather prediction [40]. On the one hand, the horizontal viscosity ν_v is much larger than the vertical viscosity ν_z in the atmosphere and the limiting case $\nu_z = 0$ is considered a good approximation. On the other hand, numerical (hyper-)viscosity acting only in the horizontal directions is often used in the computer simulations for the hydrostatic Euler equations.

Cao, Li and Titi [7, 8] were the first to study the primitive equations with only horizontal viscosity and diffusivity analytically. They tackled this problem in a periodic setting by considering a vanishing vertical viscosity limit, i.e.

$$-\Delta_H - \varepsilon\partial_z^2 \quad \text{for} \quad \varepsilon \to 0.$$

Using this strategy, they obtained a remarkable global strong well-posedness result for the initial value problem with initial data of regularity near H^1, and local well-posedness for initial data in H^1. Recently, the first author applied a more direct approach considering the system without vanishing viscosity limit [51] where local well-posedness results even for less partial viscosities has been proven and, in the case of only horizontal viscosity, unnecessary boundary conditions on the bottom and top have been avoided. The construction of weak solutions, namely the compactness argument, is difficult in this case due to the lack of dissipation in the vertical direction. In [34], the existence of z-weak solutions, i.e. weak solutions with additional regularity in the vertical direction, and the existence of time-periodic solutions is shown for initial data with $v_0, \partial_z v_0 \in L^2(M)$. Additionally, the global existence of solutions is shown for less regular data than in [51]. For more results and further references on the deterministic primitive equations with horizontal and full viscosity, we refer the reader to the surveys [41] and [32].

Results on local well-posedness of the Navier-Stokes equations with only horizontal viscosity can be found in [1, Chapter 6].

The primitive equations without any viscosity and the equation for the temperature are called the hydrostatic Euler equations. One important reason to consider this system is the understanding of the behaviour of numerical weather and climate models because the time scale associated with viscous dissipation is beyond the current computational capability [40, Chapter 2]. The primitive equations with only horizontal viscosity are closer to this situation than the ones with full viscosities. For the hydrostatic Euler equations, blow-up results were established by Wong [59], see also [6], and ill-posedness results in Sobolev spaces were established by Han-Kwan and Nguyen [30]. Local well-posedness was proven only for analytical data by Kukavica et al. [38].

For more information on previous results and the geophysical applications of the primitive equations, we refer to the works of Washington and Parkinson [58], Pedlosky [50], Majda [45] and Vallis [57].

Stochastic results. Stochastic modelling plays an important role in meteorology and climatology. Current models in these fields consist of a deterministic dynamical core based on equations of continuum mechanics and thermodynamics, complemented by stochastic elements at several levels: random initial conditions, reflecting partial knowledge of the initial state, and random inputs distributed in space-time, related for instance to sub-grid stochastic parametrizations. Predictions are probabilistic in the sense that they aim to produce a range of scenarios with associated probabilities, usually by the method called ensemble forecasting system. For more information on this field, see the review articles [19, 48] and the references therein. Of particular interest in fluid dynamics and also for geophysical flows is a noise of transport type which appears naturally when stochastic models are derived from Hamiltonian principles as proposed in [33] (see also [4] for a brief description) and yield a physically relevant randomization [2] with energy conservation.

Recently, the importance of transport noise was discussed in the connection with unresolved small scales and stochastic model reduction in [17, 18], see also the references therein. Let $v = v_L + v_S$ be the decomposition of the velocity into parts describing interactions at large and small scales, respectively. In climate modelling, one is interested mainly in the large-scale interactions captured in v_L. Loosely speaking, the stochastic model reduction replaces the small-scales interaction $(v_S \cdot \nabla)v_S$ in the equation for v_S by additive noise of the form $\varepsilon^{-\alpha}\,\mathrm{d}\,W$ for some $\alpha > 0$ and studies the limiting equation for v_L as $\varepsilon \to 0$. Under suitable conditions, it can be shown that the limiting equation for v_L contains transport noise arising from the term $(v_S \cdot \nabla)v_L$.

Stochastic primitive equations with full viscosities were studied by several authors. In two space dimensions, i.e. neglecting one of the horizontal directions, the so-called weak-strong solutions for multiplicative white noise in time were constructed by a Galerkin approach by Glatt-Holtz and Ziane [25], where the weak-strong solutions are weak in the PDE sense and strong in the stochastic sense. In particular, these solutions can be interpreted in a pathwise sense. For initial data v_0 with $v_0, \partial_z v_0 \in L^p(\Omega, L^2(M))$ where Ω is the probability space and M the spatial domain, the authors show the existence and uniqueness of such solutions for $p \geq 4$, so these are z-weak solutions. They make use of a special cancellation related to the assumed periodic boundary conditions. The long term behaviour of weak solutions of the stochastic two-dimensional primitive equations is studied in [46]. The existence of global pathwise strong solutions is shown in [23] for initial data in $v_0 \in L^2(\Omega, H^1(M))$ also by means of Galerkin approximation. The authors use continuous martingale theory and stopping time arguments to treat the primitive equations with physical boundary conditions without the above-mentioned cancellation by establishing stronger convergence of the Galerkin approximations. Furthermore, a large deviation principle [20] and a central limit theorem [29] are known to hold.

Regarding the 3D problem, Debussche, Glatt-Holtz, Temam and Ziane established a global well-posedness result for pathwise strong solutions for multiplicative white noise in time in [12] and the related work [11] by a different method than in the 2D case.

They first show the existence of martingale solutions and pathwise uniqueness, which then leads to the existence of pathwise solutions by a Yamada-Watanabe type argument. The assumptions on the noise for global existence in [12] require certain smoothing properties that fail for transport noise. The second author with Brzeźniak [5] established global existence for noise allowing transport by the vertical average of the horizontal velocity $\overline{v}$. In the case of additive noise, the existence of a random pull-back attractor is known [28]. Logarithmic moment bounds in $H^2(M)$ are obtained in [22] and used to prove the existence of ergodic invariant measures supported in $H^1(M)$. A construction of weak-martingale solutions, i.e. martingale solutions whose regularity in space and time is the one of a weak solution, by an implicit Euler scheme is given in [24]. Large deviation principles [13] and moderate deviation principles [55] are known to hold for small multiplicative noise and short times [15]. The existence of a Markov selection is proven in [14] for additive noise.

All these above results are shown for the primitive equations with both horizontal and (i.e. full) viscosity and diffusivity. To the best of our knowledge, there are no results on the stochastic system with only horizontal viscosity and diffusivity or stochastic hydrostatic Euler equations.

1.3 Main results

Let us now present a simplified version of the main results. Full (precise and more technical) statements can be found in Section 2.7.

For separable Hilbert spaces $\mathcal{U}$ and X, $L_2(\mathcal{U},X)$ denotes the space of Hilbert-Schmidt operators from $\mathcal{U}$ to X. Let $\Delta_H = \partial_{xx}+\partial_{yy}$ be the horizontal Laplace operator. For $k,l\in\mathbb{N}_0$, we define the anisotropic spaces $H^k_zH^l_{xy}=H^k(-h,0;H^l(G))$. An additional subscript $_D$ or $_N$ in the symbol for anisotropic spaces denotes additional Dirichlet or Neumann boundary condition, respectively, on the appropriate part of the boundary. For example, $L^2_zH^1_{D,xy}=L^2(-h,0;H^1_0(G))$ introduces the Dirichlet boundary condition to the lateral part of the boundary and, similarly, $H^2_{N,z}L^2_{xy}=\{f\in H^2(-h,0;L^q(G)):\partial_z f=0 \text{ on } \Gamma_b\cup\Gamma_u\}$ imposes the Neumann boundary condition on the top and bottom parts of the boundary. The notation is explained in more detail in Section 2.1 below. For simplicity, we also assume $\nu=\nu_v=\nu_T$. The symbol c_{BDG} denotes the optimal constant in the Burkholder-Davis-Gundy inequality, see Section 2.5.

Theorem (Maximal existence)**.** Let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be a stochastic basis with filtration $\mathbb{F}=\{\mathcal{F}_t\}_{t\geq 0}$. Let $\mathcal{U}$ be a separable Hilbert space and let $\sigma: L^2_zH^1_{xy}\to L_2(\mathcal{U},L^2)$

1. satisfy the growth bound
$$\left\|(\partial_z)^k(-\Delta_H)^{\frac{l}{2}}\sigma(U)\right\|^2_{L_2(\mathcal{U},L^2)}\leq c\left(1+\|U\|^2_{H^k_zH^l_{xy}}\right)+\eta^2\left\|(\partial_z)^k(-\Delta_H)^{\frac{l+1}{2}}U\right\|^2_{L^2}$$
for all $U\in H^k_zH^{l+1}_{xy}$, all $(k,l)\in\mathcal{K}:=\{(0,0),(1,0),(1,1),(2,0)\}$ and some $\eta>0$ such that $\eta^2<2\nu/(3+8c^2_{BDG})$,
2. be Lipschitz as a map $\sigma: H^k_zH^{l+1}_{xy}\to L_2(\mathcal{U},H^k_zH^l_{xy})$ for all $(k,l)\in\mathcal{K}$ with constant $\gamma>0$ such that $\gamma^2<4\nu/(4c^2_{BDG}+1)$,
3. satisfy suitable bounds on the boundary $\Gamma_u\cup\Gamma_b$.

Let $(f_v,f_T)^T\in L^4(\Omega;L^2(0,t;H^2_{N,z}L^2_{xy}\times H^2_{D,z}L^2_{xy}))$. Then for all initial data $U_0=(v_0,T_0)\in L^2(\Omega;(L^2_zH^1_{D,xy}\times L^2_zH^1_{xy})\cap(H^2_{N,z}L^2_{xy}\times H^2_{D,z}L^2_{xy}))$ there exists a unique maximal pathwise (i.e. strong in the stochastic sense) strong (in the PDE sense) solution of (1.1)-(1.4).

The proof of the above theorem follows the method from [11]. First, the global existence of martingale solutions is established for a modified problem with a cut-off. Compared to [11], where the cut-off acts on the H^1-norm of the solution, we need to use a weaker cut-off acting on the $L_z^\infty L_{xy}^4$-norm due to the lack of vertical smoothing which requires more involved estimates. After a standard argument using the theorems by Prokhorov and Skorokhod (relying on compactness in the anisotropic spaces), the local existence of strong (in the stochastic sense) solutions is established by the Gyöngy-Krylov theorem.

However, for the Gyöngy-Krylov theorem, we need the to establish strong uniqueness of martingale solutions in the space $L^2(0,T;L_z^2H_{xy}^1 \cap L_z^\infty L_{xy}^4)$. The required uniqueness is a consequence Proposition 3.7 where we provide estimates on the difference of two martingale solutions in the space $H_z^1L_{xy}^2$. In this setting, the cancellation property of the nonlinear term cannot be employed and higher order estimates, in particular estimate on $\partial_{zz}U$, seem to be needed to control the nonlinear term. This additional regularity of the solution requires additional regularity of the initial data, namely $U_0 \in H_z^2L_{xy}^2 \times H_z^2L_{xy}^2$. Compared to the deterministic case studied in [34], where uniqueness of z-weak solutions is established in L^2, the procedure requires additional boundary conditions for v and T in the vertical direction. Similarly as in [7], we chose the homogeneous Neumann boundary condition for the velocity field and the homogeneous Dirichlet boundary conditions for the temperature. These boundary conditions are preserved by the primitive equations, see Remark 2.3 for more details.

The smallness assumption on η and γ come from the estimates in Lemma 3.1 (where it suffices to take $q=4$ as will be made clear from the proof of existence) and Lemma 3.7.

Given a map $u: M \to \mathbb{R}^2$, let $\overline{u} = h^{-1}\int_{-h}^0 u(\cdot,z)\,\mathrm{d}\,z$ be the vertical average and let $\widetilde{u} = u - \bar{u}$.

Theorem (Global existence)**.** Let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$, $\mathcal{U}$, U_0 and σ be as the Theorem above. Additionally, let σ have the transport form

$$\begin{aligned}\sigma_1(v)e_k &= \Psi_k \cdot \nabla_H \widetilde{v} + \Phi_k \cdot \nabla_H \overline{v} + h_k(v),\\ \sigma_2(U)e_k &= \Psi_k^T \cdot \nabla_H T + g_k(T,v),\end{aligned}$$

for some $\Psi_k: G \to \mathbb{R}^2$, $\Phi_k: (-h,0) \to \mathbb{R}^2$, $\Psi_k^T: M \to \mathbb{R}^2$ and $h_k(v), g_k(v,T): M \to \mathbb{R}^2$ such that $(\Psi_k^T)_k, (\overline{\Psi}_k)_k$ and $(\widetilde{\Psi}_k)_k$ are controlled by η in $\ell^2(L^\infty)$ and h_k and g_k satisfy suitable sublinear growth bounds. Let

$$U_0 = (v_0,T_0) \in L^{16/3}(\Omega;H^1) \cap L^{8/3}(\Omega;L^{132}), \quad \partial_z v_0 \in L^6(\Omega;L^6),$$

and

$$\begin{aligned}(f_v,f_T)^T \in & L^4(\Omega;L^2_{\mathrm{loc}}(0,\infty;H^2_{N,z}L^2_{xy} \times H^2_{D,z}L^2_{xy})) \cap L^{16/3}(\Omega;L^2_{\mathrm{loc}}(0,\infty;H_z^1L_{xy}^2))\\ & \cap L^{8/3}(\Omega;L^2_{\mathrm{loc}}(0,\infty;L^{132})).\end{aligned}$$

Let $\eta^2 < 2\nu/(131 + 2c_{BDG}^2)$. Then the solution from the above theorem is global.

Global existence of solutions is established combining the (deterministic) estimates, the logarithmic Sobolev inequality and logarithmic Gronwall lemma from [8] and an iterated stopping time argument from [12, 5].

In [8], one of the key steps is an estimate on the asymptotic behaviour of $\|v\|_{L^q}/q^{1/2}$ w.r.t. $q \to \infty$ which later leads to a bound on v in L^∞ and $\partial_z v$ in L^2 using the logarithmic Sobolev embedding and the logarithmic Gronwall lemma. However, due to the asymptotic nature of the estimate, it seems impossible to get a similar bound in the stochastic setting with noise acting on $\nabla_H u$, see Remark 4.9. The issue is resolved by a straightforward yet useful modification of the logarithmic Sobolev inequality allowing to substitute the asymptotic bound (which would require Ψ_k and Ψ_k^T to be zero) by bounds in L^q with q sufficiently large. It turns out that we can choose $q = 132$, see the discussion below

Proposition 4.1 for more details. With a bound in L^{132}, we are able to adapt the rest of the argument from [8] to the stochastic setting and obtain the desired L^2-integrability in time of the L^∞-norm and, in turn, a bound on $\partial_z v$ using an argument similar to the logarithmic Gronwall lemma.

In contrast to previous deterministic results [8, 34], we do not require the initial data to be essentially bounded thanks to the modified version of the logarithmic Sobolev inequality. Compared to previous results for the stochastic primitive equations with full viscosity and diffusivity [12, 5], our approach allows for full transport noise. The key difference to [12] are pressure estimates in $L^q(M)$ in Lemma 4.7 necessary to deduce $L^q(M)$-estimates for the velocity field in Proposition 4.8. In [12], the authors consider an auxiliary Stokes problem with noise term from the original equation and then prove estimates for the difference of the solution of the full non-linear problem and the solution of the auxiliary Stokes problem. The difference solves a random PDE and standard analytic tools can be used to estimate the pressure term. The disadvantage of this approach is that it requires the solution of the Stokes problem to be rather smooth and transport noise cannot be included for this reason. We follow the approach of [5] where the conditions on the noise allow to obtain a random PDE for the pressure by using a hydrostatic Leray-Helmholtz projection. Using the linear structure of the transport part of the noise, we can go beyond the results in [5] and consider transport noise acting not only on the vertical average of the velocity $\overline{v}$ but also on the remainder $\widetilde{v} = v - \overline{v}$.

The additional smallness assumption on η comes from the L^q-estimates in Proposition 4.8, see also Remark 4.9.

1.4 Organization of the paper

In Section 2, we define the function spaces used throughout the rest of the paper and reformulate the primitive equations as an abstract functional problem. In particular, the discussion on the assumptions on the noise term σ and the definition of solution can be found in Sections 2.4 and 2.6, respectively. In Section 3, we prove the maximal existence theorem above. After defining the Galerkin approximations of a modified problem in Section 3.1 and establishing bounds on the approximations in Section 3.2, we prove the tightness of the corresponding measures and the global existence of martingale solutions of the modified problem in Section 3.3. Uniqueness is established in Section 3.4 for more regular initial data due to the use of the Gyöngy-Krylov theorem. Finally, Section 4 contains the proof of the global existence of strong solutions using estimates on the barotropic and baroclinic modes of the velocity, the logarithmic Sobolev embedding and the logarithmic Gronwall inequality.

2 Preliminaries

2.1 Function spaces and notations

By $L^2(M)$, we denote the standard real Lebesgue space with scalar product

$$\langle f, g\rangle_M := \int_M f(x,y,z)g(x,y,z)\,\mathrm{d(x,y,z)},$$

with $L^2(G)$ and $\langle f, g\rangle_G$ defined analogously. We denote the induced norms by $\|f\|_{L^2(M)}$ and $\|f\|_{L^2(G)}$, respectively. If there is no ambiguity, we will not specify the domain in the notation and write e.g. $< f, g >$ instead of $< f, g >_M$.

For $k \in \mathbb{N}$, the space $H^k(M)$ consists of $f \in L^2(M)$ such that $\nabla^\alpha f \in L^2(M)$ for all multi-indices $|\alpha| \leq k$ endowed with the norm

$$\|f\|_{H^k(M)} = \sum_{|\alpha|\leq k} \|\nabla^\alpha f\|_{L^2(M)},$$

where $\nabla^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_z^{\alpha_3}$ for $\alpha \in \mathbb{N}_0^3$. The space $H^k(G)$ is defined analogously and we set $H_0^1(G) := \{f \in H^1(G) \mid f|_{\partial G} = 0\}$. Again, we will write $\|f\|_{H^k}$ if there is no ambiguity.

For non-integer $s \geq 0$, the Bessel potential spaces H^s are defined by complex interpolation, see [56] for details. Analogous definitions hold for spaces $H^{s,p}$ for $p \in [1,\infty]$ and for Banach space-valued function spaces such as the Bochner space $H^1(0,t;L^p(M))$ and the anisotropic space $H^{s,p}(-h,0;H^{r,q}(G))$ for which we will often use the notation $H_z^{s,p}H_{xy}^{r,q}$. We identify $H^s = H^{s,2}$. Furthermore, for $s \geq 1$, we write

$$L_z^p H_{D,xy}^s := L^2(-h,0;H^s(G) \cap H_0^1(G)), \tag{2.1}$$

$$L_z^2 H_{N,xy}^2 := \{f \in L_z^2 H_{xy}^2 \mid \partial_{n_G} T = 0 \text{ on } \Gamma_l\} \tag{2.2}$$

and

$$H_{D,z}^2 L_{xy}^q := \{f \in H^2(-h,0;L^q(G)) : f = 0 \text{ on } \Gamma_b \cup \Gamma_u\}, \tag{2.3}$$

$$H_{N,z}^2 L_{xy}^q := \{f \in H^2(-h,0;L^q(G)) : \partial_z f = 0 \text{ on } \Gamma_b \cup \Gamma_u\}. \tag{2.4}$$

for the spaces encoding the boundary condition on the lateral and the vertical boundary. The boundary condition $\partial_{n_G} T = 0$ on Γ_l has to be understood as a condition for the trace of $\nabla_H T$ on Γ_l for almost every $z \in (-h,0)$.

For the sake of completeness, let us recall the definition of fractional Sobolev spaces $W^{\alpha,p}(0,t;X)$ for some Banach space X and $\alpha \in (0,1]$. For $\alpha = 1$, let

$$W^{1,p}(0,t;X) = \left\{u \in L^p(0,t;X) \mid \tfrac{\mathrm{d}}{\mathrm{dt}}u \in L^p(0,t;X)\right\},$$

$$\|U\|_{W^{1,p}(0,t;X)}^p = \int_0^t \|U\|_X^p + \left\|\tfrac{\mathrm{d}}{\mathrm{dt}}u\right\|_X^p \mathrm{ds}.$$

For $\alpha \in (0,1)$ and $p \in (1,\infty)$, the fractional Sobolev spaces $W^{\alpha,p}(0,t;X)$ are defined by real interpolation. By [54], we have $H^{\alpha,p}(0,t;X) \subset W^{\alpha,p}(0,t;X)$ for $\alpha \in (0,1)$ and $p \in (1,\infty)$. We can now state the following compactness result. For proofs see [53, Theorem 5] and [16, Theorem 2.1], respectively.

Lemma 2.1. *a)* (the Aubin-Lions-Simon lemma) *Let $X_2 \subset X \subset X_1$ be Banach spaces such that the embedding $X_2 \hookrightarrow\hookrightarrow X$ is compact and the embedding $X \hookrightarrow X_1$ is continuous. Let $p \in [1,\infty]$ and $\alpha \in (0,1]$. Then the following embedding is compact:*

$$L^p(0,t;X_2) \cap W^{\alpha,p}(0,t;X_1) \hookrightarrow\hookrightarrow L^p(0,t;X).$$

b) Let $X_2 \subset X$ be Banach spaces such that X_2 is reflexive and the embedding $X_2 \hookrightarrow\hookrightarrow X$ is compact. Let $\alpha \in (0,1)$ and $p \in (1,\infty)$ be such that $\alpha p > 1$. Then the following embedding is compact:

$$W^{\alpha,p}(0,t;X_2) \hookrightarrow\hookrightarrow C([0,t],X).$$

Let us now summarize some of the properties of the anisotropic spaces on a cylindric domain.

Proposition 2.2. *The anisotropic spaces have the following properties:*

1. *Embeddings in the spaces $H_z^{s,p}H_{xy}^{r,q}$ can be performed separately:*

$$H_z^{s,p}H_{xy}^{r,q} \hookrightarrow H_z^{s',p'}H_{xy}^{r,q} \quad \text{if} \quad H^{s,p}((-h,0)) \hookrightarrow H^{s',p'}((-h,0)),$$

$$H_z^{s,p}H_{xy}^{r,q} \hookrightarrow H_z^{s,p}H_{xy}^{r',q'} \quad \text{if} \quad H^{r,q}(G) \hookrightarrow H^{r',q'}(G).$$

2. *The following anisotropic Hölder inequality holds: Let* $p, p_1, p_2, q, q_1, q_2 \in [1, \infty]$ *with* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}, \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$ *and* $f \in L^{p_1}_z L^{q_1}_{xy}, g \in L^{p_2}_z L^{q_2}_{xy}$*. Then* $fg \in L^p_z L^q_{xy}$ *and*

$$\|fg\|_{L^p_z L^q_{xy}} \leq \|f\|_{L^{p_1}_z L^{q_1}_{xy}} \|g\|_{L^{p_2}_z L^{q_2}_{xy}} . \tag{2.5}$$

3. *Let* $v \in H^1_z L^p_{xy}$*. Then* $v \in L^\infty_z L^p_{xy}$ *and*

$$\|v\|_{L^\infty_z L^p_{xy}} \leq c \|v\|^{1/2}_{L^2_z L^p_{xy}} \|v\|^{1/2}_{H^1_z L^p_{xy}} . \tag{2.6}$$

4. *Let* $v \in H^1$*. Then* $v \in L^\infty_z L^4_{xy}$ *and*

$$\|v\|_{L^\infty_z L^4_{xy}} \leq c \|v\|^{1/2}_{H^1_z L^2_{xy}} \|v\|^{1/2}_{L^2_z H^1_{xy}} . \tag{2.7}$$

5. *Let* $\alpha, \beta > 1/2$*. The following embeddings are compact:*

$$L^2_z H^2_{xy} \cap H^1_z H^1_{xy} \hookrightarrow\hookrightarrow L^2_z H^1_{xy}, \qquad H^\beta_z H^\alpha_{xy} \hookrightarrow\hookrightarrow L^\infty_z L^4_{xy}.$$

Proof. Checking the embedding properties of the anisotropic spaces and the anisotropic Hölder inequality is straightforward. The interpolation inequality (2.6) has been established in [27, Lemma 3.3(a)] for $p = 2$. The more general case follows in the same manner. The interpolation inequality (2.7) has been proven in [8, Lemma 2.3].

It remains to establish the compact embeddings. The first follows directly from the Aubin-Lions lemma, see Lemma 2.1 a). The second embedding is a consequence of the Aubin-Lions lemma, the continuous embedding $H^\beta_z H^\alpha_{xy} \hookrightarrow L^\infty_z H^\alpha_{xy} \cap H^{\gamma,\infty}_z L^4_{xy}$ holding for some $\gamma > 0$ sufficiently small and the compact embedding $H^\alpha(G) \hookrightarrow\hookrightarrow L^4(G)$. □

2.2 Hydrostatic-solenoidal vector fields

We reformulate the primitive equations (1.1) and (1.2) as a system containing only two-dimensional surface pressure p_s and prognostic variables v and T. The divergence free condition $\partial_z w + \mathrm{div}_H\, v = 0$ and the boundary condition $w = 0$ on $\Gamma_b \cup \Gamma_u$ are equivalent to

$$w(v)(t,x,y,z) = w(t,x,y,z) = -\mathrm{div}_H \int_{-h}^{z} v(t,x,y,z')\,\mathrm{d}z', \tag{2.8}$$

$$\mathrm{div}_H \int_{-h}^{0} v(t,x,y,z')\,\mathrm{d}z' = 0, \tag{2.9}$$

for v sufficiently smooth, e.g. with $\mathrm{div}_H\, v \in L^1(M)$. The vertical velocity w is thus a function of the horizontal velocity v. If we denote the vertical average and its complement by $\overline{v}$ and $\tilde{v}$, respectively, i.e.

$$\overline{v}(t,x,y) := \frac{1}{h} \int_{-h}^{0} v(t,x,y,z')\,\mathrm{d}z' \quad \text{and} \quad \tilde{v} := v - \overline{v}, \tag{2.10}$$

then (2.9) implies that the vertical average $\overline{v}$ is divergence free, i.e. $\mathrm{div}_H\, \overline{v} = 0$. To ease the notation from the typographical point of view, we sometimes write $\mathcal{A}v$ and $\mathcal{R}v$ instead of $\overline{v}$ and $\widetilde{v}$, respectively. Hence, one identifies a suitable hydrostatic-solenoidal space as

$$L^2_{\overline{\sigma}}(M) := \overline{\{v \in C^\infty_c(M)^2 \mid \mathrm{div}_H\, \overline{v} = 0\}}^{\|\cdot\|_{L^2}},$$

where C^∞_c stands for smooth compactly supported functions. Note that this space admits the decomposition

$$L^2_{\overline{\sigma}}(M) = L^2_\sigma(G) \oplus \{v \in L^2(M)^2 \mid \overline{v} = 0\}, \tag{2.11}$$

where $L^2_\sigma(G) = \overline{\{\varphi \in C_c^\infty(G)^2 \colon \operatorname{div}_H \varphi = 0\}}^{||\cdot||_{L^2}}$ is the space of solenoidal vector fields over G. The hydrostatic Helmholtz projection thereon is therefore defined by

$$P_\sigma \colon L^2(M)^2 \to L^2_{\overline{\sigma}}(M), \quad P_\sigma v = \tilde{v} + P_G \overline{v}, \tag{2.12}$$

where P_G the classical Leray-Helmholtz projection on $L^2(G)$. More precisely, since due to the product structure $L^2(M) = \overline{L^2(G) \otimes L^2(-h,0)}$, one obtains by applying P_σ that

$$L^2_{\overline{\sigma}}(M) = \overline{L^2_\sigma(G) \otimes \operatorname{span}\{1\}} \oplus \overline{L^2(G)^2 \otimes L^2_0(-h,0)}, \tag{2.13}$$

where $L^2_0(-h,0) = \{v \in L^2(-h,0) \mid \int_{-h}^0 v(z)\,\mathrm{dz} = 0\}$ and $1 \in L^2(-h,0)$ is a constant function.

We now may reformulate the system (1.1, 1.2) as (see [49, Section 2.1] for more details)

$$\begin{aligned} &\partial_t v - \nu_v \Delta_H v + k \times v \\ + \frac{1}{\rho_0} \nabla_H p_s - \beta_T g \int_z^0 &\nabla_H T(x,y,z')\,\mathrm{dz}' \\ &+ v \cdot \nabla_H v - w(v)\partial_z v = f_v + \sigma_1(v, \nabla_H v, T, \nabla_H T)\dot{W}_1, \\ \partial_t T - \nu_T \Delta_H T + v \cdot \nabla_H T &+ w(v)\partial_z T = f_T + \sigma_2(v, \nabla_H v, T, \nabla_H T)\dot{W}_2 \end{aligned} \tag{2.14}$$

in $M \times (0,t)$, where p_s is the surface pressure, and

$$v(s) \in L^2_{\overline{\sigma}}(M) \quad \text{for } s \in (0,t) \tag{2.15}$$

complemented by the initial conditions (1.3) and boundary conditions (1.4). The system is thus closed. We can reconstruct the full pressure p from the surface pressure p_s by

$$p(x,y,z) = p_s(x,y) + g \int_z^0 \rho(x,y,z')\,\mathrm{dz}', \tag{2.16}$$

where $\rho = \rho_0(1 - \beta_T(T - T_r))$ as in (1.1). We emphasize that p_s is independent of the vertical variable z.

Remark 2.3. In order to show the existence of strong solutions, we assume $T_0 = 0$ on $\Gamma_b \cup \Gamma_u$. Note that the homogeneous Dirichlet boundary condition can be relaxed to $T_0 = c_b$ on Γ_b and $T_0 = c_u$ on Γ_u for constants c_b, c_u. Indeed, applying the linear transformation $\hat{T} = T - c_u \frac{z+h}{h} + c_b \frac{z}{h}$, one immediately observes that $\hat{T}_0 = 0$ on $\Gamma_b \cup \Gamma_u$ and that $(v, \hat{T})$ satisfies (2.14) with an additional term $w(v)\frac{c_u - c_b}{h}$ and $\sigma_2(v, \nabla_H v, \hat{T}, \nabla_H \hat{T})$ replaced by $\sigma_2(v, \nabla_H v, \hat{T} + c_u \frac{z+h}{h} - c_b \frac{z}{h}, \nabla_H \hat{T})$. The additional deterministic term can be handled in the same way as $-\beta_T g \int_z^0 \nabla_H T(x,y,z')\,\mathrm{d}z'$ in the equation for v and it vanishes on $\Gamma_b \cup \Gamma_u$. Since the inhomogenous Dirichlet condition does not introduce any additional difficulties, we consider only the homogenous Dirichlet condition to simplify the presentation.

2.3 Functional formulation

Let us now formulate the original equations (1.1)-(1.4) in an abstract functional form. We also provide estimates on the nonlinear term demonstrating the importance of the anisotropic spaces defined in the previous section. For simplicity, we assume that $\nu = \nu_v = \nu_T$.

We call the operator $-\nu P_\sigma \Delta_H$ the *hydrostatic Stokes operator*. Let

$$D(A_H) = \left(L_z^2 H_{D,xy}^2 \cap L_\sigma^2\right) \times L_z^2 H_{N,xy}^2.$$

For $U \in D(A_H)$, we define the operator A_H by

$$A_H\, U = \begin{pmatrix} -\nu P_\sigma \Delta_H\, v \\ -\nu \Delta_H\, T \end{pmatrix}.$$

The representation of P_σ given in the previous subsection and (2.13) allow us to introduce an orthonormal system of eigenfunctions of A_H using the known orthonormal systems associated with the 2D Stokes operator the 3D horizontal Laplacian. To this end, let $(\varphi_m)_m \subset C^\infty(\overline{G})^2 \cap H_0^1(G)^2$ and $(\hat{\varphi}_m)_m \subset C^\infty(\overline{G})^2 \cap H_0^1(G)^2 \cap L_\sigma^2(G)$ be orthonormal bases of eigenfunctions of the Dirichlet Laplacian in $L^2(G)^2$ and the Stokes operator in $L_\sigma^2(G)$, respectively, with corresponding increasing sequences of eigenvalues $(\mu_m)_m$ and $(\hat{\mu}_m)_m$. Moreover, recall that $\{\cos(\frac{k\pi}{h}(z+h))\}_{k\in\mathbb{N}_0}$ forms a basis of eigenfunctions of the Neumann Laplacian on $L^2((-h,0))$ and, by the first representation theorem, a basis on $H^1((-h,0))$ and of $H_N^2((-h,0)) := \{v \in H^2((-h,0)) \mid \partial_z v(-h) = \partial_z v(0) = 0\}$ as well.

For $m \in \mathbb{N}$ and $k \in \mathbb{N}_0$, we define the functions $\Phi_{m,k} \in C^\infty(\overline{M})$ by

$$\Phi_{m,k}(x,y,z) := \begin{cases} \frac{2}{h}\varphi_m(x,y)\cos\left(\frac{k\pi}{h}(z+h)\right) & \text{for} \quad k>0, \\ \frac{1}{h}\hat{\varphi}_m(x,y) & \text{for} \quad k=0. \end{cases} \tag{2.17}$$

Then $\operatorname{span}\{\Phi_{m,k} | m\in\mathbb{N}, k\in\mathbb{N}_0\}$ is dense in L_σ^2, in particular $\operatorname{div}_H \overline{\Phi}_{m,k} = 0$, because for $k>0$ we have $\overline{\Phi}_{m,k} = 0$ and $\operatorname{div}_H \overline{\Phi}_{m0} = \operatorname{div}_H \hat{\varphi}_m = 0$.

For the temperature, note that the functions $\{\sin(\frac{k\pi}{h}(z+h))\}_{k\in\mathbb{N}}$ form a basis of eigenfunctions of the Dirichlet Laplacian on $L^2((-h,0))$ and a basis on $H_0^1((-h,0))$ and of $H_D^2((-h,0)) := H^2((-h,0)) \cap H_0^1((-h,0))$. Let $(\psi_m)_m \subset C^\infty(\overline{G})^2$ be an orthonormal basis of eigenfunctions of the Neumann Laplacian in $L^2(G)^2$ with an increasing sequence of eigenvalues $(\lambda_m)_m$ and define for $m,k\in\mathbb{N}$ the functions $\Psi_{m,k} \in C^\infty(\overline{M})$ by

$$\Psi_{m,k}(x,y,z) := \frac{2}{h}\psi_m(x,y)\sin\left(\frac{k\pi}{h}(z+h)\right). \tag{2.18}$$

Then $\operatorname{span}\{\Psi_{m,k} | m,k \in \mathbb{N}\}$ is dense in L^2. Hence, $\operatorname{span}\{\Phi_{m,k} | m\in\mathbb{N}, k\in\mathbb{N}_0\} \times \operatorname{span}\{\Psi_{m,k} | m,k\in\mathbb{N}\}$ is dense in $L_\sigma^2 \times L^2(M)$ and from the construction we observe that it is also dense in $D(A_H)$. Additionally,

$$-\Delta_H\, \Psi_{m,k} = \lambda_m \Psi_{m,k} \quad \text{and} \quad -P_\sigma \Delta_H\, \Phi_{m,k} = \begin{cases} \mu_m \Phi_{m,k} & k>0, \\ \tilde{\mu}_m \Phi_{m,k}, & k=0, \end{cases} \tag{2.19}$$

therefore the sequences $(\Psi_{m,k})_{m,k}$ and $(\Phi_{m,k})_{m,k}$ are indeed eigenfunctions of $-\Delta_H$ and $-P_\sigma\Delta_H$, respectively. Note that the eigenspaces of all the eigenvalues have infinite dimension.

Proposition 2.4. *The operator A_H is a self-adjoint unbounded operator. In particular, $D(A_H) \subset L_z^2 H_{xy}^2$, the domains of fractional powers $D(A_H^\alpha)$ are dense in $L_\sigma^2 \times L^2$ for $\alpha \geq 0$. Moreover, let*

$$\tilde{P}_n : L_\sigma^2 \times L^2 \to \operatorname{span}\{\Phi_{m,k} \mid m \leq n, k\in\mathbb{N}_0\} \times \operatorname{span}\{\Psi_{m,k} \mid m\leq n, k\in\mathbb{N}\},$$

where $\Phi_{m,k}$ and $\Psi_{m,k}$ are defined in (2.17) *and* (2.18)*, be the orthogonal projection onto its range, $Q_n := I - \tilde{P}_n$ and $\|\cdot\|_\alpha := \|A_H \cdot\|_{L_\sigma^2\times L^2}$. Then the following Poincaré* *inequalities hold*

$$\|\tilde{P}_n U\|_{\alpha_2} \leq \overline{\lambda}_n^{\alpha_2-\alpha_1} \|\tilde{P}_n U\|_{\alpha_1}, \quad \|Q_n U\|_{\alpha_1} \leq \overline{\lambda}_n^{\alpha_1-\alpha_2} \|Q_n U\|_{\alpha_2}, \tag{2.20}$$

for $n \in \mathbb{N}$, $0 \leq \alpha_1 < \alpha_2$ *and* $\overline{\lambda}_n := \max\{\mu_n, \hat{\mu}_n, \lambda_n\}$.

Proof. Density of the domains of fractional powers is well-known, see e.g. [52, Theorem 37.2]. It is easy to see that the operator A_H is positive and symmetric, hence there exists a self-adjoint extension. Having already found an orthonormal system of eigenfunctions for which the corresponding eigenvalues don't have an accumulation point, we conclude that A_H itself is this extension. The inequalities (2.20) follow from the estimate

$$\begin{aligned}
\|\tilde{P}_n U\|_{\alpha_2}^2 &= \int_{-h}^0 \|\tilde{P}_n U(z)\|_{H_{xy}^{\alpha_2}}^2 \,\mathrm{dz} \\
&\leq \int_{-h}^0 \max\{\mu_n, \tilde{\mu}_n, \lambda_n\}^{2(\alpha_2-\alpha_1)} \|\tilde{P}_n U(z)\|_{H_{xy}^{\alpha_2-\alpha_1}}^2 \,\mathrm{dz} \\
&= \max\{\mu_n, \tilde{\mu}_n, \lambda_n\}^{2(\alpha_2-\alpha_1)} \|\tilde{P}_n U\|_{\alpha_2-\alpha_1}^2,
\end{aligned}$$

where we used the inequality

$$\|\tilde{P}_n U(z)\|_{H_{xy}^{\alpha_2}} \leq \max\{\mu_n, \tilde{\mu}_n, \lambda_n\}^{2(\alpha_2-\alpha_1)} \|\tilde{P}_n U(z)\|_{H_{xy}^{\alpha_2-\alpha_1}},$$

which can be derived similarly as in [26, Lemma 2.1] by considering the operator A_H in $L^2(G)$. ☐

Let $B : H^1 \times H^1 \to H^{-2}$ be the bilinear operator

$$B(U, U^\flat) = \begin{pmatrix} P_\sigma \left[(v \cdot \nabla_H)v^\flat + w(v)\partial_z v^\flat\right] \\ (v \cdot \nabla_H)T^\flat + w(v)\partial_z T^\flat \end{pmatrix}, \qquad U = (v, T), U^\flat = (v^\flat, T^\flat) \in H^1.$$

The operator B is continuous by e.g. [49, Lemma 2.1]. In fact, by (2.24) in Proposition 2.5 below, $B(U, U^\flat)$ is well defined in L^2 for $U, U^\flat \in L_z^2 H_{xy}^2 \cap H_z^1 H_{xy}^1$. The standard cancellation property

$$\left\langle B(U, U^\sharp), |U^\sharp|^q U^\sharp \right\rangle = 0,$$

holds for $q \geq 0$, $U = (v, T)$ and $U^\sharp = (v^\sharp, T^\sharp)$ sufficiently regular. Indeed, denoting $b(v^\sharp, v) := v^\sharp \cdot \nabla_H v + w(v^\sharp)\partial_z v$ and $b(v^\sharp, T) := v^\sharp \cdot \nabla_H T + w(v^\sharp)\partial_z T$ with a slight abuse of notation, we have

$$\begin{aligned}
\left\langle b(v, v^\sharp), |v^\sharp|^q v^\sharp \right\rangle &= \int_M (v \cdot \nabla_H v^\sharp)|v^\sharp|^q v^\sharp \,\mathrm{d(x,y,z)} \\
&\quad - \int_M \left(\int_{-h}^z \mathrm{div}_H\, v(x,y,z') \,\mathrm{dz'}\right) \partial_z v^\sharp \; |v^\sharp|^q v^\sharp \,\mathrm{d(x,y,z)} \\
&= -\frac{1}{q+2} \int_M (\mathrm{div}_H\, v)|v^\sharp|^{q+2} - (\mathrm{div}_H\, v)|v^\sharp|^{q+2} \,\mathrm{d(x,y,z)} \\
&= 0,
\end{aligned}$$

where one uses the divergence-free vertical average condition (2.9). The identity $\left\langle b(v, T^\sharp), |T^\sharp|^q T^\sharp \right\rangle = 0$ can be obtained similarly.

Proposition 2.5. *We have the following estimates on the nonlinear term:*

1. *For* $U \in H^1$, $U^\flat \in H_z^1 H_{xy}^1$ *and* $U^\sharp \in L_z^2 H_{xy}^1$, *we have*

$$\left|\left\langle B(U, U^\flat), U^\sharp \right\rangle\right| \leq c\, \|U\|_{H^1} \, \|U^\flat\|_{H_z^1 H_{xy}^1} \|U^\sharp\|_{L_z^2 H_{xy}^1}. \tag{2.21}$$

2. *For* $U \in L^2_z H^1_{xy} \cap L^\infty_z L^4_{xy}$*,* $U^\flat \in L^2_z H^1_{xy} \cap L^\infty_z L^4_{xy}$*,* $U^\sharp \in H^1_z L^4_{xy}$*, we have*

$$\left|\left\langle B(U, U^\flat), U^\sharp \right\rangle\right| \\ \leq c(\|U\|_{L^\infty_z L^4_{xy}} \|\nabla_H U^\flat\|_{L^2} + \|\nabla_H U\|_{L^2}\|U^\flat\|_{L^\infty_z L^4_{xy}})\|U^\sharp\|_{H^1_z L^4_{xy}}. \quad (2.22)$$

3. *For* $U \in H^1_z H^1_{xy}$ *with* $v \in L^2_z H^1_{D,xy}$*, we have*

$$|\langle \partial_z B(U, U), \partial_z U\rangle| \\ \leq c\,\|U\|_{L^\infty_z L^4_{xy}} \left(\|\nabla_H \partial_z U\|_{L^2}\|\partial_z U\|_{L^2} + \|\nabla_H \partial_z U\|^{3/2}_{L^2}\|\partial_z U\|^{1/2}_{L^2}\right). \quad (2.23)$$

4. *For* $U \in L^\infty_z L^4_{xy} \cap L^2_z H^2_{xy}$*,* $U^\flat \in L^2_z H^2_{xy} \cap H^1_z H^1_{xy}$*, we have*

$$\|B(U, U^\flat)\|^2_{L^2} \leq c\,\|U\|^2_{L^\infty_z L^4_{xy}} \|\nabla_H U^\flat\|_{L^2} \left(\|\nabla_H U^\flat\|_{L^2} + \|\Delta_H U^\flat\|_{L^2}\right) \\ + c\|\nabla_H U\|_{L^2}\|\Delta_H U\|_{L^2}\|\partial_z U^\flat\|_{L^2} \left(\|\partial_z U^\flat\|_{L^2} + \|\nabla_H \partial_z U^\flat\|_{L^2}\right). \quad (2.24)$$

5. *For* $U \in H^1_z H^1_{xy}$ *with* $(v, T) \in L^2_z H^2_{D,xy} \times L^2_z H^2_{N,xy}$*, we have*

$$|\langle B(U, U), \Delta_H U\rangle| \leq c\|U\|_{L^\infty_z L^4_{xy}}\|U\|^{1/2}_{L^2_z H^1_{xy}}\|U\|^{3/2}_{L^2_z H^2_{xy}} \\ + c\|U\|_{L^\infty_z L^4_{xy}}\|U\|^{1/2}_{L^2_z H^1_{xy}}\|U\|^{1/2}_{L^2_z H^2_{xy}}\|U\|_{H^1_z H^1_{xy}}. \quad (2.25)$$

Proof. The estimates are established using the anisotropic Hölder inequality (2.5). For simplicity of the presentation, we prove the estimates only for the temperature.

To prove (2.21), we employ (2.7) to obtain

$$\left|\left\langle v \cdot \nabla_H T^\flat, T^\sharp \right\rangle\right| \leq \|v\|_{L^\infty_z L^4_{xy}} \|\nabla_H T^\flat\|_{L^2}\|T^\sharp\|_{L^2_z L^4_{xy}} \\ \leq \|v\|_{H^1} \|T^\flat\|_{H^1}\|T^\sharp\|_{L^2_z H^1_{xy}}.$$

Similarly, using $\|w(v)\|_{L^\infty_z L^2_{xy}} \leq \|\nabla_H v\|_{L^2}$ which follows from the definition of w (2.8), we get

$$\left|\left\langle w(v)\partial_z T^\flat, T^\sharp \right\rangle\right| \leq c\|\nabla_H v\|_{L^2}\|\partial_z T^\flat\|_{L^2_z L^4_{xy}}\|T^\sharp\|_{L^2_z L^4_{xy}}.$$

Continuing to (2.22), we treat the first term as above by

$$\left|\left\langle v \cdot \nabla_H T^\flat, T^\sharp \right\rangle\right| \leq \|v\|_{L^\infty_z L^4_{xy}} \|\nabla_H T^\flat\|_{L^2}\|T^\sharp\|_{L^2_z L^4_{xy}}.$$

Recalling that $w = 0$ on $\Gamma_u \cup \Gamma_b$, we use integration by parts and the definition of w (2.8) to get

$$\left|\left\langle w(v)\partial_z T^\flat, T^\sharp \right\rangle\right| = \left|\left\langle \mathrm{div}_H\, v T^\flat, T^\sharp \right\rangle + \left\langle w(v) T^\flat, \partial_z T^\sharp \right\rangle\right| \\ \leq \|\mathrm{div}_H\, v\|_{L^2}\|T^\flat\|_{L^\infty_z L^4_{xy}}\|T^\sharp\|_{L^2_z L^4_{xy}} + \|w(v)\|_{L^2}\|T^\flat\|_{L^\infty_z L^4_{xy}}\|\partial_z T^\sharp\|_{L^2_z L^4_{xy}} \\ \leq \|\nabla_H v\|_{L^2}\|T^\flat\|_{L^\infty_z L^4_{xy}}\|T^\sharp\|_{H^1_z L^4_{xy}}.$$

For (2.23), we use the cancellation property $\langle B(U, \partial_z U), \partial_z U\rangle = 0$ and integration by parts with $v = 0$ on Γ_l to obtain

$$\langle \partial_z b(v, T), \partial_z T\rangle = \langle \partial_z v \nabla_H T, \partial_z T\rangle - \langle \mathrm{div}\,_H v \partial_z T, \partial_z T\rangle \\ = -\langle \mathrm{div}_H\, \partial_z v T, \partial_z T\rangle - \langle \partial_z v T, \nabla_H \partial_z T\rangle + 2\,\langle v \cdot \nabla_H \partial_z T, \partial_z T\rangle.$$

Therefore, by the Hölder inequality and the Ladyzhenskaya inequality, we get

$$\begin{aligned}|\langle \partial_z b(v,T), \partial_z T\rangle| &\le c\,\|U\|_{L^\infty_z L^4_{xy}}\,\|\partial_z U\|_{L^2_z L^4_{xy}}\|\nabla_H\,\partial_z U\|_{L^2}\\ &\le c\,\|U\|_{L^\infty_z L^4_{xy}}\left(\|\nabla_H\,\partial_z U\|_{L^2}\|\partial_z U\|_{L^2}+\|\nabla_H\,\partial_z U\|^{3/2}_{L^2}\|\partial_z U\|^{1/2}_{L^2}\right).\end{aligned}$$

To establish (2.24), we use the 2D Gagliardo-Nirenberg inequality to deduce

$$\begin{aligned}\|v\cdot\nabla_H\,T^\flat\|^2_{L^2} &\le \|v\|^2_{L^\infty_z L^4_{xy}}\,\|\nabla_H\,T^\flat\|^2_{L^2_z L^4_{xy}}\\ &\le c\,\|v\|^2_{L^\infty_z L^4_{xy}}\,\|\nabla_H\,T^\flat\|_{L^2}\left(\|\nabla_H\,T^\flat\|_{L^2}+\|\Delta_H\,T^\flat\|_{L^2}\right).\end{aligned}$$

Similarly, from the definition of w (2.8) and the interpolation inequality (2.7), we have

$$\begin{aligned}\|w(v)\partial_z T^\flat\|^2_{L^2} &\le \|w(v)\|^2_{L^\infty_z L^4_{xy}}\|\partial_z T^\flat\|^2_{L^2_z L^4_{xy}}\\ &\le c\|\nabla_H\,v\|_{L^2}\|\Delta_H\,v\|_{L^2}\|\partial_z T^\flat\|_{L^2}\left(\|\partial_z T^\flat\|_{L^2}+\|\nabla_H\,\partial_z T^\flat\|_{L^2}\right).\end{aligned}$$

Finally, regarding (2.25), we integrate by parts to get

$$\begin{aligned}\langle b(v,T),\Delta_H\,T\rangle &= -\langle\nabla_H\,b(v,T),\nabla_H\,T\rangle\\ &= -\langle b(v,\nabla_H\,T),\nabla_H\,T\rangle\\ &\quad -\langle\nabla_H\,v_1\partial_x T+\nabla_H\,v_2\partial_y T+\nabla_H\,w(v)\partial_z T,\nabla_H\,T\rangle\\ &= -\langle\nabla_H\,v_1\partial_x T,\nabla_H\,T\rangle-\langle\nabla_H\,v_2\partial_y T,\nabla_H\,T\rangle-\langle\nabla_H\,w(v)\partial_z T,\nabla_H\,T\rangle\\ &= \langle v_1\nabla_H\,\partial_x T,\nabla_H\,T\rangle+\langle v_1\partial_x T,\Delta_H\,T\rangle\\ &\quad +\langle v_2\nabla_H\,\partial_y T,\nabla_H\,T\rangle+\langle v_2\partial_y T,\Delta_H\,T\rangle\\ &\quad -\langle\nabla_H\,\mathrm{div}_H\,vT,\nabla_H\,T\rangle+\langle w(v)T,\nabla_H\,\partial_z T\rangle\,,\end{aligned}$$

where we used the boundary conditions for v, T and w. Hence, we have

$$\begin{aligned}|\,\langle b(v,T),\Delta_H\,T\rangle\,| &\le c\|T\|_{L^2_z H^2_{xy}}\|v\|_{L^\infty_z L^4_{xy}}\|\nabla_H\,T\|_{L^2_z L^4_{xy}}\\ &\quad + c\|w(v)\|_{L^2_z L^4_{xy}}\|T\|_{L^\infty_z L^4_{xy}}\|\nabla_H\,\partial_z T\|_{L^2}\\ &\le c\|U\|_{L^\infty_z L^4_{xy}}\|U\|^{1/2}_{L^2_z H^1_{xy}}\|U\|^{3/2}_{L^2_z H^2_{xy}}\\ &\quad + c\|U\|_{L^\infty_z L^4_{xy}}\|U\|^{1/2}_{L^2_z H^1_{xy}}\|U\|^{1/2}_{L^2_z H^2_{xy}}\|U\|_{H^1_z H^1_{xy}}.\end{aligned}$$ ☐

Next, we turn to a higher order estimate in the vertical direction used in Section 3.4 to establish the existence of solutions regular enough to prove uniqueness.

Proposition 2.6. *For $U,\partial_z U\in H^1_z H^1_{xy}$ with $(v,T)\in L^2_z H^1_{D,xy}\times L^2_z H^2_{N,xy}$, we have*

$$|\langle\partial_{zz}B(U,U),\partial_{zz}U\rangle|\le c\,\|U\|_{H^1_z L^4_{xy}}\,\|\partial_{zz}U\|^{1/2}_{L^2}\,\|U\|^{3/2}_{H^2_z H^1_{xy}}\,. \tag{2.26}$$

Proof. Similarly as above, we only show the estimates for the temperature to keep the presentation concise. We prove the claim by the anisotropic Hölder inequality (2.5). Using the cancellation $\langle B(v,\partial_{zz}T),\partial_{zz}T\rangle=0$, we get

$$|\langle\partial_{zz}b(v,T),\partial_{zz}T\rangle|\le|\langle b(\partial_{zz}v,T),\partial_{zz}T\rangle|+2\,|\langle b(\partial_z v,\partial_z T),\partial_{zz}T\rangle|\,.$$

Integration by parts and the Ladyzhenskaya inequality yield

$$\begin{aligned}|\langle \partial_{zz} v \cdot \nabla_H T, \partial_{zz} T\rangle| &= |\langle T \partial_{zz} \mathrm{div}_H\, v, \partial_{zz} T\rangle + \langle T \partial_{zz} v, \nabla_H \partial_{zz} T\rangle| \\ &\le c \, \|\partial_{zz} U\|_{L^2_z L^4_{xy}} \, \|U\|_{L^\infty_z L^4_{xy}} \, \|\nabla_H \partial_{zz} U\|_{L^2} \\ &\le c \, \|U\|_{L^\infty_z L^4_{xy}} \, \|U\|^{1/2}_{H^2_z L^2_{xy}} \, \|U\|^{3/2}_{H^2_z H^1_{xy}}\end{aligned}$$

and

$$|\langle \mathrm{div}_H\, v \partial_{zz} T, \partial_{zz} T\rangle| \le \|U\|_{L^\infty_z L^4_{xy}} \, \|\partial_{zz} U\|^{1/2}_{L^2} \, \|\partial_{zz} U\|^{3/2}_{L^2_z H^1_{xy}} .$$

From (2.5) and the interpolation inequality (2.6), we deduce

$$\begin{aligned}&|\langle \mathrm{div}_H\, \partial_z v \partial_z T, \partial_{zz} T\rangle| \\ &\quad\le c \, \|\mathrm{div}_H\, \partial_z v\|_{L^\infty_z L^2_{xy}} \, \|\partial_z T\|_{L^2_z L^4_{xy}} \, \|\partial_{zz} T\|_{L^2_z L^4_{xy}} \\ &\quad\le c \, \|\mathrm{div}_H\, \partial_z U\|^{1/2}_{L^2} \, \|\mathrm{div}_H\, \partial_z U\|^{1/2}_{H^1_z L^2_{xy}} \, \|U\|_{H^1_z L^4_{xy}} \, \|\partial_{zz} U\|^{1/2}_{L^2} \, \|\partial_{zz} U\|^{1/2}_{L^2_z H^1_{xy}} \\ &\quad\le c \, \|U\|_{H^1_z L^4_{xy}} \, \|U\|^{1/2}_{H^2_z L^2_{xy}} \, \|U\|^{3/2}_{H^2_z H^1_{xy}}\end{aligned}$$

and

$$|\langle \partial_z v \cdot \partial_z \nabla_H T, \partial_{zz} T\rangle| \le c \, \|U\|_{H^1_z L^4_{xy}} \, \|U\|^{1/2}_{H^2_z L^2_{xy}} \, \|U\|^{3/2}_{H^2_z H^1_{xy}} \square$$

We denote the hydrostatic contribution of the pressure by

$$A_{\mathrm{pr}}\, U = \begin{pmatrix} -P_\sigma \beta_T g \int_z^0 \nabla_H T(x,y,z')\, \mathrm{d}z' \\ 0 \end{pmatrix}, \qquad U \in L^2_z H^1_{xy},$$

the Coriolis forcing by

$$EU = \begin{pmatrix} P_\sigma k \times v \\ 0 \end{pmatrix}, \qquad U \in L^2,$$

and the regular forcing by

$$F_U = \begin{pmatrix} P_\sigma f_v \\ f_T \end{pmatrix} \in H^1 \quad \text{a.e. in } [0,t].$$

To ease up the notation, we define

$$F(U) \equiv \begin{pmatrix} F_v(U) \\ F_T(U) \end{pmatrix} = A_{\mathrm{pr}}\, U + EU - F_U, \qquad U \in L^2_z H^1_{xy}.$$

Clearly, $F(U)$ satisfies a sublinear growth condition and is Lipschitz continuous, that is

$$\begin{aligned}\|F(U)\|_{L^2} &\le c \left(\|F_U\|_{L^2} + \|U\|_{L^2_z H^1_{xy}} \right), && U \in L^2_z H^1_{xy}, \\ \|F(U) - F(U^\sharp)\|_{L^2} &\le c \|U - U^\sharp\|_{L^2_z H^1_{xy}}, && U, U^\sharp \in L^2_z H^1_{xy}.\end{aligned}$$

Let $\mathcal{U}$ be a separable Hilbert space. For another Hilbert space X, let $L_2(\mathcal{U}, X)$ be the space of Hilbert-Schmidt operators $G : \mathcal{U} \to X$. We define the noise term $\sigma : L^2_z H^1_{xy} \to L_2(\mathcal{U}, L^2)$ by

$$\sigma(U) = \begin{pmatrix} P_\sigma \sigma_1(v, \nabla_H v, T, \nabla_H T) \\ \sigma_2(v, \nabla_H v, T, \nabla_H T) \end{pmatrix}, \qquad U \in L^2_z H^1_{xy}.$$

Assumptions on the noise term σ are discussed in Section 2.4.

We may now reformulate equation (2.14) as

$$\mathrm{d}U + [A_H U + B(U) + F(U)]\, \mathrm{dt} = \sigma(U)\, \mathrm{dW}, \qquad U(t=0) = U_0. \tag{2.27}$$

To establish bounds in better spaces required for global existence, we need to use the particular structure of the primitive equations, in particular the possibility of the decomposition into the barotropic and baroclinic modes.

Recalling the notation in (2.10) and below, we follow [9] and split the momentum equation in (2.14) into equations for the barotropic mode $\overline{v}$

$$\partial_t\overline{v} - \nu\Delta_H\,\overline{v} + \frac{1}{\rho_0}\nabla_H\,p_s + \overline{v}\cdot\nabla_H\,\overline{v} = -N(\widetilde{v}) + \mathcal{A}F_v(U) + \mathcal{A}\sigma_1(U)\dot{W}_1, \tag{2.28}$$

$$\operatorname{div}_H\overline{v} = 0, \tag{2.29}$$

where

$$\mathcal{A}F_v(U) = -k\times\overline{v} + \frac{\beta_T g}{h}\int_{-h}^{0}\int_{z}^{0}\nabla_H\,T(x,y,z')\,\mathrm{d}z'\,\mathrm{d}z + \overline{f_v},$$

$$N(\widetilde{v}) = \frac{1}{h}\int_{-h}^{0}(\widetilde{v}\cdot\nabla_H\widetilde{v} + (\operatorname{div}_H\widetilde{v})\,\widetilde{v})\,\mathrm{d}z,$$

and the baroclinic mode $\widetilde{v}$

$$\partial_t\widetilde{v} - \nu\Delta_H\,\widetilde{v} + \widetilde{v}\cdot\nabla_H\,\widetilde{v} = -\overline{v}\cdot\nabla_H\widetilde{v} - \tilde{v}\cdot\nabla_H\overline{v} + N(\widetilde{v}) + \mathcal{R}F_v(U) + \mathcal{R}\sigma_1\dot{W}_1, \tag{2.30}$$

where

$$\mathcal{R}F_v(U) = F_v(U) - \mathcal{A}F_v(U) = -k\times\widetilde{v} + \widetilde{f_v} + \beta_T g\left[\int_{z}^{0}\nabla_H\,T(x,y,z')\,\mathrm{d}z' - \frac{1}{h}\int_{-h}^{0}\int_{z}^{0}\nabla_H\,T(x,y,z')\,\mathrm{d}z'\,\mathrm{d}z\right].$$

2.4 Assumptions on noise

We assume that σ satisfies the growth conditions

$$\|\sigma(U)\|^2_{L_2(\mathcal{U},L^2)} \le c(1+\|U\|^2_{L^2}) + \eta^2\|(-\Delta_H)^{1/2}U\|^2_{L^2}, \tag{2.31}$$

$$\|\partial_z\sigma(U)\|^2_{L_2(\mathcal{U},L^2)} \le c(1+\|U\|^2_{H^1_zL^2_{xy}}) + \eta^2\|(-\Delta_H)^{1/2}\partial_zU\|^2_{L^2}, \tag{2.32}$$

$$\|(-\Delta_H)^{1/2}\sigma(U)\|^2_{L_2(\mathcal{U},L^2)} \le c(1+\|U\|^2_{L^2_zH^1_{xy}}) + \eta^2\|\Delta_H\,U\|^2_{L^2}, \tag{2.33}$$

$$\|\partial_{zz}\sigma(U)\|^2_{L_2(\mathcal{U},L^2)} \le c(1+\|U\|^2_{H^2_zL^2_{xy}}) + \eta^2\|\partial_{zz}(-\Delta_H)^{1/2}U^n\|^2_{L^2}, \tag{2.34}$$

for $U\in L^2_zH^1_{xy}$, $H^1_zH^1_{xy}$, $L^2_zH^2_{xy}$ and $H^2_zH^1_{xy}$, respectively, with $\eta>0$, and is Lipschitz continuous

$$\|\sigma(U)-\sigma(U^\sharp)\|_{L_2(\mathcal{U},L^2)} \le \gamma\|U-U^\sharp\|_{L^2_zH^1_{xy}}, \qquad U,U^\sharp\in L^2_zH^1_{xy}, \tag{2.35}$$

$$\|\partial_z\left[\sigma(U)-\sigma(U^\sharp)\right]\|_{L_2(\mathcal{U},L^2)} \le \gamma\|U-U^\sharp\|_{H^1_zH^1_{xy}}, \qquad U,U^\sharp\in H^1_zH^1_{xy}, \tag{2.36}$$

$$\|(-\Delta_H)^{1/2}\left[\sigma(U)-\sigma(U^\sharp)\right]\|_{L_2(\mathcal{U},L^2)} \le \gamma\|U-U^\sharp\|_{L^2_zH^2_{xy}}, \qquad U,U^\sharp\in L^2_zH^2_{xy}, \tag{2.37}$$

$$\|\partial_{zz}\left[\sigma(U)-\sigma(U^\sharp)\right]\|_{L_2(\mathcal{U},L^2)} \le \gamma\|U-U^\sharp\|_{H^2_zH^1_{xy}}, \qquad U,U^\sharp\in H^2_zH^1_{xy} \tag{2.38}$$

for some $\gamma>0$. An example of a noise term σ satisfying the above can be constructed similarly as in [5, Section 2.5]. Furthermore, let on $\Gamma_b\cup\Gamma_u$,

$$\|\partial_z\sigma_1(U)\|^2_{L_2(\mathcal{U},L^2(G))} \le c(\|\partial_zv\|^2_{L^2(G)} + \|T\|^2_{L^2(G)}) + \eta^2\|(-\Delta_H)^{1/2}\partial_zv\|^2_{L^2(G)}, \tag{2.39}$$

$$\|\sigma_2(U)\|^2_{L_2(\mathcal{U},L^2(G))} \le c(\|\partial_zv\|^2_{L^2(G)} + \|T\|^2_{L^2(G)}) + \eta^2\|(-\Delta_H)^{1/2}T\|^2_{L^2}. \tag{2.40}$$

Under the above conditions, we show the existence of a maximal solution from Theorem 2.11 in Section 3. However, to obtain global solutions from Theorem 2.12, we need stronger assumptions using the split into (2.28) and (2.30). Thus, in Section 4, we will consider noise of the form

$$\begin{aligned} \sigma_1(v)e_k &= \Psi_k \cdot \nabla_H \widetilde{v} + \Phi_k \cdot \nabla_H \overline{v} + h_k(v), \\ \sigma_2(U)e_k &= \Psi_k^T \cdot \nabla_H T + g_k(T,v), \end{aligned} \tag{2.41}$$

where $(e_k)_k$ is a basis of the underlying Hilbert space $\mathcal{U}$, the functions $\Psi_k : G \to \mathbb{R}^2$, $\Phi_k : (-h,0) \to \mathbb{R}^2$, $\Psi_k^T : M \to \mathbb{R}^2$ satisfy

$$\Psi_k \in W^{1,\infty}(G), \quad \Phi_k \in W^{2,\infty}(-h,0), \quad \Psi_k^T \in W_z^{2,\infty} W_{xy}^{1,\infty},$$

$\partial_z \Phi_k = 0$ on $\Gamma_b \cup \Gamma_u$ and $h_k(v), g_k(v,T) : M \to \mathbb{R}^2$ are such that

$$\operatorname{div}_H \mathcal{A} h_k(v) = 0 \text{ for } \operatorname{div}_H \overline{v} = 0, \tag{2.42}$$

$$\sum_{k=1}^{\infty} \|D h_k(v)\|_{L^q}^2 \le c\left(1 + \|v\|_{L^q}^2 + \|Dv\|_{L^q}^2\right), \quad D \in \{1, \nabla_H, \partial_z\}, \tag{2.43}$$

$$\sum_{k=1}^{\infty} \|g_k(v,T)\|_{L^q}^2 \le c\left(1 + \|T\|_{L^q}^2 + \|\nabla_H \overline{v}\|_{L^q}^2 + \|v\|_{L^q}^2\right), \tag{2.44}$$

$$\sum_{k=1}^{\infty} \|\Psi_k^T\|_{L^\infty}^2 \le \eta^2, \quad \sum_{k=1}^{\infty} \|\mathcal{A}\Phi_k\|_{L^\infty}^2 \le \eta^2, \quad \sum_{k=1}^{\infty} \|\mathcal{R}\Psi_k\|_{L^\infty}^2 \le \eta^2. \tag{2.45}$$

Moreover, let on $\Gamma_b \cup \Gamma_u$

$$\|\partial_z h_k(v)\|_{L^2(G)} \le c\|\partial_z v\|_{L^2(G)}, \quad \|g_k(v)\|_{L^2(G)} \le c(\|\partial_z v\|_{L^2(G)} + \|T\|_{L^2(G)})$$

The above assumptions imply

$$\partial_z \sigma_1(v)e_k = \Psi_k \cdot \nabla_H \partial_z v + \partial_z \phi_k \cdot \nabla_H \overline{v} + \partial_z h_k(v), \tag{2.46}$$

$$\mathcal{A}\sigma_1(v)e_k = (\mathcal{A}\Phi_k) \cdot \nabla_H \overline{v} + \mathcal{A} h_k(v), \tag{2.47}$$

$$\mathcal{R}\sigma_1(v)e_k = \Psi_k \cdot \nabla_H \widetilde{v} + (\mathcal{R}\Phi_k) \cdot \nabla_H \overline{v} + \mathcal{R} h_k(v). \tag{2.48}$$

Moreover, since $(\mathcal{A}\Phi_k)$ is constant, (2.42) yields $\operatorname{div}_H \mathcal{A}\sigma_1(v) = 0$, in other words

$$(1 - P_G)\mathcal{A}\sigma_1(v) = 0 \text{ in } L_2(\mathcal{U}, L^2), \tag{2.49}$$

where P_G is the standard Leray-Helmholtz projection on the 2D domain G. By (2.43), (2.45), (2.47) and the boundary conditions for v, we get

$$\|(-\Delta_H)^{1/2}\mathcal{A}\sigma_1(U)\|_{L_2(\mathcal{U},L^2)}^2 \le \eta^2 \|\Delta_H \overline{v}\|_{L^2}^2 + c\left(1 + \|(-\Delta_H)^{1/2} v\|_{L^2}^2\right). \tag{2.50}$$

Example 2.7. The additional assumptions (2.42)-(2.45) are satisfied for

$$\begin{aligned} h_k(v) &= \zeta_k \overline{v} + \nu_k \widetilde{v} + \chi_k, \\ g_k(v) &= \gamma_k T + \Theta_k \cdot \nabla_H \overline{v} + \hat{\zeta}_k \overline{v} + \hat{\nu}_k \widetilde{v} + \hat{\chi}_k \end{aligned}$$

for sufficiently regular functions ζ_k, ν_k, χ_k, γ_k, Θ_k, $\hat{\zeta}_k$, $\hat{\nu}_k$ and $\hat{\chi}_k$ such that $\partial_z \zeta_k$, $\partial_z \nu_k$, $\partial_z \chi_k$, Θ_k, $\hat{\zeta}_k$, $\hat{\nu}_k$ and $\hat{\chi}_k$ vanish on $\Gamma_b \cup \Gamma_u$.

2.5 Stochastic preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a stochastic basis with filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$. Let $\mathcal{U}$ be a separable Hilbert space and let W be an $\mathbb{F}$-cylindrical Wiener process with reproducing kernel Hilbert space $\mathcal{U}$. It is well-known that if $\mathcal{U}_0$ is a Hilbert space such that the embedding $\mathcal{U} \hookrightarrow \mathcal{U}_0$ is Hilbert-Schmidt, the trajectories of W are continuous in time in $\mathcal{U}_0$.

Let X be a Hilbert space. For a predictable process $\Phi : (0,t)\times\Omega \to L_2(\mathcal{U}, X)$ satisfying $\|\Phi\|_{L^2(0,t;L_2(\mathcal{U},X))} < \infty$ $\mathbb{P}$-a.s. the stochastic integral $\int_0^{\cdot} G \,\mathrm{dW}$ is well defined, see e.g. [10, Section 4]. We will often use the following two variants of the Burkholder-Davis-Gundy inequality. Let $\Phi \in L^p(\Omega; L^2(0,t; L_2(\mathcal{U}, X)))$ be predictable for some $p \geq 1$. Then

$$\mathbb{E} \sup_{s\in[0,t]} \left\| \int_0^s \Phi \,\mathrm{dW} \right\|_X^p \leq c_{BDG} \mathbb{E} \left(\int_0^t \|\Phi\|^2_{L_2(\mathcal{U},X)} \,\mathrm{ds} \right)^{p/2}. \tag{2.51}$$

For proof see e.g. [36, Theorem 3.28, p. 166]. We note that the constant c_{BDG} depends on p, even though we will tacitly omit the dependence since it does not significantly affect the results in this paper. A fractional variant of the Burkholder-Davis-Gundy inequality has been established in [16, Lemma 2.1]. Let $p \geq 2$, $\Phi \in L^p(\Omega; L^p(0,t; L_2(\mathcal{U}, X)))$ be predictable and let $\alpha \in [0, 1/2)$. Then

$$\mathbb{E} \left\| \int_0^t \Phi \,\mathrm{dW} \right\|^p_{W^{\alpha,p}(0,t;X)} \leq c_{BDG} \mathbb{E} \int_0^t \|\Phi\|^p_{L_2(\mathcal{U},X)} \,\mathrm{ds}\,. \tag{2.52}$$

2.6 Notion of solution

We adapt the definitions from [11]. All the solutions considered here are strong in the PDE sense. The definitions can be changed in a straightforward way to cover modified variants of equation (2.27) which we will study in Section 3.

For the definitions below, let $\mathcal{U}$ be a fixed separable Hilbert space.

Definition 2.8. *Let μ_0 be a probability measure on H^1 such that*

$$\int_{H^1} \|U\|^2_{H^1} \,\mathrm{d}\mu_0(\mathrm{U}) < \infty.$$

1. *A tuple $(\mathcal{S}, U, \tau, W)$ is a* local martingale solution *of* (2.27) *if $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a stochastic basis, W is an $\mathbb{F}$-cylindrical Wiener process with reproducing kernel Hilbert space $\mathcal{U}$, τ is a strictly positive $\mathbb{F}$-stopping time and $U = U(\cdot\wedge\tau) : \Omega \times [0,\infty) \to H^1$ is an $\mathbb{F}$-adapted stochastic process satisfying*

$$\begin{aligned} U(\cdot\wedge\tau) &\in L^2\left(\Omega; C([0,\infty), H^1)\right), \\ \mathbb{1}_{[0,\tau]} U &\in L^2\left(\Omega; L^2_{\mathrm{loc}}(0,\infty; H^1_z H^1_{xy} \cap D(A_H))\right), \end{aligned} \tag{2.53}$$

the law of $U(0)$ is μ_0 and the process U satisfies

$$U(s\wedge\tau) + \int_0^s A_H U + B(U,U) + F(U) \,\mathrm{dr} = U(0) + \int_0^s \sigma(U) \,\mathrm{dW} \tag{2.54}$$

for all $s \geq 0$ in H.

2. *The martingale solution $(\mathcal{S}, U, \tau)$ is* global *if $\tau = \infty$ $\mathbb{P}$-almost surely.*

Definition 2.9. *Let $U_0 \in L^2(\Omega; H^1)$ be an $\mathcal{F}_0$-measurable random variable and let $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a stochastic basis. Let W be an $\mathbb{F}$-cylindrical Wiener process with reproducing kernel Hilbert space $\mathcal{U}$.*

1. *A pair (U, τ) is a* local pathwise solution *if τ is a strictly positive $\mathbb{F}$-stopping time and $U = U(\cdot\wedge\tau) : \Omega\times[0,\infty) \to H^1$ is an $\mathbb{F}$-adapted stochastic process such that* (2.53) *and* (2.54) *hold w.r.t. the stochastic basis $\mathcal{S}$.*

2. *Let (τ_n) be an increasing sequence of $\mathbb{F}$-stopping times converging $\mathbb{P}$-a.s. to an $\mathbb{F}$-stopping time ξ. The triple $(U, \xi, (\tau_n))$ is called a* maximal pathwise solution *if (U, τ_n) is a local pathwise solution for all $n \in \mathbb{N}$ and*

$$\sup_{s\in[0,\xi)} \|U\|_{H^1}^2 + \int_0^\xi \|A_H U\|_{L^2}^2 + \|U\|_{H_z^1 H_{xy}^1}^2 \,\mathrm{ds} = \infty \tag{2.55}$$

for a.a. $\omega \in \{\xi < \infty\}$.

3. *The maximal pathwise solution $(U, \xi, (\tau_n))$ is called* global *if $\xi = \infty$ $\mathbb{P}$-a.s.*

Remark 2.10. The notions of martingale and pathwise solutions from Definitions 2.8 and 2.9 are the weak and strong solutions, respectively, in the stochastic sense. We chose to adopt the terminology from e.g. [23, 11] to avoid any confusion with weak or strong solutions in the PDE sense.

2.7 Main results revisited

The following theorems will be proved in Sections 3.5 and 4.4, respectively.

Theorem 2.11 (Maximal existence)**.** *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a stochastic basis with filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $\mathcal{U}$ be a separable Hilbert space and let σ satisfy* (2.31)-(2.40)*. Let $(f_v, f_T)^T \in L^4(\Omega; L^2(0, t; H^2_{N,z} L^2_{xy} \times H^2_{D,z} L^2_{xy}))$. Let $\eta^2 < 2\nu/(3 + 8c^2_{BDG})$ and $\gamma^2 < 4\nu/(4c^2_{BDG} + 1)$. Then for all initial data $U_0 = (v_0, T_0) \in L^2(\Omega; (L^2_z H^1_{D,xy} \times L^2_z H^1_{xy}) \cap (H^2_{N,z} L^2_{xy} \times H^2_{D,z} L^2_{xy}))$ there exists a unique maximal pathwise solution (U, τ) of* (1.1)-(1.4) *in the sense of Definition 2.9. Moreover, the solution satisfies*

$$\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_N]} \|U\|^2_{H^2_z L^2_{xy}} + \int_0^{t\wedge\tau_N} \|U\|^2_{H^2_z H^1_{xy}} \,\mathrm{ds}\right] < \infty \tag{2.56}$$

for all $N \in \mathbb{N}$ and $t > 0$ and $U(s) \in H^2_{N,z} L^2_{xy} \times H^2_{D,z} L^2_{xy}$ for all $0 < s < \tau$ $\mathbb{P}$-a.s.

Theorem 2.12 (Global existence)**.** *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, $\mathcal{U}$, U_0 and σ_1 be as in Theorem 2.11. Additionally, let σ satisfy* (2.41)-(2.45) *and let*

$$U_0 = (v_0, T_0) \in L^{16/3}(\Omega; H^1) \cap L^{8/3}(\Omega; L^{132}), \quad \partial_z v_0 \in L^6(\Omega; L^6),$$

and

$$\begin{aligned}(f_v, f_T)^T \in & L^4(\Omega; L^2_{\mathrm{loc}}(0, \infty; H^2_{N,z} L^2_{xy} \times H^2_{D,z} L^2_{xy})) \cap L^{16/3}(\Omega; L^2_{\mathrm{loc}}(0, \infty; H^1_z L^2_{xy})) \\ & \cap L^{8/3}(\Omega; L^2_{\mathrm{loc}}(0, \infty; L^{132})).\end{aligned}$$

Let $\eta^2 < 2\nu/(131 + 2c^2_{BDG})$. Then there exists a unique global pathwise solution $(U, \xi, (\tau_n))$ of (1.1)-(1.4) *in the sense of Definition 2.9. Moreover, the solution satisfies* (2.56) *for all $N \in \mathbb{N}$ and $t > 0$.*

Remark 2.13. Even though we omit the dependence of the constant c_{BDG} in the Burkholder-Davis-Gundy inequality (2.51) in the rest of the paper, let us note that the conditions on ν and γ in the theorems above should hold for $c_{BDG} = c_{BDG}(p)$ with $p \in [1, 2]$ for both Theorem 2.11 and Theorem 2.12.

3 Existence of maximal solutions

To establish local existence of strong solutions of the system (2.14), we study a modified equation with a cut-off to make all the nonlinear transport terms globally Lipschitz in suitable spaces. Local existence will then follow by a localization argument.

Let $\theta \in C^\infty(\mathbb{R})$ be a fixed function such that

$$\mathbb{1}_{[-1/2,1/2]} \leq \theta \leq \mathbb{1}_{[-1,1]},$$

i.e. $\theta(r) = 1$ for $r \in [-1/2, 1/2]$ and $\theta(r) = 0$ for $|r| \geq 1$. Let $\theta_\lambda(\cdot) = \theta(\cdot/\lambda)$ for $\lambda > 0$. For $\rho > 0$ fixed, we define

$$\theta(U(t)) = \theta_\rho \left(\|U(t)\|_{L_z^\infty L_{xy}^4} \right). \tag{3.1}$$

We are looking for a solution $U = (v, T)$ to the modified system

$$\mathrm{d}U + [A_H U + \theta(U) B(U,U) + F(U)]\,\mathrm{d}t = \sigma(U)\,\mathrm{dW}, \quad U(0) = U_0. \tag{3.2}$$

As we have already described in the introduction, the original system (2.14) doesn't provide any direct control over vertical derivatives $\partial_z v, \partial_z T$. To work around this issue, we use the basis defined in (2.17) and (2.18) and consider the spaces

$$H := L^2_{\overline{\sigma}} \times L^2 \text{ and } V := \left(L^2_z H^1_{D,xy} \cap L^2_{\overline{\sigma}} \right) \times L^2_z H^1_{xy}$$

with the corresponding inner products $\langle \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle_V$. By V_1', V_2' we denote the dual spaces

$$V' := V_1' \times V_2' = L^2_z H^{-1}_{xy} \times L^2_z H^{-1}_{xy},$$

where, with slight abuse of notation, H^{-1}_{xy} denotes the duals of $H^1_0(G)$ and $H^1(G)$, respectively. We denote the dual pairing in $V \times V'$ by $\langle \cdot, \cdot \rangle_H$ to keep the notation simple.

3.1 Galerkin scheme

To define a suitable basis of $H_{\overline{\sigma}}$ for a Galerkin scheme, one can take advantage of the direct sum (2.13). In Section 2.3 we have defined for $m, l \in \mathbb{N}, k \in \mathbb{N}_0$ the functions $\Phi_{m,k}$ and $\Psi_{m,k}$ in (2.17) and (2.18) to be eigenfunctions of the hydrostatic Stokes operator and the horizontal Laplacian, see (2.19). Recall that these functions are dense in $H_{\overline{\sigma}}$ and H_2, respectively. We set

$$H_{\overline{\sigma},n} := \operatorname{span}\{\Phi_{m,k} | m,k \leq n\},\ H_{2,n} := \operatorname{span}\{\Psi_{m,l} | m,l \leq n\},\ H_n := H_{\overline{\sigma},n} \times H_{2,n}$$

and define $P_n : H \to H_n$ to be the orthogonal projection onto H_n, see Proposition 2.4. We remark that this step is only possible because we consider a cylindrical domain. In the case of a more realistic topography, one has to transform the domain into a cylindrical one which leads to additional lower order terms in both equations.

Another consequence of having a cylindrical domain is the following important property of the projector P_n: For $k, m \in \mathbb{N}$ and for any function $g \in C^\infty(\overline{M})$, it holds

$$\langle g, \Phi_{m,k} \rangle = -\frac{h^2}{k^2 \pi^2} \langle g, -\partial_{zz} \Phi_{m,k} \rangle = \frac{h^2}{k^2 \pi^2} \langle \partial_z g, \partial_z \Phi_{m,k} \rangle .$$

Similarly, we have $\langle g, \Phi_{m,k} \rangle = \frac{h^4}{k^4 \pi^4} \langle \partial_{zz} g, \partial_{zz} \Phi_{m,k} \rangle$ if additionally $\partial_z g = 0$ on $\Gamma_b \cup \Gamma_u$ and we obtain the same equalities for $\Psi_{m,k}$ and $g \in C^\infty(\overline{M})$ with $g = 0$ on $\Gamma_b \cup \Gamma_u$. Hence, when projecting the system (2.14) onto H_n, the projected equations do not only hold for U but also for $\partial_z U$ and $\partial_{zz} U$ because of (2.39), (2.40) and $w = 0$ on $\Gamma_b \cup \Gamma_u$.

We now project the primitive equation onto the finite-dimensional space H_n and we look for a solution $U^n = (v^n, T^n) : [0, T] \to V_n$ of the system of stochastic differential equations

$$\mathrm{d}U^n + [A_H U^n + \theta(U^n) B^n(U^n, U^n) + F^n(U^n)]\,\mathrm{d}t = \sigma^n(U^n)\,\mathrm{dW} \tag{3.3}$$

with the initial condition

$$U^n(t=0) = U_0^n := (P_n v_0, P_n T_0), \tag{3.4}$$

where

$$B^n(\cdot,\cdot) = P_n B(\cdot,\cdot), \qquad F^n(\cdot) = P_n F(\cdot) \quad \text{and} \quad \sigma^n(\cdot)(\cdot) = P_n[\sigma(\cdot)(\cdot)].$$

Existence of solutions U^n of the finite-dimensional system (3.3) follows by a standard fixed point argument if $U_0 \in H$ since the nonlinear term $\theta(U^n)B^n(U^n,U^n)$ is globally Lipschitz and satisfies a sublinear growth condition thanks the cut-off function θ and the form of the eigenvectors $\{\Phi_{m,k}, \Psi_{m,l} \mid m,l \in \mathbb{N}, k \in \mathbb{N}_0\}$. Note that such a solution with enough regularity to have the trace of $\nabla_H T$ defined on Γ_l satisfies the Neumann boundary condition for T.

Note that $v_n(t) \in H_{\overline{\sigma},n}$ encodes a divergence free condition and determines the pressures p_s^n and that the projected equations hold also for the vertical derivatives. Thus, we have for $\partial_z U^n$ the system

$$\begin{aligned} \mathrm{d}\partial_z U^n + [A_H \partial_z U^n + \theta(U^n)\partial_z B^n(U^n,U^n) + \partial_z F(U^n)]\,\mathrm{dt} \\ = \partial_z F_U^n \,\mathrm{dt} + \partial_z \sigma^n(U^n)\,\mathrm{dW}, \end{aligned} \tag{3.5}$$

with the initial condition $\partial_z U^n(t=0) = \partial_z U_0^n$ and for $\partial_{zz} U^n$ the system

$$\begin{aligned} \mathrm{d}\partial_{zz} U^n + [A_H \partial_{zz} U^n + \theta(U^n)\partial_{zz} B^n(U^n,U^n) + \partial_{zz} F(U^n)]\,\mathrm{dt} \\ = \partial_{zz} F_U^n \,\mathrm{dt} + \partial_{zz} \sigma^n(U^n)\,\mathrm{dW}, \end{aligned} \tag{3.6}$$

with the initial condition $\partial_{zz} U^n(t=0) = \partial_{zz} U_0^n$.

3.2 Estimates

In this section, we establish the main estimates needed to pass to the limit in the Galerkin approximations.

Lemma 3.1. *Let $t > 0$, $q \geq 2$ and let $U_0 = (v_0, T_0) \in L^q(\Omega; H^1)$ be an $\mathcal{F}_0$-measurable random variable. Let σ satisfy* (2.31)-(2.33), (2.35)-(2.37) *and let*

$$F_U \in L^q\left(\Omega; L^q(0,t; H_z^1 L_{xy}^2)\right).$$

Assume $\nu > \eta^2\left(\frac{q-1}{2} + q c_{BDG}^2\right)$. Then the following holds:

1. *The sequence U^n is bounded in*

$$L^q\left(\Omega; L^\infty(0,t; H^1)\right), \quad L^q\left(\Omega; L^2(0,t; H_z^1 H_{xy}^1 \cap L_z^2 H_{xy}^2)\right).$$

2. *Let $\alpha \in [0, 1/2)$. Then $\int_0^\cdot \sigma^n(U^n)\,\mathrm{dW}$ is bounded in*

$$L^q\left(\Omega; W^{\alpha,q}(0,t; L^2)\right).$$

3. *Let $q \geq 4$ and $p \geq q/2$. Then $U^n - \int_0^\cdot \sigma^n(U^n)\,\mathrm{dW}$ is bounded in*

$$L^p\left(\Omega; W^{1,2}(0,t; L^2)\right)$$

Proof. We divide the proof into eight steps. Due to the length of the estimates, we do not try make the estimates as sharp as possible and rather aim for readability. In particular, some of the constants below do depend on η even though it is not always necessary.

Step 1: U^n is bounded in $L^q(\Omega; L^\infty(0,t;L^2))$.
Applying the finite-dimensional Itô formula to (3.3) and estimating the trace term, we have

$$\begin{aligned} &\mathrm{d}\|U^n\|_{L^2}^q + q\nu\|U^n\|_{L^2}^{q-2}\|(-\Delta_H)^{1/2}U^n\|_{L^2}^2\,\mathrm{dt} \\ &\le -q\|U^n\|_{L^2}^{q-2}\langle U^n, F^n(U^n)\rangle\,\mathrm{dt} + \frac{q(q-1)}{2}\|U^n\|_{L^2}^{q-2}\|\sigma^n(U^n)\|_{L_2(U,L^2)}^2\,\mathrm{dt} \\ &\quad + q\|U^n\|_{L^2}^{q-2}\langle U^n, \sigma^n(U^n)\,\mathrm{dW}\rangle \\ &= \sum_{i=1}^2 I_i^n\,\mathrm{dt} + I_3^n\,\mathrm{dW} \end{aligned} \tag{3.7}$$

where we already used the cancellation property $\langle B(U^n,U^n),U^n\rangle = 0$. The linear part is straightforward. For any $\varepsilon > 0$, we have

$$\begin{aligned} \int_0^t |I_1^n|\,\mathrm{ds} \le& q\varepsilon\int_0^t \|U^n\|_{L^2}^{q-2}\|(-\Delta_H)^{1/2}U^n\|_{L^2}^2\,\mathrm{ds} + c_\varepsilon\|U^n\|_{L^2}^q \\ &+ \frac{1}{4}\sup_{s\in[0,t]}\|U^n\|_{L^2}^q + \left(\int_0^t \|F_U^n\|_{L^2}^2\,\mathrm{ds}\right)^{q/2}. \end{aligned}$$

By the growth estimate (2.31) of σ in $L_2(\mathcal{U}, L^2)$, we obtain

$$\begin{aligned} I_2^n &\le \frac{q(q-1)}{2}\|U^n\|_{L^2}^{q-2}\|\sigma^n(U^n)\|_{L_2(\mathcal{U},L^2)}^2 \\ &\le c\left(1+\|U^n\|_{L^2}^q\right) + \frac{q(q-1)}{2}\eta^2\|U^n\|_{L^2}^{q-2}\|(-\Delta_H)^{1/2}U^n\|_{L^2}^2. \end{aligned}$$

The stochastic integral is estimated by the Burkholder-Davis-Gundy inequality (2.51) by

$$\begin{aligned} &\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s q\|U^n\|_{L^2}^{q-2}\langle U^n, \sigma^n(U^n)\,\mathrm{dW}\rangle\right| \\ &\le qc_{BDG}\mathbb{E}\left(\int_0^t \|U^n\|_{L^2}^{2q-2}\|\sigma^n(U^n)\|_{L_2(U,L^2)}^2\,\mathrm{ds}\right)^{1/2} \\ &\le qc_{BDG}\mathbb{E}\left(\int_0^t c\left(1+\|U^n\|_{L^2}^{2q}\right) + \eta^2\|U^n\|_{L^2}^{2q-2}\|(-\Delta_H)^{1/2}U^n\|_{L^2}^2\,\mathrm{ds}\right)^{1/2} \\ &\le \frac{1}{4}\mathbb{E}\sup_{s\in[0,t]}\|U^n\|_{L^2}^q + q^2c_{BDG}^2\eta^2\mathbb{E}\int_0^t \|U^n\|_{L^2}^{q-2}\|(-\Delta_H)^{1/2}U^n\|_{L^2}^2\,\mathrm{ds} \\ &\quad + c\mathbb{E}\int_0^t 1 + \|U^n\|_{L^2}^q\,\mathrm{ds}. \end{aligned}$$

Collecting the estimates above yields

$$\begin{aligned} \frac{1}{4}\mathbb{E}\sup_{s\in[0,t]}\|U^n\|_{L^2}^q &+ c(q,\nu,\varepsilon,\eta)\mathbb{E}\int_0^t \|U^n\|_{L^2}^{q-2}\|(-\Delta_H)^{1/2}U^n\|_{L^2}^2\,\mathrm{ds} \\ &\le c_\varepsilon\mathbb{E}\left[\|U^n(0)\|_{L^2}^q + 1 + \int_0^t \|U^n\|_{L^2}^q\,\mathrm{ds} + \left(\int_0^t \|F_U^n\|_{L^2}^2\,\mathrm{ds}\right)^{q/2}\right], \end{aligned}$$

where $c(q,\nu,\varepsilon,\eta) = q[\nu - \varepsilon - \eta^2(\frac{q-1}{2} + qc_{BDG}^2)]$. Choosing ε sufficiently small, we employ

Gronwall's lemma to get

$$\mathbb{E}\sup_{s\in[0,t]}\|U^n\|_{L^2}^q+\mathbb{E}\int_0^t\|U^n\|_{L^2}^{q-2}\|(-\Delta_H)^{1/2}U^n\|_{L^2}^2\,\mathrm{ds}$$
$$\leq c\mathbb{E}\left[\|U^n(0)\|_{L^2}^q+1+\left(\int_0^t\|F_U^n\|_{L^2}^2\,\mathrm{ds}\right)^{q/2}\right]. \quad (3.8)$$

Step 2: U^n is bounded in $L^q(\Omega;L^2(0,t;L_z^2H_{xy}^1))$.
Since $\nu>\eta^2/2$ by the assumptions, the gradient terms in the estimates of the correction term I_2^n can be absorbed to the left-hand side of (3.7). Then (3.7), the estimates from Step 1 with $q=2$ and the growth estimate (2.31) on σ in $L_2(\mathcal{U},L^2)$ imply

$$\left(\int_0^t\|(-\Delta_H)^{1/2}U^n\|_{L^2}^2\,\mathrm{ds}\right)^{q/2}\leq c\|U^n(0)\|_{L^2}^q+\sup_{s\in[0,t]}\|U^n\|_{L^2}^q+c$$
$$+c\left(\int_0^t\|F_U^n\|_{L^2}^2\,\mathrm{ds}\right)^{q/2}+c\left(\int_0^t\langle U^n,\sigma^n(U^n)\,\mathrm{dW}\rangle\right)^{q/2}.$$

By the Burkholder-Davis-Gundy inequality (2.51) we have

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s\langle U^n,\sigma^n(U^n)\,\mathrm{dW}\rangle\right|^{q/2}&\leq c_{BDG}\mathbb{E}\left(\int_0^t\|U^n\|_{L^2}^2\|\sigma^n(U^n)\|_{L_2(\mathcal{U},L^2)}^2\,\mathrm{ds}\right)^{q/4}\\&\leq c\mathbb{E}\left(\int_0^t c\left(1+\|U^n\|_{L^2}^4\right)+\eta^2\|U^n\|_{L^2}^2\|(-\Delta_H)^{1/2}U^n\|_{L^2}^2\,\mathrm{ds}\right)^{q/4}\\&\leq\varepsilon\mathbb{E}\left(\int_0^t\|(-\Delta_H)^{1/2}U^n\|_{L^2}^2\,\mathrm{ds}\right)^{q/2}+c_\varepsilon\mathbb{E}\left[1+\sup_{s\in[0,t]}\|U^n\|^q\right].\end{aligned}$$

Choosing ε sufficiently small, we use the bound (3.8) from Step 1 to get

$$\begin{aligned}&\mathbb{E}\left(\int_0^t\|(-\Delta_H)^{1/2}U^n\|_{L^2}^2\,\mathrm{ds}\right)^{q/2}\\&\quad\leq c\mathbb{E}\left[\|U^n(0)\|_{L^2}^q+\sup_{s\in[0,t]}\|U^n\|_{L^2}^q+1+\left(\int_0^t\|F_U^n\|_{L^2}^2\,\mathrm{ds}\right)^{q/2}\right]\\&\quad\leq c\mathbb{E}\left[\|U^n(0)\|_{L^2}^q+1+\left(\int_0^t\|F_U^n\|_{L^2}^2\,\mathrm{ds}\right)^{q/2}\right].\end{aligned} \quad (3.9)$$

Step 3: U^n is bounded in $L^q(\Omega;L^\infty(0,t;H_z^1L_x^2))$.
To obtain the estimates for vertical derivative, we use the properties of the projector P, in particular (3.5). Employing the Itô formula and estimates for the trace term, we obtain

$$\begin{aligned}&\mathrm{d}\|\partial_zU^n\|_{L^2}^q+q\nu\|\partial_zU^n\|_{L^2}^{q-2}\|(-\Delta_H)^{1/2}\partial_zU^n\|_{L^2}^2\,\mathrm{dt}\\&\quad\leq -q\|\partial_zU^n\|_{L^2}^{q-2}\langle\partial_zU^n,\partial_zF^n(U^n)+\theta(U^n)P_n\partial_zB(U^n,U^n)\rangle\,\mathrm{dt}\\&\qquad+\frac{q(q-1)}{2}\|\partial_zU^n\|_{L^2}^{q-2}\|\partial_z\sigma^n(U^n)\|_{L_2(U,L^2)}^2\,\mathrm{dt}\\&\qquad+q\|\partial_zU^n\|_{L^2}^{q-2}\langle\partial_zU^n,\partial_z\sigma^n(U^n)\,\mathrm{dW}\rangle\\&\quad=\sum_{i=1}^3I_i^n\,\mathrm{dt}+I_4^n\,\mathrm{dW}.\end{aligned}$$

Similarly as in Step 1, we estimate the lower order linear term as

$$\int_0^t |I_1^n| \,\mathrm{ds} \le \frac{1}{4} \sup_{s\in[0,t]} \|\partial_z U^n\|_{L^2}^q + c\int_0^t \|\partial_z U^n\|_{L^2}^q \,\mathrm{ds} + c\left(\int_0^t \|\nabla_H U^n\|_{L^2}^2 \,\mathrm{ds}\right)^{q/2} + c\left(\int_0^t \|\partial_z F_U\|_{L^2}^2 \,\mathrm{ds}\right)^{q/2},$$

the correction terms as

$$I_3^n \le c\left(1+\|\partial_z U^n\|_{L^2}^q\right) + \frac{q(q-1)}{2}\eta^2 \|U^n\|_{H_z^1 L_{xy}^2}^{q-2} \|(-\Delta_H)^{1/2}\partial_z U^n\|_{L^2}^2,$$

and the stochastic term by the Burkholder-Davis-Gundy inequality (2.51)

$$\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s I_4^n \,\mathrm{dW}\right| \le \frac{1}{4}\mathbb{E}\sup_{s\in[0,t]} \|\partial_z U^n\|_{L^2}^q + c\mathbb{E}\int_0^t 1 + \|U^n\|_{H_z^1 L^2 xy}^q \,\mathrm{ds} + q^2 c_{BDG}^2 \eta^2 \mathbb{E}\int_0^t \|\partial_z U^n\|_{L^2}^{q-2} \|(-\Delta_H)^{1/2}\partial_z U^n\|_{L^2}^2 \,\mathrm{ds}.$$

The nonlinear term is dealt with by the estimate (2.23). Recalling the form of the cut-off θ (3.1), for $\varepsilon > 0$, we have

$$\begin{aligned}
|I_2^n| &\le cq\theta(U^n)\,\|U\|_{L_z^\infty L_{xy}^4} \|\nabla_H \partial_z U\|_{L^2}^{3/2} \|\partial_z U\|_{L^2}^{q-3/2} \\
&\le cq\rho \|\nabla_H \partial_z U\|_{L^2}^{3/2} \|\partial_z U\|_{L^2}^{q-3/2} \\
&\le q\varepsilon \|\nabla_H \partial_z U^n\|_{L^2}^2 \|\partial_z U^n\|_{L^2}^{q-2} + c_\varepsilon q\rho^4 \|\partial_z U^n\|_{L^2}^q.
\end{aligned}$$

Collecting the above estimates, we obtain

$$\begin{aligned}
\mathbb{E}&\left[\frac{1}{2}\sup_{s\in[0,t]} \|\partial_z U^n\|_{L^2}^q + c(q,\nu,\varepsilon,\eta)\int_0^t \|\partial_z U^n\|_{L^2}^{q-2} \|(-\Delta_H)^{1/2}\partial_z U^n\|_{L^2}^2 \,\mathrm{ds}\right] \\
&\le c\mathbb{E}\left[\|\partial_z U^n(0)\|_{L^2}^q + 1 + c\left(\int_0^t \|\partial_z F_U\|_{L^2}^2 \,\mathrm{ds}\right)^{q/2} + \left(\int_0^t \|\nabla_H U^n\|_{L^2}^2 \,\mathrm{ds}\right)^{q/2}\right] \\
&\qquad + c\mathbb{E}\int_0^t \|U^n\|_{L^2}^q + \|\partial_z U^n\|_{L^2}^q \,\mathrm{ds},
\end{aligned}$$

where again $c(q,\nu,\varepsilon,\eta) = q[\nu - \varepsilon - \eta^2(qc_{BDG}^2 + \frac{q-1}{2})]$. With ε sufficiently small, Gronwall's lemma and the bound (3.9) yield

$$\mathbb{E}\left[\sup_{s\in[0,t]} \|\partial_z U^n\|_{L^2}^q + \int_0^t \|\partial_z U^n\|_{L^2}^{q-2}(\|(-\Delta_H)^{1/2}\partial_z U^n\|_{L^2}^2) \,\mathrm{ds}\right] \le c\mathbb{E}\left[\|U^n(0)\|_{H_z^1 L_{xy}^2}^q + 1 + \left(\int_0^t \|F_U\|_{H_z^1 L_{xy}^2}^2 \,\mathrm{ds}\right)^{q/2}\right]. \quad (3.10)$$

Step 4: U^n is bounded in $L^q(\Omega; L^2(0,t; H_z^1 H_{xy}^1))$.
The desired estimate

$$\mathbb{E}\left(\int_0^t \|\nabla_H \partial_z U^n\|_{L^2}^2 \,\mathrm{ds}\right)^{q/2} \le c\mathbb{E}\left[\|U^n(0)\|_{H_z^1 L_{xy}^2}^q + 1 + \left(\int_0^t \|F_U\|_{H_z^1 L_{xy}^2}^2 \,\mathrm{ds}\right)^{q/2}\right] \quad (3.11)$$

can be obtained from (3.9) and (3.10) with $q = 2$ similarly as in Step 2.

Step 5: U^n is bounded in $L^q(\Omega; L^\infty(0,t; L^2_z H^1_{xy}))$.

Since our basis consists of eigenvectors of the linear operator A_H, we may use the Itô formula on (3.3) to obtain

$$\begin{aligned}
&\mathrm{d}\|(-\Delta_H)^{1/2} U^n\|_{L^2}^q + q\nu \|(-\Delta_H)^{1/2} U^n\|_{L^2}^{q-2} \|\Delta_H U^n\|_{L^2}^2 \,\mathrm{dt} \\
&\quad \le -q \|(-\Delta_H)^{1/2} U^n\|_{L^2}^{q-2} \left\langle (-\Delta_H)^{1/2} U^n, (-\Delta_H)^{1/2} F^n(U^n) \right\rangle \mathrm{dt} \\
&\qquad - q \|(-\Delta_H)^{1/2} U^n\|_{L^2}^{q-2} \left\langle (-\Delta_H)^{1/2} U^n, \theta(U^n) P_n (-\Delta_H)^{1/2} B(U^n, U^n) \right\rangle \mathrm{dt} \\
&\qquad + \frac{q(q-1)}{2} \|(-\Delta_H)^{1/2} U^n\|_{L^2}^{q-2} \|(-\Delta_H)^{1/2} \sigma^n(U^n)\|_{L_2(\mathcal{U}, L^2)}^2 \,\mathrm{dt} \\
&\qquad + q \|(-\Delta_H)^{1/2} U^n\|_{L^2}^{q-2} \left\langle (-\Delta_H)^{1/2} U^n, (-\Delta_H)^{1/2} \sigma^n(U^n) \,\mathrm{dW} \right\rangle \\
&\quad = \sum_{i=1}^3 I_i^n \,\mathrm{dt} + I_4^n \,\mathrm{dW}\,.
\end{aligned}$$

Similarly as above, for $\varepsilon > 0$, we estimate the lower order term and the correction term by

$$\begin{aligned}
\int_0^t I_1^n \,\mathrm{ds} &\le \frac{\varepsilon}{4} \int_0^t \|\Delta_H U^n\|_{L^2}^2 \|\nabla_H U^n\|_{L^2}^{q-2} \,\mathrm{ds} + c_\varepsilon \int_0^t \|\nabla_H U^n\|_{L^2}^q \,\mathrm{ds} \\
&\quad + \frac{1}{4} \sup_{s\in[0,t]} \|\nabla_H U^n\|_{L^2}^q + c \left(\int_0^t \|F_U\|_{L^2}^2 \,\mathrm{ds} \right)^{q/2}, \\
I_3^n &\le \frac{q(q-1)}{2} \eta^2 \|\Delta_H U^n\|_{L^2}^2 \|\nabla_H U^n\|_{L^2}^{q-2} + c \left(1 + \|U^n\|_{L^2_z H^1_{xy}}^q \right).
\end{aligned}$$

From the Burkholder-Davis-Gundy inequality (2.51), we deduce

$$\begin{aligned}
\mathbb{E} \sup_{s\in[0,t]} \left| \int_0^s I_4^n \,\mathrm{dW} \right| &\le \frac{1}{4} \mathbb{E} \sup_{s\in[0,t]} \|\nabla_H U^n\|_{L^2}^q + c\mathbb{E} \int_0^t 1 + \|U^n\|_{L^2_z H^1_{xy}}^q \,\mathrm{ds} \\
&\quad + q^2 c_{BDG}^2 \eta^2 \mathbb{E} \int_0^t \|\nabla_H U^n\|_{L^2}^{q-2} \|\Delta_H U^n\|_{L^2}^2 \,\mathrm{ds}\,.
\end{aligned}$$

To deal with the nonlinear term I_2^n, we use (2.25). We estimate the first term on the right-hand side of (2.25) multiplied by $\|\nabla_H U^n\|_{L^2}^{q-2}$ using the form of the cut-off θ (3.1) by

$$\begin{aligned}
&\theta(U^n) \|U\|_{L^\infty_z L^4_{xy}} \|U\|_{L^2_z H^1_{xy}}^{1/2} \|U\|_{L^2_z H^2_{xy}}^{3/2} \|\nabla_H U^n\|_{L^2}^{q-2} \\
&\qquad \le \frac{\varepsilon}{4} \|\Delta_H U^n\|_{L^2}^2 \|\nabla_H U^n\|_{L^2}^{q-2} + c_\varepsilon \rho^4 \left(1 + \|U^n\|_{L^2}^q + \|\nabla_H U^n\|_{L^2}^q \right).
\end{aligned}$$

For the second term on the right-hand side of (2.25), we have

$$\begin{aligned}
&\theta(U^n) \|U\|_{L^\infty_z L^4_{xy}} \|U\|_{L^2_z H^1_{xy}}^{1/2} \|U\|_{L^2_z H^2_{xy}}^{1/2} \|U\|_{H^1_z H^1_{xy}} \|\nabla_H U^n\|_{L^2}^{q-2} \\
&\quad \le q \frac{\varepsilon}{4} \|\Delta_H U^n\|_{L^2}^2 \|\nabla_H U^n\|_{L^2}^{q-2} \\
&\qquad + c_\varepsilon \rho^{4/3} \left(1 + \|U^n\|_{H^1_z L^2_{xy}}^q + \|\nabla_H U^n\|_{L^2}^q + \|\nabla_H \partial_z U^n\|_{L^2}^2 \|\nabla_H U^n\|_{L^2}^{q-2} \right).
\end{aligned}$$

Collecting the above and using the estimate

$$\begin{aligned}
&\int_0^t \|\nabla_H \partial_z U^n\|_{L^2}^2 \|\nabla_H U^n\|_{L^2}^{q-2} \,\mathrm{ds} \\
&\qquad \le \frac{1}{4} \sup_{s\in[0,t]} \|\nabla_H U^n\|_{L^2}^q + c \left(\int_0^t \|\nabla_H \partial_z U^n\|_{L^2}^2 \,\mathrm{ds} \right)^{q/2},
\end{aligned}$$

we deduce

$$\mathbb{E}\left[\frac{1}{4}\sup_{s\in[0,t]}\|\nabla_H U^n\|_{L^2}^q + c(p,\nu,\varepsilon,\eta)\int_0^t \|\nabla_H U^n\|_{L^2}^{q-2}\|\Delta_H U^n\|_{L^2}^2\,\mathrm{ds}\right]$$
$$\leq c\mathbb{E}\left[\|\nabla_H U^n(0)\|_{L^2}^q + 1 + \left(\int_0^t \|F_U\|_{L^2}^2\,\mathrm{ds}\right)^{q/2} + \left(\int_0^t \|\nabla_H\,\partial_z U^n\|_{L^2}^2\,\mathrm{ds}\right)^{q/2}\right]$$
$$+ c\mathbb{E}\int_0^t \|U^n\|_{H_z^1 L_{xy}^2}^q + \|U^n\|_{L_z^2 H_{xy}^1}^q\,\mathrm{ds},$$

where $c(q,\nu,\varepsilon,\eta) = q[\nu-\varepsilon-\eta^2(\frac{q-1}{2}+qc_{BDG}^2)]$. Choosing ε sufficiently small, we may employ Gronwall's lemma and (3.10) to get

$$\mathbb{E}\left[\sup_{s\in[0,t]}\|\nabla_H U^n\|_{L^2}^q + \int_0^t \|\nabla_H U^n\|_{L^2}^{q-2}\|\Delta_H U^n\|_{L^2}^2\,\mathrm{ds}\right]$$
$$\leq c\mathbb{E}\left[\|U^n(0)\|_{H^1}^q + 1 + \left(\int_0^t \|F_U\|_{H_z^1 L_{xy}^2}^2\,\mathrm{ds}\right)^{q/2}\right]. \quad (3.12)$$

Step 6: U^n is bounded in $L^q(\Omega; L^2(0,t; L_z^2 H_{xy}^2))$.
The estimate

$$\mathbb{E}\left(\int_0^t \|\Delta_H U^n\|_{L^2}^2\,\mathrm{ds}\right)^{q/2}$$
$$\leq c\mathbb{E}\left[\|U^n(0)\|_{H^1}^q + 1 + \left(\int_0^t \|F_U\|_{H_z^1 L_{xy}^2}^2\,\mathrm{ds}\right)^{q/2}\right] \quad (3.13)$$

follows from (3.10) and (3.12) with $q=2$ similarly as in Step 2.

Step 7: $\int_0^\cdot \sigma^n(U^n)\,\mathrm{dW}$ is bounded in $L^q(\Omega; W^{\alpha,q}(0,t;L^2))$ for $\alpha\in[0,1/2)$.
Using the fractional version of the Burkholder-Davis-Gundy inequality (2.52) and (3.12), we get

$$\mathbb{E}\left\|\int_0^\cdot \sigma^n(U^n)\,\mathrm{dW}\right\|_{W^{\alpha,q}(0,t;L^2)}^q \leq c\mathbb{E}\int_0^t \|\sigma^n(U^n)\|_{L_2(\mathcal{U},L^2)}^q\,\mathrm{ds}$$
$$\leq c\mathbb{E}\left[\|U^n(0)\|_{H^1}^q + 1 + \left(\int_0^t \|F_U\|_{H_z^1 L_{xy}^2}^2\,\mathrm{ds}\right)^{q/2}\right]. \quad (3.14)$$

Step 8: $U^n(\cdot) - \int_0^\cdot \sigma^n(U^n)\,\mathrm{dW}$ is bounded in $L^p(\Omega; W^{1,2}(0,t;L^2))$.
The boundedness follows from the definition of the $W^{1,2}$-norm, the estimate (2.24) and the bounds from previous steps thanks to the assumption $q\geq 4$. □

Remark 3.2. By interpolation inequality (2.7), we immediately observe that U^n is bounded in the space $L^q(\Omega; L^\infty(0,t; L_z^\infty L_{xy}^4))$.

3.3 Convergence of finite-dimensional approximations

The convergence of Galerkin approximations is established in a similar manner as in [11]. We will thus only briefly summarize the argument and provide details only in the parts where the absence of vertical dissipation plays an important role.

We first establish the existence of martingale solutions.

Proposition 3.3. *Let* $U_0 \in L^q(\Omega; H^1)$ *with* $v_0 \in L^q(\Omega; L_z^2 H_{D,xy}^1 \cap L_{\overline{\sigma}}^2)$ *for some* $q\geq 4$. *Then there exists a global martingale solution of the modified equation* (3.2).

Proof. Let $\mathcal{U}_0$ be a Hilbert space such that the embedding $\mathcal{U} \hookrightarrow \mathcal{U}_0$ is Hilbert-Schmidt. Let

$$\mathcal{X}_U = L^2(0,t;L^2_zH^1_{xy} \cap L^\infty_z L^4_{xy}) \cap C([0,t],H^{-1}(M)),$$
$$\mathcal{X}_W = C([0,t],\mathcal{U}_0), \qquad \mathcal{X} = \mathcal{X}_U \times \mathcal{X}_W.$$

Let μ^n_U, μ^n_W and μ^n be the probability measures on $\mathcal{X}_U$, $\mathcal{X}_W$ and $\mathcal{X}$, respectively, defined by

$$\mu^n_U(\cdot) = \mathbb{P}(U^n \in \cdot), \qquad \mu^n_W(\cdot) = \mathbb{P}(W \in \cdot), \qquad \mu^n = \mu^n_U \times \mu^n_W.$$

Recall that the embeddings

$$L^2(0,t;L^2_zH^2_{xy} \cap H^1_zH^1_{xy}) \cap W^{1/4,2}(0,t;L^2) \hookrightarrow\hookrightarrow L^2(0,t;L^2_zH^1_{xy} \cap L^\infty_z L^4_{xy}),$$

and

$$W^{\alpha,p}(0,t;L^2) \hookrightarrow\hookrightarrow C([0,t];H^{-1})$$

are compact by Lemma 2.1 a) and b) if $\alpha p > 1$, respectively. We may follow the argument of [11, Lemma 4.1] and use the bounds from Lemma 3.1 the Prokhorov theorem to establish that the sequence of measures μ^n on $\mathcal{X}$ is tight and therefore weakly compact.

By the Skorokhod theorem, there exists a probability space $(\tilde{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$, an increasing sequence $(n_k)_k$ and $\mathcal{X}$-valued random variables $(\hat{U}^{n_k}, \hat{W}^{n_k})$ and $(\hat{U}, \hat{W})$ such that $(\hat{U}^{n_k}, \hat{W}^{n_k}) \to (\hat{U}, \hat{W})$ $\hat{\mathbb{P}}$-a.s. in the space $\mathcal{X}$. The processes $\hat{W}^{n_k}$ are cylindrical Wiener processes with respect to the filtration $\hat{\mathbb{F}}^{n_k} = \{\hat{\mathcal{F}}^{n_k}_t\}_{t\geq 0}$, where $\hat{\mathcal{F}}^{n_k}_t$ is the completion of $\sigma(\{\hat{U}^{n_k}(s), \hat{W}^{n_k}(s) \mid 0 \leq s \leq t\})$. Moreover, by the Bensoussan argument from [3, Section 4.3.4], the couple $(\hat{U}^{n_k}, \hat{W}^{n_k})$ solves the equation

$$\mathrm{d}\hat{U}^{n_k} + [A_H\hat{U}^{n_k} + \theta(\hat{U}^{n_k})B^{n_k}(\hat{U}^{n_k}, \hat{U}^{n_k}) + F^{n_k}(\hat{U}^{n_k})]\,\mathrm{dt} = \sigma^{n_k}(\hat{U}^{n_k})\,\mathrm{d}\hat{\mathrm{W}}^{\mathrm{n_k}}$$

with the initial condition

$$\hat{U}^{n_k}(0) = U^{n_k}_0.$$

Similarly as in [11, Section 7.1], we can establish

$$\begin{aligned} \hat{U}^{n_k} &\in L^2\left(\tilde{\Omega}; L^2(0,t;L^2_zH^2_{xy} \cap H^1_zH^1_{xy}) \cap L^\infty(0,t;H^1)\right), \\ \hat{U}^{n_k} &\rightharpoonup \hat{U} \text{ in } L^2\left(\tilde{\Omega}; L^2(0,t;L^2_zH^2_{xy} \cap H^1_zH^1_{xy})\right), \end{aligned} \tag{3.15}$$

and by the Vitali convergence theorem and the convergence $\hat{\mathbb{P}}$-a.s. in $\mathcal{X}$

$$\hat{U}^{n_k} \to \hat{U} \text{ in } L^2\left(\tilde{\Omega}; L^2(0,t;L^2_zH^1_{xy} \cap L^\infty_z L^4_{xy})\right). \tag{3.16}$$

In particular, by further thinning the sequence we may assume that

$$\|\hat{U}^{n_k} - \hat{U}\|_{L^2_zH^1_{xy}}, \|\hat{U}^{n_k} - \hat{U}\|_{L^\infty_z L^4_{xy}} \to 0 \quad \text{a.s. in } [0,t] \times \tilde{\Omega}. \tag{3.17}$$

The limiting process of [11, Section 7.2] can be divided into four steps: First, one needs to establish convergence of the deterministic terms in the equation a.s. in $[0,t] \times \tilde{\Omega}$. This is followed by showing that the deterministic terms converge in $L^p([0,t] \times \tilde{\Omega})$ is shown for $1 \leq p < 2$. Next, convergence of the stochastic term $\int_0^\cdot \sigma^n(\hat{U}^{n_k})\,\mathrm{d}\hat{\mathrm{W}}^{\mathrm{n_k}}$ in $L^2(\tilde{\Omega}; L^2(0,t;L_2(\mathcal{U}, L^2)))$ is established. Finally, one combines the previous steps and shows that one can take the limit of the variational formulation of the equation provided the set of test functions is sufficiently smooth and dense in $L^2_\sigma \times L^2$.

The above argument from [11] carries over to this case almost completely with the following two exceptions – the a.s. convergence of the deterministic nonlinear term and convergence of gradient-dependent stochastic term. The latter has been established in [5, Sectiom 3.4] and therefore it remains to show only the a.s. convergence.

Let $s \in [0,t]$ be fixed and let $U^\sharp \in D(A_H) \cap H^1_z L^4_{xy}$. Then

$$\begin{aligned}
&\left| \int_0^s \left\langle \theta(\hat{U}^{n_k}) B^{n_k}(\hat{U}^{n_k}, \hat{U}^{n_k}) - \theta(\hat{U}) B(\hat{U},\hat{U}), U^\sharp \right\rangle \mathrm{dr} \right| \\
&\quad \le \left| \int_0^s \theta(\hat{U}^{n_k}) \left\langle B(\hat{U}^{n_k}, \hat{U}^{n_k}) - B(\hat{U},\hat{U}), P_n U^\sharp \right\rangle \mathrm{dr} \right| \\
&\quad + \left| \int_0^s \theta(\hat{U}^{n_k}) \left\langle B(\hat{U},\hat{U}), Q_n U^\sharp \right\rangle \mathrm{dr} \right| + \left| \int_0^s \left(\theta(\hat{U}^{n_k}) - \theta(\hat{U}) \right) \left\langle B(\hat{U},\hat{U}), U^\sharp \right\rangle \mathrm{dr} \right| \\
&\quad = I_1^n + I_2^n + I_3^n .
\end{aligned}$$

Since B is bilinear, we have

$$I_1^n \le \int_0^s \left| \left\langle B(\hat{U}^{n_k} - \hat{U}, \hat{U}), P_n U^\sharp \right\rangle \right| + \left| \left\langle B(\hat{U}, \hat{U}^{n_k} - \hat{U}), P_n U^\sharp \right\rangle \right| \mathrm{dr} .$$

Hence, by (2.22), we deduce

$$\begin{aligned}
I_1^n &\le c \int_0^s \left(\|\hat{U}^{n_k} - \hat{U}\|_{L^\infty_z L^4_{xy}} + \|\nabla_H (\hat{U}^{n_k} - \hat{U})\|_{L^2} \right) \|\hat{U}^{n_k}\|_{H^1} \|U^\sharp\|_{H^1_z L^4_{xy}} \mathrm{dr} \\
&\le c \left(\|\hat{U}^{n_k} - \hat{U}\|_{L^2(0,s;L^\infty_z L^4_{xy})} + \|\hat{U}^{n_k} - \hat{U}\|_{L^2(0,s;L^2_z H^1_{xy})} \right) \\
&\quad \cdot \|\hat{U}^{n_k}\|_{L^2(0,s;H^1)} \|U^\sharp\|_{H^1_z L^4_{xy}} \to 0
\end{aligned}$$

We treat the term I_2^n using the Poincaré inequality (2.20) and the bound (2.21) as

$$\begin{aligned}
I_2^n &\le c \|Q_n U^\sharp\|_{L^2_z H^1_{xy}} \int_0^s \|\hat{U}\|_{H^1} \|\hat{U}\|_{H^1_z H^1_{xy}} \mathrm{ds} \\
&\le c \|Q_n U^\sharp\|_{D(A_H^{1/2})} \le \frac{c}{\overline{\lambda}_n^{1/2}} \|U^\sharp\|_{D(A_H)} \to 0,
\end{aligned}$$

where $\overline{\lambda}_n$ and Q_n are as in Proposition 2.4. Regarding I_3^n, (2.22) and (2.6) yield

$$\mathbb{E} \int_0^t I_3^n(s) \,\mathrm{ds} \le c \|U^\sharp\|_{H^1_z L^4_{xy}} \mathbb{E} \int_0^t \|\hat{U}\|^2_{H^1} \,\mathrm{ds} < \infty .$$

By the Dominated convergence theorem with the convergence (3.17) and Lipschitz continuity of θ, the estimate above implies $I_3^n \to 0$ a.s. in $[0,t] \times \tilde{\Omega}$. ☐

3.4 Uniqueness

To establish strong uniqueness of martingale solutions, we need additional regularity in the vertical direction which can be obtained provided the initial data is additionally in $H^2_{N,z} L^2_{xy} \times H^2_{D,z} L^2_{xy}$. By the particular structure of the primitive equations, the solution also takes values in $H^2_{N,z} L^2_{xy} \times H^2_{D,z} L^2_{xy}$.

The requirement of higher regularity introduces a minor technical drawback as we need a stronger cut-off to show the desired estimate. Let $\theta \in C^\infty(\mathbb{R})$ be as in Section 3. For $\mu > 0$ fixed, we define

$$\tilde{\theta}(U(s)) = \theta_\mu \left(\|U(s)\|_{H^1_z L^4_{xy}} \right) . \tag{3.18}$$

We are looking for a solution $U = (v, T)$ of the modified system

$$\mathrm{d}U + [A_H U + \tilde{\theta}(U)B(U,U) + F(U)]\,\mathrm{dt} = \sigma(U)\,\mathrm{dW}, \quad U(0) = U_0. \tag{3.19}$$

Approximated solutions U^n for this system are constructed as in Section 3.1. Moreover, the estimates from Lemma 3.1 also hold for these approximations since the cut-off is stronger than the one considered in Lemma 3.1.

Lemma 3.4. *Let the assumptions of Lemma 3.1 hold. Additionally, let $\partial_{zz}U_0 \in L^q(\Omega; L^2)$, $\partial_{zz}F_U \in L^q\left(\Omega; L^q(0,t;L^2)\right)$ and let σ satisfy* (2.31)-(2.38) *and consider the sequence of approximating solutions $(U^n)_n$ of* (3.19)*. Then $\partial_{zz}U^n$ is bounded in*

$$L^q\left(\Omega; C([0,t], L^2)\right) \cap L^q\left(\Omega; L^2(0,t; L^2_z H^1_{xy})\right). \tag{3.20}$$

Remark 3.5. Assuming the solution U has the regularity from Definition 2.9 and (3.20), in particular it is continuous in $H^2_z L^2_{xy}$ and $L^2_z H^1_{xy}$, one may use a variant of the mixed derivative theorem [47, Proposition 3.2] to establish $U \in C([0,t], H^1_z H^{1/2}_{xy}) \subset C([0,t], H^1_z L^4_{xy})$. The argument of the cut-off function $\tilde{\theta}$ is therefore bounded on $[0,t]$.

Proof. We only show $\partial_{zz}U^n \in L^q\left(\Omega; L^\infty(0,t;L^2)\right)$, the other conclusion follows as in Lemma 3.1 above.

Similarly as in Step 3 of Lemma 3.1, we apply Itô's formula and use (3.6) to get

$$\begin{aligned}
&\mathrm{d}\|\partial_{zz}U^n\|^q_{L^2} + q\nu\|\partial_{zz}U^n\|^{q-2}_{L^2}\|(-\Delta_H)^{1/2}\partial_{zz}U^n\|^2_{L^2}\,\mathrm{dt} \\
&\le -q\|\partial_{zz}U^n\|^{q-2}_{L^2}\left\langle \partial_{zz}U^n, \partial_{zz}F^n(U^n) + \tilde{\theta}(U^n)P_n\partial_{zz}B(U^n,U^n)\right\rangle \mathrm{dt} \\
&\quad + \frac{q(q-1)}{2}\|\partial_{zz}U^n\|^{q-2}_{L^2}\|\partial_{zz}\sigma^n(U^n)\|^2_{L_2(U,L^2)}\,\mathrm{dt} \\
&\quad + q\|\partial_{zz}U^n\|^{q-2}_{L^2}\langle \partial_{zz}U^n, \partial_{zz}\sigma^n(U^n)\,\mathrm{dW}\rangle \\
&= \sum_{i=1}^{3} I^n_i\,\mathrm{dt} + I^n_4\,\mathrm{dW}.
\end{aligned}$$

We estimate the lower order linear term as

$$\begin{aligned}
\int_0^t |I^n_1|\,\mathrm{ds} \le \frac{1}{4}\sup_{s\in[0,t]}\|\partial_{zz}U^n\|^q_{L^2} + c\int_0^t \|\partial_{zz}U^n\|^q_{L^2}\,\mathrm{ds} \\
+ c\left(\int_0^t \|\partial_z \nabla_H U^n\|^2_{L^2}\,\mathrm{ds}\right)^{q/2} + c\left(\int_0^t \|\partial_{zz}F^n_U\|^2_{L^2}\,\mathrm{ds}\right)^{q/2}
\end{aligned}$$

the correction terms as

$$I^n_3 \le c\left(1 + \|U^n\|^q_{H^2_z L^2_{xy}}\right) + \frac{q(q-1)}{2}\eta^2\|\partial_{zz}U^n\|^{q-2}_{L^2}\|(-\Delta_H)^{1/2}\partial_{zz}U^n\|^2_{L^2},$$

and the stochastic term by the Burkholder-Davis-Gundy inequality (2.51)

$$\begin{aligned}
\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s I^n_4\,\mathrm{dW}\right| \le \frac{1}{4}\mathbb{E}\sup_{s\in[0,t]}\|\partial_{zz}U^n\|^q_{L^2} + c\mathbb{E}\int_0^t 1 + \|U^n\|^q_{H^2_z L^2_{xy}}\,\mathrm{ds} \\
+ q^2 c^2_{BDG}\eta^2\mathbb{E}\int_0^t \|\partial_{zz}U^n\|^{q-2}_{L^2}\|(-\Delta_H)^{1/2}\partial_{zz}U^n\|^2_{L^2}\,\mathrm{ds}.
\end{aligned}$$

The nonlinear term is dealt with by the estimate (2.26). For $\varepsilon > 0$, we have

$$\begin{aligned}
|I^n_2| &\le c\tilde{\theta}(U^n)\|\partial_{zz}U^n\|^{q-3/2}_{L^2}\|U\|_{H^1_z L^4_{xy}}\|U^n\|^{3/2}_{H^2_z H^1_{xy}} \\
&\le \varepsilon\|\partial_{zz}U^n\|^{q-2}_{L^2}\left\|(-\Delta_H)^{1/2}\partial_{zz}U^n\right\|^2_{L^2} + c_\varepsilon \mu^4\|\partial_{zz}U^n\|^q_{L^2}.
\end{aligned}$$

The estimates above lead to

$$\mathbb{E}\left[\frac{1}{2}\sup_{s\in[0,t]}\|\partial_{zz}U^n\|_{L^2}^q + c(q,\nu,\varepsilon,\eta)\int_0^t \|\partial_{zz}U^n\|_{L^2}^{q-2}\|(-\Delta_H)^{1/2}\partial_{zz}U^n\|_{L^2}^2\,\mathrm{ds}\right]$$
$$\leq c\mathbb{E}\left[\|\partial_{zz}U^n(0)\|_{L^2}^q + 1 + \left(\int_0^t \|\partial_{zz}F_U^n\|_{L^2}^2\,\mathrm{ds}\right)^{q/2} + \left(\int_0^r \|U^n\|_{H_z^1H_{xy}^1}^2\,\mathrm{ds}\right)^{q/2}\right]$$
$$+ c_\varepsilon\mathbb{E}\int_0^t \|U^n\|_{H_z^2L_{xy}^2}^q + \mu^4\|\partial_{zz}U^n\|_{L^2}^q\,\mathrm{ds}$$

where $c(q,\nu,\varepsilon,\eta) = q[\nu-\varepsilon-\eta^2(qc_{BDG}^2+\frac{q-1}{2})]$. For ε small enough, from Gronwall's inequality and the bound (3.11), it follows

$$\mathbb{E}\left[\sup_{s\in[0,t]}\|\partial_{zz}U^n\|_{L^2}^q + \int_0^t \|\partial_{zz}U^n\|_{L^2}^{q-2}(\|(-\Delta_H)^{1/2}\partial_{zz}U^n\|_{L^2}^2)\,\mathrm{ds}\right]$$
$$\leq c\mathbb{E}\left[\|\partial_{zz}U^n(0)\|_{L^2}^q + \|U^n(0)\|_{H^1}^q + 1 + \left(\int_0^t \|F_U^n\|_{H_z^2L_{xy}^2}^2\,\mathrm{ds}\right)^{q/2}\right]. \quad (3.21)$$

□

Remark 3.6. The existence of martingale solutions of the modified system (3.19) follows similarly as in Proposition 3.3 using the compact embedding $H_z^2H_{xy}^1 \hookrightarrow\hookrightarrow H_z^1L_{xy}^4$.

We may now state the result on strong uniqueness of martingale solutions of the modified problem (3.19).

Proposition 3.7. *Let $0<\gamma<\sqrt{4\nu/(4c_{BDG}^2+1)}$. Let $\partial_{zz}F_U \in L^q\left(\Omega;L^q(0,t;L^2)\right)$. Let $(\mathcal{S},W,U_1)$ and $(\mathcal{S},W,U_2)$ be two global martingale solutions of the modified problem* (3.19) *over the same stochastic basis $\mathcal{S}$ with the same cylindrical Wiener process W. Let $\Omega_0=\{U_1(0)=U_2(0)\}$. Let also $\partial_{zz}U_1(0),\partial_{zz}U_2(0)\in L^2(\Omega;L^2)$. Then U_1 and U_2 are indistinguishable on Ω_0 in the sense*

$$\mathbb{P}\left(\{\mathbb{1}_{\Omega_0}(U_1(t)-U_2(t))=0 \textit{ for all } t\geq 0\}\right)=1.$$

Proof. The following proposition is established similarly as [11, Proposition 5.1] using stopping times

$$\tau_n = \{t\geq 0 \mid \int_0^t \|U_1\|_{H_z^2L_{xy}^2}^{2/3}\|U_1\|_{L_z^2H_{xy}^1}^{2/3}\|U_1\|_{H_z^2H_{xy}^1}^{2/3}\|U_1\|_{L_z^2H_{xy}^2}^{2/3}$$
$$+\|U_1\|_{L^2}^{2/3}\|U_1\|_{L_z^2H_{xy}^2}^{2/3}\|U_1\|_{H_z^2H_{xy}^1}^{4/3} + \|U_1\|_{H_z^1L_{xy}^2}^2\|U_1\|_{H_z^1H_{xy}^1}^2$$
$$+\|U_2\|_{L_z^2H_{xy}^2}^{4/3} + \|U_2\|_{H_z^2H_{xy}^1}^2 + \|U_2\|_{H_z^2L_{xy}^2}^2\|U_2\|_{H_z^2H_{xy}^1}^2\,\mathrm{ds}\geq n\},$$

and the stochastic Gronwall lemma, see Lemma A.2. From the estimates (2.23), (2.24) and (3.21), we observe $\tau_n\to\infty$ almost surely. Defining $R=(v_R,T_R)=U_1-U_2$ and $\hat{R}=\mathbb{1}_{\Omega_0}R$, we have

$$\mathrm{d}R + [A_HR + \tilde{\theta}(U_1)B(U_1,U_1) + F(U_1) - \tilde{\theta}(U_2)B(U_2,U_2) - F(U_2)]\,\mathrm{dt}$$
$$= (\sigma(U_1)-\sigma(U_2))\,\mathrm{dW}.$$

For fixed $n\in\mathbb{N}$, we apply the Itô formula to $\|R\|_{H_z^1L_{xy}^2}^2$ and estimate the trace term, which yields

$$\begin{aligned}
&\mathrm{d}\|R\|^2_{H^1_zL^2_{xy}} + 2\nu\|(-\Delta_H)^{1/2}R\|^2_{H^1_zL^2_{xy}}\,\mathrm{dt}\\
&\quad\le 2\langle R, F(U_1)-F(U_2)\rangle + 2\langle\partial_z R, \partial_z(F(U_1)-F(U_2))\rangle\,\mathrm{dt}\\
&\quad\quad + 2\left\langle R, \tilde\theta(U_1)B(U_1,U_1)-\tilde\theta(U_2)B(U_2,U_2)\right\rangle\mathrm{dt}\\
&\quad\quad + 2\left\langle \partial_z R, \partial_z(\tilde\theta(U_1)B(U_1,U_1)-\tilde\theta(U_2)B(U_2,U_2))\right\rangle\mathrm{dt}\\
&\quad\quad + \|\sigma(U_1)-\sigma(U_2)\|^2_{L_2(U,H^1_zL^2_{xy})}\,\mathrm{dt}\\
&\quad\quad + 2\langle R, \sigma(U_1)-\sigma(U_2)\,\mathrm{dW}\rangle + 2\langle\partial_z R, \partial_z(\sigma(U_1)-\sigma(U_2))\,\mathrm{dW}\rangle\,.
\end{aligned}$$

For arbitrary $\Upsilon > 0$ fixed and arbitrary stopping times $0 \le \tau_a \le \tau_b \le \tau_n \wedge \Upsilon$, we have

$$\begin{aligned}
&\mathbb{E}\int_{\tau_a}^{\tau_b} \mathbb{1}_{\Omega_0}\left|\langle R, F(U_1)-F(U_2)\rangle + \langle\partial_z R, \partial_z(F(U_1)-F(U_2))\rangle\right|\mathrm{dt}\\
&\qquad\le c\mathbb{E}\int_{\tau_a}^{\tau_b}\|\hat R\|_{H^1_zL^2_{xy}}\|\hat R\|_{H^1_zH^1_{xy}}\,\mathrm{dt}\\
&\qquad\le \frac{\varepsilon}{6}\mathbb{E}\int_{\tau_a}^{\tau_b}\|(-\Delta_H)^{1/2}\hat R\|^2_{H^1_zL^2_{xy}}\,\mathrm{dt} + c_\varepsilon\mathbb{E}\int_{\tau_a}^{\tau_b}\|\hat R\|^2_{H^1_zL^2_{xy}}\,\mathrm{dt}
\end{aligned}$$

for all $\varepsilon > 0$ and

$$\begin{aligned}
&\mathbb{E}\int_{\tau_a}^{\tau_b}\mathbb{1}_{\Omega_0}\|\sigma(U_1)-\sigma(U_2)\|^2_{L_2(U,H^1_zL^2_{xy})}\,\mathrm{dt}\\
&\qquad\le c\mathbb{E}\int_{\tau_a}^{\tau_b}\|\hat R\|^2_{H^1_zL^2_{xy}}\,\mathrm{dt} + \eta^2\mathbb{E}\int_{\tau_a}^{\tau_b}\mathbb{1}_{\Omega_0}\|(-\Delta_H)^{\frac12}\hat R\|^2_{H^1_zL^2_{xy}}\,\mathrm{dt}\,.
\end{aligned}$$

By the Burkholder-Davis-Gundy inequality (2.51) and estimates similar to the ones in Lemma 3.1, we obtain

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[\tau_a,\tau_b]}\left|\int_{\tau_a}^{t}\left\langle \hat R, \sigma(U_1)-\sigma(U_2)\,\mathrm{dW}\right\rangle + \int_{\tau_a}^{t}\left\langle\partial_z\hat R, \partial_z(\sigma(U_1)-\sigma(U_2))\,\mathrm{dW}\right\rangle\right|\\
&\quad\le c\mathbb{E}\left(\int_{\tau_a}^{\tau_b}\|\hat R\|^2_{H^1_zL^2_{xy}}\|\sigma(U_1)-\sigma(U_2)\|^2_{H^1_zL^2_{xy}}\,\mathrm{dt}\right)^{1/2}\\
&\quad\le \frac14\mathbb{E}\sup_{t\in[\tau_a,\tau_b]}\|\hat R\|^2_{H^1_zL^2_{xy}} + c\gamma^2\mathbb{E}\int_{\tau_a}^{\tau_b}\|(-\Delta_H)^{\frac12}\overline{R}\|^2_{H^1_zL^2_{xy}}\,\mathrm{dt} + c\mathbb{E}\int_{\tau_a}^{\tau_b}\|\hat R\|^2_{H^1_zL^2_{xy}}\,\mathrm{dt}\,.
\end{aligned}$$

For the nonlinear part, we use the cancellation property to get

$$\begin{aligned}
&\left\langle R, \tilde\theta(U_1)B(U_1,U_1)-\tilde\theta(U_2)B(U_2,U_2)\right\rangle\\
&\quad= \left\langle R, (\tilde\theta(U_1)-\tilde\theta(U_2))B(U_1,U_1)\right\rangle + \left\langle R, \tilde\theta(U_2)(B(U_1,U_1)-B(U_2,U_2))\right\rangle\\
&\quad= \left\langle R, (\tilde\theta(U_1)-\tilde\theta(U_2))B(U_1,U_1)\right\rangle + \left\langle R, \tilde\theta(U_2)B(U_1,R)\right\rangle + \left\langle R, \tilde\theta(U_2)B(R,U_2)\right\rangle\\
&\quad= (\tilde\theta(U_1)-\tilde\theta(U_2))\langle R, B(U_1,U_1)\rangle + \tilde\theta(U_2)\langle R, B(R,U_2)\rangle
\end{aligned}$$

and similarly

$$
\begin{aligned}
&\Big\langle \partial_z R, \partial_z(\tilde\theta(U_1)B(U_1,U_1) - \tilde\theta(U_2)B(U_2,U_2))\Big\rangle \\
&= \Big\langle \partial_z R, (\tilde\theta(U_1) - \tilde\theta(U_2))\partial_z B(U_1,U_1)\Big\rangle + \Big\langle \partial_z R, \tilde\theta(U_2)\partial_z(B(U_1,U_1) - B(U_2,U_2))\Big\rangle \\
&= (\tilde\theta(U_1) - \tilde\theta(U_2)) \langle \partial_z R, \partial_z B(U_1,U_1)\rangle + \tilde\theta(U_2) \langle \partial_z R, \partial_z B(U_1,R)\rangle \\
&\quad + \tilde\theta(U_2) \langle \partial_z R, \partial_z B(R,U_2)\rangle \\
&= (\tilde\theta(U_1) - \tilde\theta(U_2)) \langle \partial_z R, \partial_z B(U_1,U_1)\rangle + \tilde\theta(U_2) \langle \partial_z R, B(\partial_z U_1,R)\rangle \\
&\quad + \tilde\theta(U_2) \langle \partial_z R, \partial_z B(R,U_2)\rangle .
\end{aligned}
$$

We concentrate only on the estimates for the part of nonlinearity with vertical derivatives since the ones without vertical derivatives are easier to obtain. As above, we provide details only for the estimates for the temperature for the simplicity of presentation. Using the anisotropic Hölder inequality (2.5) and the Ladyzhenskaya inequality, we get

$$
\begin{aligned}
&\|\partial_z B(v_R, T_2)\|_{L^2_z L^{4/3}_{xy}} \\
&\quad = \|\partial_z v_R \cdot \nabla_H T_2 - \mathrm{div}_H\, v_R \partial_z T_2 + v_R \cdot \nabla_H \partial_z T_2 + w(R)\partial_{zz} T_2\|_{L^2_z L^{4/3}_{xy}} \\
&\quad \le c\big(\|\partial_z R\|_{L^2_z L^2_{xy}} \|\nabla_H U_2\|_{L^2_z L^4_{xy}} + \|\mathrm{div}_H R\|_{L^\infty_z L^2_{xy}} \|\partial_z U_2\|_{L^2_z L^4_{xy}} \\
&\qquad + \|R\|_{L^2_z L^4_{xy}} \|\nabla_H \partial_z U_2\|_{L^\infty_z L^2_{xy}} + \|w(R)\|_{L^\infty_z L^2_{xy}} \|\partial_{zz} U_2\|_{L^2_z L^4_{xy}}\big) \\
&\quad \le c\big(\|R\|_{H^1_z L^2_{xy}} \|U_2\|_{L^2_z H^2_{xy}} + \|R\|_{L^\infty_z H^1_{xy}} \|\partial_z U_2\|_{L^2_z L^4_{xy}} \\
&\qquad + \|R\|_{L^2_z L^4_{xy}} \|U_2\|_{H^2_z H^1_{xy}} + \|\mathrm{div}_H R\|_{L^2_z L^2_{xy}} \|U_2\|_{H^2_z L^4_{xy}}\big) \\
&\quad \le c\big(\|R\|_{H^1_z L^2_{xy}} \|U_2\|_{L^2_z H^2_{xy}} + \|R\|_{H^1_z H^1_{xy}} \|U_2\|^{1/2}_{H^1_z L^2_{xy}} \|U_2\|^{1/2}_{H^1_z H^1_{xy}} \\
&\qquad + \|R\|^{1/2}_{L^2} \|R\|^{1/2}_{L^2_z H^1_{xy}} \|U_2\|_{H^2_z H^1_{xy}} + \|R\|_{L^2_z H^1_{xy}} \|v\|^{1/2}_{H^2_z L^2_{xy}} \|U_2\|^{1/2}_{H^2_z H^1_{xy}}\big) \\
&\quad \le c\big(\|R\|_{H^1_z L^2_{xy}} \|U_2\|_{L^2_z H^2_{xy}} + \|R\|_{H^1_z H^1_{xy}} \|v\|^{1/2}_{H^2_z L^2_{xy}} \|U_2\|^{1/2}_{H^2_z H^1_{xy}} \\
&\qquad + \|R\|^{1/2}_{L^2} \|R\|^{1/2}_{L^2_z H^1_{xy}} \|U_2\|_{H^2_z H^1_{xy}}\big)
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\Big|\tilde\theta(U_2) \langle \partial_z R, \partial_z B(R,U_2)\rangle\Big| \le \|\partial_z R\|_{L^2_z L^4_{xy}} \|\partial_z B(R,U_2)\|_{L^2_z L^{4/3}_{xy}} \\
&\le \|R\|^{1/2}_{H^1_z L^2_{xy}} \|R\|^{1/2}_{H^1_z H^1_{xy}} \|\partial_z B(R,U_2)\|_{L^2_z L^{4/3}_{xy}} \\
&\le c\|R\|^{3/2}_{H^1_z L^2_{xy}} \|R\|^{1/2}_{H^1_z H^1_{xy}} \|U_2\|_{L^2_z H^2_{xy}} + c\|R\|_{H^1_z L^2_{xy}} \|R\|_{H^1_z H^1_{xy}} \|U_2\|_{H^2_z H^1_{xy}} \\
&\quad + c\|R\|^{1/2}_{H^1_z L^2_{xy}} \|R\|^{3/2}_{H^1_z H^1_{xy}} \|U_2\|^{1/2}_{H^2_z L^2_{xy}} \|U_2\|^{1/2}_{H^2_z H^1_{xy}} \\
&\le \frac{\varepsilon}{6} \|(-\Delta_H)^{1/2} R\|^2_{H^1_z L^2_{xy}} \\
&\quad + c_\varepsilon\big(\|U_2\|^{4/3}_{L^2_z H^2_{xy}} + \|U_2\|^2_{H^2_z H^1_{xy}} + \|U_2\|^2_{H^2_z L^2_{xy}} \|U_2\|^2_{H^2_z H^1_{xy}}\big) \|R\|^2_{H^1_z L^2_{xy}} .
\end{aligned}
$$

Similarly as above, we deduce

$$
\begin{aligned}
&\|(\partial_z v_1, T_R)\|_{L^2_z L^{4/3}_{xy}} = \|\partial_z v_1 \cdot \nabla_H T_R + \mathrm{div}_H\, v_1 \partial_z T_R\|_{L^2_z L^{4/3}_{xy}} \\
&\quad \le \|\partial_z v_1\|_{L^2_z L^4_{xy}} \|\nabla_H T_R\|_{L^\infty_z L^2_{xy}} + \|\nabla_H v_1\|_{L^\infty_z L^2_x} \|\partial_z T_R\|_{L^2_z L^{4/3}_{xy}} \\
&\quad \le c\big(\|U_1\|^{1/2}_{H^1_z L^2_{xy}} \|U_1\|^{1/2}_{H^1_z H^1_{xy}} \|R\|_{H^1_z H^1_{xy}} \\
&\qquad + \|U_1\|^{1/2}_{L^2_z H^1_{xy}} \|U_1\|^{1/2}_{H^1_z H^1_{xy}} \|R\|^{1/2}_{H^1_z L^2_{xy}} \|R\|^{1/2}_{H^1_z H^1_{xy}}\big),
\end{aligned}
$$

which leads to

$$\begin{aligned}\left|\tilde\theta(U_2)\,\langle\partial_z R, B(\partial_z U_1,T)\rangle\right| &\le \|\partial_z R\|_{L^2_zL^4_{xy}}\,\|\partial_z B(\partial_z U_1,R)\|_{L^2_zL^{4/3}_{xy}}\\ &\le \frac{\varepsilon}{6}\|R\|^2_{H^1_zH^1_{xy}} + c_\varepsilon\|R\|^2_{H^1_zL^2_{xy}}\|U_1\|^2_{H^1_zL^2_{xy}}\|U_1\|^2_{H^1_zH^1_{xy}}.\end{aligned}$$

Furthermore,

$$\begin{aligned}&\|\partial_z B(v_1,T_1)\|_{L^2} = \|\partial_z v_1\cdot\nabla_H T_1 - \mathrm{div}_H\, v_1\partial_z T_1 + v_1\cdot\nabla_H\,\partial_z T_1 + w(U_1)\partial_{zz}T_1\|_{L^2}\\ &\le c\big(\|\partial_z U_1\|_{L^\infty_zL^4_{xy}}\|\nabla_H U_1\|_{L^2_zL^4_{xy}} + \|\mathrm{div}_H U_1\|_{L^2_zL^4_{xy}}\|\partial_z U_1\|_{L^\infty_zL^4_{xy}}\\ &\quad + \|U_1\|_{L^2_zL^\infty_{xy}}\|\nabla_H\,\partial_z U_1\|_{L^\infty_zL^2_{xy}} + \|w(U_1)\|_{L^\infty_zL^4_{xy}}\|\partial_{zz}U_1\|_{L^2_zL^4_{xy}}\big)\\ &\le c\big(\|\partial_z U_1\|^{1/2}_{L^\infty_zL^2_{xy}}\|\partial_z U_1\|^{1/2}_{L^\infty_zH^1_{xy}}\|\nabla_H U_1\|^{1/2}_{L^2_zL^2_{xy}}\|\nabla_H U_1\|^{1/2}_{L^2_zH^1_{xy}}\\ &\quad + \|U_1\|^{1/2}_{L^2_zH^2_{xy}}\|U_1\|^{1/2}_{L^2_zH^1_{xy}}\|\partial_z U_1\|^{1/2}_{L^\infty_zL^2_{xy}}\|\partial_z U_1\|^{1/2}_{L^\infty_zH^1_{xy}}\\ &\quad + \|U_1\|^{1/2}_{L^2_zL^2_{xy}}\|U_1\|^{1/2}_{L^2_zH^2_{xy}}\|\partial_z U_1\|_{L^\infty_zH^1_{xy}}\\ &\quad + \|w(U_1)\|^{1/2}_{L^\infty_zL^2_{xy}}\|w(U_1)\|^{1/2}_{L^\infty_zH^1_{xy}}\|\partial_{zz}U_1\|^{1/2}_{L^2_zL^2_{xy}}\|\partial_{zz}U_1\|^{1/2}_{L^2_zH^1_{xy}}\big)\\ &\le c\big(\|U_1\|^{1/2}_{H^2_zL^2_{xy}}\|U_1\|^{1/2}_{L^2_zH^1_{xy}}\|U_1\|^{1/2}_{H^2_zH^1_{xy}}\|U_1\|^{1/2}_{L^2_zH^2_{xy}} + \|U_1\|^{1/2}_{L^2}\|U_1\|^{1/2}_{L^2_zH^2_{xy}}\|U_1\|_{H^2_zH^1_{xy}}\big).\end{aligned}$$

Hence, by the Lipschitz continuity of the cut-off $\tilde\theta$, we have

$$\begin{aligned}&|(\tilde\theta(U_1)-\tilde\theta(U_2))\,\langle\partial_z R,\partial_z B(U_1,U_1)\rangle\,| \le c\|R\|_{H^1_zL^4_{xy}}\|\partial_z R\|_{L^2}\|\partial_z B(U_1,U_1)\|_{L^2}\\ &\le c\|R\|^{3/2}_{H^1_zL^2_{xy}}\|R\|^{1/2}_{H^1_zH^1_{xy}}\big(\|U_1\|^{1/2}_{H^2_zL^2_{xy}}\|U_1\|^{1/2}_{H^2_zH^1_{xy}}\|U_1\|^{1/2}_{L^2_zH^1_{xy}}\|U_1\|^{1/2}_{L^2_zH^2_{xy}}\\ &\qquad + \|U_1\|^{1/2}_{L^2}\|U_1\|^{1/2}_{L^2_zH^2_{xy}}\|U_1\|_{H^2_zH^1_{xy}}\big)\\ &\le \frac{\varepsilon}{6}\|(-\Delta_H\,R)^{1/2}\|^2_{H^1_zL^2_{xy}} + c_\varepsilon\big(\|U_1\|^{2/3}_{H^2_zL^2_{xy}}\|U_1\|^{2/3}_{L^2_zH^1_{xy}}\|U_1\|^{2/3}_{H^2_zH^1_{xy}}\|U_1\|^{2/3}_{L^2_zH^2_{xy}}\\ &\qquad + \|U_1\|^{2/3}_{L^2}\|U_1\|^{2/3}_{L^2_zH^2_{xy}}\|U_1\|^{4/3}_{H^2_zH^1_{xy}}\big)\|R\|^2_{H^1_zL^2_{xy}}.\end{aligned}$$

We conclude

$$\begin{aligned}&\mathbb{E}\left[\frac{1}{2}\sup_{s\in[\tau_a,\tau_b]}\|\hat R\|^2_{H^1_zL^2_{xy}} + c(\nu,\varepsilon,\gamma)\int_{\tau_a}^{\tau_b}\|(-\Delta_H\,)^{1/2}\hat R\|^2_{H^1_zL^2_{xy}}\,\mathrm{ds}\right] \le c\mathbb{E}\|\hat R(\tau_a)\|^2_{H^1_zL^2_{xy}}\\ &\quad + c_\varepsilon\mathbb{E}\Bigg[\int_{\tau_a}^{\tau_b}\big(1 + \|U_1\|^{2/3}_{H^2_zL^2_{xy}}\|U_1\|^{2/3}_{L^2_zH^1_{xy}}\|U_1\|^{2/3}_{H^2_zH^1_{xy}}\|U_1\|^{2/3}_{L^2_zH^2_{xy}}\\ &\qquad + \|U_1\|^{2/3}_{L^2}\|U_1\|^{2/3}_{L^2_zH^2_{xy}}\|U_1\|^{4/3}_{H^2_zH^1_{xy}} + \|U_2\|^{4/3}_{L^2_zH^2_{xy}} + \|U_2\|^2_{H^2_zH^1_{xy}}\\ &\qquad + \|U_1\|^2_{H^1_zL^2_{xy}}\|U_1\|^2_{H^1_zH^1_{xy}} + \|U_2\|^2_{H^2_zL^2_{xy}}\|U_2\|^2_{H^2_zH^1_{xy}}\big)\|\hat R\|^2_{H^1_zL^2_{xy}}\,\mathrm{dt}\Bigg],\end{aligned}$$

where $c(\nu,\varepsilon,\gamma) = 2[\nu-\varepsilon-\gamma^2(2c^2_{BDG}+\frac{1}{2})]$. From this, the claim follows by applying the stochastic Gronwall lemma, see Lemma A.2, and recalling that Υ was arbitrary. □

Remark 3.8. The last estimate in the above proof also identifies $U_1 = U_2$ in the space $L^2(0,\Upsilon;H^1_zH^1_{xy})$ for all $\Upsilon > 0$ $\mathbb{P}$-almost surely. The identification therefore holds also in the space $\mathcal{X}_U = L^2(0,\Upsilon;L^2_zH^1_{xy}\cap L^\infty_zL^4_{xy})$ from Proposition 3.3 which justifies the use of the Gyöngy-Krylov theorem in the proof of Theorem 2.11 below.

3.5 Proof of Theorem 2.11

The existence of global strong solutions of the modified problem (3.19) for initial data $U_0 \in L^q(\Omega;H^1\cap H^2_zL^2_{xy})$ for some $q\ge 4$ follows similarly as in [11, Section 5.2] from the Gyöngy-Krylov theorem and the pathwise uniqueness established established in Proposition 3.7, see also Remark 3.8. Thus, we omit the proof.

In contrast to the approximations in [11], where the cut-off function in (3.19) does not start at 0, the cut-off in (3.19) may be active from the initial time $t=0$. A localization procedure that also establishes existence of local solutions of the original equation (2.27) for initial data in $L^2(\Omega; H^1\cap H^2_zL^2_{xy})$ is therefore required. For $n\in\mathbb{N}$, let

$$\Omega_n := \{n-1 \le \|U_0\|_{H^1} + \|U_0\|_{H^2_zL^2_{xy}} < n\}, \qquad U_{0,n} = U_0\mathbb{1}_{\Omega_n},$$

and let U_n be the global pathwise solution of equation (3.19) with cut-off $\tilde\theta$ with $\mu = 2c_e n$, where $c_e > 0$ a fixed number such that $\|f\|_{H^1_zL^4_{xy}} \le c_e(\|f\|_{H^2_zL^2_{xy}} + \|f\|_{L^2_zH^1_{xy}})$ for all $f\in H^2_zL^2_{xy}\cap L^2_zH^1_{xy}$, and initial data $U_{0,n}\in L^\infty(\Omega; H^1\cap H^2_zL^2_{xy})$. Let $U=\sum_{n=1}^\infty \mathbb{1}_{\Omega_n}U_n$. Let us fix $M>0$ and define the stopping times

$$\tau_n^\mu = \inf\left\{t\ge 0 \mid \|U_n\|_{H^1} + \|U_n\|_{H^2_zL^2_{xy}} \ge 2n\right\},$$
$$\tau_n^M = \inf\left\{t\ge 0 \mid \sup_{s\in[0,t]}\|U_n\|^2_{H^1} + \int_0^t \|(-\Delta_H\,)U_n\|^2_{L^2}\,\mathrm{ds} \ge M + \|U_{0,n}\|^2_{H^1}\right\}.$$

Let $\tau = \sum_{n=1}^\infty \mathbb{1}_{\Omega_n}\left(\tau_n^\mu\wedge\tau_n^M\right)$. Clearly, $\tau>0$ $\mathbb{P}$-almost surely. It is now straightforward to check that (U,τ) has the desired integrability and is indeed a local strong solution.

Existence of maximal solutions follows similarly as in [5, Section 3.4]. This concludes the proof of Theorem 2.11.

4 Global existence

In the whole section, let (U,ξ) be the maximal solution established in Theorem 2.11 and let U_0, σ and f_v, f_T satisfy the assumptions of Theorem 2.12. For the simplicity of presentation, we will not list list the smallness requirements on η in the statements of the auxiliary results in this section since they are easily recoverable from the proofs.

The proof of global existence combines the technique from [12] with the use of the logarithmic Sobolev inequality from [8]. In particular, our goal is to find a sequence of stopping times ρ_K satisfying $\rho_K\to\infty$ and $\rho_K\le\xi$ for all $K\in\mathbb{N}$. In the process, we obtain an inequality of the form

$$f(t_1) \le f(t_0) + \int_{t_0}^{t_1} \|v\|^2_{L^\infty} f(s)\,\mathrm{ds},$$

where f contains certain Sobolev norms of the solution, see Lemma 4.11. To control $\|v\|_\infty$, we need the logarithmic Sobolev inequality. After that, we employ an argument similar to the logarithmic Gronwall inequality from [8, Lemma 2.5], see also the references therein.

Proposition 4.1 (Logarithmic Sobolev inequality)**.** *Let $p_1,p_2\in(1,\infty)$ satisfy $\frac{2}{p_1}+\frac{1}{p_2}<1$. Then there exist $r_1,r_2\in[2,\infty)$ such that for all $F\in L^{p_1}_zH^{1,p_1}_{0,xy}$ with $\partial_zF\in L^{p_2}$ we have*

$$\|F\|_{L^\infty} \le C_{p,\lambda}\max\left\{1, \max_{i=1,2}\frac{\|F\|_{L^{r_i}}}{{r_i}^\lambda}\right\} \cdot \log^\lambda\left(e + \|F\|_{L^{p_1}} + \|\nabla_H\,F\|_{L^{p_1}} + \|F\|_{L^{p_2}} + \|\partial_zF\|_{L^{p_2}}\right), \quad (4.1)$$

for any $\lambda>0$ provided all the norms are finite.

Proposition 4.1 can be proved in exactly the same manner as [8, Lemma A.1], the only difference being using the maximum of several L^p-norms rather than $\sup_{p\in[2,\infty)}\|f\|_{L^p}$.

The numbers r_i can be computed explicitly. From the proof of [8, Lemma A.1], we observe $r_i = (q-1)\kappa_i$ for some $q \geq 3$, where

$$\alpha_i = \frac{1}{p_i}, \quad \kappa_i = \frac{p_i\left(1+\sum_{j=1}^{2}\alpha_j\right)}{1-\sum_{j=1}^{2}\alpha_j}.$$

Below, we apply the inequality with $p_1 = 6$ and $p_2 = 2$, choosing $q = 3$ we get

$$r_1 = (3-1)\kappa_1 = 132, \qquad r_2 = (3-1)\frac{22}{5} = \frac{44}{5}.$$

Moreover, since $r_1 > r_2$, we observe

$$\max\left\{1, \max_{i=1,2}\frac{\|F\|_{L^{r_i}}}{{r_i}^{\lambda}}\right\} \leq 1 + \max_{i=1,2}\frac{\|F\|_{L^{r_i}}}{{r_i}^{\lambda}} \leq c\left(1+\frac{\|F\|_{L^{r_1}}}{{r_1}^{\lambda}}\right),$$

and therefore (4.1) reads as

$$\|F\|_{L^\infty} \leq c\,(1+\|F\|_{L^{r_1}})\log^{\lambda}(e+\|\nabla_H F\|_6+\|F\|_6+\|\partial_z F\|_{L^2}+\|F\|_{L^2}). \tag{4.2}$$

Let us start with the standard energy estimate.

Lemma 4.2 (L^2 bounds)**.** *Let $q \geq 2$ and $U_0 \in L^q(\Omega; L^2(M))$. Let the forcing F_U satisfy $F_U \in L^q(\Omega; L^2_{loc}(0,\infty; L^2(M)))$. Then the stopping times $\tau_K^{w,q}$ defined for $K \in \mathbb{N}$ by*

$$\tau_K^{w,q} = \inf\left\{s \geq 0 \mid \sup_{s\in[0,t\wedge\xi)}\|U\|_{L^2}^q + \int_0^{t\wedge\xi}\|U\|_{L^2}^{q-2}\|U\|_{L_z^2H_{xy}^1}^2\,\mathrm{ds} \right.$$
$$\left. + \left(\int_0^{t\wedge\xi}\|U\|_{L_z^2H_{xy}^1}^2\,\mathrm{ds}\right)^{q/2} \geq K\right\}$$

satisfy $\tau_K^{w,q} \to \infty$ $\mathbb{P}$-a.s. as $K \to \infty$.

Note that the assumptions of Theorem 2.12 are such that the above Lemma can be applied for $q \leq 16/3$.

Proof. Similarly as in Step 1 of Lemma 3.1, we deduce

$$\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_N]}\|U\|_{L^2}^q + \int_0^{t\wedge\tau_N}\|U\|_{L^2}^{q-2}\|U\|_{L_z^2H_{xy}^1}^2\,\mathrm{ds}\right]$$
$$\leq C_t\mathbb{E}\left[\|U_0\|_{L^2}^q + 1 + \left(\int_0^{t\wedge\tau_N}\|F_U\|_{L^2}^2\,\mathrm{ds}\right)^{q/2}\right].$$

By the monotone convergence theorem, we get

$$\mathbb{E}\left[\sup_{s\in[0,t\wedge\xi)}\|U\|_{L^2}^q + \int_0^{t\wedge\xi}\|U\|_{L^2}^{q-2}\|U\|_{L_z^2H_{xy}^1}^2\,\mathrm{ds}\right]$$
$$\leq C_t\mathbb{E}\left[\|U_0\|_{L^2}^q + 1 + \left(\int_0^{t\wedge\xi}\|F_U\|_{L^2}^2\,\mathrm{ds}\right)^{q/2}\right].$$

The a.s. convergence to infinity of the auxiliary stopping time τ_K^1 defined by

$$\tau_K^1 = \inf\left\{s \geq 0 \mid \sup_{r\in[0,s\wedge\xi)}\|U\|_{L^2}^q + \int_0^{s\wedge\xi}\|U\|_{L^2}^{q-2}\|U\|_{L_z^2H_{xy}^1}^2\,\mathrm{dr} \geq K\right\}$$

follows by the first claim in Lemma A.1. Hence, the second claim in Lemma A.1 with $\Psi(s) = s^{q/2}$ leads to the convergence of the auxiliary stopping time τ_K^2 defined by

$$\tau_K^2 = \inf\left\{ s \geq 0 \mid \left(\int_0^{s\wedge\xi} \|U\|^2_{L^2_z H^1_{xy}} \,\mathrm{dr} \right)^{q/2} \geq K \right\}.$$

The proof is concluded by setting $\tau_K^{w,q} = \tau_K^1 \wedge \tau_K^2$. □

4.1 Estimates by splitting

Next, we decompose the momentum equation into equations for barotropic and baroclinic modes $\overline{v}$ and $\widetilde{v}$ in (2.28) and (2.30), respectively, to establish estimates later leading to control of the L^p-norm of the solution. Note that the baroclinic mode is mean-value free in the vertical direction and $\partial_z \widetilde{v} = \partial_z v$. Thus, the vertical Poincaré inequality

$$\int_{-h}^0 |\widetilde{v}|^q \,\mathrm{dz} \leq c_q \|\partial_z \widetilde{v}\|^q_{L^q(-h,0)} = c_q \|\partial_z v\|^q_{L^q(-h,0)}$$

holds. This implies

$$\|v\|_{L^6} \leq \|\overline{v}\|_{L^6} + \|\widetilde{v}\|_{L^6} \leq c\left(\|\nabla_H \overline{v}\|_{L^2} + \|\partial_z v\|_{L^6}\right). \tag{4.3}$$

It is straightforward to check that $\|\mathcal{A}v\|_{L^q(G)} \leq h^{-1/q}\|v\|_{L^q(M)}$ and $\|\mathcal{R}v\|_{L^q} \leq (1 + h^{-1/q})\|v\|_{L^q}$.

We begin with an open estimate for $\overline{v}$ similarly as in [31]. In Lemma 4.4, we will combine the estimate with an estimate for $\widetilde{v}$ to obtain a useful bound.

Lemma 4.3. *For any $N \in \mathbb{N}$, $\varepsilon > 0$ and any two stopping times $0 \leq \tau_a \leq \tau_b \leq t \wedge \tau_N$, it holds*

$$\begin{aligned} &\mathbb{E}\left[\frac{1}{2} \sup_{s\in[\tau_a,\tau_b]} \|(-\Delta_H)^{1/2}\overline{v}\|^2_{L^2} + c(\nu,\varepsilon,\eta) \int_{\tau_a}^{\tau_b} \|\Delta_H \overline{v}\|^2_{L^2} \,\mathrm{ds} \right] \\ &\quad \leq c\left(\mathbb{E}\|(-\Delta_H)^{1/2}\overline{v}(\tau_a)\|^2_{L^2} + 1 \right) \\ &\qquad + c\mathbb{E} \int_{\tau_a}^{\tau_b} (1 + \|U\|^2_{L^2} \|U\|^2_{L^2_z H^1_{xy}}) \|(-\Delta_H)^{1/2}\overline{v}\|^2_{L^2} \,\mathrm{ds} \\ &\qquad + c_\varepsilon \mathbb{E} \int_{\tau_a}^{\tau_b} \|U\|^2_{L^2_z H^1_{xy}} + \left\|\overline{f_v}\right\|^2_{L^2} + \||\widetilde{v}|\nabla_H \widetilde{v}\|^2_{L^2} \,\mathrm{ds}, \end{aligned} \tag{4.4}$$

where $c(\nu,\varepsilon,\eta) = 2[\nu - \varepsilon - \eta^2(\frac{1}{2} + 2c^2_{BDG})]$ and the constants c, c_ε are independent of N, τ_a and τ_b.

Proof. We apply the Itô formula to $\left\|(-\Delta_H)^{-1/2} P_G \cdot\right\|^2_{L^2}$ and obtain

$$\begin{aligned} &\mathrm{d}\|(-\Delta_H)^{1/2}\overline{v}\|^2_{L^2} + 2\nu\|\Delta_H \overline{v}\|^2_{L^2} \,\mathrm{dt} \\ &\quad \leq 2\|(-\Delta_H)^{1/2} \left\langle (-\Delta_H)^{1/2}\overline{v}, (-\Delta_H)^{1/2} P_G \left[\mathcal{A}F_v(U) - (\overline{v}\cdot\nabla_H \overline{v}) - N(\widetilde{v})\right] \right\rangle \mathrm{dt} \\ &\qquad + 2\|(-\Delta_H)^{1/2}\mathcal{A}\sigma_1(U)\|^2_{L_2(\mathcal{U},L^2)} \,\mathrm{dt} \\ &\qquad + 2\left\langle (-\Delta_H)^{1/2}\overline{v}, (-\Delta_H)^{1/2}\mathcal{A}\sigma_1(U) \,\mathrm{dW} \right\rangle, \end{aligned}$$

where we already used $P_G \overline{v} = \overline{v}$ and $P_G \mathcal{A}\sigma_1(U) = \mathcal{A}\sigma_1(U)$ from (2.29) and (2.49), respectively. In [34, Lemma 5.3], it was shown that, for $\varepsilon > 0$, we have

$$\begin{aligned} &\left| \left\langle (-\Delta_H)^{1/2}\overline{v}, (-\Delta_H)^{1/2}(\overline{v}\cdot\nabla_H \overline{v}) + (-\Delta_H)^{1/2} N(\widetilde{v}) \right\rangle \right| \\ &\qquad \leq C_\varepsilon \left(\|\overline{v}\|^2_{L^2} \|v\|^2_{L^2_z H^1_{xy}} \|(-\Delta_H)^{1/2}\overline{v}\|^2_{L^2} + \||\widetilde{v}|\nabla_H \widetilde{v}\|^2_{L^2} \right) + \frac{\varepsilon}{2} \|\Delta_H \overline{v}\|^2_{L^2} \end{aligned}$$

and

$$\begin{aligned}&\left|\left\langle (-\Delta_H)^{1/2}\overline{v}, (-\Delta_H)^{1/2}\mathcal{A}F_v(U)\right\rangle\right|\\&\qquad\leq C_\varepsilon\left(\|\nabla_H T\|_{L^2}^2+\|v\|_{L^2}^2+\left\|\overline{f_v}\right\|_{L^2}^2\right)+\frac{\varepsilon}{2}\|\Delta_H \overline{v}\|_{L^2}^2\,.\end{aligned}$$

By the sub-linear growth of $\mathcal{A}\sigma_1$ (2.50) and the Burkholder-Davis-Gundy inequality (2.51), we deduce

$$\begin{aligned}&2\mathbb{E}\sup_{s\in[\tau_a,\tau_b]}\left|\int_0^s\left\langle (-\Delta_H)^{1/2}\overline{v}, (-\Delta_H)^{1/2}\mathcal{A}\sigma_1(U)\right\rangle \mathrm{dW}\right|\\&\leq \frac{1}{2}\mathbb{E}\sup_{s\in[\tau_a,\tau_b]}\|(-\Delta_H)^{1/2}\overline{v}\|_{L^2}^2+2c_{BDG}^2\eta^2\mathbb{E}\int_{\tau_a}^{\tau_b}\|\Delta_H\overline{v}\|_{L^2}^2\,\mathrm{ds}\\&\quad+C\mathbb{E}\int_0^t 1+\|(-\Delta_H)^{1/2}v\|_{L^2}^2\,\mathrm{ds}\,.\end{aligned}$$

We deal with the Itô correction term by (2.50), bounds from Lemma 4.2 and $\|\nabla_H\overline{v}\|_{L^2}\leq c\|\nabla_H v\|_{L^2}\leq c\|U\|_{L^2_zH^1_{xy}}$. The claim follows by collecting the above estimates. ☐

Lemma 4.4 (H^1 bound for $\overline{v}$ and L^4 bound for $\widetilde{v}$)**.** *The stopping times $\tau_K^{\widetilde{v},4}$ and $\tau_K^{\overline{v},2}$ defined for $K\in\mathbb{N}$ by*

$$\begin{aligned}\tau_K^{\widetilde{v},4}&=\inf\left\{s\geq 0 \mid \sup_{r\in[0,s\wedge\xi)}\|\widetilde{v}\|_{L^4}^4+\int_0^{s\wedge\xi}\||\widetilde{v}|\nabla_H\widetilde{v}\|_{L^2}^2\,\mathrm{dr}\geq K\right\},\\ \tau_K^{\overline{v},2}&=\inf\left\{s\geq 0 \mid \sup_{r\in[0,s\wedge\xi)}\|(-\Delta_H)^{1/2}\overline{v}\|_{L^2}^2+\int_0^{s\wedge\xi}\|\Delta_H\overline{v}\|_{L^2}^2\,\mathrm{ds}\geq K\right\},\end{aligned}$$

satisfy $\tau_K^{\widetilde{v}}\to\infty$ $\mathbb{P}$-a.s. as $K\to\infty$.

Proof. Recalling the cancellation property of the nonlinear term, we apply the Itô formula to $\|\widetilde{v}\|_{L^4}^4$ and obtain

$$\begin{aligned}&\mathrm{d}\|\widetilde{v}\|_{L^4}^4+4\nu\left(\||\widetilde{v}|\nabla_H\widetilde{v}\|_{L^2}^2+2\||\widetilde{v}|\nabla_H|\widetilde{v}|\|_{L^2}^2\right)\mathrm{dt}\\&\quad\leq -4\left\langle|\widetilde{v}|^2\widetilde{v},\mathcal{R}F_v(U)-\widetilde{v}\cdot\nabla_H\overline{v}+N(\widetilde{v})\right\rangle\mathrm{dt}+6\sum_{k=1}^\infty\left\langle|\widetilde{v}|^2,(\mathcal{R}\sigma_1(U)e_k)^2\right\rangle\mathrm{dt}\\&\quad+4\left\langle|\widetilde{v}|^2\widetilde{v},\mathcal{R}\sigma_1(U)\,\mathrm{dW}\right\rangle\end{aligned}$$

almost surely. Integrating by parts, we get

$$\begin{aligned}4\left|\left\langle|\widetilde{v}|^2\widetilde{v},\beta_T g\int_\cdot^0\nabla_H T(x,y,z')\,\mathrm{dz}'\right\rangle\right|&\leq c\left\|\int_\cdot^0 T(x,y,z')\,\mathrm{dz}'\right\|_{L^4}\||\widetilde{v}|\nabla_H\widetilde{v}\|_{L^2}\|\widetilde{v}\|_{L^4}\\&\leq c_\varepsilon\|T\|_{L^2_zL^4_{xy}}^2\|\widetilde{v}\|_{L^4}^2+\varepsilon\||\widetilde{v}|\nabla_H\widetilde{v}\|_{L^2}^2\\&\leq c_\varepsilon\|T\|_{L^2_zH^1_{xy}}^2\|\widetilde{v}\|_{L^4}^2+\varepsilon\||\widetilde{v}|\nabla_H\widetilde{v}\|_{L^2}^2\end{aligned}$$

for $\varepsilon>0$ arbitrary. A similar estimate for the term with double vertical integral can be established in a similar manner. Hence, by the Young inequality, we have

$$
\begin{aligned}
&4\int_{\tau_a}^{\tau_b} \left|\left\langle |\widetilde{v}|^2\widetilde{v}, \mathcal{R}F_v(U)\right\rangle\right| \mathrm{ds} \\
&\quad\le \int_{\tau_a}^{\tau_b} c_\varepsilon \left(\|T\|^2_{L^2_z H^1_{xy}} \|\widetilde{v}\|^2_{L^4} + \left\|\widetilde{f_v}\right\|_{L^4} \|\widetilde{v}\|^3_{L^4} + \|\widetilde{v}\|^4_{L^4}\right) + \varepsilon \, \||\widetilde{v}|\nabla_H \widetilde{v}\|^2_{L^2} \mathrm{ds} \\
&\quad\le \varepsilon \int_{\tau_a}^{\tau_b} \||\widetilde{v}|\nabla_H \widetilde{v}\|^2_{L^2} \mathrm{ds} + c_\varepsilon \int_{\tau_a}^{\tau_b} \left(1+\|T\|^2_{L^2_z H^1_{xy}}\right)(1+\|\widetilde{v}\|^4_{L^4}) \mathrm{ds} \\
&\qquad + \frac{1}{4}\sup_{s\in[\tau_a,\tau_b]} \|\widetilde{v}\|^4_{L^4} + c\left(\int_{\tau_a}^{\tau_b} \left\|\widetilde{f_v}\right\|^2_{L^4} \mathrm{ds}\right)^2
\end{aligned}
$$

for any stopping times $0 \le \tau_a \le \tau_b$. Following [31, Proof of Theorem 1.1 in Section 6, Step 3, integrals I_7 and I_8], we obtain

$$
4\left|\left\langle |\widetilde{v}|^2\widetilde{v}, -\tilde{v}\cdot\nabla_H \overline{v} + N(\tilde{v})\right\rangle\right| \le c_\varepsilon \|\nabla_H \overline{v}\|^2_{L^2} \|\tilde{v}\|^4_{L^4} + \varepsilon \||\tilde{v}|\nabla_H \tilde{v}\|^2_{L^2}.
$$

Using (2.48), (2.43) and (2.45), the boundedness of the operator $\mathcal{R}$ and the Hölder and Young inequalities, we get

$$
\begin{aligned}
&6\sum_{k=1}^{\infty} \left\langle |\widetilde{v}|^2, (\mathcal{R}\sigma_1(U)e_k)^2\right\rangle \\
&\quad= 6\sum_{k=1}^{\infty} \left\langle |\widetilde{v}|^2, (\Psi_k\cdot\nabla_H \widetilde{v} + (\mathcal{R}\Phi_k)\cdot\nabla_H \overline{v} + \mathcal{R}h_k(v))^2\right\rangle \\
&\quad\le 6\sum_{k=1}^{\infty} \left\langle |\widetilde{v}|^2, (1+\frac{\varepsilon}{3})|\Psi_k\cdot\nabla_H \widetilde{v}|^2 + c_\varepsilon|(\mathcal{R}\Phi_k)\cdot\nabla_H \overline{v}|^2 + c_\varepsilon|\mathcal{R}h_k(v)|^2\right\rangle \\
&\quad\le (6\eta^2+2\varepsilon)\||\widetilde{v}|\nabla_H \widetilde{v}\|^2_{L^2} + c_\varepsilon\left(\|\widetilde{v}\|^2_{L^4}\|\nabla_H \overline{v}\|^2_{L^4} + \|\widetilde{v}\|^2_{L^4}\|v\|^2_{L^4} + \|\widetilde{v}\|^2_{L^4}\right) \\
&\quad\le (6\eta^2+2\varepsilon)\||\widetilde{v}|\nabla_H \widetilde{v}\|^2_{L^2} + c_\varepsilon\|\widetilde{v}\|^2_{L^4}\left(\|\nabla_H \overline{v}\|^2_{L^4} + (\|\widetilde{v}\|_{L^4} + \|\overline{v}\|_{L^4})^2 + 1\right) \\
&\quad\le (6\eta^2+2\varepsilon)\||\widetilde{v}|\nabla_H \widetilde{v}\|^2_{L^2} + c_\varepsilon\|\widetilde{v}\|^2_{L^4}\|\nabla_H \overline{v}\|_{L^2}\|\Delta_H \overline{v}\|_{L^2} \\
&\qquad + c_\varepsilon\|\widetilde{v}\|^2_{L^4}\left(\|\widetilde{v}\|^2_{L^4} + \|\nabla_H \overline{v}\|^2_{L^2} + 1\right) \\
&\quad\le (6\eta^2+2\varepsilon)\||\widetilde{v}|\nabla_H \widetilde{v}\|^2_{L^2} + \frac{\overline{\varepsilon}}{2}\|\Delta_H \overline{v}\|^2_{L^2} + c_{\varepsilon,\overline{\varepsilon}}(1+\|\nabla_H \overline{v}\|^2_{L^2})(1+\|\widetilde{v}\|^4_{L^4}).
\end{aligned}
\tag{4.5}
$$

Similarly, we deduce

$$
\begin{aligned}
&\sum_{k=1}^{\infty} |\left\langle |\widetilde{v}|^2\widetilde{v}, \mathcal{R}\sigma_1(U)e_k\right\rangle| \\
&\quad= \sum_{k=1}^{\infty} |\left\langle |\widetilde{v}|^2\widetilde{v}, \Psi_k\cdot\nabla_H \widetilde{v} + (\mathcal{R}\Phi_k)\cdot\nabla_H \overline{v} + \mathcal{R}h_k(v)\right\rangle| \\
&\quad\le \eta\||\widetilde{v}|\nabla_H \widetilde{v}\|_{L^2}\|\widetilde{v}\|^2_{L^4} + c(\|\widetilde{v}\|^3_{L^4}\|\nabla_H \overline{v}\|_{L^4} + \|\widetilde{v}\|^3_{L^4}\|v\|_{L^4} + \|\widetilde{v}\|^2_{L^4}) \\
&\quad\le \eta\||\widetilde{v}|\nabla_H \widetilde{v}\|_{L^2}\|\widetilde{v}\|^2_{L^4} + c\|\widetilde{v}\|^3_{L^4}\|\nabla_H \overline{v}\|^{1/2}_{L^2}\|\Delta_H \overline{v}\|^{1/2}_{L^2} \\
&\qquad + c\left(\|\widetilde{v}\|^4_{L^4} + \|\widetilde{v}\|^3_{L^4}\|\nabla_H \overline{v}\|_{L^2} + 1\right).
\end{aligned}
\tag{4.6}
$$

Let $K, N \in \mathbb{N}$ be fixed and let $0 \le \tau_a \le \tau_b \le t\wedge\tau_N\wedge\tau_K^{w,4}$. By the Burkholder-Davis-Gundy inequality (2.51) and (4.6), we obtain

$$
\begin{aligned}
&4\mathbb{E}\sup_{s\in[\tau_a,\tau_b]}\left|\int_{\tau_a}^{s}\left\langle|\widetilde{v}|^2\widetilde{v},\mathcal{R}\sigma_1(U)\right\rangle \mathrm{dW}\right| \\
&\le 4c_{BDG}\mathbb{E}\left(\int_{\tau_a}^{\tau_b}\left(\sum_{k=1}^{\infty}|\left\langle|\widetilde{v}|^2\widetilde{v},\mathcal{R}\sigma_1(U)e_k\right\rangle|\right)^2 \mathrm{ds}\right)^{1/2} \\
&\le 4c_{BDG}\mathbb{E}\biggl(\int_{\tau_a}^{\tau_b}\sqrt{2}\eta^2\||\widetilde{v}|\nabla_H\widetilde{v}\|_{L^2}^2\|\widetilde{v}\|_{L^4}^4+c\|\widetilde{v}\|_{L^4}^6\|\nabla_H\overline{v}\|_{L^2}\|\Delta_H\overline{v}\|_{L^2} \\
&\qquad\qquad +\|\widetilde{v}\|_{L^4}^8+\|\widetilde{v}\|_{L^4}^6\|\nabla_H\overline{v}\|_{L^2}^2+1\,\mathrm{ds}\biggr)^{1/2} \\
&\le \frac{1}{2}\mathbb{E}\sup_{s\in[\tau_a,\tau_b]}\|\widetilde{v}\|_{L^4}^4+16c_{BDG}^2\eta^2\mathbb{E}\int_{\tau_a}^{\tau_b}\||\widetilde{v}|\nabla_H\widetilde{v}\|_{L^2}^2\,\mathrm{ds} \\
&\quad + c\mathbb{E}\int_{\tau_a}^{\tau_b}\|\widetilde{v}\|_{L^4}^2\|\nabla_H\overline{v}\|_{L^2}\|\Delta_H\overline{v}\|_{L^2}+\|\widetilde{v}\|_{L^4}^4+\|\widetilde{v}\|_{L^4}^2\|\nabla_H\overline{v}\|_{L^2}^2+1\,\mathrm{ds} \\
&\le \frac{1}{2}\mathbb{E}\sup_{s\in[\tau_a,\tau_b]}\|\widetilde{v}\|_{L^4}^4+16c_{BDG}^2\eta^2\mathbb{E}\int_{\tau_a}^{\tau_b}\||\widetilde{v}|\nabla_H\widetilde{v}\|_{L^2}^2\,\mathrm{ds}+\frac{\overline{\varepsilon}}{2}\mathbb{E}\int_{\tau_a}^{\tau_b}\|\Delta_H\overline{v}\|_{L^2}^2\,\mathrm{ds} \\
&\quad + c_{\overline{\varepsilon}}\mathbb{E}\int_{\tau_a}^{\tau_b}(1+\|\nabla_H\overline{v}\|_{L^2}^2)(1+\|\widetilde{v}\|_{L^4}^4)\,\mathrm{ds}\,.
\end{aligned}
$$

Collecting the above, we deduce

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[\tau_a,\tau_b]}\|\widetilde{v}\|_{L^4}^4+4\left(\nu-\varepsilon-\eta^2\left(4c_{BDG}^2+\frac{3}{2}\right)\right)\int_{\tau_a}^{\tau_b}\||\widetilde{v}|\nabla_H\widetilde{v}\|_{L^2}^2\,\mathrm{ds}\right] \\
&\quad\le \mathbb{E}\left[c\|\widetilde{v}(\tau_a)\|_{L^4}^4+\overline{\varepsilon}\int_{\tau_a}^{\tau_b}\|\Delta_H\overline{v}\|_{L^2}^2\,\mathrm{ds}\right] \\
&\qquad\quad + c_{\varepsilon,\overline{\varepsilon}}\mathbb{E}\int_{\tau_a}^{\tau_b}(1+\|U\|_{L_z^2H_{xy}^1}^2)(1+\|\widetilde{v}\|_{L^4}^4)\,\mathrm{ds}+c\left(\int_{\tau_a}^{\tau_b}\left\|\widetilde{f}_v\right\|_{L^4}^2\mathrm{ds}\right)^2.
\end{aligned}
$$

Next, multiply (4.4) by $b>0$ precisely determinetd later and add it to the estimate above to obtain

$$
\begin{aligned}
&\mathbb{E}\left[\frac{b}{2}\sup_{s\in[\tau_a,\tau_b]}\|(-\Delta_H)^{1/2}\overline{v}\|_{L^2}^2+2b\left(\nu-\widetilde{\varepsilon}-\eta^2\left(\frac{1}{2}+2c_{BDG}^2\right)\right)\int_{\tau_a}^{\tau_b}\|\Delta_H\overline{v}\|_{L^2}^2\,\mathrm{ds}\right] \\
&\quad+\mathbb{E}\left[\sup_{s\in[\tau_a,\tau_b]}\|\widetilde{v}\|_{L^4}^4+4\left(\nu-\varepsilon-\eta^2\left(4c_{BDG}^2+\frac{3}{2}\right)\right)\int_{\tau_a}^{\tau_b}\||\widetilde{v}|\nabla_H\widetilde{v}\|_{L^2}^2\,\mathrm{ds}\right] \\
&\le cb\mathbb{E}\left[\|(-\Delta_H)^{1/2}\overline{v}(\tau_a)\|_{L^2}^2+\int_{\tau_a}^{\tau_b}(1+\|U\|_{L^2}^2\|U\|_{L_z^2H_{xy}^1}^2)\|(-\Delta_H)^{1/2}\overline{v}\|_{L^2}^2\,\mathrm{ds}\right] \\
&\quad+c_{\widetilde{\varepsilon}}b\mathbb{E}\left[\int_{\tau_a}^{\tau_b}1+\|U\|_{L_z^2H_{xy}^1}^2+\left\|\overline{f_v}\right\|_{L^2}^2+\||\widetilde{v}|\nabla_H\widetilde{v}\|_{L^2}^2\,\mathrm{ds}\right]+\overline{\varepsilon}\mathbb{E}\int_{\tau_a}^{\tau_b}\|\Delta_H\overline{v}\|_{L^2}^2\,\mathrm{ds} \\
&\quad+c\mathbb{E}\|\widetilde{v}(\tau_a)\|_{L^4}^4+c\mathbb{E}\int_{\tau_a}^{\tau_b}(1+\|U\|_{L_z^2H_{xy}^1}^2)(1+\|\widetilde{v}\|_{L^4}^4)\,\mathrm{ds}+c\left(\int_{\tau_a}^{\tau_b}\left\|\widetilde{f}_v\right\|_{L^4}^2\mathrm{ds}\right)^2.
\end{aligned}
$$

With $b = \frac{4\nu\varepsilon}{c_{\widetilde{\varepsilon}}}$ and $\overline{\varepsilon} = 2\nu b\widetilde{\varepsilon}$, the above reads as

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[\tau_a,\tau_b]}\|(-\Delta_H)^{1/2}\overline{v}\|_{L^2}^2 + 2b\left(\nu - 2\widetilde{\varepsilon} - \eta^2\left(\frac{1}{2}+2c_{BDG}^2\right)\right)\int_{\tau_a}^{\tau_b}\|\Delta_H\overline{v}\|_{L^2}^2\,\mathrm{ds}\right] \\
&\quad + \mathbb{E}\left[\sup_{s\in[\tau_a,\tau_b]}\|\widetilde{v}\|_{L^4}^4 + 4\left(\nu - 2\varepsilon - \eta^2\left(4c_{BDG}^2 - \frac{3}{2}\right)\right)\int_{\tau_a}^{\tau_b}\||\widetilde{v}|\nabla_H\widetilde{v}\|_{L^2}^2\,\mathrm{ds}\right] \\
&\le c\mathbb{E}\left[\|(-\Delta_H)^{1/2}\overline{v}(\tau_a)\|_{L^2}^2 + \|\widetilde{v}(\tau_a)\|_{L^4}^4 + \left(\int_{\tau_a}^{\tau_b}\left\|\widetilde{f}_v\right\|_{L^4}^2\mathrm{ds}\right)^2 + \int_{\tau_a}^{\tau_b}\|\overline{f_v}\|_{L^2}^2\,\mathrm{ds}\right] \\
&\quad + c\mathbb{E}\int_{\tau_a}^{\tau_b}1 + (1+\|U\|_{L^2}^2\|U\|_{L_z^2H_{xy}^1}^2)\|(-\Delta_H)^{1/2}\overline{v}\|_{L^2}^2\,\mathrm{ds} \\
&\quad + c\mathbb{E}\int_{\tau_a}^{\tau_b}\|U\|_{L_z^2H_{xy}^1}^2 + (1+\|U\|_{L_z^2H_{xy}^1}^2)\|\widetilde{v}\|_{L^4}^4\,\mathrm{ds},
\end{aligned}$$

Choosing ε and $\tilde{\varepsilon}$ sufficiently small, we may apply the stochastic Gronwall lemma (Lemma A.2) to get

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t\wedge\tau_K^{w,4}\wedge\tau_N]}\|\widetilde{v}\|_{L^4}^4 + \mathbb{E}\sup_{s\in[0,t\wedge\tau_K^{w,4}\wedge\tau_N]}\|(-\Delta_H)^{1/2}\overline{v}\|_{L^2}^2 \\
&\quad + \mathbb{E}\int_0^{t\wedge\tau_K^{w,4}\wedge\tau_N}\||\widetilde{v}|\nabla_H\widetilde{v}\|_{L^2}^2 + \|\Delta_H\overline{v}\|_{L^2}^2\,\mathrm{ds} \\
&\le C_{t,K}\mathbb{E}\left[\|\widetilde{v}(0)\|_{L^4}^4 + \|(-\Delta_H)^{1/2}\overline{v}(0)\|_{L^2}^2 + 1 + \int_0^{t\wedge\tau_K^{w,4}\wedge\tau_N}\|U\|_{L_z^2H_{xy}^1}^2\,\mathrm{ds}\right] \\
&\quad + C_{t,K}\mathbb{E}\left[\left(\int_0^{t\wedge\tau_K^{w,4}\wedge\tau_N}\left\|\widetilde{f}_v\right\|_{L^4}^2\mathrm{ds}\right)^2 + \int_0^{t\wedge\tau_K^{w,4}\wedge\tau_N}\|\overline{f_v}\|_{L^2}^2\,\mathrm{ds}\right].
\end{aligned}$$

Clearly, the right-hand side of the above estimate is uniformly bounded w.r.t. N. Passing to the limit w.r.t. $N\to\infty$, we observe

$$\begin{aligned}
&\sup_{s\in[0,t\wedge\tau_K^{w,4}\wedge\xi)}\|\widetilde{v}\|_{L^4}^4 + \sup_{s\in[0,t\wedge\tau_K^{w,4}\wedge\xi)}\|(-\Delta_H)^{1/2}\overline{v}\|_{L^2}^2 \\
&\qquad + \int_0^{t\wedge\tau_K^{w,4}\wedge\xi}\||\widetilde{v}|\nabla_H\widetilde{v}\|_{L^2}^2 + \|\Delta_H\overline{v}\|_{L^2}^2\,\mathrm{ds} < \infty
\end{aligned}\tag{4.7}$$

almost surely. Recalling $\tau_K^{w,4}\to\infty$ a.s. as $K\to\infty$ and (4.7), we deduce

$$\begin{aligned}
&\sup_{s\in[0,t\wedge\xi)}\|\widetilde{v}\|_{L^4}^4 + \sup_{s\in[0,t\wedge\xi)}\|(-\Delta_H)^{1/2}\overline{v}\|_{L^2}^2 \\
&\qquad + \int_0^{t\wedge\xi}\||\widetilde{v}|\nabla_H\widetilde{v}\|_{L^2}^2 + \|(-\Delta_H)^{1/2}\overline{v}\|_{L^2}^2\,\mathrm{ds} < \infty
\end{aligned}$$

almost surely. The proof is concluded by Lemma A.1 similarly as in the proof of Lemma 4.2. □

Lemma 4.5 (improved L^4 bound for $\widetilde{v}$)**.** *Let $q\ge 4$ and $\widetilde{v}_0\in L^q(\Omega;L^4(M))$. Let $\widetilde{f}_v\in L^q(\Omega;L^2_{loc}(0,\infty;L^4(M)))$. Then the stopping time $\tau_K^{\widetilde{v},q}$ defined for $K\in\mathbb{N}$ and $q\in[4,\infty)$*

by

$$\tau_K^{\widetilde{v},q} = \inf\Bigg\{ s \geq 0 \mid \sup_{r\in[0,s\wedge\xi)} \|\widetilde{v}\|_{L^4}^q + \int_0^{s\wedge\xi} \|\widetilde{v}\|_{L^4}^{q-4} \, \||\widetilde{v}|\nabla_H \widetilde{v}\|_{L^2}^2 \,\mathrm{dr} + \left(\int_0^{s\wedge\xi} \||\widetilde{v}|\nabla_H \widetilde{v}\|_{L^2}^2 \,\mathrm{ds} \right)^{q/2} \geq K \Bigg\}$$

satisfies $\tau_K^{\widetilde{v},q} \to \infty$ $\mathbb{P}$*-a.s. as* $K \to \infty$.

Proof. Recalling the cancellation property of the nonlinear term, the Itô formula applied to $\|\widetilde{v}\|_{L^4}^q$ yields

$$\begin{aligned}
\mathrm{d}\|\widetilde{v}\|_{L^4}^q &+ q\nu\|\widetilde{v}\|_{L^4}^{q-4} \left(\||\widetilde{v}|\nabla_H \widetilde{v}\|_{L^2}^2 + 2\||\widetilde{v}|\nabla_H |\widetilde{v}|\|_{L^2}^2 \right) \mathrm{dt} \\
&\leq -q\|\widetilde{v}\|_{L^4}^{q-4} \left\langle |\widetilde{v}|^2\widetilde{v}, \mathcal{R}F_v(U) - \widetilde{v}\cdot\nabla_H \overline{v} + N(\widetilde{v}) \right\rangle \mathrm{dt}, \\
&\quad + \frac{q(q-4)}{2}\|\widetilde{v}\|_{L^4}^{q-8} \left(\sum_{k=1}^\infty \left\langle |\widetilde{v}|^2\widetilde{v}, \mathcal{R}\sigma_1(U)e_k \right\rangle \right)^2 \mathrm{dt} \\
&\quad + \frac{3q}{2}\|\widetilde{v}\|_{L^4}^{q-4} \sum_{k=1}^\infty \left\langle |\widetilde{v}|^2, (\mathcal{R}\sigma_1(U)e_k)^2 \right\rangle \mathrm{dt} + q\|\widetilde{v}\|_{L^4}^{q-4} \left\langle |\widetilde{v}|^2\widetilde{v}, \mathcal{R}\sigma_1(U)\,\mathrm{dW} \right\rangle
\end{aligned}$$

almost surely. Nearly all the terms can be handled as in the previous proof, only the correction term requires further treatment. Due to Lemma 4.4, we can now handle the mixed term $\|\Delta_H \overline{v}\|_{L^2}^2 \|\widetilde{v}\|_{L^4}^q$. Recalling (4.5), we have

$$\sum_{k=1}^\infty \left\langle |\widetilde{v}|^2, (\mathcal{R}\sigma_1(U)e_k)^2 \right\rangle \leq (\eta^2+\varepsilon)\||\widetilde{v}|\nabla_H \widetilde{v}\|_{L^2}^2 + c(1+\|\Delta_H \overline{v}\|_{L^2}^2)(1+\|\widetilde{v}\|_{L^4}^4),$$

and, from (2.48), (2.43) and (2.45), we deduce

$$\begin{aligned}
&\sum_{k=1}^\infty \left| \left\langle |\widetilde{v}|^2\widetilde{v}, \mathcal{R}\sigma_1(U)e_k \right\rangle \right| \\
&\quad\leq \eta\||\widetilde{v}|\nabla_H \widetilde{v}\|_{L^2}\|\widetilde{v}\|_{L^4}^2 \\
&\qquad + c(\|\widetilde{v}\|_{L^4}^3 \|\nabla_H \overline{v}\|_{L^2}^{1/2} \|\Delta_H \overline{v}\|_{L^2}^{1/2} + \|\widetilde{v}\|_{L^4}^4 + \|\widetilde{v}\|_{L^4}^3 \|\nabla_H \overline{v}\|_{L^2} + 1) \\
&\quad\leq \eta\||\widetilde{v}|\nabla_H \widetilde{v}\|_{L^2}\|\widetilde{v}\|_{L^4}^2 + c(1+\|\Delta_H \overline{v}\|_{L^2})(1+\|\widetilde{v}\|_{L^4}^4).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{q(q-4)}{2}\|\widetilde{v}\|_{L^4}^{q-8} \left(\sum_{k=1}^\infty \left\langle |\widetilde{v}|^2\widetilde{v}, \mathcal{R}\sigma_1(U)e_k \right\rangle \right)^2 + \frac{3q}{2}\|\widetilde{v}\|_{L^4}^{q-4} \sum_{k=1}^\infty \left\langle |\widetilde{v}|^2, (\mathcal{R}\sigma_1(U)e_k)^2 \right\rangle \\
&\leq \frac{q(q-1)}{2}\eta^2 \|\widetilde{v}\|_{L^4}^{q-4} \||\widetilde{v}|\nabla_H \widetilde{v}\|_{L^2}^2 + c(1+\|\Delta_H \overline{v}\|_{L^2}^2)(1+\|\widetilde{v}\|_{L^4}^q).
\end{aligned}$$

Dealing with the remaining terms as in Lemma 4.4 and using the stochastic Gronwall lemma (Lemma A.2), we obtain

$$\sup_{s\in[0,t\wedge\xi)} \|\widetilde{v}\|_{L^4}^q + \int_0^{t\wedge\xi} \|\widetilde{v}\|_{L^4}^{q-4} \, \||\widetilde{v}|\nabla_H \widetilde{v}\|_{L^2}^2 \,\mathrm{ds} < \infty \quad \text{a.s.}$$

The proof is concluded by Lemma A.1 similarly as in the proof of Lemma 4.2. □

Analogously, adapting the estimates of the stochastic terms in Lemma 4.4, we may also deduce higher integrability in the probability space for $\overline{v}$. The proof is omitted.

Lemma 4.6 (improved H^1 bound for $\overline{v}$)**.** *Let $q \geq 2$ and $\overline{v_0} \in L^q(\Omega; H^1_0(G))$. Let $\overline{f_v} \in L^q(\Omega; L^2_{loc}(0, \infty; L^2(G)))$. Then the stopping time $\tau_K^{\nabla_H \overline{v}, q}$ defined for $K \in \mathbb{N}$ by*

$$\tau_K^{\nabla_H \overline{v}, q} = \inf \left\{ s \geq 0 \mid \sup_{r \in [0, s \wedge \xi)} \|\overline{v}\|^q_{H^1} + \int_0^{s \wedge \xi} \|\nabla_H \overline{v}\|^{q-2}_{L^2} \|\Delta_H \overline{v}\|^2_{L^2} \, \mathrm{dr} \right. \\ \left. + \left(\int_0^{s \wedge \xi} \|\Delta_H \overline{v}\|^2_{L^2} \, \mathrm{ds} \right)^{q/2} \geq K \right\}$$

satisfies $\tau_K^{\nabla_H \overline{v}, q} \to \infty$ $\mathbb{P}$-a.s. as $K \to \infty$.

4.2 Estimate for $\|U\|^p_{L^q}$

Before we proceed to estimates in L^q, we establish an estimate on the pressure. We emphasize that the estimate is possible by (2.49) since then there is no noise present in the equation for the pressure and the desired bound can thus be obtained using a standard (and essentially deterministic) argument.

Lemma 4.7 (Pressure bound)**.** *Let $\tau_K^{\nabla_H p}$ be the stopping time defined for $K \in \mathbb{N}$ by*

$$\tau_K^{\nabla_H p} = \inf \left\{ s \geq 0 \mid \int_0^{s \wedge \xi} \|\nabla_H p_s\|^2_{L^2} \, \mathrm{dr} \geq K \right\}$$

satisfies $\tau_K^{\nabla_H p} \to \infty$ $\mathbb{P}$-a.s. as $K \to \infty$.

Proof. Let $K, N \in \mathbb{N}$. Applying $1 - P_G$ to (2.28) and recalling (2.49), we obtain

$$\frac{1}{\rho_0} \nabla_H p_s = (1 - P_G)(\nu \Delta_H \overline{v} - \overline{v} \cdot \nabla_H \overline{v} - N(\widetilde{v}) + \mathcal{A} F_v(U)).$$

Using the bounds

$$\|\overline{v} \cdot \nabla_H \overline{v}\|^2_{L^2} \leq c \|\Delta_H \overline{v}\|^2_{L^2} \|\nabla_H \overline{v}\|^2_{L^2}, \quad \|N(\widetilde{v})\|^2_{L^2} \leq c \||\widetilde{v}| \nabla_H \widetilde{v}\|^2_{L^2}, \\ \|\mathcal{A} F_v(U)\|^2 \leq c \left(\|f\|^2_{L^2} + \|U\|^2_{L^2_z H^1_{xy}} \right),$$

integrating from 0 to $s \wedge \tau_N \wedge \tau_K^{w,2} \wedge \tau_K^{\widetilde{v},2} \wedge \tau_K^{\nabla_H \overline{v},2} \wedge \tau_K^{\nabla_H \overline{v},4}$, passing to the limit w.r.t. $N \to \infty$ and recalling the convergences of the stopping times established above, we get $\int_0^{s \wedge \xi} \|\nabla_H p_s\|^2_{L^2} \, \mathrm{dr} < \infty$ for a.a. $s \geq 0$. The claim follows by Lemma A.1. □

Proposition 4.8 (L^q bound for v)**.** *Let $2 \leq p \leq q < \infty$, $U_0 \in L^{2p}\left(\Omega; L^2_z H^1_{xy}\right) \cap L^p\left(\Omega; L^q\right)$ and $f_v \in L^{2p}\left(\Omega; L^2(0, t; H^1_z L^2_{xy})\right) \cap L^p\left(\Omega; L^2(0, t; L^q)\right)$. Then the stopping time $\tau_K^{v,q,p}$ defined for $K \in \mathbb{N}$ by*

$$\tau_K^{v,q,p} = \inf \left\{ s \geq 0 \mid \sup_{r \in [0, s \wedge \xi)} \|v\|^p_{L^q} \geq K \right\}$$

satisfies $\tau_K^{v,q,p} \to \infty$ $\mathbb{P}$-a.s. as $K \to \infty$.

Proof. Assume for the moment that

$$\mathbb{E} \sup_{s \in [0, t \wedge \tau_N]} \|U\|^p_{L^q} < \infty \tag{4.8}$$

holds for all $N \in \mathbb{N}$ and let $A(v) = 1 + \|v\|_{L^q}^q$. The Itô formula applied to $A(\cdot)^{p/q}$ and the cancellation property of the nonlinear term yields

$$\begin{aligned}
&\mathrm{d}A(v)^{p/q} + p\nu A(v)^{(p-q)/q}\left[\||v|^{(q-2)/2}\nabla_H v\|_{L^2}^2 + (q-2)\||v|^{(q-2)/2}\nabla_H |v|\|_{L^2}^2\right]\mathrm{dt}\\
&\leq -pA(v)^{(p-q)/q}\left\langle |v|^{q-2}v, F_v(U) - \nabla_H p_s\right\rangle \mathrm{dt}\\
&\quad + \frac{p}{2}(q-1)\sum_{k=1}^{\infty} A(v)^{(p-q)/q}\left\langle |v|^{q-2}, (\sigma_1(U)e_k)^2\right\rangle \mathrm{dt}\\
&\quad + \frac{p}{2}(p-q)\sum_{k=1}^{\infty} A(v)^{(p-2q)/q}\left\langle |v|^{q-2}v, \sigma_1(U)e_k\right\rangle^2 \mathrm{dt}\\
&\quad + pA(v)^{(p-q)/q}\left\langle |v|^{q-2}v, \sigma_1(U)\,\mathrm{dW}\right\rangle.
\end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
&\left|\left\langle |v|^{q-2}v, \beta_T g\int_{\cdot}^{0}\nabla_H T(x,y,z')\,\mathrm{dz}'\right\rangle\right|\\
&\quad\leq c\left\||v|^{(q-2)/2}\int_{\cdot}^{0}T(x,y,z')\,\mathrm{dz}\right\|_{L^2}^2 + \varepsilon\||v|^{(q-2)/2}\nabla_H v\|_{L^2}^2\\
&\quad\leq c\left\|\int_{\cdot}^{0}T(x,y,z')\,\mathrm{dz}\right\|_{L^{2q/(q+2)}}^2\|v\|_{L^q}^{q-2} + \varepsilon\||v|^{(q-2)/2}\nabla_H v\|_{L^2}^2\\
&\quad\leq c\,\|U\|_{L_z^2H_{xy}^1}^2(1+\|v\|_{L^q}^q) + \nu\varepsilon\||v|^{(q-2)/2}\nabla_H v\|_{L^2}^2.
\end{aligned}$$

Using $\|v\|_{L^q}^{q'} \leq A(v)^{q'/q}$ for $q' \geq 0$, we get

$$\begin{aligned}
A(v)^{(p-q)/q}|\left\langle |v|^{q-2}v, f_v\right\rangle| &\leq A(v)^{(p-q)/q}\|v\|_{L^q}^{q-1}\|f_v\|_{L^q}\\
&\leq A(v)^{(p-1)/q}\|f_v\|_{L^q}.
\end{aligned}$$

Hence, for $0 \leq \tau_a \leq \tau_b$ stopping times specified later, we obtain

$$\begin{aligned}
&\int_{\tau_a}^{\tau_b} A(v)^{(p-q)/q}|\left\langle |v|^{q-2}v, F_v(U)\right\rangle|\,\mathrm{ds}\\
&\leq \nu\varepsilon\int_{\tau_a}^{\tau_b} A(v)^{(p-q)/q}\||v|^{(q-2)/2}\nabla_H v\|_{L^2}^2\,\mathrm{ds} + \frac{1}{4}\sup_{s\in[\tau_a,\tau_b]}A(v)^{p/q}\\
&\quad + c_\varepsilon\int_{\tau_a}^{\tau_b}(1+\|U\|_{L_z^2H_{xy}^1}^2)(1+A(v)^{p/q})\,\mathrm{ds} + c\left(\int_{\tau_a}^{\tau_b}\|f_v\|_{L^q}^2\,\mathrm{ds}\right)^{p/2}.
\end{aligned}$$

Before we estimate the pressure term, we recall that the surface pressure p_s is independent of z and that we may shift it by a constant so that $\int_G p_s(t,x,y)\,\mathrm{d}(x,y) = 0$. Therefore, we get

$$\begin{aligned}
|\left\langle |v|^{q-2}v, \nabla_H p_s\right\rangle| &\leq (q-1)|\left\langle |v|^{q-2}|\nabla_H v|, p_s\right\rangle|\\
&\leq (q-1)\||v|^{(q-2)/2}\nabla_H v\|_{L^2}\|v^{(q-2)/2}\|_{L_z^2L_{xy}^{2q/(q-2)}}\|p_s\|_{L_z^\infty L_{xy}^q}\\
&\leq \nu\varepsilon\||v|^{(q-2)/2}\nabla_H v\|_{L^2}^2 + c_{q,\varepsilon}\|\nabla_H p_s\|_{L^2}^2\|v\|_{L^q}^{q-2},
\end{aligned}$$

with $c_{q,\varepsilon} \approx q^{3/2}\varepsilon^{-1}$. The second part of the correction term is non-positive and thus it can be dropped in the estimates. For the first part, we use (2.41), (2.43) and (2.45) to get

$$
\begin{aligned}
&\sum_{k=1}^{\infty} \left\langle |v|^{q-2}, (\sigma_1(U)e_k)^2 \right\rangle \\
&\quad = \sum_{k=1}^{\infty} \left\langle |v|^{q-2}, (\Psi_k \cdot \nabla_H v + (\Phi_k - \Psi_k) \cdot \nabla_H \overline{v} + h_k(v))^2 \right\rangle \\
&\quad \leq \sum_{k=1}^{\infty} \left\langle |v|^{q-2}, (1 + \frac{\varepsilon}{q-2}) |\Psi_k \cdot \nabla_H v|^2 + c|(\Phi_k - \Psi_k) \cdot \nabla_H \overline{v}|^2 + c|h_k(v)|^2 \right\rangle \\
&\quad \leq (1 + \frac{\varepsilon}{q-1}) \eta^2 \| |v|^{(q-2)/2} \nabla_H v\|_{L^2}^2 + c \| |v|^{q-2} \|_{L^{q/(q-2)}} \| |\nabla_H \overline{v}|^2 \|_{L^{q/2}} \\
&\qquad + c \| |v|^{q-2} \|_{L^{q/(q-2)}} (1 + \| |v|^2 \|_{L^{q/2}}) \\
&\quad \leq (1 + \frac{\varepsilon}{q-1}) \eta^2 \| |v|^{(q-2)/2} \nabla_H v\|_{L^2}^2 + c(1 + \|v\|_{L^q}^q) + c \|\Delta_H \overline{v}\|_{L^2}^2 \|v\|_{L^q}^{q-2}.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\frac{p}{2}(q-1) A(v)^{(p-q)/q} \sum_{k=1}^{\infty} \left\langle |v|^{q-2}, (\sigma_1(U)e_k)^2 \right\rangle \\
&\quad \leq \frac{p}{2}(q - 1 + \varepsilon) \eta^2 A(v)^{(p-q)/q} \| |v|^{(q-2)/2} \nabla_H v\|_{L^2}^2 + c(A(v)^{p/q} + \|\Delta_H \overline{v}\|_{L^2}^2 A(v)^{(p-2)/q}).
\end{aligned}
$$

For $K \in \mathbb{N}$, let $\Upsilon_K = \tau_K^{w,2} \wedge \tau_K^{\nabla_H \overline{v},2} \wedge \tau_K^{\nabla_H p,2}$. Let $N \in \mathbb{N}$ and let $0 \leq \tau_a \leq \tau_b \leq t \wedge \tau_N \wedge \Upsilon_K$ be stopping times. With (2.41), (2.43) and (2.45) and the Young inequality, we deduce

$$
\begin{aligned}
&\sum_{k=1}^{\infty} \left\langle |v|^{q-1} |\sigma_1(U)e_k| \right\rangle^2 \\
&\quad \leq \left(\eta \| |v|^{(q-2)/2} \nabla_H v\|_{L^2} \|v\|_{L^q}^{q/2} + c \|\nabla_H \overline{v}\|_{L^q} \|v\|_{L^q}^{q-1} + c(1 + \|v\|_{L^q}^q) \right)^2 \\
&\quad \leq (1 + \frac{\varepsilon}{q-1}) \eta^2 \| |v|^{(q-2)/2} \nabla_H v\|_{L^2}^2 A(v) + c_q (A(v)^2 + \|\Delta_H \overline{v}\|_{L^2}^2 A(v)^{2-2/q}).
\end{aligned}
$$

Hence, employing the Burkholder-Davis-Gundy inequality (2.51) and assumptions (2.41) and (2.45), we get

$$
\begin{aligned}
&\mathbb{E} \sup_{s \in [\tau_a, \tau_b]} \left| \int_{\tau_a}^{s} A(v)^{(p-q)/q} \left\langle |v|^{q-2} v, \sigma_1(U) \right\rangle \mathrm{dW} \right| \\
&\quad \leq c_{BDG} \mathbb{E} \left(\int_{\tau_a}^{\tau_b} A(v)^{2(p-q)/q} \sum_{k=1}^{\infty} \left\langle |v|^{q-1} |\sigma_1(U)e_k| \right\rangle^2 \mathrm{ds} \right)^{1/2} \\
&\quad \leq c_{BDG} \mathbb{E} \Bigg(\left(1 + \frac{\varepsilon}{q-1}\right) \eta^2 \int_{\tau_a}^{\tau_b} A(v)^{p/q + (p-q)/q} \| |v|^{(q-2)/2} \nabla_H v\|_{L^2}^2 \mathrm{ds} \\
&\qquad\qquad + c_q \int_{\tau_a}^{\tau_b} A(v)^{2p/q} + \|\Delta_H \overline{v}\|_{L^2}^2 A(v)^{2(p-1)/q} \mathrm{ds} \Bigg)^{1/2} \\
&\quad \leq \frac{1}{4} \mathbb{E} \sup_{s \in [\tau_a, \tau_b]} A(v)^{p/q} + c_q \mathbb{E} \int_{\tau_a}^{\tau_b} (1 + \|\Delta_H \overline{v}\|_{L^2}^2) A(v)^{p/q} \mathrm{ds} \\
&\qquad + \left(1 + \frac{\varepsilon}{q-1}\right) c_{BDG}^2 \eta^2 \mathbb{E} \int_{\tau_a}^{\tau_b} A(v)^{(p-q)/q} \| |v|^{(q-2)/2} \nabla_H v\|_{L^2}^2 \mathrm{ds}.
\end{aligned}
$$

The claim follows by collecting the above and the stochastic Gronwall lemma (Lemma A.2) similarly as in the previous lemmata provided we establish (4.8).

To that end, we consider the Galerkin approximations $U^n = (v^n, T^n)$ from Section 3. Since $\Phi_{m,k} \in L^q(M)$ for all $k \in \mathbb{N}_0$, $m \in \mathbb{N}$ and $q \in [2, \infty]$, one has $v^n \in L^\infty(0, t; L^q)$.

Using similar estimates as above, boundedness of the operator $\mathcal{A}$ and the deterministic Gronwall lemma, we obtain

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]} A(v^n)^{p/q} \le \mathbb{E}A(v^n(0))^{p/q} + c_q \\ + c_q\mathbb{E}\left[\sup_{s\in[0,t]}\|U^n\|^p_{L^2_zH^1_{xy}} + \left(\int_0^t \|\Delta_H v^n\|^2_{L^2}\,\mathrm{ds}\right)^{p/2}\right] \\ + c_q\mathbb{E}\left[\left(\int_0^t \|f^n_v\|^2_{L^q}\,\mathrm{ds}\right)^{p/2} + \left(\int_0^t \|\nabla_H p^n_s\|^2_{L^2}\,\mathrm{ds}\right)^{p/2}\right].\end{aligned}$$

Since U^n in $L^p(\Omega; L^2(0,t; L^2_zH^2_{xy}))$ and $L^p(\Omega; L^\infty(0,t;H^1))$ can be controlled by Lemma 3.1, it remains to deal with the pressure term. First, we observe that $H^1_zH^1_{xy} \cap L^2_zH^2_{xy} \subset H^{3/4}_zH^{5/4}_{xy} \subset L^\infty$ by the mixed derivative theorem [47, Proposition 3.2]. Thus, since the operator $\mathcal{R}$ is bounded, we have

$$\begin{aligned}\int_0^t \||\widetilde{v}^n|\nabla_H \widetilde{v}^n\|^2_{L^2}\,\mathrm{ds} &\le c\sup_{s\in[0,t]}\|v^n\|^2_{L^2_zH^1_{xy}}\int_0^t \|v^n\|^2_{L^\infty}\,\mathrm{ds} \\ &\le c\sup_{s\in[0,t]}\|v^n\|^4_{L^2_zH^1_{xy}} + c\left(\int_0^t \|v^n\|^2_{L^2_zH^2_{xy}}\,\mathrm{ds}\right)^2 + c\left(\int_0^t \|v^n\|^2_{H^2_zL^2_{xy}}\,\mathrm{ds}\right)^2.\end{aligned}$$

The estimates in the proof of Lemma 4.7 and boundedness of $\mathcal{A}$ now imply

$$\begin{aligned}\int_0^t \|\nabla_H p^n_s\|^2_{L^2}\,\mathrm{ds} \le c\left(1+\sup_{s\in[0,t]}\|U^n\|^4_{L^2_zH^1_{xy}}\right) + c\left(\int_0^t \|v^n\|^2_{L^2_zH^2_{xy}}\,\mathrm{ds}\right)^2 \\ + c\left(\int_0^t \|v^n\|^2_{H^1_zH^1_{xy}}\,\mathrm{ds}\right)^2 + c\int_0^t \|f_v\|^2_{L^2}\,\mathrm{ds}.\end{aligned}$$

Hence, by the above and the Young inequality, we get

$$\begin{aligned}\mathbb{E}\sup_{s\in[0,t]} A(v^n)^{p/q} \le \mathbb{E}A(v^n(0))^{p/q} + c_{p,q} \\ + c_{p_q}\mathbb{E}\left[\sup_{s\in[0,t]}\|U^n\|^{2p}_{L^2_zH^1_{xy}} + \left(\int_0^t \|v^n\|^2_{L^2_zH^2_{xy}}\,\mathrm{ds}\right)^{p}\right] \\ + c_{p,q}\mathbb{E}\left[\left(\int_0^t \|v^n\|^2_{H^1_zH^1_{xy}}\,\mathrm{ds}\right)^{p} + \left(\int_0^t \|f^n_v\|^2_{L^q}\,\mathrm{ds}\right)^{p/2}\right].\end{aligned}$$

Now, (4.8) follows from the construction of the solution provided

$$U_0 \in L^{2p}\left(\Omega; L^2_zH^1_{xy}\right) \cap L^p\left(\Omega; L^q\right),\ f \in L^{2p}\left(\Omega; L^2(0,t; H^1_zL^2_{xy})\right) \cap L^p\left(\Omega; L^2(0,t;L^q)\right). \quad \square$$

Remark 4.9. We note that, after collecting the estimates in the above proof, the dissipation term $A(v)^{(p-q)/q)}\||v|^{(q-2)/2}\nabla_H v\|^2_{L^2}$ is present with coefficient

$$p\nu - \frac{p}{2}(q-1)\left(1+\frac{\varepsilon}{q-1}\right)\eta^2 - p\left(1+\frac{\varepsilon}{q-1}\right)c^2_{BDG}\eta^2.$$

Therefore, it is not possible to find $\eta > 0$ such that the estimates go through uniformly w.r.t. $q \in [2,\infty)$ for fixed p. This fact underlines the importance of the modified logarithmic Sobolev inequality, see Proposition 4.1, in the stochastic setting with transport noise. We also recall that the coefficient above also explains the additional smallness assumption on η in the global existence result.

We conclude this section with an $L^q(M)$-regularity result for the temperature.

Lemma 4.10. *Let* $2 \le p \le q < \infty$ *and* $U_0 \in L^{3p/2}\left(\Omega; L^2_z H^1_{xy}\right) \cap L^p\left(\Omega; L^q\right)$*. Let* $f_T \in L^{3p/2}\left(\Omega; L^2(0,t; H^1_z L^2_{xy})\right) \cap L^p\left(\Omega; L^2(0,t;L^q)\right)$*. Then the stopping time* $\tau_K^{T,q,p}$ *defined for* $K \in \mathbb{N}$ *by*

$$\tau_K^{T,q,p} = \inf\left\{ s \ge 0 \mid \sup_{r\in[0,s\wedge\xi)} \|T\|^p_{L^q} \ge K \right\}$$

satisfies $\tau_K^{T,q,p} \to \infty$ $\mathbb{P}$*-a.s. as* $K \to \infty$*.*

Proof. For $K \in \mathbb{N}$, let $\Upsilon_K = \tau_K^{w,2} \wedge \tau_K^{\nabla_H \overline{v},2} \wedge \tau_K^{v,q,2}$. Let $N \in \mathbb{N}$ and let $0 \le \tau_a \le \tau_b \le t \wedge \tau_N \wedge \Upsilon_K$ be stopping times. Similarly as above, we use (2.41) and (2.44) to deduce

$$\sum_{k=1}^{\infty} \left\langle |T|^{q-2}, (\sigma_2(U)e_k)^2 \right\rangle \\ \le (\eta^2+\varepsilon)\||T|^{(q-2)/2}\nabla_H T\|^2_{L^2} + (1+\|\Delta_H \overline{v}\|^2_{L^2} + \|v\|^2_{L^q})(1+\|T\|^q_{L^q})\,\mathrm{ds}\,.$$

Thus, by the Burkholder-Davis-Gundy ineuquality (2.51), we get

$$\begin{aligned} &q\mathbb{E}\sup_{s\in[\tau_a,\tau_b]}\left|\int_0^s \left\langle |T|^{q-1}T, \sigma_2(U)\right\rangle \mathrm{dW}\right| \\ &\le \mathbb{E}\left[\frac{1}{2}\sup_{s\in[\tau_a,\tau_b]}\|T\|^q_{L^q} + (q^2c^2_{BDG}\eta^2+\varepsilon)\int_{\tau_a}^{\tau_b}\||T|^{(q-2)/2}\nabla_H T\|^2_{L^2}\,\mathrm{ds}\right] \\ &\quad + c\mathbb{E}\int_{\tau_a}^{\tau_b}(1+\|\Delta_H \overline{v}\|^2_{L^2}+\|v\|^2_{L^q})(1+\|T\|^q_{L^q})\,\mathrm{ds}\,. \end{aligned}$$

With $\int_{\tau_a}^{\tau_b} | \left\langle |T|^{q-2}T, f_T\right\rangle | \,\mathrm{ds} \le c\left(\int_{\tau_a}^{\tau_b}\|f_T\|^2_{L^q}\right)^{q/2} + c\int_{\tau_a}^{\tau_b}\|T\|^q_{L^q}\,\mathrm{ds}$, the claim follows as in the previous proof by applying the Itô formula to $(1+\|T\|^q_{L^q})^{p/q}$, the stochastic Gronwall lemma (Lemma A.2) and Lemma A.1. □

4.3 Higher order estimates

Lemma 4.11 (L^∞ bound for v)**.** *The stopping time* $\tau_K^{v,\infty}$ *defined for* $K \in \mathbb{N}$ *by*

$$\tau_K^{v,\infty} = \inf\left\{ s \ge 0 \mid \int_0^{s\wedge\xi} \|v\|^2_{L^\infty}\,\mathrm{ds} \ge K \right\}$$

satisfies $\tau_K^{v,\infty} \to \infty$ $\mathbb{P}$*-a.s. as* $K \to \infty$*.*

Proof. We apply the Itô formula to $\log\left(e + \|(-\Delta_H)^{1/2}v\|^2_{L^2} + \|\partial_z v\|^2_{L^2} + \|\partial_z v\|^6_{L^6}\right)$. Denoting $A(t) = e + \|(-\Delta_H)^{1/2}v\|^2_{L^2} + \|\partial_z v\|^2_{L^2} + \|\partial_z v\|^6_{L^6}$, we obtain

$$\begin{aligned} &\mathrm{d}\log(A(t)) + \frac{2\nu}{A(t)}\|\Delta_H v\|^2_{L^2} + \frac{2\nu}{A(t)}\|\nabla_H \partial_z v\|^2_{L^2}\,\mathrm{dt} \\ &\quad + \frac{6\nu}{A(t)}\left[\||\partial_z v|^2\nabla_H \partial_z v\|^2_{L^2} + 4\||\partial_z v|^2\nabla_H |\partial_z v|\|^2_{L^2}\right]\mathrm{dt} \\ &\le \frac{2}{A(t)}\left\langle (-\Delta_H)^{1/2}v, (-\Delta_H)^{1/2}F_v(U) - (-\Delta_H)^{1/2}b(v,v)\right\rangle \mathrm{dt} \\ &\quad + \frac{1}{A(t)}\|(-\Delta_H)^{1/2}\sigma_1(U)\|^2_{L_2(\mathcal{U},L^2)}\,\mathrm{dt} + \frac{2}{A(t)}\left\langle (-\Delta_H)^{1/2}v, (-\Delta_H)^{1/2}\sigma_1(U)\,\mathrm{dW}\right\rangle \\ &\quad + \frac{2}{A(t)}\left\langle \partial_z v, \partial_z F_v(U) - \partial_z b(v,v)\right\rangle \mathrm{dt} \\ &\quad + \frac{1}{A(t)}\|\partial_z\sigma_1(U)\|^2_{L_2(\mathcal{U},L^2)}\,\mathrm{dt} + \frac{2}{A(t)}\left\langle \partial_z v, \partial_z\sigma_1(U)\,\mathrm{dW}\right\rangle \end{aligned}$$

$$
- \frac{6}{A(t)} \left\langle |\partial_z v|^4 \partial_z v, \partial_z F_v(U) + b(\partial_z v, v) \right\rangle \mathrm{dt} \\
+ \frac{15}{A(t)} \sum_{k=1}^{\infty} \left\langle |\partial_z v|^4, (\partial_z \sigma_1(U) e_k)^2 \right\rangle \mathrm{dt} + \frac{6}{A(t)} \left\langle |\partial_z v|^4 \partial_z v, \partial_z \sigma_1(U) \, \mathrm{dW} \right\rangle
$$

Integrating by parts, we have

$$
\begin{aligned}
\frac{6}{A(t)} \left| \left\langle |\partial_z v|^4 \partial_z v, b(\partial_z v, v) \right\rangle \right| &\leq \frac{c}{A(t)} \int_M |\partial_z v|^5 |(-\Delta_H)^{1/2} \partial_z v| |v| \, \mathrm{d(x,y,z)} \\
&\leq \frac{\varepsilon}{A(t)} \| |\partial_z v|^2 (-\Delta_H)^{1/2} \partial_z v \|_{L^2}^2 + \frac{c_\varepsilon}{A(t)} \|v\|_{L^\infty}^2 \|\partial_z v\|_{L^6}^6 \\
&\leq \frac{\varepsilon}{A(t)} \| |\partial_z v|^2 (-\Delta_H)^{1/2} \partial_z v \|_{L^2}^2 + c_\varepsilon \|v\|_{L^\infty}^2
\end{aligned}
$$

and

$$
\frac{2}{A(t)} \left| \langle \partial_z v, b(\partial_z v, v) \rangle \right| \leq \frac{c}{A(t)} \|\partial_z v\|_{L^4}^2 \|\nabla_H v\|_{L^2} \leq c.
$$

Similarly, since $v = 0 = \partial_z v$ on Γ_ℓ, the Hölder, Young and two-dimensional Gagliardo-Nirenberg inequalities yield

$$
\begin{aligned}
&\frac{2}{A(t)} \left| \left\langle (-\Delta_H)^{1/2} v, (-\Delta_H)^{1/2} b(v,v) \right\rangle \right| = \frac{2}{A(t)} \left| \langle \Delta_H v, v \cdot \nabla_H v + w(v) \partial_z v \rangle \right| \\
&\quad \leq \frac{c}{A(t)} \|\Delta_H v\|_{L^2} \left(\|v\|_{L^\infty} \|\nabla_H v\|_{L^2} + \|\mathrm{div}_H v\|_{L_z^2 L_{xy}^3} \|\partial_z v\|_{L_z^2 L_{xy}^6} \right) \\
&\quad \leq \frac{c}{A(t)} \|\Delta_H v\|_{L^2} \left(\|v\|_{L^\infty} \|\nabla_H v\|_{L^2} + \|\Delta_H v\|_{L^2}^{1/3} \|\nabla_H v\|_{L^2}^{2/3} \|\partial_z v\|_{L^6} \right) \qquad (4.9) \\
&\quad \leq \frac{\varepsilon}{A(t)} \|\Delta_H v\|_{L^2}^2 + \frac{c_\varepsilon}{A(t)} (\|v\|_{L^\infty}^2 \|\nabla_H v\|_{L^2}^2 + \|\partial_z v\|_{L^6}^6 + \|\nabla_H v\|_{L^2}^4) \\
&\quad \leq \frac{\varepsilon}{A(t)} \|\Delta_H v\|_{L^2}^2 + c_\varepsilon (\|v\|_{L^\infty}^2 + 1 + \|\nabla_H v\|_{L^2}^2).
\end{aligned}
$$

The linear terms are straightforward. We estimate

$$
\begin{aligned}
&\frac{2}{A(t)} \left| \left\langle (-\Delta_H)^{1/2} v, (-\Delta_H)^{1/2} F_v(U) \right\rangle \right| \\
&\qquad \leq \frac{\varepsilon}{A(t)} \|\Delta_H v\|_{L^2}^2 + \frac{c_\varepsilon}{A(t)} \left(\|T\|_{L_z^2 H_{xy}^1}^2 + \|f_v\|_{L^2}^2 + \left\| (-\Delta_H)^{1/2} v \right\|_{L^2}^2 \right) \\
&\qquad \leq \frac{\varepsilon}{A(t)} \|\Delta_H v\|_{L^2}^2 + c_\varepsilon \left(\|T\|_{L_z^2 H_{xy}^1}^2 + \|f_v\|_{L^2}^2 + 1 \right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{2}{A(t)} \langle \partial_z v, \partial_z F_v(U) \rangle &\leq \frac{c}{A(t)} \left(\|\partial_z v\|_{L^2}^2 + \|\partial_z v\|_{L^2} \|\partial_z f_v\|_{L^2} + \|\partial_z v\|_{L^2} \|T\|_{L_z^2 H_{xy}^1} \right) \\
&\leq c \left(1 + \|\partial_z f_v\|_{L^2} + \|T\|_{L_z^2 H_{xy}^1} \right),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{6}{A(t)} \left| \left\langle |\partial_z v|^4 \partial_z v, \partial_z F_v(U) \right\rangle \right| \\
&\qquad \leq \frac{\varepsilon}{A(t)} \| |\partial_z v|^2 \nabla_H \partial_z v \|_{L^2}^2 + \frac{c_\varepsilon}{A(t)} \left(1 + \|T\|_{L^6}^2 + \|\partial_z f_v\|_{L^6} \right) (1 + \|\partial_z v\|_{L^6}^6) \\
&\qquad \leq \frac{\varepsilon}{A(t)} \| |\partial_z v|^2 \nabla_H \partial_z v \|_{L^2}^2 + c_\varepsilon \left(1 + \|T\|_{L^6}^2 + \|\partial_z f_v\|_{L^6} \right),
\end{aligned}
$$

where we used

$$
\left| \left\langle |\partial_z v|^4 \partial_z v, \int_\cdot^0 \nabla_H T(x,y,z') \, \mathrm{dz'} \right\rangle \right| \leq \| |\partial_z v|^2 \nabla_H \partial_z v \|_{L^2} \|\partial_z v\|_{L^6}^2 \|T\|_{L^6}.
$$

For $K \in \mathbb{N}$, let $\Upsilon_K = \tau_K^{v,132,8/3}$. Let $N \in \mathbb{N}$ and let $0 \leq \tau_a \leq \tau_b \leq t \wedge \tau_N \wedge \Upsilon_K$ be stopping times. By the sub-linear growth of σ_1 (2.33) and the Burkholder-Davis-Gundy inequality (2.51), we deduce

$$
\begin{aligned}
&2\mathbb{E}\sup_{s\in[\tau_a,\tau_b]}\left|\int_{\tau_a}^{\tau_b}\frac{1}{A(s)}\left\langle(-\Delta_H)^{1/2}v,(-\Delta_H)^{1/2}\sigma_1(U)\,\mathrm{dW}\right\rangle\right| \\
&\quad\le c\mathbb{E}\left(\int_{\tau_a}^{\tau_b}\frac{\|(-\Delta_H)^{1/2}v\|_{L^2}^2}{A(s)}\frac{\|(-\Delta_H)^{1/2}\sigma_1(U)\|_{L_2(U,L^2)}^2}{A(s)}\,\mathrm{ds}\right)^{1/2} \\
&\quad\le c\mathbb{E}\left(\int_{\tau_a}^{\tau_b}\frac{1+\|\Delta_H v\|_{L^2}^2}{A(s)}\,\mathrm{ds}\right)^{1/2} \\
&\quad\le \varepsilon\mathbb{E}\int_{\tau_a}^{\tau_b}\frac{\|\Delta_H v\|_{L^2}^2}{A(s)}\,\mathrm{ds}+c_\varepsilon\mathbb{E}\int_{\tau_a}^{\tau_b}\frac{1}{A(s)}\,\mathrm{ds}+c_\varepsilon \\
&\quad\le \varepsilon\mathbb{E}\int_{\tau_a}^{\tau_b}\frac{\|\Delta_H v\|_{L^2}^2}{A(s)}\,\mathrm{ds}+c_\varepsilon
\end{aligned}
$$

and, similarly, we estimate the first correction term by

$$
\frac{1}{A(t)}\|(-\Delta_H)^{1/2}\sigma_1(U)\|_{L_2(\mathcal{U},L^2)}^2\le\eta^2\frac{1}{A(t)}\|\Delta_H v\|_{L^2}^2+c.
$$

In the same way, we obtain

$$
2\mathbb{E}\sup_{s\in[\tau_a,\tau_b]}\left|\int_{\tau_a}^{\tau_b}\frac{1}{A(s)}\langle\partial_z v,\partial_z\sigma_1(U)\,\mathrm{dW}\rangle\right|\le\varepsilon\mathbb{E}\int_{\tau_a}^{\tau_b}\frac{\|\nabla_H\,\partial_z v\|_{L^2}^2}{A(s)}\,\mathrm{ds}+c_\varepsilon
$$

and

$$
\frac{1}{A(t)}\|(-\Delta_H)^{1/2}\sigma_1(U)\|_{L_2(\mathcal{U},L^2)}^2\le\eta^2\frac{1}{A(t)}\|\nabla_H\,\partial_z v\|_{L^2}^2+c.
$$

Recalling assumption on noise (2.46) and (2.43), we have

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\left|\left\langle|\partial_z v|^4\partial_z v,\partial_z\sigma_1(U)e_k\right\rangle\right|^2 \\
&\quad\le c\big(\||\partial_z v|^2\nabla_H\,\partial_z v\|_{L^2}\||\partial_z v|^3\|_{L^2}+\||\partial_z v|^5\|_{L^{6/5}}\|\nabla_H\,\overline{v}\|_{L^6}+\|\partial_z v\|_{L^6}^6 \\
&\qquad+\||\partial_z v|^5\|_{L^{6/5}}\|v\|_{L^6}+\||\partial_z v|^5\|_{L^1}\big)^2 \\
&\quad\le c\left(\||\partial_z v|^2\nabla_H\,\partial_z v\|_{L^2}^2\|\partial_z v\|_{L^6}^6+(1+\|\nabla_H\,\overline{v}\|_{L^2}^2+\|\nabla_H\,\overline{v}\|_{L^6}^2)(1+\|\partial_z v\|_{L^6}^{12})\right),
\end{aligned}
$$

and hence, by the Burkholder-Davis-Gundy inequality (2.51) and (4.3),

$$
\begin{aligned}
&6\mathbb{E}\sup_{s\in[\tau_a,\tau_b]}\left|\int_0^s\frac{1}{A(s)}\left\langle|\partial_z v|^4\partial_z v,\partial_z\sigma_1(U)\right\rangle\mathrm{dW}\right| \\
&\quad\le c\mathbb{E}\left(\int_{\tau_a}^{\tau_b}\frac{1}{A(s)^2}\sum_{k=1}^{\infty}|\left\langle|\partial_z v|^4\partial_z v,\partial_z\sigma_1(U)e_k\right\rangle|^2\,\mathrm{ds}\right)^{1/2} \\
&\quad\le c\mathbb{E}\bigg(\int_{\tau_a}^{\tau_b}\frac{\||\partial_z v|^2\nabla_H\,\partial_z v\|_{L^2}^2\|\partial_z v\|_{L^6}^6}{A(s)^2} \\
&\qquad\qquad+\frac{(1+\|\partial_z v\|_{L^6}^{12})}{A(s)^2}(1+\|\nabla_H\,\overline{v}\|_{L^2}^2+\|\nabla_H\,\overline{v}\|_{L^6}^2)\,\mathrm{ds}\bigg)^{1/2} \\
&\quad\le c\mathbb{E}\left(\int_{\tau_a}^{\tau_b}\frac{\||\partial_z v|^2\nabla_H\,\partial_z v\|_{L^2}^2}{A(s)}+1+\|\Delta_H\,\overline{v}\|_{L^2}^2\,\mathrm{ds}\right)^{1/2} \\
&\quad\le\varepsilon\mathbb{E}\int_{\tau_a}^{\tau_b}\frac{\||\partial_z v|^2\nabla_H\,\partial_z v\|_{L^2}^2}{A(s)}\,\mathrm{ds}+c_\varepsilon\mathbb{E}\int_{\tau_a}^{\tau_b}1+\|\Delta_H\,\overline{v}\|_{L^2}^2\,\mathrm{ds}+c_\varepsilon.
\end{aligned}
$$

For the remaining correction term, we proceed similarly. By (4.3), (2.46), (2.43) and (2.45), we obtain

$$
\begin{aligned}
&\frac{15}{A(t)}\sum_{k=1}^{\infty}\left\langle|\partial_z v|^4,(\partial_z\sigma_1(U)e_k)^2\right\rangle\\
&\quad\le\left(15\eta^2+\varepsilon\right)\frac{\||\partial_z v|^2\nabla_H\,\partial_z v\|_{L^2}^2}{A(t)}+\frac{c_\varepsilon}{A(t)}\big(\||\partial_z v|^4\|_{L^{3/2}}\||\nabla_H\,\overline{v}|^2\|_{L^3}+\|\partial_z v\|_{L^6}^6\\
&\qquad+\||\partial_z v|^4\|_{L^{3/2}}\||v|^2\|_{L^3}+\|\partial_z v\|_{L^4}^4\big)\\
&\quad\le\left(15\eta^2+\varepsilon\right)\frac{\||\partial_z v|^2\nabla_H\,\partial_z v\|_{L^2}^2}{A(t)}\\
&\qquad+\frac{c_\varepsilon}{A(t)}\left(\|\partial_z v\|_{L^6}^4\left(1+\|\Delta_H\,\overline{v}\|_{L^2}^2+\|v\|_{L^6}^2\right)+\|\partial_z v\|_{L^6}^6+1\right)\\
&\quad\le\left(15\eta^2+\varepsilon\right)\frac{\||\partial_z v|^2\nabla_H\,\partial_z v\|_{L^2}^2}{A(t)}+c_\varepsilon\left(1+\|\Delta_H\,\overline{v}\|_{L^2}^2\right).
\end{aligned}
$$

Collecting the estimates above, we have

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[\tau_a,\tau_b]}\log(A(s))+\mathbb{E}\int_{\tau_a}^{\tau_b}\frac{2\nu-\eta^2-3\varepsilon}{A(s)}\|\Delta_H\,v\|_{L^2}^2\,\mathrm{ds}\\
&\quad+\mathbb{E}\int_{\tau_a}^{\tau_b}\frac{2\nu-\eta^2-3\varepsilon}{A(s)}\|\nabla_H\,\partial_z v\|_{L^2}^2+\frac{(6\nu-15\eta^2-4\varepsilon)}{A(s)}\||\partial_z v|^2\nabla_H\,\partial_z v\|_{L^2}^2\,\mathrm{ds}\\
&\le\mathbb{E}\log(A(\tau_a))+c_\varepsilon\\
&\quad+c_\varepsilon\mathbb{E}\int_{\tau_a}^{\tau_b}\|v\|_{L^\infty}^2+\|\Delta_H\,\overline{v}\|_{L^2}^2+\|T\|_{L^6}^2+\|\partial_z f_v\|_{L^6}\,\mathrm{ds}.
\end{aligned}
\tag{4.10}
$$

Let $B(t)=e+\|\Delta_H\,v\|_{L^2}^2+\|\nabla_H\,\partial_z v\|_{L^2}^2+\|\partial_z v\|_{L^6}^6$. Now we employ an argument similar to the logarithmic Sobolev inequality. By the Poincaré inequality, (4.3), the logarithmic Sobolev inequality (4.2) with $\lambda=1/2$ and $r_1=132$ and the inequality $\log z\le cz^{1/4}$ holding for $z\ge 1$, we obtain

$$
\begin{aligned}
\|v\|_{L^\infty}^2&\le c\left(1+\|v\|_{L^{132}}^2\right)\log\left(e+\|\nabla_H\,v\|_{L^6}+\|v\|_{L^6}+\|\partial_z v\|_{L^2}+\|v\|_{L^2}\right)\\
&\le c\left(1+\|v\|_{L^{132}}^2\right)\log B(t)\\
&\le c\left(1+\|v\|_{L^{132}}^2\right)\left(\log A(t)+\log\frac{B(t)}{A(t)}\right)\\
&\le c\left(1+\|v\|_{L^{132}}^2\right)\left(\log A(t)+\left(\frac{B(t)}{A(t)}\right)^{1/4}\right)\\
&\le c_\varepsilon\left(1+\|v\|_{L^{132}}^{8/3}\right)(\log A(t)+1)+\varepsilon\frac{\|\Delta_H\,v\|_{L^2}^2}{A(t)}+\varepsilon\frac{1+\|\partial_z v\|_{L^6}^6}{A(t)}\\
&\le c_\varepsilon\left(1+\|v\|_{L^{132}}^{8/3}\right)(\log A(t)+1)+\varepsilon\frac{\|\Delta_H\,v\|_{L^2}^2}{A(t)}.
\end{aligned}
$$

Thus, from (4.10), it follows

$$
\begin{aligned}
&\mathbb{E}\sup_{s\in[\tau_a,\tau_b]}\log(A(s))+\mathbb{E}\int_{\tau_a}^{\tau_b}\frac{2\nu-\eta^2-4\varepsilon}{A(s)}\|\Delta_H\,v\|_{L^2}^2\,\mathrm{ds}\\
&\quad+\mathbb{E}\int_{\tau_a}^{\tau_b}\frac{2\nu-\eta^2-3\varepsilon}{A(s)}\|\nabla_H\,\partial_z v\|_{L^2}^2+\frac{(6\nu-15\eta^2-4\varepsilon)}{A(s)}\||\partial_z v|^2\nabla_H\,\partial_z v\|_{L^2}^2\,\mathrm{ds}\\
&\le\mathbb{E}\left[\log(A(\tau_a))+c_\varepsilon\int_{\tau_a}^{\tau_b}1+\|T\|_{L^6}^2+\|\partial_z f_v\|_{L^6}+\|\Delta_H\,\overline{v}\|_{L^2}^2\,\mathrm{ds}\right]\\
&\quad+c_\varepsilon\mathbb{E}\int_{\tau_a}^{\tau_b}\left(1+\|v\|_{L^{132}}^{8/3}\right)(1+\log A(s))\,\mathrm{ds}
\end{aligned}
$$

By a similar argument as in the previous proofs relying on the stochastic Gronwall lemma (Lemma A.2) and Lemma A.1, we deduce that the stopping time $\tau_K^{\log A}$ defined for $K \in \mathbb{N}$ by

$$\tau_K^{\log A} = \left\{ s \geq 0 \mid \sup_{r\in[0,s\wedge\xi)} \log A(r) + \int_0^{s\wedge\xi} \frac{B(r)}{A(r)}\,\mathrm{dr} \geq K \right\}$$

satisfies $\tau_K^{\log A} \to \infty$ $\mathbb{P}$-almost surely as $K \to \infty$.

Finally, similarly as above, for $\hat{\Upsilon}_K = \tau_K^{\log A} \wedge \tau_K^{v,132,8/3}$ and $s \geq 0$, we get

$$\mathbb{E}\int_0^{s\wedge\tau_N\wedge\hat{\Upsilon}_K} \|v\|_{L^\infty}^2\,\mathrm{dr} \leq c\mathbb{E}\int_0^{s\wedge\tau_N\wedge\hat{\Upsilon}_K} \left(1+\|v\|_{L^{132}}^{8/3}\right)(1+\log A(r)) + \frac{B(r)}{A(r)}\,\mathrm{dr}$$

and the claim follows by Lemma A.1. □

Remark 4.12. The above Lemma can be shown by taking $A(t) = e + \|(-\Delta_H)^{1/2}v\|_{L^2}^2 + \|\partial_z v\|_{L^2}^2 + \|\partial_z v\|_{L^q}^{4q/(q-2)}$ for $q \in (2,6)$ instead of the case $q = 6$. Taking q large means stronger integrability in the spacial variable, but less in the probability space. The only estimate that has to be substantially adapted is (4.9) which is replaced it by

$$\begin{aligned}
&\frac{2}{A(t)}\left|\left\langle(-\Delta_H)^{1/2}v,(-\Delta_H)^{1/2}b(v,v)\right\rangle\right| = \frac{2}{A(t)}\left|\langle\Delta_H v, v\cdot\nabla_H v + w(v)\partial_z v\rangle\right| \\
&\leq \frac{c}{A(t)}\|\Delta_H v\|_{L^2}\left(\|v\|_{L^\infty}\|\nabla_H v\|_{L^2} + \|\mathrm{div}_H v\|_{L_z^2 L_{xy}^{2q/(q-2)}}\|\partial_z v\|_{L_z^2 L_{xy}^q}\right) \\
&\leq \frac{c}{A(t)}\|\Delta_H v\|_{L^2}\left(\|v\|_{L^\infty}\|\nabla_H v\|_{L^2} + \|\Delta_H v\|_{L^2}^{2/q}\|\nabla_H v\|_{L^2}^{(q-2)/q}\|\partial_z v\|_{L^q}\right) \\
&\leq \frac{\varepsilon}{A(t)}\|\Delta_H v\|_{L^2}^2 + \frac{c_\varepsilon}{A(t)}(\|v\|_{L^\infty}^2\|\nabla_H v\|_{L^2}^2 + \|\partial_z v\|_{L^q}^{4q/(q-2)} + \|\nabla_H v\|_{L^2}^4) \\
&\leq \frac{\varepsilon}{A(t)}\|\Delta_H v\|_{L^2}^2 + c_\varepsilon(\|v\|_{L^\infty}^2 + 1 + \|\nabla_H v\|_{L^2}^2).
\end{aligned}$$

We chose to consider the case $q = 6$ to keep the statement of Theorem 2.12 simpler.

Having established the L^∞ bound, we can prove the boundedness of $\partial_z U$ and deduce the boundedness of $\Delta_H U$ with estimates close to the ones used in Section 3 for the local existence. In most cases, only the estimates on the nonlinear terms have to be changed.

Lemma 4.13 (L^2 bound for $\partial_z v$)**.** *Let $q \geq 2$, $\partial_z v_0 \in L^q(\Omega; L^2)$ and assume that $f_v \in L^q(\Omega; L^2_{loc}(0,\infty; H_z^1 L_{xy}^2))$. Then the stopping time $\tau_K^{\partial_z v,q}$ defined for $K \in \mathbb{N}$ by*

$$\tau_K^{\partial_z v,q} = \inf\left\{ s \geq 0 \mid \sup_{r\in[0,s\wedge\xi)} \|\partial_z v\|_{L^2}^q + \int_0^{s\wedge\xi} \|\partial_z v\|_{L^2}^{q-2}\|\nabla_H \partial_z v\|_{L^2}^2\,\mathrm{dr} \right.$$
$$\left. + \left(\int_0^{s\wedge\xi}\|\nabla_H \partial_z v\|_{L^2}^2\,\mathrm{dr}\right)^{q/2} \geq K \right\}$$

satisfies $\tau_K^{\partial_z v,q} \to \infty$ $\mathbb{P}$-a.s. as $K \to \infty$.

Proof. Following Step 3 in Lemma 3.1, we employ the Itô formula and get

$$\begin{aligned}
&\mathrm{d}\|\partial_z v\|_{L^2}^q + q\nu\|\partial_z v\|_{L^2}^{q-2}\|(-\Delta_H)^{1/2}\partial_z v\|_{L^2}^2\,\mathrm{dt} \\
&\leq -q\|\partial_z v\|_{L^2}^{q-2}\langle\partial_z v, \partial_z f(U) + b(\partial_z v, v)\rangle\,\mathrm{dt} \\
&\quad + \frac{q(q-1)}{2}\|\partial_z v\|_{L^2}^{q-2}\|\partial_z\sigma_1(U)\|_{L_2(U,L^2)}^2\,\mathrm{dt} \\
&\quad + q\|\partial_z v\|_{L^2}^{q-2}\langle\partial_z v, \partial_z\sigma(U)\,\mathrm{dW}\rangle.
\end{aligned}$$

For the nonlinear term, similarly as in (4.9), we get

$$|\langle \partial_z v, b(\partial_z v, v)\rangle| \le \varepsilon \|\mathrm{div}_H\, \partial_z v\|_{L^2}^2 + c\|v\|_{L^\infty}^2 \|\partial_z v\|_{L^2}^2.$$

For $K \in \mathbb{N}$ and $N \in \mathbb{N}$, let $0 \le \tau_a \le \tau_b \le t \wedge \tau_N \wedge \tau_K^{w,q} \wedge \tau_K^{v,\infty}$ be stopping times. By the sub-linear growth of σ_1 (2.32) and the Burkholder-Davis-Gundy inequality (2.51), we follow the estimates in Step 3 in Lemma 3.1 and deduce

$$\begin{aligned} q\mathbb{E} \sup_{s\in[\tau_a,\tau_b]} \left| \int_{\tau_a}^{\tau_b} \|\partial_z v\|_{L^2}^{q-2} \langle \partial_z v, \partial_z \sigma(U)\rangle \,\mathrm{dW} \right| &\le \frac{1}{4}\mathbb{E} \sup_{s\in[\tau_a,\tau_b]} \|\partial_z v\|_{L^2}^q \\ &+ c\mathbb{E}\int_{\tau_a}^{\tau_b} 1 + \|v\|_{H_z^1 L_{xy}^2}^q \,\mathrm{ds} + q^2 c_{BDG}^2 \eta^2 \mathbb{E} \int_{\tau_a}^{\tau_b} \|v\|_{H_z^1 L_{xy}^2}^{q-2} \|(-\Delta_H)^{1/2} \partial_z v\|_{L^2}^2 \,\mathrm{ds} \end{aligned}$$

and

$$\begin{aligned} \int_{\tau_a}^{\tau_b} \|\partial_z v\|_{L^2}^{q-2} |\langle \partial_z v, \partial_z f(U)\rangle| \,\mathrm{ds} &\le \frac{1}{4} \sup_{s\in[0,t]} \|\partial_z v\|_{L^2}^q \\ &+ c\left(\int_{\tau_a}^{\tau_b} \|\nabla_H T\|_{L^2}^2 \,\mathrm{ds}\right)^{q/2} + c\int_{\tau_a}^{\tau_b} \|\partial_z v\|_{L^2}^q \,\mathrm{ds} + c\left(\int_{\tau_a}^{\tau_b} \|\partial_z f_v\|_{L^2}^2 \,\mathrm{ds}\right)^{q/2}. \end{aligned}$$

Collecting the above and using (2.32) for the correction term, we get

$$\begin{aligned} &\frac{1}{2}\mathbb{E} \sup_{s\in[\tau_a,\tau_b]} \|\partial_z v\|_{L^2}^q + c(q,\nu,\varepsilon,\eta)\mathbb{E}\int_{\tau_a}^{\tau_b} \|\partial_z v\|_{L^2}^{q-2} \|(-\Delta_H)^{1/2}\partial_z v\|_{L^2}^2 \,\mathrm{ds} \\ &\le c\mathbb{E}\left[\|\partial_z v(\tau_a)\|_{L^2}^q + 1 + \int_{\tau_a}^{\tau_b} \|v\|_{L^2}^q + (1+\|v\|_{L^\infty}^2)\|\partial_z v\|_{L^2}^q \,\mathrm{ds}\right] \\ &\quad + c\mathbb{E}\left(\int_{\tau_a}^{\tau_b} \|U\|_{L_z^2 H_{xy}^1}^2 \,\mathrm{ds}\right)^{q/2} + c\left(\int_{\tau_a}^{\tau_b} \|\partial_z f_v\|_{L^2}^2 \,\mathrm{ds}\right)^{q/2} \end{aligned}$$

where $c(q,\nu,\varepsilon,\eta) = q[\nu - \varepsilon - \eta^2(qc_{BDG}^2 + \frac{q-1}{2})]$. The claim follows as in the above proofs. □

Lemma 4.14 (L^2 **bound for** $\partial_z T$**).** *Let $q \ge 2$ and $\partial_z T_0 \in L^q(\Omega; L^2)$. Let the temperature forcing satisfy $f_T \in L^q(\Omega; L^2_{loc}(0,\infty; H_z^1 L_{xy}^2))$. Then the stopping time $\tau_K^{\partial_z T,q}$ defined for $K \in \mathbb{N}$ by*

$$\begin{aligned} \tau_K^{\partial_z T,q} = \inf \Bigg\{ s \ge 0 \mid \sup_{r\in[0,s\wedge\xi)} \|\partial_z T\|_{L^2}^q + \int_0^{s\wedge\xi} \|\partial_z T\|_{L^2}^{q-2} \|\nabla_H \partial_z T\|_{L^2}^2 \,\mathrm{dr} \\ + \left(\int_{\tau_a}^{\tau_b} \|\nabla_H \partial_z T\|_{L^2}^2 \,\mathrm{ds}\right)^{q/2} \ge K \Bigg\} \end{aligned}$$

satisfies $\tau_K^{\partial_z T,q} \to \infty$ $\mathbb{P}$-a.s. as $K \to \infty$.

Proof. As in the previous Lemma, the estimates in Step 3 in Lemma 3.1 can be carried over with the exception of the estimate of the nonlinear term. We replace it by

$$\begin{aligned} |\langle \partial_z T, b(\partial_z v, T)\rangle| &= |\langle \partial_z v \nabla_H T, \partial_z T\rangle - \langle \mathrm{div}_H v \partial_z T, \partial_z T\rangle| \\ &\le c(\|\partial_z v\|_{L_z^2 L_{xy}^4} \|\nabla_H T\|_{L_z^\infty L_{xy}^2} \|\partial_z T\|_{L_z^2 L_{xy}^4} + \|\mathrm{div}_H v\|_{L_z^\infty L_{xy}^2} \|\partial_z T\|_{L_z^2 L_{xy}^4}^2) \\ &\le c\|\partial_z v\|_{L^2}^{1/2} \|\partial_z \nabla_H v\|_{L^2}^{1/2} (\|\nabla_H T\|_{L^2}^{1/2} \|\nabla_H \partial_z T\|_{L^2}^{1/2} + \|\nabla_H T\|_{L^2}) \\ &\quad \cdot (\|\partial_z T\|_{L^2}^{1/2} \|\nabla_H \partial_z T\|_{L^2}^{1/2} + \|\partial_z T\|_{L^2}) \\ &\quad + c\|\nabla_H v\|_{L^2}^{1/2} \|\partial_z \nabla_H v\|_{L^2}^{1/2} (\|\partial_z T\|_{L^2} \|\nabla_H \partial_z T\|_{L^2} + \|\partial_z T\|_{L^2}^2) \\ &\le \varepsilon \|\nabla_H \partial_z T\|_{L^2}^2 + c(1 + \|\nabla_H T\|_{L^2}^2 + \|\partial_z v\|_{L^2}^2 \|\partial_z \nabla_H v\|_{L^2}^2 \\ &\quad + \|\nabla_H v\|_{L^2} \|\partial_z \nabla_H v\|_{L^2})(1 + \|\partial_z T\|_{L^2}^2) \\ &\le \varepsilon \|\nabla_H \partial_z T\|_{L^2}^2 + c(1 + (1 + \|\partial_z v\|_{L^2}^2)\|\partial_z \nabla_H v\|_{L^2}^2 + \|U\|_{L_z^2 H_{xy}^1}^2)(1 + \|\partial_z T\|_{L^2}^2). \end{aligned}$$

For $K \in \mathbb{N}$, let $\Upsilon_K = \tau_K^{w,2} \wedge \tau_K^{\partial_z v,4}$. Let $N \in \mathbb{N}$ and let $0 \le \tau_a \le \tau_b \le t \wedge \tau_N \wedge \Upsilon_K$ be stopping times. We get

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{2}\sup_{s\in[\tau_a,\tau_b]}\|\partial_z T\|_{L^2}^q + c(q,\nu,\varepsilon,\eta)\int_{\tau_a}^{\tau_b}\|\partial_z T\|_{L^2}^{q-2}\|(-\Delta_H)^{1/2}\partial_z T\|_{L^2}^2\,\mathrm{ds}\right]\\
&\qquad\le c\mathbb{E}\Bigg[\|\partial_z T(\tau_a)\|_{L^2}^q + 1 + \int_{\tau_a}^{\tau_b}\|U\|_{L^2}^2 + \|\partial_z v\|_{L^2}^2\,\mathrm{ds}\\
&\quad + \left(\int_{\tau_a}^{\tau_b}\|U\|_{L_z^2H_{xy}^1}^2\,\mathrm{ds}\right)^{q/2} + \left(\int_{\tau_a}^{\tau_b}\|\nabla_H\,\partial_z v\|_{L^2}^2\,\mathrm{ds}\right)^{q/2} + \left(\int_{\tau_a}^{\tau_b}\|\nabla_H\,\partial_z f_T\|_{L^2}^2\,\mathrm{ds}\right)^{q/2}\Bigg]\\
&\qquad + c\mathbb{E}\int_{\tau_a}^{\tau_b}(1 + (1+\|\partial_z v\|_{L^2}^2)\|\partial_z\nabla_H\,v\|_{L^2}^2 + \|U\|_{L_z^2H_{xy}^1}^2)(1+\|\partial_z T\|_{L^2}^q)\,\mathrm{ds}
\end{aligned}$$

where again $c(q,\nu,\varepsilon,\eta) = q[\nu - \varepsilon - \eta^2(qc_{BDG}^2 + \frac{q-1}{2})]$. The claim follows as above. □

Lemma 4.15 (L^2 bound for $\nabla_H U$)**.** *Let $q \ge 2$ and $U_0 \in L^q(\Omega; L_z^2H^1)$. Let the forcing satisfy $F_U \in L^q(\Omega; L_{loc}^2(0,\infty;L^2))$. Then the stopping time $\tau_K^{\nabla_H U,q}$ defined for $K \in \mathbb{N}$ by*

$$\begin{aligned}
\tau_K^{\nabla_H U,q} = \inf\Bigg\{s \ge 0 \mid \sup_{r\in[0,s\wedge\xi)}\|\nabla_H\,U\|_{L^2}^q + \int_0^{s\wedge\xi}\|\nabla_H\,U\|_{L^2}^{q-2}\,\|\Delta_H\,U\|_{L^2}^2\,\mathrm{dr}\\
+ \left(\int_{\tau_a}^{\tau_b}\|\Delta_H\,U\|_{L^2}^2\,\mathrm{ds}\right)^{q/2} \ge K\Bigg\}
\end{aligned}$$

satisfies $\tau_K^{\nabla_H U,q} \to \infty$ $\mathbb{P}$-a.s. as $K \to \infty$.

Proof. We use estimates from Step 5 in Lemma 3.1 with the exception of the bound of the nonlinear term which, recalling (2.25) and (2.7), we replace by

$$\begin{aligned}
&\|\nabla_H\,U\|_{L^2}^{q-2}|\langle B(U,U),\Delta_H\,U\rangle|\\
&\le c\|\nabla_H\,U\|_{L^2}^{q-2}\|U\|_{L_z^\infty L_{xy}^4}\left(\|U\|_{L_z^2H_{xy}^1}^{1/2}\|U\|_{L_z^2H_{xy}^2}^{3/2} + \|U\|_{L_z^2H_{xy}^1}^{1/2}\|U\|_{H_z^1H_{xy}^1}\|U\|_{L_z^2H_{xy}^2}^{1/2}\right)\\
&\le \varepsilon\|U\|_{L_z^2H_{xy}^2}^2\|\nabla_H\,U\|_{L^2}^{q-2} + c_\varepsilon\|U\|_{L_z^\infty L_{xy}^4}^4\|U\|_{L_z^2H_{xy}^1}^2\|\nabla_H\,U\|_{L^2}^{q-2}\\
&\quad + c_\varepsilon\|U\|_{L_z^\infty L_{xy}^4}^{4/3}\|U\|_{L_z^2H_{xy}^1}^{2/3}\|U\|_{H_z^1H_{xy}^1}^{4/3}\|\nabla_H\,U\|_{L^2}^{q-2}\\
&\le \varepsilon\|U\|_{L_z^2H_{xy}^2}^2\|\nabla_H\,U\|_{L^2}^{q-2} + c_\varepsilon\|U\|_{H_z^1L_{xy}^2}^2\|U\|_{L_z^2H_{xy}^1}^2\|U\|_{L_z^2H_{xy}^1}^2\|\nabla_H\,U\|_{L^2}^{q-2}\\
&\quad + c_\varepsilon\|U\|_{H_z^1L_{xy}^2}^{2/3}\|U\|_{L_z^2H_{xy}^1}^{4/3}\|U\|_{H_z^1H_{xy}^1}^{4/3}\|\nabla_H\,U\|_{L^2}^{q-2}\\
&\le \varepsilon\|\Delta_H\,U\|_{L^2}^2\|\nabla_H\,U\|_{L^2}^{q-2}\\
&\quad + c_\varepsilon\left(1+\|U\|_{L^2}^2\right)\left(1+\|U\|_{H_z^1L_{xy}^2}^2\right)\left(1+\|U\|_{L_z^2H_{xy}^1}^2 + \|U\|_{H_z^1H_{xy}^1}^2\right)\\
&\qquad\cdot\left(1+\|\nabla_H\,U\|_{L^2}^q\right).
\end{aligned}$$

Accordingly, we deduce a suitable inequality from the stochastic Gronwall lemma (Lemma A.2) for stopping times $0 \le \tau_a \le \tau_b \le t \wedge \tau_N \wedge \Upsilon_K$ for $K, N \in \mathbb{N}$ and $\Upsilon_K = \tau_K^{w,2} \wedge \tau_K^{\partial_z v,2}$. The claim then follows similarly as in the above proofs. □

We remark that above Lemma can be established in an easier manner if one considers $\Upsilon_K = \tau_K^{w,6} \wedge \tau_K^{\partial_z v,6}$ instead of the stopping time Υ_K above.

The last auxiliary step is to bound $\|\partial_{zz}U\|_{L^2}$, which also follows similarly as the corresponding bound for the finite-dimensional approximations.

Lemma 4.16 (L^2 bound for $\partial_{zz}U$). *Let $q \geq 2$ and $U_0 \in L^q(\Omega; H^2_z L^2_{xy})$. Let $F_U \in L^q(\Omega; L^2_{loc}(0,\infty; H^2_z L^2_{xy}))$. Then the stopping time $\tau_K^{\partial_{zz}U,q}$ defined for $K \in \mathbb{N}$ by*

$$\tau_K^{\partial_{zz}U,q} = \inf \Bigg\{ s \geq 0 \mid \sup_{r\in[0,s\wedge\xi)} \|\partial_{zz}U\|^q_{L^2} + \int_0^{s\wedge\xi} \|\partial_{zz}U\|^{q-2}_{L^2} \|\nabla_H \partial_{zz}U\|^2_{L^2} \,\mathrm{dr} + \left(\int_{\tau_a}^{\tau_b} \|\nabla_H \partial_{zz}U\|^2_{L^2} \,\mathrm{ds}\right)^{q/2} \geq K \Bigg\}$$

satisfies $\tau_K^{\partial_{zz}U,q} \to \infty$ $\mathbb{P}$-a.s. as $K \to \infty$.

Proof. We can proceed as in Lemma 3.4 by replacing the factor μ^4 by $\|U\|^4_{H^1_z L^4_{xy}}$ in the estimate for I^n_2. Using $\|U\|^4_{H^1_z L^4_{xy}} \leq c\|U\|^2_{H^1_z L^2_{xy}} \|U\|^2_{H^1_z H^1_{xy}}$, we can argue as in the proof above by taking stopping times $0 \leq \tau_a \leq \tau_b \leq t \wedge \tau_N \wedge \tau_K^{\partial_z v,q}$. □

4.4 Proof of Theorem 2.12

The proof closely follows the proof of [12, Theorem 3.2] and we include it mainly for completeness. Let ρ_K be the stopping time (collected from Lemmata 4.13, 4.14 and 4.15) defined for $K \in \mathbb{N}$ by

$$\rho_K = \tau_K^{\partial_z v,2} \wedge \tau_K^{\partial_z T,2} \wedge \tau_K^{\nabla_H U,2}.$$

Our aim is to show that $\rho_K \leq \xi$ for all $K \in \mathbb{N}$. Since $\rho_K \to \infty$ as $K \to \infty$ a.s. by the results above, this will yield global existence.

From the proofs of Lemmata 4.13, 4.14 and 4.15 (see the final part of the proof of Lemma 4.4 for a detailed explanation), we get

$$\sup_{s\in[0,t\wedge\rho_K\wedge\xi)} \|U\|^2_{H^1} + \int_0^{t\wedge\rho_K\wedge\xi} \|U\|^2_{L^2_z H^2_{xy}} + \|U\|^2_{H^1_z H^1_{xy}} \,\mathrm{ds} < \infty \quad \mathbb{P}\text{-a.s.} \tag{4.11}$$

For contradiction, let $\mathbb{P}(\{\rho_K > \xi\}) > 0$ for some $K \in \mathbb{N}$. Then, since $\{\rho_K > \xi\} = \cup_{t\geq 0}\{\rho_K \wedge t > \xi\}$, there exists $t \geq 0$ such that $\mathbb{P}(\{\rho_K \wedge t > \xi\}) > 0$. By Definition 2.9 and (2.55) in particular, this leads to

$$\sup_{s\in[0,t\wedge\rho_K\wedge\xi)} \|U\|^2_{H^1} + \int_0^{t\wedge\rho_K\wedge\xi} \|U\|^2_{L^2_z H^2_{xy}} + \|U\|^2_{H^1_z H^1_{xy}} \,\mathrm{ds} \geq \sup_{s\in[0,t\wedge\xi)} \|U\|^2_{H^1} + \int_0^{t\wedge\xi} \|U\|^2_{L^2_z H^2_{xy}} + \|U\|^2_{H^1_z H^1_{xy}} \,\mathrm{ds} = \infty$$

on a set of positive measure, which contradicts (4.11). This concludes the proof of Theorem 2.12.

A Auxiliary results

Lemma A.1. *Let $f : \Omega \times [0,\infty) \to [0,\infty]$ be such that $f(\cdot,\omega)$ is non-decreasing and continuous for a.a. $\omega \in \Omega$ and $f(t,\omega)$ is measurable and a.s. finite for all $t \geq 0$. For $K \in \mathbb{N}$, let $\rho_K = \inf\{s \geq 0 \mid f(s,\omega) \geq K\}$. Then $\rho_K \to \infty$ as $K \to \infty$ almost surely.*

Moreover, if $\Phi : [0,\infty) \to [0,\infty)$ is a non-decreasing function such that $\Phi(s) \to \infty$ as $s \to \infty$, then $\rho_K^\Phi \to \infty$ as $K \to \infty$ almost surely, where $\rho_K^\Phi = \{s \geq 0 \mid \Phi(f(s,\omega)) \geq K\}$.

Proof. The first claim is established in e.g. [5, Lemma 4.1], see also [23, Proposition A.1]. To prove the second claim, we argue by contradiction. Assuming $\rho := \lim_K \rho_K^\Phi < \infty$ on a measurable set $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) > 0$, we observe $f(\rho_K,\omega) \geq \Phi^{-1}(K)$. In particular, $f(\cdot,\omega)$ is unbounded on the (random) interval $[0,\rho]$ a.s. on Ω_0. On the other hand, from $\rho_K \to \infty$, we deduce that $f(\cdot,\omega)$ is a.s. bounded on $[0,\rho]$, a contradiction. □

Lemma A.2 (stochastic Gronwall lemma, [26, Lemma 5.3]). *Let* $t > 0$ *and let* $X, Y, Z, R : [0,\infty) \times \Omega \to [0,\infty)$ *be stochastic processes on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. *Let* $\tau : \Omega \to [0,t)$ *be a stopping time such that* $\mathbb{E}\int_0^\tau RX + Z\,\mathrm{ds} < \infty$ *and* $\int_0^\tau R\,\mathrm{ds} < \kappa$ $\mathbb{P}$*-a.s. for some* $\kappa > 0$. *Assume that there exists a constant* $c_0 > 0$ *such that*

$$\mathbb{E}\left[\sup_{s\in[\tau_a,\tau_b]} X + \int_{\tau_a}^{\tau_b} Y\,\mathrm{ds}\right] \le c_0\mathbb{E}\left[X(\tau_a) + \int_{\tau_a}^{\tau_b} RX + Z\,\mathrm{ds}\right]$$

for all stopping times τ_a, τ_b *satisfying* $0 \le \tau_a \le \tau_b \le \tau$. *Then*

$$\mathbb{E}\left[\sup_{s\in[0,\tau]} X + \int_0^\tau Y\,\mathrm{ds}\right] \le c_{c_0,t,\kappa}\mathbb{E}\left[X(0) + \int_0^\tau Z\,\mathrm{ds}\right].$$

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