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Addendum to "Isomorphism theorems, extended Markov processes and random interlacements"*

Nathalie Eisenbaum† Haya Kaspi‡

Abstract

This addendum clarifies Definition 5.1 in section 5.3 of the previously published paper Electron. J. Probab. 27, 1–27 (2022).

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This addendum clarifies Definition 5.1 in section 5.3. Definition 5.1, which defines random interlacements for continuous transient Borel processes in weak duality, should be replaced by a new version presented below. We keep the notation of the original paper and in particular of section 5.3.

In the original paper, Definition 5.1 makes use of the law of $(\hat{X}_t, t \geq 0)$ under $\hat{I\!\!P}_x[\ .\ |\hat{X}(0,\infty)\cap B = \emptyset]$ for B compact subset of E. For x in $E \setminus B$, this law is well defined and not null, but this is not always so for x in ∂B. To replace this failing expression we will use a result of Getoor (Theorem 2.12 in *Splitting time and Shift functionals Z.W. 47, 69-81(1979)*), according to which one has in particular:

$$\hat{I\!\!P}_\nu[F(\hat{X}_{\hat{L}_B+s}, s>0) f(\hat{X}_{\hat{L}_B});\ 0<\hat{L}_B\leq t] = \hat{I\!\!P}_\nu[\Gamma(\hat{X}_{\hat{L}_B}, F) f(\hat{X}_{\hat{L}_B});\ 0<\hat{L}_B\leq t], \tag{1}$$

for every $t>0$, where $\hat{I\!\!P}_\nu$ denotes $\int_E \nu(dx)\hat{I\!\!P}_x$ and $(\Gamma(x,A), x\in E, A\in\mathcal{F})$ is a Markov kernel (see Getoor's paper for a full description of Γ) independent of ν.

To introduce the new version of Definition 5.1, we first set, for any path ω of $\mathcal{W}$: $\lambda_B(\omega) = \sup\{s\in(b(\omega),d(\omega)) : \omega(s)\in B\}$, with $\sup\emptyset = -\infty$ (we remind that $b(\omega)$ and $d(\omega)$ denote the birth and death times of ω). Note from Proposition 13.11 in Getoor and Sharpe [17] that $\hat{\mathbf{Q}}_\nu[\ .\ ; 0<\lambda_B\leq 1]$ is a finite measure ($\hat{\mathbf{Q}}_\nu$ denotes the Kuznetsov measure of the dual process $\hat{X}$).

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†CNRS, Université Paris Cité. E-mail: nathalie.eisenbaum@parisdescartes.fr
‡Department of Industrial Engineering and Management, Technion.
E-mail: iehaya@tx.tecnion.ac.il

The capacitary measure $\hat{e}_B$ of B with respect to $\hat{X}$, can also be expressed as follows:

$$\hat{e}_B(f) = \lim_{t\to 0} \frac{1}{t} \hat{\mathbf{Q}}_\nu [Z_0 \in E, f(Z_{\lambda_B}); 0 < \lambda_B \leq t]$$

for every nonnegative measurable function f. Since one also has ((5.18) with (5.6)) $\hat{e}_B(f) = \hat{\mathbf{Q}}_\nu[f(Z_{\lambda_B}); 0 < \lambda_B \leq 1]$, one obtains:

$$\lim_{t\to 0} \frac{1}{t} \hat{\mathbf{Q}}_\nu [0 \leq b < \lambda_B \leq t] = 0. \tag{2}$$

Since λ_B is a stationary time, one has thanks to (2.1) and (2.3) in [14] for every $t > 0$, and every functional F:

$$\frac{1}{t} \hat{\mathbf{Q}}_\nu [F(Z_{\lambda_B+s}, s \geq 0); 0 < \lambda_B \leq t] = \hat{Q}_\nu [F(Z_{\lambda_B+s}, s \geq 0); 0 < \lambda_B \leq 1],$$

which leads together with (2) to

$$\hat{\mathbf{Q}}_\nu [F(Z_{\lambda_B+s}, s > 0) f(Z_{\lambda_B}); 0 < \lambda_B \leq 1] = \lim_{t\to 0} \frac{1}{t} \hat{I\!P}_\nu [F(\hat{X}_{\hat{L}_B+s}, s > 0) f(\hat{X}_{\hat{L}_B}); 0 < \hat{L}_B \leq t].$$

Hence using (1), one has:

$$\hat{\mathbf{Q}}_\nu [F(Z_{\lambda_B+s}, s > 0) f(Z_{\lambda_B}); 0 < \lambda_B \leq 1] = \hat{e}_B(\Gamma(., F) f),$$

which leads, $\hat{e}_B(dx)$ a.e x, to

$$\hat{\mathbf{Q}}_\nu [F(Z_{\lambda_B+s}, s > 0); 0 < \lambda_B \leq 1 | Z_{\lambda_B} = x] = \hat{I\!P}_\nu [F(\hat{X}_{\hat{L}_B+s}, s > 0) | \hat{X}_{\hat{L}_B} = x].$$

We then define for $\hat{e}_B(dx)$ a.e. x, the probability measure $\hat{I\!P}_x^B$ on the set of E-valued paths indexed by $I\!R_+$ by

$$\hat{I\!P}_x^B [F(Z_s, s \geq 0)] = \int_E \nu(dy) \hat{I\!P}_y [F(\hat{X}_{\hat{L}_B+s}, s \geq 0) \mid \hat{X}_{\hat{L}_B} = x].$$

Remark that $\hat{I\!P}_x^B$ is independent of the choice of ν and that for every $\varepsilon > 0$, one has: $\hat{I\!P}_x^B [\{Z_s, s \geq \varepsilon\} \cap B \neq \emptyset] = 0$.

One finally sets the following definition for random interlacements.

Definition 5.1 *For $u > 0$ the random interlacements at level u associated to $\{\nu, ((P_t)_{t\geq 0}, (\hat{P}_t)_{t\geq 0})\}$ is a PPP with intensity measure $u\mu_\nu$ where μ_ν is the measure on $(\mathcal{W}, \mathcal{A})$ such that $\mu_\nu(\omega \equiv \Delta) = 0$, characterized by the following properties:*

- *for any compact subset B of E, define $H_B = \inf\{t \in (b(\omega), d(\omega)) : \omega(t) \in B\}$ with $\inf \emptyset = +\infty$, then*

$$\mu_\nu [\omega_{H_B} \in dx; H_B < \infty] = \hat{e}_B(dx) \tag{5.13}$$

  where $\hat{e}_B$ is the capacitary measure of B associated to $\hat{X}$ with respect to ν;
- *for every couple of $\mathcal{A}$ measurable functionals (F_1, F_2)*

$$\begin{aligned} &\mu_\nu [F_1(\omega(H_B + t), t \geq 0);\ F_2(\omega(H_B - t), t \geq 0);\ H_B < \infty] \\ &= \int_E \hat{e}_B(dx) I\!P_x [F_1(X_t, t \geq 0)] \hat{I\!P}_x^B [F_2(Z_t, t \geq 0)]. \end{aligned} \tag{5.14}$$

With this modified Definition 5.1, the rest of the paper is unchanged except for the proof of (5.15) in Theorem 5.1. We now take the proof of (5.15) up from the equation (page 21 lines 16 and 17):

$$\begin{aligned} \mathbf{Q}_\nu & [F_1(Z(H_B+t), t \geq 0); F_2(Z(H_B-t), t \geq 0); 0 < H_B \leq 1] \\ = \quad & \mathbf{Q}_\nu[0 < H_B \leq 1, I\!P_{Z(H_B)})[F_1(X_s, s \geq 0)]F_2(Z(H_B-s), s \geq 0)], \end{aligned}$$

which gives using (5.17), then (2.1), (2.3) in [14] and (5.6):

$$\begin{aligned} \mathbf{Q}_\nu & [F_1(Z(H_B+t), t \geq 0); F_2(Z(H_B-t), t \geq 0); 0 < H_B \leq 1] \\ = \quad & \int_B \hat{e}_B(dx) P_x[F_1(X_s, s \geq 0)]\mathbf{Q}_\nu[0 < H_B \leq 1, F_2(Z_{H_B-s}, s \geq 0)|Z_{H_B} = x] \\ = \quad & \int_B \hat{e}_B(dx) I\!P_x[F_1(X_s, s \geq 0)]\hat{\mathbf{Q}}_\nu[F_2(Z_{\lambda_B+s}, s \geq 0);\ 0 < \lambda_B \leq 1 \mid Z_{\lambda_B} = x]. \end{aligned}$$

To finish the proof of (5.15) one finally uses the definition of $\hat{I\!P}_x^B$.

Besides the formulation of the open question presented in Remark 5.4 has to be reformulated accordingly.

Finally we remind that in [30], for $\hat{X}$ Brownian motion on $I\!R^d$ $(d \geq 3)$, B closed ball of $I\!R^d$ and $x \in \partial B$, the law of the "Brownian motion avoiding B and starting from x" is defined as the weak limit of the law of $\hat{X}$ under $\hat{I\!P}_y[\ .\ |\hat{X}(0,\infty) \cap B = \emptyset]$ as y tends to x with y in $I\!R^d \setminus B$. This law is the one used then to set the definition of random interlacements of Brownian motion in $I\!R^d$. As a consequence of Theorem 5.1 and [5] (Theorem 10), one obtains that in this particular case this law coincides with $\hat{I\!P}_x^B$ and that Definition 5.1 is indeed an extension of the definition of random interlacements of Brownian motion.

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