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A note on some critical thresholds of Bernoulli percolation*

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Abstract

Consider Bernoulli bond percolation on a locally finite, connected graph G and let p_{cut} be the threshold corresponding to a "first-moment method" lower bound. Kahn (*Electron. Comm. Probab. Volume 8, 184-187.* (2003)) constructed a counter-example to Lyons' conjecture of $p_{\mathrm{cut}} = p_c$ and proposed a modification. Here we give a positive answer to Kahn's modified question. The key observation is that in Kahn's modification, the new expectation quantity also appears in the differential inequality of one-arm events. This links the question to a lemma of Duminil-Copin and Tassion (*Comm. Math. Phys. Volume 343, 725-745.* (2016)). We also study some applications for Bernoulli percolation on periodic trees.

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1 Introduction

Let $G = (V, E)$ be a locally finite (i.e., each vertex has finite degree), connected, infinite graph. For $p \in [0,1]$, ***Bernoulli***(p) ***bond percolation*** studies the random subgraph ω of G formed by keeping each edge with probability p and removing otherwise, independently of each other. The edges kept in ω are called ***open edges*** and the edges removed are called ***closed edges***. The connected components are called (open) clusters. For background on Bernoulli percolation, see Chapter 7 of [12] or [5]. For $p \in [0,1]$, let $\mathbb{P}_p$ denote the law of Bernoulli(p) bond percolation and $\mathbb{E}_p$ the corresponding expectation.

Let $C(x)$ denote the open cluster of x in Bernoulli percolation. Let $|C(x)|_V, |C(x)|_E$ denote the number of vertices and edges in the cluster $C(x)$ respectively. Let $A \longleftrightarrow B$ denote the event that there is an open path connecting some vertex $x \in A$ and $y \in B$. Let $x \longleftrightarrow \infty$ denote the event that the diameter of $C(x)$ is infinite. The ***critical probability*** p_c is defined as

$$p_c = p_c(G) := \sup\{p \geq 0 \colon \mathbb{P}_p(x \longleftrightarrow \infty) = 0\}.$$

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Since for a locally finite graph G, the three events $x \longleftrightarrow \infty$, $|C(x)|_V = \infty$ and $|C(x)|_E = \infty$ are actually the same event, one can also define

$$p_c = p_c(G) := \sup\{p \geq 0 \colon \mathbb{P}_p(|C(x)|_V = \infty) = 0\}$$

or

$$p_c = p_c(G) := \sup\{p \geq 0 \colon \mathbb{P}_p(|C(x)|_E = \infty) = 0\}.$$

Let x be a vertex in G. We say that Π_E is a ***edge cutset*** separating x from infinity, if Π_E is a set of edges such that the connected component of x in $G \backslash \Pi_E$ is finite. Similarly one can define ***vertex cutsets***.

Definition 1.1. *Suppose G is a locally finite, connected, infinite graph. Define*

$$p_{\mathrm{cut,E}} = p_{\mathrm{cut,E}}(G) := \sup\{p \geq 0 \colon \inf_{\Pi_E} \mathbb{E}_p[|C(x) \cap \Pi_E|] = 0\},$$

*where the infimum is taken over all **edge cutsets** Π_E separating x from infinity and $C(x) \cap \Pi_E$ denotes the intersection of the edge set of $C(x)$ with Π_E.*

Define

$$p_{\mathrm{cut,V}} = p_{\mathrm{cut,V}}(G) := \sup\{p \geq 0 \colon \inf_{\Pi_V} \mathbb{E}_p[|C(x) \cap \Pi_V|] = 0\},$$

*where the infimum is taken over all **vertex cutsets** Π_V separating x from infinity and $C(x) \cap \Pi_V$ denotes the intersection of the vertex set of $C(x)$ with Π_V.*

For any edge (or vertex) cutset Π separating x from infinity, if the event $\{x \longleftrightarrow \infty\}$ occurs, then $C(x) \cap \Pi$ is nonempty. Hence

$$\mathbb{P}_p(x \longleftrightarrow \infty) \leq \mathbb{P}_p(|C(x) \cap \Pi| \geq 1) \leq \mathbb{E}_p[|C(x) \cap \Pi|]. \tag{1.1}$$

Thus one has that

$$p_{\mathrm{cut,E}} \leq p_c \quad \text{and} \quad p_{\mathrm{cut,V}} \leq p_c. \tag{1.2}$$

Historically another critical value p_T is also of great interest (coincide with the notation $p_{\mathrm{T,V}}$ below).

Definition 1.2. *Suppose G is a locally finite, connected, infinite graph. Define*

$$p_{\mathrm{T,V}} = p_{\mathrm{T,V}}(G) := \sup\{p \geq 0 \colon \mathbb{E}_p[|C(x)|_V] < \infty\}$$

and

$$p_{\mathrm{T,E}} = p_{\mathrm{T,E}}(G) := \sup\{p \geq 0 \colon \mathbb{E}_p[|C(x)|_E] < \infty\}.$$

If $p < p_{\mathrm{T,V}}$, then $\sum_{n=1}^{\infty} \mathbb{E}_p[|C(x) \cap \Pi_n|] \leq \mathbb{E}_p[|C(x)|_V] < \infty$, where $\Pi_n := \{y \colon d_G(y, x) = n\}$ is the cutset consisting of vertices at graph distance n to x. Hence $p < p_{\mathrm{T,V}}$ implies that $p \leq p_{\mathrm{cut,V}}$. Thus

$$p_{\mathrm{T,V}} \leq p_{\mathrm{cut,V}}. \tag{1.3}$$

Similarly one has that

$$p_{\mathrm{T,E}} \leq p_{\mathrm{cut,E}}. \tag{1.4}$$

It is easy to see that these critical values $p_c, p_{\mathrm{cut,E}}, p_{\mathrm{cut,V}}, p_{\mathrm{T,E}}, p_{\mathrm{T,V}}$ do not depend on the choice of x by Harris' inequality [12, Section 5.8].

By (1.2), (1.3) and (1.4) we now have

$$p_{\mathrm{T,E}} \leq p_{\mathrm{cut,E}} \leq p_c.$$

and

$$p_{\mathrm{T,V}} \leq p_{\mathrm{cut,V}} \leq p_c.$$

Lyons showed that $p_{\mathrm{c}} = p_{\mathrm{cut,V}}$ holds for trees [11] and tree-like graphs [10] and pointed out $p_{\mathrm{cut,V}} = p_{\mathrm{c}}$ for transitive graphs in [11] since $p_{\mathrm{T,V}} = p_{\mathrm{c}}$ for such graphs [1, 13]; and these results for $p_{\mathrm{cut,V}}$ applied equally to $p_{\mathrm{cut,E}}$ on these graphs. In view of these examples Lyons conjectured that $p_{\mathrm{c}} = p_{\mathrm{cut,V}}$ for general graphs (lines 11–12 on page 955 of [11]).

Later Kahn [8] constructed a family of counterexamples to Lyons' conjecture. Kahn's examples exhibited a sequences of vertex cutsets Π_n such that the quantity $|C(x)\cap\Pi_n| = \sum_{v\in\Pi_n} \mathbf{1}_{(x\longleftrightarrow v)}$ is usually zero but has a large expectation for some $p < p_{\mathrm{c}}(G)$. That was achieved by large correlation among the events $\{x \longleftrightarrow v\}$ for $v \in \Pi_n$, i.e., conditioned on the event that v is connected to x via an open path, with high probability a lot of other vertices in Π_n are also connected to x via v. In light of this Kahn proposed the following modification of Lyons' conjecture:

Question 1.3. Does $p_{\mathrm{c}}(G) = p'_{\mathrm{cut,V}}(G)$ hold for every locally finite, connected, infinite graph G?

Here the notation $p'_{\mathrm{cut,V}} = p'_{\mathrm{cut,V}}(G)$ from [8] (there it was denoted by p'_{cut}) is defined as follows.

Definition 1.4. *Suppose G is a locally finite, connected, infinite graph. For $x \in V$, let Π_V be a vertex cutset which separates x from infinity. For each $v \in \Pi_V$, let $A(x, v, \Pi_V)$ denote the event that x is connected to v via an open path without using vertices in $\Pi_V\backslash\{v\}$. Define*

$$p'_{\mathrm{cut,V}} = p'_{\mathrm{cut,V}}(G) := \sup\Big\{p \geq 0\colon \inf_{\Pi_V} \sum_{v\in\Pi_V} \mathbb{P}_p[A(x, v, \Pi_V)] = 0\Big\},$$

where the infimum is taken over all ***vertex cutsets*** *Π_V separating x from infinity.*

Similarly for an edge cutset Π_E separating x from infinity and $e \in \Pi_E$, let $A(x, e, \Pi_E)$ denote the event that x is connected to e via an open path without using edges in $\Pi_E\backslash\{e\}$ (Here we assume e itself is also open on $A(x, e, \Pi_E)$.) Define

$$p'_{\mathrm{cut,E}} = p'_{\mathrm{cut,E}}(G) := \sup\Big\{p \geq 0\colon \inf_{\Pi_E} \sum_{e\in\Pi_E} \mathbb{P}_p[A(x, e, \Pi_E)] = 0\Big\},$$

where the infimum is taken over all ***edge cutsets*** *Π_E separating x from infinity.*

Similarly one can ask ([12, Question 5.16]):

Question 1.5. Does $p_{\mathrm{c}}(G) = p'_{\mathrm{cut,E}}(G)$ hold for every locally finite, connected, infinite graph G?

Our main result is the following affirmative answer to Question 1.3 and 1.5 for Bernoulli bond percolation.

Theorem 1.6. *For Bernoulli bond percolation on every locally finite, connected, infinite graph G, one has that*

$$p'_{\mathrm{cut,E}} = p'_{\mathrm{cut,V}} = p_{\mathrm{c}}.$$

The same result holds for Bernoulli site percolation on a locally finite, connected, infinite graph with **bounded degree** if one defines $p'_{\mathrm{cut,E}}, p'_{\mathrm{cut,V}}$ accordingly using Bernoulli site percolation; see Remark 5.1 and Conjecture 5.2 for more discussions.

2 Some relations of the critical thresholds

For any edge cutset Π separating x from infinity, if the event $\{x \longleftrightarrow \infty\}$ occurs, then there is at least one edge e such that the event $A(x, e, \Pi)$ occurs. Hence by union bounds,

$$\mathbb{P}_p(x \longleftrightarrow \infty) \leq \sum_{e\in\Pi} \mathbb{P}_p[A(x, e, \Pi)] \tag{2.1}$$

Thus one has that

$$p'_{\mathrm{cut,E}} \le p_{\mathrm{c}}. \tag{2.2}$$

Similarly one has that

$$p'_{\mathrm{cut,V}} \le p_{\mathrm{c}}. \tag{2.3}$$

Also obviously for any edge cutset Π one has that $\sum_{e\in\Pi} \mathbb{P}_p[A(x,e,\Pi)] \le \sum_{e\in\Pi} \mathbb{P}_p[e \in C(x)] = \mathbb{E}_p[|C(x)\cap\Pi|]$. Hence one has that

$$p_{\mathrm{cut,E}} \le p'_{\mathrm{cut,E}}. \tag{2.4}$$

Similarly one has that

$$p_{\mathrm{cut,V}} \le p'_{\mathrm{cut,V}}. \tag{2.5}$$

By (1.3), (1.4), (2.2), (2.3), (2.4) and (2.5) one has that

$$p_{\mathrm{T,V}} \le p_{\mathrm{cut,V}} \le p'_{\mathrm{cut,V}} \le p_{\mathrm{c}} \tag{2.6}$$

and

$$p_{\mathrm{T,E}} \le p_{\mathrm{cut,E}} \le p'_{\mathrm{cut,E}} \le p_{\mathrm{c}}. \tag{2.7}$$

We also have the following relations.

Lemma 2.1. *Suppose G is a locally finite, connected, infinite graph. Then*

$$p_{\mathrm{cut,E}} \le p_{\mathrm{cut,V}} \tag{2.8}$$

If moreover G has bounded degree, then the equality holds in (2.8).

Lemma 2.2. *Suppose G is a locally finite, connected, infinite graph. Then*

$$p_{\mathrm{T,E}} \le p_{\mathrm{T,V}} \tag{2.9}$$

If moreover G has bounded degree, then the equality holds in (2.9).

Proof of Lemma 2.2. For (2.9), if $p > p_{\mathrm{T,V}}$, then $\mathbb{E}_p[|C(x)|_V] = \infty$. Since $C(x)$ is connected, $|C(x)|_E \ge |C(x)|_V - 1$. Hence $\mathbb{E}_p[|C(x)|_E] = \infty$. Therefore if $p > p_{\mathrm{T,V}}$, then $p \ge p_{\mathrm{T,E}}$. Thus $p_{\mathrm{T,E}} \le p_{\mathrm{T,V}}$ as desired.

If G has bounded degree, i.e., $D(G) := \sup\{\deg(v)\colon v \in V\} < \infty$, then by $|C(x)|_E \le D(G)|C(x)|_V$ one can get the other direction similarly. Hence if G has bounded degree, then $p_{\mathrm{T,V}}(G) = p_{\mathrm{T,E}}(G)$. □

Example 2.3. Here we give an example G with unbounded degree and such that $p_{\mathrm{T,E}} < p_{\mathrm{T,V}}$. Let $M > 1$ be an integer. Let C_n be a complete graph with M^n vertices. Let $o = (0,0)$ be the origin of $\mathbb{Z}^2$ and let $(n,0) \in \mathbb{Z}^2, n \ge 1$ be the points on the x-axis. For each $n \ge 1$, add an edge from $(n,0)$ to each vertex of C_n. Let G be the graph obtained in this way; see Figure 1. Then obviously $p_{\mathrm{c}}(G) = p_{\mathrm{c}}(\mathbb{Z}^2) = \frac{1}{2}$. Note that for $p \in (0, p_{\mathrm{c}})$, $\mathbb{P}_p[o \longleftrightarrow (n,0) \text{ in } G] = \mathbb{P}_p[o \longleftrightarrow (n,0) \text{ in } \mathbb{Z}^2] \approx e^{-n\varphi(p)}$, where $\varphi(p)$ is the reciprocal of the correlation length (see Proposition 6.47 in [5] for example.) When computing $\mathbb{E}_p[|C_o|_V]$, each clique C_n contributes roughly $p \cdot e^{-n\varphi(p)} \cdot M^n$ but when computing $\mathbb{E}_p[|C_o|_E]$, each clique C_n contributes roughly $p^2 \cdot e^{-n\varphi(p)} \cdot M^{2n}$. Using the properties of $\varphi(p)$ (Theorem 6.14 in [5]) it is easy to show that $0 < p_{\mathrm{T,E}}(G) = \varphi^{-1}(2\log M) < p_{\mathrm{T,V}}(G) = \varphi^{-1}(\log M) < p_{\mathrm{c}}(G)$ and we omit the details.

Before proving Lemma 2.1, we recall the definitions of boundaries of a set of vertices.

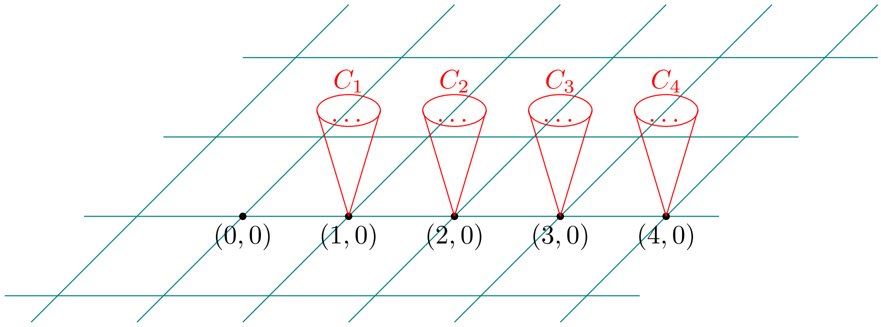

Figure 1: An example with $0 < p_{\mathrm{T,E}} < p_{\mathrm{T,V}} < p_{\mathrm{c}} < 1$.

Definition 2.4. *For a nonempty set of vertices $K \subset V$, we define its inner vertex boundary, outer vertex boundary and edge boundary as follows. The **inner vertex boundary** $\partial_{\mathrm{V}}^{\mathrm{in}} K$ is*

$$\partial_{\mathrm{V}}^{\mathrm{in}} K := \{y \in K \colon \exists\, z \notin K \text{ s.t. } y \sim z\},$$

*where $y \sim z$ denotes that y and z are neighbors in G. The **outer vertex boundary** $\partial_{\mathrm{V}} K$ is*

$$\partial_{\mathrm{V}} K := \{z \notin K \colon \exists\, y \in K \text{ s.t. } y \sim z\}.$$

*The **edge boundary** ΔK is defined as*

$$\Delta K := \{e = (y,z) \in E \colon\ y \in K, z \notin K\}.$$

Lemma 2.5. *If Π is a minimal vertex cutset separating x from infinity, then $\partial_V S(\Pi) = \Pi$, where $S(\Pi)$ is the connected component of x in the subgraph $G \backslash \Pi$. Similarly, if Π is a minimal edge cutset separating x from infinity, then $\Delta S(\Pi) = \Pi$, where $S(\Pi)$ is the connected component of x in the subgraph $G \backslash \Pi$.*

Definition 2.6. *Suppose G is a locally finite, connected, infinite graph. Fix $x \in V(G)$. Let $\mathscr{B}_E = \mathscr{B}_E(x)$ be the collection of edge cutsets that are the edge boundary of some finite cluster of x. In other words,*

$$\mathscr{B}_E = \{\Delta S \colon S \text{ is a finite connected subgraph containing } x\}.$$

*We call $\mathscr{B}_E$ the family of **boundary edge cutset**. Similarly we denote by $\mathscr{B}_V$ the collection of vertex cutsets that arise as outer vertex boundary of some finite cluster of x and call $\mathscr{B}_V$ the family of **boundary vertex cutset**.*

Proof of Lemma 2.5. Suppose Π is a minimal vertex cutset separating x from infinity. The connected component $S(\Pi)$ of x in $G \backslash \Pi$ is finite.

First we show that $\partial_V S(\Pi) \subset \Pi$. For any $z \in \partial_V S(\Pi)$, by definition of $\partial_V S(\Pi)$, there is some vertex $y \in S(\Pi)$ such that $y \sim z$. Then if $z \notin \Pi$, then by definition of $S(\Pi)$, then z can be connected to x via a path from x to y in $G \backslash \Pi$ and the edge (y,z). This implies that $z \in S(\Pi)$ if $z \notin \Pi$, which contradicts with the choice of $z \in \partial_V S(\Pi)$. Hence $\partial_V S(\Pi) \subset \Pi$.

On the other hand, since $\partial_V S(\Pi)$ is also a vertex cutset and Π is minimal with respect to inclusion, one has that $\partial_V S(\Pi) \supset \Pi$.

The case of minimal edge cutset can be proved similarly and we omit the details. $\square$

Remark 2.7. The reverse of Lemma 2.5 is not true. For example consider the half integer line $G = (\mathbb{N}, E)$, where $E = \{(n, n+1) \colon n \in \mathbb{N}\}$. Let $S = \{10, 11, \ldots, 100\}$, then $\Delta S = \{(9,10), (100,101)\}$ is not minimal.

Proof of Lemma 2.1. For any edge cutset Π_E separating x from infinity, let $S(\Pi_E)$ be the connected component of x in $G\backslash\Pi_E$. Since Π_E is a cutset, $S(\Pi_E)$ is finite. Let $\Pi_V = \Pi_V(\Pi_E)$ be the endpoints of edges in $\Delta S(\Pi_E)$ that are not in $S(\Pi_E)$. Then Π_V is a vertex cutset, since every infinite path from x to infinity has to leave $S(\Pi_E)$, the first vertex on the path that is not in $S(\Pi_E)$ must be a vertex in Π_V. For each $v \in \Pi_V$, pick an arbitrary edge $e = e(v) \in \Delta S(\Pi_E)$ such that e is incident to v. Note that

1. $\Delta S(\Pi_E) \subset \Pi_E$;
2. for distinct $v \in \Pi_V$, the edges $e(v)$ are also distinct (since each such edge $e(v)$ has exactly one endpoints not in $S(\Pi_E)$, i.e., v);
3. for such v and $e = e(v)$, $\mathbb{P}[e \in C(x)] \geq p\mathbb{P}_p[v \in C(x)]$ (By insertion-tolerance of Bernoulli bond percolation, see [12, Exercise 7.1]).

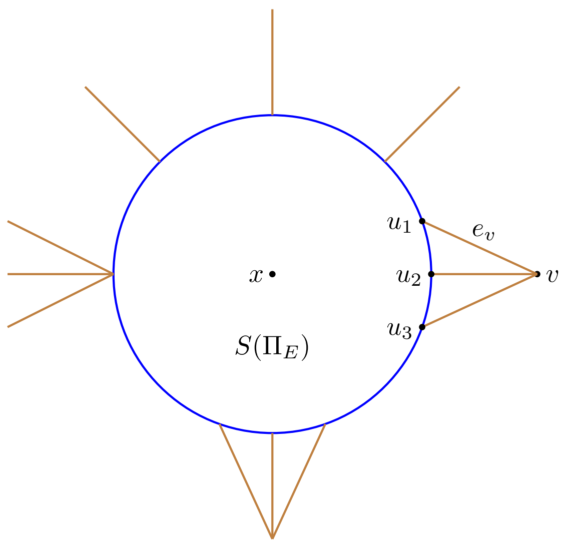

Figure 2: A systematic drawing of $S(\Pi_E)$, $v \in \Pi_V(\Pi_E)$ and $e = e(v)$; edges in $\Delta S(\Pi_E)$ are colored brown.

Hence

$$\mathbb{E}_p[|C(x) \cap \Pi_V|] = \sum_{v \in \Pi_V} \mathbb{P}_p[v \in C(x)] \leq \frac{1}{p} \sum_{e \in \Pi_E} \mathbb{P}_p[e \in C(x)] = \frac{1}{p}\mathbb{E}_p[|C(x) \cap \Pi_E|] \quad (2.10)$$

If $0 < p < p_{\text{cut,E}}$, then $\inf_{\Pi_E} \mathbb{E}_p[|C(x) \cap \Pi_E|] = 0$. Thus by (2.10), for $p < p_{\text{cut,E}}$

$$\inf_{\Pi_V} \mathbb{E}_p[|C(x) \cap \Pi_V|] = 0.$$

Hence if $p < p_{\text{cut,E}}$, then $p \leq p_{\text{cut,V}}$. This implies (2.8).

Now we assume that G has bounded degree. Let Π_V be a vertex cutset and without loss of generality we assume that $x \notin \Pi_V$. Let $S(\Pi_V)$ denote the connected component of x in $G\backslash\Pi_V$. Since Π_V is a cutset, the connected component of $S(\Pi_V)$ is finite. Let $\Pi_E = \Pi_E(\Pi_V)$ be the edge boundary $\Delta S(\Pi_V)$. Now for each edge $e \in \Delta S(\Pi_V)$, there is a unique vertex $v = v(e) \in \Pi_V$ associated to e: v is incident to e (the other endpoint of e is in $S(\Pi_V)$, which is disjoint from Π_V by its definition). Note that

1. for each $v \in \Pi_V$, there are at most $D = D(G)$ edges in $\Delta S(\Pi_V)$ associated to it;
2. for each $e \in \Delta S(\Pi_V)$ and its associated vertex $v = v(E) \in \Pi_V$, $\mathbb{P}_p[v \in C(x)] \geq \mathbb{P}_p[e \in C(x)]$.

Hence

$$\mathbb{E}_p[|C(x) \cap \Pi_E|] = \sum_{e \in \Pi_E} \mathbb{P}_p[e \in C(x)] \leq D \sum_{v \in \Pi_V} \mathbb{P}_p[v \in C(x)] = D\mathbb{E}_p[|C(x) \cap \Pi_V|] \quad (2.11)$$

Thus when $D < \infty$, by (2.11) one has that

$$\forall\ p < p_{\mathrm{cut,V}},\ \inf_{\Pi_E} \mathbb{E}_p[|C(x) \cap \Pi_E|] = 0.$$

Hence when $D < \infty$, $p < p_{\mathrm{cut,V}} \Rightarrow p \leq p_{\mathrm{cut,E}}$. Together with (2.8) one has the equality when $D < \infty$. □

When considering $p_{\mathrm{cut,V}}$ and $p_{\mathrm{cut,E}}$, obviously it suffices to consider minimal cutsets (with respect to inclusion). However a priori it is not clear whether it is sufficient to consider minimal cutsets for $p'_{\mathrm{cut,E}}$ and $p'_{\mathrm{cut,V}}$. This consideration for minimal cutsets together with Lemma 2.5 motivates us to consider $p''_{\mathrm{cut,E}}$ and $p''_{\mathrm{cut,V}}$ in Definition 2.8 and it turns out they all coincides with each other; see Theorem 2.9.

Definition 2.8. *Suppose G is a locally finite, connected, infinite graph. Fix $x \in V(G)$. Define*

$$p''_{\mathrm{cut,V}} = p''_{\mathrm{cut,V}}(G) := \sup\Big\{p \geq 0 \colon \inf_{\Pi \in \mathscr{B}_V} \sum_{v \in \Pi} \mathbb{P}_p[A(x,v,\Pi)] = 0\Big\},$$

where the infimum is taken over all ***boundary vertex cutsets*** Π *that separate x from infinity.*

Similarly, we define

$$p''_{\mathrm{cut,E}} = p''_{\mathrm{cut,E}}(G) := \sup\Big\{p \geq 0 \colon \inf_{\Pi_E \in \mathscr{B}_E} \sum_{e \in \Pi_E} \mathbb{P}_p[A(x,e,\Pi_E)] = 0\Big\},$$

where the infimum is taken over all ***boundary edge cutsets*** Π_E *that separate x from infinity.*

By Definition 1.4 and 2.8, and inequalities (2.2), (2.3) one has that

$$p''_{\mathrm{cut,E}} \leq p'_{\mathrm{cut,E}} \leq p_{\mathrm{c}} \text{ and } p''_{\mathrm{cut,V}} \leq p'_{\mathrm{cut,V}} \leq p_c. \quad (2.12)$$

Theorem 1.6 is contained in the following more general theorem.

Theorem 2.9. *For Bernoulli bond percolation on every locally finite, connected, infinite graph G, one has that*

$$p''_{\mathrm{cut,E}} = p'_{\mathrm{cut,E}} = p''_{\mathrm{cut,V}} = p'_{\mathrm{cut,V}} = p_{\mathrm{c}}.$$

Lemma 2.10. *Suppose G is a locally finite, connected infinite graph. Then*

$$p''_{\mathrm{cut,E}} \leq p''_{\mathrm{cut,V}}. \quad (2.13)$$

Proof. Let $\Pi_E \in \mathscr{B}_E$ be a boundary edge cutset separating x from infinity. Let S be the finite connected component of x in $G \backslash \Pi_E$. By definition $\Pi_E = \Delta S$. Let $\Pi_V = \partial_V S$ be the outer vertex boundary of S. Then $\Pi_V \in \mathscr{B}_V$ is a boundary vertex cutset separating x from infinity.

For each $v \in \Pi_V$, if the event $A(x,v,\Pi_V)$ occurs, then there is a self-avoiding open path $\gamma_{x,v}$ from x to v only using v in Π_V. Hence this path uses only one edge e in ΔS, namely the edge e on $\gamma_{x,v}$ that is incident to v. Hence the event $A(x,e,\Pi_E)$ occurs for this edge e on the path $\gamma_{x,v}$. Thus

$$\mathbb{P}_p(A(x,v,\Pi_V)) \leq \sum_{e \in \Delta S \colon e \sim v} \mathbb{P}_p(A(x,e,\Pi_E)),$$

where $e \sim v$ denotes that v is an endpoint of e.

Note that for any two distinct vertices $v, v' \in \Pi_V$, the two sets $\{e \in \Delta S \colon e \sim v\}$ and $\{e' \in \Delta S \colon e' \sim v'\}$ are disjoint. Hence summing the above inequality over $v \in \Pi_V = \partial S$, one has that

$$\sum_{v \in \Pi_V} \mathbb{P}_p(A(x, v, \Pi_V)) \leq \sum_{e \in \Pi_E} \mathbb{P}_p(A(x, e, \Pi_E)).$$

From this we have that $p < p''_{\mathrm{cut,E}} \Rightarrow p \leq p''_{\mathrm{cut,V}}$ and then we have the desired inequality (2.13). ☐

3 Proof of Theorem 2.9

Duminil-Copin and Tassion [3] gave a new proof of the sharpness of the phase transition [1, 13]. For our purpose, we just need to look at the short version for Bernoulli percolation [4].

For $p \in [0, 1]$, $x \in V$ and a finite set S with $x \in S \subset V$, define

$$\varphi_p(x, S) := p \sum_{y \in S} \sum_{z \notin S, (y,z) \in E} \mathbb{P}_p(x \stackrel{S}{\longleftrightarrow} y), \tag{3.1}$$

where $\{x \stackrel{S}{\longleftrightarrow} y\}$ denotes the event that there is an open path connecting x and y only using vertices lying in S. Recall that the edge boundary ΔS of S is the set of edges that connect S to its complement. So $\varphi_p(x, S)$ is the expected number of open edges on the edge boundary ΔS which has an endpoint is connected to x via an open path entirely lying in S. For transitive graphs, Duminil-Copin and Tassion defined

$$\widetilde{p}_c := \sup\{p \geq 0 \colon \inf\{\varphi_p(x, S) \colon x \in S, S \text{ is finite}\} < 1\}$$

and showed that $\widetilde{p}_c = p_c$ for transitive graphs.

The main new ingredient of the proof of Theorem 2.9 is the following observation.

Proposition 3.1. *For Bernoulli bond percolation on a locally finite, connected, infinite graph G one has that*

$$\inf_{\Pi \in \mathscr{B}_E} \sum_{e \in \Pi} \mathbb{P}_p(A(x, e, \Pi)) = \inf_S \varphi_p(x, S), \tag{3.2}$$

where the infimum on the left hand side of (3.2) *is over all the **boundary edge cutsets** separating x from infinity and the infimum on the right is over all finite sets containing x.*

We have that $p_c(G) = \sup\{p \geq 0 \colon \inf\{\varphi_p(x, S) \colon x \in S, S \text{ is finite}\} = 0\}$ for all locally finite, connected, infinite graphs in light of Proposition 3.1 and Theorem 2.9.

Proof of Proposition 3.1. On the one hand, for any finite set S containing x, let S' be the connected component of x in the induced subgraph of S. Then $\Pi(S) := \Delta S'$ is a boundary edge cutset separating x from infinity. For each edge $e = (y, z) \in \Delta S'$, say $y \in S', z \notin S'$, it is easy to see that $z \notin S$ and

$$\mathbb{P}_p(A(x, e, \Pi(S))) = p \cdot \mathbb{P}_p[x \stackrel{S}{\longleftrightarrow} y].$$

Summing this over all edges $e \in \Delta S'$, one has that

$$\inf_{\Pi \in \mathscr{B}_E} \sum_{e \in \Pi} \mathbb{P}_p(A(x, e, \Pi)) \leq \sum_{e \in \Pi(S)} \mathbb{P}_p(A(x, e, \Pi(S))) = \varphi_p(x, S') = \varphi_p(x, S),$$

where the last equality is a simple observation from the definition of S'.

Hence

$$\inf_{\Pi\in\mathscr{B}_E}\sum_{e\in\Pi}\mathbb{P}_p(A(x,e,\Pi))\le\inf_S\varphi_p(x,S).$$

On the other hand, for any boundary edge cutset Π separating x from infinity, let $S=S(\Pi)$ be the connected component of x in the graph $G\backslash\Pi$. By Definition 2.6, $\Delta S=\Pi$. For each edge $e=(y,z)\in\Delta S=\Pi$, say $y\in S,z\notin S$, one has that

$$\mathbb{P}_p(A(x,e,\Pi))=p\cdot\mathbb{P}_p[x\overset{S}{\longleftrightarrow}y].$$

Summing this over all edges $e\in\Delta S$, one has that for a boundary edge cutset Π and $S=S(\Pi)$

$$\sum_{e\in\Pi}\mathbb{P}_p(A(x,e,\Pi))=\varphi_p(x,S(\Pi))\ge\inf_S\varphi_p(x,S).$$

Hence one has the other direction

$$\inf_{\Pi\in\mathscr{B}_E}\sum_{e\in\Pi}\mathbb{P}_p(A(x,e,\Pi))\ge\inf_S\varphi_p(x,S).$$ □

Next we recall a lemma from [4]. For a finite set Λ, let Λ^c denote its complement in V. Let Λ_n denote the ball $\{y:d(x,y)\le n\}$ of radius n centered at x, where d denotes the graph distance on G.

Lemma 3.2 (Lemma 2.1 of [4]). *For $x\in V$ and ball Λ_n with $n\ge1$, one has*

$$\frac{d}{dp}\mathbb{P}_p(x\longleftrightarrow\Lambda_n^c)\ge\frac{1}{p(1-p)}\cdot\inf_{S\subset\Lambda_n,x\in S}\varphi_p(x,S)\cdot[1-\mathbb{P}_p(x\longleftrightarrow\Lambda_n^c)]\tag{3.3}$$

Proof of Theorem 2.9. By (2.12) and Lemma 2.10, it suffices to show $p''_{\mathrm{cut,E}}\ge p_{\mathrm{c}}$.

Suppose $p''_{\mathrm{cut,E}}<p_{\mathrm{c}}$. Pick p_0,p_1 such that $p''_{\mathrm{cut,E}}<p_0<p_1<p_{\mathrm{c}}$.

By the definition of $p''_{\mathrm{cut,E}}$ and Proposition 3.1, there is a constant $\kappa>0$ such that for any $p\in[p_0,p_1]$,

$$\inf_S\varphi_p(x,S)\ge\kappa.$$

Write $\theta_x(p):=\mathbb{P}_p(x\longleftrightarrow\infty)$ and $\theta_{x,n}(p):=\mathbb{P}_p(x\longleftrightarrow\Lambda_n^c)$. By (3.3) one has that for $p\in[p_0,p_1]$,

$$\frac{\theta'_{x,n}(p)}{1-\theta_{x,n}(p)}\ge\frac{\kappa}{p(1-p)}.$$

Integrating this inequality from p_0 to p_1, one has that

$$\theta_{x,n}(p_1)\ge1-\left(\frac{1-p_1}{p_1}\cdot\frac{p_0}{1-p_0}\right)^\kappa+\theta_{x,n}(p_0)\left(\frac{1-p_1}{p_1}\cdot\frac{p_0}{1-p_0}\right)^\kappa\ge1-\left(\frac{1-p_1}{p_1}\cdot\frac{p_0}{1-p_0}\right)^\kappa.\tag{3.4}$$

Letting $n\to\infty$ one has that

$$\theta_x(p_1)\ge1-\left(\frac{1-p_1}{p_1}\cdot\frac{p_0}{1-p_0}\right)^\kappa>0,$$

which contradicts with the choice that $p_1<p_{\mathrm{c}}$. Hence $p''_{\mathrm{cut,E}}\ge p_{\mathrm{c}}$ and we are done. □

4 Percolation probability for subperiodic trees

To highlight the importance of Proposition 3.1, in this section we discuss some applications of it to subperiodic trees.

For transitive graphs, Duminil-Copin and Tassion pointed out that $\inf\{\varphi_p(x,S):x\in S,\ S\text{ is finite}\}\ge1$ at $p=p_{\mathrm{c}}$. Using this they obtained a lower bound for percolation probability on transitive graphs with $p_c \in (0,1)$: $\theta(p) \geq \frac{p-p_c}{p(1-p_c)}$ for $p \geq p_c$. Here for transitive graphs, the percolation probability $\theta(x,p)$ does not depend on x and we simply write it as $\theta(p)$. This lower bound can be extended to 0-subperiodic trees. We first adopt some notations and then recall the definitions of periodic and subperiodic trees as in [12, Section 3.3].

Notation. Suppose T is an infinite, locally finite tree with a distinguished vertex o, called the ***root*** of T. Write $|x|$ for the graph distance from x to o; $x \leq y$ if x is on the shortest path from o to y; $x < y$ if $x \leq y$ and $x \neq y$; $x \to y$ if $x \leq y$ and $|y| = |x| + 1$ and in this case we call x the parent of y; and T^x for the subtree of T containing the vertices $y \geq x$. For Bernoulli(p) percolation on the tree T with root o, let $\theta(p) := \mathbb{P}_p[o \longleftrightarrow \infty]$ be the probability that o is in an infinite cluster.

Definition 4.1 (Definition on page 82 of [12])**.** *Let $N \geq 0$ be an integer. An infinite, locally finite tree T with root x is called N**-periodic** (resp., N**-subperiodic**), if $\forall x \in T$ there exists an adjacency-preserving bijection (resp., injection) $f : T^x \to T^{f(x)}$ with $|f(x)| \leq N$. A tree is **periodic** (resp., **subperiodic**) if there is some $N \geq 0$ for which it is N-periodic (resp., N-subperiodic).*

Remark 4.2. For a 0-subperiodic tree T with root o, one has that $\inf_\Pi \sum_{e\in\Pi} \mathrm{br}(T)^{-|e|} \geq 1$ (formula (3.7) on page 85 of [12]). It is well-known that $p_c(T) = 1/\mathrm{br}(T)$ for every locally finite infinite tree T (for instance see [12, Theorem 5.15]). Hence

$$\inf_{\Pi\in\mathscr{B}_E} \sum_{e\in\Pi} \mathbb{P}_{p_c}(A(o,e,\Pi)) = \inf_{\Pi\in\mathscr{B}_E} \sum_{e\in\Pi} p_c^{|e|} \geq \inf_{\Pi} \sum_{e\in\Pi} \mathrm{br}(T)^{-|e|} \geq 1.$$

Then by Proposition 3.1 one can set $p_0 = p_c$ and $\kappa = 1$ in (3.4) and letting $n \to \infty$ to get

$$\theta(p) = \mathbb{P}_p(o \longleftrightarrow \infty) \geq \frac{p - p_c}{p(1-p_c)},\ p \geq p_c$$

for every 0-subperiodic tree T.

Actually this is true for all subperiodic trees with nontrivial p_c.

Proposition 4.3. *Consider Bernoulli percolation on a subperiodic tree T with $p_c(T) < 1$. Then the lower right Dini derivative of the percolation probability $\theta(p)$ at p_c belong to $(0,\infty]$.*

Proof. Theorem 3.8 in [12] says that $\inf_\Pi \sum_{e\in\Pi} \mathrm{br}(T)^{-|e|} > 0$ for a general subperiodic tree T with $p_c(T) < 1$, where Π runs over all edge cutsets separating the root of T from infinity. Define

$$\alpha(o,p) := \inf\{\varphi_p(o,S) : o \in S,\ S \text{ is finite}\}.$$

Then as before, one has that for a subperiodic tree T with root o and $p_c < 1$,

$$\alpha(o,p_c) \overset{\text{Prop.3.1}}{=} \inf_{\Pi\in\mathscr{B}_E} \sum_{e\in\Pi} \mathbb{P}_{p_c}(A(o,e,\Pi)) = \inf_{\Pi\in\mathscr{B}_E} \sum_{e\in\Pi} p_c^{|e|} \geq \inf_{\Pi} \sum_{e\in\Pi} \mathrm{br}(T)^{-|e|} > 0.$$

By the definition of φ in (3.1) it is obvious that $\alpha(o,p)$ is increasing in p. Setting $p_0 = p_c$ in (3.4) and letting $n \to \infty$ one has that

$$\frac{\theta(p) - \theta(p_c)}{1 - \theta(p_c)} \geq 1 - \left(\frac{1-p}{p} \cdot \frac{p_c}{1-p_c}\right)^{\alpha(o,p_c)}.$$

This implies that the lower right Dini derivative of the percolation probability $\theta(p)$ at p_c is positive:

$$D_+\theta(p_c) := \liminf_{p\to p_c^+} \frac{\theta(p) - \theta(p_c)}{p - p_c} \geq \frac{\alpha(o,p_c)(1-\theta(p_c))}{p_c(1-p_c)} > 0. \qquad \square$$

Without of the restriction of (sub)periodicity, it is possible to have the lower right Dini derivative of the percolation probability being equal to zero. See the following two remarks.

Remark 4.4. It is easy to construct trees with the property that at $p = p_{\mathrm{c}}$,

$$\inf\{\varphi_p(o, S) : o \in S,\ S \text{ is finite}\} = 0.$$

Indeed, we construct a spherically symmetric tree T with root o as follows. Let T_n denote the set of vertices with graph distance n to the root o. If $n = 2^k$ for some $k \geq 0$, let each vertex in T_n have exactly one child; otherwise, let each vertex in T_n have exactly two children. Then it is easy to see that

$$|T_n| \asymp \frac{2^n}{n}.$$

Here for two positive function f, g on $\mathbb{Z}^+$, $f(n) \asymp g(n)$ means that there exist constants $c_1, c_2 > 0$ such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n > 0$.

Hence $\mathrm{br}(T) = \liminf_n |T_n|^{1/n} = 2$ by Exercise 1.2 in [12]. Thus $p_{\mathrm{c}}(T) = 1/\mathrm{br}(T) = 1/2$. Let S_n be the ball of radius n and center x. Then at $p = p_{\mathrm{c}} = 1/2$,

$$\varphi_p(o, S_n) = |T_{n+1}| \cdot \frac{1}{2^{n+1}} \leq \frac{c_2}{n+1}.$$

Thus $\inf\{\varphi_p(o, S) : o \in S,\ S \text{ is finite}\} = 0$.

Remark 4.5. For the spherically symmetric tree T in Remark 4.4, one also has

$$\theta(p) \asymp \big(p - \frac{1}{2}\big)^2,\ p \geq p_{\mathrm{c}}$$

and in particular, the upper right Dini derivative $D^+\theta(p_{\mathrm{c}}) := \limsup_{p \to p_{\mathrm{c}}^+} \frac{\theta(p) - \theta(p_{\mathrm{c}})}{p - p_{\mathrm{c}}} = 0$.

In fact, let $c(e) = (1-p)^{-1} p^{|e|}$ be the conductance of edge e. Then formula (5.12) on page 142 of [12] is satisfied with $\mathbf{P} = \mathbb{P}_p$. Since T is spherically symmetric, for $p > p_{\mathrm{c}} = 1/2$, the effective resistance is

$$\mathscr{R}(o \longleftrightarrow \infty) = \sum_{n=1}^{\infty} (1-p) p^{-n} / |T_n| \asymp \sum_{n=1}^{\infty} \frac{(1-p)n}{(2p)^n} \asymp \frac{1-p}{(2p-1)^2}.$$

Then by Theorem 5.24 [12] one has

$$\theta(p) \asymp \frac{\mathscr{C}(o \longleftrightarrow \infty)}{1 + \mathscr{C}(o \longleftrightarrow \infty)} = \frac{1}{1 + \mathscr{R}(o \longleftrightarrow \infty)} \asymp \big(p - \frac{1}{2}\big)^2,\ p > p_{\mathrm{c}}.$$

Proposition 4.3 states that the lower right Dini derivative of the percolation probability on a subperiodic tree with $p_c \in (0,1)$ is positive at p_c and it might equal to infinity in some cases (Example 4.10). This leads us to the following question:

Question 4.6. What kind of subperiodic trees have the property that the right Dini derivatives of $\theta(p)$ at p_{c} are finite? What kind of subperiodic trees have the property that $\lim_{p \downarrow p_{\mathrm{c}}} \frac{\theta(p) - \theta(p_{\mathrm{c}})}{p - p_{\mathrm{c}}} \in (0, \infty)$?

The critical exponent β for Bernoulli percolation is characterized by $\theta(p) - \theta(p_{\mathrm{c}}) \approx (p - p_{\mathrm{c}})^{\beta}$. For $\mathbb{Z}^2$, it is conjectured that $\theta(p) - \theta(p_{\mathrm{c}}) \approx (p - p_{\mathrm{c}})^{\beta}$ for $\beta = \frac{5}{36}$ [5, Table 10.1 on page 279], in particular in this case the lower right Dini derivative at p_{c} is infinite [9]. Indeed for site percolation on the triangular lattice in the plane, one does have $\theta(p) - \theta(p_{\mathrm{c}}) = (p - p_{\mathrm{c}})^{\frac{5}{36} + o(1)}$ [14, Theorem 1.1]. Question 4.6 asks what kind of subperiodic trees have $\beta = 1$.

A partial answer for Question 4.6 is Theorem 4.7 which considers directed covers of strongly connected graphs. An oriented graph G is called ***strongly connected*** if for any two vertices u, v of G, there is a directed path in G from u to v. Suppose that G is a finite oriented graph and v is any vertex in G. The ***directed cover*** of G based at v is the tree T whose vertices are the finite paths of edges $\langle e_1, e_2, \dots, e_n\rangle$ in G that start at v. We take the root of T to be the empty path and we join two vertices in T by an edge when a path is an extension of the other path by one more edge in G. Every periodic tree is a directed cover of a finite directed graph G; for a proof see pages 82-83 of [12]. Also not all periodic trees have finite right derivatives for $\theta(p)$ at p_{c}; see item 3 in Example 4.10.

Theorem 4.7. *If T is a periodic tree with root o and $p_{\mathrm{c}}(T) \in (0,1)$ and it is the directed cover of some strongly connected graph, then the right derivative of $\theta(p)$ exists at p_{c} and this derivative is positive and finite.*

We first outline the ideas of the proof of Theorem 4.7. Suppose T is the directed cover of some finite strongly connected directed graph $G = (V, E)$ based at some vertex $v_1 \in V$, where $V = \{v_1, \dots, v_n\}$. Let $\theta_i(p)$ be the percolation probability of the tree T_i that is the directed cover of G based at v_i. Then these quantities $\theta_i(p)$ are related by a family of algebraic equations (4.1). We will then proceed as follows:

1. We first show that $\theta_i(p)$ are continuous at $p_c(T_i)$ (Lemma 4.8).
2. Then we show that these trees T_i have the same critical probability p_c and the upper right Dini derivatives of $\theta_i(p)$ are finite at p_c (Lemma 4.9).
3. We finish the preparation by showing that the right derivatives of $\theta_i(p)$ at p_c exist via a convexity/concavity argument in Lemma 4.12. The proof of Lemma 4.12 is a little bit involved and we will need two more lemmas for its proof:
   - the functions $\theta_i(p)$ are analytic on $(p_c, 1)$ (Lemma 4.13) and
   - the uniqueness of $\theta_i(p)$ as solutions of (4.1) (Lemma 4.14).
4. We then finish the proof of Theorem 4.7: the existence is from Step 3 (Lemma 4.12), the positiveness is from Proposition 4.3 and the finiteness is from Step 2 (Lemma 4.9).

Lemma 4.8. *Suppose T is a periodic tree with root o and $p_{\mathrm{c}}(T) < 1$. Then $\theta(p_{\mathrm{c}}) = 0$.*

Proof. At $p = p_{\mathrm{c}} = \frac{1}{\mathrm{br}(T)}$, if we put conductance $c(e(x)) = (1-p_{\mathrm{c}})^{-1} p_{\mathrm{c}}^{|x|}$, where $e(x)$ is the edge from x to its parent, then (5.12) on page 142 of [12] is satisfied. As noted on page 142 line 17 of [12], these conductances correspond to the homesick random walk $\mathrm{RW}_{\mathrm{br}(T)}$. If we put resistance $\Phi(e(x)) = \lambda^{|x|-1}$ for the edge $e(x)$ instead, then it is known that as $\lambda \uparrow \lambda_* = \frac{1}{p_c}$, the effective resistance from the root to infinity of the corresponding network is tend to infinity [11, Theorem 5.1]. This implies that the homesick random walk $\mathrm{RW}_{\mathrm{br}(T)}$ is recurrent. Hence by Corollary 5.25 of [12] we know $\theta(p_{\mathrm{c}}) = 0$. $\square$

Now we restrict to a subset of periodic trees that are directed covers of finite strongly connected oriented graphs.

Lemma 4.9. *Suppose $G = (V(G), E(G))$ is a finite, strongly connected directed graph and $V(G) = \{v_1, \dots, v_n\}$. Let λ_* be the largest eigenvalue of the adjacency matrix A_G of G. Let T_i be the directed cover of G based at v_i and denote its root by o_i. Then $p_{\mathrm{c}}(T_1) = \cdots = p_{\mathrm{c}}(T_n) = \frac{1}{\lambda_*}$.*

Moreover if $\lambda_* > 1$, then the upper right Dini derivative of $\theta_i(p)$ at p_{c} is finite for every $i \in \{1, \cdots, n\}$, where $\theta_i(p) := \mathbb{P}_p[o_i \longleftrightarrow \infty \text{ in } T_i]$ denotes the probability that the root o_i of T_i is in an infinite open cluster.

Proof. The first part is a standard result. See the discussion on pages 83-84 of [12] for example.

Since G is strongly connected, A_G is irreducible. Hence by the Perron-Frobenius theorem (e.g. see [6, Theorem 8.4.4]) there is a left λ_*-eigenvector $v_* = (v_1, \dots, v_n)$ all of whose entries are positive. We also normalize v_* such that its l_2-norm is 1.

Since $o_i \not\leftrightarrow \infty$ in T_i if and only if o_i can't connect to infinity via any of its children, we have the following relations for these percolation probabilities:

$$1 - \theta_i(p) = \prod_{j=1}^{n} [1 - p\theta_j(p)]^{a_{ij}}, \ i \in \{1, \dots, n\},$$

i.e.,

$$\theta_i(p) = 1 - \prod_{j=1}^{n} [1 - p\theta_j(p)]^{a_{ij}}, \ i \in \{1, \dots, n\}, \tag{4.1}$$

where a_{ij} is the (i, j)-entry of the matrix A_G, i.e., the number of directed edges in G from vertex v_i to v_j.

Denote by $\theta_{\max}(p) = \max\{\theta_1(p), \cdots, \theta_n(p)\}$. Since G is strongly connected, there exists $M > 0$ such that there is a directed path with length at most M from v_i to v_j for any pair $v_i, v_j \in V(G)$. Hence $\theta_i(p) \geq p^M \theta_{\max}(p)$ for all $i = 1, \dots, n$. Thus for $p > p_c$,

$$0 < \theta_i(p) \asymp \theta_{\max}(p), \ i \in \{1, \dots, n\}, . \tag{4.2}$$

By Lemma 4.8 and the right continuity of $\theta_i(p)$ (e.g., see [12, Exercise 7.33]), one has that

$$0 < \theta_i(p) = o(p - p_c), \ 0 < p - p_c \ll 1. \tag{4.3}$$

Using (4.3) when $0 < p - p_c \ll 1$ and $i \in \{1, \dots, n\}$, we can rewrite (4.1) as

$$\theta_i(p) = p \sum_{j=1}^{n} a_{ij}\theta_j(p) - p^2 \sum_{j=1}^{n} \binom{a_{ij}}{2} \theta_j^2(p) - p^2 \sum_{j \neq k} a_{ij}a_{ik}\theta_j(p)\theta_k(p) + \theta_{\max}^2(p) \cdot o(1), \tag{4.4}$$

where we use the convention that $\binom{a_{ij}}{2} = 0$ if $a_{ij} = 0, 1$.

Multiplying v_i on both sides of (4.4) and adding them up, one has that

$$\begin{aligned} \sum_{i=1}^{n} v_i\theta_i(p) \quad &= \quad p \sum_{i=1}^{n} v_i \sum_{j=1}^{n} a_{ij}\theta_j(p) + \theta_{\max}^2(p) \cdot o(1) \\ &\quad -p^2 \sum_{i=1}^{n} v_i \left[\sum_{j=1}^{n} \binom{a_{ij}}{2} \theta_j^2(p) + \sum_{j \neq k} a_{ij}a_{ik}\theta_j(p)\theta_k(p) \right] \end{aligned} \tag{4.5}$$

Since $p_c(T_i) = \frac{1}{\lambda_*} < 1$, there exists some i such that either $a_{ij} \geq 2$ for some j or $a_{ij}a_{ik} \geq 1$ for some $j \neq k$. Therefore by (4.2) and (4.5) there exists $c > 0$ such that

$$\sum_{i=1}^{n} v_i\theta_i(p) \leq p \sum_{i=1}^{n} v_i \sum_{j=1}^{n} a_{ij}\theta_j(p) - cp^2\theta_{\max}^2(p), \ 0 < p - p_c \ll 1. \tag{4.6}$$

Since v_* is a left λ_*-eigenvector of A_G, one has that

$$\sum_{i=1}^{n} v_i \sum_{j=1}^{n} a_{ij}\theta_j(p) = v_* A_G \boldsymbol{\theta}(p) = \lambda_* v_* \cdot \boldsymbol{\theta}(p) = \lambda_* \sum_{i=1}^{n} v_i\theta_i(p), \tag{4.7}$$

where $\boldsymbol{\theta}(p) = (\theta_1(p), \cdots, \theta_n(p))^T$ is the vector of percolation probabilities in $\mathbb{R}^n$.

Plugging (4.7) into (4.6) and using $\lambda_* = \frac{1}{p_c}$ and (4.2) one has that for $0 < p - p_c \ll 1$,

$$cp^2\theta_{\max}^2(p) \le \frac{p-p_c}{p_c}\sum_{i=1}^{n} v_i\theta_i(p) \le c'\theta_{\max}(p)(p-p_c),$$

for some constant $c' > 0$. This implies that $\theta_i(p) \le \theta_{\max}(p) \le c''(p - p_c)$ for $0 < p - p_c \ll 1$ and then we have the desired result on the upper right Dini derivative. $\square$

The strongly connectedness of G is needed for the finiteness of the right Dini derivative. See the following example.

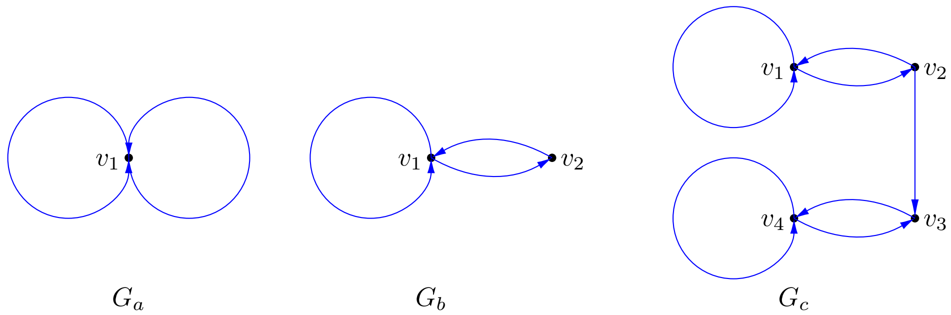

Figure 3: Directed graphs G_a, G_b, G_c from left to right.

Example 4.10. Let G_a, G_b, G_c be as illustrated in Figure 3. Let $T_s, s \in \{a, b, c\}$ be the directed cover of G_s based at $v_1(G_s)$.

1. The tree T_a is a binary tree with root o. It is easy to see that $p_c(T_a) = \frac{1}{2}$ and for $p \ge \frac{1}{2}$, $\theta(p) = \frac{2p-1}{p^2}$. In this case $\theta'_+(p_c) = 8$ and $\theta(p)$ is concave on $(p_c, 1)$.
2. The tree T_b is a Fibonacci tree with root o and $\deg(o) = 2$. See Figure 3.2 on page 83 of [12] for an illustration of the Fibonacci tree. It is easy to see that $p_c(T_b) = \frac{\sqrt{5}-1}{2}$. Writing $\theta(p) = \mathbb{P}_p[o \longleftrightarrow \infty]$ and using (4.1) one has that $\theta(p) = \frac{p^2+p-1}{p^3}$ for $p \ge \frac{\sqrt{5}-1}{2}$. Hence in this case $\theta'_+(p_c) = 5 + \sqrt{5}$ and $\theta(p)$ is also concave on $(p_c, 1)$.
3. The tree T_c also has $p_c(T_c) = \frac{\sqrt{5}-1}{2}$. Actually if we define $T_i, \theta_i(p)$ as in Lemma 4.9, then it is easy to check that $p_c(T_i) = \frac{\sqrt{5}-1}{2}, i = 1, 2, 3, 4$. Solving (4.1) one can find that for $p \in (p_c, 1)$,

$$\begin{cases} \theta_1(p) &= \frac{(1-2p)(p^2+p-1)+\sqrt{(p^2+p-1)(-3p^2+5p-1)}}{2p^2(1-p^2)} \\ \theta_2(p) &= \frac{p^2+p-1+\sqrt{(p^2+p-1)(-3p^2+5p-1)}}{2p^2} \\ \theta_3(p) &= \frac{p^2+p-1}{p^2} \\ \theta_4(p) &= \frac{p^2+p-1}{p^3} \end{cases} \tag{4.8}$$

   In particular, $\theta(p) = \theta_1(p) = \Theta(\sqrt{p - p_c})$ for $0 < p - p_c \ll 1$ and thus the right Dini derivative at p_c is infinite. One can also check that $\theta_1(p)$ and $\theta_2(p)$ are concave on $(p_c, 1)$.

Example 4.10 and the fact that $\theta(p) - \theta(p_c) \approx (p - p_c)^{5/36}$ on the triangle lattice on the plane [14, Theorem 1.1] lead us to the following question:

Question 4.11. For a transitive graph or a periodic tree with root o, is the percolation probability $\theta(p)$ concave on $(p_c, 1)$?

We now proceed to show the right derivatives of $\theta_i(p)$ exist at p_c.

Lemma 4.12. *The right derivative of $\theta_i(p)$ exists at p_c for all $i \in \{1, \ldots, n\}$.*

As mentioned earlier, the proof of Lemma 4.12 is somewhat long and we begin by showing that $\theta_i(p)$ is analytic on $(p_c, 1)$ for all $i \in \{1, \ldots, n\}$.

Lemma 4.13. *Under the same assumptions as Lemma 4.9 one has that $\theta_i(p)$ is analytic on $(p_c, 1)$ for all $i \in \{1, \cdots, n\}$.*

Proof. Recall that the percolation probabilities satisfy (4.1):

$$\theta_i(p) = 1 - \prod_{j=1}^{n} [1 - p\theta_j(p)]^{a_{ij}},\ i \in \{1, \ldots, n\},$$

where a_{ij} is the number of directed edges in G from vertex v_i to v_j.

Define $f_i : \mathbb{R}^{1+n} \to \mathbb{R}$ for $i \in \{1, \ldots, n\}$ by

$$f_i((x, y_1, \ldots, y_n)^T) = y_i - 1 + \prod_{j=1}^{n} [1 - xy_j]^{a_{ij}}.$$

Write $\boldsymbol{f} = (f_1, \cdots, f_n)^T$. By (4.1), we know that $(p, \theta_1(p), \ldots, \theta_n(p))$ is a positive solution of $\boldsymbol{f} = \mathbf{0}$ when $p > p_c$.

Note that when $i \neq j$,

$$\frac{\partial f_i}{\partial y_j}(p, \theta_1(p), \ldots, \theta_n(p)) = -pa_{ij}[1 - p\theta_j(p)]^{a_{ij}-1} \cdot \prod_{j' \neq j} [1 - p\theta_{j'}(p)]^{a_{ij'}} \overset{(4.1)}{=} \frac{-pa_{ij}(1 - \theta_i(p))}{1 - p\theta_j(p)}$$

and

$$\frac{\partial f_i}{\partial y_i}(p, \theta_1(p), \ldots, \theta_n(p)) = 1 - pa_{ii}[1 - p\theta_i(p)]^{a_{ii}-1} \cdot \prod_{j' \neq i} [1 - p\theta_{j'}(p)]^{a_{ij'}} \overset{(4.1)}{=} 1 - \frac{pa_{ii}(1 - \theta_i(p))}{1 - p\theta_i(p)}.$$

Therefore the Jacobi matrix $J = \left[\frac{\partial f_i}{\partial y_j}(p, \theta_1(p), \ldots, \theta_n(p))\right]_{1 \le i,j \le n}$ can be written as

$$J = I - BC \tag{4.9}$$

where I is the identity matrix and B is a diagonal matrix with $b_{ii} = 1 - \theta_i(p)$ and C is a matrix with (i, j)-entry $c_{ij} = \frac{pa_{ij}}{1 - p\theta_j(p)}$.

Notice that Bernoulli(p) percolation on T_i can also be viewed as a multi-type Galton–Watson tree Z. Each vertex u on the tree corresponds to a directed path on G. If the endpoint of the path is v_j, then we say that u has type j. In particular, we view the root of T_i is of type i. The number of type j children of a type i vertex has Binomial distribution $\mathrm{Bin}(a_{ij}, p)$. The percolation probability $\theta_i(p)$ is just the non-extinction probability for such a n-type Galton–Watson tree started with a single type i vertex. Let $\mathbb{P}_s$ and $\mathbb{E}_s$ denote the probability measure and corresponding expectation for such an n-type Galton-Watson tree started with a single ancestor with type $s \in \{1, \ldots, n\}$.

Now let Ext denote the event that the n-type Galton–Watson tree is extinct. Then $\mathbb{P}_i[\mathsf{Ext}] = 1 - \theta_i(p)$. Let Z_{1j} denote the number of children of type j of Z_0. For a nonnegative integer sequence $(t_1,\ldots,t_n)$ with $t_i \le a_{ij}$, one has that

$$\begin{aligned}
& \mathbb{P}_i\big[Z_{1j}=t_j, j=1,2,\ldots,n\big|\mathsf{Ext}\big] \\
= \quad & \frac{1}{1-\theta_i(p)}\cdot\prod_{j=1}^{n}\binom{a_{ij}}{t_j}p^{t_j}(1-p)^{a_{ij}-t_j}\cdot(1-\theta_j(p))^{t_j} \\
= \quad & \frac{1}{1-\theta_i(p)}\cdot\prod_{j=1}^{n}\binom{a_{ij}}{t_j}(1-p)^{a_{ij}-t_j}\cdot(p-p\theta_j(p))^{t_j} \\
\overset{(4.1)}{=} \quad & \prod_{j=1}^{n}\binom{a_{ij}}{t_j}\Big(\frac{1-p}{1-p\theta_j(p)}\Big)^{a_{ij}-t_j}\cdot\Big(\frac{p-p\theta_j(p)}{1-p\theta_j(p)}\Big)^{t_j} \\
= \quad & \prod_{j=1}^{n}\binom{a_{ij}}{t_j}\Big(\frac{p-p\theta_j(p)}{1-p\theta_j(p)}\Big)^{t_j}\cdot\Big(1-\frac{p-p\theta_j(p)}{1-p\theta_j(p)}\Big)^{a_{ij}-t_j}.
\end{aligned} \tag{4.10}$$

By [7] we know conditioned on extinction, the n-type Galton-Watson tree is still a multi-type Galton–Watson tree. Let $\widetilde{\mathbb{P}}_s$ and $\widetilde{\mathbb{E}}_s$ denote the probability measure and corresponding expectation for the n-type Galton-Watson tree started with a single ancestor with type s **conditioned on extinction**. By (4.10), conditioned on extinction, the number of type j children of a type i vertex has Binomial distribution $\mathrm{Bin}(a_{ij},\frac{p-p\theta_j(p)}{1-p\theta_j(p)})$. Hence the mean offspring matrix M has (i,j)-entry $m_{ij}=a_{ij}\frac{p-p\theta_j(p)}{1-p\theta_j(p)}=(1-\theta_j(p))\cdot\frac{pa_{ij}}{1-p\theta_j(p)}$. Observe that

$$M = CB \tag{4.11}$$

Let $q:=\max_{1\le j\le n}[1-\theta_j(p)]$ be the maximum of the extinction probability. For $p>p_{\mathrm{c}}$, we know $q<1$. Let Z_k denote the size of k-th generation of the multi-type Galton-Watson tree. As the last displayed inequality on page 547 of [7], one has that

$$\widetilde{\mathbb{E}}_s[Z_k]\le\frac{1}{1-\theta_s(p)}\cdot\mathbb{E}_s\big[Z_k q^{Z_k}\big]\to 0 \text{ as } k\to\infty.$$

Hence the largest eigenvalue $\lambda_1(M)$ for the mean offspring matrix M satisfies $\lambda_1(M)<1$. By [6, Theorem 1.3.22], the largest eigenvalue of BC satisfies that $\lambda_1(BC)=\lambda_1(CB)=\lambda_1(M)<1$. Therefore by (4.9) the Jacobi matrix J is invertible for $p\in(p_{\mathrm{c}},1)$. Hence by the analytic implicit function theorem, we obtain that the functions $\theta_i(p)$ are analytic on $(p_{\mathrm{c}},1)$. □

Lemma 4.14. *The solution $(p,\theta_1(p),\ldots,\theta_n(p))$ of (4.1) in $(p_{\mathrm{c}},1)\times(0,1)^n$ is unique.*

The following Proposition 4.15 will be needed for Lemma 4.14.

For $p\in[0,1]$, we define the operator $\boldsymbol{B}_p:[0,1]^n\longrightarrow[0,1]^n$ as given by (4.1):

$$\boldsymbol{B}_p(\boldsymbol{\alpha})_i = 1-\prod_{j=1}^{n}[1-p\alpha_j]^{a_{ij}}, \tag{4.12}$$

where $\boldsymbol{\alpha}=(\alpha_1,\cdots,\alpha_n)^T\in[0,1]^n$. For example, $\boldsymbol{B}_p(\mathbf{0})=\mathbf{0}$.

For $\boldsymbol{\alpha},\boldsymbol{\beta}\in[0,1]^n$, write $\boldsymbol{\alpha}\le\boldsymbol{\beta}$ if $\alpha_i\le\beta_i$ for all $i\in\{1,\cdots,n\}$ and write $\boldsymbol{\alpha}\prec\boldsymbol{\beta}$ if $\boldsymbol{\alpha}\le\boldsymbol{\beta}$ and $\boldsymbol{\alpha}\ne\boldsymbol{\beta}$.

Proposition 4.15. *We have the following properties for the operator $\boldsymbol{B}_p$.*

(a) The operator $\boldsymbol{B}_p$ is increasing in the sense that if $\boldsymbol{\alpha}\le\boldsymbol{\beta}$, then $\boldsymbol{B}_p(\boldsymbol{\alpha})\le\boldsymbol{B}_p(\boldsymbol{\beta})$.

(b) Moreover, if $p\in(0,1)$, then $\boldsymbol{B}_p$ is strictly increasing in the sense that if $\boldsymbol{\alpha}\prec\boldsymbol{\beta}$, then $\boldsymbol{B}_p(\boldsymbol{\alpha})\prec\boldsymbol{B}_p(\boldsymbol{\beta})$.

(c) If $0\le p_1<p_2\le 1$ and $\boldsymbol{\alpha}\in[0,1]^n$, then $\boldsymbol{B}_{p_1}(\boldsymbol{\alpha})\le\boldsymbol{B}_{p_2}(\boldsymbol{\alpha})$.

(d) *Moreover, if* $\boldsymbol{\alpha} \in [0,1]^n$ *and* $\boldsymbol{\alpha} \neq \mathbf{0}$*, then for* $0 \leq p_1 < p_2 < 1$*, one has that* $\boldsymbol{B}_{p_1}(\boldsymbol{\alpha}) \prec \boldsymbol{B}_{p_2}(\boldsymbol{\alpha})$*.*

(e) *For* $p > 0$*, if* $\mathbf{0} \neq \boldsymbol{\alpha} \in [0,1]^n$ *is a fixed point of* $\boldsymbol{B}_p$*, i.e.,* $\boldsymbol{B}_p(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$*, then* $\alpha_i > 0$ *for all* $i \in \{1, \cdots, n\}$*.*

Proof of Proposition 4.15. The items (a) and (c) are obvious from the definition of $\boldsymbol{B}_p$. For item (b), suppose $\alpha_j < \beta_j$ for some $j \in \{1, \cdots, n\}$. Since G is strongly connected, there exists some i such that $a_{ij} \geq 1$. Then

$$[1 - p\alpha_j]^{a_{ij}} > [1 - p\beta_j]^{a_{ij}}$$

and $[1 - p\alpha_{j'}]^{a_{ij'}} \geq [1 - p\beta_{j'}]^{a_{ij'}} \geq [1 - p]^{a_{ij'}} > 0$ for $j' \neq j$. Therefore $\boldsymbol{B}_p(\boldsymbol{\alpha})_i < \boldsymbol{B}_p(\boldsymbol{\beta})_i$. Together with item (a) we know $\boldsymbol{B}_p(\boldsymbol{\alpha}) \prec \boldsymbol{B}_p(\boldsymbol{\beta})$.

For item (d), the proof is similar to item (b) and we omit it.

For item (e), if $a_{ij} \geq 1$, by (4.12) and the fact that $\boldsymbol{\alpha}$ is a fixed point,

$$\alpha_i = \boldsymbol{B}_p(\boldsymbol{\alpha})_i \geq 1 - [1 - p\alpha_j]^{a_{ij}} \geq 1 - [1 - p\alpha_j] = p\alpha_j.$$

Repeating this argument, we get

$$\alpha_i = \boldsymbol{B}_p(\boldsymbol{\alpha})_i \geq p^M \alpha_{j'}, \forall\, j' \in \{1, \cdots, n\},$$

where M is the maximum of the lengths of the shortest oriented paths connecting two points in G. Therefore since $\alpha_j > 0$ for some j, then $\alpha_i > 0$ for all $i \in \{1, \cdots, n\}$. $\square$

Proof of Lemma 4.14. Write $\theta_{i,k}(p) := \mathbb{P}_p[o_i$ is connected to level k of $T_i]$, where by level k we mean the set of vertices in T_i with graph distance k to the root. Let $\boldsymbol{\theta}_k(p) = (\theta_{1,k}(p), \cdots, \theta_{n,k}(p))^T \in [0,1]^n$. In particular, $\boldsymbol{\theta}_0(p) = \mathbf{1}$. Then as (4.1), one has that

$$\boldsymbol{\theta}_{k+1}(p) = \boldsymbol{B}_p(\boldsymbol{\theta}_k(p))$$

and thus $\boldsymbol{\theta}_k(p) = \boldsymbol{B}_p^{\circ k}(\mathbf{1})$. By the definition of $\theta_{i,k}(p), \theta_i(p)$,

$$\boldsymbol{\theta}(p) = \lim_{k \to \infty} \boldsymbol{\theta}_k(p) = \lim_{k \to \infty} \boldsymbol{B}_p^{\circ k}(\mathbf{1}).$$

Suppose $\boldsymbol{\alpha} \in [0,1]^n$ is some fixed point of $\boldsymbol{B}_p$, i.e., $\boldsymbol{B}_{\boldsymbol{p}}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$. Then $\boldsymbol{\alpha} \leq \mathbf{1}$. By item (a) of Proposition 4.15, one has that $\boldsymbol{\alpha} = \boldsymbol{B}_{\boldsymbol{p}}(\boldsymbol{\alpha}) \leq \boldsymbol{B}_p(\mathbf{1}) = \boldsymbol{\theta}_1(p)$, and then $\boldsymbol{\alpha} = \boldsymbol{B}_{\boldsymbol{p}}(\boldsymbol{\alpha}) \leq \boldsymbol{B}_p(\boldsymbol{\theta}_1(p)) = \boldsymbol{\theta}_2(p), \cdots$ In the end, we have

$$\boldsymbol{\alpha} \leq \lim_{k \to \infty} \boldsymbol{\theta}_k(p) = \boldsymbol{\theta}(p),$$

i.e., $\boldsymbol{\theta}(p)$ is the largest fixed point for the operator $\boldsymbol{B}_p$ in $[0,1]^n$.

Now suppose $p \in (p_{\mathrm{c}}, 1)$ and we have some solution $\boldsymbol{\alpha} \in [0,1]^n \backslash \{\mathbf{0}\}$ for (4.1), i.e., $\boldsymbol{\alpha}$ is a nonzero fixed point of $\boldsymbol{B}_p$ in $[0,1]^n$. Since $\boldsymbol{\alpha} \in [0,1]^n$ is a fixed point of $\boldsymbol{B}_p$, we have showed that $\boldsymbol{\alpha} \leq \boldsymbol{\theta}(p)$. Since $\boldsymbol{\theta}(p) \geq \boldsymbol{\alpha} \neq \mathbf{0}$, by item (e) in Proposition 4.15, $\boldsymbol{\alpha} \in (0,1)^n$. Define $p_1 := \sup\{t \geq p_{\mathrm{c}} \colon \boldsymbol{\theta}(t) \leq \boldsymbol{\alpha}\}$.

Since $\boldsymbol{\alpha} \leq \boldsymbol{\theta}(p)$ and $\lim_{t \downarrow p_{\mathrm{c}}} \boldsymbol{\theta}(t) \to \mathbf{0}$ (Lemma 4.8), one has that $p_1 \in (p_{\mathrm{c}}, p]$. Since $\boldsymbol{\theta}(p)$ is infinite differentiable (Lemma 4.13) and increasing in $(p_{\mathrm{c}}, 1)$, one has that $\boldsymbol{\theta}(p_1) \leq \boldsymbol{\alpha}$ and for some $i \in \{1, \ldots, n\}$, $\theta_i(p_1) = \alpha_i$.

Since $\boldsymbol{\theta}(p_1)$ is a fixed point of $\boldsymbol{B}_{p_1}$ and $\boldsymbol{\alpha}$ is a fixed point of $\boldsymbol{B}_p$, one has that

$$\theta_i(p_1) = 1 - \prod_{j=1}^{n} [1 - p_1 \theta_j(p_1)]^{a_{ij}} = \alpha_i = 1 - \prod_{j=1}^{n} [1 - p\alpha_j]^{a_{ij}}. \tag{4.13}$$

Since G is strongly connected, there is some j such that $a_{ij} \geq 1$. As in the proof of item (e) in Proposition 4.15, to satisfy (4.13), by $p_1 \leq p$ and $\boldsymbol{\theta}(p_1) \leq \boldsymbol{\alpha}$ one must have

$$1 - p_1\theta_j(p_1) = 1 - p\alpha_j.$$

Since $0 < \theta_j(p_1) \leq \alpha_j \leq \theta_j(p)$ and $p_c < p_1 \leq p < 1$, one must have $p_1 = p$ and $\alpha_j = \theta_j(p_1)$. Therefore by $p_1 = p$ and the continuity of $\boldsymbol{\theta}$ one has the other direction $\boldsymbol{\alpha} \geq \lim_{t\uparrow p_1} \boldsymbol{\theta}(t) = \boldsymbol{\theta}(p_1) = \boldsymbol{\theta}(p)$. Hence $\alpha = \boldsymbol{\theta}(p)$ is the unique solution of (4.1) in $(0,1)^n$ for $p \in (p_c, 1)$. □

Now we are ready to prove Lemma 4.12.

Proof of Lemma 4.12. First by (4.1) and Lemma 4.14 we know that the set

$$\{(p, \theta_1(p), \ldots, \theta_n(p)) \in (p_c, 1) \times (0,1)^n\}$$

is semi-algebraic (Definition 2.1.4 of [2]). By Theorem 2.2.1 of [2], its projection $S := \{(\theta_1(p), \ldots, \theta_n(p)) \colon p \in (p_c, 1)\}$ is also semi-algebraic set and by Definition 2.2.5 the map $p \mapsto (\theta_1(p), \ldots, \theta_n(p))$ is a semi-algebraic map from $(p_c, 1)$ to S in view of (4.1).

By Theorem 2.2.1 and Proposition 2.2.6 of [2], the maps $p \mapsto \theta_i(p)$ are also semi-algebraic functions on $(p_c, 1)$.

By Lemma 4.13 we already know that the functions $p \mapsto \theta_i(p)$ are infinitely differentiable on $(p_c, 1)$. Hence by Proposition 2.9.1 of [2], we know the second derivatives $\theta_i''(p)$ are also semi-algebraic functions on $(p_c, 1)$, i.e., the sets $\{(p, \theta_i''(p)) \colon p \in (p_c, 1)\}$ are semi-algebraic sets for all $i \in \{1, \ldots, n\}$.

Hence for every $i \in \{1, \ldots, n\}$, $\{(p, \theta_i''(p)) \colon p \in (p_c, 1), \theta_i''(p) = 0\}$ is a semi-algebraic set since it is the intersection of two semi-algebraic sets, $\{(p, \theta_i''(p)) \colon p \in (p_c, 1)\}$ and $\{(p, 0) \colon p \in (p_c, 1)\}$. Thus by Theorem 2.2.1 of [2] for every $i \in \{1, \ldots, n\}$, the projection $\{p \colon p \in (p_c, 1), \theta_i''(p) = 0\}$ is a semi-algebraic set. By Proposition 2.1.7 of [2], for every $i \in \{1, \ldots, n\}$, the set $\{p \colon p \in (p_c, 1), \theta_i''(p) = 0\}$ is a **finite** union of points and open intervals. Hence there exists some $\varepsilon_i > 0$ such that $\theta_i''(p)$ cannot change its sign on $(p_c, p_c + \varepsilon_i)$. Taking $\varepsilon = \min\{\varepsilon_i \colon i = 1, 2, \ldots, n\}$ we have that the function $\theta_i''(p)$ does not change its sign (i.e. remains nonnegative or nonpositive) on $(p_c, p_c + \varepsilon)$ for all $i \in \{1, 2, \ldots, n\}$.

By Lemma 4.12 and the right continuity of the functions $\theta_i(p)$ at p_c, we know these functions $\theta_i(p)$ are either convex or concave on $[p_c, p_c + \varepsilon)$. Hence $\lim_{p\downarrow p_c} \frac{\theta_i(p) - \theta_i(p_c)}{p - p_c} = \lim_{p\downarrow p_c} \frac{\theta_i(p) - 0}{p - p_c}$ exists, i.e., the right derivative of $\theta_i(p)$ at p_c exists. □

Proof of Theorem 4.7. The existence of the right derivative of $\theta(p)$ follows from Lemma 4.12. The positiveness and finiteness of the right derivatives follow from Proposition 4.3 and Lemma 4.9 respectively. □

5 Concluding remarks and questions

5.1 Remark on Bernoulli site percolation

For $p \in [0, 1]$, if we instead keep each vertex with probability p and remove it otherwise. Call the vertices kept ***open vertices*** and those removed ***closed vertices***. ***Bernoulli***(p) ***site percolation*** studies the random subgraph ξ of G induced by the open vertices. For Bernoulli site percolation, an edge is call open if and only if its two endpoints are open. When talking about Bernoulli site percolation, we will use $\mathbb{P}_p^{\text{site}}$ and $\mathbb{P}_p^{\text{site}}$ to stress that.

Remark 5.1. If the connected graph G has bounded degree, say, with a upper bound D, then for Bernoulli site percolation, the following analogue of Lemma 3.2 holds:

$$\frac{d}{dp}\mathbb{P}_p^{\text{site}}(x \longleftrightarrow \Lambda_n^c) \geq \frac{1}{1-p} \min(1, \frac{\inf_{S\subset\Lambda_n, x\in S} \varphi_p(x, S)}{D-1}) \cdot [1 - \mathbb{P}_p^{\text{site}}(x \longleftrightarrow \Lambda_n^c)] \quad (5.1)$$

where $\varphi_p(x,S) := \sum_{y\in S}\sum_{z\notin S,(y,z)\in E}\mathbb{P}_p^{\mathrm{site}}[x \overset{S}{\longleftrightarrow} y]$. If one defines $p'_{\text{cut,site}}$ accordingly for site percolation, one can prove $p_{c,\text{site}} = p'_{\text{cut,site}}$ similarly as the bond percolation case.

This leads us to the following conjecture:

Conjecture 5.2. The answers for Question 1.3 and 1.5 are also positive for Bernoulli site percolation.

5.2 Is there an example with $p_{\mathrm{cut,E}} < p_{\mathrm{cut,V}}$?

We have seen examples with $p_{\mathrm{T,E}} < p_{\mathrm{T,V}}$ (Example 2.3). One can ask the same question for $p_{\mathrm{cut,E}}$ and $p_{\mathrm{cut,V}}$:

Question 5.3. Is there a locally finite, connected, infinite graph G such that $p_{\mathrm{cut,E}} < p_{\mathrm{cut,V}}$?

In view of Lemma 2.1, if there is a graph G with $p_{\mathrm{cut,E}} < p_{\mathrm{cut,V}}$, then it must have unbounded degree and for a vertex cutset Π_V, for "most" $v\in\Pi_V$ there should be a lot of edges in the corresponding edge cutset $\Pi_E = \Delta S(\Pi_V)$ incident to v. One might first want to consider certain 1-dimensional multigraphs. However there is no simple 1-dimensional example; see Proposition 5.5.

Definition 5.4. *Let $(a_n)_{n\geq 0}$ be a sequence of positive integers. Let $G = G((a_n)_{n\geq 0})$ be the graph with vertex set $V = \mathbb{N} = \{0,1,2,\cdots\}$ and edge set $E = \bigcup_n E_n$, where $E_n = \{e_{n,j}\colon j = 1,\ldots,a_n\}$ is the set of a_n parallel edges from n to $n+1$.*

Proposition 5.5. *There is no sequence of positive integers $(a_n)_{n\geq 0}$ such that $G = G((a_n)_{n\geq 0})$ has the property of $p_{\mathrm{cut,E}}(G) < p_{\mathrm{cut,V}}(G)$.*

We need a technical lemma.

Lemma 5.6. *There is no sequence of positive integers $(a_n)_{\geq 0}$ that satisfies both*

$$\sum_{i=0}^{\infty}(1-p_2)^{a_i} = \infty \tag{5.2}$$

and

$$a_n \geq \frac{c}{\prod_{i=0}^{n-1}\left[1-(1-p_1)^{a_i}\right]}, \forall\, n\geq 1. \tag{5.3}$$

for some constants $0<p_1<p_2<1$ and $c>0$.

Proof of Proposition 5.5 assuming Lemma 5.6. Notice that

$$\mathbb{P}_p[0\longleftrightarrow n] = \prod_{i=0}^{n-1}\left[1-(1-p)^{a_i}\right] \tag{5.4}$$

Thus

$$p<p_{\mathrm{c}} \;\Rightarrow\; \lim_{n\to\infty}\prod_{i=0}^{n-1}\left[1-(1-p)^{a_i}\right] = 0 \;\Leftrightarrow\; \sum_{i=0}^{\infty}(1-p)^{a_i} = \infty \tag{5.5}$$

and

$$\sum_{i=0}^{\infty}(1-p)^{a_i} < \infty \;\Leftrightarrow\; \lim_{n\to\infty}\prod_{i=0}^{n-1}\left[1-(1-p)^{a_i}\right] > 0 \;\Rightarrow\; p\geq p_{\mathrm{c}} \tag{5.6}$$

Notice that the minimal vertex cutsets are $\{\Pi_n\}$ where $\Pi_n = n$ and the minimal edge cutsets are $E_n = \{e_{n,j}\colon j=1,\cdots,a_n\}$, the a_n parallel edges from n to $n+1$.

Hence $\mathbb{E}_p[|C(0)\cap\Pi_n|] = \mathbb{P}_p[0\longleftrightarrow n] = \prod_{i=0}^{n-1}\left[1-(1-p)^{a_i}\right]$. Thus $p_{\mathrm{cut,V}} = p_{\mathrm{c}}$.

Suppose there is some sequence $(a_n)_{n\geq 0}$ such that $p_{\mathrm{cut,E}} < p_{\mathrm{cut,V}}$. Pick p_1, p_2 such that $p_{\mathrm{cut,E}} < p_1 < p_2 < p_{\mathrm{cut,V}}$.

Since $p_2 < p_{\mathrm{cut,V}} = p_{\mathrm{c}}$, by (5.5) one has that

$$\sum_{i=0}^{\infty}(1-p_2)^{a_i} = \infty$$

Since we choose $p_1 > p_{\mathrm{cut,E}}$ and noting that $\mathbb{E}_p[|C(0) \cap E_n|] = pa_n\mathbb{P}_p[0 \longleftrightarrow n]$, one has that

$$\inf_n p_1 a_n \prod_{i=0}^{n-1} \left[1-(1-p_1)^{a_i}\right] > 0 \tag{5.7}$$

i.e., there exists $c > 0$ s.t.

$$a_n \geq \frac{c}{\prod_{i=0}^{n-1}\left[1-(1-p_1)^{a_i}\right]}, \forall\, n \geq 1.$$

But this contradicts with Lemma 5.6 and hence Proposition 5.5 holds. □

Proof of Lemma 5.6. First we reduce to the case of increasing sequence. If there is some sequence satisfies both (5.2) and (5.3) for some $0 < p_1 < p_2 < 1$, then

$$\prod_{i=0}^{n-1}\left[1-(1-p_1)^{a_i}\right] \leq \prod_{i=0}^{n-1}\left[1-(1-p_2)^{a_i}\right] \overset{(5.2)}{\to} 0 \text{ as } n \to \infty.$$

Thus by (5.3), one has that $a_n \to \infty$. In the following for simplicity we write $p = p_1$.

Now we consider the sequence (a'_n), the rearrangement of a_n in the non-decreasing order. Obviously, (a'_n) also satisfies (5.2). As for (5.3), let $m = m(n)$ be the last index such that $a_m \leq a'_n$, i.e., $m = \max\{k\colon a_k \leq a'_n\}$. Obviously $m \geq n$. Since $a_k \to \infty$ as $k \to \infty$, $m < \infty$. Then we claim that there exists a constant $c_0 > 0$ such that

$$a'_n \prod_{i=0}^{n-1}\left[1-(1-p)^{a'_i}\right] \geq c_0 a_m \prod_{i=0}^{m-1}\left[1-(1-p)^{a_i}\right] \overset{(5.3)}{\geq} cc_0, \tag{5.8}$$

Write $A = \{v_i\colon v_1 < v_2 < \cdots\}$ for the all the values of the sequence (a_n). For each $v \in A$, let $N(v) = |\{j\colon a_j = v\}| \geq 1$ be the number of times the sequence taking the value v.

Case one: $a_m = a'_n$. By the definition of a'_n, we assume $a'_n = v_k$ and then

$$\prod_{i=0}^{n-1}\left[1-(1-p)^{a'_i}\right] \geq \left[1-(1-p)^{v_k}\right]^{N(v_k)-1} \times \prod_{i=1}^{k-1}\left[1-(1-p)^{v_i}\right]^{N(v_i)} \tag{5.9}$$

By the choice of m, the multi-set $\{a_0, \cdots, a_{m-1}\}$ contains at least $N(v_k) - 1$'s v_k and all the $N(v_i)$'s v_i for $i < k$. Hence

$$\left[1-(1-p)^{v_k}\right]^{-1} \times \prod_{i=1}^{k}\left[1-(1-p)^{v_i}\right]^{N(v_i)} \geq \prod_{i=0}^{m-1}\left[1-(1-p)^{a_i}\right]. \tag{5.10}$$

By $a'_n = a_m$ and the above two inequalities (5.9), (5.10) we have (5.8) for all $c_0 \leq 1$.

Case two: $a_m < a'_n$, say $a'_n = v_k$ and $a_m = v_j$ for some $j < k$. Then by the choice of m, the multi-set $\{a_0, \cdots, a_{m-1}\}$ contains at least $N(v_j) - 1$'s v_j and all the other $N(v_i)$'s v_i for $i \leq k, i \neq j$. Hence

$$\left[1-(1-p)^{v_j}\right]^{-1} \times \prod_{i=1}^{k}\left[1-(1-p)^{v_i}\right]^{N(v_i)} \geq \prod_{i=0}^{m-1}\left[1-(1-p)^{a_i}\right]. \tag{5.11}$$

By (5.9) and (5.11) we have that

$$\frac{a_n' \prod_{i=0}^{n-1}\left[1-(1-p)^{a_i'}\right]}{a_m \prod_{i=0}^{m-1}\left[1-(1-p)^{a_i}\right]} \geq \frac{v_k\left[1-(1-p)^{v_k}\right]^{-1}}{v_j\left[1-(1-p)^{v_j}\right]^{-1}} \geq p\frac{v_k}{v_j} \geq p.$$

where in the second inequality we use the fact that

$$f(x) = \frac{x}{1-(1-p)^x} \in (x, \frac{x}{p}], x \geq 1.$$

Hence in this case (5.8) holds with $c_0 = p = p_1$.

Combining the two cases one has that (5.8) holds with $c_0 = p = p_1$.

By the reduction, we can assume $(a_n)_{n\geq 0}$ is increasing. Thus there is a strictly increasing sequence (n_j) such that for $n \in [n_j, n_{j+1}-1]$, $a_n = v_j$. In particular, (5.2) becomes

$$\sum_{j=1}^{\infty}(n_{j+1}-n_j)(1-p_2)^{v_j} = \infty \tag{5.12}$$

and (5.3) becomes (only needs to look at times $n_{j+1}-1$)

$$v_j \geq \frac{c\left[1-(1-p_1)^{v_j}\right]}{\prod_{i=1}^{j}\left[1-(1-p_1)^{v_i}\right]^{n_{i+1}-n_i}} \tag{5.13}$$

By (5.13) one has that

$$\frac{c}{v_j\left[1-(1-p_1)^{v_j}\right]^{n_{j+1}-n_j-1}} \leq \prod_{i=1}^{j-1}\left[1-(1-p_1)^{v_i}\right]^{n_{i+1}-n_i} \leq 1.$$

Hence

$$v_j\left[1-(1-p_1)^{v_j}\right]^{n_{j+1}-n_j-1} \geq c.$$

Taking logarithm one has that

$$\log v_j + (n_{j+1}-n_j-1)\log[1-(1-p_1)^{v_j}] \geq \log c$$

Hence

$$n_{j+1}-n_j-1 \leq \frac{\log c - \log v_j}{\log[1-(1-p_1)^{v_j}]} \leq \frac{c' \log v_j}{(1-p_1)^{v_j}}, \text{ when } v_j > 1.$$

But this contradicts with (5.12): (noting $\{v_j\}$ is a strictly increasing subsequence of $\mathbb{N}$ and $1-p_2 < 1-p_1$)

$$\sum_{j=1}^{\infty}(n_{j+1}-n_j)(1-p_2)^{v_j} \leq (n_2-n_1)(1-p_2)^{v_1} + \sum_{j=2}^{\infty}(1-p_2)^{v_j} + \sum_{j=2}^{\infty}\frac{c'\log v_j}{(1-p_1)^{v_j}}(1-p_2)^{v_j} < \infty.$$

This contradiction implies Lemma 5.6. □

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