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Green function estimates for second order elliptic operators in non-divergence form with Dini continuous coefficients*

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Abstract

Two-sided sharp Green function estimates are obtained for second order uniformly elliptic operators in non-divergence form with Dini continuous coefficients in bounded $C^{1,1}$ domains, which are shown to be comparable to that of the Dirichlet Laplace operator in the domain. The first and second order derivative estimates of the Green functions are also derived. Moreover, boundary Harnack inequality with an explicit boundary decay rate and interior Schauder's estimates for these differential operators are established, which may be of independent interest.

Keywords: Green function; harmonic function; Martin integral representation; boundary Harnack principle; interior Schauder's estimate.
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1 Introduction

Consider the following elliptic operator in non-divergence form on $\mathbb{R}^d$ with $d \geq 3$:

$$\mathcal{L} f(x) = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} f(x), \tag{1.1}$$

where $(a_{ij}(x))_{1\leq i,j\leq d}$ is a symmetric $d \times d$ matrix-valued function on $\mathbb{R}^d$ that is uniform bounded and elliptic; that is, there exists a constant $\lambda_0 \geq 1$ such that for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$,

$$\lambda_0^{-1} |\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j \leq \lambda_0 |\xi|^2. \tag{1.2}$$

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Throughout this paper, we assume that the entries $a_{ij}(x)$, $1 \le i, j \le d$, are Dini continuous in the sense that

$$\sum_{i,j=1}^{d} |a_{ij}(x) - a_{ij}(y)| \le \ell(|x-y|) \quad \text{for all } x, y \in \mathbb{R}^d \text{ and } 1 \le i, j \le d. \tag{1.3}$$

Here $\ell(\cdot) : [0, \infty) \to [0, \infty)$ is an increasing continuous function with $\ell(0) = 0$ and $\int_0^1 \ell(t)/t \, dt < \infty$ such that there are positive constants c_0 and $\alpha \in (0, 1]$ so that

$$\ell(R)/\ell(r) \le c_0 (R/r)^\alpha \quad \text{for all } 0 < r < R \le 1. \tag{1.4}$$

Examples of such functions include $\ell(t) := \log^\beta(2/(t \wedge 1))$ for any $\beta < -1$. Here and in what follows, we use $:=$ as a way of definition. For $a, b \in \mathbb{R}$, $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$.

The main purpose of this paper is to establish sharp two-sided estimates as well as first and second order derivative estimates with explicit boundary decay rates for Green functions of such an $\mathcal{L}$ in bounded $C^{1,1}$ domains in $\mathbb{R}^d$. In particular, we are interested in whether the Green function of $\mathcal{L}$ in a bounded $C^{1,1}$ domain D is comparable to that of the Laplacian in D.

1.1 Prior results

In analysis and PDE, the Green function of an elliptic operator $\mathcal{L}$ in D is the fundamental solution of the elliptic Poisson equation $\mathcal{L}u = -f$ in D with zero boundary condition, while in probability theory, the Green function is the occupation density of the diffusion process associated with $\mathcal{L}$ stayed in D before exiting. Many times in the literature, one of these two notions of the Green function is used but without being properly and rigorously identified to the other. There is an extensive literature on Green functions of elliptic differential operators, for instance, [34] and the references therein. Here we concentrate on those related to pointwise bounds of Green functions, which are the main topics of of this paper.

When $A(x) := (a_{ij}(x))$ is the identity matrix, that is, when $\mathcal{L}$ is the Laplace operator Δ, upper and lower bound estimates of the Green function for Δ on a bounded $C^{1,1}$ domain are derived in Grüter and Widman [23] and Zhao [39], respectively. These estimates are sharp in the sense that the upper bound is a constant multiple of the lower bound, and the boundary decay rates are given explicitly in terms of the distance function to the boundary. Sharp two-sided Green function estimates for the Laplacian in bounded Lipschitz domains are derived in [6], where the boundary decay rate function is implicit and is given in terms of the Green function itself with one variable fixed.

In fact, in Grüter and Widman [23, Theorems 1.1 and 3.3], the existence of Green function is established for any divergence form operator $\mathcal{L}^{(s)}$ on any bounded domain in $\mathbb{R}^d$ for $d \ge 3$ with measurable coefficients that is uniformly elliptic and bounded. The following upper bound estimate is obtained in [23] for such $\mathcal{L}^{(s)}$ with coefficients satisfying Dini condition (1.3)-(1.4) in a bounded domain satisfying a uniform exterior sphere condition:

$$G_D(x,y) \le \frac{c}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right) \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right) \quad \text{for } x, y \in D, \tag{1.5}$$

where $\delta_D(x) = \inf\{|x-z| : z \in D^c\}$ is the Euclidean distance from x to D^c. Upper bounds for the derivatives of $G_D(x,y)$ up to the second order are also given in [23, Theorem 3.3]. These results improve an earlier well-known result of Littman, Stampacchia and Weinberger [33]. When $\mathcal{L}^{(s)} = \Delta$, its Green function G_D in a bounded $C^{1,1}$-domain D, as mentioned above, is in fact comparable to the function on the right hand side of (1.5). However, this comparable lower bound estimate in general does not hold for the Green function of $\mathcal{L}^{(s)}$, even on smooth domains such as balls. In other words, the Green function $G_D^{\mathcal{L}^{(s)}}$ for $\mathcal{L}^{(s)}$ is in general not comparable to that of G_D^{Δ} even when D is a ball. This is because were it true, then the harmonic measures of $\mathcal{L}^{(s)}$ and the Laplacian Δ on ∂D would be mutually absolutely continuous to each other; see the proof of Proposition 7.3 in Ancona [1] or [9, Theorem 2.2]. The latter is true if and only if the coefficients of $\mathcal{L}^{(s)}$ are globally continuous and the modulus of continuity along some non-tangential direction at each boundary point of the ball is bounded uniformly by a function η that satisfies the Dini-type condition $\int_0^1 \eta(t)^2 t^{-1} dt < \infty$; see [8, 20]. Thus for a divergence form elliptic operator $\mathcal{L}^{(s)}$, if its Green function in a ball is comparable to the classical Green function (of the Laplacian), then the diffusion coefficients of $\mathcal{L}^{(s)}$ should at least be globally continuous and satisfy certain Dini-continuity condition.

We also mention that for a divergence form elliptic operator $\mathcal{L}^{(s)}$, under a certain local energy condition (LH) with parameters $\mu_0 \in (0,1]$ and $R_{\max} > 0$ on weak $\mathcal{L}$-harmonic functions in D vanishing on part of the boundary, it is shown in [27, Theorem 3.13] that

$$G_D(x,y) \le \frac{c}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)^{\mu_0} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right)^{\mu_0} \quad \text{for } x,y \in D \text{ with } |x-y| \le R_{\max}. \tag{1.6}$$

This estimate is weaker than that of (1.5). No information is given in [27] on the optimal value of μ_0 in the condition (LH) even when D is C^2-smooth. According to [27, Corollary 4.4], the (LH) condition holds for $\mathcal{L}$ in those bounded domains D that there exist some constants C and $R > 0$ so that

$$|B(z,r) \setminus D| \ge C\, |B(z,r)| \quad \text{for all } z \in \partial D \text{ and } r \in (0,R).$$

Here $B(z,r)$ denotes the open ball with radius r cenerted at z, and for a Lebesgue measurable $A \subset \mathbb{R}^d$, we use $|A|$ to denote its Lebesgue measure.

Unlike the case of divergence form operators, the Green function $G_D^{\mathcal{L}}(x,y)$ for non-divergence form operator $\mathcal{L}$ on $\mathbb{R}^d$ in a smooth domain D can be locally unbounded away from the pole even when $A(x)$ is uniformly continuous, bounded and elliptic; see Bauman [5]. Consequently, without additional regularity assumption on $A(x)$, $G_D^{\mathcal{L}}(x,y)$ may not be bounded by $c|x-y|^{2-d}$. When $A(x)$ satisfying (1.2) is Hölder continuous, it is shown in Hueber and Sieveking [24] that the Green functions of the non-divergence form operator $\mathcal{L}$ on bounded $C^{1,1}$ domains are comparable to that of the Laplace operator. This comparability result fails on bounded Lipschitz domains even for elliptic operators with constant coefficients. Indeed, it is illustrated in [24, Section 4.2] that when D is a unit square, there is an elliptic operator $\mathcal{L}$ on $\mathbb{R}^2$ having constant coefficients so that $G_D^{\mathcal{L}}$ is not comparable to G_D^{Δ}.

Recently, Hwang and Kim [25] showed that the non-divergence form elliptic operator $\mathcal{L}$ of (1.1) with coefficients satisfying a Dini mean oscillation condition admits a non-negative function $G_D^{\mathcal{L}}(x,y)$ on $D \times D \setminus \mathrm{diag}$ for any bounded $C^{1,1}$ domain $D \subset \mathbb{R}^d$ with $d \ge 3$ as the fundamental solution to its adjoint operator $\mathcal{L}^*$ in D so that

$$G_D^{\mathcal{L}}(x,y) \le c|x-y|^{2-d} \quad \text{and} \quad |\nabla_x G_D^{\mathcal{L}}(x,y)| \le c|x-y|^{1-d} \tag{1.7}$$

on any bounded $C^{1,1}$ domain D and $|D_x^2 G_D^{\mathcal{L}}(x,y)| \le c|x-y|^{-d}$ on any bounded $C^{2,\mathrm{Dini}}$ domain D. Here diag stands for the diagonal set $\{(x,x) : x \in D\}$ of $D \times D$, and for a function f, $\nabla_x f := (\frac{\partial}{\partial x_1} f, \cdots, \frac{\partial}{\partial x_d} f)$ denotes its gradient, $D_x^2 f = \big(\frac{\partial^2}{\partial x_i \partial x_j} f\big)_{1 \le i,j \le d}$ is the Hessian matrix of second derivatives and $|D_x^2 f| = \big(\sum_{i,j=1}^d |\frac{\partial^2}{\partial x_i \partial x_j} f|^2\big)^{1/2}$. Here the diffusion coefficients $(a_{ij}(x))_{1 \le i,j \le d}$ of $\mathcal{L}$ is said to satisfy a Dini mean oscillation condition if

$$\int_0^1 \frac{\omega_{a_{ij}}(r)}{r} dr < \infty \quad \text{for every } 1 \le i,j \le d, \tag{1.8}$$

where

$$\omega_{a_{ij}}(r) := \sup_{x\in\mathbb{R}^d} \frac{1}{|B(x,r)|} \int_{B(x,r)} \left| a_{ij}(y) - \frac{1}{|B(x,r)|} \int_{B(x,r)} a_{ij}(z)dz \right| dy.$$

Dini mean oscillation condition is weaker than Dini condition but stronger than the uniform continuity. In fact, by [25, Lemma A.1], there are continuous functions $a^*_{ij}(x)$ so that $a^*_{ij}(x) = a_{ij}(x)$ for a.e. on $\mathbb{R}^d$ and

$$\sum_{i,j=1}^d |a^*_{ij}(x) - a^*_{ij}(y)| \le \rho(|x-y|) \quad \text{for all } x, y \in \mathbb{R}^d, \tag{1.9}$$

where

$$\rho(r) := c \sum_{i,j=1}^d \int_0^r \frac{\omega_{a_{ij}}(s)}{s} ds.$$

Note that

$$\int_0^1 \frac{\rho(r)}{r} dr = c \sum_{i,j=1}^d \int_0^1 \frac{\omega_{a_{ij}}(s)\ln(1/s)}{s} ds. \tag{1.10}$$

If the right hand side is integrable, then the functions $\{a^*_{ij}(x), 1 \le i,j \le d\}$ are Dini continuous. The above function $G^{\mathcal{L}}_D(x,y)$ from [25] is the Green function of $\mathcal{L}$ in D in the sense that for any $f \in L^p(D)$ with $p > d/2$, $G^{\mathcal{L}}_D f(x) := \int_D G^{\mathcal{L}}_D(x,y)f(y)dy$ is in $W^{2,p}(D) \cap W^{1,p}_0(D)$ and is a strong solution for $\mathcal{L}u = -f$ in D with zero boundary value on ∂D; see [25, Remark 1.14]. Its relation to the occupation density in D of the diffusion process associated with $\mathcal{L}$ is not given in [25]. Similar results have been obtained in Dong and Kim [18] for Green functions of $\mathcal{L}$ on bounded $C^{1,1}$ domains D in $\mathbb{R}^2$ but with $1 + \log \frac{\operatorname{diam}(D)}{|x-y|}$ in place of $|x-y|^{2-d}$ for the upper bound estimate of $G^{\mathcal{L}}_D(x,y)$. These upper bounds on $G^{\mathcal{L}}_D(x,y)$ are not sharp as they do not give boundary decay information near ∂D. We remark that from the gradient estimate in (1.7) for bounded $C^{1,1}$-domain D in $\mathbb{R}^d$ with $d \ge 3$ and the property that $x \mapsto G^{\mathcal{L}}_D(x,y)$ vanishes on ∂D from Theorem 2.3(i) below, one can deduce that

$$G^{\mathcal{L}}_D(x,y) \le \frac{c}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right) \quad \text{for } x, y \in D \tag{1.11}$$

for some constant c that depends on d, ℓ, λ_0 and D; see Remark 2.1(iii).

As one can see from the above, obtaining sharp two-sided Green functions of elliptic operators either with less regular coefficients or in less smooth domains is a challenging and delicate problem. In this paper, we are concerned with sharp two-sided Green function estimates for non-divergence form elliptic operators with less regular coefficients in bounded $C^{1,1}$-domains. A natural and interesting question is whether for non-divergence form elliptic differential operator $\mathcal{L}$ with Dini-continuous coefficients, $G^{\mathcal{L}}_D(x,y)$ is comparable to that of the Laplacian in every bounded $C^{1,1}$ domain $D \subset \mathbb{R}^d$. In this paper, we give an affirmative answer to this question. As an important consequence, we show that the harmonic measure for such an operator $\mathcal{L}$ is comparable to that of classical harmonic measure (of the Laplacian) in any bounded $C^{1,1}$ domain. We further derive its derivative estimates that contain the explicit boundary decay information; see Theorem 1.1 below. The crux of the study is on deriving boundary decay of $G^{\mathcal{L}}_D(x,y)$ in y-variable and the comparable lower bounds of $G^{\mathcal{L}}_D(x,y)$ as well as the dependence of the comparison constants.

1.2 Main results of this paper

Throughout this paper, we assume for simplicity that the dimension $d \geq 3$. The approach of this paper works for $d = 2$ as well but one needs to replace the Newtonian potential kernel $|x-y|^{2-d}$ for $d \geq 3$ by a suitable logarithm potential kernel in $d = 2$. Hereafter, unless otherwise specified, $\mathcal{L}$ is the non-divergence form operator of (1.1) on $\mathbb{R}^d$ with Dini continuous diffusion coefficients $\{a_{ij}(x); 1 \leq i, j \leq d\}$ satisfying (1.2)-(1.3). Since $A(x)$ is uniformly continuous, bounded and uniformly elliptic, it is well known (see [37, Theorem 7.2.4]) that the martingale problem for $(\mathcal{L}, C_c^\infty(\mathbb{R}^d))$ is well posed and its solution forms a conservative strong Markov process X that has continuous sample paths and strong Feller property. Moreover, by [37, Lemma 9.2.2], X has a transition density function $p(t, x, y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$. For any open set D, define $\tau_D := \inf\{t > 0 : X_t \notin D\}$ the first exit time from D by the process X. Denote by X^D the subprocess of X killed upon leaving D. We show that for any bounded $C^{1,1}$ domain D in $\mathbb{R}^d$, there is a unique jointly continuous non-negative function $G_D(x, y)$ on $D \times D \setminus \mathrm{diag}$ so that for any non-negative $f \in C_c(D)$,

$$\mathbb{E}_x \left[\int_0^{\tau_D} f(X_s) ds \right] = \int_D G_D(x, y) f(y) dy \quad \text{for every } x \in D. \tag{1.12}$$

We call $G_D(x, y)$ the Green function of X in D. It is shown in Theorem 2.3 that $G_D(x, y)$ is the same as the Green kernel $G_D^{\mathcal{L}}(x, y)$ defined analytically in [25]. Thus, $G_D(x, y)$ can also be called the Green function of $\mathcal{L}$ in D.

Recall that an open set D in $\mathbb{R}^d$ is said to be $C^{1,1}$ if there exist a localization radius $R_0 > 0$ and a constant $\Lambda_0 > 0$ such that for every $Q \in \partial D$, there exist a $C^{1,1}$ function $\phi = \phi_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = \nabla\phi(0) = 0$, $\|\nabla\phi\|_\infty \leq \Lambda_0$, $|\nabla\phi(x) - \nabla\phi(y)| \leq \Lambda_0|x-y|$, and an orthonormal coordinate system $CS_Q : y = (y_1, \cdots, y_{d-1}, y_d) =: (\tilde{y}, y_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ with its origin at Q such that

$$B(Q, R_0) \cap D = \{y = (\tilde{y}, y_d) \in B(0, R_0) \text{ in } CS_Q : y_d > \phi(\tilde{y})\}.$$

The pair (R_0, Λ_0) is called the characteristics of the $C^{1,1}$ open set D. Without loss of generality, throughout this paper, we assume that the characteristics (R_0, Λ_0) of a $C^{1,1}$ open set satisfies $R_0 \leq 1$ and $\Lambda_0 \geq 1$. An open connected set will be called a domain in this paper.

Suppose D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with $C^{1,1}$ characteristics (R_0, Λ_0). We denote by $G_D^\Delta(x, y)$ the Green function of the Laplacian Δ in D. It is shown in [23, Theorem 3.3] and [39, Theorem 1] that there is a constant $C = C(d, D) > 1$ such that for $x \neq y$ in D,

$$\frac{C^{-1}}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right) \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right) \leq G_D^\Delta(x, y) \leq \frac{C}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right) \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right). \tag{1.13}$$

In fact, the constant $C = C(d, D)$ above can be taken to be dependent on D through d, Λ_0, R_0 and $\mathrm{diam}(D)$ only. Here $\mathrm{diam}(D)$ stands for the diameter of the domain D. We state this as Theorem 5.1 and give a proof in the Appendix of this paper.

The following is the main result of this paper, which is the summary of Theorems 3.7, 4.4 and 4.8.

Theorem 1.1. Let $\mathcal{L}$ be a second order differential operator on $\mathbb{R}^d$ of non-divergence form (1.1) satisfying the conditions (1.2)-(1.3), and X the diffusion process associated with it. Suppose D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R_0, Λ_0). There is a unique jointly continuous non-negative function $G_D(x, y)$ on $D \times D \setminus \mathrm{diag}$ so that (1.12) holds for any $f \in C_c(D)$. Moreover, the following holds.

(i) $G_D(x,y)$ is jointly continuous in $D \times D \setminus \mathrm{diag}$, and for each $y \in D$, $x \mapsto G_D(x,y)$ is C^2 in $D \setminus \{y\}$.

(ii) There exists $K_1 = K_1(d, \lambda_0, \ell, \Lambda_0, R_0, \mathrm{diam}(D)) > 1$ such that for every $x \neq y$ in D,

$$\begin{aligned} &\frac{K_1^{-1}}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)\left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right) \\ &\leq G_D(x,y) \leq \frac{K_1}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)\left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right). \end{aligned} \tag{1.14}$$

(iii) There exists $K_2 = K_2(d, \lambda_0, \ell, \Lambda_0, R_0, \mathrm{diam}(D))$ such that

$$|\nabla_x G_D(x,y)| \leq \frac{K_2}{|x-y|^{d-1}} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right) \quad \text{for any } x \neq y \text{ in } D. \tag{1.15}$$

(iv) There exists $K_3 = K_3(d, \lambda_0, \ell, \Lambda_0, R_0, \mathrm{diam}(D))$ such that for any $1 \leq i,j \leq d$,

$$\left|\frac{\partial^2}{\partial x_i \partial x_j} G_D(x,y)\right| \leq \frac{K_3}{|x-y|^d} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right)\left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)^{-1} \quad \text{for any } x \neq y \text{ in } D. \tag{1.16}$$

Remark 1.1. We point out that the dependence of the above coefficients K_i, $1 \leq i \leq 3$, on $\mathrm{diam}(D)$ (as well as in other places and cases throughout the paper) is on an upper bound of $\mathrm{diam}(D)$ rather than on the exact value of $\mathrm{diam}(D)$. This can be seen through the following scaling argument. For each $\lambda > 0$, let $X_t^{(\lambda)} := \lambda X_{t/\lambda^2}$. It is easy to check that the infinitesimal generator of $X^{(\lambda)}$ is

$$\mathcal{L}^{(\lambda)} := \sum_{i,j=1}^d a_{ij}(x/\lambda) \frac{\partial^2}{\partial x_i \partial x_j}.$$

Denote by $G_D^{(\lambda)}$ the Green function of $X^{(\lambda)}$ in the domain D. Then the Green function $G_D(x,y)$ satisfies the following scaling property

$$G_D(x,y) = \lambda^{d-2} G_{\lambda D}^{(\lambda)}(\lambda x, \lambda y), \quad x,y \in D.$$

For any bounded $C^{1,1}$ domain D_1 with characteristics (R_0, Λ_0) and $\mathrm{diam}(D_1) \leq \mathrm{diam}(D)$, let $\lambda_1 := \frac{\mathrm{diam}(D)}{\mathrm{diam}(D_1)}$. It follows from the scaling formula above with D_1 in place of D that

$$G_{D_1}(x,y) = \lambda_1^{d-2} G_{\lambda_1 D_1}^{(\lambda_1)}(\lambda_1 x, \lambda_1 y), \quad x,y \in D_1. \tag{1.17}$$

Note that $\lambda_1 \geq 1$, then $a_{ij}(\cdot/\lambda_1)$ satisfies the conditions (1.2)-(1.3) with the same modulo of continuity function ℓ, and $\lambda_1 D_1$ is a bounded $C^{1,1}$ domain with characteristics (R_0, Λ_0) and $\mathrm{diam}(\lambda_1 D_1) = \mathrm{diam}(D)$. Applying (1.14)-(1.16) to $G_{\lambda_1 D}^{(\lambda_1)}(\lambda_1 x, \lambda_1 y)$ and together with (1.17), this yields that the constants $K_i, i = 1,2,3$ in Theorem 1.1 hold uniformly for any bounded $C^{1,1}$ domain D_1 with characteristics (R_0, Λ_0) and $\mathrm{diam}(D_1) \leq \mathrm{diam}(D)$ in $\mathbb{R}^d$.

The following is an immediate consequence of Theorem 1.1(i) by the proof of [1, Proposition 7.3] or [9, Theorem 2.2]; see Theorem 3.8 and its proof for more details. Other implications of the Green function estimate (1.14) are given below in Corollary 3.9 and Theorem 3.10.

Corollary 1.2. Under the setting of Theorem 1.1, the harmonic measure of X in D is comparable to that of Brownian motion in D. More specifically, there is a constant $C_0 = C_0(d, \lambda_0, \ell, \Lambda_0, R_0, \mathrm{diam}(D)) > 1$ such that for every $x \in D$,

$$C_0^{-1} \frac{\delta_D(x)}{|x-z|^d} \sigma(dz) \leq \mathbb{P}_x(X_{\tau_D} \in dz) \leq C_0 \frac{\delta_D(x)}{|x-z|^d} \sigma(dz) \quad \text{on } \partial D,$$

where σ is the Lebesgue surface measure on ∂D.

The Green function estimates in Theorem 1.1 significantly improve the results in Hwang and Kim [25], where some upper bounds on $G_D(x,y)$ and its derivatives are derived but without the boundary decay factor, at the expense of assuming a slightly stronger Dini continuous condition on the diffusion coefficients $\{a_{ij}(x)\}_{1\le i,j\le d}$ here. It also extends the main result of Hueber and Sieveking [24], where the two-sided Green function estimates (1.14) are obtained for non-divergence form operator $\mathcal{L}$ with Hölder continuous coefficients under the framework of Brelot spaces and harmonic sheafs. Our estimates give sharp pointwise two-sided global bounds of Green functions, up to the boundary. In order to get precise boundary decay rates, our current approach use the pointwise Dini condition (1.3) on the diffusion coefficients $\{a_{ij}(x)\}_{1\le i,j\le d}$ in a crucial way; see, for example, the proof of Lemmas 2.6, 2.7 and 2.9. It is an open problem whether the main results of this paper can be extended to non-divergence form elliptic operators whose coefficients satisfy Dini mean oscillation condition. As mentioned earlier, for second order elliptic operators of divergence form with Dini continuous coefficients, the corresponding Green function upper bound estimates as well as the derivative estimates have been obtained in Grüter and Widman [23, Theorem 3.3] on bounded domains satisfying uniform exterior sphere condition.

1.3 Methodology and novelties of our approach

In this paper, we employ both analytic and probabilistic methods, including Levi's freezing coefficient method. We first show in Theorem 2.3 that the Green function $G_D(x,y)$ of $\mathcal{L}$ constructed analytically in [25] coincides with the occupation density of the $\mathcal{L}$-diffusion X in D as given in (1.12). This identification enables us to study the properties of $G_D(x,y)$ by both analytic and probabilistic techniques.

The Levi's freezing coefficient method is an effective tool in the heat kernel analysis for non-divergence elliptic operators in $\mathbb{R}^d$; see [21]. But it seems that this method has rarely been used in the study of sharp Green function estimates in bounded domains. Different from heat kernel in the parabolic case, the Green function $G_D^{(z)}(x,y)$ of the differential operator $\sum_{i,j=1}^d a_{ij}(z)\frac{\partial^2}{\partial x_i \partial x_j}$ in D with z frozen and its derivatives blow up along the diagonal of $D\times D$. This causes challenges in identifying the kernel constructed by the Levi's freezing coefficient method in the elliptic case with the Green function defined as the fundamental solution for the elliptic operator $\mathcal{L}$. We overcome this difficulty by first considering $C^{1,1}$ domains λD for small $\lambda>0$ and carrying out an approximation scheme by smoothing out the singularity along the diagonal and establishing a weak convergence locally in the Sobolev space $W^{2,p}(\lambda D)$. We are then able to identify the kernel obtained via the Levi's freezing coefficient method with the analytic notation of the Green function $G_{\lambda D}$ in Theorem 2.8 for sufficiently small $\lambda>0$. The Levi's freezing coefficient method combined with a testing function method and a probabilistic argument allows us to obtain in Theorem 2.12 sharp two-sided Green function estimates on λD for sufficiently small $\lambda>0$.

To obtain the two-sided estimates of $G_D(x,y)$ of $\mathcal{L}$ in a general bounded $C^{1,1}$ domain D, we patch the "interior" Green function estimate with the boundary decay rate of $G_D(x,y)$ in x and y. We mainly use a suitable dual process of the subprocess X^D of X in D, together with the two sided Green function estimates in $C^{1,1}$ domains with small diameters to establish that $G_D(x,y)$ is comparable to the Green function of Brownian motion in D. Moreover, the two-sided Green function estimates enable us to obtain a Poisson integral representation for non-negative $\mathcal{L}$-harmonic functions in a bounded $C^{1,1}$ domain, to directly establish the boundary Harnack principle for $\mathcal{L}$ with explicit boundary decay rate, and to identify the Martin boundary and minimal Martin boundary with the Euclidean boundary for any bounded $C^{1,1}$ domain; see Theorems 3.8-3.10.

Derivative estimates of Green functions with an explicit boundary decay rate play a crucial role in the study of sharp two-sided Green function estimates with explicit boundary decay rates of diffusion operators under drift perturbations and non-local perturbations; see, e.g. [15] and [11]. As $x \mapsto G_D(x,y)$ is $\mathcal{L}$-harmonic in $D \setminus \{y\}$, when $A(x)$ is Hölder continuous, derivative estimates on $x \mapsto G_D(x,y)$ can be deduced from the upper bound of $G_D(x,y)$ and the interior Schauder's estimates for $\mathcal{L}$-harmonic functions (see [22, Theorem 6.2]). However, to the best of the authors' knowledge, there is no readily available literature on the interior Schauder's estimates for $\mathcal{L}$ with Dini-continuous coefficients. We derive the gradient estimate on $G_D(x,y)$ through the identity from Levi's freezing coefficient method. But this method does not work for deriving the second order derivative estimates on $G_D(x,y)$ as some integrals involved become divergent. To overcome this difficulty, we use a variant of the integral representation of the Green function $G_B(x,y)$ of $\mathcal{L}$ on small balls in terms of the Poisson kernel of Brownian motion starting from y in a $C^{1,1}$ subdomain of B in Lemma 4.5. This combined with the second derivative estimate $|\nabla_x^2 G_B(x,y)| \leq c|x-y|^{-d}$ on balls from [25] enables us to obtain the second order derivative estimates on $G_D(x,y)$. Along the way, we establish the interior Schauder's estimates for harmonic functions of $\mathcal{L}$ in Theorem 4.9, which extends the known results in literature for non-divergence form elliptic operators with Hölder coefficients to those with Dini coefficients.

The rest of the paper is organized as follows. Let D be a bounded $C^{1,1}$ domain in $\mathbb{R}^d$. In Section 2, we identify the Green kernel $G_D^{\mathcal{L}}(x,y)$ defined analytically in [25] with the occupation density function $G_D(x,y)$ of the $\mathcal{L}$-diffusion process in D. We next show that the Green function can also be constructed recursively by the Levi's freezing method on λD with sufficiently small $\lambda > 0$. This property together with a testing function method and a probabilistic argument enables us to obtain in Theorem 2.12 sharp two-sided Green function estimates on λD for sufficiently small $\lambda > 0$. In Section 3, we derive two-sided estimates on the Green function $G_D(x,y)$ of $\mathcal{L}$ in any bounded $C^{1,1}$ domain D. Using this estimates, we derive a Poisson integral representation for non-negative $\mathcal{L}$-harmonic functions on bounded $C^{1,1}$ domains and establish the boundary Harnack principle for $\mathcal{L}$ with explicit boundary decay rate. We further identify the Martin boundary and minimal Martin boundary of $\mathcal{L}$ in D with its Euclidean boundary for any bounded $C^{1,1}$ domain D. In Section 4, we derive the first and second derivative estimates of $G_D(x,y)$, and establish interior Schauder's estimates for harmonic functions of $\mathcal{L}$ with Dini coefficients.

Notation. Throughout this paper, we use the capital letters $C_1, C_2, \cdots$ to denote constants in the statement of the results, and their labeling will be fixed. The lowercase constants $c_1, c_2, \cdots$ will denote generic constants used in the proofs, whose exact values are not important and can change from one appearance to another. For any open set $D \subset \mathbb{R}^d$, $C_c(D)$ and $C_c^2(D)$ denotes the space of continuous functions with compact in D and the space of C^2 smooth functions with compact support in D. For $p \geq 1$, $L^p(D)$ (resp. $L^p_{loc}(D)$) denotes the space of L^p-integrable (resp. locally L^p-integrable) functions on D with respect to the Lebesgue measure on D. For integer $k \geq 1$ and real number $p \geq 1$, $W^{k,p}(D)$ (resp. $W^{k,p}_{loc}(D)$) is the space of all L^p-integrable (resp. locally L^p-integrable) functions on D whose distributional derivatives up to and including order k are also L^p-integrable (resp. locally L^p-integrable). We use diag to denote the diagonal of the product domain $D \times D$.

2 Green function estimates for small $C^{1,1}$ domains

Let D be a bounded $C^{1,1}$ domain in $\mathbb{R}^d$. In this section, we first identify the Green function $G_D^{\mathcal{L}}(x,y)$ analytically constructed in [25] with the Green function $G_D(x,y)$ defined as the occupation density of the diffusion process X associated with $\mathcal{L}$ in D. We then use the Levi's freezing coefficient method and a testing function method together with a probabilistic approach to derive sharp estimates on $G_{\lambda D}(x,y)$ uniformly for sufficiently small $\lambda > 0$.

The next two lemmas holds for any non-divergence form operator $\mathcal{L}$ of (1.1) with bounded, continuous and uniformly elliptic diffusion coefficients. Recall that there is a conservative diffusion process $X = \{X_t, t \geq 0; \mathbb{P}_x, x \in \mathbb{R}^d\}$ associated with $\mathcal{L}$, as the unique solution to the martingale problem $(\mathcal{L}, C_c^\infty(\mathbb{R}^d))$. For an open set $D \subset \mathbb{R}^d$, denote by $\tau_D := \inf\{t > 0 : X_t \notin D\}$ the first exit time from D by X. We first prepare two lemmas that will be needed later. These two lemmas hold for any uniformly elliptic operators $\mathcal{L}$ with continuous diffusion coefficients $A(x)$. Recall that $\lambda_0 \geq 1$ is the ellipticity constant in (1.2).

Lemma 2.1. For each $x_0 \in \mathbb{R}^d$ and any ball $B(x_0, r)$ in $\mathbb{R}^d$ with radius $r > 0$,

$$\mathbb{E}_x\left[\tau_{B(x_0,r)}\right] \leq \lambda_0 r^2 \quad \text{for every } x \in B(x_0, r).$$

Proof. This result is well known. For reader's convenience, we provide a proof here. Fix $x_0 \in \mathbb{R}^d$ and $r > 0$. Define $f \in C_b^2(\mathbb{R}^d)$ so that $f(x) = |x - x_0|^2$ on $B(x_0, r)$ and $f(x) \geq r^2$ on $B(x_0, r)^c$. Since X is the solution to the martingale problem $(\mathcal{L}, C_c^\infty(\mathbb{R}^d))$, we have by (1.2) that for every $x \in B(x_0, r)$ and $t > 0$,

$$\begin{aligned}
&\mathbb{E}_x\left[f(X_{t\wedge\tau_{B(x_0,r)}})\right] \geq \mathbb{E}_x\left[f(X_{t\wedge\tau_{B(x_0,r)}})\right] - f(x) \\
&= \mathbb{E}_x\left[\int_0^{t\wedge\tau_{B(x_0,r)}} \mathcal{L}f(X_s)ds\right] \geq \lambda_0^{-1}\mathbb{E}_x\left[t \wedge \tau_{B(x_0,r)}\right].
\end{aligned}$$

Taking $t \to \infty$, it follows from the monotone convergence theorem that

$$\mathbb{E}_x\left[\tau_{B(x_0,r)}\right] \leq \lambda_0 \liminf_{t\to\infty} \mathbb{E}_x\left[f(X_{t\wedge\tau_{B(x_0,r)}})\right] \leq \lambda_0 r^2.$$

□

Lemma 2.2. Let D be a bounded $C^{1,1}$ domain in $\mathbb{R}^d$. Then for any $f \in C_c(D)$ and $\varphi \in C(\partial D)$ and for every $p > d/2$,

$$u(x) := \mathbb{E}_x\left[\varphi(X_{\tau_D})\right] + \mathbb{E}_x\left[\int_0^{\tau_D} f(X_s)ds\right], \quad x \in D, \tag{2.1}$$

is the unique function in $W^{2,p}_{loc}(D) \cap C(\overline{D})$ so that $\mathcal{L}u = -f$ a.e. in D and $u = \varphi$ on ∂D. When $\varphi = 0$, the above unique solution is in $W^{2,p}(D) \cap C(\overline{D})$ with $u = 0$ on ∂D.

Proof. For any $f \in C_c(D)$, $\varphi \in C(\partial D)$ and every $p > d/2$, by [22, Corollary 9.18], $\mathcal{L}u = -f$ in D and $u = \varphi$ on ∂D has a unique strong solution $u \in W^{2,p}_{loc}(D) \cap C(\overline{D})$. Let $\psi \in C_c^\infty(\mathbb{R}^d)$ be non-negative with support in $B(0,1)$ and $\int_{\mathbb{R}^d} \psi(x)dx = 1$. For $\varepsilon > 0$, define for $x \in D_\varepsilon := \{x \in D : \delta_D(x) > \varepsilon\}$,

$$u_\varepsilon(x) := \int_D \varepsilon^{-d}\psi((x-y)/\varepsilon)u(y)dy = \int_D \varepsilon^{-d}\psi(y/\varepsilon)u(x-y)dy$$

and

$$f_\varepsilon(x) := \int_D \varepsilon^{-d}\psi((x-y)/\varepsilon)f(y)dy = \int_D \varepsilon^{-d}\psi(y/\varepsilon)f(x-y)dy.$$

Note that $u_\varepsilon, f_\varepsilon \in C^\infty(D_\varepsilon) \cap C(\overline{D}_\varepsilon)$ and $\mathcal{L}u_\varepsilon = -f_\varepsilon$ on D_ε. Let $\varepsilon_0 > 0$ be fixed and small. For any $\varepsilon \in (0, \varepsilon_0)$ and $t > 0$, by Ito's formula,

$$\mathbb{E}_x\left[u_\varepsilon(X_{t\wedge\tau_{D_{\varepsilon_0}}})\right] - u_\varepsilon(x) = \mathbb{E}_x\left[\int_0^{t\wedge\tau_{D_{\varepsilon_0}}} \mathcal{L}u_\varepsilon(X_s)ds\right] = -\mathbb{E}_x\left[\int_0^{t\wedge\tau_{D_{\varepsilon_0}}} f_\varepsilon(X_s)ds\right]$$

for every $x \in D_{\varepsilon_0}$. Note that f_ε converges uniformly to f on D_{ε_0} as $\varepsilon \to 0$. In view of Lemma 2.1, we have by taking $t \to \infty$ and then letting ε approaching to zero along a decreasing sequence in the above display,

$$u(x) = \mathbb{E}_x \left[\int_0^{\tau_{D_{\varepsilon_0}}} f(X_s) ds \right] + \mathbb{E}_x \left[u(X_{\tau_{D_{\varepsilon_0}}}) \right] \quad \text{for every } x \in D_{\varepsilon_0}.$$

Now letting $\varepsilon_0 \to 0$ along a decreasing sequence shows that

$$u(x) = \mathbb{E}_x \left[u(X_{\tau_D}) \right] + \mathbb{E}_x \left[\int_0^{\tau_D} f(X_s) ds \right] = \mathbb{E}_x \left[\varphi(X_{\tau_D}) \right] + \mathbb{E}_x \left[\int_0^{\tau_D} f(X_s) ds \right]$$

for every $x \in D$.

When $\varphi = 0$, it is known (see [22, Theorem 9.15]) that the Dirichlet problem $\mathcal{L}u = -f$ has a unique solution in $W^{2,p}(D) \cap W_0^{1,p}(D)$ for any $p > 1$. Since D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$, $W^{2,p}(D) \subset C(\overline{D})$ for any $p > d/2$ by the Sobolev embedding theorem [22, Theorem 7.26]. It follows that u is continuous on $\overline{D}$ with $u = 0$ on ∂D. So this solution has a representation (2.1) with $\varphi = 0$. □

Denote by $\mathcal{L}^*$ the adjoint operator of $\mathcal{L}$. A solution of $\mathcal{L}^* v = f$ in D is defined to be a function v in $L^1_{loc}(D)$ such that for any $\varphi \in C_c^\infty(D)$,

$$\int_D v(y) \mathcal{L}\varphi(y)\, dy = \int_D f(y)\varphi(y)\, dy.$$

Theorem 2.3. Suppose the diffusion coefficient $(a_{ij}(x))_{1 \le i,j \le d}$ of $\mathcal{L}$ is of Dini mean oscillation, and D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R_0, Λ_0). There is a unique jointly continuous function $G_D(\cdot,\cdot)$ on $D \times D \setminus \mathrm{diag}$ so that

$$\mathbb{E}_x \left[\int_0^{\tau_D} f(X_s) ds \right] = \int_D G_D(x,y) f(y) dy, \quad x \in D, \tag{2.2}$$

for every $f \in C_c(D)$. Moreover, $G_D(x,y)$ has the following properties.

(i) For each $z \in \partial D$ and $y \in D$, $\lim_{\substack{x \to z \\ x \in D}} G_D(x,y) = 0$.

(ii) For each subdomain V with $D \setminus \overline{V} \neq \emptyset$,

$$G_D(x,y) = \mathbb{E}_x G_D \left[(X_{\tau_V}, y) \right] \quad \text{for every } x \in V \text{and} \quad y \in D \setminus \overline{V}. \tag{2.3}$$

(iii) There exists $C = C(d, \lambda_0, \ell, R_0, \Lambda_0, \mathrm{diam}(D)) > 0$ such that

$$G_D(x,y) \le C|x-y|^{2-d} \quad \text{for } x, y \in D. \tag{2.4}$$

(iv) For each fixed $x, y \in D$,

$$\mathcal{L} G_D(\cdot, y) = -\delta_{\{y\}} \quad \text{and} \quad \mathcal{L}^* G_D(x, \cdot) = -\delta_{\{x\}} \tag{2.5}$$

in the sense that for every $\phi \in L^p(D)$ with $p > d/2$, $G_D\phi(x) := \int_D G_D(x,y)\phi(y)dy$ is in $W^{2,p}_{loc}(D)$ for every $p > d/2$ satisfying $\mathcal{L} G_D \phi = -\phi$ a.e. in D, and $G_D^*\phi(y) := \int_D G_D(x,y)\phi(x)dx$ satisfies $\mathcal{L}^* G_D^* \phi = -\phi$ a.e. in D.

(v) $G_D(x,y) = G_D^{\mathcal{L}}(x,y)$ for any $x \neq y \in D$, where $G_D^{\mathcal{L}}(x,y)$ is the Green function of $\mathcal{L}$ in D defined analytically in [25].

Proof. Since $A(x) = (a_{ij}(x))$ is uniformly elliptic and bounded, and is of Dini mean oscillation, by [25, Theorem 1.9 and p.34], there is a unique non-negative function $G_D^{\mathcal{L}}(x,y)$ that is jointly continuous on $D \times D \setminus \mathrm{diag}$ so that for each $f \in L^p(D)$ with $p > d/2$,

$$u(y) = \int_D G_D^{\mathcal{L}}(x,y) f(x)\, dx. \tag{2.6}$$

is the unique solution of

$$\mathcal{L}^* u = -f \quad \text{in } D \quad \text{with } \lim_{y \to z \in \partial D} u(y) = 0. \tag{2.7}$$

Moreover, $G_D^{\mathcal{L}}(x,y)$ has the following property: there is a constant $c = c(d, \lambda_0, \ell, D) > 0$ so that

$$G_D^{\mathcal{L}}(x,y) \le c|x-y|^{2-d} \quad \text{and} \quad |\nabla_x G_D^{\mathcal{L}}(x,y)| \le c|x-y|^{1-d} \quad \text{for every } x, y \in D \text{ with } x \neq y, \tag{2.8}$$

and when D is $C^{2,\mathrm{Dini}}$,

$$|D_x^2 G_D^{\mathcal{L}}(x,y)| \le c|x-y|^{-d} \quad \text{for } x, y \in D \text{ with } x \neq y. \tag{2.9}$$

(The continuity of $G_D^{\mathcal{L}}(\cdot, y)$ in $D \setminus \{y\}$ and $G_D^{\mathcal{L}}(x, \cdot)$ in $D \setminus \{x\}$ are shown in [25, p.34 and p.36], which combined with (2.8) yields that $G_D^{\mathcal{L}}(x,y)$ is jointly continuous on $D \times D \setminus \mathrm{diag}$.)

By [25, Remark 1.14], for each $g \in L^q(D)$ with $q > d$,

$$v(x) := G_D^{\mathcal{L}} g(x) := \int_D G_D^{\mathcal{L}}(x,y) g(y)\, dy$$

is in $W^{2,q}(D) \cap W_0^{1,q}(D)$ and is a strong solution of

$$\mathcal{L} v = -g \quad \text{a.e. in } D \quad \text{with } \lim_{x \to z \in \partial D} v(x) = 0. \tag{2.10}$$

Note that by the Sobolev embedding theorem (see, e.g., [22, Theorem 7.10]), $W_0^{1,q}(D) \subset C_\infty(D)$, the space of continuous functions on $\overline{D}$ that vanishes on ∂D. This together with Lemma 2.2 yields that for any $f \in C_c(D)$,

$$\int_D G_D^{\mathcal{L}}(x,y) f(y)\, dy = \mathbb{E}_x \left[\int_0^{\tau_D} f(X_s) ds \right] \quad \text{for every } x \in D. \tag{2.11}$$

When D is a bounded $C^{1,1}$ domain. Let B be an open ball in $\mathbb{R}^d$ with radius $\mathrm{diam}(D)$ that contains $\overline{D}$ and define

$$G_D(x,y) = G_B^{\mathcal{L}}(x,y) - \mathbb{E}_x \left[G_B^{\mathcal{L}}(X_{\tau_D}, y) \right] \quad \text{for } x, y \in D \text{ with } x \neq y. \tag{2.12}$$

By Lemma 2.2, for each fixed $y \in D$, $x \mapsto v(x,y) := \mathbb{E}_x \left[G_B^{\mathcal{L}}(X_{\tau_D}, y) \right]$ is a function in $W^{2,p}_{loc}(D) \cap C(\overline{D})$ with $\mathcal{L}_x v(x,y) = 0$ in D and $v(x,y) = G_B^{\mathcal{L}}(x,y)$ on ∂D. Hence $x \mapsto G_D(x,y)$ is continuous on $D \setminus \{y\}$ and for each $z \in \partial D$ and $y \in D$, $\lim_{\substack{x \to z \\ x \in D}} G_D(x,y) = 0$. That is, $x \mapsto G_D(x,y)$ is continuous on $\overline{D} \setminus \{y\}$ after we set $G_D(x,y) = 0$ for $x \in \partial D$. In view of (2.8), there is a constant $c = c(d, \lambda_0, \ell, \mathrm{diam}(B)) > 0$ so that

$$G_D(x,y) \le G_B^{\mathcal{L}}(x,y) \le c|x-y|^{2-d} \quad \text{for every } x \neq y \in D.$$

Since $G_B^{\mathcal{L}}(x,y)$ is jointly continuous on $B \times B \setminus \mathrm{diag}$, by the dominated convergence theorem, $y \mapsto \mathbb{E}_x \left[G_B^{\mathcal{L}}(X_{\tau_D}, y) \right]$ is continuous on $D \setminus \{x\}$. It follows that $y \mapsto G_D(x,y)$ is continuous on $D \setminus \{x\}$.

We next identify G_D with $G_D^{\mathcal{L}}$ and show that $G_D(x,y)$ has the property (2.3). For every $f \in C_c(D)$, it follows from the strong Markov property of X and property (2.11) with B in place of D that for every $x \in D$,

$$\begin{aligned}
\int_D G_B^{\mathcal{L}}(x,y)f(y)dy &= \mathbb{E}_x\left[\int_0^{\tau_D} f(X_s)ds + \int_{\tau_D}^{\tau_B} f(X_s)ds\right] \\
&= \mathbb{E}_x\left[\int_0^{\tau_D} f(X_s)ds\right] + \mathbb{E}_x\left[\mathbb{E}_{X_{\tau_D}}\int_0^{\tau_B} f(X_r)dr\right] \\
&= \mathbb{E}_x\left[\int_0^{\tau_D} f(X_s)ds\right] + \mathbb{E}_x\int_D G_B^{\mathcal{L}}(X_{\tau_D},y)f(y)dy.
\end{aligned}$$

That is,

$$\mathbb{E}_x\left[\int_0^{\tau_D} f(X_s)ds\right] = \int_D G_D(x,y)f(y)dy \quad \text{for every } x \in D.$$

We conclude from (2.11) that

$$G_D(x,y) = G_D^{\mathcal{L}}(x,y) \text{ pointwise on } D\times D\setminus \mathrm{diag} \tag{2.13}$$

as both functions are continuous in y there. Thus we have established that $G_D(x,y)$ has all the desired properties stated in the theorem except (ii).

Suppose V is a subdomain of D with $D\setminus\overline{V} \neq \emptyset$. For any $f \in C_c(D\setminus\overline{V})$, we have by (2.2), the strong Markov property of X and the Fubini theorem,

$$\begin{aligned}
&\int_D G_D(x,z)f(z)dz = \mathbb{E}_x\left[\int_0^{\tau_D} f(X_s)ds\right] = \mathbb{E}_x\left[\int_{\tau_V}^{\tau_D} f(X_s)ds\right] \\
= \;& \mathbb{E}_x\left[\mathbb{E}_{X_{\tau_V}}\int_0^{\tau_D} f(X_s)ds\right] = \mathbb{E}_x\left[\int_D G_D(X_{\tau_V},z)f(z)dz\right] = \int_D \mathbb{E}_x\left[G_D(X_{\tau_V},z)\right]f(z)dz.
\end{aligned}$$

Consequently, $G_D(x,z) = \mathbb{E}_x\left[G_D(X_{\tau_V},z)\right]$ for a.e. and hence for every $z \in D\setminus\overline{V}$ as the functions $z \mapsto G_D(x,z)$ and $\mathbb{E}_x\left[G_D(X_{\tau_V},z)\right]$ are both continuous in $z \in D\setminus\overline{V}$ in view of the upper bound estimate of $G_D^{\mathcal{L}}(x,y)$ in (2.8). Taking $z=y$ establishes (2.3). This completes the proof of the theorem. □

In the following, we will use Levi's freezing coefficient method to derive Green function estimates on $G_{\lambda D}(x,y)$ uniformly for sufficiently small $\lambda > 0$. For this, define for each fixed $z \in \mathbb{R}^d$,

$$\mathcal{L}^{(z)} := \sum_{i,j=1}^d a_{ij}(z)\frac{\partial^2}{\partial x_i \partial x_j}. \tag{2.14}$$

Let U be a connected open subset of $\mathbb{R}^d$. For each fixed $z \in \mathbb{R}^d$, denote by $G_U^{(z)}$ the Green function of $\mathcal{L}^{(z)}$ in U. We search for Green function $G_U(x,y)$ of $\mathcal{L}$ in U of the following form

$$G_U(x,y) = G_U^{(y)}(x,y) + \int_U G_U^{(z)}(x,z)g_U(z,y)dz \tag{2.15}$$

for some function $g_U(x,y)$. Formally applying $\mathcal{L}$ on both sides in x, we have

$$\begin{aligned}
-\delta_{\{y\}}(x) &= \mathcal{L}^{(x)}G_U^{(y)}(\cdot,y)(x) + \int_U \mathcal{L}^{(x)}G_U^{(z)}(\cdot,z)(x)g_U(z,y)dy \\
&= -\delta_y(x) + (\mathcal{L}^{(x)} - \mathcal{L}^{(y)})G_U^{(y)}(\cdot,y)(x) - g_U(x,y) \\
&\quad + \int_U (\mathcal{L}^{(x)} - \mathcal{L}^{(z)})G_U^{(z)}(\cdot,z)(x)g_U(z,y)dy.
\end{aligned}$$

Here $\delta_{\{y\}}$ stands for the Dirac measure concentrated at y. Setting

$$g_U^{(0)}(x,y) := (\mathcal{L}^{(x)} - \mathcal{L}^{(y)}) G_U^{(y)}(\cdot, y)(x), \tag{2.16}$$

we see that $g_U(x,y)$ should satisfy the following integral equation

$$g_U(x,y) = g_U^{(0)}(x,y) + \int_U g_U^{(0)}(x,z) g_U(z,y) dz. \tag{2.17}$$

Applying the above equation recursively, it indicates that $g_U(x,y)$ is given by

$$g_U(x,y) = \sum_{k=0}^{\infty} g_U^{(k)}(x,y) \tag{2.18}$$

whenever it converges, where

$$g_U^{(k+1)}(x,y) := \int_U g_U^{(0)}(x,z) g_U^{(k)}(z,y) dz \quad \text{for } k \geq 0. \tag{2.19}$$

We will show in Lemma 2.6 and Theorem 2.8 that for $U = \lambda D$ when λ is sufficiently small, $g_{\lambda D}$ defined by (2.18) is convergent absolutely, and the function defined by the right hand side of (2.15) is indeed the Green function of $\mathcal{L}$ in λD.

Recall that

$$P_r(x,z) := \frac{r^2 - |x|^2}{\omega_d r \, |x-z|^d}, \quad x \in B(0,r), \; z \in \partial B(0,r),$$

is the Poisson kernel of the Laplacian (or equivalently, of Brownian motion) in the ball $B(0,r)$ in $\mathbb{R}^d$, where ω_d is the surface area of the unit sphere in $\mathbb{R}^d$.

Lemma 2.4. Let $x_0 \in \mathbb{R}^d$ and $r > 0$ and h be a harmonic function with respect to Δ in $B(x_0, r)$, then there exist positive constants $c_k = c_k(d) \geq 1$, $k = 1, 2$ such that

$$|\nabla h(x)| \leq c_1 h(x)/r \quad \text{and} \quad |D_x^2 h(x)| \leq c_2 h(x)/r^2 \quad \text{for } x \in B(x_0, r/2).$$

Proof. Without loss of generality, we assume $x_0 = 0$. There are $c_k = c_k(d) \geq 1$, $k = 1, 2$ such that for $x \in B(0, r/2)$ and $z \in \partial B(0, 3r/4)$,

$$|\nabla_x P_{3r/4}(x,z)| \leq c_1 \frac{P_r(x,z)}{3r/4 - |x|}, \quad |D_x^2 P_{3r/4}(x,z)| \leq c_2 \frac{P_r(x,z)}{(3r/4 - |x|)^2}. \tag{2.20}$$

Note that $h(x) = \int_{\partial B(0,3r/4)} P_{3r/4}(x,z) h(z) \sigma(dz)$, where $\sigma(\cdot)$ is the surface measure of $\partial B(0, 3r/4)$. Hence, by (2.20), the conclusion is obtained. □

Suppose D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R_0, Λ_0). By the Brownian scaling, it is well known that for any $\lambda > 0$,

$$G_{\lambda D}^{\Delta}(x,y) = \lambda^{2-d} G_D^{\Delta}(x/\lambda, y/\lambda) \quad \text{for every } x, y \in \lambda D, \; x \neq y. \tag{2.21}$$

By Theorem 5.1 and (2.21), there is a constant $C = C(d, \Lambda_0, R_0, \operatorname{diam}(D)) > 1$ such that for any $\lambda > 0$

$$\begin{aligned} &\frac{C^{-1}}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_{\lambda D}(x)}{|x-y|}\right) \left(1 \wedge \frac{\delta_{\lambda D}(y)}{|x-y|}\right) \leq G_{\lambda D}^{\Delta}(x,y) \\ &\leq \frac{C}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_{\lambda D}(x)}{|x-y|}\right) \left(1 \wedge \frac{\delta_{\lambda D}(y)}{|x-y|}\right) \end{aligned} \tag{2.22}$$

for all $x \neq y$ in λD.

Lemma 2.5. Suppose D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R_0, Λ_0). For each $z \in D$, the Green function $G_D^{(z)}(x,y)$ of $\mathcal{L}^{(z)}$ exists and is a smooth function in $D \times D \setminus \mathrm{diag}$. Moreover, there exist constants $C_k = C_k(d, \lambda_0, \Lambda_0, R_0, \mathrm{diam}(D)) \geq 1$, $k = 1, 2, 3$, such that for every $\lambda > 0$, $z \in \mathbb{R}^d$ and $x \neq y$ in λD,

$$\begin{aligned} \frac{C_1^{-1}}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_{\lambda D}(x)}{|x-y|}\right) \left(1 \wedge \frac{\delta_{\lambda D}(y)}{|x-y|}\right) &\leq G_{\lambda D}^{(z)}(x,y) \\ \leq \frac{C_1}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_{\lambda D}(x)}{|x-y|}\right) &\left(1 \wedge \frac{\delta_{\lambda D}(y)}{|x-y|}\right); \end{aligned} \tag{2.23}$$

and

$$|\nabla_x G_{\lambda D}^{(z)}(x,y)| \leq \frac{C_2}{|x-y|^{d-1}} \left(1 \wedge \frac{\delta_{\lambda D}(y)}{|x-y|}\right); \tag{2.24}$$

$$\left|\frac{\partial^2}{\partial x_i \partial x_j} G_{\lambda D}^{(z)}(x,y)\right| \leq \frac{C_3}{|x-y|^d} \left(1 \wedge \frac{\delta_{\lambda D}(y)}{|x-y|}\right) \left(1 \wedge \frac{\delta_{\lambda D}(x)}{|x-y|}\right)^{-1}. \tag{2.25}$$

Proof. Denote by $A(x) = (a_{ij}(x))$ and $A^{1/2}(x)$ its symmetric square root. Let W be a standard Brownian motion in $\mathbb{R}^d$. Fix $z \in \mathbb{R}^d$. It is easy to prove that the generator of the affined transform $A^{1/2}(z)W$ of W is $\mathcal{L}^{(z)}$. Let $D_z := \{A^{-1/2}(z)y : y \in D\}$. Then

$$G_{\lambda D}^{(z)}(x,y) := |A(z)|^{-d/2} G_{\lambda D_z}^{\Delta}(A^{-1/2}(z)x, A^{-1/2}(z)y), \quad x, y \in \lambda D, \tag{2.26}$$

is the Green function of $\mathcal{L}^{(z)}$ in D. Note that D_z is a $C^{1,1}$ domain in $\mathbb{R}^d$ whose $C^{1,1}$-characteristics depend only on R_0, Λ_0 and the uniform ellipticity constant λ_0 of the matrix $A(z)$, while the bounds on the diameter of D_z depend only on that of the diameter of D and λ_0. the desired estimate (2.23) follows from (2.26) and (2.22) for $G_{\lambda D_z}^{\Delta}$.

For each fixed $y \in \lambda D_z$, $x \mapsto G_{\lambda D_z}^{\Delta}(x,y)$ is harmonic with respect to Δ in $B(x, (\delta_{\lambda D_z}(x) \wedge |x-y|)/2)$. Hence, for $x, y \in \lambda D_z$ with $x \neq y$, we have by (2.22) and Lemma 2.4 with $r = (\delta_{\lambda D_z}(x) \wedge |x-y|)/2$ and $x_0 = x$ that

$$|\nabla_x G_{\lambda D_z}^{\Delta}(x,y)| \leq \frac{c_2}{|x-y|^{d-1}} \left(1 \wedge \frac{\delta_{\lambda D_z}(y)}{|x-y|}\right), \quad x \neq y \in \lambda D_z; \tag{2.27}$$

$$\left|\frac{\partial^2}{\partial x_i \partial x_j} G_{\lambda D_z}^{\Delta}(x,y)\right| \leq \frac{c_3}{|x-y|^d} \left(1 \wedge \frac{\delta_{\lambda D_z}(y)}{|x-y|}\right) \left(1 \wedge \frac{\delta_{\lambda D_z}(x)}{|x-y|}\right)^{-1}, \quad x \neq y \in \lambda D_z, \tag{2.28}$$

where $c_k = c_k(d, \lambda_0, \Lambda_0, R_0, \mathrm{diam}(D))$ for $k = 2, 3$. The estimates (2.24)-(2.25) then follow from (2.26) and (2.27)-(2.28). $\square$

Let $g_D^{(0)}$, $g_D^{(k)}$ and g_D be defined by (2.16), (2.19) and (2.18). In the following, we assume the diffusion coefficients $(a_{ij}(x))_{1 \leq i,j \leq d}$ of the non-divergence form operator $\mathcal{L}$ of (1.1) are ℓ-Dini continuous satisfying condition (1.3). Recall that $\ell(\cdot) : [0, \infty) \to [0, \infty)$ is an increasing continuous function with $\ell(0) = 0$ and $\int_0^1 \ell(t)/t \, dt < \infty$ such that there are positive constants c_0 and $\alpha \in (0, 1]$ satisfying (1.4).

Lemma 2.6. Suppose D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R_0, Λ_0). There exist positive constants $\theta_1 = \theta_1(d, \lambda_0, \ell, \Lambda_0, R_0, \mathrm{diam}(D)) \in (0, \frac{1}{2\mathrm{diam}(D)})$ and $C_4 = C_4(d, \lambda_0, \ell, \Lambda_0, R_0, \mathrm{diam}(D))$ such that for any $\lambda \in (0, \theta_1]$, $g_{\lambda D}(x,y) := \sum_{k=0}^{\infty} g_{\lambda D}^{(k)}(x,y)$ converges absolutely and locally uniformly on $(\lambda D) \times (\lambda D) \setminus \mathrm{diag}$ with

$$|g_{\lambda D}(x,y)| \leq \sum_{k=0}^{\infty} |g_{\lambda D}^{(k)}(x,y)| \leq C_4 \frac{\ell(|x-y|)}{|x-y|^d} \frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(x)} \quad \text{for } x \neq y \in \lambda D. \tag{2.29}$$

In particular, $g_{\lambda D}(x,y)$ is jointly continuous on $(\lambda D)\times(\lambda D)\setminus \mathrm{diag}$. Moreover,

$$g_{\lambda D}(x,y) = g^{(0)}_{\lambda D}(x,y) + \int_{\lambda D} g^{(0)}_{\lambda D}(x,z) g_{\lambda D}(z,y)dz \quad \text{for } x\neq y\in \lambda D, \tag{2.30}$$

where $g^{(0)}_{\lambda D}(x,y) = (\mathcal{L}^{(x)}-\mathcal{L}^{(y)})G^{(y)}_{\lambda D}(\cdot,y)(x)$.

Proof. Let $r_{\lambda D}(x,y) := \delta_{\lambda D}(x)+\delta_{\lambda D}(y)+|x-y|$. Then

$$r_{\lambda D}(x,y) \asymp |x-y| + \delta_{\lambda D}(x) \asymp |x-y| + \delta_{\lambda D}(y) \tag{2.31}$$

and

$$\delta_{\lambda D}(y) + |x-y| \le r_{\lambda D}(x,y) \le 2(\delta_{\lambda D}(y)+|x-y|) \quad \text{for every } x,y\in\lambda D.$$

Note that since for $a,b>0$,

$$\frac{a}{a+b} \le 1\wedge \frac{a}{b} \le \frac{2a}{a+b}, \tag{2.32}$$

we have by (2.25) that

$$\left|\frac{\partial^2}{\partial x_i \partial x_j} G^{(y)}_{\lambda D}(\cdot,y)(x)\right| \le \frac{4C_3}{|x-y|^d}\,\frac{\delta_{\lambda D}(y)}{r_{\lambda D}(x,y)}\cdot \frac{r_{\lambda D}(x,y)}{\delta_{\lambda D}(x)} = 4C_3|x-y|^{-d}\,\frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(x)}, \tag{2.33}$$

where C_3 is the positive constant in (2.25). Hence,

$$\begin{aligned} |g^{(0)}_{\lambda D}(x,y)| &= |(\mathcal{L}^{(x)}-\mathcal{L}^{(y)})G^{(y)}_{\lambda D}(\cdot,y)(x)| \\ &= \sum_{i,j=1}^d |a_{ij}(x)-a_{ij}(y)| \left|\frac{\partial^2}{\partial x_i\partial x_j} G^{(y)}_{\lambda D}(\cdot,y)(x)\right| \\ &\le 4C_3 \frac{\ell(|x-y|)}{|x-y|^d}\,\frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(x)}. \end{aligned} \tag{2.34}$$

For simplicity, let $c_1 := 4C_3$. We claim that for $n\ge 0$,

$$|g^{(n)}_{\lambda D}(x,y)| \le c_1\, 2^{-n}\frac{\ell(|x-y|)}{|x-y|^d}\frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(x)}, \quad x\neq y\in\lambda D. \tag{2.35}$$

Suppose that (2.35) holds for $n=k\ge 0$. Note that by (1.4), we have for $|x_1|<|x_2|<1$,

$$\frac{|x_1|^d}{|x_2|^d} \le \frac{|x_1|^\alpha}{|x_2|^\alpha} \le c_0\frac{\ell(|x_1|)}{\ell(|x_2|)};$$

that is,

$$\frac{|x_1|^d}{\ell(|x_1|)} \le c_0\frac{|x_2|^d}{\ell(|x_2|)} \quad \text{for} \quad |x_1|<|x_2|<1. \tag{2.36}$$

Thus, we have for $x,y,z\in\lambda D$ with $\mathrm{diam}(\lambda D)<1/2$,

$$\frac{|x-y|^d}{\ell(|x-y|)} \le c_0\frac{(|x-z|+|z-y|)^d}{\ell(|x-z|+|z-y|)} \le c_0\frac{(2|x-z|\vee 2|z-y|)^d}{\ell(2|x-z|\vee 2|z-y|)} \le c_0 2^d\left(\frac{|x-z|^d}{\ell(|x-z|)}+\frac{|z-y|^d}{\ell(|z-y|)}\right).$$

Hence, for $x,y,z\in\lambda D$ with $\mathrm{diam}(\lambda D)<1/2$,

$$\frac{\ell(|x-z|)}{|x-z|^d}\cdot\frac{\ell(|z-y|)}{|z-y|^d} \le c_0 2^d\left(\frac{\ell(|x-z|)}{|x-z|^d}+\frac{\ell(|z-y|)}{|z-y|^d}\right)\frac{\ell(|x-y|)}{|x-y|^d}. \tag{2.37}$$

By (2.34), (2.35) with $n=k$, and (2.37), for $\lambda < 1/(2\mathrm{diam}(D))$,

$$\begin{aligned} |g^{(0)}_{\lambda D}(x,z) g^{(k)}_{\lambda D}(z,y)| &\le c_1^2\, 2^{-k}\frac{\ell(|x-z|)}{|x-z|^d}\frac{\delta_{\lambda D}(z)}{\delta_{\lambda D}(x)}\frac{\ell(|z-y|)}{|z-y|^d}\frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(z)} \\ &\le c_1^2\, 2^{-k}c_0 2^d\frac{\ell(|x-y|)}{|x-y|^d}\frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(x)}\left(\frac{\ell(|x-z|)}{|x-z|^d}+\frac{\ell(|z-y|)}{|z-y|^d}\right) \end{aligned} \tag{2.38}$$

Hence,

$$
\begin{aligned}
|g^{(k+1)}_{\lambda D}(x,y)| &\le \int_{\lambda D} |g^{(0)}_{\lambda D}(x,z) g^{(k)}_{\lambda D}(z,y)|\, dz \\
&\le \omega_d c_1^2 c_0 2^{d+1-k} \frac{\ell(|x-y|)}{|x-y|^d} \frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(x)} \int_0^{\lambda \mathrm{diam}(D)} \frac{\ell(r)}{r}\, dr
\end{aligned}
$$

Let θ_1 be small enough such that for $\lambda \le \theta_1 \wedge \frac{1}{2\mathrm{diam}(D)}$,

$$
\omega_d c_1 c_0 2^{d+2} \int_{0<r<\lambda \mathrm{diam}(D)} \frac{\ell(r)}{r}\, dr < 1.
$$

Then for $\lambda \in (0, \theta_1]$,

$$
|g^{(k+1)}_{\lambda D}(x,y)| \le c_1 2^{-(k+1)} \frac{\ell(|x-y|)}{|x-y|^d} \frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(x)}, \quad x \ne y \in \lambda D. \tag{2.39}
$$

Thus (2.35) holds for $n = k+1$. In view of (2.34) and the mathematical induction, we have proved the claim (2.35), which holds for any bounded $C^{1,1}$ domain D with characteristics (Λ_0, R_0) and $\lambda \in (0, \theta_1]$. Consequently,

$$
|g_{\lambda D}(x,y)| \le \sum_{k=0}^{\infty} |g^{(k)}_{\lambda D}(x,y)| \le 2c_1 \frac{\ell(|x-y|)}{|x-y|^d} \frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(x)} \quad \text{for } x \ne y \in \lambda D.
$$

This proves (2.29) with $C_4 := 2c_1 = 8C_3$.

Note that in view of (2.16), $g^{(0)}_{\lambda D}(x,y) := (\mathcal{L}^{(x)} - \mathcal{L}^{(y)}) G^{(y)}_{\lambda D}(\cdot, y)(x)$ is jointly continuous on $(\lambda D) \times (\lambda D) \setminus \mathrm{diag}$. Fix $x_0, y_0 \in \lambda D$. Let $\varepsilon_0 = (\delta_{\lambda D}(x_0) \wedge \delta_{\lambda D}(y_0) \wedge |x_0 - y_0|)/8$ and $\delta \in (0, \varepsilon_0)$. By (2.38),

$$
\begin{aligned}
&\sup_{\substack{x \in B(x_0,\varepsilon_0) \\ y \in B(y_0,\varepsilon_0)}} \int_{\lambda D \cap B(x,\delta)} |g^{(0)}_{\lambda D}(x,z) g^{(k)}_{\lambda D}(z,y)|\, dz \\
\le & c_1^2\, 2^{-k} c_0 2^d \sup_{\substack{x \in B(x_0,\varepsilon_0) \\ y \in B(y_0,\varepsilon_0)}} \left(\frac{\ell(|x-y|)}{|x-y|^d} \frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(x)} \int_{\lambda D \cap B(x,\delta)} \left(\frac{\ell(|x-z|)}{|x-z|^d} + \frac{\ell(|z-y|)}{|z-y|^d} \right) dz \right) \\
\le & c_1^2 c_0 2^{d-k} \frac{\ell(2|x_0-y_0|)}{(|x_0-y_0|/2)^d} \frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(x_0)} \sup_{x \in B(x_0,\varepsilon_0)} \left(\int_{B(x,\delta)} \frac{\ell(|x-z|)}{|x-z|^d}\, dz + \int_{B(x,\delta)} \frac{\ell(2|x_0-y_0|)}{(|x_0-y_0|/2)^d}\, dz \right) \\
\le & c_1^2 c_0 2^{2d-k} \frac{\ell(2|x_0-y_0|)}{|x_0-y_0|^d} \frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(x_0)} \left(\omega_d \int_0^{\delta} \frac{\ell(s)}{s}\, ds + 2^d \frac{\ell(2|x_0-y_0|)}{|x_0-y_0|^d} |B(0,\delta)| \right)
\end{aligned} \tag{2.40}
$$

converges to 0 as $\delta \to 0$, where the second inequality holds due to $|x_0 - y_0|/2 \le |w - y| \le 2|x_0 - y_0|$ for $w \in B(x_0, \varepsilon_0)$ and $y \in B(y_0, \varepsilon_0)$. In the same way, we can show that

$$
\lim_{\delta \to 0} \sup_{\substack{x \in B(x_0,\varepsilon_0) \\ y \in B(y_0,\varepsilon_0)}} \int_{\lambda D \cap B(y,\delta)} |g^{(0)}_{\lambda D}(x,z) g^{(k)}_{\lambda D}(z,y)|\, dz = 0. \tag{2.41}
$$

Moreover, in view of (2.35),

$$
\lim_{\delta \to 0} \sup_{\substack{x \in B(x_0,\varepsilon_0) \\ y \in B(y_0,\varepsilon_0)}} \int_{\{z \in \lambda D: \delta_{\lambda D}(z) \le \delta\}} |g^{(0)}_{\lambda D}(x,z) g^{(k)}_{\lambda D}(z,y)|\, dz = 0. \tag{2.42}
$$

On the other hand, it follows from (2.35), Hölder's inequality and the bounded convergence theorem that for any $w \in B(x_0, \varepsilon_0)$ and $\delta \in (0, \varepsilon_0)$,

$$
\begin{aligned}
&\lim_{x \to w} \int_{B(x,\delta)^c \cap B(y,\delta)^c} \mathbb{1}_{\{\delta_{\lambda D}(z) > \delta\}} g^{(0)}_{\lambda D}(x,z) g^{(k)}_{\lambda D}(z,y)\, dz \\
&= \int_{B(w,\delta)^c \cap B(y,\delta)^c} \mathbb{1}_{\{\delta_{\lambda D}(z) > \delta\}} g^{(0)}_{\lambda D}(w,z) g^{(k)}_{\lambda D}(z,y)\, dz
\end{aligned} \tag{2.43}
$$

uniformly in $y \in B(y_0, \varepsilon_0)$. Since $g^{(0)}_{\lambda D}(x,y)$ is jointly continuous on $(\lambda D) \times (\lambda D) \setminus \mathrm{diag}$, we conclude from (2.40)-(2.43) by mathematical induction that $g^{(k+1)}_{\lambda D}(x,y)$ is continuous in $x \in B(x_0, \varepsilon_0)$ uniformly in $y \in B(y_0, \varepsilon_0)$ for every $k \geq 0$. Similarly, one can prove by induction that $g^{(k+1)}_{\lambda D}(x,y)$ is continuous in $y \in B(y_0, \varepsilon)$ uniformly in $x \in B(x_0, \varepsilon)$ for every $k \geq 0$. Consequently, $g^{(n)}_{\lambda D}(x,y)$ is jointly continuous on $(\lambda D) \times (\lambda D) \setminus \mathrm{diag}$ for every $n \geq 0$. As $\sum_{k=0}^\infty g^{(k)}_{\lambda D}(x,y)$ converges to $g_{\lambda D}(x,y)$ locally uniformly on $(\lambda D) \times (\lambda D) \setminus \mathrm{diag}$ by (2.39), $g_{\lambda D}(x,y)$ is jointly continuous on $(\lambda D) \times (\lambda D) \setminus \mathrm{diag}$. Identity (2.30) follows directly from the estimate (2.39) and recursive definition (2.19) of $g^{(k+1)}_{\lambda D}(x,y)$. $\square$

Lemma 2.7. Suppose D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R_0, Λ_0). Let θ_1 be the positive constant in Lemma 2.6. There exists a positive constant $C_5 = C_5(d, \lambda_0, \ell, \Lambda_0, R_0, \mathrm{diam}(D))$ such that for any $\lambda \in (0, \theta_1]$,

$$\int_{\lambda D} G^{(z)}_{\lambda D}(x,z) |g_{\lambda D}(z,y)|\, dz \leq \frac{C_5}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_{\lambda D}(y)}{|x-y|} \right) \quad \text{on } (\lambda D) \times (\lambda D) \setminus \mathrm{diag}. \tag{2.44}$$

Moreover, $\int_{\lambda D} G^{(z)}_{\lambda D}(x,z) g_{\lambda D}(z,y)\, dz$ is jointly continuous in $(x,y) \in (\lambda D) \times (\lambda D) \setminus diag$ and for each $y \in \lambda D$,

$$\lim_{x \to \partial(\lambda D)} \int_{\lambda D} G^{(z)}_{\lambda D}(x,z) g_{\lambda D}(z,y)\, dz = 0.$$

Proof. Let $r_{\lambda D}(x,y) := \delta_{\lambda D}(x) + \delta_{\lambda D}(y) + |x-y|$. Note that

$$\delta_{\lambda D}(x) + |x-y| \leq r_{\lambda D}(x,y) \leq 2(\delta_{\lambda D}(x) + |x-y|) \quad \text{for every } x, y \in \lambda D. \tag{2.45}$$

By (2.31)-(2.32), (2.23) and Lemma 2.6, for $\lambda \in (0, \theta_1]$,

$$\begin{aligned} G^{(z)}_{\lambda D}(x,z) |g_{\lambda D}(z,y)| &\leq 4 C_1 C_4 \frac{1}{|x-z|^{d-2}} \frac{\delta_{\lambda D}(z)}{r_{\lambda D}(x,z)} \frac{\ell(|z-y|)}{|z-y|^d} \frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(z)} \\ &\leq \frac{4 C_1 C_4}{|x-y|^{d-2}} \frac{\delta_{\lambda D}(y)}{r_{\lambda D}(x,y)} \left(\frac{|x-y|^{d-2}}{|x-z|^{d-2}} \frac{\ell(|z-y|)}{|z-y|^d} \frac{r_{\lambda D}(x,y)}{r_{\lambda D}(x,z)} \right) \end{aligned} \tag{2.46}$$

Note that by (2.45)

$$\begin{aligned} & \frac{|x-y|^{d-2}}{|x-z|^{d-2}} \frac{\ell(|z-y|)}{|z-y|^d} \frac{r_{\lambda D}(x,y)}{r_{\lambda D}(x,z)} \\ \leq\ & 2 \frac{|x-y|^{d-2}}{|x-z|^{d-2}} \frac{\ell(|z-y|)}{|z-y|^d} \frac{|x-z| + |z-y| + \delta_{\lambda D}(x)}{|x-z| + \delta_{\lambda D}(x)} \\ \leq\ & 4 \frac{(2\max\{|x-z|, |z-y|\})^{d-2}}{|x-z|^{d-2}} \frac{\ell(|z-y|)}{|z-y|^d} \frac{|x-z| + |z-y|}{|x-z|} \\ \leq\ & 4 \frac{(2\max\{|x-z|, |z-y|\})^{d-1}}{|x-z|^{d-1}} \frac{\ell(|z-y|)}{|z-y|^d} \\ \leq\ & 2^{d+1} \ell(|z-y|) \left(\frac{1}{|z-y|^d} + \frac{1}{|x-z|^{d-1} |z-y|} \right) \\ \leq\ & 2^{d+1} \left(\frac{2\ell(|z-y|)}{|z-y|^d} + 1_{\{|z-y| > |x-z|\}} \frac{\ell(|x-z|)}{|x-z|^d} \frac{\ell(|z-y|)}{\ell(|x-z|)} \frac{|x-z|}{|z-y|} \right) \\ \leq\ & 2^{d+1} \left(\frac{2\ell(|z-y|)}{|z-y|^d} + 1_{\{|z-y| > |x-z|\}} c_0 \frac{\ell(|x-z|)}{|x-z|^d} \left(\frac{|z-y|}{|x-z|} \right)^\alpha \frac{|x-z|}{|z-y|} \right) \\ \leq\ & 2^{d+1} \left(\frac{2\ell(|z-y|)}{|z-y|^d} + c_0 \frac{\ell(|x-z|)}{|x-z|^d} \right), \end{aligned} \tag{2.47}$$

where the second to the last inequality is due to (1.4). Hence, we have by (2.46) and (2.47),

$$
\begin{aligned}
\int_{\lambda D} G^{(z)}_{\lambda D}(x,z)|g_{\lambda D}(z,y)|\,dz &\le \frac{2^{d+3}C_1C_4}{|x-y|^{d-2}}\frac{\delta_{\lambda D}(y)}{r_{\lambda D}(x,y)}\int_{\lambda D}\left(2\frac{\ell(|z-y|)}{|z-y|^d}+c_0\frac{\ell(|x-z|)}{|x-z|^d}\right)dz \\
&\le \frac{2^{d+3}(2+c_0)C_1C_4}{|x-y|^{d-2}}\frac{\delta_{\lambda D}(y)}{r_{\lambda D}(x,y)}\int_0^{\lambda \mathrm{diam}(D)}\omega_d\frac{\ell(r)}{r}\,dr \\
&\le \frac{2^{d+3}(2+c_0)C_1C_4}{|x-y|^{d-2}}\frac{\delta_{\lambda D}(y)}{r_{\lambda D}(x,y)}\int_0^1\omega_d\frac{\ell(r)}{r}\,dr \qquad \text{for } x\neq y\in\lambda D.
\end{aligned}
\tag{2.48}
$$

This establishes (2.44).

It follows from (2.46) and (2.47) that

$$
\begin{aligned}
G^{(z)}_{\lambda D}(x,z)|g_{\lambda D}(z,y)| &\le \frac{2^{d+3}(2+c_0)C_1C_4}{|x-y|^{d-2}}\frac{\delta_{\lambda D}(y)}{r_{\lambda D}(x,y)}\left(\frac{\ell(|z-y|)}{|z-y|^d}+\frac{\ell(|x-z|)}{|x-z|^d}\right) \\
&\le \frac{2^{d+3}(2+c_0)C_1C_4}{|x-y|^{d-2}}\left(\frac{\ell(|z-y|)}{|z-y|^d}+\frac{\ell(|x-z|)}{|x-z|^d}\right)
\end{aligned}
\tag{2.49}
$$

By a similar argument as that for (2.40)-(2.43), one can show that $\int_{\lambda D} G^{(z)}_{\lambda D}(x,z)g_{\lambda D}(z, y)\,dz$ is jointly continuous in $(x,y)\in(\lambda D)\times(\lambda D)\setminus\mathrm{diag}$.

Finally, we prove that for each $y\in\lambda D$ and $Q\in\partial(\lambda D)$,

$$
\lim_{x\to Q}\int_{\lambda D} G^{(z)}_{\lambda D}(x,z)g_{\lambda D}(z,y)\,dz=0 \tag{2.50}
$$

Fix $y\in\lambda D$ and $Q\in\partial(\lambda D)$. Let $\varepsilon\in(0,\delta_{\lambda D}(y)/2)$. By (2.49) and the dominated convergence theorem,

$$
\begin{aligned}
&\lim_{\lambda D\cap B(Q,\varepsilon/2)\ni x\to Q}\int_{\lambda D\setminus B(Q,\varepsilon)} G^{(z)}_{\lambda D}(x,z)g_{\lambda D}(z,y)\,dz \\
&=\int_{\lambda D\setminus B(Q,\varepsilon)}\lim_{x\to Q}G^{(z)}_{\lambda D}(x,z)g_{\lambda D}(z,y)\,dz=0.
\end{aligned}
\tag{2.51}
$$

On the other hand, by (2.49),

$$
\begin{aligned}
&\sup_{x\in\lambda D\cap B(Q,\varepsilon)}\int_{\lambda D\cap B(Q,\varepsilon)} G^{(z)}_{\lambda D}(x,z)|g_{\lambda D}(z,y)|dz \\
&\le \sup_{x\in\lambda D\cap B(Q,\varepsilon)}\frac{2^{d+3}(2+c_0)C_1C_4}{|x-y|^{d-2}}\int_{\lambda D\cap B(Q,\varepsilon)}\frac{\ell(|x-z|)}{|x-z|^d}+\frac{\ell(|z-y|)}{|z-y|^d}\,dz \\
&\le 2^{d+3}(2+c_0)C_1C_4\left(\frac{\delta_{\lambda D}(y)}{2}\right)^{2-d}\sup_{x\in\lambda D\cap B(Q,\varepsilon)}\int_{|x-z|<2\varepsilon}\frac{\ell(|x-z|)}{|x-z|^d}\,dz \\
&\quad+2^{d+3}(2+c_0)C_1C_4\left(\frac{\delta_{\lambda D}(y)}{2}\right)^{2-d}\int_{\lambda D\cap B(Q,\varepsilon)}\frac{\ell(|z-y|)}{|z-y|^d}\,dz \\
&\le 2^{d+3}(2+c_0)C_1C_4\left(\frac{\delta_{\lambda D}(y)}{2}\right)^{2-d}\left(\int_0^{2\varepsilon}\omega_d\frac{\ell(s)}{s}\,ds+\frac{\ell(\mathrm{diam}(\lambda D))}{(\delta_{\lambda D}(y)/2)^d}\,|B(Q,\varepsilon)|\right)\to 0
\end{aligned}
$$

as $\varepsilon\to 0$. This combined with (2.51) establishes (2.50). □

The following result is a key step of this paper. It shows that the kernel obtained via Levi's freezing coefficient method defined on the right hand side of (2.15) through (2.16)-(2.19) with λD in place of U is indeed the Green function $G_{\lambda D}$ when $\lambda>0$ is sufficiently small.

Theorem 2.8. Suppose D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R_0, Λ_0). Let θ_1 be the positive constant in Lemma 2.6. Then for any $\lambda \in (0, \theta_1]$, the Green function $G_{\lambda D}$ of $\mathcal{L}$ in λD satisfies equation (2.15) with λD in place of U; that is,

$$G_{\lambda D}(x,y) = G^{(y)}_{\lambda D}(x,y) + \int_{\lambda D} G^{(z)}_{\lambda D}(x,z) g_{\lambda D}(z,y) dz \quad \text{for } x \neq y \in \lambda D. \tag{2.52}$$

Proof. For $\lambda \in (0, \theta_1]$, define

$$\widetilde{G}_{\lambda D}(x,y) := G^{(y)}_{\lambda D}(x,y) + \int_{\lambda D} G^{(z)}_{\lambda D}(x,z) g_{\lambda D}(z,y) dz, \quad x \neq y \in \lambda D. \tag{2.53}$$

The crux of this proof is to show that for each $C_c^\infty(\lambda D)$ function ψ, $\widetilde{G}_{\lambda D}\psi(x) := \int_{\lambda D} \widetilde{G}_{\lambda D}(x, y)\psi(y)dy$ is in $W^{2,p}_{loc}(\lambda D)$ for any $p > 1$ and $\mathcal{L}\widetilde{G}_{\lambda D}\psi = -\psi$ a.e. on λD. Once this is established, we can use mollifier and Ito's formula to show that $\widetilde{G}_{\lambda D}(x,y)$ is the occupation density of the diffusion process X associated with $\mathcal{L}$ in λD and hence identify it with the Green function $G_{\lambda D}(x,y)$. However, due to the singularity of the second derivative of $G^{(z)}_{\lambda D}(x,z)$ along the diagonal and the intertwined z in the second derivative of $G^{(z)}_{\lambda D}(x,z)$, it is difficult to show directly the weak differentiability of $\widetilde{G}_{\lambda D}\psi(x)$ and to establish the property $\mathcal{L}\widetilde{G}_{\lambda D}\psi = -\psi$ a.e. on λD. To overcome these difficulties, we use an approximation procedure to smooth out the singularity of $G^{(z)}_{\lambda D}(x,z)$ along the diagonal. Some ideas of this approximation are motivated by those from [22, Lemma 4.2] which is for the C^2-derivatives of the Newtonian potentials of the Laplace operator. But in our case, since the differential operator $\mathcal{L}$ has less smooth coefficients and the form of $\tilde{G}_{\lambda D}(x,y)$ is much more involved, our argument is significantly different from that of [22, Lemma 4.2] and the proof here is much more delicate.

The proof of this theorem is pretty long. We divided it into three parts and part (I) is further divided into three steps. Let $\eta \in [0,2]$ be a $C^\infty(\mathbb{R})$ function satisfying

$$\eta(t) = 0 \quad \text{for} \quad t \leq 1, \quad \text{and} \quad \eta(t) = 1 \quad \text{for} \quad t \geq 2$$

and $0 \leq \frac{d}{dt}\eta(t) \leq 2, |\frac{d}{dt^2}\eta(t)| \leq 4$ for $t \in \mathbb{R}$. We define for $\varepsilon \in (0,1)$, $\eta_\varepsilon(x,y) := \eta(|x-y|/\varepsilon)$ in $\mathbb{R}^d \times \mathbb{R}^d$. Let $(G^{(y)}_{\lambda D}\eta_\varepsilon)(x,y) := G^{(y)}_{\lambda D}(x,y)\eta_\varepsilon(x,y)$. Then $(G^{(y)}_{\lambda D}\eta_\varepsilon)(x,y) = 0$ for $|x-y| < \varepsilon$ and $(G^{(y)}_{\lambda D}\eta_\varepsilon)(x,y) = G^{(y)}_{\lambda D}(x,y)$ for $|y-x| > 2\varepsilon$. Define

$$\widetilde{G}^{(\varepsilon)}_{\lambda D}(x,y) := (G^{(y)}_{\lambda D}\eta_\varepsilon)(x,y) + \int_{\lambda D} (G^{(z)}_{\lambda D}\eta_\varepsilon)(x,z) g_{\lambda D}(z,y) dz, \quad x \neq y \in \lambda D.$$

(I) We claim that for any $\phi, \psi \in C_c^\infty(\lambda D)$,

$$\lim_{\varepsilon \to 0} \int_{\lambda D} \phi(x) \mathcal{L}_x \tilde{G}^{(\varepsilon)}_{\lambda D}\psi(x)\, dx = -\int_{\lambda D} \phi(x)\psi(x)\, dx. \tag{2.54}$$

We prove this through the following three steps.

Step 1: We prove that for any $\psi, \phi \in C_c^\infty(\lambda D)$,

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int_{\lambda D} \mathcal{L}_x \left(\int_{\lambda D} (G^{(y)}_{\lambda D}\eta_\varepsilon)(x,y)\psi(y)\, dy \right) \phi(x)\, dx \\ &= \int_{\lambda D}\int_{\lambda D} (\mathcal{L}^{(x)}_x - \mathcal{L}^{(y)}_x) G^{(y)}_{\lambda D}(x,y)\psi(y)\, dy \phi(x)\, dx - \int_{\lambda D} \psi(x)\phi(x)\, dx. \end{aligned} \tag{2.55}$$

Note that by the estimates (2.23)-(2.24) and (2.33), there exists $c_1 = c_1(d, \lambda_0, R_0, \Lambda_0,$

$\mathrm{diam}(D)) > 1$ such that for each $x, y \in \lambda D$,

$$\begin{aligned}
&\sup_{|h|<\delta_{\lambda D}(x)/4} \sum_{i,j=1}^{d} |D_{ij}(G_{\lambda D}^{(y)} \cdot \eta_\varepsilon)(x+h,y)| \\
=&\sup_{|h|<\delta_{\lambda D}(x)/4} \sum_{i,j=1}^{d} |(D_{ij}G_{\lambda D}^{(y)} \cdot \eta_\varepsilon)(x+h,y) + D_i G_{\lambda D}^{(y)}(x+h,y) \cdot D_j \eta_\varepsilon(x+h,y) \\
&\qquad + G_{\lambda D}^{(y)}(x+h,y) \cdot D_{ij}\eta_\varepsilon(x+h,y)| \\
\le& c_1 \sup_{|h|<\delta_{\lambda D}(x)/4} \left(|x+h-y|^{-d} \frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(x+h)} + \frac{2}{\varepsilon}|x+h-y|^{1-d} + \frac{4}{\varepsilon^2}|x+h-y|^{2-d} \right) \mathbb{1}_{|x+h-y|>\varepsilon/2} \\
\le& 2^d c_1 \varepsilon^{-d} \left(2\frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(x)} + 2 \right).
\end{aligned}$$

Thus, by the dominated convergence theorem,

$$D_{ij} \int_{\lambda D} (G_{\lambda D}^{(y)} \eta_\varepsilon)(x,y)\psi(y)\, dy = \int_{\lambda D} D_{ij}(G_{\lambda D}^{(y)} \eta_\varepsilon)(\cdot, y)(x)\psi(y)\, dy.$$

Hence, $\mathcal{L}_x \int_{\lambda D} G_{\lambda D}^{(y)} \eta_\varepsilon(x,y)\psi(y)\, dy = \int_{\lambda D} \mathcal{L}_x (G_{\lambda D}^{(y)} \eta_\varepsilon)(x,y)\psi(y)\, dy$. We divide the integral into two parts

$$\begin{aligned}
&\int_{\lambda D}\int_{\lambda D} \mathcal{L}_x (G_{\lambda D}^{(y)} \eta_\varepsilon)(x,y)\psi(y)\, dy \phi(x)\, dx \\
=&\int_{\lambda D}\int_{\lambda D} (\mathcal{L}_x^{(x)} - \mathcal{L}_x^{(y)})(G_{\lambda D}^{(y)} \eta_\varepsilon)(x,y)\psi(y)\, dy \phi(x)\, dx \\
&+ \int_{\lambda D}\int_{\lambda D} \mathcal{L}_x^{(y)} (G_{\lambda D}^{(y)} \eta_\varepsilon)(x,y)\psi(y)\, dy \phi(x)\, dx \\
=:& I_1 + I_2
\end{aligned} \tag{2.56}$$

For the first term, by the Dini condition (1.3) of a_{ij}, (2.23)-(2.24) and (2.33), there exists $c_2 = c_(d, \lambda_0, R_0, \Lambda_0, \mathrm{diam}(D))$ such that

$$\begin{aligned}
&\int_{\lambda D} |(\mathcal{L}_x^{(x)} - \mathcal{L}_x^{(y)})((G_{\lambda D}^{(y)} \eta_\varepsilon)(x,y) - G_{\lambda D}^{(y)}(x,y))||\psi(y)|\, dy \\
\le& \sum_{i,j=1}^{d} \int_{\lambda D \cap \{|y-x|\le 2\varepsilon\}} |a_{ij}(x) - a_{ij}(y)| \cdot |D_{ij}(G_{\lambda D}^{(y)} \eta_\varepsilon)(x,y) - D_{ij}G_{\lambda D}^{(y)}(x,y)| \cdot |\psi(y)|\, dy \\
\le& \sum_{i,j=1}^{d} \|\psi\|_\infty \int_{\lambda D \cap \{|y-x|\le 2\varepsilon\}} \ell(|x-y|)(|D_{ij}(G_{\lambda D}^{(y)} \eta_\varepsilon)(x,y)| + |D_{ij}G_{\lambda D}^{(y)}(x,y)|)\, dy \\
\le& \sum_{i,j=1}^{d} \|\psi\|_\infty \int_{\lambda D \cap \{|y-x|\le 2\varepsilon\}} \ell(|x-y|)(|D_{ij}\eta_\varepsilon(x,y)|G_{\lambda D}^{(y)}(x,y) + |D_i \eta_\varepsilon(x,y)||D_i G_{\lambda D}^{(y)}(x,y)|)\, dy \\
&+ 2\sum_{i,j=1}^{d} \|\psi\|_\infty \int_{\lambda D \cap \{|y-x|\le 2\varepsilon\}} \ell(|x-y|)|D_{ij}G_{\lambda D}^{(y)}(x,y)|)\, dy \\
\le& c_2 \|\psi\|_\infty \int_{\lambda D \cap \{|y-x|\le 2\varepsilon\}} \ell(|x-y|) \left(4\varepsilon^{-2}|x-y|^{2-d} + 2\varepsilon^{-1}|x-y|^{1-d} + |x-y|^{-d} \frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(x)} \right) dy \\
\le& c_2 \left(20 + \frac{1}{\delta_{\lambda D}(x)} \right) \|\psi\|_\infty \int_{\{|x-y|\le 2\varepsilon\}} \frac{\ell(|x-y|)}{|x-y|^d}\, dy \\
\le& c_2 \left(20 + \frac{1}{\delta_{\lambda D}(x)} \right) \|\psi\|_\infty \int_{\{0\le s\le 2\varepsilon\}} \omega_d \frac{\ell(s)}{s}\, ds,
\end{aligned} \tag{2.57}$$

which converges to 0 as $\varepsilon \to 0$ uniformly on any compact subset of λD, where the second to the last inequality holds due to $\delta_{\lambda D}(y) \le \mathrm{diam}(\lambda D) \le 1$. Hence, for each $\phi \in C_c^\infty(\lambda D)$,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\lambda D}\int_{\lambda D}(\mathcal{L}_x^{(x)} - \mathcal{L}_x^{(y)})(G_{\lambda D}^{(y)}\eta_\varepsilon)(x,y)\psi(y)\,dy\phi(x)\,dx \\ &= \int_{\lambda D}\int_{\lambda D}(\mathcal{L}_x^{(x)} - \mathcal{L}_x^{(y)})G_{\lambda D}^{(y)}(x,y)\psi(y)\,dy\phi(x)\,dx.\end{aligned} \tag{2.58}$$

Next, we consider I_2 in (2.56). For any $\psi, \phi \in C_c^\infty(\lambda D)$, by Fubini theorem and integration by parts,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\lambda D}\int_{\lambda D}\mathcal{L}_x^{(y)}(G_{\lambda D}^{(y)}\eta_\varepsilon)(x,y)\psi(y)\,dy\phi(x)\,dx \\ =\;& \lim_{\varepsilon\to 0}\int_{\lambda D}\left(\int_{\lambda D}\mathcal{L}_x^{(y)}(G_{\lambda D}^{(y)}\eta_\varepsilon)(x,y)\phi(x)\,dx\right)\psi(y)\,dy \\ =\;& \lim_{\varepsilon\to 0}\int_{\lambda D}\left(\int_{\lambda D}\mathcal{L}_x^{(y)}\phi(x)(G_{\lambda D}^{(y)}\eta_\varepsilon)(x,y)dx\right)\psi(y)\,dy \\ =\;& \int_{\lambda D}\left(\int_{\lambda D}\mathcal{L}_x^{(y)}\phi(x)G_{\lambda D}^{(y)}(x,y)dx\right)\psi(y)\,dy \\ =\;& \int_{\lambda D}\lim_{\varepsilon\to 0}\frac{1}{|B(y,\varepsilon)\cap(\lambda D)|}\left(\int_{\lambda D}\mathcal{L}_x^{(y)}\phi(x)G_{\lambda D}^{(y)}1_{B(y,\varepsilon)}(x)dx\right)\psi(y)\,dy \\ =\;& \int_{\lambda D}\lim_{\varepsilon\to 0}\frac{1}{|B(y,\varepsilon)\cap(\lambda D)|}\left(\int_{\lambda D}\phi(x)(-1_{B(y,\varepsilon)})(x)dx\right)\psi(y)\,dy \\ =\;& -\int_{\lambda D}\phi(y)\,\psi(y)dy.\end{aligned} \tag{2.59}$$

In the fourth equality we used the dominated convergence theorem and the fact that for $y \ne x \in \lambda D$, $\lim_{\varepsilon\to 0}\frac{1}{|B(y,\varepsilon)\cap(\lambda D)|}G_{\lambda D}^{(y)}1_{B(y,\varepsilon)}(x) = G_{\lambda D}^{(y)}(x,y)$ and

$$\frac{1}{|B(y,\varepsilon)\cap(\lambda D)|}G_{\lambda D}^{(y)}1_{B(y,\varepsilon)}(x) \le \frac{c_1}{|B(y,\varepsilon)\cap(\lambda D)|}\int_{\lambda D}\frac{1}{|x-w|^{d-2}}1_{B(y,\varepsilon)}(w)dw \le \frac{c_2}{|x-y|^{d-2}},$$

by considering two cases separately: (i) when $|x-y| > 2\varepsilon$, then $|x-w| \asymp |x-y|$ for $w \in B(y,\varepsilon)$; (ii) when $|x-y| \le 2\varepsilon$, then $B(y,\varepsilon) \subset B(x,3\varepsilon)$ and so

$$\begin{aligned}&\frac{c_1}{|B(y,\varepsilon)\cap(\lambda D)|}\int_{\lambda D}\frac{1}{|x-w|^{d-2}}1_{B(y,\varepsilon)}(w)dw \\ \le\;& \frac{c_1}{|B(y,\varepsilon)\cap(\lambda D)|}\int_{B(x,3\varepsilon)\cap\lambda D}\frac{1}{|x-w|^{d-2}}dw \\ \le\;& \frac{c_2}{|B(y,\varepsilon)\cap(\lambda D)|}\varepsilon^2 \le \frac{c_3}{\varepsilon^{d-2}} \le \frac{c_4}{|x-y|^{d-2}}.\end{aligned}$$

In the fifth equality, we used the fact that for $f \in L^p(\lambda D)$, $G_{\lambda D}^{(y)}f$ is a weak solution to $\mathcal{L}^{(y)}u = -f$ in λD. Combing (2.58) with (2.59) yields the desired property (2.55).

Step 2: We prove that for any $\psi, \phi \in C_c^\infty(\lambda D)$,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\lambda D\times\lambda D}\mathcal{L}_x\left[\int_{\lambda D}(G_{\lambda D}^{(z)}\eta_\varepsilon)(x,z)g_{\lambda D}(z,y)dz\right]\psi(y)\phi(x)\,dy\,dx \\ &= \int_{\lambda D\times\lambda D}\left[\int_{\lambda D}(\mathcal{L}_x^{(x)} - \mathcal{L}_x^{(z)})G_{\lambda D}^{(z)}(x,z)g_{\lambda D}(z,y)dz\right]\psi(y)\phi(x)\,dy\,dx \\ &\quad - \int_{\lambda D}\int_{\lambda D}g_{\lambda D}(x,y)\psi(y)\,dy\phi(x)\,dx.\end{aligned} \tag{2.60}$$

By a similar argument in (2.57),

$$\begin{aligned}
&\int_{\lambda D}\int_{\lambda D} |(\mathcal{L}_x^{(x)}-\mathcal{L}_x^{(z)})((G_{\lambda D}^{(z)}\eta_\varepsilon)(x,z)-G_{\lambda D}^{(z)}(x,z))|\cdot|g_{\lambda D}(z,y)||\psi(y)|\,dz\,dy\\
\le&\|\psi\|_\infty\int_{\lambda D}\int_{\lambda D\cap\{|z-x|\le 2\varepsilon\}}\ell(|x-z|)\left(\frac{4}{\varepsilon^2}G_{\lambda D}^{(z)}(x,z)+\frac{2}{\varepsilon}|D_iG_{\lambda D}^{(z)}(x,z)|+2|D_{ij}G_{\lambda D}^{(z)}(x,z)|\right)\\
&\quad\cdot|g_{\lambda D}(z,y)|\,dz\,dy
\end{aligned}\tag{2.61}$$

Let $r_{\lambda D}(x,y):=\delta_{\lambda D}(x)+\delta_{\lambda D}(y)+|x-y|$. In view of (2.23), (2.32), (2.45) and Lemma 2.6, for any $\varepsilon\in(0,(\delta_{\lambda D}(x)\wedge|x-y|)/4)$, the first integral in the right side of (2.61) satisfies

$$\begin{aligned}
&\int_{\lambda D}\int_{\lambda D\cap\{|z-x|\le 2\varepsilon\}}\ell(|x-z|)\frac{4}{\varepsilon^2}G_{\lambda D}^{(z)}(x,z)|g_{\lambda D}(z,y)|\,dz\,dy\\
\le&\frac{16}{\varepsilon^2}C_1C_4\int_{\lambda D}\int_{\lambda D\cap\{|z-x|\le 2\varepsilon\}}\ell(|x-z|)|x-z|^{2-d}\frac{\delta_{\lambda D}(z)}{r_{\lambda D}(x,z)}\frac{\ell(|z-y|)}{|z-y|^d}\frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(z)}\,dz\,dy\\
\le&C_1C_4\frac{16}{\varepsilon^2}\frac{1}{\delta_{\lambda D}(x)}\int_{\lambda D}\int_{\lambda D\cap\{|z-x|\le 2\varepsilon\}}|x-z|^2\frac{\ell(|x-z|)}{|x-z|^d}\frac{\ell(|z-y|)}{|z-y|^d}\,dz\,dy\\
\le&64C_1C_4\frac{1}{\delta_{\lambda D}(x)}\int_{\lambda D\cap\{|z-x|\le 2\varepsilon\}}\frac{\ell(|x-z|)}{|x-z|^d}\left(\int_{\lambda D}\frac{\ell(|z-y|)}{|z-y|^d}\,dy\right)dz\\
\le&64C_1C_4\frac{1}{\delta_{\lambda D}(x)}\int_0^1\omega_d\frac{\ell(r)}{r}\,dr\int_0^{2\varepsilon}\omega_d\frac{\ell(s)}{s}\,ds,
\end{aligned}$$

where the second inequality is due to $r_{\lambda D}(x,z)\ge\delta_{\lambda D}(x)$ and $\delta_{\lambda D}(y)\le\mathrm{diam}(\lambda D)\le 1$. Similarly, by (2.24) and Lemma 2.6, the second integral in the right side of (2.61) satisfies for any $\varepsilon\in(0,(\delta_{\lambda D}(x)\wedge|x-y|)/4)$,

$$\begin{aligned}
&\int_{\lambda D}\int_{\lambda D\cap\{|z-x|\le 2\varepsilon\}}\ell(|x-z|)\frac{2}{\varepsilon}|D_iG_{\lambda D}^{(z)}(x,z)||g_{\lambda D}(z,y)|\,dz\,dy\\
\le&\frac{8}{\varepsilon}C_2C_4\int_{\lambda D}\int_{\lambda D\cap\{|z-x|\le 2\varepsilon\}}\ell(|x-z|)|x-z|^{1-d}\frac{\delta_{\lambda D}(z)}{r_{\lambda D}(x,z)}\frac{\ell(|z-y|)}{|z-y|^d}\frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(z)}\,dz\,dy\\
\le&C_2C_4\frac{8}{\varepsilon}\frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(x)}\int_{\lambda D}\int_{\lambda D\cap\{|z-x|\le 2\varepsilon\}}|x-z|\frac{\ell(|x-z|)}{|x-z|^d}\frac{\ell(|z-y|)}{|z-y|^d}\,dz\,dy\\
\le&16C_2C_4\frac{1}{\delta_{\lambda D}(x)}\int_0^1\omega_d\frac{\ell(r)}{r}\,dr\int_0^{2\varepsilon}\omega_d\frac{\ell(s)}{s}\,ds.
\end{aligned}$$

By (2.33) and Lemma 2.6, the third therm in the right side of (2.61) satisfies for any $\varepsilon\in(0,(\delta_{\lambda D}(x)\wedge|x-y|)/4)$,

$$\begin{aligned}
&\int_{\lambda D}\int_{\lambda D\cap\{|z-x|\le 2\varepsilon\}}\ell(|x-z|)|D_{ij}G_{\lambda D}^{(z)}(x,z)g_{\lambda D}(z,y)|\,dz\,dy\\
\le&\ C_3C_4\int_{\lambda D}\int_{\lambda D\cap\{|z-x|\le 2\varepsilon\}}\frac{\ell(|x-z|)}{|x-z|^d}\frac{\delta_{\lambda D}(z)}{\delta_{\lambda D}(x)}\frac{\ell(|z-y|)}{|z-y|^d}\frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(z)}dzdy\\
\le&\ C_3C_4\frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(x)}\int_{\lambda D}\int_{\lambda D\cap\{|z-x|\le 2\varepsilon\}}\frac{\ell(|x-z|)}{|x-z|^d}\frac{\ell(|z-y|)}{|z-y|^d}\,dz\,dy\\
\le&\ C_3C_4\frac{1}{\delta_{\lambda D}(x)}\int_0^1\omega_d\frac{\ell(r)}{r}\,dr\int_0^{2\varepsilon}\omega_d\frac{\ell(s)}{s}\,ds.
\end{aligned}$$

Hence, by the three inequalities above and (2.61),

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\lambda D}\int_{\lambda D}(\mathcal{L}_x^{(x)}-\mathcal{L}_x^{(z)})(G_{\lambda D}^{(z)}\eta_\varepsilon)(x,z)g_{\lambda D}(z,y)\psi(y)\,dz\,dy\\
=&\int_{\lambda D}\int_{\lambda D}(\mathcal{L}_x^{(x)}-\mathcal{L}_x^{(z)})G_{\lambda D}^{(z)}(x,z)g_{\lambda D}(z,y)\psi(y)\,dz\,dy
\end{aligned}$$

uniformly on any compact set of λD. Therefore,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\lambda D}\left[\int_{\lambda D}\int_{\lambda D}(\mathcal{L}_x^{(x)}-\mathcal{L}_x^{(z)})(G_{\lambda D}^{(z)}\eta_\varepsilon)(x,z)g_{\lambda D}(z,y)\psi(y)\,dz\,dy\right]\phi(x)\,dx\\&=\int_{\lambda D}\left[\int_{\lambda D}\int_{\lambda D}(\mathcal{L}_x^{(x)}-\mathcal{L}_x^{(z)})G_{\lambda D}^{(z)}(x,z)g_{\lambda D}(z,y)\psi(y)\,dz\,dy\right]\phi(x)\,dx.\end{aligned}\tag{2.62}$$

By a similar argument of (2.59), for any $\phi,\psi\in C_c^\infty(\lambda D)$,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\lambda D}\int_{\lambda D}\int_{\lambda D}\mathcal{L}_x^{(z)}(G_{\lambda D}^{(z)}\eta_\varepsilon)(x,z)g_{\lambda D}(z,y)dz\psi(y)\,dy\phi(x)\,dx\\
=\;&\lim_{\varepsilon\to 0}\int_{\lambda D}\int_{\lambda D}\left(\int_{\lambda D}\mathcal{L}_x^{(z)}(G_{\lambda D}^{(z)}\eta_\varepsilon)(x,z)\phi(x)\,dx\right)g_{\lambda D}(z,y)dz\psi(y)\,dy\\
=\;&\lim_{\varepsilon\to 0}\int_{\lambda D}\int_{\lambda D}\left(\int_{\lambda D}\mathcal{L}_x^{(z)}\phi(x)(G_{\lambda D}^{(z)}\eta_\varepsilon)(x,z)dx\right)g_{\lambda D}(z,y)\psi(y)\,dzdy\\
=\;&\int_{\lambda D}\int_{\lambda D}\left(\int_{\lambda D}\mathcal{L}_x^{(z)}\phi(x)G_{\lambda D}^{(z)}(x,z)dx\right)g_{\lambda D}(z,y)\psi(y)\,dzdy\\
=\;&\int_{\lambda D}\int_{\lambda D}\lim_{\varepsilon\to 0}\frac{1}{|B(z,\varepsilon)\cap(\lambda D)|}\left(\int_{\lambda D}\mathcal{L}_x^{(z)}\phi(x)G_{\lambda D}^{(z)}1_{B(z,\varepsilon)}(x)dx\right)g_{\lambda D}(z,y)\psi(y)\,dzdy\\
=\;&\int_{\lambda D}\int_{\lambda D}\lim_{\varepsilon\to 0}\frac{1}{|B(z,\varepsilon)\cap(\lambda D)|}\left(\int_{\lambda D}\phi(x)(-1_{B(z,\varepsilon)})(x)dx\right)g_{\lambda D}(z,y)\psi(y)\,dzdy\\
=\;&-\int_{\lambda D}\left(\int_{\lambda D}\phi(z)\,g_{\lambda D}(z,y)dz\right)\psi(y)dy.\end{aligned}\tag{2.63}$$

The desired property (2.60) follows from (2.62) and (2.63).

Step 3: Recall that

$$\widetilde{G}_{\lambda D}^{(\varepsilon)}(x,y)=(G_{\lambda D}^{(y)}\eta_\varepsilon)(x,y)+\int_{\lambda D}(G_{\lambda D}^{(z)}\eta_\varepsilon)(x,z)g_{\lambda D}(z,y)dz,\quad x\neq y\in\lambda D.$$

By Steps 1 and 2 and the definition of $g_{\lambda D}(x,y)$, for any $\psi,\phi\in C_c^\infty(\lambda D)$,

$$\lim_{\varepsilon\to 0}\int_{\lambda D}\phi(x)\mathcal{L}\tilde{G}_{\lambda D}^{(\varepsilon)}\psi(x)dx\tag{2.64}$$

$$\begin{aligned}=\;&\lim_{\varepsilon\to 0}\int_{\lambda D}\mathcal{L}_x\int_{\lambda D}\tilde{G}_{\lambda D}^{(\varepsilon)}(x,y)\psi(y)\,dy\phi(x)\,dx\\
=\;&\lim_{\varepsilon\to 0}\int_{\lambda D}\mathcal{L}_x\int_{\lambda D}(G_{\lambda D}^{(y)}\eta_\varepsilon)(x,y)\psi(y)\,dy\phi(x)\,dx\\
&+\lim_{\varepsilon\to 0}\int_{\lambda D}\mathcal{L}_x\int_{\lambda D}\int_{\lambda D}(G_{\lambda D}^{(z)}\eta_\varepsilon)(x,z)g_{\lambda D}(z,y)\,dz\psi(y)\,dy\phi(x)\,dx\\
=\;&-\int_{\lambda D}\psi(x)\phi(x)\,dx+\int_{\lambda D}\int_{\lambda D}\left((\mathcal{L}_x^{(x)}-\mathcal{L}_x^{(y)})G_{\lambda D}^{(y)}(x,y)\right)\psi(y)\,dy\phi(x)\,dx\\
&-\int_{\lambda D}g_{\lambda D}(x,y)\psi(y)\,dy\phi(x)\,dx\\
&+\int_{\lambda D}\int_{\lambda D}\left(\int_{\lambda D}(\mathcal{L}_x^{(x)}-\mathcal{L}_x^{(z)})G_{\lambda D}^{(z)}(x,z)g_{\lambda D}(z,y)dz\right)\psi(y)\,dy\phi(x)\,dx\\
=\;&-\int_{\lambda D}\psi(x)\phi(x)\,dx.\end{aligned}\tag{2.65}$$

This establishes the claim (2.54).

(II) Next we claim that there exists $c=c(d,\lambda_0,\ell,R_0,\Lambda_0,\mathrm{diam}(D))$ such that for any $\varepsilon>0$,

$$\int_{\lambda D}|\mathcal{L}_x\tilde{G}_{\lambda D}^{(\varepsilon)}(x,y)|\,dy\le c\frac{1}{\delta_{\lambda D}(x)}.\tag{2.66}$$

Due to the cut-off of singularity along the diagonal by η_ε, we have by Lemma 2.5 that $x \mapsto \widetilde{G}^{(\varepsilon)}_{\lambda D}(x,y)$ is C^∞ in $\lambda D \setminus \{y\}$ for each $y \in \lambda D$ and

$$\mathcal{L}\widetilde{G}^{(\varepsilon)}_{\lambda D}(x,y) = \mathcal{L}_x(G^{(y)}_{\lambda D}\eta_\varepsilon)(x,y) + \int_{\lambda D} \mathcal{L}_x(G^{(z)}_{\lambda D}\eta_\varepsilon)(x,z) g_{\lambda D}(z,y)dz.$$

To prove (2.66), we first prove that there exists a constant $c = c(d, \lambda_0, \ell, R_0, \Lambda_0, \mathrm{diam}(D))$ such that for any $\varepsilon > 0$,

$$\int_{\lambda D\times\lambda D} |\mathcal{L}_x(G^{(z)}_{\lambda D}\eta_\varepsilon)(x,z) g_{\lambda D}(z,y)|\, dz\, dy \le c\frac{1}{\delta_{\lambda D}(x)}. \tag{2.67}$$

Note that

$$\begin{aligned}\mathcal{L}_x(G^{(z)}_{\lambda D}\eta_\varepsilon)(x,z) &= \eta_\varepsilon(x,z)\mathcal{L}_x G^{(z)}_{\lambda D}(x,z) + G^{(z)}_{\lambda D}(x,z)\mathcal{L}_x\eta_\varepsilon(x,z) \\ &\quad + \sum_{i,j=1}^d a_{ij}(x)\frac{\partial G^{(z)}_{\lambda D}(x,z)}{\partial x_i}\frac{\partial \eta_\varepsilon(x,z)}{\partial x_j}.\end{aligned} \tag{2.68}$$

Note that $G^{(z)}_{\lambda D}(\cdot, z)$ is C^2 and harmonic with respect to $\mathcal{L}^{(z)}$ on $\lambda D \setminus \overline{B}(z,\varepsilon)$. Thus $\eta_\varepsilon(x,z)\mathcal{L}^{(z)}_x G^{(z)}_{\lambda D}(x,z) = 0$. Hence, for the integral of the first term in (2.68) multiplying by $g_{\lambda D}(z,y)$, we have by (2.33), (2.37) and Lemma 2.6 that,

$$\begin{aligned}
& \int_{\lambda D\times\lambda D} |\eta_\varepsilon(x,z)\mathcal{L}_x G^{(z)}_{\lambda D}(x,z) g_{\lambda D}(z,y)| dz\, dy \\
\le\ & \int_{\lambda D\times\lambda D} |(\mathcal{L}^{(x)}_x - \mathcal{L}^{(z)}_x) G^{(z)}_{\lambda D}(x,z) g_{\lambda D}(z,y)| dz\, dy \\
+\ & \int_{\lambda D\times\lambda D} |\eta_\varepsilon(x,z)\mathcal{L}^{(z)}_x G^{(z)}_{\lambda D}(x,z) g_{\lambda D}(z,y)|\, dz\, dy \\
\le\ & \sum_{i,j=1}^d \int_{\lambda D\times\lambda D} |a_{ij}(x) - a_{ij}(z)||D_{ij}G^{(z)}_{\lambda D}(\cdot,z)(x) g_{\lambda D}(z,y)| dz\, dy + 0 \\
\le\ & \sum_{i,j=1}^d \int_{\lambda D\times\lambda D} \ell(|x-z|)|D_{ij}G^{(z)}_{\lambda D}(\cdot,z)(x) g_{\lambda D}(z,y)| dz\, dy \\
\le\ & 4C_3C_4 \int_{\lambda D\times\lambda D} \frac{\ell(|x-z|)}{|x-z|^d}\frac{\delta_{\lambda D}(z)}{\delta_{\lambda D}(x)}\frac{\ell(|z-y|)}{|z-y|^d}\frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(z)} dz\, dy \\
\le\ & 2^{d+2}c_0C_3C_4 \int_{\lambda D}\frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(x)}\frac{\ell(|x-y|)}{|x-y|^d}\int_{\lambda D}\left(\frac{\ell(|x-z|)}{|x-z|^d} + \frac{\ell(|z-y|)}{|z-y|^d}\right) dz\, dy \\
\le\ & 2^{d+3}c_0C_3C_4\frac{1}{\delta_{\lambda D}(x)}\left(\int_{0<s<1}\omega_d\frac{\ell(s)}{s}\, ds\right)^2.
\end{aligned}$$

For the integral of the second term in (2.68) multiplying by $g_{\lambda D}(z,y)$, we have by (2.23), (2.37) and Lemma 2.6 that,

$$\begin{aligned}
&\int_{\lambda D}\int_{\lambda D} G^{(z)}_{\lambda D}(x,z)|\mathcal{L}_x\eta_\varepsilon(x,z)g_{\lambda D}(z,y)|\,dz\,dy\\
\le&\sum_{i,j=1}^d|a_{ij}(x)|\int_{\lambda D}\int_{\lambda D} G^{(z)}_{\lambda D}(x,z)|D_{ij}\eta_\varepsilon(\cdot,z)(x)g_{\lambda D}(z,y)|\,dz\,dy\\
\le&4d^2\lambda_0\varepsilon^{-2}\int_{\lambda D}\int_{\lambda D\cap\{|z-x|\le 2\varepsilon\}} G^{(z)}_{\lambda D}(x,z)|g_{\lambda D}(z,y)|\,dz\,dy\\
\le&16d^2\lambda_0C_1C_4\varepsilon^{-2}\int_{\lambda D\cap\{|z-x|\le 2\varepsilon\}}\int_{\lambda D}|x-z|^{2-d}\frac{\delta_{\lambda D}(z)}{r_{\lambda D}(x,z)}\frac{\ell(|z-y|)}{|z-y|^d}\frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(z)}\,dy\,dz\\
\le&16d^2\lambda_0C_1C_4\varepsilon^{-2}\frac{1}{\delta_{\lambda D}(x)}\int_{\lambda D\cap\{|z-x|\le 2\varepsilon\}}|x-z|^{2-d}\int_{\lambda D}\frac{\ell(|z-y|)}{|z-y|^d}\,dy\,dz\\
\le&64d^2\lambda_0C_1C_4\frac{1}{\delta_{\lambda D}(x)}\int_0^1\omega_d\frac{\ell(s)}{s}\,ds,
\end{aligned}$$

where the second to the last inequality is due to $r_{\lambda D}(x,z)\ge\delta_{\lambda D}(x)$. By a similar argument, for the integral of the third term in (2.68) multiplying by $g_{\lambda D}(z,y)$, we have by (2.24), (2.37) and Lemma 2.6 that,

$$\begin{aligned}
&\int_{\lambda D}\int_{\lambda D}\sum_{i,j=1}^d|a_{ij}(x)D_iG^{(z)}_{\lambda D}(\cdot,z)(x)D_j\eta_\varepsilon(\cdot,z)(x)g_{\lambda D}(z,y)|\,dz\,dy\\
\le&\sum_{i,j=1}^d|a_{ij}(x)|\int_{\lambda D}\int_{\lambda D}|D_iG^{(z)}_{\lambda D}(\cdot,z)(x)D_j\eta_\varepsilon(\cdot,z)(x)g_{\lambda D}(z,y)|\,dz\,dy\\
\le&2d^2\lambda_0\varepsilon^{-1}\int_{\lambda D}\int_{\lambda D\cap\{|z-x|\le 2\varepsilon\}}|D_iG^{(z)}_{\lambda D}(\cdot,z)(x)g_{\lambda D}(z,y)|\,dz\,dy\\
\le&8d^2\lambda_0\varepsilon^{-1}\int_{\lambda D\cap\{|z-x|\le 2\varepsilon\}}\int_{\lambda D}|x-z|^{1-d}\frac{\delta_{\lambda D}(z)}{r_{\lambda D}(x,z)}\frac{\ell(|z-y|)}{|z-y|^d}\frac{\delta_{\lambda D}(y)}{\delta_{\lambda D}(z)}\,dy\,dz\\
\le&8d^2\lambda_0\varepsilon^{-1}C_2C_4\frac{1}{\delta_{\lambda D}(x)}\int_{\lambda D\cap\{|z-x|\le 2\varepsilon}|x-z|^{1-d}\int_{\lambda D}\frac{\ell(|z-y|)}{|z-y|^d}\,dy\,dz\\
\le&16d^2\lambda_0C_2C_4\frac{1}{\delta_{\lambda D}(x)}\int_0^1\omega_d\frac{\ell(s)}{s}\,ds.
\end{aligned}$$

Consequently, by the three displays above and (2.68), (2.67) is established. By a very similar but simpler argument, one can also prove that

$$\int_{\lambda D}|\mathcal{L}_x(G^{(y)}_{\lambda D}\eta_\varepsilon)(x,y)|\,dy\le c\frac{1}{\delta_{\lambda D}(x)},\tag{2.69}$$

where $c=c(d,\lambda_0,\ell,R_0,\Lambda_0,\mathrm{diam}(D))$. The proof is omitted here. Thus, by (2.69) and (2.67), (2.66) holds.

(III) Recall that $\widetilde{G}_{\lambda D}(x,y)$ is defined by (2.53). We claim that for any $\psi\in C_c^\infty(\lambda D)$, $\widetilde{G}_{\lambda D}\psi\in W^{2,p}_{loc}(\lambda D)$ for any $p>1$ and

$$\mathcal{L}\widetilde{G}_{\lambda D}\psi=-\psi\quad\text{a.e. on}\quad\lambda D.\tag{2.70}$$

This will then imply that $G_{\lambda D}(x,y)=\tilde{G}_{\lambda D}(x,y)$, establishing (2.52).

Fix a function $\psi\in C_c^\infty(\lambda D)$ and define $\tilde{G}^{(\varepsilon)}_{\lambda D}\psi(x):=\int_{\lambda D}\tilde{G}^{(\varepsilon)}_{\lambda D}(x,y)\psi(y)dy$. Due to the cut-off of singularity along the diagonal by η_ε, it is easy to show that for any $p>1$,

$\widetilde{G}^{(\varepsilon)}_{\lambda D}\psi \in W^{2,p}(\lambda D) \cap C(\lambda D)$. It follows from (2.66) in (II) that for any relatively compact open subset D' of D and for any $p > 1$

$$\sup_{\varepsilon>0} \|\mathcal{L}\widetilde{G}^{(\varepsilon)}_{\lambda D}\psi\|_{L^p(\lambda D')} < +\infty.$$

By Lemma 2.7 and (2.23),

$$\sup_{\varepsilon>0} G^{(\varepsilon)}_{\lambda D}(x,y) \le G^{(y)}_{\lambda D}(x,y) + \int_{\lambda D} G^{(z)}_{\lambda D}(x,z)|g_{\lambda D}(z,y)|dz \le (C_1+C_5)|x-y|^{2-d}, \quad x \neq y \in \lambda D.$$

Thus $\sup_{\varepsilon>0}\sup_{x\in\lambda D} |\widetilde{G}^{(\varepsilon)}_{\lambda D}\psi(x)| < \infty$ and whence $\sup_{\varepsilon>0} \|\widetilde{G}^{(\varepsilon)}_{\lambda D}\psi\|_{L^p(\lambda D)} < \infty$ for any $p > 1$. Consequently, by [22, Theorem 9.11], for any relative compact subdomain D'' of D', there exists a constant $c = c(d, p, \lambda_0, \ell, \lambda D'', \lambda D')$ so that for $p > 1$,

$$\sup_{\varepsilon} \|\widetilde{G}^{(\varepsilon)}_{\lambda D}\psi\|_{W^{2,p}(\lambda D'')} \le c \sup_{\varepsilon} \|\mathcal{L}\widetilde{G}^{(\varepsilon)}_{\lambda D}\psi\|_{L^p(\lambda D')} + c \sup_{\varepsilon} \|\widetilde{G}^{(\varepsilon)}_{\lambda D}\psi\|_{L^p(\lambda D')} < +\infty. \tag{2.71}$$

By (2.71), the weak compactness of bounded sets in $W^{2,p}(\lambda D')$ for any relative compact subdomain $\lambda D'$ of λD and a diagonal selection procedure, there exists a sequence of positive numbers $\{\varepsilon_i; i \ge 1\}$ decreasing to 0 and a function $\Psi \in W^{2,p}(\lambda D)$ such that for any compact $C^{1,1}$ subdomain $\lambda D'$ of λD, $\widetilde{G}^{(\varepsilon_i)}_{\lambda D}\psi$ converges to Ψ weakly in $W^{2,p}(\lambda D')$ as $i \to \infty$. In particular, for any compact subset $\lambda D'$ of λD and any $g \in L^{p/(p-1)}(\lambda D')$,

$$\lim_{i\to\infty} \left(g, D^k \widetilde{G}^{(\varepsilon_i)}_{\lambda D}\psi\right)_{L^2(\lambda D')} = \left(g, D^k\Psi\right)_{L^2(\lambda D')}$$

for all $|k| \le 2$, where $\left(f, g\right)_{L^2(\lambda D')} := \int_{\lambda D'} f(x)g(x)\,dx$. Note that each $|a_{ij}(x)| \le \lambda_0$, $g \cdot a_{ij} \in L^{p/(p-1)}(\lambda D')$ for every $g \in L^{p/(p-1)}(\lambda D')$. It follows that

$$\lim_{i\to\infty} (g, \mathcal{L}\widetilde{G}^{(\varepsilon_i)}_{\lambda D}\psi)_{L^2(\lambda D')} = (g, \mathcal{L}\Psi)_{L^2(\lambda D')}.$$

This together with (2.64) yields that $\mathcal{L}\Psi = -\psi$ a.e. on $\lambda D'$. On the other hand, as $\widetilde{G}^{(\varepsilon)}_{\lambda D}\psi$ converges to $\widetilde{G}_{\lambda D}\psi$ uniformly on $\lambda D'$, $\widetilde{G}^{(\varepsilon)}_{\lambda D}\psi$ converges to $\widetilde{G}_{\lambda D}\psi$ weakly in $L^p(\lambda D')$ as $\varepsilon \to 0$. By the uniqueness of weak limit, $\widetilde{G}_{\lambda D}\psi = \Psi$ a.e. on $\lambda D'$. Since this holds for any relatively compact subdomain $\lambda D'$ of λD, we get the desired conclusion (2.70).

Let $\rho \in C_c^\infty(\mathbb{R}^d)$ be non-negative with $\mathrm{supp}[\rho] \subset B(0,1)$ and $\int_{\mathbb{R}^d} \rho(x)\,dx = 1$. Define $\rho_n(x) := n^d\rho(nx)$. For any $\psi \in C_c^\infty(\lambda D)$, let $f := \widetilde{G}_{\lambda D}\psi$ and $f_n(x) := \rho_n * f(x) := \int_{\lambda D} \rho_n(x-y)f(y)dy$. Clearly, $f_n \in C^\infty(\lambda D)$ and f_n converges uniformly to f on λD. Let $\{D_k; k \ge 1\}$ be an increasing sequence of relatively compact subdomains of D that increases to D. For each fixed $k \ge 1$ and $x \in \lambda D_k$, by Ito's formula, when n is sufficiently large,

$$\mathbb{E}_x[f_n(X_{t\wedge\tau_{\lambda D_k}})] - f_n(x) = \mathbb{E}_x \int_0^{t\wedge\tau_{\lambda D_k}} \mathcal{L}f_n(X_s)\,ds = -\mathbb{E}_x \int_0^{t\wedge\tau_{\lambda D_k}} \rho_n * \psi(X_s)\,ds.$$

Letting $n \to \infty$ and then $t \to \infty$, we get

$$\mathbb{E}_x[f(X_{\tau_{\lambda D_k}})] - f(x) = -\mathbb{E}_x \int_0^{\tau_{\lambda D_k}} \psi(X_s)\,ds \quad \text{for every } x \in \lambda D_k. \tag{2.72}$$

Recall that

$$\widetilde{G}_{\lambda D}(x,y) = G^{(y)}_{\lambda D}(x,y) + \int_{\lambda D} G^{(z)}_{\lambda D}(x,z)g_{\lambda D}(z,y)dz, \quad x \neq y \in \lambda D.$$

By (2.23) and Lemma 2.7, $\widetilde{G}_{\lambda D}(x,y) \leq (C_1 + C_5)|x-y|^{2-d}$ for $x \neq y \in \lambda D$ and for each $y \in \lambda D$, $\widetilde{G}_{\lambda D}(x,y)$ converges to 0 as $x \to \partial(\lambda D)$. Hence, by the dominated convergence theorem,

$$\lim_{x \to \partial(\lambda D)} f(x) = \lim_{x \to \partial(\lambda D)} \widetilde{G}_{\lambda D}\psi(x) = 0.$$

Sending $k \to \infty$ in (2.72), we have by Theorem 2.3 that for each $x \in \lambda D$

$$\widetilde{G}_{\lambda D}\psi(x) = f(x) = \mathbb{E}_x \int_0^{\tau_{\lambda D}} \psi(X_t)\,dt = \int_{\lambda D} G_{\lambda D}(x,y)\psi(y)dy.$$

This shows that for each $x \in \lambda D$, $\widetilde{G}_{\lambda D}(x,y) = G_{\lambda D}(x,y)$ for a.e. $y \in \lambda D$. Since by Theorem 2.3 and Lemma 2.7, both $G_{\lambda D}(x,y)$ and $\widetilde{G}_{\lambda D}(x,y)$ are jointly continuous on $(\lambda D) \times (\lambda D) \setminus \mathrm{diag}$, we conclude that $\widetilde{G}_{\lambda D}(x,y) = G_{\lambda D}(x,y)$ holds for every $(x,y) \in (\lambda D) \times (\lambda D) \setminus \mathrm{diag}$. This establishes (2.52). □

Lemma 2.9. Suppose D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R_0, Λ_0). Let C_1, θ_1 and C_5 be the positive constants in Lemmas 2.5, 2.6 and 2.7. Then for any $\lambda \in (0, \theta_1]$,

$$G_{\lambda D}(x,y) \leq \frac{C_1 + C_5}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_{\lambda D}(y)}{|x-y|}\right) \quad \text{on } (\lambda D) \times (\lambda D) \setminus \mathrm{diag}. \tag{2.73}$$

Moreover, for any $\gamma \geq 1$, there are positive constants $\theta_2 = \theta_2(d, \lambda_0, \ell, \Lambda_0, R_0, \mathrm{diam}(D), \gamma) \in (0, \theta_1]$ and $C_6 = C_6(d, \lambda_0, \Lambda_0, R_0, \mathrm{diam}(D), \gamma) \geq 1$ such that for any $\lambda \in (0, \theta_2]$,

$$C_6^{-1} G^{\Delta}_{\lambda D}(x,y) \leq G_{\lambda D}(x,y) \leq C_6 G^{\Delta}_{\lambda D}(x,y) \quad \text{for } x \neq y \in \lambda D \text{ with } |x-y| \leq \gamma \delta_{\lambda D}(x). \tag{2.74}$$

Proof. By (2.23) and (2.44), (2.73) is established. Let $r_{\lambda D}(x,y) := \delta_{\lambda D}(x) + \delta_{\lambda D}(y) + |x-y|$. Note that

$$\delta_{\lambda D}(x) + |x-y| \leq r_{\lambda D}(x,y) \leq 2(\delta_{\lambda D}(x) + |x-y|) \quad \text{for every } x, y \in \lambda D.$$

For $x, y \in \lambda D$ with $|x-y| \leq \gamma \delta_{\lambda D}(x)$, we have $\delta_{\lambda D}(y) \leq |y-x| + \delta_{\lambda D}(x) \leq (1+\gamma)\delta_{\lambda D}(x)$ and so

$$\delta_{\lambda D}(x) \leq r_{\lambda D}(x,y) \leq 2(1+\gamma)\delta_{\lambda D}(x).$$

In this case, the two-sided estimates (2.23) for $G^{(y)}_{\lambda D}(x,y)$ can be written as

$$G^{(y)}_{\lambda D}(x,y) \asymp \frac{1}{|x-y|^{d-2}} \frac{\delta_{\lambda D}(y)}{r_{\lambda D}(x,y)}, \quad |x-y| \leq \gamma \delta_{\lambda D}(x), \tag{2.75}$$

where the comparison constants depend only on $(d, \lambda_0, \Lambda_0, R_0, \mathrm{diam}(D), \gamma)$. By the second inequality in (2.48),

$$\int_{\lambda D} G^{(z)}_{\lambda D}(x,z)|g_{\lambda D}(z,y)|\,dz \leq \frac{2^{d+3}(2+c_0)C_1C_4}{|x-y|^{d-2}} \frac{\delta_{\lambda D}(y)}{r_{\lambda D}(x,y)} \int_{0<r\leq\lambda \mathrm{diam}(D)} \omega_d \frac{\ell(r)}{r}\,dr$$

for $x \neq y \in \lambda D$. Hence, there is a constant $\theta_2 = \theta_2(d, \lambda_0, \ell, \Lambda_0, R_0, \mathrm{diam}(D), \gamma) \in (0, \theta_1)$ so that for any $\lambda \in (0, \theta_2]$,

$$\int_{\lambda D} G^{(z)}_{\lambda D}(x,z)|g_{\lambda D}(z,y)|\,dz \leq \frac{1}{2} G^{(y)}_{\lambda D}(x,y) \tag{2.76}$$

for any $x \neq y \in \lambda D$ with $|x-y| \leq \gamma \delta_{\lambda D}(x)$. Consequently, this together with (2.52) and (2.22)-(2.23) yields that there exists a constant $C_6 = C_6(d, \lambda_0, \Lambda_0, R_0, \mathrm{diam}(D)) \geq 1$ so that for any $\lambda \in (0, \theta_2]$,

$$G_{\lambda D}(x,y) = G^{(y)}_{\lambda D}(x,y) + \int_{\lambda D} G^{(z)}_{\lambda D}(x,z) g_{\lambda D}(z,y)dy \leq \frac{3}{2} G^{(y)}_{\lambda D}(x,y) \leq C_6 G^{\Delta}_{\lambda D}(x,y) \tag{2.77}$$

and

$$G_{\lambda D}(x,y) \geq G^{(y)}_{\lambda D}(x,y) - \int_{\lambda D} G^{(z)}_{\lambda D}(x,z)|g_{\lambda D}(z,y)|dy \geq \frac{1}{2} G^{(y)}_{\lambda D}(x,y) \geq C_6^{-1} G^{\Delta}_{\lambda D}(x,y)$$

for any $x \neq y$ in λD with $|x-y| \leq \gamma \delta_{\lambda D}(x)$. This completes the proof of the theorem. □

Note that it follows from (1.2) (by taking $\xi_k = \delta_{ik}$ and $\xi_k = \delta_{ik} + \delta_{jk}$ for $1 \leq k \leq d$ there respectively) that

$$\lambda_0^{-1} \leq a_{ii}(x) \leq \lambda_0 \quad \text{and} \quad |a_{ij}(x)| = |a_{ij}(x) + a_{ji}(x)|/2 \leq \lambda_0 \quad \text{for every } 1 \leq i,j \leq d. \tag{2.78}$$

Recall that for each $Q \in \partial D$, there exist a $C^{1,1}$ function $\phi = \phi_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = \nabla\phi(0) = 0, \|\nabla\phi\|_\infty \leq \Lambda_0, |\nabla\phi(x) - \nabla\phi(y)| \leq \Lambda_0|x-y|$, and an orthonormal coordinate system CS_Q:

$$y = (\tilde{y}, y_d) := (y_1, \cdots, y_{d-1}, y_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$$

with its origin at Q such that

$$B(Q,R_0) \cap D = \{y = (\tilde{y}, y_d) \in B(0,R_0) \text{ in } CS_Q : y_d > \phi(\tilde{y})\}.$$

For the simplicity of notation, let $Q = 0 \in \partial D$. For $r > 0$, define $\phi^{(r)}_Q(\tilde{y}) := r\phi_Q(r^{-1}\tilde{y})$. Then $\phi^{(r)}$ is the $C^{1,1}$ function representing rD in the coordinate system centered at $Q \in \partial(rD)$ so that

$$(rD) \cap B(Q, rR_0) = \left\{(\tilde{y}, y_d) \in (rD) \cap B(Q, rR_0) : y_d > \phi^{(r)}_Q(\tilde{y})\right\}.$$

Let $\rho^{(r)}_Q(x) := x_d - \phi^{(r)}_Q(\tilde{x})$ for $x \in (rD) \cap B(Q, rR_0)$. Define for $r_1, r_2 > 0$,

$$D^{(r)}_Q(r_1, r_2) := \left\{y \in rD \cap B(Q, rR_0) :\ 0 < \rho^{(r)}_Q(y) < r_1, |\tilde{y}| < r_2\right\} \tag{2.79}$$

and

$$U^{(r)}_Q(r_1, r_2) := \left\{y \in rD \cap B(Q, rR_0) : \rho^{(r)}_Q(y) = r_1, |\tilde{y}| < r_2\right\}. \tag{2.80}$$

The next lemma holds for any non-divergence form operator $\mathcal{L}$ of (1.1) with bounded, continuous and uniformly elliptic diffusion coefficients.

Lemma 2.10. Suppose D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R_0, Λ_0). Let $r_0 := \frac{R_0}{4\sqrt{1+\Lambda_0^2}}$. There are constants $\delta_0 = \delta_0(d, \lambda_0, R_0, \Lambda_0) \in (0, r_0)$ and $C_k = C_k(d, \lambda_0, R_0, \Lambda_0)$, $k = 7, 8$, such that for any $r \in (0,1]$, $\lambda_1 > 1, Q \in \partial(rD)$, and $x \in D^{(r)}_Q(r\delta_0/\lambda_1, rr_0/\lambda_1)$ with $\tilde{x} = 0$,

$$\mathbb{P}_x\left(X_{\tau_{D^{(r)}_Q(r\delta_0/\lambda_1, rr_0/\lambda_1)}} \in rD\right) \leq C_7 \frac{\lambda_1}{r} \delta_{rD}(x),$$

$$\mathbb{P}_x\left(X_{\tau_{D^{(r)}_Q(r\delta_0/\lambda_1, rr_0/\lambda_1)}} \in U^{(r)}_Q(r\delta_0/\lambda_1, rr_0/\lambda_1)\right) \geq C_8 \frac{\lambda_1}{r} \delta_{rD}(x).$$

Proof. The proof adopts some ideas from the proof of [10, Lemma 3.4], which in turn are motivated by that of [7, Theorems 5.10 and 6.4]. Without loss of generality, we assume $Q = 0$. Let $\phi = \phi_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ be the $C^{1,1}$ function representing D with $\phi(\tilde{0}) = 0$, $\|\nabla\phi\|_\infty \leq \Lambda_0$ and $\|\nabla\phi(\tilde{y}) - \nabla\phi(\tilde{z})| \leq \Lambda_0|\tilde{y} - \tilde{z}|$ and

$$D \cap B(Q, R_0) = \{(\tilde{y}, y_d) \in D \cap B(Q, R_0) : y_d > \phi(\tilde{y}) \quad \text{in} \quad CS_Q\}.$$

Let $\lambda_1 > 1$. Then $\lambda_1 D$ is a $C^{1,1}$ domain in $\mathbb{R}^d$ with the same $C^{1,1}$-characteristics (Λ_0, R_0). Let $\phi_{\lambda_1}(\tilde{y}) := \lambda_1 \phi_Q(\lambda_1^{-1}\tilde{y})$. Then ϕ_{λ_1} is the $C^{1,1}$ function representing $\lambda_1 D$ centered at Q so that

$$(\lambda_1 D) \cap B(Q, R_0) = \{(\tilde{y}, y_d) \in (\lambda_1 D) \cap B(Q, R_0) : y_d > \phi_{\lambda_1}(\tilde{y})\}.$$

Let $r \in (0, 1]$ and let $\lambda_2 := \lambda_1/r$ and $X_t^{(\lambda_2)} := \lambda_2 X_{t/\lambda_2^2}$. It is easy to check that the infinitesimal generator of $X^{(\lambda_2)}$ is

$$\mathcal{L}^{(\lambda_2)} := \sum_{i,j=1}^{d} a_{ij}(x/\lambda_2) \frac{\partial^2}{\partial x_i \partial x_j}.$$

Note that $x \mapsto \{a_{ij}(x/\lambda_2); 1 \le i, j \le d\}$ has the uniform ellipticity constant λ_0.

Let $p = 3/2$. Define

$$\begin{aligned}
\rho_{\lambda_1}(y) &:= y_d - \phi_{\lambda_1}(\tilde{y}) \\
h_{\lambda_1}(y) &:= \rho_{\lambda_1}(y) \mathbb{1}_{\lambda_1 D \cap B(0,4r_0)}(y) \\
h_{\lambda_1,p}(y) &:= \rho_{\lambda_1}^p(y) \mathbb{1}_{\lambda_1 D \cap B(0,4r_0)}(y) \\
D(\lambda_1, r_1, r_2) &:= \{y = (\tilde{y}, y_d) \in \lambda_1 D : \ 0 < \rho_{\lambda_1}(y) < r_1, |\tilde{y}| < r_2\} \\
U(\lambda_1, r_1, r_2) &:= \{y = (\tilde{y}, y_d) \in \lambda_1 D : \rho_{\lambda_1}(y) = r_1, |\tilde{y}| < r_2\}.
\end{aligned}$$

It is easy to see that $D(\lambda_1, r_1, r_2)$ is contained in $D \cap B(Q, R_0)$ for every $r_1, r_2 \le r_0$. Note that for $y \in \lambda_1 D \cap B(0, 4r_0)$,

$$\mathcal{L}^{(\lambda_2)} h_{\lambda_1}(y) = -\sum_{i,j=1}^{d-1} a_{ij}(y/\lambda_2) \frac{\partial^2}{\partial y_i \partial y_j} \phi_{\lambda_1}(\tilde{y}) - 2\sum_{i=1}^{d-1} a_{id}(y/\lambda_2) \frac{\partial}{\partial y_i} \phi_{\lambda_1}(\tilde{y}).$$

Hence, by (2.78),

$$|\mathcal{L}^{(\lambda_2)} h_{\lambda_1}(y)| \le (d-1)(d+1)\lambda_0 \Lambda_0, \quad y \in (\lambda_1 D) \cap B(Q, 4r_0) \tag{2.81}$$

and

$$\begin{aligned}
& \mathcal{L}^{(\lambda_2)} h_{\lambda_1,p}(y) \\
= \ & p(p-1)\rho_{\lambda_1}(y)^{p-2} \sum_{i,j=1}^{d} a_{ij}(y/\lambda_2) \partial_{y_i} \rho_{\lambda_1}(y) \partial_{y_j} \rho_{\lambda_1}(y) \\
& - \sum_{i,j=1}^{d} a_{ij}(y/\lambda_2) p \rho_{\lambda_1}(y)^{p-1} \partial^2_{y_i y_j} \rho_{\lambda_1}(y) \\
\ge \ & p(p-1)\lambda_0^{-1} \rho_{\lambda_1}(y)^{p-2} \left(|\nabla_{\tilde{y}} \phi_{\lambda_1}(\tilde{y})|^2 + 1\right) \\
& - p\lambda_0 \rho_{\lambda_1}(y)^{p-1} \left(\sum_{i=1}^{d-1} |\partial^2_{y_i^2} \phi_{\lambda_1}(\tilde{y})| + 2 \sum_{1 \le i < j \le d-1} |\partial^2_{y_i y_j} \phi_{\lambda_1}(\tilde{y})| \right) \\
\ge \ & \lambda_0^{-1} p(p-1)\rho_{\lambda_1}(y)^{p-2} - (d-1)^2 p \lambda_0 \Lambda_0 \rho_{\lambda_1}(y)^{p-1}, \quad \text{for} \quad y \in \lambda_1 D \cap B(0, 4r_0).
\end{aligned}$$

Since $p \in (1, 2)$, there is some $\delta_1 \in (0, r_0)$ depending only on $(\lambda_0, \Lambda_0, p)$ so that

$$\mathcal{L}^{(\lambda_2)} h_{\lambda_1,p}(y) > c_1 \rho_{\lambda_1}(y)^{p-2} > 0 \quad \text{for every } y \in D(\lambda_1, \delta_1, r_0). \tag{2.82}$$

Let ψ be a smooth positive function with bounded first and second order partial derivatives such that $\psi(y) = 2^{p+1}|\tilde{y}|^2/r_0^2$ for $|y| < r_0/4$ and $2^{p+1} < \psi(y) < 2^{p+2}$ for $|y| \ge r_0/2$. Define

$$u_{1,\lambda_1}(y) := h_{\lambda_1}(y) + h_{\lambda_1,p}(y) \quad \text{and} \quad u_{2,\lambda_1}(y) := h_{\lambda_1}(y) + \psi(y) - h_{\lambda_1,p}(y).$$

Thus, in view of (2.81) and (2.82), there is some $\delta_0 \in (0, \delta_1)$ depending only on (λ_0, Λ_0) so that

$$\mathcal{L}^{(\lambda_2)} u_{1,\lambda_1}(y) > 0 \quad \text{and} \quad \mathcal{L}^{(\lambda_2)} u_{2,\lambda_1}(y) < 0 \qquad \text{for every } y \in D(\lambda_1, \delta_0, r_0). \tag{2.83}$$

For any open set $U \subset \mathbb{R}^d$, denote by $\tau_U^{(\lambda_2)}$ the first exit time from U by the process $X^{(\lambda_2)}$. It follows from (2.83) that $t \mapsto u_{1,\lambda_1}(X^{(\lambda_2)}_{t\wedge\tau^{(\lambda_2)}_{D(\lambda_1,\delta_0,r_0)}})$ is a bounded submartingale and $t \mapsto u_{2,\lambda_1}(X^{(\lambda_2)}_{t\wedge\tau^{(\lambda_2)}_{D(\lambda_1,\delta_0,r_0)}})$ is a bounded supermartingale. Then for $x \in D(\lambda_1, \delta_0, r_0)$,

$$\rho_{\lambda_1}(x) \le u_{1,\lambda_1}(x) \le \mathbb{E}_x u_{1,\lambda_1}\big(X^{(\lambda_2)}_{\tau^{(\lambda_2)}_{D(\lambda_1,\delta_0,r_0)}}\big) \le 2\mathbb{P}_x\Big(X^{(\lambda_2)}_{\tau^{(\lambda_2)}_{D(\lambda_1,\delta_0,r_0)}} \in \lambda_1 D\Big).$$

Note that $\psi \ge 2^{p+1}$ on $|\tilde{y}| > r_0$, it follows that for $x \in D(\lambda_1, \delta_0, r_0)$,

$$\rho_{\lambda_1}(x) \ge u_{2,\lambda_1}(x) \ge \mathbb{E}_x u_{2,\lambda_1}\big(X^{(\lambda_2)}_{\tau^{(\lambda_2)}_{D(\lambda_1,\delta_0,r_0)}}\big) \ge (2^{p+1}-1)\mathbb{P}_x\Big(X^{(\lambda_2)}_{\tau^{(\lambda_2)}_{D(\lambda_1,\delta_0,r_0)}} \in \lambda_1 D \setminus U(\lambda_1, \delta_0, r_0)\Big).$$

Hence, for $x \in D(\lambda_1, \delta_0, r_0)$,

$$\mathbb{P}_x\Big(X^{(\lambda_2)}_{\tau^{(\lambda_2)}_{D(\lambda_1,\delta_0,r_0)}} \in U(\lambda_1, \delta_0, r_0)\Big) \ge \big(2^{-1} - (2^{p+1}-1)^{-1}\big)\, \rho_{\lambda_1}(x). \tag{2.84}$$

Recall that $0 < h_{\lambda_1,p} \le 1$ and $\psi(y) \ge 2^{p+1}$ if $|y| > r_0/2$, then we have

$$u_{2,\lambda_1}(y) \ge \psi(y) - h_{\lambda_1,p}(y) \ge 2^p \ge 1, \quad |y| > r_0/2.$$

Furthermore, for $y \in B(0, 4r_0)$ with $\delta_0 \le \rho_{\lambda_1}(y) \le 4r_0$,

$$u_{2,\lambda_1}(y) \ge h_{\lambda_1}(y) - h_{\lambda_1,p}(y) \ge \rho_{\lambda_1}(y) - \rho_{\lambda_1,p}(y) \ge c_2,$$

where $c_2 \in (0,1)$ depends on δ_0 and r_0. Hence, for $x \in D(\lambda_1, \delta_0, r_0)$,

$$\rho_{\lambda_1}(x) \ge u_{2,\lambda_1}(x) \ge \mathbb{E}_x u_{2,\lambda_1}\big(X^{(\lambda_2)}_{\tau^{(\lambda_2)}_{D(\lambda_1,\delta_0,r_0)}}\big) \ge c_2\mathbb{P}_x\Big(X^{(\lambda_2)}_{\tau^{(\lambda_2)}_{D(\lambda_1,\delta_0,r_0)}} \in \lambda_1 D\Big).$$

That is, for $x \in D(\lambda_1, \delta_0, r_0)$,

$$\mathbb{P}_x\Big(X^{(\lambda_2)}_{\tau^{(\lambda_2)}_{D(\lambda_1,\delta_0,r_0)}} \in \lambda_1 D\Big) \le c_2^{-1}\rho_{\lambda_1}(x). \tag{2.85}$$

Hence, by (2.84) and (2.85), for $x \in D_Q^{(r)}(r\delta_0/\lambda_1, rr_0/\lambda_1)$ with $\tilde{x} = 0$,

$$\begin{aligned}
&\mathbb{P}_x\Big(X_{\tau_{D_Q^{(r)}(r\delta_0/\lambda_1, rr_0/\lambda_1)}} \in rD\Big)\\
&= \mathbb{P}_{\lambda_1 x/r}\Big(X^{(\lambda_2)}_{\tau^{(\lambda_2)}_{D(\lambda_1,\delta_0,r_0)}} \in \lambda_1 D\Big)\\
&\le c_2^{-1}\rho_{\lambda_1}(\lambda_1 x/r) \le c_2^{-1}\frac{\lambda_1}{r}\delta_{rD}(x),
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}_x\left(X_{\tau_{D_Q^{(r)}(r\delta_0/\lambda_1, rr_0/\lambda_1)}} \in U_Q^{(r)}(r\delta_0/\lambda_1, rr_0/\lambda_1)\right)\\
&= \mathbb{P}_{\lambda_1 x/r}\Big(X^{(\lambda_2)}_{\tau^{(\lambda_2)}_{D(\lambda_1,\delta_0,r_0)}} \in U(\lambda_1, \delta_0, r_0)\Big)\\
&\ge \big(2^{-1} - (2^{p+1}-1)^{-1}\big)\, \rho_{\lambda_1}(\lambda_1 x/r) \ge \big(2^{-1} - (2^{p+1}-1)^{-1}\big)\, \frac{\lambda_1}{r}\delta_{rD}(x).
\end{aligned}$$

This completes the proof of the Lemma. □

Lemma 2.11. Suppose that D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R_0, Λ_0), there exists a constant $C_9 = C_9(d, \lambda_0, \ell, R_0, \Lambda_0, \mathrm{diam}(D)) > 1$ such that

$$G_D(x,y) \le \frac{C_9}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right), \quad x \neq y \in D. \tag{2.86}$$

Proof. Recall that δ_0 is the constant in Lemma 2.10. If $\delta_D(x) \ge \delta_0(|x-y| \wedge 1)$, then $\frac{\delta_D(x)}{|x-y|} \ge \delta_0(1 \wedge (\mathrm{diam}(D))^{-1})$, hence the conclusion holds by (2.4). Hence, it suffices to prove the case when $\delta_D(x) < \delta_0(|x-y| \wedge 1)$.

Let Q_x be the point on ∂D such that $|x - Q_x| = \delta_D(x)$. Let ϕ_{Q_x} be the local boundary function in the coordinate system CS_{Q_x}. Let $\rho_{Q_x}(y) := y_d - \phi_{Q_x}(\tilde{y})$ and define for $r_1, r_2 > 0$,

$$D_{Q_x}(r_1, r_2) := \{y = (\tilde{y}, y_d) : 0 < \rho_{Q_x}(y) < r_1 \text{ and } |\tilde{y}| < r_2 \text{ in } CS_{Q_x}\}. \tag{2.87}$$

Let $a = |x-y| \wedge 1$. By (2.4) and Lemma 2.10 with $r = 1$, there exists a constant $c_1 = c_1(d, \lambda_0, \ell, R_0, \Lambda_0, \mathrm{diam}(D)) > 1$ such that for $\delta_D(x) < \delta_0(|x-y| \wedge 1)$,

$$\begin{aligned} G_D(x,y) &= \mathbb{E}_x \left[G_D(X_{\tau_{D_{Q_x}(a\delta_0, ar_0)}}, y); X_{\tau_{D_{Q_x}(ar_0, a\delta_0)}} \in D \right] \\ &\le c_1 |x-y|^{2-d} \mathbb{P}_x \left(X_{\tau_{D_{Q_x}(a\delta_0, ar_0)}} \in D \right) \\ &\le c_1 C_7 |x-y|^{2-d} \frac{\delta_D(x)}{1 \wedge |x-y|} \\ &\le c_1 C_7 (1 \vee \mathrm{diam}(D)) |x-y|^{2-d} \frac{\delta_D(x)}{|x-y|} \end{aligned}$$

where the first inequality is due to $r_0 \in (0, \frac{1}{4})$ and so $|u-y| \ge \frac{1}{2}|x-y|$ for $u \in D_{Q_x}(a\delta_0, ar_0)$. Note that $\frac{\delta_D(x)}{|x-y|} < \delta_0(1 \wedge |x-y|^{-1}) < \delta_0 < 1$ for $\delta_D(x) < \delta_0(|x-y| \wedge 1)$. The proof is complete. □

Remark 2.1.

(i) Note that the proof of Lemma 2.10 does not use the Dini continuous assumption on $\{a_{ij}(x); 1 \le i,j \le d\}$ of $\mathcal{L}$. So Lemma 2.10 holds for any non-divergence form operator $\mathcal{L}$ with continuous diffusion coefficients that satisfy (1.2).

(ii) In view of the above remark, we see from the proof of Lemma 2.11 that the estimate (2.86) holds with constant $C_9 = \tilde{c}_0 \widetilde{C}_9(d, \lambda_0, R_0, \Lambda_0)$ as long as we have $G_D(x,y) \le \tilde{c}_0 |x-y|^{2-d}$ for $x, y \in D$. Thus by Theorem 2.3(iii), Lemma 2.11 holds with $C_9 = C_9(d, \lambda_0, \ell, R_0, \Lambda_0, \mathrm{diam}(D))$ for any operator $\mathcal{L}$ with coefficients $\{a_{ij}(x); 1 \le i,j \le d\}$ satisfying the Dini mean oscillation condition as well as (1.2).

(iii) One can also derive the upper bound estimate (2.86) from the gradient estimate on $G_D(x,y)$ obtained in [25] for non-divergence operator $\mathcal{L}$ with diffusion coefficients that are Dini mean oscillation condition and satisfy (1.2) at the expense that the constant $C_9 = C_9(d, \lambda_0, \ell, D)$ depends on the bounded $C^{1,1}$ domain D rather than through its $C^{1,1}$-characteristics (R_0, Λ_0) and its diameter $\mathrm{diam}(D)$. Here are the details. Let $x, y \in D$ with $x \neq y$. Let z_x be a boundary point on ∂D so that $|x - z_x| = \delta_D(x)$. If $|y-x| > 2\delta_D(x)$, since $x \mapsto G_D(x,y)$ vanishes continuously on ∂D by Theorem 2.3(i), it follows from the mean value theorem, the first derivative estimate in (2.8) and (2.13) that

$$\begin{aligned} G_D(x,y) &\le \delta_D(x) \sup_{\theta \in (0,1)} |\nabla_x G_D(x + \theta(x - z_x), y)| \\ &\le c_1 \delta_D(x) |x-y|^{1-d} \\ &\le \frac{2c_1}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right) \end{aligned}$$

for some $c_1 = c_1(d, \lambda_0, \ell, D) > 0$, where the second inequality is due to that

$$|x + \theta(x - z_x) - y| > |x - y| - \delta_D(x) > |x - y|/2.$$

If $|y - x| \le 2\delta_D(x)$, then by (2.8),

$$G_D(x, y) \le c_2|x - y|^{2-d} \le \frac{2c_2}{|x - y|^{d-2}} \left(1 \wedge \frac{\delta_D(x)}{|x - y|}\right)$$

for some $c_2 = c_2(d, \lambda_0, \ell, D) > 0$. Thus there is a constant $c = c(d, \lambda_0, \ell, D) > 0$ so that

$$G_D(x, y) \le \frac{c}{|x - y|^{d-2}} \left(1 \wedge \frac{\delta_D(x)}{|x - y|}\right) \quad \text{for } x, y \in D.$$

Theorem 2.12. Suppose D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R_0, Λ_0). Let θ_2 be the positive constant from Lemma 2.9, which depends only on $d, \lambda_0, \ell, \Lambda_0, R_0$ and $\mathrm{diam}(D)$. There exists a positive constant $C_{10} = C_{10}(d, \lambda_0, \ell, \Lambda_0, R_0, \mathrm{diam}(D)) \ge 1$ such that for any $r \in (0, \theta_2]$,

$$C_{10}^{-1} G^{\Delta}_{rD}(x, y) \le G_{rD}(x, y) \le C_{10} G^{\Delta}_{rD}(x, y) \quad \text{for } x \ne y \in rD. \tag{2.88}$$

Proof. Let $r_0 := \frac{R_0}{4\sqrt{1+\Lambda_0^2}}$. Let δ_0 be the constant in Lemma 2.10. In view of (2.74) by taking $\gamma = 4/\delta_0$, it suffices to prove (2.88) for $\delta_{rD}(x) < \delta_0|x - y|/4$.

Fix $x, y \in rD$ with $\delta_{rD}(x) < \delta_0|x - y|/4$. Let Q_x be the point on $\partial(rD)$ such that $|x - Q_x| = \delta_{rD}(x)$. Recall that $D^{(r)}_{Q_x}(r_1, r_2)$ and $U^{(r)}_{Q_x}(r_1, r_2)$ are defined in (2.79) and (2.80). For the simplicity of notation, we denote

$$D^{(r)}_{Q_x}(\delta_0(|x - y| \wedge r)/4, r_0(|x - y| \wedge r)/4) \quad \text{and} \quad U^{(r)}_{Q_x}(\delta_0(|x - y| \wedge r)/4, r_0(|x - y| \wedge r)/4)$$

by $D^{(r)}(1, 1)$ and $U^{(r)}(1, 1)$, respectively. Note that for every $z \in \partial D^{(r)}(1, 1)$,

$$|x-y|/4 \le |x-y|-|x-Q_x|-|Q_x-z| \le |z-y| \le |z-Q_x|+|Q_x-x|+|x-y| \le 3|x-y|/2. \tag{2.89}$$

So by Theorem 2.3(ii), (2.73) and Lemma 2.10, for any $r \in (0, \theta_1]$,

$$\begin{aligned}
G_{rD}(x, y) &= \mathbb{E}_x \left[G_{rD}(X_{\tau_{D^{(r)}(1,1)}}, y); X_{\tau_{D^{(r)}(1,1)}} \in rD \right] \\
&\le 4^{d-1}(C_1 + C_5)|x - y|^{2-d} \left(1 \wedge \frac{\delta_{rD}(y)}{|x - y|}\right) \mathbb{P}_x(X_{\tau_{D^{(r)}(1,1)}} \in rD) \\
&\le 4^{d-1}(C_1 + C_5)C_7|x - y|^{2-d} \left(1 \wedge \frac{\delta_{rD}(y)}{|x - y|}\right) \left(1 \wedge \frac{\delta_{rD}(x)}{r \wedge |x - y|}\right) \\
&\le 4^{d-1}(C_1 + C_5)C_7 \left(1 \vee \mathrm{diam}(D)^{-1}\right) |x - y|^{2-d} \left(1 \wedge \frac{\delta_{rD}(y)}{|x - y|}\right) \left(1 \wedge \frac{\delta_{rD}(x)}{|x - y|}\right).
\end{aligned}$$

On the other hand, since $|x - y| \le r\mathrm{diam}(D)$, one has for every $z \in U^{(r)}(1, 1)$,

$$\delta_{rD}(z) \ge \frac{\delta_0(|x - y| \wedge r)}{4} \ge \frac{\delta_0|x - y|}{4(\mathrm{diam}(D) \vee 1)} \ge \frac{\delta_0|z - y|}{6(\mathrm{diam}(D) \vee 1)},$$

where the last inequality is due to (2.89). Hence by (2.22) and (2.74) of Lemma 2.9 for $\gamma = 6(\mathrm{diam}(D) \vee 1)/\delta_0$, there exists a positive constant $c = c(d, \lambda_0, \ell, \Lambda_0, R_0, \mathrm{diam}(D))$

such that for any $r \in (0, \theta_2]$,

$$\begin{aligned} G_{rD}(x,y) &= \mathbb{E}_x \left[G_{rD}(X_{\tau_{D^{(r)}(1,1)}}, y); X_{\tau_{D^{(r)}(1,1)}} \in rD \right] \\ &\geq \mathbb{E}_x \left[G_{rD}(X_{\tau_{D^{(r)}(1,1)}}, y); X_{\tau_{D^{(r)}(1,1)}} \in U^{(r)}(1,1) \right] \\ &\geq c|x-y|^{2-d} \left(1 \wedge \frac{\delta_{rD}(y)}{|x-y|} \right) \mathbb{P}_x \left(X_{\tau_{D^{(r)}(1,1)}} \in U^{(r)}(1,1) \right) \\ &\geq cC_8 |x-y|^{2-d} \left(1 \wedge \frac{\delta_{rD}(y)}{|x-y|} \right) \frac{\delta_{rD}(x)}{r \wedge |x-y|} \\ &\geq cC_8 |x-y|^{2-d} \left(1 \wedge \frac{\delta_{rD}(y)}{|x-y|} \right) \left(1 \wedge \frac{\delta_{rD}(x)}{|x-y|} \right), \end{aligned}$$

where the second to the last inequality is due to Lemma 2.10. By comparing with (2.22) of the estimates of $G^{\Delta}_{rD}(x,y)$, this establishes the Lemma. $\square$

Using the two-sided Green function estimates in Theorem 2.12, we can give an alternative proof of Krylov-Safonov's Harnack inequality for non-negative X-harmonic functions. We point out that the Harnack inequality established in [29] is more general, holding for any uniformly elliptic non-divergence form elliptic operators with measurable coefficients; see also [3, Theorem V.7.6].

Definition 2.13. Suppose U is an open subset of $\mathbb{R}^d$. A Borel function u defined on U is said to be X-harmonic in U if for every bounded open set B with $\overline{B} \subset U$,

$$\mathbb{E}_x |u(X_{\tau_B})| < \infty \quad \text{and} \quad u(x) = \mathbb{E}_x u(X_{\tau_B}) \quad \text{for every } x \in B.$$

Remark 2.2. Note that if h is C^2 and $\mathcal{L}u = 0$ in U, then by using Ito's formula, h is X-harmonic in U. In fact using molifier, we can further show in a similar way as that in Lemma 2.2 that if $h \in W^{2,1}_{loc}(U) \cap C(U)$ and $\mathcal{L}u = 0$ a.e. in U, then h is X-harmonic in U. So the definition of X-harmonic function given above is consistent with the notion of $\mathcal{L}$-harmonic in analysis but Definition 2.13 does not require a priori the existence of second order (distributional) derivative of h. In the rest of this paper, $\mathcal{L}$-harmonicity will be understood in the sense of Definition 2.13; that is, we say h is $\mathcal{L}$-harmonic in an open set $U \subset \mathbb{R}^d$ if it is X-harmonic in U. $\square$

Theorem 2.14 (Scale invariant Harnack inequality)**.** There exist positive constants $c = c(d, \lambda_0, \ell)$ and $\varepsilon_0 = \varepsilon_0(d, \lambda_0, \ell)$ such that for any $x_0 \in \mathbb{R}^d, r \in (0, \varepsilon_0)$ and any non-negative X-harmonic function h in $B(x_0, r)$,

$$h(x_1) \leq ch(x_2) \quad \text{for any } x_1, x_2 \in B(x_0, r/2).$$

Proof. Fix $x_0 \in \mathbb{R}^d$. For the simplicity of notation, we denote $B(x_0, r)$ by B_r. By Theorem 2.12 with B_1 and r in place of D and λ, there exist positive constants $\varepsilon_0 = \varepsilon_0(d, \lambda_0, \ell)$ and $c_1 = c_1(d, \lambda_0, \ell)$ such that for any $r \in (0, \varepsilon_0)$,

$$c_1^{-1} G^{\Delta}_{B_r}(x,y) \leq G_{B_r}(x,y) \leq c_1 G^{\Delta}_{B_r}(x,y), \quad x, y \in B_r. \tag{2.90}$$

Let h be a non-negative harmonic function in B_r with $r \in (0, \varepsilon_0)$. Let $T_{B_{2r/3}} := \inf\{t > 0 : X^{B_r}_t \in B_{2r/3}\}$. Define $h_1(x) := \mathbb{E}_x h(X^{B_r}_{T_{B_{2r/3}}})$. Then $h(x) = h_1(x)$ for $x \in B_{2r/3}$. By Corollary 1 to Theorem 2 in [13], there exists a Radon measure ν on $\overline{B}_{2r/3}$ such that

$$h_1(x) = \int_{\overline{B}_{2r/3}} G_{B_r}(x,z)\, \nu(dz), \quad x \in B_r.$$

Since $h_1 = h$ is harmonic in $B_{2r/3}$, then ν is supported in $\partial B_{2r/3}$. Hence,

$$h(x) = h_1(x) = \int_{\partial B_{2r/3}} G_{B_r}(x,z)\,\nu(dz), \quad x \in B_{2r/3}. \tag{2.91}$$

By (2.90) and (2.22), there exists a positive constant $c_2 = c_2(d, \lambda_0, \ell) > 1$ such that for any $r \in (0, \varepsilon_0)$,

$$c_2^{-1} r^{2-d} \le G_{B_r}(x,z) \le c_2 r^{2-d} \quad \text{for every } x \in B_{r/2} \text{ and } z \in \partial B_{2r/3}.$$

This together (2.91) yields the desired comparability result. □

3 Two-sided estimates of Green function on a bounded $C^{1,1}$ domain D

Throughout this section, D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with $C^{1,1}$-characteristics (R_0, Λ_0). In this section, we derive two-sided sharp estimates on the Green function $G_D(x,y)$ of $\mathcal{L}$ in D. For this, we first construct a dual process of X^D with respect to an excessive measure of X^D and use it to establish an integral representation formula of Green function $G_D(x,y)$ in Theorem 3.3, which will play a key role in deriving the explicit decay rate of $G_D(x,y)$ in y.

Let B be a ball with radius $\mathrm{diam}(D)$ centered at x_0 so that $D \subset B(x_0, 2\mathrm{diam}(D)/3)$. Define

$$h_B(x) := \int_B G_B(y,x)\,dy \quad \text{and} \quad \xi_B(dx) := h_B(x)\,dx.$$

It follows from Theorem 2.3 that h_B is bounded strictly positive and continuous on B. Suppose, in addition, that $\{a_{ij}(x); 1 \le i,j \le d\}$ are C^1 on $\mathbb{R}^d$. Then according to [28, Theorem 3.1], there exists a transient continuous Hunt process $\widehat{X}^B = \{\widehat{X}^B_t, t \ge 0; \widehat{\mathbb{P}}_x, x \in B\}$ in B such that $\widehat{X}^B$ is a strong dual of X^B with respect to the measure $\xi_B(dx)$ in the sense that

$$\int_B f(x) P^B_t g(x) \xi_B(dx) = \int_B g(x) \widehat{P}^B_t f(x) \xi_B(dx) \quad \text{for all} \quad f, g \in L^2(B; \xi_B),$$

where P^B_t and $\widehat{P}^B_t$ are the transition semigroups of X^B and $\widehat{X}^B$, respectively. The dual process $\widehat{X}^B$ has joint continuous transition density function $\widehat{p}^B(t,x,y) := p_B(t,y,x)\frac{h_B(y)}{h_B(x)}$ with respect to the Lebesgue measure on B. Denote by $\widehat{X}^{B,D}$ the subprocess of $\widehat{X}^B$ killed upon exiting D. By [35, Theorem 2 and its Remark 2], $\widehat{X}^{B,D}$ and X^D are in duality with respect to the measure $\xi_B(dx)$ restricted to D. The transition density function and the Green function of $\widehat{X}^{B,D}$ with respect to the Lebesgue measure on D are

$$\widehat{p}^B_D(t,x,y) := \frac{p_D(t,y,x) h_B(y)}{h_B(x)} \quad \text{and} \quad \widehat{G}^B_D(x,y) := \frac{G_D(y,x) h_B(y)}{h_B(x)}. \tag{3.1}$$

It is easy to check that for $x, y \in D$,

$$\widehat{G}^B_D(y,x) = \widehat{G}_B(y,x) - \widehat{\mathbb{E}}_y\left[\widehat{G}_B(\widehat{X}^B_{\widehat{\tau}_D}, x); \widehat{\tau}_D < \widehat{\zeta}\right], \tag{3.2}$$

where $\widehat{\tau}_D$ is the first exit time of the process $\widehat{X}^B$ from D and $\widehat{\zeta}$ is the lifetime of $\widehat{X}^B$. By the joint continuity of $\widehat{G}_B(\cdot,\cdot)$ on $B \times B \setminus \mathrm{diag}$ and [28, Propositions 4.5 and 4.6], for $x \in D$ and $z \in \partial D$,

$$\widehat{G}_B(z,x) = \lim_{y \to z} \mathbb{E}_y\left[\widehat{G}_B(\widehat{X}^B_{\widehat{\tau}_D}, x)\right]. \tag{3.3}$$

For any $C^{1,1}$ domain D in $\mathbb{R}^d$ with characteristics (R_0, Λ_0), it is well known (see, for instance, [36, Lemma 2.2]) that there exists $L = L(d, R_0, \Lambda_0) > 0$ such that for any $Q \in \partial D$ and $r \in (0, R_0)$, there is a $C^{1,1}$ connected open set $U_{Q,r} \subset D$ with characteristics $(rR_0/L, \Lambda_0 L/r)$ such that

$$D \cap B(Q, r/2) \subset U_{Q,r} \subset D \cap B(Q, r). \tag{3.4}$$

Note that $r^{-1}U_{Q,r}$ is a $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics $(R_0/L, \Lambda_0 L)$ and $\mathrm{diam}(r^{-1}U_{Q,r}) \leq 2$. In the remainder of this paper, we always use $U_{Q,r}$ to denote such a $C^{1,1}$ open subset of D. By Theorem 2.12 with $r^{-1}U_{Q,r}$ in place of D, we have the following result.

Corollary 3.1. There exist positive constants $\theta = \theta(d, \lambda_0, \ell, \Lambda_0, R_0)$ and $C_{11} = C_{11}(d, \lambda_0, \ell, \Lambda_0, R_0) \geq 1$ such that for any $Q \in \partial D$ and $r \in (0, \theta \wedge R_0)$,

$$C_{11}^{-1} G^{\Delta}_{U_{Q,r}}(x,y) \leq G_{U_{Q,r}}(x,y) \leq C_{11} G^{\Delta}_{U_{Q,r}}(x,y) \quad \text{for any } x \neq y \in U_{Q,r}.$$

For an open subset $U \subset B$, denote by $\widehat{\tau}_U := \inf\{t > 0 : \widehat{X}^B_t \notin U\}$ the exist time from U by $\widehat{X}^B$.

Lemma 3.2. Let θ be the constant in Corollary 3.1. Suppose that $\{a_{ij}; 1 \leq i, j \leq d\}$ are C^1 on $\mathbb{R}^d$ and satisfy the conditions (1.2) and (1.3). There exists a positive constant $M_1 = M_1(d, \lambda_0, \ell, R_0, \Lambda_0) > 1$ such that for any $Q \in \partial D, r \in (0, \theta \wedge R_0)$ and $U := U_{Q,r}$,

$$\begin{aligned}
&\frac{M_1^{-1}}{h_B(x)} \int_{\partial U \cap D} h_B(z) \mathbb{P}_x\left(W_{\tau_U} \in dz\right) \leq \mathbb{P}_x\left(\widehat{X}^B_{\widehat{\tau}_U} \in dz; \widehat{\tau}_U < \widehat{\tau}_D\right) \\
&\leq \frac{M_1}{h_B(x)} \int_{\partial U \cap D} h_B(z) \mathbb{P}_x\left(W_{\tau_U} \in dz\right) \qquad \text{on } D \cap \partial U \text{ for every } x \in U
\end{aligned}$$

where W is a Brownian motion on $\mathbb{R}^d$ and τ_U is its first exit time from U.

Proof. The proof is similar to that of [28, Theorem 4.7] (cf. [9, Theorem 2.2]). For the convenience of the reader, we spell out its details here. Let φ be a non-negative continuous function with compact support on $\partial U \cap D$ and

$$u(x) := \mathbb{E}_x\left[\varphi(\widehat{X}^B_{\widehat{\tau}_U})\right] = \mathbb{E}_x\left[\varphi(\widehat{X}^B_{\widehat{\tau}_U}); \widehat{\tau}_U < \widehat{\tau}_D\right].$$

Then u is harmonic for $\widehat{X}^B$ in U. Let $\{U_n\}_{n\geq 1}$ be an increasing sequence of open sets so that $\overline{U}_n \subset U_{n+1}$ and $\cup_{n=1}^\infty U_n = U$. For each $n \geq 1$, by [28, Proposition 4.2], there exists a Radon measures ν_n supported on ∂U_n such that

$$u(x) = \frac{1}{h_B(x)} \int_{\partial U_n} G_U(y,x) \nu_n(dy) \quad \text{for } x \in U_n.$$

Define

$$v_n(x) := \int_{\partial U_n} G^{\Delta}_U(y,x) \nu_n(dy), \quad x \in U_n.$$

By Corollary 3.1,

$$C_{11}^{-1} G^{\Delta}_U(x,y) \leq G_U(x,y) \leq C_{11} G^{\Delta}_U(x,y) \quad \text{for any } x \neq y \in U.$$

Consequently,

$$C_{11}^{-1} v_n(x) \leq h_B(x) u(x) \leq C_{11} v_n(x), \quad x \in U_n.$$

For each $n \geq 1$, $\{v_k; k \geq n\}$ is a sequence of bounded classical harmonic function with respect to the Laplacian on U_n. So by the equi-Hölder-continuity of $\{v_k; k \geq n\}$ and a diagonal selection procedure, there is a subsequence n_k such that v_{n_k} converges uniformly on each U_n to a harmonic function v in U. Clearly,

$$C_{11}^{-1} v(x) \le h_B(x) u(x) \le C_{11} v(x), \quad x \in U. \tag{3.5}$$

For an open subset V of B with $\overline{V} \subset B$ and $z \in \partial V$, if there is a cone A with vertex z so that $A \cap B(z,r) \subset B \setminus V$ for some $r > 0$, then z is a regular point of V for $\widehat{X}^B$ by [28, Proposition 4.5], that is, $\mathbb{P}_z(\widehat{\tau}_V^B = 0) = 1$. Here $\widehat{\tau}_V^B := \inf\{t > 0 : \widehat{X}_t^B \notin V\}$. Hence every boundary point of U is regular for $\widehat{X}^B$. Since $\varphi \in C_c(\partial U \cap D) \subset C_c(\partial U)$, it follows from [28, Proposition 4.6] that

$$\lim_{U \ni x \to z} u(x) = \varphi(z) \quad \text{for every } z \in \partial U.$$

Consequently, we have by (3.5) that for every $z \in \partial U$,

$$C_{11}^{-1} h_B(z) \varphi(z) \le \liminf_{U \ni x \to z} v(x) \le \limsup_{U \ni x \to z} v(x) \le C_{11} h_B(z) \varphi(z). \tag{3.6}$$

Define

$$w(x) := \mathbb{E}_x[h_B(W_{\tau_U}) \varphi(W_{\tau_U})], \quad x \in U.$$

Then w is a harmonic function with respect to W in U with the boundary $h_B(z)\varphi(z)$. By the maximum principle and (3.6), we obtain that $C_{11}^{-1} w(x) \le v(x) \le C_{11} w(x)$ in U. Hence, combined with (3.5), we have $C_{11}^{-2} w(x) \le h_B(x) u(x) \le C_{11}^2 w(x)$ in U. That is, for each non-negative $\varphi \in C_c(\partial U \cap D)$,

$$\begin{aligned} C_{11}^{-2} \int_{D \cap \partial U} h_B(z) \varphi(z) \mathbb{P}_x\left(W_{\tau_U} \in dz\right) &\le h_B(x) \int_{D \cap \partial U} \varphi(z) \mathbb{P}_y\left(\widehat{X}_{\widehat{\tau}_U}^B \in dz;\, \widehat{\tau}_U < \widehat{\tau}_D\right) \\ &\le C_{11}^2 \int_{D \cap \partial U} h_B(z) \varphi(z) \mathbb{P}_x\left(W_{\tau_U} \in dz\right), \quad x \in U. \end{aligned}$$

Observe that on $\left\{\widehat{\tau}_U < \widehat{\tau}_D\right\}$, $\widehat{X}_{\widehat{\tau}_U}^B \in D \cap \partial U$. Let $M_1 := C_{11}^2$. This establishes the lemma. □

Theorem 3.3. Suppose that $\{a_{ij}; 1 \le i, j \le d\}$ are C^1 and satisfy the conditions (1.2) and (1.3). Let θ be the constant in Corollary 3.1 and let $M_1 > 1$ be the constant in Lemma 3.2. There exists a measurable function ψ_1 on $\mathbb{R}^d \times \mathbb{R}^d$ that is bounded between M_1^{-1} and M_1 such that for any $Q \in \partial D$ and $U := U_{Q,r}$ with $r \in (0, \theta \wedge R_0)$,

$$G_D(x,y) = \int_{\partial U \cap D} G_D(x,z) \psi_1(y,z) K_U^{\Delta}(y,z) \sigma(dz), \quad x \in D \setminus \overline{U}, \quad y \in U,$$

where $K_U^{\Delta}(y,z) := \frac{\partial}{\partial \vec{n}_z} G_U^{\Delta}(y,z)$ is the Poisson kernel of Brownian motion. Here $\vec{n}_z$ is the unit inward normal vector $z \in \partial U$ for U, and σ is the surface measure of ∂U.

Proof. By the strong Markov property of $\widehat{X}^B$, the Green function $\widehat{G}_D^B(x,y)$ of $\widehat{X}^{B,D}$ has the property that for $y \in U$ and $x \in D \setminus \overline{U}$,

$$\widehat{G}_D^B(y,x) = \mathbb{E}_y\left[\widehat{G}_D^B(\widehat{X}_{\widehat{\tau}_U}^B, x); \widehat{\tau}_U < \widehat{\tau}_D\right].$$

Thus by Lemma 3.2, we have for $y \in U$ and $x \in D \setminus \overline{U}$,

$$\begin{aligned} &\frac{M_1^{-1}}{h_B(y)} \mathbb{E}_y\left[h_B(W_{\tau_U}) \widehat{G}_D^B(W_{\tau_U}, x); W_{\tau_U} \in D\right] \\ &\le \widehat{G}_D^B(y,x) \\ &\le \frac{M_1}{h_B(y)} \mathbb{E}_y\left[h_B(W_{\tau_U}) \widehat{G}_D^B(W_{\tau_U}, x); W_{\tau_U} \in D\right]. \end{aligned}$$

Recall from (3.1) that

$$\widehat{G}_D^B(y,x) = \frac{G_D(x,y)h_B(x)}{h_B(y)} \quad \text{on } D \times D \setminus \text{diag}.$$

Thus we have for $x \in D \setminus \overline{U}$ and $y \in U$,

$$M_1^{-1}\mathbb{E}_y[G_D(x,W_{\tau_U}); W_{\tau_U} \in D \cap \partial U] \le G_D(x,y) \le M_1 \mathbb{E}_y[G_D(x,W_{\tau_U}); W_{\tau_U} \in D \cap \partial U]. \tag{3.7}$$

It is well-known (see, e.g., [14, Proposition 5.13]) that $\mathbb{P}_y(W_{\tau_U} \in dz) = K_U^\Delta(y,z)\sigma(dz)$ on ∂U, where σ is the surface measure of ∂U. This together with (3.7) yields the desired conclusion. □

Theorem 3.4. Suppose that $\{a_{ij}; 1 \le i,j \le d\}$ are C^1 on $\mathbb{R}^d$ and satisfy the conditions (1.2) and (1.3). There exists a constant $C_{12} = C_{12}(d,\lambda_0,\ell,R_0,\Lambda_0,\text{diam}(D)) > 1$ such that

$$G_D(x,y) \le C_{12}|x-y|^{2-d}\left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)\left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right), \quad x \ne y \in D.$$

Proof. Let θ be the constant in Corollary 3.1, and set $\kappa_0 := \theta \wedge R_0$. By Lemma 2.11, it suffices to prove theorem for $\delta_D(y) < (|x-y| \wedge \kappa_0)/8$.

Fix $x,y \in D$ with $\delta_D(y) < (|x-y| \wedge \kappa_0)/8$. Let Q_y be the point on ∂D such that $|y - Q_y| = \delta_D(y)$. Denote by $U := U_{Q_y,|x-y|\wedge\kappa_0}$ a $C^{1,1}$ connected open set such that $D \cap B(Q_y, (|x-y| \wedge \kappa_0)/2) \subset U \subset D \cap B(Q_y, |x-y| \wedge \kappa_0)$. Let $y_0 := Q_y + \frac{(|x-y|\wedge\kappa_0)}{4}(y - Q_y)/|y-Q_y|$. It is well known (see e.g. by (2.22)) that there exists $c_1 = c_1(d,R_0,\Lambda_0)$ such that for $z \in \partial U \cap D$,

$$K_U^\Delta(y,z) \le c_1\frac{\delta_D(y)}{\delta_D(y_0)}K_U^\Delta(y_0,z) = 4c_1\frac{\delta_D(y)}{|x-y| \wedge \kappa_0}K_U^\Delta(y_0,z) \le 4c_1\frac{\text{diam}(D)}{\kappa_0}\frac{\delta_D(y)}{|x-y|}K_U^\Delta(y_0,z).$$

Let $c_2 = 4c_1\dfrac{\text{diam}(D)}{\kappa_0}$. It follows from Theorem 3.3 that

$$\begin{aligned} G_D(x,y) &\le M_1 \int_{\partial U \cap D} G_D(x,z)K_U^\Delta(y,z)\sigma(dz) \\ &\le c_2 M_1 \frac{\delta_D(y)}{|x-y|}\int_{\partial U \cap D} G_D(x,z)K_U^\Delta(y_0,z)\sigma(dz) \\ &\le c_2 M_1^2 \frac{\delta_D(y)}{|x-y|} G_D(x,y_0) \\ &= c_2 M_1^2\left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right) G_D(x,y_0) \end{aligned} \tag{3.8}$$

Note that $\frac{1}{2}|x-y| \le |x-y_0| \le 2|x-y|$, we have by Lemma 2.11 that

$$G_D(x,y_0) \le C_9|x-y_0|^{2-d}\left(1 \wedge \frac{\delta_D(x)}{|x-y_0|}\right) \le C_9 2^{d-1}|x-y|^{2-d}\left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right). \tag{3.9}$$

The desired result follows from (3.8) and (3.9). □

For $r > 0$, let $D_r := \{z \in D : \text{dist}(z,\partial D) \le r\}$.

Lemma 3.5. For each $a \in (0,1]$, there exists a constant $C_{13} = C_{13}(d,a,\lambda_0,\ell,R_0,\Lambda_0,\text{diam}(D)) \in (0,1)$ such that

$$G_D(x,y) \ge C_{13}|x-y|^{2-d}\left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right), \quad x \in D, y \in D \setminus D_a.$$

Proof. Fix a positive constant $a \in (0,1]$. Suppose $y \in D \setminus D_a$. Recall that δ_0 is the constant in Lemma 2.10.

(i) We first consider $x \in D \setminus D_{a\delta_0/2}$. By Theorem 2.12 and (2.22), there exist positive constants $a_0 \in (0, a/2)$ and $c_1 = c_1(d, \lambda_0, \ell)$ such that for $r \in (0, a_0)$ and $y \in \mathbb{R}^d$,

$$G_{B(y,r)}(u,y) \geq c_1|u-y|^{2-d}, \quad u \in B(y, r/2). \tag{3.10}$$

Then by the standard chain argument and Harnack principle Theorem 2.14 of the operator $\mathcal{L}$, there exists $c_2 = c_2(d, \lambda_0, \ell, R_0, \Lambda_0, \mathrm{diam}(D))$ such that for $x \in D \setminus D_{a\delta_0/2}$ and $y \in D \setminus D_a$,

$$G_D(x,y) \geq c_2|x-y|^{2-d}.$$

(ii) Now we consider $x \in D_{a\delta_0/2}$. Let Q_x be the point on ∂D such that $|x - Q_x| = \delta_D(x)$. Let ϕ_{Q_x} be the local boundary function in the coordinate system CS_{Q_x}. Let $\rho_{Q_x}(y) := y_d - \phi_{Q_x}(\tilde{y})$. For $r_1, r_2 > 0$, let

$$D_{Q_x}(r_1, r_2) := \{y = (\tilde{y}, y_d) \in D : 0 < \rho_{Q_x}(y) < r_1 \text{ and } |\tilde{y}| < r_2 \text{ in } CS_{Q_x}\}$$

$$U_{Q_x}(r_1, r_2) := \{y = (\tilde{y}, y_d) \in D : \rho_{Q_x}(y) = r_1 \text{ and } |\tilde{y}| < r_2 \text{ in } CS_{Q_x}\}$$

By step (i) and Lemma 2.10 with $r = 1$, there exists $c_3 = c_3(d, \lambda_0, \ell, R_0, \Lambda_0, \mathrm{diam}(D))$ such that

$$\begin{aligned} G_D(x,y) &\geq \mathbb{E}_x\left[G_D(X_{\tau_{D_{Q_x}(a\delta_0, ar_0)}}, y); X_{\tau_{D_{Q_x}(a\delta_0, ar_0)}} \in U_{Q_x}(a\delta_0, ar_0)\right] \\ &\geq c_3|x-y|^{2-d}\mathbb{P}_x\left(X_{\tau_{D_{Q_x}(a\delta_0, ar_0)}} \in U_{Q_x}(a\delta_0, ar_0)\right) \\ &\geq c_3 C_8|x-y|^{2-d}\frac{\delta_D(x)}{a} \geq \frac{1}{2}c_3 C_8|x-y|^{2-d}\left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right), \end{aligned}$$

where in the last inequality we used the fact that $|x-y| > a/2$. □

Theorem 3.6. Suppose that $\{a_{ij}; 1 \leq i, j \leq d\}$ are C^1 on $\mathbb{R}^d$ and satisfy the conditions (1.2) and (1.3). There exists a constant $C_{14} = C_{14}(d, \lambda_0, \ell, \Lambda_0, R_0, \mathrm{diam}(D))$ such that

$$G_D(x,y) \geq C_{14}|x-y|^{2-d}\left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)\left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right) \quad \text{on } D \times D \setminus \mathrm{diag}. \tag{3.11}$$

Proof. Let θ be the constant in Corollary 3.1. Let $\kappa_0 := \theta \wedge R_0$. By Lemma 3.5, it suffices to prove the result for $x \in D$ and $y \in D_{\kappa_0/32}$.

(1) Suppose $y \in D_{\kappa_0/32}$ and $|x-y| \leq \kappa_0/8$. Let Q_y be the point on ∂D such that $|y - Q_y| = \delta_D(y)$. Let $U_1 := U_{Q_y, \kappa_0/2}$ be a $C^{1,1}$ connected open set such that $D \cap B(Q_y, \kappa_0/4) \subset U_1 \subset D \cap B(Q_y, \kappa_0/2)$. Then $x, y \in U_1$ and $\delta_{U_1}(x) = \delta_D(x), \delta_{U_1}(y) = \delta_D(y)$. Then by Corollary 3.1, there exists a constant $c_1 = c_1(d, \lambda_0, \ell, \Lambda_0, R_0)$ such that

$$G_D(x,y) \geq G_{U_1}(x,y) \geq c_1|x-y|^{2-d}\left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)\left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right).$$

(2) If $y \in D_{\kappa_0/32}$ and $|x-y| > \kappa_0/8$. Let Q_y be the point on ∂D such that $|y - Q_y| = \delta_D(y)$. Let $y_0 := Q_y + \frac{\kappa_0}{32}(y - Q_y)/|y - Q_y|$. Let $U_2 := U_{Q_y, \kappa_0/8}$ be a $C^{1,1}$ connected open set such that $D \cap B(Q_y, \kappa_0/16) \subset U_2 \subset D \cap B(Q_y, \kappa_0/8)$. It is well known (see e.g. by (2.22)) that there exists $c_2 = c_2(d, R_0, \Lambda_0)$ such that for $z \in \partial U_2 \cap D$,

$$K^{\Delta}_{U_2}(y,z) \geq c_2\frac{\delta_D(y)}{\delta_D(y_0)}K^{\Delta}_{U_2}(y_0, z) \geq c_2\frac{\delta_D(y)}{|x-y|}K^{\Delta}_{U_2}(y_0, z).$$

Then by Theorem 3.3,

$$
\begin{aligned}
G_D(x,y) &\geq M_1^{-1} \int_{\partial U \cap D} G_D(x,z) K^{\Delta}_{U_2}(y,z)\sigma(dz) \\
&\geq c_2 M_1^{-1} \frac{\delta_D(y)}{|x-y|} \int_{\partial U \cap D} G_D(x,z) K^{\Delta}_{U_2}(y_0,z)\sigma(dz) \\
&\geq c_2 M_1^{-2} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right) G_D(x,y_0).
\end{aligned}
\tag{3.12}
$$

By Lemma 3.5 with $a = \kappa_0/64$,

$$
G_D(x,y_0) \geq C_{13}|x-y_0|^{2-d}\left(1 \wedge \frac{\delta_D(x)}{|x-y_0|}\right) \geq C_{13}2^{1-d}|x-y|^{2-d}\left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right). \tag{3.13}
$$

The desired result follows from (3.12) and (3.13). □

We now removed the C^1 smoothness assumption on $\{a_{ij}(x); 1 \leq i,j \leq d\}$ from Theorem 3.4 and Theorem 3.6.

Theorem 3.7. There exists a constant $C_{15} = C_{15}(d, \lambda_0, \ell, \Lambda_0, R_0, \mathrm{diam}(D))$ such that

$$
C_{15}^{-1} G^{\Delta}_D(x,y) \leq G_D(x,y) \leq C_{15} G^{\Delta}_D(x,y) \quad \text{on } D \times D \setminus \mathrm{diag}. \tag{3.14}
$$

Proof. Let $\phi \in C_c^\infty(\mathbb{R}^d)$ with $\phi \geq 0$, $\mathrm{supp}[\phi] \subset B(0,1)$ and $\int_{\mathbb{R}^d} \phi(x)\,dx = 1$. For each integer $k \geq 1$, define $\phi_k(x) := k^d \phi(kx)$ and $a^{(k)}_{ij}(x) := \phi_k * a_{ij}(x) := \int_{\mathbb{R}^d} \phi_k(x-y)a_{ij}(y)dy$. Then $a^{(k)}_{ij} \in C^\infty(\mathbb{R}^d)$ satisfying the conditions (1.2)-(1.3) with the same ellipticity constant $\lambda_0 \geq 1$ and Dini modulo of continuity function ℓ, and $a^{(k)}_{ij}$ converges uniformly to a_{ij} on any compact set of $\mathbb{R}^d$. Denote by $\mathcal{L}^{(k)}$ the non-divergence operator $\mathcal{L}$ but with diffusion coefficients $a^{(k)}_{ij}$ in place of a_{ij}. Let $X^{(k)}$ be the diffusion process having $\mathcal{L}^{(k)}$ as its infinitesimal generator, and $G^{(k)}_D(x,y)$ its Green function on D. That is, $G^{(k)}_D(x,y)$ is the unique jointly continuous function on $D \times D \setminus \mathrm{diag}$ so that for every $f \in C_c(D)$,

$$
\mathbb{E}_x\left[\int_0^{\tau^{(k)}_D} f(X^{(k)}_s)ds\right] = \int_D G^{(k)}_D(x,y)f(y)dy, \quad x \in D,
$$

where $\tau^{(k)}_D$ is the first exit time of the process $X^{(k)}$ from D.

We first show that for each $x \in D$ and any $f \in C_c(D)$, $\lim_{k\to\infty} G^{(k)}_D f(x) = G_D f(x)$. For any $f \in C_c(D)$, define $u^{(k)}(x) := G^{(k)}_D f(x)$ and $u(x) := G_D f(x)$. By Lemma 2.2, $u^{(k)}$ and u are the unique solutions in $W^{2,p}(D) \cap C(\overline{D})$ with $p > d$ for

$$
\mathcal{L}^{(k)} u^{(k)}(x) = -f \quad \text{in } D \quad \text{and} \quad u^{(k)} = 0 \quad \text{on } \partial D,
$$

and for

$$
\mathcal{L} u = -f \quad \text{in } D \quad \text{and} \quad u = 0 \quad \text{on } \partial D,
$$

respectively. The function $v^{(k)} := u^{(k)} - u \in W^{2,p}(D) \cap C(\overline{D})$ satisfies

$$
\mathcal{L}^{(k)} v^{(k)}(x) = -g^{(k)} \quad \text{in } D \quad \text{and} \quad v^{(k)} = 0 \quad \text{on } \partial D.
$$

where $g^{(k)}(x) := f(x) + \mathcal{L}^{(k)}u(x) = (\mathcal{L}^{(k)} - \mathcal{L})u(x)$, which is in $L^p(D)$. Thus by [25, Remark 1.14],

$$
v^{(k)}(x) = G^{(k)}_D g^{(k)}(x) = \int_D G^{(k)}_D(x,y) g^{(k)}(y)\,dy \quad \text{for } x \in D.
$$

By Hölder's inequality and (2.4), there is $c = c(d, \lambda_0, \ell, R_0, \Lambda_0, \mathrm{diam}(D), p) > 0$ so that

$$\sup_{x \in D} |v^{(k)}(x)| \le \sup_{x \in D} \|G_D^{(k)}(x, \cdot)\|_{L^{p/(p-1)}(D)} \|g^{(k)}\|_{L^p(D)} \le c \|g^{(k)}\|_{L^p(D)}.$$

Since $u \in W^{2,p}(D)$,

$$\lim_{k \to \infty} \|g^{(k)}\|_{L^p(D)} \le \lim_{k \to \infty} \sum_{i,j=1}^{d} \|a_{ij}^{(k)} - a_{ij}\|_{L^\infty(D)} \|\partial^2_{x_i x_j} u\|_{L^p(D)} = 0.$$

Therefore, $u^{(k)}(x)$ converges uniformly to $u(x)$ on D as $k \to \infty$. This in particular implies that for each $x \in D$,

$$\lim_{k \to \infty} \int_D G_D^{(k)}(x, y) f(y) dy = \int_D G_D(x, y) f(y) dy \quad \text{for every } f \in C_c(D). \tag{3.15}$$

From Theorems 3.4 and 3.6, we know that there exists a constant $C_{15} = C_{15}(d, \lambda_0, \ell, \Lambda_0, R_0, \mathrm{diam}(D))$ such that for any $k \ge 1$,

$$C_{15}^{-1} G_D^{\Delta}(x, y) \le G_D^{(k)}(x, y) \le C_{15} G_D^{\Delta}(x, y) \quad \text{on } D \times D \setminus \mathrm{diag}. \tag{3.16}$$

Consequently, for each $x \in D$ and every non-negative $f \in C_c(D)$, $C_{15}^{-1} G_D^{\Delta} f(x) \le G_D^{(k)} f(x) \le C_{15} G_D^{\Delta} f(x)$. Passing $k \to \infty$, (3.15) and (3.16) yield

$$C_{15}^{-1} G_D^{\Delta} f(x) \le G_D f(x) \le C_{15} G_D^{\Delta} f(x).$$

Hence,

$$C_{15}^{-1} G_D^{\Delta}(x, y) \le G_D(x, y) \le C_{15} G_D^{\Delta}(x, y) \quad \text{for} \quad \text{a.e. } y.$$

By the continuity of $G_D(x, y)$ in $y \in D \setminus \{x\}$, we get

$$C_{15}^{-1} G_D^{\Delta}(x, y) \le G_D(x, y) \le C_{15} G_D^{\Delta}(x, y) \quad \text{for every } x \ne y \in D.$$

The proof is complete. □

In the last part of this section, we derive a Poisson integral representation for non-negative $\mathcal{L}$-harmonic functions on a bounded $C^{1,1}$ domain D and establish a comparison result between non-negative $\mathcal{L}$-harmonic functions and non-negative classical harmonic functions on D. The latter is then used to give a direct proof of the boundary Harnack principle in bounded $C^{1,1}$ domains with an explicit boundary decay rate. Boundary Harnack principle for $\mathcal{L}$ has previously been established in [4, 19] for Lipschitz domains without an explicit boundary decay rate. We next use it to identify the Martin boundary and minimal Martin boundary with the Euclidean boundary of D for non-divergence form differential operator $\mathcal{L}$ with Dini coefficients.

Theorem 3.8. Let $C_{15} \ge 1$ be the constant in Theorem 3.7.

(i) For every non-negative $\mathcal{L}$-harmonic function u in D, there exists a classical harmonic function v in D so that $C_{15}^{-1} v \le u \le C_{15} v$ on D;

(ii) There exists a function ψ on $D \times \partial D$ that is bounded between $1/C_{15}^2$ and C_{15}^2 so that for each non-negative function φ on ∂D,

$$\mathbb{E}_x \left[\varphi(X_{\tau_D}) \right] = \int_{\partial D} K_D^{\Delta}(x, z) \psi(x, z) \varphi(z) \sigma(dz), \quad x \in D, \tag{3.17}$$

where $K_D^{\Delta}(x, z)$ is the Poisson kernel of Brownian motion on $D \times \partial D$ and σ is the surface measure of ∂D. Consequently, there is a constant $C_0 = C_0(d, \lambda_0, \ell, \Lambda_0, R_0, \mathrm{diam}(D)) > 1$ such that for every $x \in D$,

$$C_0^{-1} \frac{\delta_D(x)}{|x - z|^d} \sigma(dz) \le \mathbb{P}_x(X_{\tau_D} \in dz) \le C_0 \frac{\delta_D(x)}{|x - z|^d} \sigma(dz) \quad \text{on } \partial D.$$

Proof. The proof is similar to that of [9, Theorem 2.2]. For readers' convenience, we spell out the details here.

(i) Suppose that u is a non-negative $\mathcal{L}$-harmonic function on D. Let $\{D_n\}_{n\geq 1}$ be an increasing sequence of open sets so that $\overline{D}_n \subset D_{n+1}$ and $\cup_{n=1}^{\infty} D_n = D$. Let $T_{D_n} := \inf\{t > 0 : X_t \in D_n\}$. For each $n \geq 1$, by Corollary 1 to Theorem 2 in [13], there exists a Radon measure ν_n on $\overline{D}_n$ so that

$$u(x) = \mathbb{E}_x u(X^D_{T_{D_n}}) = \int_{\overline{D}_n} G_D(x,y)\nu_n(dy) \quad \text{for } x \in D_n.$$

Define

$$v_n(x) := \int_{\overline{D}_n} G^{\Delta}_D(x,y)\nu_n(dy), \quad x \in D_n.$$

By Theorem 3.7,

$$C_{15}^{-1} v_n(x) \leq u(x) \leq C_{15} v_n(x), \quad x \in D_n.$$

By a similar argument as that for Lemma 3.2, there is a subsequence n_k such that v_{n_k} converges uniformly on each D_n to a Δ-harmonic function v in D. Clearly,

$$C_{15}^{-1} v(x) \leq u(x) \leq C_{15} v(x) \quad \text{for } x \in D. \tag{3.18}$$

(ii) Let $\varphi \in C(\partial D)$ be non-negative and take $u(x) := \mathbb{E}_x\left[\varphi(X_{\tau_D})\right]$ for $x \in D$, which is a non-negative $\mathcal{L}$-harmonic function in D. Let v be a classical harmonic function in D obtained in (i) so that $C_{15}^{-1} v \leq u \leq C_{15} v$ on D. On the other hand, by Lemma 2.2, u is the unique function in $W^{2,p}_{loc}(D) \cap C(\overline{D})$ with $p > d/2$ so that $\mathcal{L}u = 0$ in D and $u = \varphi$ on ∂D. Hence, $\lim_{D \ni x \to z} u(x) = \varphi(z)$ for every $z \in \partial D$. It follows from (3.18) that for every $z \in \partial D$,

$$C_{15}^{-1}\varphi(z) \leq \liminf_{D\ni x \to z} v(x) \leq \limsup_{D \ni x \to z} v(x) \leq C_{15}\varphi(z).$$

Define $w(x) = \mathbb{E}_x\left[\varphi(W_{\tau_D})\right]$, where W is the standard Brownian motion on $\mathbb{R}^d$. Then w is a classical harmonic function in D with $w = \varphi$ on ∂D. By the above display and the maximum principle, we have $C_{15}^{-1} w(x) \leq v(x) \leq C_{15} w(x)$ in D. Thus we have by (3.18) that $C_{15}^{-2} w(x) \leq u(x) \leq C_{15}^2 w(x)$ in D. This implies that $\mathbb{P}_x(X_{\tau_D} \in dz)$ has a density $K_D(x,z)$ with respect to the surface measure σ on ∂D and that for every non-negative $\varphi \in C(\partial D)$,

$$C_{15}^{-2}\int_{\partial D} K^{\Delta}_D(x,z)\varphi(z)\sigma(dz) \leq \int_{\partial D} K_D(x,z)\varphi(z)\sigma(dz) \leq C_{15}^2 \int_{\partial D} K^{\Delta}_D(x,z)\varphi(z)\sigma(dz).$$

This implies that for each $x \in D$, $C_{15}^{-2} K^{\Delta}_D(x,z) \leq K_D(x,z) \leq C_{15}^2 K^{\Delta}_D(x,z)$ σ-a.e. on ∂D. Thus there is a function $\psi(x,z)$ bounded between C_{15}^{-2} and C_{15}^2 so that $K_D(x,z) = K^{\Delta}_D(x,z)\psi(x,z)$ on $D \times \partial D$. This completes the proof of the theorem. □

Corollary 3.9 (Boundary Harnack principle)**.** Suppose D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R_0, Λ_0). There exists a constant $C = C(d, \lambda_0, \ell, R_0, \Lambda_0)$ such that for all $Q \in \partial D, r \in (0, R_0)$ and all function $h \geq 0$ on $\mathbb{R}^d$ that is $\mathcal{L}$-harmonic in $D \cap B(Q,r)$ and vanishes continuously on $\partial D \cap B(Q,r)$, we have

$$\frac{h(x)}{h(y)} \leq C \frac{\delta_D(x)}{\delta_D(y)} \quad \text{for } x, y \in D \cap B(Q, r/4).$$

Proof. Let θ be the constant in Corollary 3.1. By the Harnack inequality Theorem 2.14, it suffices to prove $r \in (0, \theta \wedge R_0)$. Let $Q \in \partial D$. Recall from (3.4) that $U_{Q,r}$ is a $C^{1,1}$ connected open set with characteristics $(rR_0/L, \Lambda_0 L/r)$ such that $D \cap B(Q, r/2) \subset U_{Q,r} \subset D \cap B(Q,r)$.

Let $h \geq 0$ be a non-negative harmonic function with respect to X in $D \cap B(Q, r)$ and vanishes continuously on $\partial D \cap B(Q, r)$. By the same argument as that for Theorem 3.8(i) on $U_{Q,r}$ instead of D and using Corollary 3.1 in place of Theorem 3.7 in the proof, we can conclude that there is a classical harmonic function v in $U_{Q,r}$ so that

$$c_1^{-1} v(x) \leq h(x) \leq c_1 v(x) \quad \text{for every } x \in U_{Q,r}. \tag{3.19}$$

where $c_1 > 1$ is a constant that depends only on $(d, \lambda_0, \ell, \Lambda_0, R_0)$. Since v vanishes continuously on $U_{Q,r} \cap \partial D \supset B(Q, r/2) \cap \partial D$, by the classical boundary Harnack inequality (for Laplacian) on bounded $C^{1,1}$ domains, there is a constant $c_2 = c_2(d, R_0, \Lambda_0) > 1$ such that

$$\frac{v(x)}{v(y)} \leq c_2 \frac{\delta_U(x)}{\delta_U(y)} = c_2 \frac{\delta_D(x)}{\delta_D(y)} \quad \text{for every } x, y \in B(Q, r/4) \cap D.$$

Consequently, by (3.19),

$$\frac{h(x)}{h(y)} \leq c_1^2 c_2 \frac{\delta_D(x)}{\delta_D(y)} \quad \text{for } x, y \in B(Q, r/4) \cap D.$$

This proves the lemma by taking $C = c_1^2 c_2$. □

Fix $x_0 \in D$. For each $x, y \in D$, we define the Martin kernel $M_D(x, y) := \dfrac{G_D(x, y)}{G_D(x_0, y)}$.

Theorem 3.10. Suppose D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R_0, Λ_0). Then $M_D(x, z) := \lim_{y \to z} M_D(x, y)$ exists for $x \in D$ and $z \in \partial D$. Moreover, the Martin boundary and the minimal Martin boundary of X in D can all be identified with the Eulidean boundary ∂D.

Proof. By the boundary Harnack principle Theorem 3.9 and a standard argument (see e.g. [2, Proposition 2.2,, Chapter III]), $M_D(x, z) := \lim_{y \to z} M_D(x, y)$ exists for $x \in D$ and $z \in \partial D$, and $M_D(x, z)$ is a continuous function on $D \times \partial D$. Note that by Theorems 3.7 and 5.1, there exists a constant $c = c(d, \lambda_0, \ell, R_0, \Lambda_0, \operatorname{diam}(D)) > 1$ such that for any $x \in D$ and $z \in \partial D$,

$$c^{-1} \frac{\delta_D(x)}{|x - z|^d} \leq M_D(x, z) \leq c \frac{\delta_D(x)}{|x - z|^d}. \tag{3.20}$$

Hence $M_D(\cdot, z_1) \neq M_D(\cdot, z_2)$ for $z_1 \neq z_2 \in \partial D$. It is easy to check that for each $z \in \partial D$, $x \mapsto M_D(x, z)$ is X-harmonic in D, and so the Martin boundary of X in D is identified with the Eulidean boundary ∂D. We next show that for each $z_0 \in \partial D$, $x \mapsto M_D(x, z_0)$ is a minimal harmonic function of X^D. Suppose that $h \geq 0$ is a non-negative X^D-harmonic function on D and $h \leq M_D(\cdot, z_0)$. By the Martin representation formula [30], there exists a unique finite measure μ on ∂D so that

$$h(x) = \int_{\partial D} M_D(x, z) \mu(dz) \quad \text{for } x \in D.$$

We claim that μ is a constant multiple of a Dirac measure concentrated at z_0. Suppose not, there is some $\varepsilon > 0$ so that $\mu_\varepsilon := \mu|_{\partial D \setminus B(z_0, \varepsilon)}$ is non-trivial. Then $h_\varepsilon(x) := \int_{\partial D} M_D(x, z) \mu_\varepsilon(dz)$ is a non-trivial non-negative X-harmonic function in D that is bounded by $M_D(x, z)$. By (3.20), $h_\varepsilon(x)$ vanishes continuous on $\partial D \setminus \{z_0\}$ as so does $M_D(x, z_0)$. On the other hand, by (3.20) and the dominated convergence theorem,

$$\limsup_{\substack{x \to z_0 \\ x \in D}} h_\varepsilon(x) \leq \limsup_{\substack{x \to z_0 \\ x \in D}} c \int_{\partial D \setminus B(z_0, \varepsilon)} \frac{\delta_D(x)}{|x - z|^d} \mu(dz) = 0.$$

Hence h_ε is a bounded non-negative X-harmonic function that vanishes continuously on ∂D. It follows from the definition of harmonicity and the bounded convergence theorem that $h_\varepsilon = 0$ on D, which is a contradiction. Hence $\mu(dz) = \lambda\delta_{\{z_0\}}$ for some $\lambda \geq 0$ and so $h(x) = \lambda M_D(x, z_0)$, proving that $M_D(x, z_0)$ is a minimal X-harmonic function in D. This shows that the minimal Martin boundary of X in D can also be identified with the Euclidean boundary ∂D. □

4 Derivative estimates

In this section, we shall prove the first and second derivatives of $G_D(\cdot, y)$ in $D \setminus \{y\}$.

Lemma 4.1. Suppose D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R_0, Λ_0). Let $C_{15} \geq 1$ be the constant in Theorem 3.7, which depends only on $(d, \lambda_0, \ell, \Lambda_0, R_0, \mathrm{diam}(D))$. Then for any $\lambda \in (0, 1]$,

$$C_{15}^{-1} G^\Delta_{\lambda D}(x, y) \leq G_{\lambda D}(x, y) \leq C_{15} G^\Delta_{\lambda D}(x, y) \quad \text{on } \lambda D \times \lambda D \setminus \mathrm{diag}. \tag{4.1}$$

In particular, there exists a positive constant $C_{16} = C_{16}(d, \lambda_0, \ell)$ such that for any $x_0 \in \mathbb{R}^d$ and $r \in (0, 1]$,

$$C_{16}^{-1} G^\Delta_{B(x_0,r)}(x, y) \leq G_{B(x_0,r)}(x, y) \leq C_{16} G^\Delta_{B(x_0,r)}(x, y) \tag{4.2}$$

for $x \neq y$ in $B(x_0, r)$.

Proof. Let $\lambda \in (0, 1]$ and $X^{(\lambda)}_t := \lambda^{-1} X_{t\lambda^2}$. It is easy to check that the infinitesimal generator of $X^{(\lambda)}$ is

$$\mathcal{L}^{(\lambda)} := \sum_{i,j=1}^d a_{ij}(\lambda x) \frac{\partial^2}{\partial x_i \partial x_j}.$$

Denote by $G^{(\lambda)}_D$ the Green function of $X^{(\lambda)}$ in D. We have

$$G_{\lambda D}(x, y) = \lambda^{2-d} G^{(\lambda)}_D(\lambda^{-1}x, \lambda^{-1}y), \quad x \neq y \in \lambda D. \tag{4.3}$$

Note that for $\lambda \in (0, 1]$, $x \mapsto \{a_{ij}(\lambda x); 1 \leq i, j \leq d\}$ is ℓ-Dini continuous and has the uniform ellipticity constant λ_0. Therefore, (4.1) is obtained by Theorem 3.7 and (4.3). In particular, (4.2) is obtained by (4.1) with $B(x_0, 1)$ and r in place of D and λ. □

In the following, we use Levi's freezing coefficient formula (2.52) in Theorem 2.8 to obtain the upper bound of the first derivative estimate $|\nabla_x G_D(x, y)|$.

Lemma 4.2. There exist $\varepsilon_0 = \varepsilon_0(d, \lambda_0, \ell) \in (0, 1]$ and $C_{17} = C_{17}(d, \lambda_0, \ell)$ such that for any $r \in (0, \varepsilon_0), x_0 \in \mathbb{R}^d$,

$$|\nabla_x G_{B(x_0,r)}(x, y)| \leq C_{17} \frac{G_{B(x_0,r)}(x, y)}{r} \quad \text{for } x \in B(x_0, r/4) \text{ and } y \in B(x_0, r) \setminus B(x_0, 3r/4). \tag{4.4}$$

Proof. For the simplicity of notation, we denote $B(x_0, r)$ by B_r. It follows from Theorem 2.8 and Lemma 2.6 with B_1 and r in place of D and λ that there exists $\varepsilon_0 = \varepsilon_0(d, \lambda_0, \ell) \in (0, 1]$ such that for any $r \in (0, \varepsilon_0)$,

$$G_{B_r}(x, y) = G^{(y)}_{B_r}(x, y) + \int_{B_r} G^{(z)}_{B_r}(x, z) g_{B_r}(z, y)\, dz, \quad x \neq y \in B_r \tag{4.5}$$

and

$$|g_{B_r}(x, y)| \leq \sum_{k=0}^\infty |g^{(k)}_{B_r}(x, y)| \leq c_1 \frac{\ell(|x - y|)}{|x - y|^d} \frac{\delta_{B_r}(y)}{\delta_{B_r}(x)} \quad \text{for } x \neq y \in B_r, \tag{4.6}$$

where c_1 is a constant depending only on (d, λ_0, ℓ). By (2.24) with B_1 and r in place of D and λ, there exists $c_2 = c_2(d, \lambda_0)$ such that for any $r \in (0, \varepsilon_0)$,

$$|\nabla_x G^{(z)}_{B_r}(x,z)| \le c_2|x-z|^{1-d}\left(1 \wedge \frac{\delta_{B_r}(z)}{|x-z|}\right) \quad \text{for } x \neq z \in B_r. \tag{4.7}$$

By (4.6) and (4.7), for $r \in (0, \varepsilon_0), x \in B_{r/4}$ and $y \in B_r \setminus B_{3r/4}$,

$$\begin{aligned}
&\int_{B_{r/2}} |\nabla_x G^{(z)}_{B_r}(x,z)||g_{B_r}(z,y)|\, dz \\
\le& c_1c_2 \int_{B_{r/2}} |x-z|^{1-d} \frac{\ell(|z-y|)}{|z-y|^d} \frac{\delta_{B_r}(y)}{\delta_{B_r}(z)}\, dz \\
\le& c_1c_2c_0^{-1} \frac{\ell(r/4)}{(r/4)^d} \frac{\delta_{B_r}(y)}{r/2} \int_{B_{r/2}} |x-z|^{1-d}\, dz \\
\le& 4^d c_1c_2c_0^{-1}\ell(1)\frac{\delta_{B_r}(y)}{r^d},
\end{aligned} \tag{4.8}$$

where the second inequality is due to (2.36) and $|z-y| \ge r/4$ for $z \in B_{r/2}$ and $y \in B_r \setminus B_{3r/4}$. Moreover, it follows from (4.6) and (4.7) that for $r \in (0, \varepsilon_0), x \in B_{r/4}$ and $y \in B_r \setminus B_{3r/4}$,

$$\begin{aligned}
&\int_{B_r \setminus B_{r/2}} |\nabla_x G^{(z)}_{B_r}(x,z)||g_{B_r}(z,y)|\, dz \\
\le& c_1c_2 \int_{B_r \setminus B_{r/2}} |x-z|^{1-d} \frac{\delta_{B_r}(z)}{|x-z|} \frac{\ell(|z-y|)}{|z-y|^d} \frac{\delta_{B_r}(y)}{\delta_{B_r}(z)}\, dz \\
\le& c_1c_2 4^d \frac{\delta_{B_r}(y)}{r^d} \int_{B_r \setminus B_{r/2}} \frac{\ell(|z-y|)}{|z-y|^d}\, dz \\
\le& c_1c_2 4^d \frac{\delta_{B_r}(y)}{r^d} \int_{|s|\le 1} \omega_d \frac{\ell(s)}{s}\, ds.
\end{aligned} \tag{4.9}$$

Hence, by (4.7)-(4.9) together with (4.5) and the dominated convergence theorem, there exists $c_3 = c_3(d, \lambda_0)$

$$|\nabla_x G_{B(x_0,r)}(x,y)| \le c_3 \frac{\delta_{B(x_0,r)}(y)}{r^d} \quad \text{for } x \in B(x_0, r/4) \text{ and } y \in B(x_0,r) \setminus B(x_0, 3r/4).$$

By Lemma 4.1 and (2.22), there exists $c_4 = c_4(d, \lambda_0, \ell) > 0$ such that $G_{B(x_0,r)}(x,y) \ge c_4 r^{1-d}\delta_{B(x_0,r)}(y)$ for $x \in B_{r/4}$ and $y \in B_r \setminus B_{3r/4}$. Therefore, the desired conclusion is obtained. □

Proposition 4.3. Let $\varepsilon_0 = \varepsilon_0(d, \lambda_0, \ell) \in (0,1]$ and $C_{17} = C_{17}(d, \lambda_0, \ell)$ be the constants in Lemma 4.2. Then for any $x_0 \in \mathbb{R}^d, r \in (0, \varepsilon_0)$ and each non-negative $\mathcal{L}$-harmonic function h in $B(x_0, r)$,

$$|\nabla h(x)| \le C_{17} \frac{h(x)}{r} \quad \text{for } x \in B(x_0, r/4). \tag{4.10}$$

Proof. For the simplicity of notation, we denote $B(x_0, r)$ by B_r. Let h be a non-negative $\mathcal{L}$-harmonic function in B_r with $r \in (0, \varepsilon_0)$. Let $T_{B_{3r/4}} := \inf\{t > 0 : X^{B_r}_t \in B_{3r/4}\}$. Define $h_1(x) := \mathbb{E}_x h(X^{B_r}_{T_{B_{3r/4}}})$. Then $h(x) = h_1(x)$ for $x \in B_{3r/4}$. By Corollary 1 to Theorem 2 in [13], there exists a Radon measure ν on $\overline{B}_{3r/4}$ such that

$$h_1(x) = \int_{\overline{B}_{3r/4}} G_{B_r}(x,z)\, \nu(dz), \quad x \in B_r.$$

Note that $h_1 = h$ is harmonic in $B_{3r/4}$, then ν is supported in $\partial B_{3r/4}$. Hence, there exists a Radon measure ν on $\partial B_{3r/4}$ such that

$$h(x) = h_1(x) = \int_{\partial B_{3r/4}} G_{B_r}(x,y)\,\nu(dy), \quad x \in B_{3r/4}. \tag{4.11}$$

Therefore, the desired conclusion is obtained by Lemma 4.2 and the dominated convergence theorem. □

Theorem 4.4. Suppose D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R_0, Λ_0). There exists $C_{18} = C_{18}(d, \lambda_0, \ell, R_0, \Lambda_0, \mathrm{diam}(D))$ such that

$$|\nabla_x G_D(x,y)| \le \frac{C_{18}}{|x-y|^{d-1}} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right) \quad \text{for } x \ne y \text{ in } D. \tag{4.12}$$

Proof. Let ε_0 be the constant in Proposition 4.3. Fix $x, y \in D$ with $x \ne y$. Note that $G_D(\cdot, y)$ is $\mathcal{L}$-harmonic in $B(x, (\delta_D(x) \wedge |x-y| \wedge \varepsilon_0)/2)$. By Proposition 4.3 with $x_0 = x, r = (\delta_D(x) \wedge |x-y| \wedge \varepsilon_0)/2$ and $h(\cdot) = G_D(\cdot, y)$, Theorems 3.7 and 5.1, there exists a positive constant $C_{18} = C_{18}(d, \lambda_0, \ell, R_0, \Lambda_0, \mathrm{diam}(D))$ such that

$$\begin{aligned}
|\nabla_x G_D(x,y)| &\le 2C_{17}(\delta_D(x) \wedge |x-y| \wedge \varepsilon_0)^{-1} G_D(x,y) \\
&\le 2C_{17}C_{15}|x-y|^{-1} \left(1 \wedge \frac{\delta_D(x)}{|x-y|} \wedge \frac{\varepsilon_0}{\mathrm{diam}(D)}\right)^{-1} G_D^{\triangle}(x,y) \\
&\le C_{18}|x-y|^{1-d} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right).
\end{aligned}$$

□

Remark 4.1. In view of Lemma 4.2, the gradient estimate on $G_{B_r}(x,y)$ (here $B_r = B(x_0, r)$) is derived mainly through the Levi's freezing coefficient formula. However, this method does not work well for the second order derivative estimates on $G_{B_r}(x,y)$. The reason is that if we use the Levi's freezing formula (4.5) on G_{B_r} and follow the argument of Lemma 4.2 similarly, then by (2.33) and (4.6) that for $x \in B_{r/4}$ and $y \in B_r \setminus B_{3r/4}$,

$$\begin{aligned}
&\int_{B_{r/2}} |D_x^2 G_{B_r}^{(z)}(x,z)||g_{B_r}(z,y)|\,dz \\
\le & c \int_{B_{r/2}} |x-z|^{-d} \frac{\delta_{B_r}(z)}{\delta_{B_r}(x)} \frac{\ell(|z-y|)}{|z-y|^d} \frac{\delta_{B_r}(y)}{\delta_{B_r}(z)}\,dz \\
\le & c \frac{\ell(r/4)}{(r/4)^d} \frac{\delta_{B_r}(y)}{\delta_{B_r}(x)} \int_{B_{r/2}} |x-z|^{-d}\,dz = +\infty.
\end{aligned}$$

In the following, we use the second derivative estimate $|\nabla_x^2 G_B(x,y)| \le c|x-y|^{-d}$ on balls from [25] and Lemma 4.5 below to obtain the second order derivative estimates on $G_D(x,y)$.

Let $x_0 \in \mathbb{R}^d$ and let $B_1 := B(x_0, 1)$. The unit ball B_1 has $C^{1,1}$ characteristics (R, Λ) with $R = 1/4$ and $\Lambda = 1$. For $Q_1 \in \partial B_1$ and $r \in (0, 1/4]$, recall from (3.4) that $U_{Q_1,r} \subset B_1$ is a $C^{1,1}$ connect open subset of B_1 with $C^{1,1}$-characteristics $(r/(4L), L/r)$ such that $B_1 \cap B(Q_1, r/2) \subset U_{Q_1,r} \subset B_1 \cap B(Q_1, r)$, where $L > 0$ is a constant that depends only on the dimension d. For each $r \in (0,1]$, $s \in (0, r/4)$ and $Q \in \partial B(x_0, r)$, define $U_{Q,s}^{(r)} := rU_{r^{-1}Q, s/r}$. Note that $r^{-1}Q \in \partial B_1$ and $U_{r^{-1}Q,s/r}$ is a connected $C^{1,1}$ open set in B_1 with characteristics $(s/(4rL), rL/s)$ such that

$$B_1 \cap B(r^{-1}Q, s/2r) \subset U_{r^{-1}Q,s/r} \subset B_1 \cap B(r^{-1}Q, s/r).$$

Then $U_{Q,s}^{(r)}$ is a $C^{1,1}$ connected open set in $B(x_0,r)$ with characteristics $(s/(4L), L/s)$ such that

$$B(x_0,r)\cap B(Q,s/2)\subset U_{Q,s}^{(r)}\subset B(x_0,r)\cap B(Q,s).$$

In the following, we use $U_{Q,s}^{(r)}$ to denote such $C^{1,1}$ subdomain of $B(x_0,r)$.

Lemma 4.5. Suppose that $\{a_{ij};1\le i,j\le d\}$ are C^1 on $\mathbb{R}^d$ and satisfy the conditions (1.2) and (1.3). There exist $M_1=M_1(d,\lambda_0,\ell)>1$ and a positive function $\psi_1\in(M_1^{-1},M_1)$ on $\mathbb{R}^d\times\mathbb{R}^d$ such that for any $r\in(0,1]$, $x_0\in\mathbb{R}^d, Q\in\partial B(x_0,r)$ and $U_{Q,s}^{(r)}$ with $s\in(0,r/4]$,

$$G_{B(x_0,r)}(x,y)=\int_{\partial U_{Q,s}^{(r)}\cap B(x_0,r)}G_{B(x_0,r)}(x,z)\psi_1(y,z)K^{\Delta}_{U_{Q,s}^{(r)}}(y,z)\sigma(dz)$$

holds for $x\in B(x_0,r/2)$ and $y\in U_{Q,s}^{(r)}$, where $K^{\Delta}_{U_{Q,s}^{(r)}}(y,z):=\frac{\partial}{\partial\vec{n}_z}G^{\Delta}_{U_{Q,s}^{(r)}}(y,z)$, $\vec{n}_z$ is the inward unit normal at $z\in\partial U_{Q,s}^{(r)}$, σ is the surface measure of $\partial U_{Q,s}^{(r)}$.

Proof. For the simplicity of notation, we denote $B(x_0,r)$ by B_r. Note that $s^{-1}U_{Q,s}^{(r)}$ is a $C^{1,1}$ connected open set in B_1 with characteristics $(1/(4L),L)$ and $\operatorname{diam}(s^{-1}U_{Q,s}^{(r)})\le 1$, where $L>0$ is a constant that depends only on the dimension d. Then by Lemma 4.1 with $s^{-1}U_{Q,s}^{(r)}$ and s in place of D and λ, there exists a constant $c=c(d,\lambda_0,\ell)$ such that for any $r\in(0,1]$ and $s\in(0,\frac14 r]$,

$$c^{-1}G^{\Delta}_{U_{Q,s}^{(r)}}(x,y)\le G_{U_{Q,s}^{(r)}}(x,y)\le cG^{\Delta}_{U_{Q,s}^{(r)}}(x,y),\quad x,y\in U_{Q,s}^{(r)}.$$

Thus, by a similar proof of Lemma 3.2 and Theorem 3.3, the conclusion can be obtained. □

Suppose D is a bounded $C^{1,1}$ domain. Note that by [25, p.34], for every $y\in D$ and $\varepsilon>0$, $G_D(\cdot,y)$ is in $W^{2,2}(D\setminus B(y,\varepsilon))$ and satisfies $\mathcal{L}G_D(\cdot,y)=0$ on $D\setminus B(y,\varepsilon)$. The C^2 regularity of the strong solution $W^{2,2}(\Omega)$ for $\mathcal{L}u=g$ with g a Dini mean oscillation function in balls and later in bounded smooth domains Ω are studied in [16] and [17]. By [16, Theorem 1.6], $G_D(x,y)$ is C^2 in $x\in D\setminus B(y,\varepsilon)$.

Lemma 4.6. Suppose that $\{a_{ij};1\le i,j\le d\}$ are C^1 on $\mathbb{R}^d$ and satisfy the conditions (1.2) and (1.3). There exists $C_{19}=C_{19}(d,\lambda_0,\ell)$ such that for any $x_0\in\mathbb{R}^d$ and $r\in(0,1]$,

$$|D_x^2G_{B(x_0,r)}(x,y)|\le C_{19}\frac{G_{B(x_0,r)}(x,y)}{r^2}\quad\text{for } x\in B(x_0,r/2)\text{ and } y\in B(x_0,r)\setminus B(x_0,\frac{15}{16}r).\tag{4.13}$$

Proof. For the simplicity of notation, we denote $B(x_0,r)$ by B_r. By Theorem 1.9 in [25], there exists $c_1=c_1(d,\lambda_0,\ell)$ such that

$$|D_x^2G_{B_1}(x,y)|\le c_1|x-y|^{-d}\quad\text{for } x\neq y\in B_1.\tag{4.14}$$

Let $G^{(r)}_{B_1}(x,y)$ be the Green function of the operator $\mathcal{L}$ with $a^{(r)}_{ij}(x)=a_{ij}(rx)$ in place of $a_{ij}(x)$. Then for each $r\in(0,1]$, $a^{(r)}_{ij}(\cdot)$ is ℓ-Dini continuous and has the same uniform ellipticity constant λ_0 as $a_{ij}(\cdot)$. By (4.14) and the scaling formula (4.3), there exists $c_1=c_1(d,\lambda_0,\ell)$ such that for any $r\in(0,1]$,

$$|D_x^2G_{B_r}(x,y)|=|r^{2-d}D_x^2G^{(r)}_{B_1}(r^{-1}x,r^{-1}y)|\le c_1|x-y|^{-d}\quad\text{for } x\neq y\in B_r.\tag{4.15}$$

Let $y \in B_r \setminus B_{\frac{15}{16}r}$ and Q_y be a point on ∂B_r with $|y - Q_y| = \delta_{B_r}(y)$, then $\delta_{B_r}(y) < r/16$. Recall that $U_y := U^{(r)}_{Q_y,\frac{1}{4}r}$ is a connected $C^{1,1}$ open set in B_r with characteristics $(r/(16L), 4L/r)$ such that

$$B_r \cap B(Q_y, r/8) \subset U_y \subset B_r \cap B(Q_y, r/4),$$

where $L > 0$ is a constant that depends only on the dimension d. By Lemma 4.5, (4.15) and the dominated convergence theorem, there exist $M_1 = M_1(d, \lambda_0, \ell) > 1$ and a positive function $\psi_1 \in (M_1, M_1)$ such that for any $r \in (0, 1]$,

$$D^2_x G_{B_r}(x,y) = \int_{\partial U_y \cap B_r} D^2_x G_{B_r}(x,z)\psi_1(y,z) K^{\Delta}_{U_y}(y,z)\sigma(dz), \quad x \in B_{r/2} \quad \text{and} \quad y \in B_r \setminus B_{\frac{15}{16}r}. \tag{4.16}$$

Thus, by (4.16) and (4.15), for any $r \in (0, 1]$, $x \in B_{r/2}$ and $y \in B_r \setminus B_{\frac{15}{16}r}$,

$$\left|D^2_x G_{B_r}(x,y)\right| \le c_1 M_1 4^d r^{-d} \int_{\partial U_y \cap B_r} K^{\Delta}_{U_y}(y,z)\sigma(dz). \tag{4.17}$$

By (2.22) with $r^{-1}U_y$ and r in place of D and λ, there exists $c_2 = c_2(d)$ such that

$$K^{\Delta}_{U_y}(y,z) = \lim_{u \to z} \frac{G^{\Delta}_{U_y}(y,u)}{\delta_{U_y}(u)} \le c_2 \frac{\delta_{U_y}(y)}{|y-z|^d} \le c_2 16^d \frac{\delta_{B_r}(y)}{r^d}, \quad \text{for} \quad z \in \partial U_y \cap B_r. \tag{4.18}$$

Therefore, by (4.17) and (4.18), there exists $c_3 = c_3(d, \lambda_0, \ell)$ such that

$$|D^2_x G_{B_r}(x,y)| \le c_3 \frac{\delta_{B_r}(y)}{r^{d+1}} \quad \text{for } x \in B_{r/2} \text{ and } y \in B_r \setminus B_{\frac{15}{16}r}.$$

By Lemma 4.1 and (2.22), there exists $c_4 = c_4(d, \lambda_0, \ell) > 0$ such that $G_{B_r}(x,y) \ge c_4 \frac{\delta_{B_r}(y)}{r^{d-1}}$ for $x \in B_{r/2}$ and $y \in B_r \setminus B_{\frac{15}{16}r}$. Thus, the desired conclusion is obtained. □

Proposition 4.7. Suppose that $\{a_{ij}; 1 \le i,j \le d\}$ are C^1 on $\mathbb{R}^d$ and satisfy the conditions (1.2) and (1.3). Let $C_{19} = C_{19}(d, \lambda_0, \ell)$ be the constant in Lemma 4.6. For any $x_0 \in \mathbb{R}^d, r \in (0, 1]$ and each non-negative $\mathcal{L}$-harmonic function h in $B(x_0, r)$,

$$|D^2_x h(x)| \le C_{19} \frac{h(x)}{r^2}, \quad \text{for } x \in B(x_0, r/2).$$

Proof. For the simplicity of notation, we denote $B(x_0, r)$ by B_r. Let h be a non-negative $\mathcal{L}$-harmonic function in B_r with $r \in (0, 1]$. By a similar argument of (4.11), there exists a Radon measure ν on $\partial B_{15r/16}$ such that

$$h(x) = \int_{\partial B_{15r/16}} G_{B_r}(x,y)\, \nu(dy) \quad \text{for } x \in B_{15r/16}.$$

Hence, the desired conclusion follows from Lemma 4.6 and the dominated convergence theorem. □

Theorem 4.8. Suppose D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ with characteristics (R_0, Λ_0). There is a positive constant $C_{20} = C_{20}(d, \lambda_0, \ell, R_0, \Lambda_0, \mathrm{diam}(D))$ such that

$$|D^2_x G_D(x,y)| \le \frac{C_{20}}{|x-y|^d} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right) \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)^{-1} \quad \text{for } x \ne y \text{ in } D. \tag{4.19}$$

Proof. Let $\phi \in C_c^\infty(\mathbb{R}^d)$ with $\phi \geq 0$, $\mathrm{supp}[\phi] \subset B(0,1)$ and $\int_{\mathbb{R}^d} \phi(x)\,dx = 1$, For each integer $k \geq 1$, define $\phi_k(x) := k^d \phi(kx)$ and $a_{ij}^{(k)}(x) := \phi_k * a_{ij}(x) := \int_{\mathbb{R}^d} \phi_k(x-y) a_{ij}(y)dy$. Then $a_{ij}^{(k)} \in C^\infty(\mathbb{R}^d)$ satisfying the conditions (1.2) and (1.3) with the same ellipticity constant $\lambda_0 \geq 1$ and Dini modulo of continuity function ℓ, and $a_{ij}^{(k)}$ converges uniformly to a_{ij} on any compact set of $\mathbb{R}^d$. Denote by $\mathcal{L}^{(k)}$ the non-divergence operator of (1.1) with diffusion coefficients $a_{ij}^{(k)}$ in place of a_{ij}. Let $X^{(k)}$ be the diffusion process having $\mathcal{L}^{(k)}$ as its infinitesimal generator, and $G_D^{(k)}(x,y)$ its Green function on D.

Note that $G_D^{(k)}(\cdot,y)$ is $\mathcal{L}^{(k)}$-harmonic in $B(x,(\delta_D(x)\wedge|x-y|\wedge 1)/2)$. By Proposition 4.7 with $h(\cdot) = G_D^{(k)}(\cdot,y)$ and $r = (\delta_D(x)\wedge|x-y|\wedge 1)/2$, Theorems 3.7 and 5.1, there exists a positive constant $C_{20} = C_{20}(d,\lambda_0,\ell,R_0,\Lambda_0,\mathrm{diam}(D))$ such that for each $k \geq 1$,

$$\begin{aligned}|D_x^2 G_D^{(k)}(x,y)| &\leq 4C_{19}(\delta_D(x)\wedge|x-y|\wedge 1)^{-2} G_D^{(k)}(x,y)\\ &\leq C_{20}|x-y|^{-d}\left(1\wedge\frac{\delta_D(y)}{|x-y|}\right)\left(1\wedge\frac{\delta_D(x)}{|x-y|}\right)^{-1}.\end{aligned}$$

Let

$$F(x,y) := |x-y|^{-d}\left(1\wedge\frac{\delta_D(y)}{|x-y|}\right)\left(1\wedge\frac{\delta_D(x)}{|x-y|}\right)^{-1}, \quad x \neq y \in D.$$

For any non-negative $\phi, f \in C_c^2(D)$, we have

$$\left|\int_{D\times D} D_x^2 G_D^{(k)}(x,y)\phi(x)f(y)dxdy\right| \leq C_{20}\int_{D\times D} F(x,y)\phi(x)f(y)dxdy. \tag{4.20}$$

On the other hand, by (3.15),

$$\begin{aligned}&\lim_{k\to\infty}\int_{D\times D} D_x^2 G_D^{(k)}(x,y)\phi(x)f(y)dxdy\\ =\ &\lim_{k\to\infty}\int_D D_x^2\phi(x)\left(\int_D G_D^{(k)}(x,y)f(y)dy\right)dx\\ =\ &\int_D D_x^2\phi(x)\left(\int_D G_D(x,y)f(y)dy\right)dx\\ =\ &\int_{D\times D} D_x^2 G_D(x,y)\phi(x)f(y)dxdy.\end{aligned}$$

Then by letting $k\to\infty$ in (4.20), it follows that

$$\begin{aligned}-C_{20}\int_{D\times D}F(x,y)\phi(x)f(y)dxdy &\leq \int_{D\times D} D_x^2 G_D(x,y)\phi(x)f(y)dxdy\\ &\leq C_{20}\int_{D\times D}F(x,y)\phi(x)f(y)dxdy.\end{aligned} \tag{4.21}$$

For any fixed $x_0 \neq y_0 \in D$, let $\varepsilon > 0$ be so that

$$B(x_0,2\varepsilon)\subset D, \quad B(y_0,2\varepsilon)\subset D \quad\text{and}\quad B(x_0,2\varepsilon)\cap B(y_0,2\varepsilon) = \emptyset.$$

As we noted earlier that by [25, p.34], for each $y \in B(y_0,\varepsilon)$, $x \mapsto G_D(x,y)$ is in $W^{2,2}(B(x_0,2\varepsilon))$ and $\mathcal{L}_x G_D(x,y) = 0$ on $B(x_0,2\varepsilon)$. Thus by [22, Theorem 9.11], there exists a positive constant $c_2 = c_2(d,\lambda_0,\ell,\varepsilon)$ such that for any $y_1,y_2 \in B(y_0,\varepsilon)$,

$$\|G_D(\cdot,y_1)-G_D(\cdot,y_2)\|_{W^{2,2}(B(x_0,\varepsilon))} \leq C\|G_D(\cdot,y_1)-G_D(\cdot,y_2)\|_{L^2(B(x_0,2\varepsilon))}. \tag{4.22}$$

Since by Theorem 2.3, $G_D(x,y) \leq C|x-y|^{2-d}$ and $G_D(x,y)$ is jointly continuous off the diagonal, $G_D(\cdot,y)$ is continuous in $L^2(B(x_0,\varepsilon))$ in $y \in B(y_0,\varepsilon)$ by the dominated convergence theorem. Consequently, $G_D(\cdot,y)$ is continuous in $W^{2,2}(B(x_0,\varepsilon))$ in $y\in B(y_0,\varepsilon)$ by (4.22). Thus for any $\phi\in C_c(B(x_0,\varepsilon))$, $\int_D D_x^2G_D(x,y)\phi(x)dx$ is a continuous function in $y\in B(y_0,\varepsilon)$. By this continuity, we deduce from (4.21) by taking all possible non-negative $f\in C_c^2(B(y_0,\varepsilon)$ that for every non-negative $\phi\in C_c(B(x_0,\varepsilon))$ and for every $y\in B(y_0,\varepsilon)$,

$$-C_{20}\int_D F(x,y)\phi(x)dx\le\int_D D_x^2G_D(x,y)\phi(x)dx\le C_{20}\int_D F(x,y)\phi(x)dx.$$

Thus we have for every $y\in B(y_0,\varepsilon)$, $-C_{20}F(x,y)\le D_x^2G_D(x,y)\le C_{20}F(x,y)$ for a.e. and hence for every $x\in B(x_0,\varepsilon)$ as $x\mapsto G_D(x,y)$ is C^2 in $D\setminus\{y\}$. This in particular shows that

$$|D_x^2G_D(x_0,y_0)|\le C_{20}F(x_0,y_0),$$

establishing the pointwise second derivative estimate for $G_D(x,y)$. □

Using scaling, we can remove the restriction of $\varepsilon_0\in(0,1]$ from Proposition 4.3 and also give the interior second derivative estimate for non-negative $\mathcal{L}$-harmonic functions.

Theorem 4.9. For any $R>0$, there exists a constant $C_{21}=C_{21}(d,\lambda_0,\ell,R)>0$ so that for any $x_0\in\mathbb{R}^d$, $r\in(0,R]$ and every non-negative $\mathcal{L}$-harmonic function h in $B_r:=B(x_0,r)$,

$$|\nabla h(x)|\le C_{21}h(x)/r\quad\text{and}\quad|D_x^2h(x)|\le C_{21}h(x)/r^2\quad\text{for all } x\in B(x_0,r/2).\tag{4.23}$$

In particular, if h is a non-negative $\mathcal{L}$-harmonic function in an open set $D\subset\mathbb{R}^d$, then there is a constant $c=c(d,\lambda_0,\ell,\mathrm{diam}(D))>0$ so that

$$|\nabla h(x)|\le ch(x)/\delta_D(x)\quad\text{and}\quad|D_x^2h(x)|\le ch(x)/\delta_D(x)^2\quad\text{for all } x\in D.\tag{4.24}$$

Proof. For any $\lambda>0$, let $\mathcal{L}^{(\lambda)}$ be the non-divergence form operator (1.1) but with $a_{ij}^{(\lambda)}(x):=a_{ij}(\lambda x)$ in place of $a_{ij}(x)$ for $1\le i,j\le d$. For any open set $U\subset\mathbb{R}^d$, denote the Green function of $\mathcal{L}^{(\lambda)}$ in U by $G_U^{(\lambda)}(x,y)$. Clearly, $\{a_{ij}^{(\lambda)}(x);1\le i,j\le d\}$ satisfies (1.2) and are ℓ_λ-Dini continuous with $\ell_\lambda(r):=\ell(\lambda r)$. By Theorems 4.4 and 4.8, there is a constant $c_1=c_1(d,\lambda_0,\ell,R)>0$ so that for any $\lambda\in(0,R],x_0\in\mathbb{R}^d$ and any $B=B(x_0,1)\subset\mathbb{R}^d$,

$$\begin{cases}|\nabla_xG_B^{(\lambda)}(x,y)|\le\frac{c_1}{|x-y|^{d-1}}\left(1\wedge\frac{\delta_B(y)}{|x-y|}\right)\\ |D_x^2G_B^{(\lambda)}(x,y)|\le\frac{c_1}{|x-y|^{d}}\left(1\wedge\frac{\delta_B(y)}{|x-y|}\right)\left(1\wedge\frac{\delta_B(x)}{|x-y|}\right)^{-1}\end{cases}\quad\text{for } x\neq y \text{ in } B.\tag{4.25}$$

By the scaling identity (4.3) for any $x_0\in\mathbb{R}^d$ and $r\in(0,R]$,

$$G_{B(x_0,r)}(x,y)=r^{2-d}G^{(r)}_{B(x_0/r,1)}(x/r,y/r)\quad\text{for any } x\neq y\in B(x_0,r).\tag{4.26}$$

This together with (4.25) implies that there is a constant $c_2=c_2(d,\lambda_0,\ell,R)>0$ so that for any $x_0\in\mathbb{R}^d$ and $r\in(0,R]$,

$$\begin{cases}|\nabla_xG_{B(x_0,r)}(x,y)|\le\frac{c_2}{|x-y|^{d-1}}\left(1\wedge\frac{\delta_{B(x_0,r)}(y)}{|x-y|}\right)\\ |D_x^2G_{B(x_0,r)}(x,y)|\le\frac{c_2}{|x-y|^{d}}\left(1\wedge\frac{\delta_{B(x_0,r)}(y)}{|x-y|}\right)\left(1\wedge\frac{\delta_{B(x_0,r)}(x)}{|x-y|}\right)^{-1}\end{cases}\quad\text{for } x\neq y \text{ in } B(x_0,r).\tag{4.27}$$

This combined with Theorem 3.7 shows that there is a constant $c_3=c_3(d,\lambda_0,\ell,R)>0$ so that for any $x_0\in\mathbb{R}^d$ and $r\in(0,R]$,

$$|\nabla_xG_{B(x_0,r)}(x,y)|\le c_3\frac{G_{B(x_0,r)}(x,y)}{r}\quad\text{and}\quad|D_x^2G_{B(x_0,r)}(x,y)|\le c_3\frac{G_{B(x_0,r)}(x,y)}{r^2}\tag{4.28}$$

for any $x\in B(x_0,r/2)$ and $y\in B(x_0,r)\setminus B(x_0,3r/4)$. The estimates in (4.23) now follows from this, (4.11) and the dominated convergence theorem, while the estimates (4.24) follows from (4.23) by taking $r=\delta_D(x)$. □

Theorems 4.9 extends the classical interior Schauder estimate (see, e.g., [22, Theorem 6.2]) for harmonic functions of non-divergence form operators $\mathcal{L}$ with Hölder continuous coefficients to Dini continuous coefficients.

5 Appendix

In this section, we show that the constant C in (1.13) can be chosen to depend only on d, Λ_0, R_0 and $\mathrm{diam}(D)$.

Theorem 5.1. Suppose that $D \subset \mathbb{R}^d$ is a bounded $C^{1,1}$ domain with characteristics (R_0, Λ_0) and $d \geq 3$. There is a constant $C = C(d, \Lambda_0, R_0, \mathrm{diam}(D)) > 1$ such that for any $x \neq y$ in D,

$$\frac{C^{-1}}{|x-y|^{d-2}}\left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)\left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right) \leq G_D^{\Delta}(x,y) \leq \frac{C}{|x-y|^{d-2}}\left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)\left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right). \tag{5.1}$$

Proof. First we mention that a $C^{1,1}$-domain D with $C^{1,1}$-characteristics (R_0, Λ_0) is a non-tangential accessible domain so in particular it satisfies the Harnack chain property in the following sense [26, p.93]: there is a constant $M > 1$ so that for any $x, y \in D$ with $|x-y| \leq K[\delta_D(x) \wedge \delta_D(y)]$, there exist at most N number of balls $B(a_i, r_i), 0 \leq i \leq N$, in D so that $a_0 = x, a_N = y$ and $B(a_i, r_i) \cap B(a_{i+1}, r_{i+1}) \neq \varnothing$ and $M^{-1} r_i \leq d(B(a_i, r_i), D^c) \leq M r_i$ for $0 \leq i \leq N-1$, where the constant M depends on (R_0, Λ_0), and N depends only on (R_0, Λ_0) and K only. See the proof of [26, Proposition 3.6] for the verification of the above statement, which is given for more general Zygmund domains. The Harnack chain property will be used several times in this proof.

(I) From the proof in Lemma 3.2 and Theorem 3.3 in [23], one can see that the constant in the upper bound of (5.1) depends on $(d, \Lambda_0, R_0, \mathrm{diam}(D))$. For the reader's convenience, we spell out the details here.

It is well known that any bounded $C^{1,1}$ domain D satisfies a uniform interior and exterior ball condition, that is, there exists a constant $r_0 > 0$ depending on the $C^{1,1}$-characteristics (R_0, Λ_0) of D such that for any $z \in \partial D$ and $r \in (0, r_0)$, there exist two balls $B_1^z(r)$ and $B_2^z(r)$ of radius r such that $B_1^z(r) \subset \overline{D}^c, B_2^z(r) \subset D$, and $\{z\} = \partial B_1^z(r) \cap \partial B_2^z(r)$. Let $x, y \in D$ with $x \neq y$. We consider three cases of x and y.

(1) Suppose that $\delta_D(x) < |x-y|/8 < r_0$. Denote by z_x the point on the boundary ∂D so that $|x - z_x| = \delta_D(x)$. Let $r = |x-y|/8$ and $x^* \in D^c$ be such that $B(x^*, r) = B_1^{z_x}(r)$. For the simplicity of notation, we denote $B_r := B(x^*, r)$. Let $E_r := B_{2r} \setminus B_r$ and let u_r be the unique solution of the Dirichlet boundary problem $\Delta u_r = 0$ in E_r with $u_r = 0$ on ∂B_r and $u_r = 1$ on ∂B_{2r}. By [23, Lemma 3.2], there is a constant $c_1 = c_1(d, r_0) > 1$ such that

$$|\nabla_x u_r(z)| \leq \frac{c_1}{r} \quad \text{for every } z \in E_r.$$

Note that $z_x \in \partial B_r$ and

$$u_r(x) = u_r(x) - u_r(z_x) \leq |x - z_x| \sup_{\theta \in (0,1)} |\nabla_x u(x + \theta(x - z_x))| \leq c_1 \frac{\delta_D(x)}{r}. \tag{5.2}$$

Set $G_D^{\Delta}(\cdot, y) = 0$ on D^c. Note that $G_D^{\Delta}(\cdot, y)$ is harmonic in $D \cap E_r$ and vanishes continuously on ∂D. Clearly, there is a constant $C = C(d) > 0$ such that $G_D^{\Delta}(x,y) \leq G_{\mathbb{R}^d}^{\Delta}(x,y) = C|x-y|^{2-d}$. As $|z-y| \geq |x-y| - |x-z| \geq |x-y| - 4r \geq |x-y|/2$ for $z \in \partial B_{2r}$ and $u_r = 1$ on ∂B_{2r}, we have

$$G_D^{\Delta}(z,y) \leq C 2^{d-2} |x-y|^{2-d} u_r(z) \quad \text{for every } z \in \partial(D \cap E_r).$$

Since $G_D^\Delta(\cdot,y)$ and $C2^{d-2}|x-y|^{2-d}u_r(\cdot)$ are both Δ-harmonic functions in $D\cap E_r$ and $x\in D\cap E_r$, one concludes by the maximum principle that $G_D^\Delta(x,y)\le C2^{d-2}|x-y|^{2-d}u_r(x)$. Hence by (5.2),

$$\begin{aligned} G_D^\Delta(x,y)\le C2^{d-2}|x-y|^{2-d}u_r(x)&\le C2^{d-2}|x-y|^{2-d}\frac{8c_1\delta_D(x)}{|x-y|}\\ &\le 2^{d+1}Cc_1|x-y|^{2-d}\left(1\wedge\frac{\delta_D(x)}{|x-y|}\right). \end{aligned}$$

(2) Suppose that $\delta_D(x)<r_0\le|x-y|/8$. By a similar argument as above in (1) but taking $r=r_0$ instead, we have

$$\begin{aligned} G_D^\Delta(x,y)&\le C2^{d-2}|x-y|^{2-d}u_r(x)\le C2^{d-2}c_1\frac{\delta_D(x)}{r_0}|x-y|^{2-d}\\ &\le 2^{d-2}Cc_1\frac{\mathrm{diam}(D)}{r_0}|x-y|^{2-d}\left(1\wedge\frac{\delta_D(x)}{|x-y|}\right). \end{aligned}$$

(3) The remaining case is $\delta_D(x)\ge\min\{|x-y|/8,r_0\}$. In this case,

$$G_D^\Delta(x,y)\le G_{\mathbb{R}^d}^\Delta(x,y)=C|x-y|^{2-d}\le 8C\frac{\mathrm{diam}(D)}{r_0}|x-y|^{2-d}\left(1\wedge\frac{\delta_D(x)}{|x-y|}\right).$$

Combining (1)-(3) gives

$$G_D^\Delta(x,y)\le 2^{d+1}Cc_1\frac{\mathrm{diam}(D)}{r_0}|x-y|^{2-d}\left(1\wedge\frac{\delta_D(x)}{|x-y|}\right)\quad\text{for any } x\neq y\in D. \tag{5.3}$$

By the symmetry of $G_D^\Delta(x,y)$ in (x,y), we have

$$G_D^\Delta(x,y)\le 2^{d+1}Cc_1\frac{\mathrm{diam}(D)}{r_0}|x-y|^{2-d}\left(1\wedge\frac{\delta_D(y)}{|x-y|}\right)\quad\text{for any } x\neq y\in D. \tag{5.4}$$

Repeating the arguments in (1)-(3) but with (5.4) in place of the bound $G_D^\Delta(x,y)\le G_{\mathbb{R}^d}^\Delta(x,y)$ there, we get

$$G_D^\Delta(x,y)\le\left(2^{d+1}Cc_1^2\frac{\mathrm{diam}(D)}{r_0}\right)^2|x-y|^{2-d}\left(1\wedge\frac{\delta_D(x)}{|x-y|}\right)\left(1\wedge\frac{\delta_D(y)}{|x-y|}\right)\quad\text{for any } x\neq y\in D.$$

(II) Next we show that the lower bound for G_D^Δ in (5.1) holds with C^{-1} depending only on $(d,\Lambda_0,R_0,\mathrm{diam}(D))$. Recall $r_0>0$ is the constant in (I) for the uniform interior and exterior ball condition of the $C^{1,1}$-domain D. By checking carefully the proof of Theorem 1 in [39], it follows from the paragraph under Theorem 1 in page 316 and (6), (17)-(18) in [39] that when $|x-y|\le\max\{\delta_D(x)/2,\delta_D(y)/2\}$ or when $|x-y|\ge\max\{\delta_D(x)/2,\delta_D(y)/2\}$ with $|x-y|\le\dfrac{r_0}{10(1+r_0\Lambda_0)}$, the constant C^{-1} in the lower bound of (5.1) depends on (d,r_0). So it remains to consider the case of $x\neq y\in D$ with $|x-y|\ge\max\{\delta_D(x)/2,\delta_D(y)/2\}$ with $|x-y|>\dfrac{r_0}{10(1+r_0\Lambda_0)}$. For simplicity, let $r_1:=\dfrac{r_0}{10(1+r_0\Lambda_0)}$. Without loss of generality, we assume $\delta_D(y)\le\delta_D(x)$. We consider its three possible scenarios.

(1) Suppose that $\delta_D(y)\le\delta_D(x)<r_1/8$. Let z_x be a point on the boundary ∂D so that $|x-z_x|=\delta_D(x)$. Since D is a bounded $C^{1,1}$ domain, there exist $\kappa\in(0,1)$ depending on (R_0,Λ_0) and a point x_0 on $D\cap\partial B(z_x,r_1/8)$ so that $\kappa r_1/8<\delta_D(x_0)<r_1/8$. Note that $|x-y|>r_1$, then $|y-z_x|>|y-x|-|x-z_x|>r_1-r_1/8=\frac{7}{8}r_1$. Hence, $G_D^\Delta(\cdot,y)$ is harmonic in $D\cap B(z_x,r_1/2)$. By the scale invariant boundary Harnack principle (BHP

in abbreviation) of the Laplacian Δ (see [10, Theorem 1.4]), there is $c_2 = c_2(d, R_0, \Lambda_0)$ such that

$$\frac{G_D^\Delta(x,y)}{G_D^\Delta(x_0,y)} \geq c_2 \frac{\delta_D(x)}{r_1}.$$

Similarly, let z_y be a point on the boundary ∂D so that $|y - z_y| = \delta_D(y)$. Let y_0 be a point on $D \cap \partial B(z_y, r_1/8)$ so that $\kappa r_1/8 < \delta_D(y_0) < r_1/8$. Note that $|x-y| > r_1$, then $|x_0 - y| > |x-y| - |x-x_0| > r_1 - r_1/4 = 3r_1/4$. Thus $|x_0 - z_y| > |x_0 - y| - |y - z_y| > 3r_1/4 - r_1/8 = \frac{5}{8}r_1$. So $G_D^\Delta(x_0, \cdot)$ is harmonic in $D \cap B(z_y, r_1/2)$. By the scale invariant BHP of Δ,

$$\frac{G_D^\Delta(x_0,y)}{G_D^\Delta(x_0,y_0)} \geq c_2 \frac{\delta_D(y)}{r_1}.$$

By [39, Lemma 3], there is a constant $c_3 = c_3(d)$ such that for any $r > 0$,

$$G^\Delta_{B(0,r)}(w,z) \geq c_3 |w-z|^{2-d}, \quad w, z \in B(0, r/2). \tag{5.5}$$

Note that $(\delta_D(x_0) \wedge \delta_D(y_0)) \geq \kappa r_1/8$. Thus by the Harnack chain property of D, the Harnack inequality for Δ and (5.5), there exists a constant $c_4 = c_4(d, R_0, \Lambda_0, \mathrm{diam}(D))$ so that

$$G_D^\Delta(x_0,y_0) \geq c_4 |x_0 - y_0|^{2-d} \geq c_4 (\mathrm{diam}(D))^{2-d}.$$

Hence, combining the above inequalities, we have

$$\begin{aligned} G_D^\Delta(x,y) &\geq \frac{G_D^\Delta(x,y)}{G_D^\Delta(x_0,y)} \frac{G_D^\Delta(x_0,y)}{G_D^\Delta(x_0,y_0)} G_D^\Delta(x_0,y_0) \geq c_2^2 c_4 \frac{\delta_D(x)}{r_1}\frac{\delta_D(y)}{r_1} (\mathrm{diam}(D))^{2-d} \\ &\geq c_2^2 c_4 \frac{(\mathrm{diam}(D))^{2-d}}{r_1^{2-d}} |x-y|^{2-d} \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)\left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right), \end{aligned}$$

where the last inequality is due to that $r_1 < |x-y|$.

(2) Suppose that $\delta_D(y) \leq r_1/8 \leq \delta_D(x)$. Note that $(\delta_D(x) \wedge \delta_D(y_0)) \geq \kappa r_1/8$. Thus by the Harnack chain property of D and Harnack inequality again, there exists $c_5 = c_5(d, R_0, \Lambda_0, \mathrm{diam}(D))$ such that

$$G_D^\Delta(x,y_0) \geq c_5 |x-y_0|^{2-d} \geq c_5 \frac{\mathrm{diam}(D)^{2-d}}{r_1^{2-d}} |x-y|^{2-d},$$

where the last inequality is due to that $|x - y_0| < \mathrm{diam}(D) \leq \dfrac{\mathrm{diam}(D)}{r_1}|x-y|$. Note that $|x - z_y| > |x-y| - |y-z_y| > r_1 - r_1/8 = 7r_1/8$, then $G_D^\Delta(x, \cdot)$ is harmonic in $D \cap B(z_y, r_1/2)$. Then by the scale invariant BHP of Δ, we have

$$\begin{aligned} G_D^\Delta(x,y) &\geq \frac{G_D^\Delta(x,y)}{G_D^\Delta(x,y_0)} G_D^\Delta(x,y_0) \geq c_2 c_5 \frac{\mathrm{diam}(D)^{2-d}}{r_1^{2-d}} \frac{\delta_D(y)}{r_1} |x-y|^{2-d} \\ &\geq \frac{c_2 c_5 \mathrm{diam}(D)^{2-d}}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)\left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right), \end{aligned}$$

where the last inequality is due to that $r_1 < |x-y|$.

(3) The remaining case is $r_1/8 \leq \delta_D(y) \leq \delta_D(x)$. In this case, note that $|x-y| < \mathrm{diam}(D) < (8/r_1)\mathrm{diam}(D)(\delta_D(x) \wedge \delta_D(y))$. By the Harnack chain property of D and Harnack inequality, there exists $c_6 = c_6(d, R_0, \Lambda_0, \mathrm{diam}(D))$ such that

$$G_D^\Delta(x,y) \geq c_6 |x-y|^{2-d} \geq \frac{c_6}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)\left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right).$$

This completes the proof of the lower bound and hence the theorem. □

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