

## Local time penalizations with various clocks for Lévy processes\*

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### Abstract

Several long-time limit theorems of one-dimensional Lévy processes weighted and normalized by functions of the local time are studied. The long-time limits are taken via certain families of random times, called clocks: exponential clock, hitting time clock, two-point hitting time clock and inverse local time clock. The limit measure can be characterized via a certain martingale expressed by an invariant function for the process killed upon hitting zero. The limit processes may differ according to the choice of the clocks when the original Lévy process is recurrent and of finite variance.

**Keywords:** one-dimensional Lévy process; limit theorem; penalization; conditioning.

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## 1 Introduction

Roynette–Vallois–Yor ([27, 26] see also [28, 29]) have studied the limit distribution for a Brownian motion, which they called a *penalization problem*, as follows. Let  $B = (B_t, t \geq 0)$  be a standard Brownian motion and  $L = (L_t, t \geq 0)$  denote its local time at 0. Then, for any positive integrable function  $f$  and any bounded adapted functional  $F_t$ , it holds that

$$\lim_{s \rightarrow \infty} \frac{\mathbb{P}[F_t f(L_s)]}{\mathbb{P}[f(L_s)]} = \mathbb{P} \left[ F_t \frac{M_t}{M_0} \right] =: \mathbb{Q}[F_t],$$

where  $M = (M_t, t \geq 0)$  is the martingale given by

$$M_t = f(L_t) |B_t| + \int_0^\infty f(L_t + u) du, \quad t \geq 0.$$

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Under the penalized probability measure  $\mathbb{Q}$ , the total local time  $L_\infty$  is finite, and in fact, a sample path behaves as the concatenation of a Brownian bridge and a three-dimensional Bessel process; see [27]. In particular,  $\mathbb{Q}$  is singular to  $\mathbb{P}$ .

This result for a Brownian motion was generalized to many other processes. In particular, we refer to Debs [7] for random walks, Najnudel–Roynette–Yor [17] for Markov chains and Bessel processes, Yano–Yano–Yor [42] for symmetric stable processes, Salminen–Vallois [30] and Profeta [23, 24] for linear diffusions. Most of these results were obtained basically under the assumption of some regular variation condition. Profeta–Yano–Yano [25] developed a general theory for one-dimensional diffusions by adopting a random clock approach. They studied the long-time limit of the form

$$\lim_{\tau \rightarrow \infty} \frac{\mathbb{P}[F_t f(L_\tau)]}{\mathbb{P}[f(L_\tau)]},$$

where  $\tau = (\tau_\lambda)$  is a certain parametrized family of random times, which they called a clock. Such a random clock approach already appeared in the problem of conditioning to avoid zero, which is a special case of our penalization with  $f(u) = 1_{\{u=0\}}$ , or in the problem of conditioning to stay positive/negative. For example, we refer to Knight [14] for Brownian motions, Chaumont [4], Chaumont–Doney [5, 6] and Doney [8, 15, Section 8] for Lévy processes conditioned to stay positive, Yano–Yano [40] for diffusions and Pantí [18] for Lévy processes conditioned to avoid zero.

Let  $X = (X_t, t \geq 0)$  be a one-dimensional Lévy process and let  $T_A$  denote the hitting time of a Borel set  $A \subset \mathbb{R}$  for  $X$ , i.e.,

$$T_A = \inf\{t > 0: X_t \in A\}$$

and we write  $T_a = T_{\{a\}}$  simply for the hitting time of a point  $a \in \mathbb{R}$ . Let  $(\eta_u^a)$  denote the right-continuous inverse of the local time at a point  $a \in \mathbb{R}$ . We adopt the random clock approach for the following four clocks:

- (i) exponential clock:  $\tau = (e_q)$  with  $q \rightarrow 0+$ ;
- (ii) hitting time clock:  $\tau = (T_a)$  with  $a \rightarrow \pm\infty$ ;
- (iii) two-point hitting time clock:  $\tau = (T_a \wedge T_{-b})$  with  $a \rightarrow \infty$  and  $b \rightarrow \infty$ ;
- (iv) inverse local time clock:  $\tau = (\eta_u^a)$  with  $a \rightarrow \pm\infty$  or with  $u \rightarrow \infty$ .

### 1.1 Main results

Let  $(X, \mathbb{P}_x)$  denote the canonical representation of a Lévy process starting from  $x$  on the càdlàg path space  $\mathcal{D}$  and set  $\mathbb{P} = \mathbb{P}_0$ . For  $t \geq 0$ , we denote by  $\mathcal{F}_t^X = \sigma(X_s, 0 \leq s \leq t)$  the natural filtration of  $X$  and write  $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s^X$ . We have

$$\mathbb{P}[e^{i\lambda X_t}] = e^{-t\Psi(\lambda)}, \quad t \geq 0, \lambda \in \mathbb{R},$$

where  $\Psi(\lambda)$  denotes the characteristic exponent of  $X$  given by the Lévy–Khintchine formula

$$\Psi(\lambda) = i\nu\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{\mathbb{R}} (1 - e^{i\lambda x} + i\lambda x 1_{\{|x|<1\}})\nu(dx)$$

for some constants  $\nu \in \mathbb{R}$  and  $\sigma \geq 0$  and some measure  $\nu$  on  $\mathbb{R}$  (called the Lévy measure) which satisfies  $\nu(\{0\}) = 0$  and

$$\int_{\mathbb{R}} (x^2 \wedge 1)\nu(dx) < \infty.$$

We denote the real and imaginary parts of  $\Psi(\lambda)$  by

$$\begin{aligned} \theta(\lambda) &= \operatorname{Re} \Psi(\lambda) = \frac{1}{2} \sigma^2 \lambda^2 + \int_{\mathbb{R}} (1 - \cos \lambda x) \nu(dx), \\ \omega(\lambda) &= \operatorname{Im} \Psi(\lambda) = v \lambda + \int_{\mathbb{R}} (\lambda x 1_{\{|x| < 1\}} - \sin \lambda x) \nu(dx). \end{aligned} \tag{1.1}$$

Note that  $\theta(\lambda) \geq 0$  for  $\lambda \in \mathbb{R}$ ,  $\theta(\lambda)$  is even and  $\omega(\lambda)$  is odd. For more details of the notation of this section, see Section 2. Throughout this paper except Sections 2, 9 and 10, we always assume  $(X, \mathbb{P})$  is recurrent, i.e.,

$$\mathbb{P} \left[ \int_0^\infty 1_{\{|X_t - a| < \varepsilon\}} dt \right] = \infty, \quad \text{for all } a \in \mathbb{R} \text{ and } \varepsilon > 0, \tag{1.2}$$

and assume the following:

**(A)** For each  $q > 0$ , it holds that

$$\int_0^\infty \left| \frac{1}{q + \Psi(\lambda)} \right| d\lambda < \infty.$$

Note that, we say  $(X, \mathbb{P})$  is transient if (1.2) does not hold. Under the assumption **(A)**, the process  $(X, \mathbb{P})$  is recurrent if and only if  $(X, \mathbb{P})$  is point recurrent, i.e.,

$$\mathbb{P}(T_a < \infty) = 1, \quad \text{for all } a \in \mathbb{R};$$

see Subsection 2.2. The assumption **(A)** implies that the  $q$ -resolvent density  $r_q$  exists for  $q > 0$ ; see Subsection 2.1. For  $q > 0$ , we define

$$h_q(x) = r_q(0) - r_q(-x) = \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left( \frac{1 - e^{i\lambda x}}{q + \Psi(\lambda)} \right) d\lambda, \quad x \in \mathbb{R}, \tag{1.3}$$

where the second identity follows from Proposition 2.3. It is obvious that  $h_q(0) = 0$ , and by (2.1), we have  $h_q(x) \geq 0$ . We denote the second moment by

$$m^2 = \mathbb{P}[X_1^2] \in (0, \infty]. \tag{1.4}$$

The following theorem plays a key role in our penalization results. Recall that  $X$  is assumed recurrent.

**Theorem 1.1.** *Suppose that **(A)** is satisfied. Then the following assertions hold.*

(i) For any  $x \in \mathbb{R}$ ,

$$h(x) := \lim_{q \rightarrow 0^+} h_q(x) \tag{1.5}$$

exists and is finite, which will be called the renormalized zero resolvent. If  $m^2 < \infty$ , then  $h$  has the following representation:

$$h(x) = \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left( \frac{1 - e^{i\lambda x}}{\Psi(\lambda)} \right) d\lambda. \tag{1.6}$$

(ii) The convergence (1.5) is uniform on compacts, and consequently  $h$  is continuous.

(iii)  $h$  is subadditive on  $\mathbb{R}$ , that is,  $h(x + y) \leq h(x) + h(y)$  for  $x, y \in \mathbb{R}$ .

The proof of Theorem 1.1 will be given in Section 3.2. The renormalized zero resolvent satisfies the following limit properties.

**Theorem 1.2.** *Suppose that (A) is satisfied. Then the following assertions hold:*

- (i)  $\lim_{x \rightarrow \pm\infty} \frac{h(x)}{|x|} = \frac{1}{m^2} \in [0, \infty)$ ;
- (ii)  $\lim_{y \rightarrow \pm\infty} \{h(x+y) - h(y)\} = \pm \frac{x}{m^2} \in \mathbb{R}$ , for all  $x \in \mathbb{R}$ .

The proof of Theorem 1.2 will be given in Section 3.2.

**Corollary 1.3.** *Suppose that (A) is satisfied. For  $-1 \leq \gamma \leq 1$ , define*

$$h^{(\gamma)}(x) = h(x) + \frac{\gamma}{m^2}x, \quad x \in \mathbb{R}. \tag{1.7}$$

Then  $h^{(\gamma)}$  is subadditive and  $h^{(\gamma)}(x) \geq 0$ .

*Proof.* By definition, we have  $h^{(\gamma)}(0) = 0$ . From (iii) of Theorem 1.1, the function  $h^{(\gamma)}$  is subadditive. From (i) of Theorem 1.2, it holds that  $\lim_{x \rightarrow \pm\infty} h^{(\gamma)}(x)/|x| = (1 + \gamma)/m^2 \geq 0$ . Suppose  $h^{(\gamma)}(x) < 0$  for some  $x \in \mathbb{R} \setminus \{0\}$ . Since  $h^{(\gamma)}$  is subadditive, we have  $h^{(\gamma)}(nx)/n \leq h^{(\gamma)}(x) < 0$  for  $n = 1, 2, \dots$ . Letting  $n \rightarrow \infty$ , we face the contradiction. Therefore we have  $h^{(\gamma)}(x) \geq 0$  for all  $x \in \mathbb{R}$ .  $\square$

We will prove in Theorem 8.1 that the function  $h^{(\gamma)}$  is invariant for the process killed upon hitting zero. Let  $\mathcal{L}_+^1$  denote the set of non-negative functions on  $[0, \infty)$  which satisfy  $\int_0^\infty f(x) dx < \infty$ . For  $f \in \mathcal{L}_+^1$ , define

$$M_t^{(\gamma)} = M_t^{(\gamma, f)} = h^{(\gamma)}(X_t)f(L_t) + \int_0^\infty f(L_t + u) du. \tag{1.8}$$

Note that, when  $m^2 = \infty$ , we have  $h^{(\gamma)} = h^{(0)} = h$  and  $M^{(\gamma)} = M^{(0)}$  for all  $\gamma$ .

**Theorem 1.4.** *Suppose that (A) is satisfied. Let  $f \in \mathcal{L}_+^1$ ,  $-1 \leq \gamma \leq 1$  and  $x \in \mathbb{R}$ . Then  $(M_t^{(\gamma)}, t \geq 0)$  is a non-negative  $(\mathcal{F}_t, \mathbb{P}_x)$ -martingale.*

Theorem 1.4 will be proved in Section 5.3. Using this martingale, we discuss our penalization problems. Let  $L = (L_t)$  denote the local time at the origin of  $X$ ; see Section 2.2.

**Theorem 1.5** (hitting time clock). *Suppose that the condition (A) is satisfied. Let  $f \in \mathcal{L}_+^1$  and  $x \in \mathbb{R}$ . Define  $h^B(a) = \mathbb{P}[L_{T_a}]$  and*

$$\begin{aligned} N_t^a &= h^B(a)\mathbb{P}_x[f(L_{T_a}); t < T_a | \mathcal{F}_t], \\ M_t^a &= h^B(a)\mathbb{P}_x[f(L_{T_a}) | \mathcal{F}_t]. \end{aligned}$$

Then it holds that

$$\lim_{a \rightarrow \pm\infty} N_t^a = \lim_{a \rightarrow \pm\infty} M_t^a = M_t^{(\pm 1)}, \quad \mathbb{P}_x\text{-a.s. and in } \mathcal{L}^1(\mathbb{P}_x).$$

Consequently, if  $M_0^{(\pm 1)} > 0$  under  $\mathbb{P}_x$ , it holds that

$$\frac{\mathbb{P}_x[F_t f(L_{T_a})]}{\mathbb{P}_x[f(L_{T_a})]} \longrightarrow \mathbb{P}_x \left[ \frac{F_t M_t^{(\pm 1)}}{M_0^{(\pm 1)}} \right], \quad \text{as } a \rightarrow \pm\infty,$$

for all bounded  $\mathcal{F}_t$ -measurable functionals  $F_t$ .

Theorem 1.5 will be proved in Section 5. If we take  $f = 1_{\{u=0\}}$ , we obtain the conditioning result.

**Corollary 1.6.** *Suppose that the condition (A) is satisfied. Let  $x \in \mathbb{R}$  with  $h^{(\pm 1)}(x) > 0$ . Then it holds that*

$$\mathbb{P}_x[F_t|T_0 > T_a] \longrightarrow \mathbb{P}_x\left[F_t \frac{h^{(\pm 1)}(X_t)}{h^{(\pm 1)}(x)}; T_0 > t\right], \quad \text{as } a \rightarrow \pm\infty,$$

for all bounded  $\mathcal{F}_t$ -measurable functionals  $F_t$ .

See also Corollary 8.2.

Let us state our penalization result with two-point hitting time clock. For  $a, b \in \mathbb{R}$ , we write  $T_{a,b} = T_{\{a,b\}} = T_a \wedge T_b$ . For  $-1 \leq \gamma \leq 1$ , we say

$$(a, b) \xrightarrow{\gamma} \infty \text{ when } a \rightarrow \infty, b \rightarrow \infty \text{ and } \frac{a-b}{a+b} \rightarrow \gamma. \tag{1.9}$$

**Theorem 1.7** (two-point hitting time clock). *Suppose that the condition (A) is satisfied. Let  $f \in \mathcal{L}_+^1$ ,  $x \in \mathbb{R}$ , and  $a, b > 0$ . Define  $h^C(a, -b) = \mathbb{P}[L_{T_{a,-b}}]$ , and*

$$N_t^{a,b} = h^C(a, -b)\mathbb{P}_x[f(L_{T_{a,-b}}); t < T_{a,-b}|\mathcal{F}_t],$$

$$M_t^{a,b} = h^C(a, -b)\mathbb{P}_x[f(L_{T_{a,-b}})|\mathcal{F}_t].$$

Then it holds that

$$\lim_{(a,b) \xrightarrow{\gamma} \infty} N_t^{a,b} = \lim_{(a,b) \xrightarrow{\gamma} \infty} M_t^{a,b} = M_t^{(\gamma)}, \quad \mathbb{P}_x\text{-a.s. and in } \mathcal{L}^1(\mathbb{P}_x).$$

Consequently, if  $M_0^{(\gamma)} > 0$  under  $\mathbb{P}_x$ , it holds that

$$\frac{\mathbb{P}_x[F_t f(L_{T_{a,-b}})]}{\mathbb{P}_x[f(L_{T_{a,-b}})]} \longrightarrow \mathbb{P}_x\left[F_t \frac{M_t^{(\gamma)}}{M_0^{(\gamma)}}\right], \quad \text{as } (a, b) \xrightarrow{\gamma} \infty,$$

for all bounded  $\mathcal{F}_t$ -measurable functionals  $F_t$ .

The proof of Theorem 1.7 will be given in Section 6.2.

**Corollary 1.8.** *Suppose that the condition (A) is satisfied. Let  $-1 \leq \gamma \leq 1$  and  $x \in \mathbb{R}$  with  $h^{(\gamma)}(x) > 0$ . Then it holds that*

$$\mathbb{P}_x[F_t|T_0 > T_{a,-b}] \longrightarrow \mathbb{P}_x\left[F_t \frac{h^{(\gamma)}(X_t)}{h^{(\gamma)}(x)}; T_0 > t\right], \quad \text{as } (a, b) \xrightarrow{\gamma} \infty,$$

for all bounded  $\mathcal{F}_t$ -measurable functionals  $F_t$ .

See also Corollary 8.2.

Note that Theorems 1.5 and 1.7 show that the limit law varies according to the chosen clock when  $m^2 < \infty$ .

### 1.2 Backgrounds of the renormalized zero resolvent

The existence of  $h$  for symmetric Lévy processes was proved by Salminen–Yor [31] under the assumption (A); see also Yano [38]. We shall review early studies of the existence of  $h$  and its limit properties for asymmetric processes.

Similar results were obtained for random walks by Spitzer [33, Chapter VII], Port–Stone [19, 20, 21] and Stone [34]. For Lévy processes, Port–Stone [22, Section 17] obtained some results which were similar to but different from Theorems 1.1 and 1.2, reducing them to the random walk case. (For the proofs of Theorems 1.1 and 1.2, we are inspired by Spitzer [33, Chapter VII], Port–Stone [19, 20, 21, 22, Section 17] and Stone [34].)

Yano [39] showed the existence of the renormalized zero resolvent  $h$  under the following two conditions:

**(Y1)**  $\int_0^\infty \frac{1}{q+\theta(\lambda)} d\lambda < \infty$  for all  $q > 0$ ;

**(Y2)**  $\theta$  and  $\omega$  have measurable derivatives on  $(0, \infty)$  which satisfy

$$\int_0^\infty \frac{(|\theta'(\lambda)| + |\omega'(\lambda)|)(\lambda^2 \wedge 1)}{\theta(\lambda)^2 + \omega(\lambda)^2} d\lambda < \infty.$$

Pantí [18] proved the existence of  $h$  under the condition

**(P)** **(Y1)** and  $\int_{\mathbb{R}} \left| \operatorname{Re} \left( \frac{1-e^{i\lambda}}{\Psi(\lambda)} \right) \right| d\lambda < \infty,$

which is weaker than **(Y1)** and **(Y2)**, and applied it to the conditioning to avoid zero with exponential clock. Tsukada [35] also proved the existence of  $h$  under the assumption

**(T)** **(A)** and  $\int_0^1 \left| \operatorname{Im} \left( \frac{\lambda}{\Psi(\lambda)} \right) \right| d\lambda < \infty,$

which is weaker than **(P)**; see [35, Proposition 15.3].

**Remark 1.9.** We do not know whether the integral representation (1.6) also holds in the case  $m^2 = \infty$ . Yano [39] showed that, if  $X$  is symmetric, then (1.6) holds. Tsukada [35] showed that **(T)** implies (1.6).

### 1.3 Organization

The remainder of this paper is organized as follows. In Section 2, we prepare certain general properties and preliminary facts of Lévy processes. In Section 3, we study the renormalized zero resolvent. In Sections 4, 5, 6 and 7, we discuss the penalization results with exponential clock, hitting time clock, two-point hitting time clock and inverse local time clock, respectively. In Section 8, we introduce certain universal  $\sigma$ -finite measures to study long time behaviors of sample paths of the penalized measure. In Section 9, we study penalization in the transient case. In Section 10 as an appendix, we study martingale property of  $(X_t f(L_t), t \geq 0)$ .

## 2 Preliminaries

### 2.1 Absolutely continuous resolvent

We now consider the following two conditions:

**(A1)** The process  $X$  is not a compound Poisson process;

**(A2)** 0 is regular for itself, i.e.,  $\mathbb{P}(T_0 = 0) = 1$ .

The next lemma is due to Kesten [13] and Bretagnolle [3].

**Lemma 2.1** ([13, 3]). *The conditions **(A1)** and **(A2)** hold if and only if the following two assertions hold:*

**(A3)** For each  $q > 0$ , the characteristic exponent  $\Psi$  satisfies

$$\int_{\mathbb{R}} \operatorname{Re} \left( \frac{1}{q + \Psi(\lambda)} \right) d\lambda < \infty;$$

**(A4)** We have either  $\sigma > 0$  or  $\int_{(-1,1)} |x| \nu(dx) = \infty$ .

Furthermore, under the condition **(A3)**, the condition **(A2)** holds if and only if the condition **(A4)** holds.

If the above conditions hold, it is known that  $X$  has a bounded continuous resolvent density. See, e.g., Theorem II.16 and Theorem II.19 of Bertoin [1]

**Lemma 2.2** ([1]). *The condition (A3) holds if and only if  $X$  has the bounded  $q$ -resolvent density  $r_q$ , for  $q > 0$ , which satisfies*

$$\int_{\mathbb{R}} f(x)r_q(x) dx = \mathbb{P} \left[ \int_0^\infty e^{-qt} f(X_t) dt \right]$$

for all non-negative measurable functions  $f$ . Moreover, under the condition (A3), the condition (A2) holds if and only if  $x \mapsto r_q(x)$  is continuous.

If  $r_q(x)$  is bounded in  $x \in \mathbb{R}$ , [1, Corollary II.18] implies that the Laplace transform of  $T_0$  can be represented as

$$\mathbb{P}_x[e^{-qT_0}] = \frac{r_q(-x)}{r_q(0)}, \quad q > 0, x \in \mathbb{R}. \tag{2.1}$$

**Proposition 2.3.** *Suppose that the condition (A) holds. Then the bounded continuous resolvent density can be expressed as*

$$r_q(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{-i\lambda x}}{q + \Psi(\lambda)} d\lambda = \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left( \frac{e^{-i\lambda x}}{q + \Psi(\lambda)} \right) d\lambda$$

for all  $q > 0$  and  $x \in \mathbb{R}$ .

Proposition 2.3 can be proved using Fourier inversion formula; see, e.g., [36, Lemma 2] and [35, Corollary 15.1]. By Lemmas 2.1 and 2.2 and Proposition 2.3, the condition (A) implies (A1)–(A4).

**Lemma 2.4** (Tsukada [35, Lemma 15.5]). *Suppose that the condition (A) holds. Then the following assertions hold:*

- (i)  $|\Psi(\lambda)| \rightarrow \infty$  as  $\lambda \rightarrow \pm\infty$ ;
- (ii)  $\int_\delta^\infty \left| \frac{1}{\Psi(\lambda)} \right| d\lambda < \infty$  for all  $\delta > 0$ ;
- (iii)  $\int_0^\delta \left| \frac{\lambda^2}{\Psi(\lambda)} \right| d\lambda < \infty$  for all  $\delta > 0$ ;
- (iv)  $\lim_{q \rightarrow 0+} \int_0^\infty \left| \frac{q}{q + \Psi(\lambda)} \right| d\lambda = 0$ . In particular,  $qr_q(x) \rightarrow 0$  as  $q \rightarrow 0+$ .

## 2.2 Local time and its excursion measure

Assume the conditions (A1) and (A2) hold. Then we can define local time at 0, which we denote by  $L = (L_t, t \geq 0)$ . Note that  $L$  is continuous in  $t$  and satisfies

$$\mathbb{P}_x \left[ \int_0^\infty e^{-qt} dL_t \right] = r_q(-x), \quad q > 0, x \in \mathbb{R}. \tag{2.2}$$

See, e.g., [1, Section V]. In particular,  $r_q(x)$  is non-decreasing as  $q \rightarrow 0+$ . Let  $\eta = (\eta_l, l \geq 0)$  denote the right-continuous inverse of  $L$  which is given as  $\eta_l = \inf\{t > 0: L_t > l\}$ . Then the process  $(\eta, \mathbb{P})$  is a possibly killed subordinator, and its Laplace transform is  $\mathbb{P}[e^{-q\eta_l}] = e^{-l/r_q(0)}$ , for  $l, q > 0$ , see, e.g., [1, Proposition V.4].

Now we can apply Itô's excursion theory. Let  $n$  denote the characteristic measure of excursions away from the origin. We denote  $e_l$  for excursion which starts at local time  $l$ . Then we see that the subordinator  $\eta$  has no drift and its Lévy measure is  $n(T_0 \in dx)$ . In particular, we have

$$e^{-l/r_q(0)} = \mathbb{P}[e^{-q\eta_l}] = \exp(-ln[1 - e^{-qT_0}]), \quad l \geq 0. \tag{2.3}$$

This implies that

$$n[1 - e^{-qT_0}] = \frac{1}{r_q(0)}, \tag{2.4}$$

which is also obtained from [42, (3.16)]. Now set

$$\kappa = \lim_{q \rightarrow 0^+} \frac{1}{r_q(0)} = n(T_0 = \infty). \tag{2.5}$$

It is known that  $\kappa = 0$  (resp.  $\kappa > 0$ ) if and only if  $X$  is recurrent (resp. transient); see, e.g., [1, Theorem I.17] and [32, Theorem 37.5]. It is also known that  $X$  is recurrent if and only if  $X$  is point recurrent; see, e.g., [32, Remark 43.12] (see also [1, Exercice II.6.4]). Under the assumption **(A)**, we can prove this fact by using Theorem 1.1; in fact, by equations (2.1) and (2.5), it holds that, for  $x \in \mathbb{R}$ ,

$$\mathbb{P}_x(T_0 < \infty) = \lim_{q \rightarrow 0^+} \mathbb{P}_x[e^{-qT_0}] = \lim_{q \rightarrow 0^+} \frac{r_q(-x)}{r_q(0)} = 1 + \lim_{q \rightarrow 0^+} \frac{h_q(x)}{r_q(0)} = 1.$$

We define  $D = \{l \geq 0: \eta_{l-} < \eta_l\}$ . Then the following formula is well-known in the excursion theory.

**Lemma 2.5** (Compensation formula; see e.g., Bertoin [1, Corollary IV.11]). *Let  $F(t, \omega, e)$  be a measurable functional on  $[0, \infty) \times \mathcal{D} \times \mathcal{D}$  such that, for every fixed  $e \in \mathcal{D}$ , the process  $(F(t, \cdot, e), t \geq 0)$  is  $(\mathcal{F}_t)$ -predictable. Then*

$$\mathbb{P} \left[ \sum_{l \in D} F(\eta_{l-}, X, e_l) \right] = \mathbb{P} \otimes \tilde{n} \left[ \int_0^\infty dL_t F(t, X, \tilde{X}) \right],$$

where the symbol  $\sim$  means independence.

Let  $L^a = (L_t^a, t \geq 0)$  denote the local time at  $a \in \mathbb{R}$  which is normalized by

$$\mathbb{P}_x \left[ \int_0^\infty e^{-qt} dL_t^a \right] = r_q(a - x), \quad q > 0, x \in \mathbb{R}.$$

We denote by  $\eta^a = (\eta_u^a, u \geq 0)$  the right-continuous inverse of  $L^a$  given by  $\eta_u^a = \inf\{t > 0: L_t^a > u\}$ . We denote by  $n^a$  the characteristic measure of excursions away from  $a$ .

### 3 The renormalized zero resolvent

Let us consider the existence and properties of the renormalized zero resolvent in Theorems 1.1 and 1.2. Recall that we assume  $X$  is recurrent, i.e.,  $\kappa = 0$ , and assume the condition **(A)**.

#### 3.1 Key lemmas for the renormalized zero resolvent

To show Theorems 1.1 and 1.2, we prepare some lemmas. Recall that  $m^2$  has been introduced in (1.4);  $m^2 = \mathbb{P}[X_1^2]$ .

**Lemma 3.1.** *The following assertions hold.*

(i) *If  $m^2 < \infty$ , then*

$$\Psi(\lambda) = \frac{1}{2} \sigma^2 \lambda^2 + \int_{\mathbb{R}} (1 - e^{i\lambda x} + i\lambda x) \nu(dx),$$

and

$$\lim_{\lambda \rightarrow 0} \frac{\Psi(\lambda)}{\lambda^2} = \lim_{\lambda \rightarrow 0} \frac{\theta(\lambda)}{\lambda^2} = \frac{m^2}{2} = \frac{1}{2} \left( \sigma^2 + \int_{\mathbb{R}} x^2 \nu(dx) \right).$$

(ii) If  $m^2 = \infty$ , then

$$\lim_{\lambda \rightarrow 0} \frac{\lambda^2}{\Psi(\lambda)} = \lim_{\lambda \rightarrow 0} \frac{\lambda^2}{\theta(\lambda)} = 0.$$

*Proof.* It is well-known (see, e.g., [15, Theorem 2.7]) that  $X$  has finite variance if and only if  $\int_{\mathbb{R}} x^2 \nu(dx) < \infty$ . We first assume that  $\int_{\mathbb{R}} x^2 \nu(dx) < \infty$ . Then we know  $\mathbb{P}[X_1]$  and  $\mathbb{P}[X_1^2]$  are finite and

$$\begin{aligned} \mathbb{P}[X_1] &= i\Psi'(0) = -v + \int_{\mathbb{R} \setminus (-1,1)} x\nu(dx), \\ \mathbb{P}[X_1^2] &= \Psi''(0) = \sigma^2 + \int_{\mathbb{R}} x^2 \nu(dx) + \mathbb{P}[X_1]^2. \end{aligned}$$

Since  $X$  is recurrent, we have  $\mathbb{P}[X_1] = 0$ ; see, e.g., [15, Problem 7.2]. This implies that

$$\Psi(\lambda) = \frac{1}{2}\sigma^2\lambda + \int_{\mathbb{R}} (1 - e^{i\lambda x} + i\lambda x)\nu(dx).$$

By l'Hôpital's rule, we obtain

$$\lim_{\lambda \rightarrow 0} \frac{\Psi(\lambda)}{\lambda^2} = \lim_{\lambda \rightarrow 0} \frac{\Psi'(\lambda)}{2\lambda} = \frac{\Psi''(0)}{2} = \frac{m^2}{2}.$$

Taking real parts on both sides, we also have

$$\lim_{\lambda \rightarrow 0} \frac{\theta(\lambda)}{\lambda^2} = \frac{m^2}{2}.$$

We next assume that  $\int_{\mathbb{R}} x^2 \nu(dx) = \infty$ . Then we know  $m^2 = \infty$ . By (1.1), we have

$$\left| \frac{\Psi(\lambda)}{\lambda^2} \right| \geq \left| \frac{\theta(\lambda)}{\lambda^2} \right| \geq \int_{\mathbb{R}} \frac{1 - \cos \lambda x}{\lambda^2} \nu(dx).$$

Using Fatou's lemma and l'Hôpital's rule, we obtain

$$\begin{aligned} \liminf_{\lambda \rightarrow 0} \left| \frac{\Psi(\lambda)}{\lambda^2} \right| &\geq \liminf_{\lambda \rightarrow 0} \left| \frac{\theta(\lambda)}{\lambda^2} \right| \geq \int_{\mathbb{R}} \liminf_{\lambda \rightarrow 0} \frac{1 - \cos \lambda x}{\lambda^2} \nu(dx) \\ &= \int_{\mathbb{R}} \frac{x^2}{2} \nu(dx) = \infty. \end{aligned}$$

Therefore the proof is complete. □

When  $m^2 < \infty$ , the next lemma is essential for the renormalized zero resolvent.

**Lemma 3.2.** Assume  $m^2 < \infty$ . Then it holds that

$$\int_{\mathbb{R}} \left| \frac{\omega(\lambda)}{\lambda^3} \right| d\lambda < \infty. \tag{3.1}$$

Consequently, Tsukada's condition **(T)** holds.

*Proof of Lemma 3.2.* By Lemma 3.1, we have

$$\omega(\lambda) = \int_{\mathbb{R}} (\lambda x - \sin \lambda x) \nu(dx).$$

Hence we have

$$\begin{aligned} \int_{\mathbb{R}} \left| \frac{\omega(\lambda)}{\lambda^3} \right| d\lambda &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \frac{\lambda x - \sin \lambda x}{\lambda^3} \nu(dx) \right| d\lambda \\ &\leq \left( \int_{-\infty}^0 + \int_0^{\infty} \right) \left( \int_{-\infty}^0 + \int_0^{\infty} \right) \left| \frac{\lambda x - \sin \lambda x}{\lambda^3} \right| \nu(dx) d\lambda. \end{aligned}$$

Since  $\lambda x - \sin \lambda x \geq 0$  for  $(x, \lambda) \in (0, \infty)^2$ , it holds that

$$\begin{aligned} \int_0^{\infty} \int_0^{\infty} \left| \frac{\lambda x - \sin \lambda x}{\lambda^3} \right| \nu(dx) d\lambda &= \int_0^{\infty} \int_0^{\infty} \frac{\lambda x - \sin \lambda x}{\lambda^3} d\lambda \nu(dx) \\ &= \int_0^{\infty} x^2 \nu(dx) \int_0^{\infty} \frac{\xi - \sin \xi}{\xi^3} d\xi \\ &= \frac{4}{\pi} \int_0^{\infty} x^2 \nu(dx) < \infty. \end{aligned}$$

Other integrals are also proved to be finite by the same discussions, and we obtain (3.1). By (i) of Lemma 3.1, we see that  $\lambda^2/\Psi(\lambda)$  is bounded near  $\lambda = 0$ . Thus we have

$$\int_0^1 \left| \operatorname{Im} \left( \frac{\lambda}{\Psi(\lambda)} \right) \right| d\lambda = \int_0^1 \left| \frac{\lambda^2}{\Psi(\lambda)} \right|^2 \left| \frac{\omega(\lambda)}{\lambda^3} \right| d\lambda < \infty.$$

Since **(A)** is assumed in this section, this implies that Tsukada’s condition **(T)** holds.  $\square$

**Lemma 3.3.** *The following assertions hold.*

- (i)  $h^S(x) := \lim_{q \rightarrow 0+} (h_q(x) + h_q(-x)) = \frac{2}{\pi} \int_0^{\infty} \operatorname{Re} \left( \frac{1 - \cos \lambda x}{\Psi(\lambda)} \right) d\lambda, \quad \text{for } x \in \mathbb{R}.$
- (ii)  $\lim_{x \rightarrow \pm\infty} \frac{h^S(x)}{|x|} = \frac{2}{m^2} \in [0, \infty).$
- (iii) For  $x, y \in \mathbb{R}$ ,

$$\begin{aligned} h^D(x, y) &:= \lim_{q \rightarrow 0+} \{h_q(y + 2x) - 2h_q(y + x) + h_q(y)\} \\ &= \frac{2}{\pi} \int_0^{\infty} \operatorname{Re} \left( e^{i\lambda(y+x)} \frac{1 - \cos \lambda x}{\Psi(\lambda)} \right) d\lambda. \end{aligned}$$

Moreover, it holds that  $\lim_{y \rightarrow \pm\infty} h^D(x, y) = 0$ .

*Proof.* (i) By (1.3), it holds that

$$h_q(x) + h_q(-x) = \frac{2}{\pi} \int_0^{\infty} \operatorname{Re} \left( \frac{1 - \cos \lambda x}{q + \Psi(\lambda)} \right) d\lambda. \tag{3.2}$$

Since  $\theta(\lambda) \geq 0$ , we have  $|q + \Psi(\lambda)| \geq |\Psi(\lambda)|$ . Hence it holds that

$$\left| \operatorname{Re} \left( \frac{1 - \cos \lambda x}{q + \Psi(\lambda)} \right) \right| \leq \left| \frac{1 - \cos \lambda x}{q + \Psi(\lambda)} \right| \leq \frac{1 - \cos \lambda x}{|\Psi(\lambda)|} \leq \frac{(\lambda x)^2 \wedge 2}{|\Psi(\lambda)|}, \tag{3.3}$$

which is integrable in  $\lambda > 0$  by Lemma 2.4. Then we may apply the dominated convergence theorem to deduce that

$$h_q(x) + h_q(-x) \longrightarrow \frac{2}{\pi} \int_0^{\infty} \operatorname{Re} \left( \frac{1 - \cos \lambda x}{\Psi(\lambda)} \right) d\lambda, \quad \text{as } q \rightarrow 0+.$$

(ii) We only consider the case  $x \rightarrow \infty$  since the case  $x \rightarrow -\infty$  can be proved in the same way. For any  $\delta > 0$ , we have

$$\left| \frac{2}{\pi x} \int_{\delta}^{\infty} \operatorname{Re} \left( \frac{1 - \cos \lambda x}{\Psi(\lambda)} \right) d\lambda \right| \leq \frac{2}{\pi |x|} \int_{\delta}^{\infty} \left| \frac{2}{\Psi(\lambda)} \right| d\lambda \rightarrow 0, \quad \text{as } x \rightarrow \infty. \quad (3.4)$$

Fix  $\varepsilon > 0$ . By Lemma 3.1, we can choose  $\delta > 0$  such that

$$\left| \frac{\lambda^2}{\Psi(\lambda)} - \frac{2}{m^2} \right| < \varepsilon, \quad \text{for } |\lambda| < \delta.$$

Then it holds that

$$\begin{aligned} \frac{2}{\pi x} \int_0^{\delta} \operatorname{Re} \left( \frac{1 - \cos \lambda x}{\Psi(\lambda)} \right) d\lambda &\leq \left( \frac{2}{m^2} + \varepsilon \right) \frac{2}{\pi x} \int_0^{\delta} \frac{1 - \cos \lambda x}{\lambda^2} d\lambda \\ &\rightarrow \left( \frac{2}{m^2} + \varepsilon \right) \frac{2}{\pi} \int_0^{\infty} \frac{1 - \cos \xi}{\xi^2} d\xi \\ &= \frac{2}{m^2} + \varepsilon, \quad \text{as } x \rightarrow \infty. \end{aligned}$$

Thus we obtain

$$\limsup_{x \rightarrow \infty} \frac{2}{\pi x} \int_0^{\delta} \operatorname{Re} \left( \frac{1 - \cos \lambda x}{\Psi(\lambda)} \right) d\lambda \leq \frac{2}{m^2} + \varepsilon. \quad (3.5)$$

In the same way, we can show that

$$\liminf_{x \rightarrow \infty} \frac{2}{\pi x} \int_0^{\delta} \operatorname{Re} \left( \frac{1 - \cos \lambda x}{\Psi(\lambda)} \right) d\lambda \geq \frac{2}{m^2} - \varepsilon. \quad (3.6)$$

By (3.4), (3.5) and (3.6), the result follows.

(iii) By (1.3), we have

$$h_q(y + 2x) - 2h_q(y + x) + h_q(y) = \frac{2}{\pi} \int_0^{\infty} \operatorname{Re} \left( e^{i\lambda(y+x)} \frac{1 - \cos \lambda x}{q + \Psi(\lambda)} \right) d\lambda.$$

By the same way as (3.3), we may apply the dominated convergence theorem to obtain

$$\begin{aligned} h^D(x, y) &= \lim_{q \rightarrow 0^+} \{h_q(y + 2x) - 2h_q(y + x) + h_q(y)\} \\ &= \frac{2}{\pi} \int_0^{\infty} \operatorname{Re} \left( e^{i\lambda(y+x)} \frac{1 - \cos \lambda x}{\Psi(\lambda)} \right) d\lambda. \end{aligned}$$

Furthermore, by the Riemann–Lebesgue lemma, we obtain  $h^D(x, y) \rightarrow 0$  as  $y \rightarrow \pm\infty$ .  $\square$

### 3.2 Proofs of Theorems 1.1 and 1.2

We separate the proof into the two cases:  $m^2 < \infty$  and  $m^2 = \infty$ . We first show the existence and properties of  $h$  in the case  $m^2 < \infty$ . In this case, we can use the dominated convergence theorem.

*Proof of (i) of Theorem 1.1 in the case  $m^2 < \infty$ .* For each  $x, \lambda \in \mathbb{R}$ , we observe that

$$\operatorname{Re} \left( \frac{1 - e^{i\lambda x}}{q + \Psi(\lambda)} \right) = \frac{(q + \theta(\lambda))(1 - \cos \lambda x) + \omega(\lambda) \sin \lambda x}{|q + \Psi(\lambda)|^2}.$$

Hence it follows from  $\theta(\lambda) \geq 0$  that

$$\begin{aligned} & \left| \operatorname{Re} \left( \frac{1 - e^{i\lambda x}}{q + \Psi(\lambda)} \right) \right| \\ & \leq \left( \frac{1 - \cos \lambda x}{|\Psi(\lambda)|} + \frac{\lambda^4}{|\Psi(\lambda)|^2} \left| \frac{\omega(\lambda)}{\lambda^3} \right| \left| \frac{\sin \lambda x}{\lambda} \right| \right) \wedge \left| \frac{1 - e^{i\lambda x}}{\Psi(\lambda)} \right| \\ & \leq \left( \left| \frac{\lambda^2 x^2}{\Psi(\lambda)} \right| + \left| \frac{\lambda^2}{\Psi(\lambda)} \right|^2 \left| \frac{\omega(\lambda)}{\lambda^3} \right| |x| \right) \wedge \left| \frac{2}{\Psi(\lambda)} \right|. \end{aligned}$$

By Lemma 2.4, (i) of Lemma 3.1 and Lemma 3.2, the last quantity is integrable in  $\lambda > 0$ . Therefore, we may apply the dominated convergence theorem to conclude that

$$\begin{aligned} h_q(x) &= \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left( \frac{1 - e^{i\lambda x}}{q + \Psi(\lambda)} \right) d\lambda \\ &\rightarrow \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left( \frac{1 - e^{i\lambda x}}{\Psi(\lambda)} \right) d\lambda, \quad \text{as } q \rightarrow 0+. \end{aligned}$$

Hence the proof is complete. □

*Proof of Theorem 1.2 in the case  $m^2 < \infty$ .* (i) We take  $\delta > 0$  sufficiently small. By (i) of Theorem 1.1, we have

$$\begin{aligned} \frac{h(x)}{x} &= \frac{1}{\pi x} \int_0^\infty \operatorname{Re} \left( \frac{1 - e^{i\lambda x}}{\Psi(\lambda)} \right) d\lambda \\ &= \frac{1}{\pi x} \left\{ \int_\delta^\infty \operatorname{Re} \left( \frac{1 - e^{i\lambda x}}{\Psi(\lambda)} \right) d\lambda + \int_0^\delta \frac{\omega(\lambda) \sin \lambda x}{|\Psi(\lambda)|^2} d\lambda + \int_0^\delta \frac{\theta(\lambda)(1 - \cos \lambda x)}{|\Psi(\lambda)|^2} d\lambda \right\}. \end{aligned} \tag{3.7}$$

For the first integral in (3.7), we have

$$\left| \frac{1}{\pi x} \int_\delta^\infty \operatorname{Re} \left( \frac{1 - e^{i\lambda x}}{\Psi(\lambda)} \right) d\lambda \right| \leq \frac{1}{\pi|x|} \int_\delta^\infty \left| \frac{2}{\Psi(\lambda)} \right| d\lambda \rightarrow 0, \quad \text{as } x \rightarrow \pm\infty.$$

For the second integral in (3.7), since we have

$$\frac{|\omega(\lambda) \sin \lambda x|}{|\Psi(\lambda)|^2 |x|} \leq \left| \frac{\lambda^2}{\Psi(\lambda)} \right|^2 \left| \frac{\omega(\lambda)}{\lambda^3} \right|, \tag{3.8}$$

which is integrable in  $\lambda \in (0, \delta)$  by (i) of Lemma 3.1 and Lemma 3.2, we can apply the dominated convergence theorem to obtain

$$\frac{1}{\pi x} \int_0^\delta \frac{\omega(\lambda) \sin \lambda x}{|\Psi(\lambda)|^2} d\lambda \rightarrow 0, \quad \text{as } x \rightarrow \pm\infty.$$

For the third integral in (3.7), we can apply the similar discussion as the proof of (ii) of Lemma 3.3. Therefore we obtain

$$\lim_{x \rightarrow \pm\infty} \frac{h(x)}{|x|} = \frac{1}{m^2}.$$

(ii) Take  $\delta > 0$  sufficiently small. Then we have

$$\begin{aligned} & h(y+x) - h(y) \\ &= \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left( e^{i\lambda y} \frac{1 - e^{i\lambda x}}{\Psi(\lambda)} \right) d\lambda \\ &= \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left( e^{i\lambda y} \frac{1 - \cos \lambda x}{\Psi(\lambda)} \right) d\lambda + \frac{1}{\pi} \left( \int_0^\delta + \int_\delta^\infty \right) \operatorname{Im} \left( e^{i\lambda y} \frac{\sin \lambda x}{\Psi(\lambda)} \right) d\lambda. \end{aligned}$$

By the Riemann–Lebesgue lemma, we obtain

$$\lim_{y \rightarrow \pm\infty} \int_0^\infty \operatorname{Re} \left( e^{i\lambda y} \frac{1 - \cos \lambda x}{\Psi(\lambda)} \right) d\lambda = \lim_{y \rightarrow \pm\infty} \int_\delta^\infty \operatorname{Im} \left( e^{i\lambda y} \frac{\sin \lambda x}{\Psi(\lambda)} \right) d\lambda = 0.$$

On the other hand, we have

$$\begin{aligned} & \frac{1}{\pi} \int_0^\delta \operatorname{Im} \left( e^{i\lambda y} \frac{\sin \lambda x}{\Psi(\lambda)} \right) d\lambda \\ &= \frac{1}{\pi} \int_0^\delta \frac{\theta(\lambda) \sin \lambda x \sin \lambda y}{|\Psi(\lambda)|^2} d\lambda - \frac{1}{\pi} \int_0^\delta \frac{\omega(\lambda) \sin \lambda x \cos \lambda y}{|\Psi(\lambda)|^2} d\lambda. \end{aligned} \tag{3.9}$$

By (3.8), we may apply the Riemann–Lebesgue lemma to the second integral and we have

$$\frac{1}{\pi} \int_0^\delta \frac{\omega(\lambda) \sin \lambda x \cos \lambda y}{|\Psi(\lambda)|^2} d\lambda \rightarrow 0, \quad \text{as } y \rightarrow \pm\infty.$$

For the first integral of (3.9), we use Lemma 3.1 and Lemma 3.4 below to show that

$$\begin{aligned} \frac{1}{\pi} \int_0^\delta \frac{\theta(\lambda) \sin \lambda x \sin \lambda y}{|\Psi(\lambda)|^2} d\lambda &= \frac{1}{\pi} \int_0^\delta \left| \frac{\lambda^2}{\Psi(\lambda)} \right|^2 \frac{\theta(\lambda) \sin \lambda x \sin \lambda y}{\lambda^2 \lambda \lambda} d\lambda \\ &\rightarrow \pm \frac{x}{m^2}, \quad \text{as } y \rightarrow \pm\infty. \end{aligned}$$

This ends the proof. □

The following lemma is an elementary calculus.

**Lemma 3.4** (Jordan’s theorem for the Dirichlet integral). *Let  $\delta > 0$  and let  $f : (0, \delta) \rightarrow \mathbb{R}$  be continuous and be of bounded variation. Then it holds that*

$$\lim_{x \rightarrow \pm\infty} \frac{2}{\pi} \int_0^\delta f(\lambda) \frac{\sin \lambda x}{\lambda} d\lambda = \pm f(0+),$$

where  $f(0+) = \lim_{\lambda \rightarrow 0+} f(\lambda)$ .

*Proof.* By integration by parts, we have

$$\begin{aligned} \frac{2}{\pi} \int_0^\delta f(\lambda) \frac{\sin \lambda x}{\lambda} d\lambda &= \frac{2}{\pi} \int_0^\delta \left( \int_0^\lambda df(\xi) \right) \frac{\sin \lambda x}{\lambda} d\lambda + \frac{2}{\pi} f(0+) \int_0^\delta \frac{\sin \lambda x}{\lambda} d\lambda \\ &= \frac{2}{\pi} \int_0^\delta \left( \int_\xi^\delta \frac{\sin \lambda x}{\lambda} d\lambda \right) df(\xi) + \frac{2}{\pi} f(0+) \int_0^\delta \frac{\sin \lambda x}{\lambda} d\lambda \\ &\rightarrow 0 \pm f(0+), \quad \text{as } x \rightarrow \pm\infty. \end{aligned}$$

Thus we obtain the desired result. □

Then let us prove the existence of  $h$  in the case  $X$  is recurrent and  $m^2 = \infty$ . Its proof is quite different from that in the case  $m^2 < \infty$ .

*Proof of (i) of Theorem 1.1 in the case  $m^2 = \infty$ .* Since  $h_q(0) = 0$ , we have the limit  $h(0) = \lim_{q \rightarrow 0+} h_q(0) = 0$ .

Fix  $a \neq 0$  and set

$$\bar{h}(a) = \limsup_{q \rightarrow 0+} h_q(a), \quad \underline{h}(a) = \liminf_{q \rightarrow 0+} h_q(a).$$

We also define  $\Delta = \bar{h}(a) - \underline{h}(a) \geq 0$  and  $A = \{2^j a : j = 0, 1, 2, 3, \dots\}$ .

It follows from  $h_q(x) \leq h_q(x) + h_q(-x)$  and (3.2)–(3.3) that  $\{h_q(x)\}_{q>0}$  is bounded for each  $x \in \mathbb{R}$ . Hence, by the diagonal argument, we can take two sequences  $\{q_n\}, \{q'_n\}$ , which satisfies the following three conditions:

- $q_n, q'_n \rightarrow 0+$  as  $n \rightarrow \infty$ ;
- $\lim_{n \rightarrow \infty} h_{q_n}(x)$  and  $\lim_{n \rightarrow \infty} h_{q'_n}(x)$  exist and are finite for each  $x \in A$ ;
- $\bar{h}(a) = \lim_{n \rightarrow \infty} h_{q_n}(a)$  and  $\underline{h}(a) = \lim_{n \rightarrow \infty} h_{q'_n}(a)$ .

Then we define, for  $x \in A$ ,

$$\bar{h}(x) = \lim_{n \rightarrow \infty} h_{q_n}(x), \quad \underline{h}(x) = \lim_{n \rightarrow \infty} h_{q'_n}(x).$$

By (iii) of Lemma 3.3, we have

$$h^D(x, 0) = \bar{h}(2x) - 2\bar{h}(x) = \underline{h}(2x) - 2\underline{h}(x), \quad x \in A.$$

This implies that  $\bar{h}(2x) - \underline{h}(2x) = 2(\bar{h}(x) - \underline{h}(x))$ . Hence it holds that  $\bar{h}(2^j a) - \underline{h}(2^j a) = 2^j \Delta$ , i.e.,

$$\frac{\bar{h}(2^j a)}{2^j a} - \frac{\underline{h}(2^j a)}{2^j a} = \frac{\Delta}{a}, \quad j = 0, 1, 2, 3, \dots \tag{3.10}$$

It follows from (ii) of Lemma 3.3 that

$$\max\left\{\frac{\bar{h}(2^j a)}{2^j a}, \frac{\underline{h}(2^j a)}{2^j a}\right\} \leq \frac{h^S(2^j a)}{2^j a} \rightarrow 0, \quad \text{as } j \rightarrow \infty.$$

Thus, by letting  $j \rightarrow \infty$  in (3.10), we obtain  $\Delta = 0$ . Therefore, we conclude that  $h(a) = \lim_{q \rightarrow 0+} h_q(a)$  exists. □

Next, we prove (ii) and (iii) of Theorem 1.1 in both cases  $m^2 = \infty$  and  $m^2 < \infty$ .

*Proof of (ii) and (iii) of Theorem 1.1.* By the Markov property, we have, for  $x, y \in \mathbb{R}$ ,

$$\mathbb{P}_{x+y}[e^{-qT_0}] = \mathbb{P}_0[e^{-qT_{-x-y}}] \geq \mathbb{P}_0[e^{-qT_{-x}} \mathbb{P}_{-x}[e^{-qT_{-x-y}}]] = \mathbb{P}_x[e^{-qT_0}] \mathbb{P}_y[e^{-qT_0}].$$

Since  $h_q(x) = r_q(0)(1 - \mathbb{P}_x[e^{-qT_0}])$  and  $(1 - \mathbb{P}_x[e^{-qT_0}])(1 - \mathbb{P}_y[e^{-qT_0}]) \geq 0$ , it holds that

$$h_q(x + y) \leq h_q(x) + h_q(y).$$

Hence  $h_q$  is subadditive. (This proof can also be found in [18, Lemma 3.3].) Since  $h_q$  is non-negative and subadditive, and by (3.2) and (3.3), it holds that

$$|h_q(x + \delta) - h_q(x)| \leq h_q(\delta) + h_q(-\delta) \leq \int_0^\infty \left| \frac{(\lambda\delta)^2 \wedge 2}{\Psi(\lambda)} \right| d\lambda.$$

Hence  $\{h_q\}_{q>0}$  is equi-continuous. Equi-continuity and pointwise-convergence imply the uniform convergence on compact subset of  $\mathbb{R}$ . The subadditivity of  $h$  follows directly from that of  $h_q$ . □

Finally, we show the properties of  $h$  in Theorem 1.2 in the case  $m^2 = \infty$ .

*Proof of Theorem 1.2 in the case  $m^2 = \infty$ .* (i) This is directly from (ii) of Lemma 3.3.  
 (ii) Since  $h$  is subadditive, we have

$$\sum_{k=1}^n \{h(kx + y) - h((k - 1)x + y)\} = h(nx + y) - h(y) \leq h(nx). \tag{3.11}$$

By (iii) of Lemma 3.3, it holds that

$$\{h(2x + y) - h(x + y)\} - \{h(x + y) - h(y)\} \rightarrow 0, \quad \text{as } y \rightarrow \pm\infty.$$

Thus we have

$$\limsup_{y \rightarrow \pm\infty} \sum_{k=1}^n \{h(kx + y) - h((k - 1)x + y)\} = n \limsup_{y \rightarrow \pm\infty} \{h(x + y) - h(y)\}. \tag{3.12}$$

Combining (3.11) and (3.12), we obtain

$$\limsup_{y \rightarrow \pm\infty} \{h(x + y) - h(y)\} \leq \frac{h(nx)}{n}.$$

Since we have  $\lim_{n \rightarrow \infty} \frac{h(nx)}{n} = 0$  by (iii) of Lemma 3.3, we have

$$\limsup_{y \rightarrow \pm\infty} \{h(x + y) - h(y)\} \leq 0.$$

Replacing  $x$  with  $-x$ , we also have

$$\liminf_{y \rightarrow \pm\infty} \{h(y) - h(y - x)\} \geq 0.$$

Therefore we obtain  $\lim_{y \rightarrow \pm\infty} \{h(x + y) - h(y)\} = 0$ . □

### 3.3 The function $h^B$

Let us compute  $\mathbb{P}[L_{T_a}]$  and  $\mathbb{P}_x(T_a < T_b)$ .

**Lemma 3.5.** (i) For  $a \in \mathbb{R}$ ,

$$h_q^B(a) := \mathbb{P} \left[ \int_0^{T_a} e^{-qt} dL_t \right] = h_q(a) + h_q(-a) - \frac{h_q(a)h_q(-a)}{r_q(0)}. \tag{3.13}$$

Consequently, it holds that

$$h^B(a) := \lim_{q \rightarrow 0^+} h_q^B(a) = \mathbb{P}[L_{T_a}] = h(a) + h(-a). \tag{3.14}$$

(ii) For  $x, a, b \in \mathbb{R}$ ,  $a \neq b$  and for  $q > 0$ , it holds that

$$\begin{aligned} & \mathbb{P}_x[e^{-qT_a}, T_a < T_b] \\ &= \frac{h_q(b - a) + h_q(x - b) - h_q(x - a) - h_q(x - b)h_q(b - a)/r_q(0)}{h_q^B(a - b)}. \end{aligned} \tag{3.15}$$

Consequently, it holds that

$$\mathbb{P}_x(T_a < T_b) = \frac{h(b - a) + h(x - b) - h(x - a)}{h^B(a - b)}. \tag{3.16}$$

*Proof.* (i) We omit the proof of (3.13), which can be found in [1, Lemma V.11]. Letting  $q \rightarrow 0+$  in (3.13), and using (2.5), we obtain (3.14).

(ii) By the strong Markov property, it holds that, for  $x, a, b \in \mathbb{R}$ ,  $a \neq b$ ,  $q > 0$ ,

$$\begin{aligned} \mathbb{P}_x[e^{-qT_a}] &= \mathbb{P}_x[e^{-qT_a}; T_a < T_b] + \mathbb{P}_x[e^{-qT_b}; T_b < T_a] \mathbb{P}_b[e^{-qT_a}], \\ \mathbb{P}_x[e^{-qT_b}] &= \mathbb{P}_x[e^{-qT_b}; T_b < T_a] + \mathbb{P}_x[e^{-qT_a}; T_a < T_b] \mathbb{P}_a[e^{-qT_b}]. \end{aligned}$$

Combining the above two equalities, we have

$$\mathbb{P}_x[e^{-qT_a}; T_a < T_b] = \frac{\mathbb{P}_x[e^{-qT_a}] - \mathbb{P}_x[e^{-qT_b}] \mathbb{P}_b[e^{-qT_a}]}{1 - \mathbb{P}_a[e^{-qT_b}] \mathbb{P}_b[e^{-qT_a}]}.$$

By (2.1), this implies that

$$\begin{aligned} &\mathbb{P}_x[e^{-qT_a}; T_a < T_b] \\ &= \frac{r_q(a-x) - r_q(b-x)r_q(a-b)/r_q(0)}{r_q(0) - r_q(b-a)r_q(a-b)/r_q(0)} \\ &= \frac{h_q(b-a) + h_q(x-b) - h_q(x-a) - h_q(x-b)h_q(b-a)/r_q(0)}{h_q^B(a-b)}. \end{aligned}$$

Hence we obtain (3.15). Letting  $q \rightarrow 0+$  in (3.15), we obtain (3.16). □

**Remark 3.6.** The formulae (3.15) and (3.16) are also discussed in Theorem 6.5 of Gettoor [12]. See also Proposition 5.3, Proposition 5.4 and Remark 5.5 of Yano–Yano–Yor [41].

The next theorem can be proved in the same way as Pantí [18, Lemma 3.10].

**Lemma 3.7** ([18, Lemma 3.10]). *It holds that  $\lim_{x \rightarrow \infty} h^B(x) = \infty$ .*

*Proof.* For completeness of this paper, we give the proof. Let  $e$  be an independent exponential time of mean 1 and set  $e_q := e/q$  for  $q > 0$ . Then we have

$$h^B(x) = \mathbb{P}[L_{T_x}] = \mathbb{P}[L_{T_x}; T_x \leq e_q] + \mathbb{P}[L_{T_x}; T_x > e_q] \geq \mathbb{P}[L_{e_q}; T_x > e_q]. \tag{3.17}$$

Letting  $x \rightarrow \infty$ , we have

$$\liminf_{x \rightarrow \infty} h^B(x) \geq \mathbb{P}[L_{e_q}] = r_q(0), \quad \text{for all } q > 0. \tag{3.18}$$

Since  $X$  is recurrent, i.e.,  $\kappa = 0$  in (2.5), we let  $q \rightarrow 0+$  to obtain  $\liminf_{x \rightarrow \infty} h^B(x) \geq \infty$ . Hence we obtain the desired result. □

The following theorem, which will be used in Section 7, is a generalization of the result in the symmetric case by Yano [38, Theorem 6.1].

**Theorem 3.8.** *For  $a \in \mathbb{R} \setminus \{0\}$ , it holds that*

$$n(T_a < T_0) = \frac{1}{h^B(a)}. \tag{3.19}$$

*Proof.* For  $l > 0$ , it holds that

$$\mathbb{P}(L_{T_a} > l) = \mathbb{P}(T_a > \eta_l) = \mathbb{P}(\sigma_{\{T_a < T_0\}} > l),$$

where  $\sigma_A = \inf\{l: e_l \in A\}$  for  $A \subset \mathcal{D}$ . Since  $\sigma_A$  is the hitting time of the set  $A$  for the Poisson point process  $((l, e_l), l \geq 0)$ , we have

$$\mathbb{P}(L_{T_a} > l) = e^{-ln(T_a < T_0)}. \tag{3.20}$$

For more details, see, e.g., [15, Lemma 6.17]. In particular,  $L_{T_a}$  is exponentially distributed. On the other hand, we know  $h^B(a) = \mathbb{P}[L_{T_a}]$  by Lemma 3.5. Hence we obtain (3.19). □

**3.4 Examples of the renormalized zero resolvent**

**Example 3.9** (Brownian motion). Assume  $X$  is a standard Brownian motion. Since  $m^2 = \sigma^2 = 1$ , it holds that

$$h^{(\gamma)}(x) = |x| + \gamma x = \begin{cases} (1 + \gamma)x & x \geq 0, \\ (1 - \gamma)x & x \leq 0, \end{cases} \text{ for } -1 \leq \gamma \leq 1.$$

**Example 3.10** (Strictly stable process). Assume that  $X$  is a strictly stable process of index  $\alpha \in (1, 2)$  with the Lévy measure

$$\nu(dx) = \begin{cases} c_+ |x|^{-\alpha-1} dx & \text{on } (0, \infty), \\ c_- |x|^{-\alpha-1} dx & \text{on } (-\infty, 0), \end{cases}$$

where  $c_+, c_- \geq 0$  and  $c_+ + c_- > 0$ . Then its characteristic exponent is given by

$$\Psi(\lambda) = c|\lambda|^\alpha \left( 1 - i\beta \operatorname{sgn}(\lambda) \tan \frac{\alpha\pi}{2} \right),$$

where  $c$  and  $\beta$  are constants defined by

$$c = \frac{(c_+ + c_-)\pi}{2\alpha\Gamma(\alpha) \sin(\pi\alpha/2)}, \quad \beta = \frac{c_+ - c_-}{c_+ + c_-}.$$

In this case, we have  $m^2 = \infty$  and the function  $h$  can be represented as

$$h(x) = \frac{1}{K(\alpha)} (1 - \beta \operatorname{sgn}(x)) |x|^{\alpha-1},$$

where

$$K(\alpha) = -2c\Gamma(\alpha) \cos \frac{\pi\alpha}{2} \left( 1 + \beta^2 \tan^2 \frac{\pi\alpha}{2} \right).$$

For more details, see [39, Section 5].

**4 Local time penalization with exponential clock**

We now start to deal with the penalization result with exponential clock. Let  $e$  be an independent exponential time of mean 1 and for  $q > 0$ , we write  $e_q := e/q$ , which has an exponential distribution of mean  $1/q$ . We compute  $\mathbb{P}_x[f(L_{e_q})]$ ,  $\mathbb{P}_x[f(L_{e_q})|\mathcal{F}_t]$  and its limit as  $q \rightarrow 0+$  to investigate  $\lim_{q \rightarrow 0+} \mathbb{P}_x[F_t f(L_{e_q})]/\mathbb{P}_x[f(L_{e_q})]$  for bounded  $\mathcal{F}_t$ -measurable functional  $F_t$ . Recall that we assume  $X$  is recurrent and assume the condition **(A)**.

**4.1 The law of the local time with exponential clock**

First, we compute  $\mathbb{P}_x[f(L_{e_q})]$ .

**Lemma 4.1.** *Let  $f$  be a non-negative measurable function. Then, for  $q > 0$  and  $x \in \mathbb{R}$ , it holds that*

$$\mathbb{P}_x[f(L_{e_q})] = \frac{1}{r_q(0)} \left\{ h_q(x) f(0) + \left( 1 - \frac{h_q(x)}{r_q(0)} \right) \int_0^\infty e^{-u/r_q(0)} f(u) du \right\}. \tag{4.1}$$

*Proof.* Using the excursion theory, we have

$$\begin{aligned} \mathbb{P}_0 \left[ \int_0^\infty f(L_t) q e^{-qt} dt \right] &= \mathbb{P}_0 \left[ \sum_{u \in D} \int_{\eta_u-}^{\eta_u} f(u) q e^{-qu} du \right] \\ &= \mathbb{P}_0 \otimes \tilde{n} \left[ \int_0^\infty dL_t f(L_t) e^{-qt} \int_0^{\tilde{T}_0} du q e^{-qu} \right], \end{aligned}$$

where the last equality follows from Lemma 2.5. By (2.3) and (2.4), we have

$$\begin{aligned} \mathbb{P}_0 \otimes \tilde{n} \left[ \int_0^\infty dL_t f(L_t) e^{-qt} \int_0^{\tilde{T}_0} du q e^{-qu} \right] &= \mathbb{P}_0 \left[ \int_0^\infty du f(u) e^{-qu} \right] n [1 - e^{-qT_0}] \\ &= \frac{1}{r_q(0)} \int_0^\infty f(u) e^{-u/r_q(0)} du. \end{aligned}$$

Applying the Markov property, we obtain

$$\begin{aligned} \mathbb{P}_x[f(L_{e_q})] &= \mathbb{P}_x \left[ \int_0^\infty f(L_t) q e^{-qt} dt \right] \\ &= \mathbb{P}_x \left[ \int_0^{T_0} f(0) q e^{-qt} dt \right] + \mathbb{P}_x[e^{-qT_0}] \mathbb{P}_0 \left[ \int_0^\infty f(L_t) q e^{-qt} dt \right] \\ &= f(0) \left( 1 - \frac{r_q(-x)}{r_q(0)} \right) + \frac{r_q(-x)}{r_q(0)} \int_0^\infty f(u) e^{-u/r_q(0)} du \\ &= \frac{1}{r_q(0)} \left\{ h_q(x) f(0) + \left( 1 - \frac{h_q(x)}{r_q(0)} \right) \int_0^\infty e^{-u/r_q(0)} f(u) du \right\}, \end{aligned}$$

here we used (2.1). Therefore we obtain the desired result. □

#### 4.2 A.s. convergence for exponential clock

To calculate  $\mathbb{P}_x[F(L_{e_q})|\mathcal{F}_t]$ , we separate into the two cases  $\{t < e_q\}$  and  $\{e_q \leq t\}$ . Let  $f \in \mathcal{L}_+^1$  and  $x \in \mathbb{R}$ . For  $q > 0$ , define

$$\begin{aligned} N_t^q &= r_q(0) \mathbb{P}_x[f(L_{e_q}); t < e_q | \mathcal{F}_t], \\ M_t^q &= r_q(0) \mathbb{P}_x[f(L_{e_q}) | \mathcal{F}_t], \\ A_t^q &= M_t^q - N_t^q = r_q(0) \mathbb{P}_x[f(L_{e_q}); e_q \leq t | \mathcal{F}_t], \end{aligned}$$

and  $h^{(\gamma)}$  and  $M_t^{(\gamma)}$  are defined in (1.7) and (1.8).

**Theorem 4.2.** For  $f \in \mathcal{L}_+^1$  and  $x \in \mathbb{R}$ , it holds that

$$\lim_{q \rightarrow 0+} N_t^q = \lim_{q \rightarrow 0+} M_t^q = M_t^{(0)}, \quad \mathbb{P}_x\text{-a.s.}$$

*Proof.* By the Markov property and the additivity of  $L$ , we have

$$\begin{aligned} N_t^q &= r_q(0) \mathbb{P}_x[f(L_{e_q}); t < e_q | \mathcal{F}_t] \\ &= r_q(0) e^{-qt} \tilde{\mathbb{P}}_{X_t}[f(L_t + \tilde{L}_{\tilde{e}_q})] \\ &= e^{-qt} \left\{ h_q(X_t) f(L_t) + \left( 1 - \frac{h_q(X_t)}{r_q(0)} \right) \int_0^\infty e^{-u/r_q(0)} f(L_t + u) du \right\}, \end{aligned}$$

here the last equality, we used Lemma 4.1. Since  $1 - \frac{h_q(X_t)}{r_q(0)} = \mathbb{P}_{X_t}[e^{-qT_0}] \rightarrow 1$ ,  $\mathbb{P}_x$ -a.s. as  $q \rightarrow 0+$  and since  $\int_0^\infty f(L_t + u) du < \infty$ , we may apply the dominated convergence theorem to deduce that  $N_t^q \rightarrow M_t^{(0)}$ ,  $\mathbb{P}_x$ -a.s. as  $q \rightarrow 0+$ . By (iv) of Lemma 2.4, we have

$$A_t^q = r_q(0) \mathbb{P}_x[f(L_{e_q}); e_q \leq t | \mathcal{F}_t] = qr_q(0) \int_0^t f(L_u) e^{-qu} du \rightarrow 0$$

$\mathbb{P}_x$ -a.s. as  $q \rightarrow 0+$ . Therefore, we obtain  $M_t^q \rightarrow M_t^{(0)}$ ,  $\mathbb{P}_x$ -a.s. as  $q \rightarrow 0+$ . □

**4.3  $\mathcal{L}^1$  convergence for exponential clock**

Now we prepare some lemma to prove the  $\mathcal{L}^1$  convergence for exponential clock. The following lemma is a part of Theorem 15.2 of Tsukada [35].

**Lemma 4.3** ([35, Theorem 15.2]). *For  $t \geq 0$ , it holds that  $h_q(X_t) \rightarrow h(X_t)$  in  $\mathcal{L}^1(\mathbb{P}_x)$  as  $q \rightarrow 0+$ .*

The next theorem is the penalization result with exponential clock.

**Theorem 4.4.** *Let  $f \in \mathcal{L}^1_+$  and  $x \in \mathbb{R}$ . Then  $(M_t^{(0)}, t \geq 0)$  is a non-negative  $((\mathcal{F}_t), \mathbb{P}_x)$ -martingale, and it holds that*

$$\lim_{q \rightarrow 0+} N_t^q = \lim_{q \rightarrow 0+} M_t^q = M_t^{(0)}, \quad \text{in } \mathcal{L}^1(\mathbb{P}_x).$$

Consequently, if  $M_0^{(0)} > 0$  under  $\mathbb{P}_x$ , it holds that

$$\frac{\mathbb{P}_x[F_t f(L_{e_q})]}{\mathbb{P}_x[f(L_{e_q})]} \rightarrow \mathbb{P}_x \left[ F_t \frac{M_t^{(0)}}{M_0^{(0)}} \right], \quad \text{as } q \rightarrow 0+, \tag{4.2}$$

for all bounded  $\mathcal{F}_t$ -measurable functionals  $F_t$ .

Note that the penalized measure in (4.2) is not the same as that of Theorems 1.5 and 1.7.

*Proof of Theorem 4.4.* We first consider the case where  $f$  is bounded. We write

$$\begin{aligned} N_t^q &= e^{-qt} \left\{ h_q(X_t) f(L_t) + \left( 1 - \frac{h_q(X_t)}{r_q(0)} \right) \int_0^\infty e^{-u/r_q(0)} f(L_t + u) du \right\} \\ &=: (I)_q + (II)_q, \\ M_t^{(0)} &= h(X_t) f(L_t) + \int_0^\infty f(L_t + u) du \\ &=: (I) + (II). \end{aligned}$$

By Lemma 4.3 and by the boundedness of  $f$ , we obtain  $(I)_q \rightarrow (I)$  in  $\mathcal{L}^1(\mathbb{P}_x)$ . Moreover, since  $\int_0^\infty f(u) du < \infty$ , it follows from the dominated convergence theorem that  $(II)_q \rightarrow (II)$  in  $\mathcal{L}^1(\mathbb{P}_x)$ . Hence we obtain  $N_t^q \rightarrow M_t^{(0)}$  in  $\mathcal{L}^1(\mathbb{P}_x)$ . By (iv) of Lemma 2.4, we have

$$\mathbb{P}_x[A_t^q] = qr_q(0) \mathbb{P}_x \left[ \int_0^t e^{-qu} f(L_u) du \right] \leq qr_q(0)t \|f\| \rightarrow 0, \quad \text{as } q \rightarrow 0+.$$

Since  $A_t^q \geq 0$ , this means that  $A_t^q \rightarrow 0$  in  $\mathcal{L}^1(\mathbb{P}_x)$ . Thus we have  $M_t^q \rightarrow M_t^{(0)}$  in  $\mathcal{L}^1(\mathbb{P}_x)$ . For  $0 \leq s \leq t$ , we know  $\mathbb{P}_x[M_t^q | \mathcal{F}_s] = M_s^q$ . Letting  $q \rightarrow 0+$  on both sides, we have

$$\mathbb{P}_x[M_t^{(0)} | \mathcal{F}_s] = M_s^{(0)}, \tag{4.3}$$

which means that  $(M_t^{(0)}, t \geq 0)$  is a non-negative  $((\mathcal{F}_t), \mathbb{P}_x)$ -martingale.

Let us consider the general  $f \in \mathcal{L}^1_+$ . We know the equality (4.3) holds for  $f \wedge n$ . Letting  $n \rightarrow \infty$ , (4.3) holds for general  $f \in \mathcal{L}^1_+$  by the monotone convergence theorem. Hence we have

$$\lim_{q \rightarrow 0+} \mathbb{P}_x[M_t^q] = \lim_{q \rightarrow 0+} M_0^q = M_0^{(0)} = \mathbb{P}_x[M_t^{(0)}],$$

and by Fatou's lemma,

$$M_0^{(0)} = \lim_{q \rightarrow 0+} \mathbb{P}_x[M_t^q] \geq \limsup_{q \rightarrow 0+} \mathbb{P}_x[N_t^q] \geq \liminf_{q \rightarrow 0+} \mathbb{P}_x[N_t^q] \geq M_0^{(0)}.$$

Thus we have  $\lim_{q \rightarrow 0+} \mathbb{P}_x[N_t^q] = \lim_{q \rightarrow 0+} \mathbb{P}_x[M_t^q] = \mathbb{P}_x[M_t^{(0)}]$ . Applying Scheffé's lemma, we obtain  $\lim_{q \rightarrow 0+} N_t^q = \lim_{q \rightarrow 0+} M_t^q = M_t^{(0)}$  in  $\mathcal{L}^1(\mathbb{P}_x)$ .  $\square$

## 5 Local time penalization with hitting time clock

We deal with the penalization result with hitting time clock  $(T_a)$ . To this aim, we compute  $\mathbb{P}_x[L_{T_a}]$  and  $\mathbb{P}_x[L_{T_a}|\mathcal{F}_t]$  and its limit as  $a \rightarrow \pm\infty$ . Recall that we assume  $X$  is recurrent and assume the condition **(A)**.

### 5.1 The law of the local time with hitting time clock

First, we compute  $\mathbb{P}_x[f(L_{T_a})]$ .

**Lemma 5.1.** For  $x, a \in \mathbb{R}$ ,  $a \neq 0$  and any non-negative measurable function  $f$ , we have

$$\mathbb{P}_x[f(L_{T_a})] = \mathbb{P}_x(T_0 > T_a)f(0) + \frac{\mathbb{P}_x(T_0 < T_a)}{h^B(a)} \int_0^\infty e^{-u/h^B(a)} f(u) du.$$

*Proof.* By (3.20),  $L_{T_a}$  is exponentially distributed with parameter  $1/h^B(a)$ . Hence we have

$$\mathbb{P}_0[f(L_{T_a})] = \frac{1}{h^B(a)} \int_0^\infty e^{-u/h^B(a)} f(u) du.$$

Applying the Markov property, we conclude that

$$\begin{aligned} \mathbb{P}_x[f(L_{T_a})] &= \mathbb{P}_x(T_0 > T_a)f(0) + \mathbb{P}_x(T_0 < T_a)\mathbb{P}_0[f(L_{T_a})] \\ &= \mathbb{P}_x(T_0 > T_a)f(0) + \frac{\mathbb{P}_x(T_0 < T_a)}{h^B(a)} \int_0^\infty e^{-u/h^B(a)} f(u) du. \end{aligned}$$

Hence the proof is complete. □

### 5.2 Proof of a.s. convergence of Theorem 1.5

We now proceed the proof of the hitting time result. We separate into the two cases  $\{t < T_a\}$  and  $\{T_a \leq t\}$ .

*Proof of a.s. convergence of Theorem 1.5.* By the strong Markov property, the additivity of  $L$  and Lemma 5.1, we have

$$\begin{aligned} N_t^a &= 1_{\{t < T_a\}} h^B(a) \tilde{\mathbb{P}}_{X_t}[f(\tilde{L}_{\tilde{T}_a} + L_t)] \\ &= 1_{\{t < T_a\}} \left\{ h^B(a) \mathbb{P}_{X_t}(T_0 > T_a) f(L_t) + \mathbb{P}_{X_t}(T_0 < T_a) \int_0^\infty e^{-u/h^B(a)} f(L_t + u) du \right\}, \end{aligned}$$

$\mathbb{P}_x$ -a.s. as  $a \rightarrow \pm\infty$ . (ii) of Theorem 1.2 and (3.16) imply that

$$\begin{aligned} h^B(a) \mathbb{P}_{X_t}(T_0 > T_a) &= h(X_t) + h(-a) - h(X_t - a) \longrightarrow h^{(\pm 1)}(X_t), \\ \mathbb{P}_{X_t}(T_0 < T_a) &\longrightarrow 1, \end{aligned}$$

$\mathbb{P}_x$ -a.s. as  $a \rightarrow \pm\infty$ . By Lemma 3.7, we obtain  $N_t^a \rightarrow M_t^{(\pm 1)}$ ,  $\mathbb{P}_x$ -a.s. as  $a \rightarrow \pm\infty$ . Furthermore, we have

$$M_t^a - N_t^a = h^B(a) \mathbb{P}_x[f(L_{T_a}); T_a \leq t | \mathcal{F}_t] \tag{5.1}$$

$$\begin{aligned} &= h^B(a) f(L_{T_a}) 1_{\{T_a \leq t\}} \\ &\rightarrow 0, \quad \mathbb{P}_x\text{-a.s. as } a \rightarrow \pm\infty. \end{aligned} \tag{5.2}$$

Hence we obtain  $M_t^a \rightarrow M_t^{(\pm 1)}$   $\mathbb{P}_x$ -a.s. as  $a \rightarrow \pm\infty$ . □

**5.3 Proof of  $\mathcal{L}^1$  convergence of Theorem 1.5**

*Proof of  $\mathcal{L}^1$  convergence of Theorem 1.5.* We first consider the case  $m^2 = \infty$ . Then we know  $M_t^{(\pm 1)} = M_t^{(0)}$ . Hence, by Theorem 4.4,  $(M_t^{(\pm 1)}, t \geq 0)$  is a non-negative  $((\mathcal{F}_t), \mathbb{P}_x)$ -martingale. Thus we have

$$\mathbb{P}_x[M_t^a] = M_0^a \rightarrow M_0^{(\pm 1)} = \mathbb{P}_x[M_t^{(\pm 1)}], \quad \text{as } a \rightarrow \pm\infty.$$

By Fatou’s lemma, we have

$$\mathbb{P}_x[M_t^{(\pm 1)}] = \lim_{a \rightarrow \pm\infty} \mathbb{P}_x[M_t^a] \geq \limsup_{a \rightarrow \pm\infty} \mathbb{P}_x[N_t^a] \geq \liminf_{a \rightarrow \pm\infty} \mathbb{P}_x[N_t^a] \geq \mathbb{P}_x[M_t^{(\pm 1)}].$$

Consequently, it holds that  $\mathbb{P}_x[N_t^a], \mathbb{P}_x[N_t^a] \rightarrow \mathbb{P}_x[M_t^{(\pm 1)}]$ , as  $a \rightarrow \pm\infty$ . Applying Scheffé’s lemma, we obtain  $N_t^a, N_t^a \rightarrow M_t^{(\pm 1)}$  in  $\mathcal{L}^1(\mathbb{P}_x)$  as  $a \rightarrow \pm\infty$ .

We next consider the case  $m^2 < \infty$ . Suppose first that  $f$  is bounded. We write

$$\begin{aligned} N_t^a &= 1_{\{t < T_a\}}(h(X_t) + h(-a) - h(X_t - a))f(L_t) \\ &\quad + 1_{\{t < T_a\}}\mathbb{P}_{X_t}(T_0 < T_a) \int_0^\infty e^{-u/h^B(a)} f(L_t + u) du \\ &=: (I)_a + (II)_a, \\ M_t^{(\pm 1)} &= h^{(\pm 1)}(X_t)f(L_t) + \int_0^\infty f(L_t + u) du \\ &=: (I) + (II). \end{aligned}$$

Since  $h$  is subadditive, we have  $h(X_t) + h(-a) - h(X_t - a) \leq h(X_t) + h(-X_t)$ . By the proof of Lemma 4.3, we know  $\mathbb{P}_x[h(X_t) + h(-X_t)] < \infty$  and thus, by the dominated convergence theorem, we have

$$h(X_t) + h(-a) - h(X_t - a) \rightarrow h^{(\pm 1)}(X_t), \quad \text{in } \mathcal{L}^1(\mathbb{P}_x) \text{ as } a \rightarrow \pm\infty.$$

The boundedness of  $f$  implies that  $(I)_a \rightarrow (I)$  in  $\mathcal{L}^1(\mathbb{P}_x)$  as  $a \rightarrow \pm\infty$ . Since  $(II)_a \leq \int_0^\infty f(u) du$ , we may apply the dominated convergence theorem to conclude  $(II)_a \rightarrow (II)$  in  $\mathcal{L}^1(\mathbb{P}_x)$  as  $a \rightarrow \pm\infty$ . Hence we obtain  $N_t^a \rightarrow M_t^{(\pm 1)}$  in  $\mathcal{L}^1(\mathbb{P}_x)$ . By Theorem 1.2 and the optional stopping theorem, we obtain

$$\begin{aligned} \mathbb{P}_x[A_t^a] &= h^B(a)\mathbb{P}_x[f(L_{T_a}); T_a \leq t] \\ &= \frac{h^B(a)}{a} \frac{a}{h(a)} \mathbb{P}_x[h(X_{T_a})f(L_{T_a}); T_a \leq t] \\ &\leq \frac{h^B(a)}{a} \frac{a}{h(a)} \mathbb{P}_x[M_{T_a}^{(0)}; T_a \leq t] \\ &= \frac{h^B(a)}{a} \frac{a}{h(a)} \mathbb{P}_x[M_t^{(0)}; T_a \leq t] \\ &\rightarrow 0, \quad \text{as } a \rightarrow \pm\infty. \end{aligned}$$

This yields that  $A_t^a \rightarrow 0$  in  $\mathcal{L}^1(\mathbb{P}_x)$  as  $a \rightarrow \pm\infty$ . Hence  $M_t^a \rightarrow M_t^{(\pm 1)}$  in  $\mathcal{L}^1(\mathbb{P}_x)$  as  $a \rightarrow \pm\infty$ . Since  $\mathbb{P}_x[M_t^a | \mathcal{F}_s] = M_s^a$  for  $0 \leq s \leq t$ , we let  $a \rightarrow \pm\infty$  to obtain

$$\mathbb{P}_x[M_t^{(\pm 1)} | \mathcal{F}_s] = M_s^{(\pm 1)}. \tag{5.3}$$

In particular,  $(M_t^{(\pm 1)}, t \geq 0)$  is a non-negative  $((\mathcal{F}_t), \mathbb{P}_x)$ -martingale. To remove the boundedness condition of  $f$ , we consider  $f \wedge n$  and let  $n \rightarrow \infty$  in (5.3). Then we have  $(M_t^{(\pm 1)}, t \geq 0)$  is a non-negative  $((\mathcal{F}_t), \mathbb{P}_x)$ -martingale. The remainder of the proof is the same as that of Theorem 4.4. So we omit it.  $\square$

**Remark 5.2.** Suppose that  $m^2 < \infty$ . Then, since  $M_t^{(0)}$  and  $M_t^{(1)}$  are different  $((\mathcal{F}_t), \mathbb{P}_x)$ -martingales,  $m^2(M_t^{(1)} - M_t^{(0)}) = X_t f(L_t)$  is also a  $((\mathcal{F}_t), \mathbb{P}_x)$ -martingale. In fact, if  $X_1$  is integrable,  $\mathbb{P}[X_1] = 0$  and  $f$  is a bounded measurable function, then  $(X_t f(L_t), t \geq 0)$  is a  $((\mathcal{F}_t), \mathbb{P}_x)$ -martingale. For more details, see Theorem 10.1.

By Remark 5.2, we can prove Theorem 1.4.

*Proof of Theorem 1.4.* Since  $h^{(\gamma)}$  is non-negative for  $-1 \leq \gamma \leq 1$ , we have  $M_t^{(\gamma)} \geq 0$ . Since  $(M_t^{(0)}, t \geq 0)$  and  $(X_t f(L_t), t \geq 0)$  are  $((\mathcal{F}_t), \mathbb{P}_x)$ -martingales, the process  $(M_t^{(\gamma)} = M_t^{(0)} + \frac{\gamma}{m^2} X_t f(L_t), t \geq 0)$  is also a  $((\mathcal{F}_t), \mathbb{P}_x)$ -martingale.  $\square$

## 6 Local time penalization with two-point hitting time clock

Let us consider the hitting time of two-point set, i.e.,  $T_{a,b} = T_{\{a,b\}} = T_a \wedge T_b$ . Recall that we assume  $X$  is recurrent and assume the condition **(A)**.

### 6.1 The law of the local time with two-point hitting time clock

First, we compute  $\mathbb{P}[L_{T_a \wedge T_b}]$ ,  $\mathbb{P}_x(T_a < T_b \wedge T_c)$  and  $\mathbb{P}_x[f(L_{T_a \wedge T_b})]$  respectively.

**Lemma 6.1.** For  $a \neq b$ , it holds that

$$\begin{aligned} h^C(a, b) &:= \mathbb{P}[L_{T_a \wedge T_b}] \\ &= \frac{1}{h^B(a - b)} \left\{ \begin{aligned} &(h(b) + h(-a))h(a - b) + (h(a) + h(-b))h(b - a) \\ &- h(a - b)h(b - a) \end{aligned} \right\}. \end{aligned}$$

*Proof.* For  $q > 0$ , by the strong Markov property, we have

$$\begin{aligned} \mathbb{P} \left[ \int_0^\infty e^{-qt} dL_t \right] &= \mathbb{P} \left[ \int_0^{T_a \wedge T_b} e^{-qt} dL_t \right] \\ &\quad + \mathbb{P}[e^{-qT_a}; T_a < T_b] \mathbb{P}_a \left[ \int_0^\infty e^{-qt} dL_t \right] \\ &\quad + \mathbb{P}[e^{-qT_b}; T_b < T_a] \mathbb{P}_b \left[ \int_0^\infty e^{-qt} dL_t \right]. \end{aligned}$$

Using (2.2) and (ii) of Lemma 3.5 and letting  $q \rightarrow 0+$ , we obtain the desired result.  $\square$

**Lemma 6.2.** For  $a, b, c, x \in \mathbb{R}$  and  $a \neq b, c$ , it holds that

$$\begin{aligned} &\mathbb{P}_x[e^{-qT_a}; T_a < T_b \wedge T_c] \\ &= \frac{\mathbb{P}_x[e^{-qT_a}; T_a < T_b] - \mathbb{P}_x[e^{-qT_c}; T_c < T_b] - \mathbb{P}_c[e^{-qT_a}; T_a < T_b]}{1 - \mathbb{P}_a[e^{-qT_c}; T_c < T_b] \mathbb{P}_c[e^{-qT_a}; T_a < T_b]}, \end{aligned} \tag{6.1}$$

and letting  $q \rightarrow 0+$ , it holds that

$$\mathbb{P}_x(T_a < T_b \wedge T_c) = \frac{\mathbb{P}_x(T_a < T_b) - \mathbb{P}_x(T_c < T_b) \mathbb{P}_c(T_a < T_b)}{1 - \mathbb{P}_a(T_c < T_b) \mathbb{P}_c(T_a < T_b)}. \tag{6.2}$$

Note that, by (3.16), we can express  $\mathbb{P}_x(T_a < T_b \wedge T_c)$  only in terms of  $h$ .

*Proof of Lemma 6.2.* Using the strong Markov property, we have

$$\begin{aligned} &\mathbb{P}_x[e^{-qT_a}; T_a < T_b] \\ &= \mathbb{P}_x[e^{-qT_a}; T_a < T_b \wedge T_c] + \mathbb{P}_x[e^{-qT_a}; T_c < T_a < T_b] \\ &= \mathbb{P}_x[e^{-qT_a}; T_a < T_b \wedge T_c] + \mathbb{P}_x[e^{-qT_c}; T_c < T_a \wedge T_b] \mathbb{P}_c[e^{-qT_a}; T_a < T_b]. \end{aligned} \tag{6.3}$$

Replacing  $a$  with  $c$ , we also have

$$\begin{aligned} & \mathbb{P}_x[e^{-qT_c}; T_c < T_b] \\ &= \mathbb{P}_x[e^{-qT_c}; T_c < T_a \wedge T_b] + \mathbb{P}_x[e^{-qT_c}; T_a < T_b \wedge T_c] \mathbb{P}_a[e^{-qT_c}; T_c < T_b]. \end{aligned} \tag{6.4}$$

Combining (6.3) and (6.4), we obtain (6.1). Letting  $q \rightarrow 0+$ , we also have (6.2). □

**Lemma 6.3.** *For  $a \neq b$  and  $x \in \mathbb{R}$ , it holds that*

$$\mathbb{P}_x[f(L_{T_a \wedge T_b})] = \mathbb{P}_x(T_0 > T_a \wedge T_b) f(0) + \frac{\mathbb{P}_x(T_0 < T_a \wedge T_b)}{h^C(a, b)} \int_0^\infty e^{-u/h^C(a, b)} f(u) du.$$

*Proof.* In the same way as (3.20), we have, for  $l > 0$ ,

$$\mathbb{P}(L_{T_a \wedge T_b} > l) = \mathbb{P}(\sigma_{\{T_a \wedge T_b < T_0\}} > l) = e^{-tn(T_a \wedge T_b < T_0)}.$$

In particular,  $L_{T_a \wedge T_b}$  is exponentially distributed. (Consequently, we obtain  $n(T_a \wedge T_b < T_0) = 1/\mathbb{P}[L_{T_a, b}] = 1/h^C(a, b)$ .) We may apply the strong Markov property and obtain the desired result. □

### 6.2 Proof of Theorem 1.7

We are now ready to prove the two-point hitting time result. Recall that  $(a, b) \xrightarrow{\gamma} \infty$  means (1.9).

*Proof of Theorem 1.7.* By Lemmas 6.1 and 6.2 and (3.16), we have

$$\begin{aligned} & h^C(a, b) \mathbb{P}_x(T_0 > T_a \wedge T_b) \\ &= h(x) + \frac{1}{h^B(a - b)} \left\{ \begin{aligned} & (h(-a) - h(x - a))h(a - b) + (h(-b) - h(x - b))h(b - a) \\ & - (h(a) - h(b))(h(-a) - h(x - a) - h(-b) + h(x - b)) \end{aligned} \right\}. \end{aligned}$$

Recall that  $h^B(a) = h(a) + h(-a)$ ; see (3.14). Replacing  $b$  with  $-b$  and using Theorem 1.2, it holds that

$$h^C(a, -b) \mathbb{P}_x(T_0 > T_a \wedge T_{-b}) \xrightarrow{(a, b) \xrightarrow{\gamma} \infty} h^{(\gamma)}(x).$$

By the strong Markov property and Lemma 6.3, we have

$$\begin{aligned} N_t^{a, b} &= 1_{\{t < T_{a, -b}\}} \{h^C(a, -b) \mathbb{P}_{X_t}(T_0 > T_{a, -b}) f(L_t)\} \\ &+ 1_{\{t < T_{a, -b}\}} \left\{ \mathbb{P}_{X_t}(T_0 < T_{a, -b}) \int_0^\infty e^{-u/h^C(a, -b)} f(L_t + u) dy \right\}. \end{aligned}$$

Hence we obtain  $N_t^{a, b} \rightarrow M_t^{(\gamma)}$ ,  $\mathbb{P}_x$ -a.s. as  $(a, b) \xrightarrow{\gamma} \infty$ . The proof of  $M_t^{a, b} \rightarrow M_t^{(\gamma)}$ ,  $\mathbb{P}_x$ -a.s. is similar to (5.1)–(5.2). Since  $(M_t^{(\gamma)}, t \geq 0)$  is a non-negative  $(\mathcal{F}_t, \mathbb{P}_x)$ -martingale (Theorem 1.4), we may apply Scheffé’s lemma to obtain the  $\mathcal{L}^1$  convergence. □

## 7 Local time penalization with inverse local time clock

We define the modified Bessel function of the first kind, which is expressed as

$$I_\nu(x) = \sum_{n=0}^\infty \frac{(x/2)^{\nu+2n}}{n! \Gamma(\nu + n + 1)}, \quad \nu \geq 0, x > 0.$$

Note that  $I_\nu(x)$  is increasing in  $x > 0$ . For more details, see e.g., [16, Section 5]. Recall that  $\eta_u^a$  denotes the inverse local time at  $a$ :  $\eta_u^a = \inf\{t \geq 0: L_t^a > u\}$ . We consider the penalization with inverse local time clock in two ways: first, we make  $a$  tend to infinity, and second,  $u$  tend to infinity. Recall that we assume  $X$  is recurrent and assume the condition **(A)**.

**7.1 The law of the local time with inverse local time clock**

**Lemma 7.1.** *Let  $a \in \mathbb{R} \setminus \{0\}$ . Then the process  $(L_{\eta_u^a}, u \geq 0)$  under  $\mathbb{P}_a$  is a compound Poisson process with Laplace transform*

$$\mathbb{P}_a[e^{-\beta L_{\eta_u^a}}] = e^{-u\beta/(1+\beta h^B(a))}, \quad \beta \geq 0. \tag{7.1}$$

Moreover, for any  $u > 0$  and  $f \in \mathcal{L}_+^1$ , it holds that

$$\mathbb{P}_a[f(L_{\eta_u^a})] = e^{-u/h^B(a)} f(0) + \int_0^\infty f(y) \rho_u^{h^B(a)}(y) dy,$$

where

$$\rho_u^a(y) = e^{-(u+y)/a} \frac{\sqrt{u/y}}{a} I_1\left(\frac{2\sqrt{uy}}{a}\right).$$

We omit the proof because it is very similar to that in the diffusion case of Profeta–Yano–Yano [25, Lemma 4.1], using Theorem 3.8. For the proof, we use  $n^a(T_0 < T_a) = n(T_{-a} < T_0) = 1/h^B(a)$  in the Lévy case, instead of  $n^a(T_0 < T_a) = n(T_a < T_0) = 1/a$  in the diffusion case. The proof of (7.1) can also be found in [1, Lemma V.13].

The proof of the next lemma is completely parallel to that of [25, Lemma 4.2]. So we omit it.

**Lemma 7.2.** *For  $u > 0, x, a \in \mathbb{R}$ , it holds that*

$$\begin{aligned} \mathbb{P}_x[f(L_{\eta_u^a})] &= \mathbb{P}_x(T_a < T_0) \mathbb{P}_a[f(L_{\eta_u^a})] + \mathbb{P}_x(T_0 < T_a) \mathbb{P}_a[f(e_{1/h^B(a)} + L_{\eta_u^a})] \\ &= \mathbb{P}_x(T_a < T_0) \mathbb{P}_a[f(L_{\eta_u^a})] + \frac{\mathbb{P}_x(T_0 < T_a)}{h^B(a)} \int_0^\infty f(y) \tilde{\rho}_u^{h^B(a)}(y) dy, \end{aligned}$$

where

$$\tilde{\rho}_u^a(y) = e^{-(u+y)/a} I_0\left(\frac{2\sqrt{uy}}{a}\right).$$

**7.2 Limit as  $a$  tends to infinity with  $u$  being fixed**

**Theorem 7.3.** *Let  $f \in \mathcal{L}_+^1$  and  $x \in \mathbb{R}$ . For any  $u > 0$  and  $a \in \mathbb{R}$ , we define*

$$\begin{aligned} N_t^{a,u} &= h^B(a) \mathbb{P}_x[f(L_{\eta_u^a}); t < \eta_u^a | \mathcal{F}_t], \\ M_t^{a,u} &= h^B(a) \mathbb{P}_x[f(L_{\eta_u^a}) | \mathcal{F}_t]. \end{aligned}$$

Then

$$\lim_{a \rightarrow \pm\infty} N_t^{a,u} = \lim_{a \rightarrow \pm\infty} M_t^{a,u} = M_t^{(\pm 1)}, \quad \mathbb{P}_x\text{-a.s. and in } \mathcal{L}^1(\mathbb{P}_x).$$

Consequently, if  $M_0^{(\pm 1)} > 0$  under  $\mathbb{P}_x$ , it holds that

$$\frac{\mathbb{P}_x[F_t f(L_{\eta_u^a})]}{\mathbb{P}_x[f(L_{\eta_u^a})]} \longrightarrow \mathbb{P}_x \left[ \frac{M_t^{(\pm 1)}}{M_0^{(\pm 1)}} \right], \quad \text{as } a \rightarrow \pm\infty,$$

for all bounded  $\mathcal{F}_t$ -measurable functionals  $F_t$ .

The proof of the theorem is very similar to that of [25, Lemma 4.4 and Theorem 4.5]. So we omit it. ([25, Lemma 4.4] states only convergence in probability but, its a.s. convergence can also be proved by the same proof.)

### 7.3 Limit as $u$ tends to infinity with $a$ being fixed

In this section, we only consider the cases  $f(x) = e^{-\beta x}$  and  $f(x) = 1_{\{x=0\}}$ . The next theorem is in the case  $f(x) = e^{-\beta x}$ .

**Theorem 7.4.** *Let  $x \in \mathbb{R}$ ,  $a \in \mathbb{R} \setminus \{0\}$ ,  $\beta > 0$  and  $t > 0$ . Define*

$$N_t^{u,\beta,a} = e^{\beta u/(1+\beta h^B(a))} \mathbb{P}_x[e^{-\beta L_{\eta_u^a}}; t < \eta_u^a | \mathcal{F}_t],$$

$$M_t^{u,\beta,a} = e^{\beta u/(1+\beta h^B(a))} \mathbb{P}_x[e^{-\beta L_{\eta_u^a}} | \mathcal{F}_t],$$

and

$$M_t^{\beta,a} = e^{-\beta L_t} \left\{ \mathbb{P}_{X_t}(T_a < T_0) + \frac{\mathbb{P}_{X_t}(T_0 < T_a)}{1 + \beta h^B(a)} e^{\beta L_t^a/(1+\beta h^B(a))} \right\}.$$

Then it holds that

$$\lim_{u \rightarrow \infty} N_t^{u,\beta,a} = \lim_{u \rightarrow \infty} M_t^{u,\beta,a} = M_t^{\beta,a}, \quad \mathbb{P}_x\text{-a.s. and in } \mathcal{L}^1(\mathbb{P}_x).$$

The proof of Theorem 7.4 is parallel to that of [25, Lemma 4.6 and Theorem 4.7]. So we omit it.

We consider the case  $f(x) = 1_{\{x=0\}}$ .

**Theorem 7.5.** *Let  $x \in \mathbb{R}$ ,  $a \in \mathbb{R} \setminus \{0\}$ ,  $\beta > 0$  and  $t > 0$ . Define*

$$N_t^{u,\infty,a} = e^{u/h^B(a)} \mathbb{P}_x(t < \eta_u^a < T_0 | \mathcal{F}_t),$$

$$M_t^{u,\infty,a} = e^{u/h^B(a)} \mathbb{P}_x(\eta_u^a < T_0 | \mathcal{F}_t),$$

and

$$M_t^{\infty,a} = e^{L_t^a/h^B(a)} \mathbb{P}_{X_t}(T_a < T_0) 1_{\{t < T_0\}}.$$

Then it holds that

$$\lim_{u \rightarrow \infty} N_t^{u,\infty,a} = \lim_{u \rightarrow \infty} M_t^{u,\infty,a} = M_t^{\infty,a}, \quad \mathbb{P}_x\text{-a.s. and in } \mathcal{L}^1(\mathbb{P}_x).$$

The proof of Theorem 7.5 is very similar to that of [25, Theorem 4.8]. So we omit it.

## 8 Universal $\sigma$ -finite measures

In this section, we shall discuss the penalized processes and  $\sigma$ -finite measures unifying the processes. We have obtained the penalized measure  $\mathbb{Q}_x^{(\gamma,f)}$ , which is given by

$$\mathbb{Q}_x^{(\gamma,f)} | \mathcal{F}_t = \frac{M_t^{(\gamma,f)}}{M_0^{(\gamma,f)}} \cdot \mathbb{P}_x | \mathcal{F}_t, \quad -1 \leq \gamma \leq 1.$$

### 8.1 Lévy processes conditioned to avoid zero

We show that  $h^{(\gamma)}$  is invariant for the killed process.

**Theorem 8.1.** *For  $-1 \leq \gamma \leq 1$ , it holds that*

$$\mathbb{P}_x[h^{(\gamma)}(X_t); T_0 > t] = h^{(\gamma)}(x), \quad n[h^{(\gamma)}(X_t); T_0 > t] = 1, \quad x \in \mathbb{R}.$$

*Proof.* We can show the case  $\gamma = 0$  by the completely same discussion as the proof of Pantí [18, (iii) of Theorem 2.2]. Combining this with Theorem 10.1, we obtain the desired result. The former equation also follows from the fact that  $(M_t^{(\gamma,1_{\{u=0\}})}, t \geq 0)$  is a  $((\mathcal{F}_t), \mathbb{P}_x)$ -martingale.  $\square$

Let  $\mathcal{H}^{(\gamma)} = \{x \in \mathbb{R} : h^{(\gamma)}(x) > 0\}$  and  $\mathcal{H}_0^{(\gamma)} = \mathcal{H}^{(\gamma)} \cup \{0\}$ . We introduce the  $h^{(\gamma)}$  transformed process given by

$$\mathbb{P}_x^{(\gamma)}|_{\mathcal{F}_t} = \begin{cases} 1_{\{T_0 > t\}} \frac{h^{(\gamma)}(X_t)}{h^{(\gamma)}(x)} \cdot \mathbb{P}_x|_{\mathcal{F}_t} & \text{if } x \in \mathcal{H}^{(\gamma)}, \\ 1_{\{T_0 > t\}} h^{(\gamma)}(X_t) \cdot n|_{\mathcal{F}_t} & \text{if } x = 0. \end{cases}$$

Since  $\mathbb{P}_x^{(\gamma)}|_{\mathcal{F}_t}$  is consistent in  $t > 0$ , the probability measure  $\mathbb{P}_x^{(\gamma)}$  can be well-defined on  $\mathcal{F}_\infty := \sigma(X_t, t \geq 0)$ , for more details, see Yano [37, Theorem 9.1]. For any  $t > 0$ , we have  $\mathbb{P}_x^{(\gamma)}(T_{\mathbb{R} \setminus \mathcal{H}^{(\gamma)}} > t) = 1$ . Consequently, we have  $\mathbb{P}_x^{(\gamma)}(T_{\mathbb{R} \setminus \mathcal{H}^{(\gamma)}} = \infty) = 1$  and in particular,  $\mathbb{P}_x^{(\gamma)}(T_0 = \infty) = 1$ . The process  $\mathbb{P}_x^{(\gamma)}$  is called a *Lévy process conditioned to avoid zero*. Note that, for  $x \in \mathcal{H}^{(\gamma)}$ , the measure  $\mathbb{P}_x^{(\gamma)}$  is absolutely continuous with respect to  $\mathbb{P}_x$  on  $\mathcal{F}_t$ , but is singular to  $\mathbb{P}_x$  on  $\mathcal{F}_\infty$  since  $\mathbb{P}_x(T_0 < \infty) = 1$ .

By Theorems 4.4, 1.5, 1.7 and 7.3 (Corollaries 1.6 and 1.8) and by taking  $f = 1_{\{u=0\}}$ , we have the following conditioning results.

**Corollary 8.2.** *Let  $t > 0$  and  $F_t$  be a bounded  $\mathcal{F}_t$ -measurable functional. Then the following assertions hold:*

- (i)  $\lim_{q \rightarrow 0^+} \mathbb{P}_x[F_t | T_0 > e_q] = \mathbb{P}_x^{(0)}[F_t]$ , for  $x \in \mathcal{H}^{(0)}$ ;
- (ii)  $\lim_{a \rightarrow \pm\infty} \mathbb{P}_x[F_t | T_0 > T_a] = \mathbb{P}_x^{(\pm 1)}[F_t]$ , for  $x \in \mathcal{H}^{(\pm 1)}$ ;
- (iii)  $\lim_{(a,b) \xrightarrow{\gamma} \infty} \mathbb{P}_x[F_t | T_0 > T_{a,-b}] = \mathbb{P}_x^{(\gamma)}[F_t]$ , for  $-1 \leq \gamma \leq 1$  and  $x \in \mathcal{H}^{(\gamma)}$ ;
- (iv)  $\lim_{a \rightarrow \pm\infty} \mathbb{P}_x[F_t | T_0 > \eta_u^a] = \mathbb{P}_x^{(\pm 1)}[F_t]$ , for  $u > 0$  and  $x \in \mathcal{H}^{(\pm 1)}$ .

Note that (i) of Corollary 8.2 generalizes Pantí [18, Theorem 2.7].

### 8.2 Universal $\sigma$ -finite measures

In this subsection, we assume that  $(X, \mathbb{P}_x)$  has a transition density  $p_t(\cdot)$ . Then we can construct the Lévy bridge. Let  $\mathbb{P}_{x,y}^u$  denote the law of bridge from  $X_0 = x$  to  $X_u = y$ . This measure can be constructed as

$$\mathbb{P}_{x,y}^u(A) = \mathbb{P}_x \left[ 1_A \frac{p_{u-t}(y - X_t)}{p_t(y - x)} \right], \quad A \in \mathcal{F}_t, \quad 0 < t < u.$$

See Fitzsimmons–Pitman–Yor [10]. We have the conditioning formula:

$$\mathbb{P}_x \left[ \int_0^t F_u \, dL_u \right] = \int_0^t \mathbb{P}_x[dL_u] \mathbb{P}_{x,0}^u[F_u], \quad t > 0,$$

for all non-negative predictable processes  $(F_u)$ , where we write symbolically  $\mathbb{P}_x[dL_u] = p_u(-x) \, du$ .

For  $x \in \mathbb{R}$  and  $-1 \leq \gamma \leq 1$ , we define

$$\mathcal{P}_x^{(\gamma)} = \int_0^\infty \mathbb{P}_x[dL_u] \left( \mathbb{P}_{x,0}^u \bullet \mathbb{P}_0^{(\gamma)} \right) + h^{(\gamma)}(x) \mathbb{P}_x^{(\gamma)},$$

where the symbol  $\bullet$  stands for the concatenation and  $h^{(\gamma)}(x) \mathbb{P}_x^{(\gamma)} = 0$  for  $x \in \mathbb{R} \setminus \mathcal{H}^{(\gamma)}$ . Then we have the following:

**Theorem 8.3.** *Let  $x \in \mathbb{R}$  and  $f \in \mathcal{L}_+^1$ . Let  $t > 0$  and  $F_t$  be a bounded  $\mathcal{F}_t$ -measurable functional. Then the following assertions hold:*

- (i)  $\lim_{q \rightarrow 0^+} r_q(0) \mathbb{P}_x[F_t f(L_{e_q})] = \mathcal{P}_x^{(0)}[F_t f(L_\infty)]$ ;

- (ii)  $\lim_{a \rightarrow \pm\infty} h^B(a) \mathbb{P}_x[F_t f(L_{T_a})] = \mathcal{P}_x^{(\pm 1)}[F_t f(L_\infty)];$
- (iii)  $\lim_{(a,b) \xrightarrow{\gamma} \infty} h^C(a, -b) \mathbb{P}_x[F_t f(L_{T_{a,-b}})] = \mathcal{P}_x^{(\gamma)}[F_t f(L_\infty)],$  for  $-1 \leq \gamma \leq 1;$
- (iv)  $\lim_{a \rightarrow \pm\infty} h^B(a) \mathbb{P}_x[F_t f(L_{\eta_u^a})] = \mathcal{P}_x^{(\pm 1)}[F_t f(L_\infty)],$  for  $u > 0.$

*Proof.* It suffices to show that

$$\mathcal{P}_x^{(\gamma)}[F_t f(L_\infty)] = \mathbb{P}_x[F_t M_t^{(\gamma)}],$$

for  $-1 \leq \gamma \leq 1.$  The proof is the same as that of Theorem 5.3 of [25]. □

Consequently, we obtain the representation of  $\mathbb{Q}_x^{(\gamma, f)}$  as follows:

$$\mathbb{Q}_x^{(\gamma, f)} = \frac{f(L_\infty)}{\mathcal{P}_x^{(\gamma)}[f(L_\infty)]} \cdot \mathcal{P}_x^{(\gamma)}.$$

Recall that  $g = \sup\{t : X_t = 0\}.$  Since  $\mathbb{Q}_x^{(\gamma, f)}(g < \infty) = \mathbb{P}_x(g = \infty) = 1,$  the two measures are singular on  $\mathcal{F}_\infty.$

### 8.3 The law of $L_\infty$ under $\mathbb{Q}_0^{(\gamma, f)}$

We assume, for simplicity, that  $f \in \mathcal{L}_+^1$  satisfies  $\int_0^\infty f(u) du = 1.$  Then we have  $M_0^{(\gamma, f)} = 1,$   $\mathbb{P}_0$ -a.s. For  $l \geq 0,$  the optional stopping theorem implies that

$$\mathbb{Q}_0^{(\gamma, f)}(L_t \geq l) = \mathbb{P}_0[M_t^{(\gamma, f)}; L_t \geq l] = \mathbb{P}_0[M_{\eta_l}^{(\gamma, f)}; \eta_l \leq t].$$

Letting  $t \rightarrow \infty,$  we may apply the monotone convergence theorem to deduce that

$$\mathbb{Q}_0^{(\gamma, f)}(L_\infty \geq l) = \mathbb{P}_0[M_{\eta_l}^{(\gamma, f)}].$$

Since  $X_{\eta_l} = 0$  and  $L_{\eta_l} = l,$  we have

$$\mathbb{P}_0[M_{\eta_l}^{(\gamma, f)}] = \int_0^\infty f(l + u) du.$$

Therefore, it holds that

$$\mathbb{Q}_0^{(\gamma, f)}(L_\infty \in du) = f(u) du, \quad u > 0.$$

## 9 The transient case

We now study penalization in the transient case. Throughout this section, we always assume  $X$  is transient and the conditions **(A1)** and **(A2)** hold. Recall that

$$\kappa := \lim_{q \rightarrow 0^+} \frac{1}{r_q(0)} = n(T_0 = \infty) > 0;$$

see (2.5).

### 9.1 The renormalized zero resolvent in the transient case

As in the recurrent case, we define  $h_q(x) = r_q(0) - r_q(-x).$

**Theorem 9.1.** *Suppose that the conditions **(A1)** and **(A2)** hold. Then the following assertions hold.*

- (i) *For any  $x \in \mathbb{R},$  it holds that  $h(x) := \lim_{q \rightarrow 0^+} h_q(x) = \kappa^{-1} \mathbb{P}_x(T_0 = \infty).$*

- (ii) The above convergence is uniform on compacts, and consequently  $h$  is continuous.
- (iii)  $h$  is subadditive on  $\mathbb{R}$ , that is,  $h(x + y) \leq h(x) + h(y)$  for  $x, y \in \mathbb{R}$ .

*Proof.* It follows from (2.1) and (2.5) that

$$h_q(x) = r_q(0)\mathbb{P}_x[1 - e^{-qT_0}] \longrightarrow \kappa^{-1}\mathbb{P}_x(T_0 = \infty), \quad \text{as } q \rightarrow 0+.$$

The proof of (ii) and (iii) are the same as that of Theorem 1.1. □

**Theorem 9.2.** *Suppose that the conditions (A1) and (A2) hold. Then the following assertions hold:*

- (i)  $\lim_{x \rightarrow \pm\infty} \frac{h(x)}{|x|} = 0$ ;
- (ii)  $\lim_{y \rightarrow \pm\infty} \{h(x + y) - h(y)\} = 0$ , for all  $x \in \mathbb{R}$ .

*Proof.* Since  $h(x) \leq \kappa^{-1}$ , it is obvious that (i) holds. The proof of (ii) is the same as that of (ii) of Theorem 1.2 in the recurrent and  $m^2 = \infty$  case. □

### 9.2 Useful equations

Before stating out penalization result, we introduce some useful equations.

**Lemma 9.3.** (i) For  $a \in \mathbb{R}$ ,

$$h^B(a) := \lim_{q \rightarrow 0+} h_q^B(a) = \mathbb{P}_0[L_{T_a}] = h(a) + h(-a) - \kappa h(a)h(-a).$$

(ii) For  $x, a, b \in \mathbb{R}$  and  $a \neq b$ ,

$$\mathbb{P}_x(T_a < T_b) = \frac{h(b - a) + h(x - b) - h(x - a) - \kappa h(x - b)h(b - a)}{h^B(a - b)}.$$

(iii) For  $x, a, b \in \mathbb{R}$  and  $a \neq b$ ,

$$\mathbb{P}[L_{T_a \wedge T_b}] = \frac{1}{h^B(a - b)} \left\{ \begin{aligned} &(h(b) + h(-a) - \kappa h(-a)h(b))h(a - b) \\ &+ (h(a) + h(-b) - \kappa h(-b)h(a))h(b - a) \\ &- h(a - b)h(b - a) \end{aligned} \right\}.$$

The proof of Lemma 9.3 is similar to that in the recurrent case. So we omit it.

**Lemma 9.4** ([18, Lemma 3.10]). *It holds that  $\lim_{x \rightarrow \infty} h^B(x) = \kappa^{-1}$ .*

*Proof.* For completeness of the paper, we give the same proof as that of [18, Lemma 3.10]. Let  $e$  be the exponentially distributed with parameter 1 and  $e_q := e/q$  for  $q > 0$ . Then we already have (3.17) and (3.18). Letting  $q \rightarrow 0+$  in (3.18), we obtain

$$\liminf_{x \rightarrow \infty} h^B(x) \geq \kappa^{-1}.$$

On the other hand, it holds that

$$h^B(x) \leq \mathbb{P}[L_{e_q}] + \mathbb{P}[L_{T_x}; T_x > e_q] = r_q(0) + \mathbb{P}[L_{T_x}; T_x > e_q].$$

Letting  $q \rightarrow 0+$ , we have  $h^B(x) \leq \kappa^{-1}$ . Therefore we obtain the desired result. □

**Theorem 9.5.** *For  $a \in \mathbb{R} \setminus \{0\}$ , it holds that*

$$n(T_a < T_0 < \infty) = \frac{1 - \kappa h^B(a)}{h^B(a)}, \quad \text{and} \quad n(T_a < T_0) = \frac{1 - \kappa h(-a)}{h^B(a)}.$$

*Proof.* For  $l > 0$ , it holds that

$$\mathbb{P}(L_{T_a} > l) = \mathbb{P}(T_a > \eta_l) = \mathbb{P}(\sigma_{\{T_0=\infty\} \cup \{T_a < T_0 < \infty\}} > l),$$

where  $\sigma_A = \inf\{l: e_l \in A\}$  for  $A \subset \mathcal{D}$ . Since  $\sigma_A$  is the hitting time of the set  $A$  for the killed Poisson point process  $((l, e_l), l \geq 0)$ , we have

$$\mathbb{P}(L_{T_a} > l) = e^{-l\{n(T_0=\infty)+n(T_a < T_0 < \infty)\}} = e^{-l\{\kappa+n(T_a < T_0 < \infty)\}}.$$

For more details, see, e.g., [15, Lemma 6.17]. In particular,  $L_{T_a}$  is exponentially distributed. On the other hand, we know  $h^B(a) = \mathbb{P}[L_{T_a}]$  by Lemma 3.5. Hence we obtain

$$n(T_a < T_0 < \infty) = \frac{1}{h^B(a)} - \kappa.$$

We use the strong Markov property of the excursion measure  $n$  (see, e.g., [2, Theorem III.3.28]) to obtain

$$n(T_a < T_0 < \infty) = n(\mathbb{P}_a(T_0 < \infty); T_a < T_0) = (1 - \kappa h(a))n(T_a < T_0). \tag{9.1}$$

It follows from (3.19) and (9.1) that

$$n(T_a < T_0) = \frac{1 - \kappa h^B(a)}{h^B(a)(1 - \kappa h(a))} = \frac{1 - \kappa h(-a)}{h^B(a)}.$$

Hence the proof is complete. □

### 9.3 Penalization result in the transient case

Let  $f$  be a non-negative function on  $[0, \infty)$  which satisfies  $\int_0^\infty e^{-\kappa u} f(u) du < \infty$ . For  $x \in \mathbb{R}$ , we introduce the process given by

$$M_t = M_t^{(f)} = h(X_t)f(L_t) + (1 - \kappa h(X_t)) \int_0^\infty e^{-\kappa u} f(L_t + u) du.$$

Then we can show that  $(M_t, t \geq 0)$  is a non-negative martingale.

**Theorem 9.6.** *Let  $x \in \mathbb{R}$  and let  $f$  be a non-negative function on  $[0, \infty)$  which satisfies  $\int_0^\infty e^{-\kappa u} f(u) du < \infty$ . Then it holds that*

$$\mathbb{P}_x[f(L_\infty)] = \kappa M_0, \quad \mathbb{P}_x\text{-a.s.}, \tag{9.2}$$

and  $(M_t, t \geq 0)$  is a non-negative  $((\mathcal{F}_t), \mathbb{P}_x)$ -martingale.

*Proof.* By the same discussion, Lemma 4.1 also holds in the transient case. Suppose first that  $f$  is bounded. We write  $g = \sup\{t: X_t = 0\}$ . Since  $X$  is transient,  $\mathbb{P}_x(g < \infty) = 1$ . We see that  $f(L_{e_q}) \rightarrow f(L_\infty)$ ,  $\mathbb{P}_x$ -a.s. as  $q \rightarrow 0+$ ; in fact, for almost every sample path,  $L_{e_q} = L_g = L_\infty$  for small  $q > 0$  (Here we do not need continuity of  $f$ ). Hence, by the dominated convergence theorem, we obtain

$$\mathbb{P}_x[f(L_{e_q})] \rightarrow \mathbb{P}_x[f(L_\infty)], \quad \text{as } q \rightarrow 0+.$$

On the other hand, by the monotone convergence theorem, we obtain

$$\int_0^\infty e^{-u/r_q(0)} f(u) du \rightarrow \int_0^\infty e^{-\kappa u} f(u) du, \quad \text{as } q \rightarrow 0+.$$

Hence (9.2) follows by letting  $q \rightarrow 0+$  in (4.1). To remove the boundedness assumption of  $f$ , we consider  $f \wedge n$  and then let  $n \rightarrow \infty$  in (9.2). Moreover, by the Markov property and the additivity of  $L$ , we have, for  $0 < s < t$ ,

$$\begin{aligned} \mathbb{P}_x[M_t|\mathcal{F}_s] &= \kappa^{-1}\mathbb{P}_x[\tilde{\mathbb{P}}_{X_t}[f(\tilde{L}_\infty + L_t)|\mathcal{F}_s]] \\ &= \kappa^{-1}\mathbb{P}_x[\mathbb{P}_x[f(L_\infty)|\mathcal{F}_t]|\mathcal{F}_s] \\ &= \kappa^{-1}\mathbb{P}_x[f(L_\infty)|\mathcal{F}_s] \\ &= \kappa^{-1}\tilde{\mathbb{P}}_{X_s}[f(\tilde{L}_\infty + L_s)] \\ &= M_s. \end{aligned}$$

This means that  $(M_t, t \geq 0)$  is a non-negative  $((\mathcal{F}_t), \mathbb{P}_x)$ -martingale. □

**Theorem 9.7.** *Suppose that the conditions (A1) and (A2) hold. Let  $f$  be a bounded non-negative function and let  $\tau$  be a random clock. Define*

$$M_t^\tau = \kappa^{-1}\mathbb{P}_x[f(L_\tau)|\mathcal{F}_t].$$

Then it holds that

$$M_t^\tau \longrightarrow M_t, \quad \mathbb{P}_x\text{-a.s. and in } \mathcal{L}^1(\mathbb{P}_x) \text{ as } \tau \rightarrow \infty.$$

Consequently, if  $M_0 > 0$  under  $\mathbb{P}_x$ , it holds that

$$\frac{\mathbb{P}_x[F_t f(L_\tau)]}{\mathbb{P}_x[f(L_\tau)]} \longrightarrow \mathbb{P}_x\left[F_t \frac{M_t}{M_0}\right], \quad \text{as } \tau \rightarrow \infty,$$

for all bounded  $\mathcal{F}_t$ -measurable functionals  $F_t$ .

*Proof.* We have  $f(L_\tau) \rightarrow f(L_\infty)$ ,  $\mathbb{P}_x$ -a.s. as  $\tau \rightarrow \infty$ ; in fact,  $L_\tau = L_q = L_\infty$  for large  $\tau$ . In addition, since  $f$  is bounded, we may apply the dominated convergence theorem to obtain  $f(L_\tau) \rightarrow f(L_\infty)$ , in  $\mathcal{L}^1(\mathbb{P}_x)$  as  $\tau \rightarrow \infty$ . This implies that  $M_t^\tau \rightarrow M_t$ ,  $\mathbb{P}_x$ -a.s. and in  $\mathcal{L}^1(\mathbb{P}_x)$  as  $\tau \rightarrow \infty$ . □

**Remark 9.8.** If  $\tau$  is exponential clock, hitting clock, two-point hitting time clock or inverse local time clock, Theorem 9.7 also holds under the assumption that  $f$  is a non-negative function which satisfies  $\int_0^\infty e^{-\kappa u} f(u) du < \infty$ .

Since we have  $M_t = \mathbb{P}_x[f(L_\infty)|\mathcal{F}_t]$ , we see that the penalized measure  $\mathbb{Q}_x^f$  can be represented as

$$\mathbb{Q}_x^f = \frac{f(L_\infty)}{\mathbb{P}_x[f(L_\infty)]} \cdot \mathbb{P}_x,$$

which shows that  $\mathbb{Q}_x^f$  is absolutely continuous with respect to  $\mathbb{P}_x$ .

## 10 Appendix: Martingale property of $X_t f(L_t)$

In Remark 5.2, we have shown that  $(X_t f(L_t), t \geq 0)$  is a  $((\mathcal{F}_t), \mathbb{P}_x)$ -martingale for  $f \in \mathcal{L}_+^1$  under the condition that  $m^2 < \infty$ . Let us remove the additional assumption  $m^2 < \infty$ . In this section, we assume  $X$  is either recurrent or transient, and assume the conditions (A1) and (A2).

**Theorem 10.1.** *Suppose the conditions (A1) and (A2) hold. Suppose, in addition, that  $\mathbb{P}[|X_1|] < \infty$  and  $\mathbb{P}[X_1] = 0$ . Then the following assertions hold:*

- (i)  $\mathbb{P}[|X_{e_q}|] < \infty$  and  $n[|X_{e_q}|; T_0 > e_q] < \infty$  for all  $q > 0$ , and  $\mathbb{P}[|X_t|] < \infty$  and  $n[|X_t|; T_0 > t] < \infty$  for all  $t > 0$ ;

- (ii)  $\mathbb{P}_x[X_t; T_0 > t] = x$ , for all  $t > 0$  and  $x \in \mathbb{R}$ ;
- (iii)  $n[X_t; T_0 > t] = 0$ , for all  $t > 0$ ;
- (iv)  $(X_t f(L_t), t \geq 0)$  is a  $((\mathcal{F}_t), \mathbb{P}_x)$ -martingale for  $x \in \mathbb{R}$  and all bounded measurable functions  $f$ .

*Proof.* (i) We have  $\mathbb{P}[|X_{t+s}|] \leq \mathbb{P}[|X_t|] + \mathbb{P}[|X_{t+s} - X_t|] = \mathbb{P}[|X_t|] + \mathbb{P}[|X_s|]$ , which implies that the function  $t \mapsto \mathbb{P}[|X_t|]$  is subadditive. Hence, for any  $k \in \mathbb{N}$ , we have  $\mathbb{P}[|X_k|] \leq k\mathbb{P}[|X_1|] < \infty$ . For  $t > 0$ , it is known that

$$\mathbb{P}\left[\sup_{0 \leq s \leq t} |X_s|\right] \leq 8\mathbb{P}[|X_t|]; \tag{10.1}$$

see Doob [9, Theorem VII.5.1] and Sato [32, Theorem 25.18 and Remark 25.19] for the proof. Hence we have  $\sup_{0 \leq t \leq k} \mathbb{P}[|X_t|] < \infty$  for all  $k \in \mathbb{N}$ . In particular, we obtain  $\mathbb{P}[|X_t|] < \infty$  for all  $t > 0$ . Again by the subadditivity of  $t \mapsto \mathbb{P}[|X_t|]$ , we have

$$\lim_{t \rightarrow \infty} \frac{\mathbb{P}[|X_t|]}{t} = \inf_{t > 0} \frac{\mathbb{P}[|X_t|]}{t} \leq \mathbb{P}[|X_1|] < \infty.$$

Thus there exist constants  $C, C' > 0$  such that  $\mathbb{P}[|X_t|] \leq C + C't$  for all  $t > 0$ . In particular, we obtain

$$\mathbb{P}[|X_{e_q}|] \leq q \int_0^\infty (C + C't)e^{-qt} dt < \infty, \quad \text{for all } q > 0.$$

By Lemma 2.5, we have

$$\begin{aligned} \mathbb{P}[|X_{e_q}|] &= \mathbb{P}\left[\sum_{u \in D} \int_{\eta_{u-}}^{\eta_u} qe^{-qt} |X_t| dt\right] \\ &= \mathbb{P} \otimes \tilde{n} \left[ \int_0^\infty dL_u qe^{-qu} \int_0^{\tilde{T}_0} dt e^{-qt} |\tilde{X}_t| \right] \\ &= \mathbb{P} \left[ \int_0^\infty dL_u e^{-qu} \right] n[|X_{e_q}|; T_0 > e_q] \\ &= r_q(0)n[|X_{e_q}|; T_0 > e_q]. \end{aligned}$$

Hence we have  $n[|X_{e_q}|; T_0 > e_q] < \infty$  for all  $q > 0$  and this implies that  $n[|X_t|; T_0 > t] < \infty$  for almost all  $t > 0$ . For any  $t > 0$ , we can take  $0 < s < t$  such that  $n[|X_s|; T_0 > s] < \infty$ . Then it follows from the Markov property of the excursion measure  $n$  that

$$\begin{aligned} n[|X_t|; T_0 > t] &= n[\mathbb{P}_{X_s}[|X_{t-s}|; T_0 > t-s]; T_0 > s] \\ &\leq n[\mathbb{P}_{X_s}[|X_{t-s}|]; T_0 > s] \\ &\leq n[|X_s| + \mathbb{P}[|X_{t-s}|]; T_0 > s] < \infty. \end{aligned}$$

Hence we obtain  $n[|X_t|; T_0 > t] < \infty$  for all  $t > 0$ .

(ii) By the Markov property, we have

$$\mathbb{P}_x[X_t] = \mathbb{P}_x[X_t; T_0 > t] + \int_{[0,t]} \mathbb{P}_x(T_0 \in ds) \mathbb{P}[X_{t-s}].$$

Since  $\mathbb{P}_x[X_t] = x$  for all  $t \geq 0$  and  $x \in \mathbb{R}$ , we obtain  $\mathbb{P}_x[X_t; T_0 > t] = x$  for all  $t > 0$ .

(iii) By Lemma 2.5, we have

$$\begin{aligned} \mathbb{P}[X_{e_q} f(L_{e_q})] &= \mathbb{P} \otimes \tilde{n} \left[ \int_0^\infty dL_u q e^{-qu} f(u) \int_0^{\tilde{T}_0} dt e^{-qt} \tilde{X}_t \right] \\ &= \mathbb{P} \left[ \int_0^\infty dL_u q e^{-qu} f(u) \right] \int_0^\infty dt e^{-qt} n[X_t; T_0 > t]. \end{aligned} \tag{10.2}$$

Here in the second equality, we use Fubini's theorem. If we take  $f \equiv 1$ , then (10.2) becomes

$$\mathbb{P}[X_{e_q}] = q r_q(0) \int_0^\infty dt e^{-qt} n[X_t; T_0 > t].$$

Since  $\mathbb{P}[X_{e_q}] = 0$  for all  $q > 0$ , we obtain  $n[X_t; T_0 > t] = 0$  for almost all  $t > 0$ . By the Markov property of the excursion measure  $n$ , we have, for  $0 < s < t$ ,

$$n[X_t; T_0 > t] = n[\mathbb{P}_{X_s}[X_{t-s}; T_0 > t-s]; T_0 > s] = n[X_s; T_0 > s],$$

which implies that  $n[X_t; T_0 > t]$  is constant in  $t > 0$ . Thus we obtain  $n[X_t; T_0 > t] = 0$  for all  $t > 0$ .

(iv) By (iii) of Theorem 10.1 and by (10.2), we have  $\mathbb{P}[X_{e_q} f(L_{e_q})] = 0$ . Hence, by the Markov property and (ii) of Theorem 10.1, it holds that,

$$\begin{aligned} \mathbb{P}_x[X_{e_q} f(L_{e_q})] &= \mathbb{P}_x \left[ \int_0^{T_0} q e^{-qt} X_t f(0) dt \right] + \mathbb{P}_x[e^{-qT_0}] \mathbb{P}[X_{e_q} f(L_{e_q})] \\ &= f(0) \int_0^\infty q e^{-qt} \mathbb{P}_x[X_t; T_0 > t] dt \\ &= x f(0). \end{aligned}$$

Thus we obtain

$$\mathbb{P}_x[X_t f(L_t)] = x f(0) \quad \text{for almost all } t > 0. \tag{10.3}$$

We show the function  $t \mapsto \mathbb{P}_x[X_t f(L_t)]$  is right-continuous for  $t > 0$ . Fix  $t > 0$ . Since  $X_t \neq 0$ ,  $\mathbb{P}_x$ -a.s., we see that, for almost every sample path, we can choose small  $\delta > 0$  such that  $L_{t+\delta} = L_t$ . Since  $t \mapsto X_t$  is right-continuous, we have  $\lim_{s \rightarrow 0+} X_{t+s} f(L_{t+s}) = X_t f(L_t)$ ,  $\mathbb{P}_x$ -a.s., where we do not require continuity of  $f$ . For  $0 < s < 1$ , it holds that  $|X_{t+s} f(L_{t+s})| \leq \sup_{0 \leq t' \leq t+1} |X_{t'}| \sup_{u \in \mathbb{R}} |f(u)|$ . By (10.1), we may apply the dominated convergence theorem to deduce that  $t \mapsto \mathbb{P}_x[X_t f(L_t)]$  is right-continuous. This and (10.3) imply that

$$\mathbb{P}_x[X_t f(L_t)] = x f(0) \quad \text{for all } t \geq 0.$$

By the Markov property and the additivity property of  $L$ , the process  $(X_t f(L_t), t \geq 0)$  is a  $((\mathcal{F}_t), \mathbb{P}_x)$ -martingale.  $\square$

By Theorems 1.4 and 10.1, we obtain the following:

**Corollary 10.2.** *Suppose that the assumptions of Theorem 10.1 are satisfied and that  $X$  is recurrent. For measurable function  $f_1$  and locally integrable function  $f_2$ , define*

$$F(X_t, L_t) = X_t f_1(L_t) + h(X_t) f_2(L_t) - \int_0^{L_t} f_2(u) du.$$

*Then the following assertions hold.*

- (i) If  $f_1$  is bounded and  $f_2$  is integrable, then  $(F(X_t, L_t), t \geq 0)$  is a  $((\mathcal{F}_t), \mathbb{P}_x)$ -martingale.
- (ii) If  $f_1$  is locally bounded and  $f_2$  is locally integrable, then  $(F(X_t, L_t), t \geq 0)$  is a local  $((\mathcal{F}_t), \mathbb{P}_x)$ -martingale.

**Remark 10.3.** For the Brownian motion  $B$  and its local time  $L$  at 0, Fitzsimmons–Wroblewski [11] proved that any local martingale of the form  $(F(B_t, L_t), t \geq 0)$  is given by the following form:

$$F(B_t, L_t) = F(0, 0) + B_t f_1(L_t) + |B_t| f_2(L_t) - \int_0^{L_t} f_2(u) du,$$

where  $f_1$  and  $f_2$  are locally integrable functions.

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