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Random walk in a birth-and-death dynamical environment*

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Abstract

We consider a particle moving in continuous time as a Markov jump process; its discrete chain is given by an ordinary random walk on $\mathbb{Z}^d$, and its jump rate at $(\mathbf{x}, t)$ is given by a fixed function φ of the state of a birth-and-death (BD) process at $\mathbf{x}$, at time t; BD processes at different sites are independent and identically distributed, and φ is assumed non-increasing and vanishing at infinity. We derive a LLN and a CLT for the particle position when the environment is "strongly ergodic". In the absence of a viable uniform lower bound for the jump rate, we resort instead to stochastic domination, as well as to a subadditive argument to control the time spent by the particle to perform n consecutive jumps; and we also impose conditions on the initial (product) environmental distribution.

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1 Introduction

In this paper, we analyse the long time behavior of random walks taking place in *an evolving field of traps.* A starting motivation is to consider a *dynamical environment* version of Bouchaud's trap model (introduced in [8]) on $\mathbb{Z}^d$. In the (simplest version of the) latter model, we have a continuous-time random walk (whose embedded chain is an ordinary random walk) on $\mathbb{Z}^d$ with spatially inhomogeneous jump rates, given by a field of iid random variables, representing traps. The greater interest is for the case where the inverses of the rates are heavy-tailed, leading to subdiffusivity of the

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particle (performing the random walk), and to the appearance of the phenomenon of aging. See [12] and [3].

In the present paper, we have again a continuous-time random walk whose embedded chain is an ordinary random walk (with various hypotheses on its jump distribution, depending on the result), but now the rates are spatially *as well as temporally* inhomogeneous, the rate at a given site and time is given by a (fixed) function, which we denote by φ, of the state of a birth-and-death chain (in continuous time; with time-homogeneous jump rates) at that site and time; birth-and-death chains for different sites are iid and ergodic.

We should not expect subdiffusivity if φ is bounded away from 0, but this should not be the only factor for diffusivity. The ergodic character of the environment should play a role.

CLT's for random walks in dynamical random environments have been, from a more general point of view, or under different motivations, previously established in a variety of situations; we mention [5, 2, 10, 19] for a few cases with fairly general environments, and [9, 18, 13] in the case of environments given by specific interacting particle systems; [6] and [7] deal with a case where the jump times of the particle are iid. There is a relatively large literature establishing strong LLN's for the position of the particle in random walks in space-time random environments; besides most of the references given above, which also establish it, we mention [1] and [4]. [20] derives large deviations for the particle in the case of an iid space-time environment.

These papers assume (or have it naturally) in their environments an ellipticity condition, from which our environment may crucially depart, in the sense that our jump rates may not be bounded away from 0. Jumps are generally also taken to be bounded, a possibly mere technical assumption in some cases, which we nevertheless forgo. It should also be said that in many other respects, these models are quite more general, or more correlated than ours.[1]

In order to go around an elipticity condition on φ away from 0, and the *strong* domination that such condition would entail on jump rates of the random walk, we develop instead a *stochastic* domination approach, for which we require φ to be non-increasing (this of course implies a boundedness from above on φ). And in the case that φ is *not* bounded away from 0, we also need to have the birth-and-death environment to be strongly enough ergodic — this can be conceptually understood as requiring the environment to be often enough close to the origin, so that we see jumps of the random walk at a steady rate; as will be seen below, this condition translates into a second moment condition on its equilibrium distribution.

The main building block for arguing our CLT, in the case where the initial environment is identically 0, is a Law of Large Numbers for the time that the particle takes to make n jumps; this in turn relies on a subadditivity argument, resorting to the Subadditive Ergodic Theorem; in order to obtain the control that the latter theorem requires on expected values, we rely on a domination of the environment left by the particle at jump times (when starting from equilibrium); this is a *stochastic* domination, rather than a strong domination, which would be provided by the infimum of φ, were it positive — but we do not require that. This is where our monotonicity requirement on φ is important. We extend our CLT to more general, product initial environments, with a uniform exponentially decaying tail (and also restricting in this case to spatially homogeneous environments), by means of coupling arguments.

We expect to be able to establish various forms of subdiffusivity in this model when the environment is either not ergodic or is ergodic but not "strongly ergodic" (with, say,

[1] This is perhaps a good point to remark that even though our environment is constituted by iid birth-and-death processes, and the embedded chain of the particle is independent of them, the continuous-time motion of the particle brings about a correlation between the particle position and the environment.

heavy-tailed equilibrium measures). This is under current investigation. [7] has results in this direction in the case where the jump times of the particle are iid.

The remainder of this paper is organized as follows. In Section 2 we define our model in detail, and discuss some of its properties. Section 3 is devoted to the formulation of our main results. In Section 4 we give proofs of the LLN and CLT under an environment started from the identically 0 configuration. The main ingredient, as mentioned above, a LLN for the time that the particle takes to give n jumps, is developed in Subsection 4.1, and the remaining subsections are devoted for the conclusion. In Section 5 we extend the CLT to more general (product) initial configurations of the environment (with a uniform exponential moment). An appendix is devoted to an auxiliary result concerning birth-and-death processes.

2 The model

We first briefly and roughly explain a specific and convenient construction of our process. Let us first describe the (potential) jump times of our random walk. We start with a rate 1 Poisson point process, say M, in $\mathbb{R}^d \times \mathbb{R}_+$, and consider half infinite square cylinders of (finite) sides 1 and centered at the points of $\mathbb{Z}^d$ (which partition $\mathbb{R}^d \times \mathbb{R}_+$) — call $C_{\mathbf{x}}$ the cylinder centered in $\mathbf{x}$. Given a realization of the birth-and-death processes indexed by $\mathbb{Z}^d$, say $\omega_{\mathbf{x}}(\cdot)$, $\mathbf{x} \in \mathbb{Z}^d$, we consider for each $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{Z}_d$ the region $\mathcal{N}_{\mathbf{x}} = \{(y_1, \dots, y_d, r)\}$ of $C_{\mathbf{x}}$ on which the $\mathbf{y}_d \in \left[x_d - \frac{1}{2}, x_d - \frac{1}{2} + \varphi\big(\omega_{\mathbf{x}}(r)\big)\right]$ — at this point we assume that φ is bounded from above, by 1. In this way we find that the projection of the points of $M \cap \mathcal{N}_{\mathbf{x}}$ onto $\{\mathbf{x}\} \times \mathbb{R}_+$ is a inhomogeneous Poisson point process of intensity $\varphi\big(\omega_{\mathbf{x}}(\cdot)\big)$. Such projected points constitute the potential jump time of our random walk, such that, given that the walk is in $\mathbf{x}$ at time t, it stays at $\mathbf{x}$ till the next projected point up from t, and then jumps according to a fixed jump distribution on $\mathbb{Z}^d$, independently from all else. The convenience of such construction will become apparent in our coupling arguments below, where we will further assume that φ is non-increasing.

Next we describe this construction in more detail. An alternative construction, also convenient, but for other purposes, will be discussed at the end of this section.

For $d \in \mathbb{N}_* := \mathbb{N} \setminus \{0\}$ and $S \subset \mathbb{R}^d$, let $\mathcal{D}(\mathbb{R}_+, S)$ denote the set of càdlàg trajectories from $\mathbb{R}_+$ to S. We represent by $\mathbf{0} \in E$ and $\mathbf{1} \in E$, $E = \mathbb{N}^d, \mathbb{Z}^d, \mathbb{N}^{\mathbb{Z}^d}$, respectively, the null element, and the element with all coordinates identically equal to 1.

We will use the notation $M \sim BDP(\mathbf{p}, \mathbf{q})$ to to indicate that M is a birth-and-death process on $\mathbb{N}$ with birth rates $\mathbf{p} = (p_n)_{n \in \mathbb{N}}$ and death rates $\mathbf{q} = (q_n)_{n \in \mathbb{N}_*}$. We will below consider independent copies of such a process, and we will assume that $p_n, q_n \in (0,1)$, $p_n + q_n \equiv 1$ for all $n \geq 1$, $p_0 \in (0,1]$, and

$$\sum_{n\geq 1} \prod_{i=1}^{n} \frac{p_{i-1}}{q_i} < \infty. \tag{2.1}$$

The latter condition is well known to be equivalent to ergodicity of such a process. We will also assume that $p_n \leq q_n$ for all $n \geq 1$, and $\inf_n p_n > 0$. See Remark 4.11 at the end of Section 3.1.

We now make an explicit construction of our process, namely, the random walk in a birth-and-death (BD) environment. Let $\omega = (\omega_{\mathbf{x}})_{\mathbf{x} \in \mathbb{Z}^d}$ be an independent family of BDP's as prescribed in the paragraph of (2.1) above, each started from its respective initial distribution $\mu_{\mathbf{x},0}$, independently of each other; we will denote by $\mu_{\mathbf{x},t}$ the distribution of $\omega_{\mathbf{x}}(t)$, $t \in \mathbb{R}_+$, $\mathbf{x} \in \mathbb{Z}^d$; ω plays the role of random dynamical environment for our random walk, which we may view as a stochastic process $(\omega(t))_{t\in\mathbb{R}_+}$ on $\Lambda := \mathbb{N}^{\mathbb{Z}^d}$ with initial distribution $\hat{\mu}_0 := \bigotimes_{\mathbf{x}\in\mathbb{Z}^d} \mu_{\mathbf{x},0}$ and trajectories living on $A := \mathcal{D}\left(\mathbb{R}_+, \mathbb{N}\right)^{\mathbb{Z}^d}$. Let $\mathrm{P}_{\hat{\mu}_0}$ denote the law of ω.

Let now π be a probability distribution on $\mathbb{Z}^d \setminus \{\mathbf{0}\}$, and let $\xi := \{\xi_n\}_{n\in\mathbb{N}_*}$ be an iid sequence of random vectors taking values in $\mathbb{Z}^d \setminus \{\mathbf{0}\}$, each distributed as π; ξ is assumed independent of ω.

Next, let $\mathcal{M}$ be a Poisson point process of rate 1 in $\mathbb{R}^d \times \mathbb{R}_+$, independent of ω and ξ. For each $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{Z}^d$, let

$$\mathcal{M}_{\mathbf{x}} = \mathcal{M} \cap (C_{\mathbf{x}} \times \mathbb{R}_+), \tag{2.2}$$

where $C_{\mathbf{x}} = \bigtimes_{i=1}^{d} [c_{x_i}, c_{x_i} + 1)$, with $c_{x_i} := x_i - 1/2$, $1 \le i \le d$. It is quite clear that

$$\mathcal{M} = \bigcup_{\mathbf{x}\in\mathbb{Z}^d} \mathcal{M}_{\mathbf{x}} \tag{2.3}$$

and that by well-known properties of Poisson point processes, $\left\{\mathcal{M}_{\mathbf{x}} : \mathbf{x} \in \mathbb{Z}^d\right\}$ is an independent collection, where $\mathcal{M}_{\mathbf{x}}$ is a Poisson point process of rate 1 in $C_{\mathbf{x}} \times [0, +\infty)$.

Given $\omega \in A$ and $\varphi : \mathbb{N} \to [0, 1]$, $\varphi \not\equiv 0$ (to avoid a special, trivial case), set

$$\mathcal{N}_{\mathbf{x}} = \{(y_1, \dots, y_d, r) \in \mathcal{M}_{\mathbf{x}} : y_d \in [\, c_{x_d}, c_{x_d} + \varphi(\omega_{\mathbf{x}}(r))\,)\}, \quad \mathbf{x} \in \mathbb{Z}^d. \tag{2.4}$$

Note that the projection of $\mathcal{N}_{\mathbf{x}}$ on $\{\mathbf{x}\} \times \mathbb{R}_+$ is a inhomogeneous Poisson point process on $\{\mathbf{x}\} \times \mathbb{R}_+$ with intensity function given by

$$\lambda_{\mathbf{x}}(r) = \varphi(\omega_{\mathbf{x}}(r)), \quad \mathbf{x} \in \mathbb{Z}^d,\ r \ge 0. \tag{2.5}$$

Let us fix $X(0) = \mathbf{x}_0$, $\mathbf{x}_0 \in \mathbb{Z}^d$, and define $X(t)$, $t \in \mathbb{R}_+$, as follows. Let $\tau_0 = 0$, and set

$$\tau_1 = \inf\{r > 0 : \mathcal{N}_{\mathbf{x}_0} \cap (C_{\mathbf{x}_0} \times (0, r]) \ne \emptyset\}, \tag{2.6}$$

where by convention $\inf\emptyset = \infty$. For $t \in (0, \tau_1)$, $X(t) = X(0)$, and, if $\tau_1 < \infty$, then

$$X(\tau_1) = X(0) + \xi_1. \tag{2.7}$$

For $n \ge 2$, we inductively define

$$\tau_n = \inf\left\{r > \tau_{n-1} : \mathcal{N}_{X(\tau_{n-1})} \cap \left(C_{X(\tau_{n-1})} \times (\tau_{n-1}, r]\right) \ne \emptyset\right\}. \tag{2.8}$$

For $t \in (\tau_{n-1}, \tau_n)$, we set $X(t) = X(\tau_{n-1})$, and, if $\tau_n < \infty$, then

$$X(\tau_n) = X(\tau_{n-1}) + \xi_n. \tag{2.9}$$

In words, $(\tau_n)_{n\in\mathbb{N}}$ are the jump times of the process $X := (X(t))_{t\in\mathbb{R}_+}$, which in turn, given $\omega \in A$, is a continuous-time random walk on $\mathbb{Z}^d$ starting from $\mathbf{x}_0$ with jump rate at $\mathbf{x}$ at time t given by $\varphi(\omega_{\mathbf{x}}(t))$, $\mathbf{x} \in \mathbb{Z}^d$. (Notice that from the ergodicity of ω and non-nullity of φ, all these times are a.s. finite.) Moreover, when at $\mathbf{x}$, the next site to be visited is given by $\mathbf{x} + \mathbf{y}$, with $\mathbf{y}$ generated from π, $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$. We adopt $\mathcal{D}\left(\mathbb{R}_+, \mathbb{Z}^d\right)$ as sample space for X.

Let us denote by $P^{\omega}_{\mathbf{x}_0}$ the conditional law of X given $\omega \in A$. We remark that, since $\mathcal{N}_{\mathbf{x}} \subset \mathcal{M}_{\mathbf{x}}$ for all $\mathbf{x} \in \mathbb{Z}^d$, it follows from the lack of memory of Poisson processes that, for each $n \in \mathbb{N}_*$, given that $\tau_{n-1} < \infty$, $P^{\omega}_{\mathbf{x}_0}$-almost surely ($P^{\omega}_{\mathbf{x}_0}$-a.s.), $\tau_n - \tau_{n-1} \ge Z_n$, with Z_n a standard exponential random variable. Thus, $\tau_n \to \infty$ $P^{\omega}_{\mathbf{x}_0}$-a.s. as $n \to \infty$, i.e., X is non-explosive. Thus, given $\omega \in A$, the inductive construction of X proposed above is well-defined for all $t \in \mathbb{R}_+$. We also notice that given the ergodicity assumption we made on ω, we also have that X takes $P^{\omega}_{\mathbf{x}_0}$-a.s. infinitely many jumps along all of its history for almost every realization of ω.

Let us denote by $\mathsf{x} = (\mathsf{x}_n)_{n\in\mathbb{N}}$ the embedded (with discrete time) chain of X. We will henceforth at times make reference to a *particle* which moves in continuous time on $\mathbb{Z}^d$, starting from $\mathbf{x}_0$, and whose trajectory is given by X; in this context, $X(t)$ is of course the position of the particle at time $t \geq 0$. For simplicity, we assume x *irreducible*.

Remark 2.1. Given ω, X is a time-inhomogeneous Markov jump process; we also have that the joint process $(X(t), \omega(t))_{t\in\mathbb{R}_+}$ is Markovian.

We may then realize our joint process in the triple $(\Omega, \mathcal{F}, \mathbf{P}_{\hat{\mu}_0,\mathbf{x}_0})$, with $\hat{\mu}_0, \mathbf{x}_0$ as above, where $\Omega = \mathcal{D}\left(\mathbb{R}_+, \mathbb{N}\right)^{\mathbb{Z}^d} \times \mathcal{D}\left(\mathbb{R}_+, \mathbb{Z}^d\right)$, $\mathcal{F}$ is the appropriate product σ-algebra on Ω, and

$$\mathbf{P}_{\hat{\mu}_0,\mathbf{x}_0}\left(M \times N\right) = \int_M d\mathrm{P}_{\hat{\mu}_0}(\omega) P^{\omega}_{\mathbf{x}_0}(N), \tag{2.10}$$

where M and N are measurable subsets from A and $\mathcal{D}\left(\mathbb{R}_+, \mathbb{Z}^d\right)$, respectively. We will call $P^{\omega}_{\mathbf{x}_0}$ the *quenched* law of X (given ω), and $\mathbf{P}_{\hat{\mu}_0,\mathbf{x}_0}$ the *annealed* law of X.

We will say that a claim about X holds $\mathbf{P}_{\mathbf{x}_0,\hat{\mu}_0}$-a.s. if for $\mathrm{P}_{\hat{\mu}_0}$-almost every ω (for $\mathrm{P}_{\hat{\mu}_0}$-a.e. ω), the claim holds $P^{\omega}_{\mathbf{x}_0}$-a.s.

We will also denote by $\mathrm{E}_{\hat{\mu}_0}$, $E^{\omega}_{\mathbf{x}_0}$ and $\mathbf{E}_{\hat{\mu}_0,\mathbf{x}_0}$ the expectations with respect to $\mathrm{P}_{\hat{\mu}_0}$, $P^{\omega}_{\mathbf{x}_0}$ and $\mathbf{P}_{\hat{\mu}_0,\mathbf{x}_0}$, respectively. We reserve the notation $\mathbb{P}_\mu$ (resp., $\mathbb{P}_n$) and $\mathbb{E}_\mu$ (resp., $\mathbb{E}_n$) for the probability and its expectation underlying a single birth-and-death process (as specified above) starting from an initial distribution μ on $\mathbb{N}$ (resp., starting from $n \in \mathbb{N}$).

Furthermore, in what follows, without loss of generality, we will adopt $\mathbf{x}_0 \equiv \mathbf{0}$, and omit such a subscript, i.e.,

$$P^{\omega} := P^{\omega}_{\mathbf{0}} \quad \text{and} \quad \mathbf{P}_{\hat{\mu}_0} := \mathbf{P}_{\hat{\mu}_0,\mathbf{0}}. \tag{2.11}$$

We will also omit the subscript $\hat{\mu}_0$ when it is irrelevant. From now on we will indicate

$$\mathbf{P}_{\mathbf{w}}, \ \mathbf{w} \in \Lambda, \tag{2.12}$$

the law of the joint process starting from $\omega(0) = \mathbf{w}$ and $\mathbf{x}_0 \equiv \mathbf{0}$.

Let now $\Delta_n := \tau_n - \tau_{n-1}$, $n \in \mathbb{N}_*$. We observe that

$$\mathbf{P}_{\hat{\mu}_0}\left(\tau_1 > t\right) = \mathrm{E}_{\hat{\mu}_0}\left[\exp\left(-\int_0^t \varphi(\omega_{\mathbf{0}}(s))\, ds\right)\right], \quad t \in \mathbb{R}_+, \tag{2.13}$$

and, for $n \in \mathbb{N}$,

$$\mathbf{P}_{\hat{\mu}_0}\left(\Delta_{n+1} > t\right) = \mathrm{E}_{\hat{\mu}_0}\left[\exp\left(-\int_{\tau_n}^{\tau_n+t} \varphi(\omega_{\mathsf{x}_n}(s))\, ds\right)\right], \quad t \in \mathbb{R}_+, \tag{2.14}$$

recalling that $(\mathsf{x}_n)_{n\in\mathbb{N}}$ denotes the jump chain of $(X(t))_{t\in\mathbb{R}_+}$. For $n \in \mathbb{N}$, let us set

$$I_n(t) := \int_{\tau_n}^{\tau_n+t} \varphi(\omega_{\mathsf{x}_n}(s))\, ds, \quad t \in \mathbb{R}_+, \tag{2.15}$$

$I_n : \mathbb{R}_+ \to \mathbb{R}_+$, $n \in \mathbb{N}$, is non-decreasing, continuous and diverges at ∞ $\mathbf{P}$-a.s. under our conditions on the parameters of ω (which ensure its recurrence). We may thus write

$$\mathbf{P}_{\hat{\mu}_0}\left(\tau_1 > t\right) = \mathrm{E}_{\hat{\mu}_0}\left[e^{-I_0(t)}\right], \quad t \in \mathbb{R}_+, \tag{2.16}$$

and

$$\mathbf{P}_{\hat{\mu}_0}\left(\Delta_{n+1} > t\right) = \mathrm{E}_{\hat{\mu}_0}\left[e^{-I_n(t)}\right], \quad t \in \mathbb{R}_+. \tag{2.17}$$

2.1 Alternative construction

We finish this section with an alternative construction of X, based in the following simple remark, which will be used further on.

Let ω and ξ as above be fixed, and set $\mathsf{T}_0 = 0$ and, for $n \in \mathbb{N}_*$, $\mathsf{T}_n = \sum_{k=0}^{n-1} I_k\left(\Delta_{k+1}\right)$.

Lemma 2.2. Under the conditions on the parameters of ω assumed in the paragraph of (2.1), we have that $\{\mathsf{T}_n : n \in \mathbb{N}_*\}$ is a rate 1 Poisson point process on $\mathbb{R}_+$, independent of ω and ξ.

Proof. It is enough to check that, given ω and ξ, $(\Delta_n)_{n\in\mathbb{N}_*}$ are the event times of a Poisson point process, which are thus independent of each other; the conclusion follows readily from the fact that

$$\mathbf{P}\big(I_n(\Delta_{n+1}) > t\big) = \mathbf{P}\big(\Delta_{n+1} > I_n^{-1}(t)\big) = \mathrm{E}\left[e^{-I_n\left(I_n^{-1}(t)\right)}\right] = e^{-t},\ t \in \mathbb{R}_+,$$

where I_n^{-1} is the right continuous inverse of I_n. $\square$

We thus have an alternative construction of X, as follows. Let ω, ξ be as described at the beginning of the section. Let also $\mathsf{V} = (\mathsf{V}_n)_{n\in\mathbb{N}}$ be an independent family of standard exponential random variables. Then, given ω, set $X(0) = \mathsf{x}_0 \equiv \mathbf{0}$ and $\tau_0 = 0$, and define

$$\tau_1 = I_0^{-1}(\mathsf{V}_1). \tag{2.18}$$

For all $t \in (0, \tau_1)$, $X(t) = X(0)$ and

$$X(\tau_1) = X(0) + \xi_1 = \mathsf{x}_1, \tag{2.19}$$

set, inductively,

$$\tau_n = \tau_{n-1} + I_{n-1}^{-1}(\mathsf{V}_n), \tag{2.20}$$

and for $t \in (\tau_{n-1}, \tau_n)$, $X(t) = X(\tau_{n-1})$ and

$$X(\tau_n) = X(\tau_{n-1}) + \xi_n = \mathsf{x}_n. \tag{2.21}$$

We have thus completed the alternative construction of X. Notice that we have made use of ω and ξ, as in the original construction, but replaced $\mathcal{M}$ of the latter construction by V as the remaining ingredient. The alternative construction comes in handy in a coupling argument we develop in order to prove a Law of Large Numbers for the jump times of X.

3 Main results

We now state our main results, a Law of Large Numbers for and central limit theorem for X_t under $\mathbf{P_0}$, and then an extension of the CLT for product initial conditions where the marginal has exponential moments with rates bounded from below.

3.1 Limit theorems under $\mathrm{P}_\mathbf{0}$

In this subsection, we state two of our main results, namely a Law of Large Numbers and a Central Limit Theorem for X under $\mathbf{P_0}$[2] and under the following extra conditions on φ:

$$\varphi \text{ is non-increasing, } \varphi(0) = 1. \tag{3.1}$$

The statements are provided shortly.

[2]We recall that $\mathbf{P_w}$ represents the law of X starting from $\omega(0) = \mathbf{w}$ and $\mathbf{x}_0 = \mathbf{0}$.

The main ingredient for these results is a Law of Large Numbers for the jump time of X. This relies on subadditivity to allow for the use of the Subadditive Ergodic Theorem, the moment control for which is obtained through a stochastic domination result for the distribution of the environment seen by the particle at jump times — this is where the monotonicity of φ plays a role. All proofs are deferred to Section 4.

In order to state the main results of this subsection, we need the following preliminaries and further conditions on $\mathbf{p}, \mathbf{q}$. Let ν denote the invariant distribution of $\omega_{\mathbf{0}}$, such that, as is well-known, $\nu_n = \text{const} \prod_{i=1}^{n} \frac{p_{i-1}}{q_i}$, for $n \in \mathbb{N}$, where the latter product is defined as equal to 1 for $n = 0$. Next set $\rho_n = \frac{p_n}{q_n}$, $R_n = \prod_{i=1}^{n} \rho_i$ and $S_n = \sum_{i \geq n} R_i$, $n \geq 1$, and let $R_0 = 1$. These quantities are well defined and, in particular, it follows from (2.1) that the latter sum is finite for all $n \geq 1$. It follows from our conditions on $\mathbf{p}, \mathbf{q}$ that $\rho_n \leq 1$ for all n, and thus that (R_n) is a non-increasing sequence.

We will require the following extra condition on $\mathbf{p}, \mathbf{q}$, in addition to those imposed in the paragraph of (2.1) above: we will assume

$$\sum_{n \geq 1} \frac{S_n^2}{R_n} < \infty. \tag{3.2}$$

We note that it follows from our previous assumptions on $\mathbf{p}, \mathbf{q}$ that (3.2) is stronger than (2.1), since $S_n \geq R_n$ for all n. The relevance of this condition is that it implies the two conditions to be introduced next.

Let w denote the embedded chain of $\omega_{\mathbf{0}}$, and, for $n \geq 0$, let T_n denote the first passage time of w by n, namely, $T_n = \inf\{i \geq 0 : \mathsf{w}_i = n\}$, with the usual convention that $\inf \emptyset = \infty$. Condition (3.2) is equivalent, as will be argued in Appendix A, to

$$\text{either } \mathbb{E}_\nu(T_0) < \infty \text{ or } \mathbb{E}_1(T_0^2) < \infty.^3 \tag{3.3}$$

In Appendix B, we will show that (3.2) is stronger than asking that ν have a finite first moment, and that a finite second moment of ν implies it, under our conditions on $\mathbf{p}, \mathbf{q}$.

Conditions (3.3) will be required in our arguments for the following main results of this subsection — they are what we meant by 'strongly ergodic' in the abstract. See Remark 4.11 at the end of next section.

Theorem 3.1 (Law of Large Numbers for X)**.** Assume the above conditions and that $\mathbf{E}(\|\xi_1\|) < \infty$. Then there exists $\theta \in (0, \infty)$ such that

$$\frac{X(t)}{t} \to \frac{\mathbf{E}(\xi_1)}{\theta} \;\; \mathbf{P_0}\text{-a.s. as } t \to \infty. \tag{3.4}$$

Here and below $\|\cdot\|$ is the sup norm in $\mathbb{Z}^d$.

Theorem 3.2 (Central Limit Theorem for X)**.** Assume the above conditions together with $\mathbf{E}(\|\xi_1\|^2) < \infty$ and $\mathbf{E}(\xi_1) = \mathbf{0}$. Then, for $\mathrm{P}_\mathbf{0}$-a.e. ω, we have that

$$\frac{X(t)}{\sqrt{t/\theta}} \Rightarrow N_d(\mathbf{0}, \Sigma) \;\text{ under }\; P^\omega, \tag{3.5}$$

where Σ is the covariance matrix of ξ_1, and μ is as in Theorem 3.1.

Going beyond the mean zero assumption in Theorem 3.2 would require substantially more work than we present here, with our approach; see Remark 4.10 at the end of this section.

[3] We note that w has the same invariant distribution as $\omega_{\mathbf{0}}$, namely ν.

3.2 Extension to other initial conditions

We now state a CLT under more general, product initial environment conditions. We will assume for simplicity that the birth-and-death environments are homogeneous, i.e., $p_n \equiv p$, with $p \in (0, 1/2)$. In this context, we use the notation $BDP(p, q)$ for the process, where $q = 1 - p$. We hope that the arguments developed for the inhomogeneous case in Section 4, as well as the ones for the homogeneous case in Section 5, are sufficiently convincing that this may be relaxed — although we do not pretend to be able to propose optimal or near optimal conditions for the validity of any of the subsequent results.

As we will see below, our argument for this extension does not go through a LLN for the position of the particle, as it did in the previous section, we do not discuss an extension for the LLN, rather focusing on the CLT.[4]

We will as before assume that the initial condition for the environment is in product form, given by $\hat{\mu}_0 = \bigotimes_{\mathbf{x}\in\mathbb{Z}^d} \mu_{\mathbf{x},0}$, and we will further assume that $\mu_{\mathbf{x},0} \preceq \bar{\mu}$, with $\bar{\mu}$ a probability measure on $\mathbb{N}$ with an exponentially decaying tail, i.e., there exists a constant $\beta > 0$ such that

$$\bar{\mu}([n, \infty)) \leq \text{const } e^{-\beta n} \tag{3.6}$$

for all $n \geq 0$. Notice that this includes $\hat{\nu}$, in the present homogeneous BDP case. Again, it should hopefully be quite clear from our arguments that these conditions can be relaxed both in terms of the homogeneity of $\bar{\mu}$, as the decay of its tail, but we do not seek to do that presently, or to suggest optimal or near optimal conditions.

Our strategy is to first couple the environment starting from $\hat{\mu}_0$ to the one starting from $\mathbf{0}$, so that for each $\mathbf{x} \in \mathbb{Z}^d$, each respective BD process evolves independently one from the other until they first meet, after which time they coalesce forever.

One natural second step is to couple two versions of the random walks, one starting from each of the two coupled environments in question, so that they jump together when they are at the same point at the same time, and see the same environment. One quite natural way to try and implement such a strategy is to have both walks have the same embedded chains, and show that they will (with high probability) eventually meet at a time at and after which they only see the same environments. Even though this looks like it should be true, we did not find a way to control the distribution of the environments seen by both walks in their evolution (in what might be seen as a *game of pursuit*) in an effective way.

So we turned to our actual subsequent strategy, which depends on the dimension (and requires different further conditions on π, the distribution of ξ, in $d \geq 2$). In $d \leq 2$, we modify the strategy proposed in the previous paragraph by letting the two walks evolve independently when separated, and relying on recurrence to ensure that they will meet in the aforementioned conditions; there is a technical issue arising in the latter point for general π (within the conditions of Theorem 3.2), which we resolve by invoking a result in the literature, which is stated for $d = 1$ only, so for $d = 2$ we need to restrict π to be symmetric. See Remark 5.3 below.

In $d \geq 3$, we of course do not have recurrence, but, rather, transience, and so we rely on this, instead, to show that our random walk will eventually find itself in a cut point of its trajectory such that the environment along its subsequent trajectory is coalesced with a suitably coupled environment starting from $\mathbf{0}$; this allows for a comparison to the situation of Theorem 3.2. The argument requires the a.s. existence of infinitely many cut points of (x_n), and, to ascertain that, we rely on the literature, which states boundedness of the support of π as a sufficient condition (but no symmetry).

[4] But the same line of argumentation below may be readily seen to yield a LLN, under the same conditions.

Theorem 3.3 (Central Limit Theorem for X)**.** Under the same conditions of Theorem 3.2, and assuming the conditions on $\hat{\mu}_0$ stipulated in the paragraph of (3.6) above hold, then we have that for $\mathrm{P}_{\hat{\mu}_0}$-a.e. ω

$$\frac{X(t)}{\sqrt{t/\theta}} \Rightarrow N_d(\mathbf{0}, \Sigma) \text{ under } P^{\omega}, \tag{3.7}$$

provided the following extra conditions on π hold, depending on d: in $d = 1$, no extra condition; in $d = 2$, π is symmetric; in $d \geq 3$, π has bounded support.

A proof of Theorem 3.3 will be presented in Section 5.

Remark 3.4. Both Theorem 3.2 and 3.3 state convergence in distribution for a.e.-realization of the environment. Averaged versions of these results (with respect to the environmental distribution) follow (by dominated convergence).

4 LLN and CLT under $\mathrm{P}_\mathbf{0}$

This section is devoted to the proofs of Theorems 3.1 and 3.2. They are both based on a LLN for the time X takes to give n jumps, the object of the next subsection.

4.1 Law of large numbers for the jump times of X

In this subsection, we prove a Law of Large Numbers for $(\tau_n)_{n\in\mathbb{N}}$ under $\mathbf{P}_\mathbf{0}$; this is the key ingredient in our arguments for the main results of this section; see Proposition 4.9 below. Our strategy for proving the latter result is to establish suitable stochastic domination of the environment by a modified environment, leading to a corresponding domination for jump times; we develop this program next.

We start by recalling some well-known definitions. Given two probability distributions on $\mathbb{N}$, υ_1 and υ_2, we indicate by $\upsilon_1 \preceq \upsilon_2$ that υ_1 is stochastically dominated by υ_2, i.e.,

$$\upsilon_1 \left(\mathbb{N} \setminus \mathsf{A}_k\right) \leq \upsilon_2 \left(\mathbb{N} \setminus \mathsf{A}_k\right), \quad \mathsf{A}_k := \{0, \ldots, k\}, \ \forall\, k \in \mathbb{N}. \tag{4.1}$$

We equivalently write in this situation, abusing notation, $X_1 \preceq \upsilon_2$, if X_1 is a random variable with distribution υ_1.

Let us assume at this point, for simplicity of exposition, that $\varphi(n) > 0$ for all n. See Remark 4.14 at the end of this section.

Now let Q denote the generator of $\omega_\mathbf{0}$ (which is a *Q-matrix*), and consider the following matrix:

$$\mathsf{Q}^{\psi} = D\mathsf{Q}, \text{ with } D = \mathrm{diag}\{\psi(n)\}_{n\in\mathbb{N}}, \tag{4.2}$$

where $\psi : \mathbb{N} \to [1, \infty)$ is such that $\psi(n) = 1/\varphi(n)$ for all n, with φ as defined in the paragraph of (2.4) above. We will write $\psi(n)$ alternatively as ψ_n many times below.

Notice that Q^{ψ} is also a Q-matrix, and that it generates a birth-and-death process on $\mathbb{N}$, say $\breve{\omega}_\mathbf{0}$, with transition rates given by

$$\mathsf{Q}^{\psi}(n, n+1) = \psi_n p_n =: p_n^{\psi},\ n \in \mathbb{N}\,; \quad \mathsf{Q}^{\psi}(n, n-1) = \psi_n q_n =: q_n^{\psi},\ n \in \mathbb{N}_*; \tag{4.3}$$

this is a positive recurrent process, with invariant distribution ν^{ψ} on $\mathbb{N}$ such that

$$\nu_n^{\psi} = \text{const} \prod_{i=1}^{n} \frac{p_{i-1}^{\psi}}{q_i^{\psi}},\ n \in \mathbb{N},$$

with a similar convention for the product as for ν. One may readily check that $\nu^{\psi} \preceq \nu$, since ψ is increasing. The relevance of $\breve{\omega}_\mathbf{0}$ in the present study issues from the following straightforward result. Recall (2.15).

Lemma 4.1. Suppose $\omega_{\mathbf{0}}(0) \sim \breve{\omega}_{\mathbf{0}}(0)$. Then

$$(\omega_{\mathbf{0}}(t),\, t \in \mathbb{R}_+) \sim (\breve{\omega}_{\mathbf{0}}(I_0(t)),\, t \in \mathbb{R}_+). \tag{4.4}$$

We have the following immediate consequence from this and Lemma 2.2.

Corollary 4.2. Let V_1 be a standard exponential random variable, independent of $\breve{\omega}_{\mathbf{0}}$. Then

$$\omega_{\mathbf{0}}(\tau_1) \sim \breve{\omega}_{\mathbf{0}}(\mathsf{V}_1). \tag{4.5}$$

Figure 1 illustrates a coupling behind (4.4), (4.5).

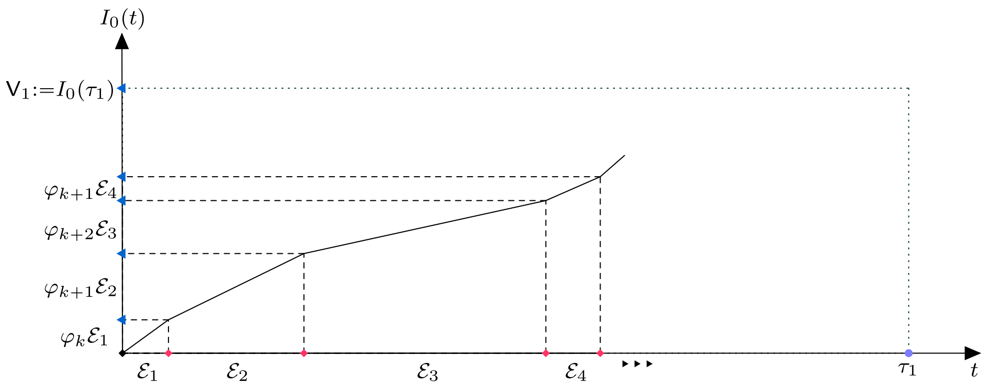

Figure 1: $\mathcal{E}_1, \mathcal{E}_2, \dots$ are iid standard exponentials; x (resp., y)-axis indicates constancy intervals of $\omega_{\mathbf{0}}$ (resp., $\breve{\omega}_{\mathbf{0}}$) in a realization where $\omega_{\mathbf{0}}(0) = \breve{\omega}_{\mathbf{0}}(0) = k \in \mathbb{N}$.

The following result is well-known, and may be obtained by a straightforward coupling argument — see e.g. discussion in paragraphs leading to Theorem 2.4 in [17].

Lemma 4.3. Let μ and μ' denote two probability distributions on $\mathbb{N}$ such that $\mu \preceq \mu'$. Then, for all $t \in \mathbb{R}_+$,

$$\mu e^{t\mathsf{Q}} \preceq \mu' e^{t\mathsf{Q}}. \tag{4.6}$$

Here and below $e^{t\mathsf{Q}'}$ denotes the semigroup associated to an irreducible and recurrent Q-matrix Q' on $\mathbb{N}$. We have an immediate consequence of Lemma 4.3, as follows.

Corollary 4.4. If μ is a probability on $\mathbb{N}$ such that $\mu \preceq \nu$, then, for all $t \in \mathbb{R}_+$,

$$\mu e^{t\mathsf{Q}} \preceq \nu. \tag{4.7}$$

We present now a few more substantial domination lemmas, leading to a key ingredient for justifying the main result of this subsection.

Lemma 4.5. Let Q^{ψ} be as in (4.2), (4.3). Then, for all $t \in \mathbb{R}_+$,

$$\nu e^{t\mathsf{Q}^{\psi}} \preceq \nu. \tag{4.8}$$

Proof. Let $\mathsf{Y} = (\mathsf{Y}_t)_{t \in \mathbb{R}_+}$ denote the birth-and-death process generated by Q^{ψ} started from ν. Set $P_{n,j}(t) := \mathbb{P}(\mathsf{Y}_t = j \mid \mathsf{Y}_0 = n)$, $t \in \mathbb{R}_+$, $n, j \in \mathbb{N}$. For $l \in \mathbb{N}$,

$$\mathbb{P}(\mathsf{Y}_t \le l) = \sum_{j \le l} \mathbb{P}(\mathsf{Y}_t = j) = \sum_{j \le l} \sum_{n \ge 0} \nu_n P_{n,j}(t). \tag{4.9}$$

By Tonelli's theorem, we write

$$\mathbb{P}(\mathsf{Y}_t \le l) = \sum_{n \ge 0} \sum_{j \le l} \nu_n P_{n,j}(t). \tag{4.10}$$

Consider now Kolmogorov's forward equations for Y, given by

$$P'_{n,0}(t) = -p_0^{\psi} P_{n,0}(t) + q_1^{\psi} P_{n,1}(t); \tag{4.11}$$

$$P'_{n,j}(t) = p_{j-1}^{\psi} P_{n,j-1}(t) - \psi_j P_{n,j}(t) + q_{j+1}^{\psi} P_{n,j+1}(t), \quad j \geq 1; \tag{4.12}$$

$n \geq 0$. It follows that

$$\left| \sum_{j \leq l} \nu_n P'_{n,j}(t) \right| = \nu_n \left| q_{l+1}^{\psi} P_{n,l+1}(t) - p_l^{\psi} P_{n,l}(t) \right| \leq \nu_n \psi_{l+1}, \tag{4.13}$$

for all t; since ν is summable, we have that

$$\mathbb{P}'(\mathsf{Y}_t \leq l) = \sum_{n \geq 0} \sum_{j \leq l} \nu_n P'_{n,j}(t). \tag{4.14}$$

We now consider Kolmogorov's backward equations for Y, given by

$$P'_{0,j}(t) = -p_0^{\psi} P_{0,j}(t) + p_0^{\psi} P_{1,j}(t) = p_0^{\psi} (P_{1,j}(t) - P_{0,j}(t)); \tag{4.15}$$

$$\begin{aligned} P'_{n,j}(t) &= q_n^{\psi} P_{n-1,j}(t) - \psi_n P_{n,j}(t) + p_n^{\psi} P_{n+1,j}(t), \\ &= q_n^{\psi} (P_{n-1,j}(t) - P_{n,j}(t)) - p_n^{\psi} (P_{n,j}(t) - P_{n+1,j}(t)), \quad n \geq 1, \end{aligned} \tag{4.16}$$

$j \geq 0$. Using Kolmogorov's backward equations for Y, given above, and setting $d_n := \mathbb{P}_n(\mathsf{Y}_t \leq l) - \mathbb{P}_{n+1}(\mathsf{Y}_t \leq l)$, $n \in \mathbb{N}$, we rewrite (4.14) as

$$\begin{aligned} \mathbb{P}'(\mathsf{Y}_t \leq l) &= \sum_{j \leq l} \nu_0 P'_{0,j}(t) + \sum_{n \geq 1} \sum_{j \leq l} \nu_n P'_{n,j}(t) \\ &= -\nu_0 p_0^{\psi} d_0 + \sum_{n \geq 1} \nu_n \big(q_n^{\psi} d_{n\text{-}1} - p_n^{\psi} d_n \big) \\ &= \sum_{n \geq 0} \nu_{n+1} q_{n+1}^{\psi} d_n - \sum_{n \geq 0} \nu_n p_n^{\psi} d_n, \end{aligned} \tag{4.17}$$

provided

$$\sum_{n \geq 1} \nu_n \psi_n (d_{n\text{-}1} \vee d_n) < \infty, \tag{4.18}$$

which we claim to hold; see justification below. We note that $d_n \geq 0$ for all n, l and t, as can be justified by a straightforward coupling argument. It follows that

$$\begin{aligned} \mathbb{P}'(\mathsf{Y}_t \leq l) &= \sum_{n \geq 0} (\nu_{n+1} q_{n+1}^{\psi} - \nu_n p_n^{\psi}) d_n \\ &= \sum_{n \geq 0} (\psi_{n+1} \nu_{n+1} q_{n+1} - \psi_n \nu_n p_n) d_n \\ &= \sum_{n \geq 0} (\psi_{n+1} \nu_n p_n - \psi_n \nu_n p_n) d_n \\ &= \sum_{n \geq 0} (\psi_{n+1} - \psi_n) \nu_n p_n d_n \geq 0, \end{aligned} \tag{4.19}$$

since ψ is non-decreasing, where the third equality follows by reversibility of Y.

We thus have that $\mathbb{P}(\mathsf{Y}_t \leq l)$ is non-decreasing in t for every l; therefore

$$\nu(\mathsf{A}_l) = \mathbb{P}(\mathsf{Y}_0 \leq l) \leq \mathbb{P}(\mathsf{Y}_t \leq l) \tag{4.20}$$

for all l, and (4.8) is established.

It remains to argue (4.18). Let $\mathsf{H}_n := \inf\{s \geq 0 : \mathsf{Y}_s = n\}$, $n \in \mathbb{N}$ be the hitting time of n by Y. For $n \geq l$, we have that

$$\begin{aligned} d_n =& \mathbb{P}_n(\mathsf{Y}_t \leq l) - \int_0^t \mathbb{P}_{n+1}(\mathsf{H}_n \in \mathrm{d}s)\mathbb{P}_n(\mathsf{Y}_{t-s} \leq l)\mathrm{d}s \\ =& \int_0^t \mathbb{P}_{n+1}(\mathsf{H}_n \in \mathrm{d}s)\Big[\mathbb{P}_n(\mathsf{Y}_t \leq l) - \mathbb{P}_n(\mathsf{Y}_{t-s} \leq l)\Big]\mathrm{d}s \\ &+ \mathbb{P}_n(\mathsf{Y}_t \leq l)\int_t^\infty \mathbb{P}_{n+1}(\mathsf{H}_n \in \mathrm{d}s)\mathrm{d}s \\ =& \int_0^t \mathbb{P}_{n+1}(\mathsf{H}_n \in \mathrm{d}s)\Big[\mathbb{P}_n(\mathsf{Y}_t \leq l, \mathsf{Y}_{t-s} > l) - \mathbb{P}_n(\mathsf{Y}_t > l, \mathsf{Y}_{t-s} \leq l)\Big]\mathrm{d}s \\ &+ \mathbb{P}_n(\mathsf{Y}_t \leq l)\mathbb{P}_{n+1}(\mathsf{H}_n > t) \\ =&: d_n' + d_n''. \end{aligned} \tag{4.21}$$

Let now $V = (V_i)_{i\in\mathbb{N}_*}$ be a sequence of independent standard exponential random variables, and consider the embedded chain $\tilde{\mathsf{Y}} = \big(\tilde{\mathsf{Y}}_k\big)_{k\geq 0}$ of $(\mathsf{Y}_t)_{t\in\mathbb{R}_+}$, and $\tilde{\mathsf{H}}_n = \inf\{k \geq 0 : \tilde{\mathsf{Y}}_k = n\}$. Notice that $\tilde{\mathsf{Y}}$ is distributed as w, and $\tilde{\mathsf{H}}_n$ is distributed as T_n, introduced at the beginning of the section. Let V and $\tilde{\mathsf{Y}}$ be independent. Let us now introduce an auxiliary random vaiable $\mathsf{H}_n' = \sum_{i=1}^{\tilde{\mathsf{H}}_n} V_i$, and note that, given that $\mathsf{Y}_0 = n+1$, $\mathsf{H}_n \overset{st}{\preceq} \varphi_{n+1}\mathsf{H}_n'$; it follows from this and the Markov inequality that

$$\mathbb{P}_{n+1}(\mathsf{H}_n > t) \leq \mathbb{P}_{n+1}(\mathsf{H}_n' > \psi_{n+1}t) \leq \frac{\varphi_{n+1}\mathcal{T}_{n+1}}{t} \leq \text{ const } \varphi_{n+1}\frac{S_n}{R_n} \tag{4.22}$$

(where $\mathcal{T}_{n+1} = \mathbb{E}_{n+1}(T_n)$; see Appendix A). It follows that

$$\sum_{n>l} \nu_n\psi_n d_{n-1}'' \leq \text{ const } \sum_{n\geq 1}\frac{\nu_n}{R_n}S_n \leq \text{ const } \sum_{n\geq 1} S_n < \infty, \tag{4.23}$$

by the ergodicity assumption on $\omega_{\mathbf{0}}$, and similarly $\sum_{n\geq 1}\nu_n\psi_n d_n'' < \infty$.

Now, by the Markov property

$$\begin{aligned} \mathbb{P}_n(\mathsf{Y}_t \leq l, \mathsf{Y}_{t-s} > l) &= \sum_{j\geq l+1}\mathbb{P}_n(\mathsf{Y}_{t-s} = j)\mathbb{P}_j(\mathsf{Y}_s \leq l) \\ &\leq \sum_{j\geq l+1}\mathbb{P}_n(\mathsf{Y}_{t-s} = j)\mathbb{P}_{l+1}(\mathsf{Y}_s \leq l) \\ &\leq \sum_{j\geq l+1}\mathbb{P}_n(\mathsf{Y}_{t-s} = j)\left(1 - e^{-\psi_{l+1}s}\right) \leq 1 - e^{-\psi_{l+1}s}. \end{aligned} \tag{4.24}$$

Thus,

$$\begin{aligned} d_n' &\leq \int_0^t \mathbb{P}_{n+1}(\mathsf{H}_n \in \mathrm{d}s)(1 - e^{-\psi_{l+1}s})\mathrm{d}s \leq \mathbb{E}_{n+1}\left(1 - e^{-\psi_{l+1}\mathsf{H}_n}\right) \\ &\leq \psi_{l+1}\mathbb{E}_{n+1}\left(H_n\right) \leq \psi_{l+1}\varphi_{n+1}\mathcal{T}_{n+1}, \end{aligned} \tag{4.25}$$

and, similarly as above, we find that $\sum_{n\geq 1}\nu_n\psi_n(d_{n-1}' + d_n') < \infty$, therefore (4.18) is established. ☐

It follows from Lemma 4.5 and preceding results that, if $\omega_{\mathbf{0}}(0) \sim \nu$, then

$$\omega_{\mathbf{0}}(\tau_1) \preceq \nu. \tag{4.26}$$

Let us now assume that $\mu_{\mathbf{x},0} \preceq \nu$ for every $\mathbf{x} \in \mathbb{Z}^d$. Based on the above domination results, we next construct a modification of the joint process (X, ω), to be denoted $(\breve{X}, \breve{\omega})$, in a coupled way to (X, ω), so that $\breve{\omega}$ has *less* spatial dependence than, and at the same time dominates ω in a suitable way. The idea is to let $\breve{X}$ have the same embedded chain as X, and jump according to $\breve{\omega}$ as X jumps according to ω; we let $\breve{\omega}$ evolve with the same law as ω between its jump times, and at jump times we replace $\breve{\omega}$ at the site where $\breve{X}$ jumped from by a suitable dominating random variable distributed as ν. Details follow.

We first construct a sequence of environments between jumps of $\breve{X}$, as follows. Let (X, ω) be as above, starting from $X(0) = 0$, $\omega(0) \sim \hat{\mu}_0$, then, enlarging the original probability space if necessary, we can find iid random variables $\omega^0_{\mathbf{x}}(0)$, $\mathbf{x} \in \mathbb{Z}^d$, distributed according to ν, such that $\omega^0_{\mathbf{x}}(0) \geq \omega_{\mathbf{x}}(0)$, $\mathbf{x} \in \mathbb{Z}^d$.

We let now ω^0 evolve for $t \geq 0$ coupled to ω in such a way that $\omega^0_{\mathbf{x}}(t) \geq \omega_{\mathbf{x}}(t)$, $\mathbf{x} \in \mathbb{Z}^d$. Let now $\breve{\tau}_1$ be obtained from ω^0 in the same way as τ_1 was obtained from ω, using the same $\mathcal{M}$ for ω^1 as for ω (recall definition from paragraph of (2.2)); $\breve{\tau}_1$ is the time of the first jump of $\breve{X}$, and set $\breve{X}(\breve{\tau}_1) = \mathsf{x}_1$. Notice that $\breve{\tau}_1 \geq \tau_1$.

Noticing as well that $\omega^0_{\mathbf{x}}(\breve{\tau}_1)$, $\mathbf{x} \neq 0$, are independent with common distribution ν, and independent of $\omega^0_{\mathbf{0}}(\breve{\tau}_1)$, and using (4.26), again enlarging the probability space if necessary, we find $\mathcal{W}_1$ with distribution ν such that $\mathcal{W}_1 \geq \omega^0_{\mathbf{0}}(\breve{\tau}_1)$, with $\mathcal{W}_1$ is independent of $\omega^0_{\mathbf{x}}(\breve{\tau}_1)$, $\mathbf{x} \neq 0$; and we make $\omega^1_{\mathbf{0}}(\breve{\tau}_1) = \mathcal{W}_1$, and $\omega^1_{\mathbf{x}}(\breve{\tau}_1) = \omega^0_{\mathbf{x}}(\breve{\tau}_1)$, $\mathbf{x} \neq 0$. Notice that $\omega^1_{\mathbf{x}}(\breve{\tau}_1)$, $\mathbf{x} \in \mathbb{Z}^d$ are iid with marginals with distribution ν.

We now iterate this construction, inductively: given ξ, let us fix $n \geq 1$, and suppose that for each $0 \leq j \leq n-1$, we have constructed $\breve{\tau}_j$, and $\omega^j(t)$, $t \geq \breve{\tau}_j$, with $\{\omega^j_{\mathbf{x}}(\breve{\tau}_j), \mathbf{x} \in \mathbb{Z}^d\}$ iid with marginals distributed as ν. We then define $\breve{\tau}_n$ from $\omega^{n-1}(\breve{\tau}_{n-1})$ in the same way as τ_1 was defined from $\omega^0(0)$, but with the random walk originating in x_{n-1}, and with the marks of $\mathcal{M}$ in the upper half-space from $\breve{\tau}_{n-1}$; $\breve{\tau}_n$ is the time of the n-th jump of $\breve{X}$, and we set $\breve{X}(\breve{\tau}_n) = \mathsf{x}_n$.

Next, from (4.26), we obtain $\mathcal{W}_n \geq \omega^{n-1}_{\mathsf{x}_{n-1}}(\breve{\tau}_n)$ such that $\{\mathcal{W}_n; \omega^{n-1}_{\mathbf{x}}(\breve{\tau}_n), \mathbf{x} \neq \mathsf{x}_{n-1}\}$ is an iid family of random variables with marginals distributed as ν, and define a $BDP(\mathbf{p}, \mathbf{q})$ $(\omega^n(t))_{t \geq \breve{\tau}_n}$ starting from $\{\omega^n_{\mathbf{x}}(\breve{\tau}_n) = \omega^{n-1}_{\mathbf{x}}(\breve{\tau}_n), \mathbf{x} \neq \mathsf{x}_{n-1}; \omega^n_{\mathsf{x}_{n-1}}(\breve{\tau}_n) = \mathcal{W}_n\}$ so that $\omega^n_{\mathsf{x}_{n-1}}(t) \geq \omega^{n-1}_{\mathsf{x}_{n-1}}(t)$, $\omega^n_{\mathbf{x}}(t) = \omega^{n-1}_{\mathbf{x}}(t)$, $\mathbf{x} \neq \mathsf{x}_{n-1}$, $t \geq \breve{\tau}_n$.

We finally define $\breve{\omega}(t) = \omega^n(t)$ for $t \in [\breve{\tau}_n, \breve{\tau}_{n+1})$, $n \geq 0$. This coupled construction of $(\omega, \breve{\omega})$ has the following properties.

Lemma 4.6.

1.
$$\breve{\omega}_{\mathbf{x}}(t) \geq \omega_{\mathbf{x}}(t) \text{ for all } \mathbf{x} \in \mathbb{Z}^d \text{ and } t \geq 0; \tag{4.27}$$

2. for each $n \geq 0$,
$$\breve{\omega}_{\mathbf{x}}(\breve{\tau}_n),\ \mathbf{x} \in \mathbb{Z}^d, \text{ are iid random variables with marginals distributed as } \nu; \tag{4.28}$$

3. for all $n \geq 0$, we have that
$$\tau_n \leq \breve{\tau}_n. \tag{4.29}$$

Proof. The first two items are quite clear from the construction, so we will argue only the third item, which is quite clear for $n = 0$ and 1 (the latter case was already pointed out in the description of the construction, above); for the remaining cases, let $n \geq 1$, and suppose, inductively, that $\tau_n \leq \breve{\tau}_n$; there are two possibilities for τ_{n+1}: either $\tau_{n+1} \leq \breve{\tau}_n$, in which case, clearly, $\tau_{n+1} \leq \breve{\tau}_{n+1}$, or $\tau_{n+1} > \breve{\tau}_n$; in this latter case, τ_{n+1} (resp., $\breve{\tau}_{n+1}$) will correspond to the earliest Poisson point (of $\mathcal{M}$) in $\mathcal{Q}_n := [c_{\mathsf{x}_n(d)}, c_{\mathsf{x}_n(d)} + \varphi(\omega_{\mathsf{x}_n}(r))_{r \geq \breve{\tau}_n}$ (resp., $\breve{\mathcal{Q}}_n := [c_{\mathsf{x}_n(d)}, c_{\mathsf{x}_n(d)} + \varphi(\breve{\omega}_{\mathsf{x}_n}(r))_{r \geq \breve{\tau}_n}$). By (4.27) and the monotonicity of φ, we have that $\breve{\mathcal{Q}}_n \subset \mathcal{Q}_n$, and it follows that $\tau_{n+1} \leq \breve{\tau}_{n+1}$. $\square$

The next result follows immediately.

Corollary 4.7. For $n \geq 1$ and any $\hat{\mu}_0 \preceq \hat{\nu}$

$$\mathbf{E}_{\hat{\mu}_0}(\tau_n) \leq \mathbf{E}_{\hat{\nu}}(\breve{\tau}_n) = n\mathbf{E}_{\hat{\nu}}(\breve{\tau}_1) = n\mathbf{E}_{\hat{\nu}}(\tau_1). \tag{4.30}$$

The following result, together with (4.30), is a key ingredient in the justification of the main result of this subsection.

Lemma 4.8.

$$\mathbf{E}_{\hat{\nu}}(\tau_1) < \infty \tag{4.31}$$

Proof. Let us write

$$\begin{aligned}
\mathbf{E}_{\hat{\nu}}(\tau_1) &= \int_0^\infty \mathbf{P}_{\hat{\nu}}(\tau_1 > t)\, dt \\
&= \int_0^\infty \mathrm{E}_{\hat{\nu}}\left(e^{-I_{\mathbf{0}}(t)}\right) dt \\
&= \int_0^{+\infty} \mathrm{E}_{\hat{\nu}}\left(e^{-I_0(t)}; I_0(t) \geq \epsilon t\right) dt + \int_0^{+\infty} \mathrm{E}_{\hat{\nu}}\left(e^{-I_0(t)}; I_0(t) < \epsilon t\right) dt \\
&\leq \epsilon^{-1} + \int_0^{+\infty} \mathbf{P}_{\hat{\nu}}(I_0(t) < \epsilon t)\, dt \\
&\leq \epsilon^{-1} + \int_0^{+\infty} \mathbf{P}_{\hat{\nu}}\left(\int_0^t \mathbb{1}\{\omega_{\mathbf{0}}(s) = 0\}\, ds < \epsilon t\right) dt.
\end{aligned} \tag{4.32}$$

For $k \in \mathbb{N}$, set $\mathbf{k} = k \times \mathbf{1}$. Conditioning in the initial state of the environment at the origin, we have, for each $\delta > 0$ and each $t \in \mathbb{R}_+$,

$$\begin{aligned}
\mathbf{P}_{\hat{\nu}}\left(\int_0^t \mathbb{1}\{\omega_{\mathbf{0}}(s) = 0\}\, ds < \epsilon t\right) &= \sum_{k=1}^{\lfloor \delta t \rfloor} \nu_k\, \mathbf{P}_{\mathbf{k}}\left(\int_0^t \mathbb{1}\{\omega_{\mathbf{0}}(s) = 0\} ds < \epsilon t\right) \\
&\quad + \sum_{k=\lceil \delta t \rceil}^{\infty} \nu_k\, \mathbf{P}_{\mathbf{k}}\left(\int_0^t \mathbb{1}\{\omega_{\mathbf{0}}(s) = 0\} ds < \epsilon t\right) \\
&\leq \sum_{k=1}^{\lfloor \delta t \rfloor} \nu_k\, \mathbf{P}_{\mathbf{k}}\left(\int_0^t \mathbb{1}\{\omega_{\mathbf{0}}(s) = 0\} ds < \epsilon t\right) \\
&\quad + \nu([\delta t, \infty)).
\end{aligned} \tag{4.33}$$

Thus,

$$\mathbf{E}_{\hat{\nu}}(\tau_1) \leq \epsilon^{-1} + \delta^{-1}\mathbb{E}(\mathcal{W}) + \int_0^{+\infty} \sum_{k=1}^{\lfloor \delta t \rfloor} \nu_k\, \mathbf{P}_{\mathbf{k}}\left(\int_0^t \mathbb{1}\{\omega_{\mathbf{0}}(s) = 0\} ds < \epsilon t\right) dt, \tag{4.34}$$

where $\mathcal{W}$ is a ν-distributed random variable; one readily checks that (3.2) implies that $\mathcal{W}$ has a first moment. It remains to consider the latter summand in (4.34).

For that, let us start by setting $W_0 = \inf\{s > 0 : \omega_{\mathbf{0}}(s) = 0\}$, and defining

$$Z_1 = \inf\{s > W_0 : \omega_{\mathbf{0}}(s) \neq 0\} - W_0, \tag{4.35}$$

$$W_1 = \inf\{s > W_0 + Z_1 : \omega_{\mathbf{0}}(s) = 0\} - (W_0 + Z_1), \tag{4.36}$$

and making $Y_1 = Z_1 + W_1$. Note that Z_1 is an exponential random variable with rate p_0, and W_1 is the hitting time of the origin by a $BDP(\mathbf{p}, \mathbf{q})$ on $\mathbb{N}$ starting from 1; under $\mathbf{P_0}$, $W_0 = 0$, clearly.

For $i \geq 1$, let us suppose defined $Y_1, \ldots, Y_{i-1}$, and let us further define

$$Z_i = \inf \left\{ s > W_0 + \sum_{j=1}^{i-1} Y_j : \omega_{\mathbf{0}}(s) \neq 0 \right\} - \left(W_0 + \sum_{j=1}^{i-1} Y_j \right), \tag{4.37}$$

$$W_i = \inf \left\{ s > W_0 + \sum_{j=1}^{i-1} Y_j + Z_i : \omega_{\mathbf{0}}(s) = 0 \right\} - \left(W_0 + \sum_{j=1}^{i-1} Y_j + Z_i \right), \tag{4.38}$$

and $Y_i = Z_i + W_i$. By the strong Markov property, it follows that Z_i e W_i are distributed as Z_1 e W_1, respectively, and Z_i, W_i, $i \geq 1$ are independent, and thus $(Y_i)_{i\geq 1}$ is iid.

Now set $T_0 = W_0$ and for $n \geq 1$, $T_n = T_{n-1} + Y_n$. Moreover, for $t \in \mathbb{R}_+$, let us define $\mathsf{C}_t = \sum\limits_{n=1}^{\infty} \mathbb{1}\{T_n \leq t\}$. Note that for $k \in \mathbb{N}$ and $a > 0$, we have

$$\begin{aligned}
\mathbf{P}_{\mathbf{k}} \left(\int_0^t \mathbb{1}\{\omega_{\mathbf{0}}(s) = 0\} ds < \epsilon t \right) &= \mathbf{P}_{\mathbf{k}} \left(\int_0^t \mathbb{1}\{\omega_{\mathbf{0}}(s) = 0\} ds < \epsilon t, \mathsf{C}_t < \lfloor at \rfloor \right) \\
&\quad + \mathbf{P}_{\mathbf{k}} \left(\int_0^t \mathbb{1}\{\omega_{\mathbf{0}}(s) = 0\} ds < \epsilon t, \mathsf{C}_t \geq \lfloor at \rfloor \right) \\
&\leq \mathbf{P}_{\mathbf{k}} \left(\mathsf{C}_t < \lfloor at \rfloor \right) + \mathbf{P} \left(\sum_{j=1}^{\lfloor at \rfloor} Z_j < \epsilon t \right)
\end{aligned} \tag{4.39}$$

and, given $\alpha \in (0,1)$,

$$\begin{aligned}
\mathbf{P}_{\mathbf{k}} \left(\mathsf{C}_t < \lfloor at \rfloor \right) &= \mathbf{P}_{\mathbf{k}} \left(\mathsf{C}_t < \lfloor at \rfloor, T_0 < \alpha t \right) + \mathbf{P}_{\mathbf{k}} \left(\mathsf{C}_t < \lfloor at \rfloor, T_0 \geq \alpha t \right) \\
&\leq \mathbf{P}_{\mathbf{0}} \left(\mathsf{C}_{(1-\alpha)t} < \lfloor at \rfloor \right) + \mathbf{P}_{\mathbf{k}} \left(T_0 \geq \alpha t \right) \\
&= \mathbf{P} \left(\sum_{j=1}^{\lfloor at \rfloor} Y_j > (1-\alpha) t \right) + \mathbf{P}_{\mathbf{k}} \left(T_0 \geq \alpha t \right).
\end{aligned} \tag{4.40}$$

By well-known elementary large deviation estimates, we have that

$$\int_0^\infty dt \, \mathbf{P} \left(\sum_{j=1}^{\lfloor at \rfloor} Z_j < \epsilon t \right) < \infty \tag{4.41}$$

as soon as $a < p_0 \epsilon$, which we assume from now on. To conclude, it then suffices to show that

$$\int_0^\infty dt \, \mathbf{P} \left(\sum_{j=1}^{\lfloor at \rfloor} Y_j > (1-\alpha) t \right) < \infty \text{ and } \int_0^\infty dt \sum_{k \geq 0} \nu_k \, \mathbf{P}_{\mathbf{k}} \left(T_0 \geq \alpha t \right) < \infty. \tag{4.42}$$

The latter integral is readily seen to be bounded above by $\alpha^{-1} \mathbb{E}_\nu(T_0)$, and the first condition in (3.3) implies the second assertion in (4.42). The first integral in (4.42) can be written as

$$\int_0^\infty dt \, \mathbf{P} \left(\frac{1}{at} \sum_{j=1}^{\lfloor at \rfloor} \bar{Y}_j > \zeta \right), \tag{4.43}$$

where $\bar{Y}_j = Y_j - b$, $b = \mathbf{E} Y_1 = \mathbf{E} Y_j$, $j \geq 1$, $\zeta = (1 - \alpha - ab)/a$. Now we have that the expression in (4.43) is finite by the Complete Convergence Theorem of Hsu and Robbins (see Theorem 1 in [14]), as soon as $a, \alpha > 0$ are close enough to 0 (so that $\zeta > 0$), and W_1 has a second moment (and thus so does Y_1), but this follows immediately from the first condition in (3.3). $\square$

We are now ready to state and prove the main result of this subsection.

Proposition 4.9. There exists a constant $\theta \in [0,\infty)$ such that

$$\frac{\tau_n}{n} \to \theta \quad \mathbf{P_0}\text{-a.s.} \quad \text{as } n \to \infty. \tag{4.44}$$

Furthermore,

$$\theta > 0. \tag{4.45}$$

Proof. We divide the argument in two parts. We first construct a superadditive triangular array of random variables $\{\mathsf{L}_{m,n} : m,n \in \mathbb{N}, m \leq n\}$ so that $\mathsf{L}_{0,n}$ equals τ_n under $\mathbf{P_0}$. Secondly, we verify that $\{-\mathsf{L}_{m,n} : m,n \in \mathbb{N}, m \leq n\}$ satisfies the conditions of Liggett's version of Kingman's Subadditive Ergodic Theorem, an application of which yields the result.

A triangular array of jump times Somewhat similarly as in the construction leading to Lemma 4.6 (see description preceding the statement of that result), we construct a sequence of environments $\mathring{\omega}^m$, $m \geq 0$, coupled to ω, in a *dominated* way (rather than *dominating*, as in the previous case), as follows.

Let $\omega(0) = \mathbf{0}$, and set $\mathring{\omega}^0 = \omega$. Consider now $\tau_1, \tau_2, \dots$, the jump times of X, as define above. For $m \geq 1$, we define $(\mathring{\omega}^m(t))_{t \geq \tau_m}$ as a $BDP(\mathbf{p},\mathbf{q})$ starting from $\mathring{\omega}^m(\tau_m) = \mathbf{0}$, coupled to ω in $[\tau_m, \infty)$ so that

$$\mathring{\omega}^m_{\mathbf{x}}(t) \leq \omega_{\mathbf{x}}(t) \tag{4.46}$$

for all $t \geq \tau_m$ and all $\mathbf{x} \in \mathbb{Z}^d$.

Let $\mathring{X}^m$ be a random walk in environment $\mathring{\omega}^m$ starting at time τ_m from $\mathbf{x}_m$, with jump times determined, besides $\mathring{\omega}^m$, the Poisson marks of $\mathcal{M}$ in the upper half space from τ_m, in the same way as the jump times of X after τ_m are determined by $(\omega(t))_{t \geq \tau_m}$ and the Poisson marks of $\mathcal{M}$ in the upper half space from τ_m, and having subsequent jump destinations given by x_j, $j \geq m$. Now set $\mathring{\tau}^m_0 = \tau_m$ and let $\mathring{\tau}^m_1, \mathring{\tau}^m_2, \dots$ be the successive jump times of $\mathring{X}^m$.

Finally, for $n \geq m$, set $\mathsf{L}_{m,n} = \mathring{\tau}^m_{n-m} - \tau_m$. $\mathsf{L}_{m,n}$ is the time $\mathring{X}$ takes to give $n-m$ jumps. Notice that $\mathsf{L}_{0,n} = \tau_n$.

Properties of $\{\mathsf{L}_{m,n},\, 0 \leq m \leq n < \infty\}$ We claim that the following assertions hold.

$$\mathsf{L}_{0,n} \geq \mathsf{L}_{0,m} + \mathsf{L}_{m,n} \;\; \mathbf{P_0}\text{-a.s.}; \tag{4.47}$$

$$\left\{\mathsf{L}_{nk,(n+1)k}, n \in \mathbb{N}\right\} \text{ is ergodic for each } k \in \mathbb{N}; \tag{4.48}$$

$$\text{the distribution of } \{\mathsf{L}_{n,n+k} : k \geq 1\} \text{ under } \mathbf{P_0} \text{ does not depend on } n \in \mathbb{N}; \tag{4.49}$$

$$\text{there exists a constant } \gamma_0 < \infty \text{ such that } \mathbf{E_0}(\mathsf{L}_{0,n}) \leq \gamma_0 n. \tag{4.50}$$

Observe that (4.44) then follows from an application of Liggett's version of Kingman's Subadditive Ergodic Theorem to $(-\mathsf{L}_{m,n})_{0 \leq m \leq n < \infty}$ (see [17], Chapter VI, Theorem 2.6).

Note that (4.49) is straightforward from definition; (4.48) follows immediately upon remarking that $\mathsf{L}_{nk,(n+1)k}$, $n \in \mathbb{N}$, are, quite clearly, independent random variables, and (4.50) follows readily from (4.30) and (4.31). So, it remains to argue (4.47), which is equivalent to

$$\mathring{\tau}^m_{n-m} \leq \tau_n,\ 0 \leq m \leq n < \infty. \tag{4.51}$$

We make this point similarly as for (4.29), above. (4.51) is immediate for $m = 0$. Let us fix $m \geq 1$. Then (4.51) is immediate for $n = m$, and for $n = m+1$ it follows readily from the fact that $\mathring{\omega}_{\mathsf{x}_m}(t) \leq \omega_{\mathsf{x}_m}(t)$, $t \geq \tau_m$.

For the remaining cases, let $n \geq m+1$, and suppose, inductively, that $\mathring{\tau}^m_{n-m} \leq \tau_n$; there are two possibilities for $\mathring{\tau}^m_{n+1-m}$: either $\mathring{\tau}^m_{n+1-m} \leq \tau_n$, in which case, clearly, $\mathring{\tau}^m_{n+1-m} \leq \tau_{n+1}$, or $\mathring{\tau}^m_{n+1-m} > \tau_n$; in this latter case, τ_{n+1} (resp., $\mathring{\tau}^m_{n+1-m}$) will correspond to the earliest Poisson point (of $\mathcal{M}$) in $\mathcal{Q}'_n := [c_{\mathsf{x}_n(d)}, c_{\mathsf{x}_n(d)} + \varphi(\omega_{\mathsf{x}_n}(r))_{r\geq\tau_n}$ (resp., $\mathring{\mathcal{Q}}_n := [c_{\mathsf{x}_n(d)}, c_{\mathsf{x}_n(d)} + \varphi(\mathring{\omega}^m_{\mathsf{x}_n}(r))_{r\geq\tau_n}$). By (4.46) and the monotonicity of φ, we have that $\mathring{\mathcal{Q}}_n \supset \mathcal{Q}'_n$, and it follows that $\mathring{\tau}^m_{n+1-m} \leq \tau_{n+1}$.

Finally, one readily checks from (4.47) that $\theta \geq \mathbb{E}_0(\tau_1)$; clearly the latter expectation is strictly positive, and the argument is complete. □

4.2 Proof of the Law of Large Numbers for X under $\mathbf{P_0}$

We may now prove Theorem 3.1. For $t \in \mathbb{R}_+$, let $\mathsf{N}_t = \inf\{n \geq 0 : \tau_n < t\}$. It follows readily from Proposition 4.9 that

$$\frac{\mathsf{N}_t}{t} \to \frac{1}{\theta}\ \mathbf{P_0}\text{-a.s. as } t \to \infty. \tag{4.52}$$

It follows from (4.52) and the Strong Law of Large Numbers for $(\mathbf{x}_n)$ that

$$\frac{X(t)}{t} = \frac{\mathsf{x}_{\mathsf{N}_t}}{t} = \frac{\mathsf{x}_{\mathsf{N}_t}}{\mathsf{N}_t} \times \frac{\mathsf{N}_t}{t} \to \frac{\mathbf{E}(\xi_1)}{\theta}\ \mathbf{P_0}\text{-a.s as } t \to \infty. \tag{4.53}$$

4.3 Proof of the Central Limit Theorem for X under $\mathbf{P_0}$

We now prove Theorem 3.2. Let $\gamma = 1/\theta$, and write

$$\frac{X(t)}{\sqrt{\gamma t}} = \frac{\mathsf{x}_{\mathsf{N}_t} - \mathsf{x}_{\lfloor \gamma t\rfloor}}{\sqrt{\gamma t}} + \frac{\mathsf{x}_{\lfloor \gamma t\rfloor}}{\sqrt{\gamma t}}. \tag{4.54}$$

By the Central Limit Theorem obeyed by (x_n), we have that, under $\mathbf{P}$, as $t \to \infty$,

$$\frac{\mathsf{x}_{\lfloor \gamma t\rfloor}}{\sqrt{\lfloor \gamma t\rfloor}} \Rightarrow N_d(\mathbf{0}, \Sigma). \tag{4.55}$$

We now claim that the first term on the right hand side of (4.54) (after multiplication by γ) vanishes in probability as $t \to \infty$ under $\mathbf{P_0}$. Indeed, let us write $\xi_k = (\xi_{k,1}, \dots, \xi_{k,d})$, $k \in \mathbb{N}$. Given $\epsilon > 0$, let us set $\delta = \epsilon^3$; we have that

$$\mathbf{P_0}\left(\left\|\frac{\mathsf{x}_{\mathsf{N}_t} - \mathsf{x}_{\lfloor \gamma t\rfloor}}{\sqrt{t}}\right\| > \epsilon\right) \leq \mathbf{P_0}\left(\left\|\mathsf{x}_{\mathsf{N}_t} - \mathsf{x}_{\lfloor \gamma t\rfloor}\right\| > \epsilon\sqrt{t},\, |\mathsf{N}_t - \gamma t| < \delta t\right) + \mathbf{P_0}\left(|\mathsf{N}_t - \gamma t| \geq \delta t\right). \tag{4.56}$$

By (4.52), it then suffices to consider the first term on the right hand side of (4.56), which may be readily seen to be bounded above by

$$\sum_{i=1}^d \left\{\mathbf{P_0}\left(\max_{0\leq\ell\leq\delta t}\left|\sum_{k=\gamma t-\ell}^{\gamma t}\xi_{k,i}\right| > \epsilon\sqrt{t}\right) + \mathbf{P_0}\left(\max_{0\leq\ell\leq\delta t}\left|\sum_{k=\gamma t}^{\gamma t+\ell}\xi_{k,i}\right| > \epsilon\sqrt{t}\right)\right\} \leq 3\,\mathrm{Tr}(\Sigma)\,\epsilon, \tag{4.57}$$

where we have used Kolmogorov's Maximal Inequality in the latter passage; the claim follows since ϵ is arbitrary. And the CLT follows readily from the claim and (4.55).

Remark 4.10. A natural extension of our arguments for the above CLT to the not mean zero case would start by writing

$$\frac{X(t) - \mathbf{E_0}(X_t)}{\sqrt{t}} = \frac{\bar{\mathsf{x}}_{\mathsf{N}_t}}{\sqrt{t}} + \mathbf{E}(\xi_1)\frac{N_t - \mathbf{E_0}(N_t)}{\sqrt{t}}. \tag{4.58}$$

A CLT for the first term on the right hand side of (4.58) follows from Theorem 3.2, but we have to treat the second term, for which a CLT presumably holds as well, and its correlation with the first term, which would require other methods than the ones developped here. A regeneration scheme might work in this case (possibly dispensing with the domination requirements of our argument for the mean zero case, in particular φ being decreasing).

Another extension is to prove a functional CLT; for the mean zero case treated above, that, we believe, requires no new ideas, and thus we refrained from presenting a standard argument to that effect (having already gone through standard steps in our justifications for the LLN and CLT for X).

Remark 4.11. It is quite clear from our arguments that all that we needed to have from our conditions on $\mathbf{p}, \mathbf{q}$ is the validity of both conditions in (3.3), and thus we may possibly relax (3.2) to some extent, and certainly other conditions imposed on $\mathbf{p}, \mathbf{q}$ (in the paragraph of (2.1)), with the same approach, but we have opted for plainness, within a measure of generality.

Remark 4.12. For the proof of Lemma 4.5, a mainstay of our approach, we relied on the reversibility of the birth-and-death process, the positivity of d_n, and the increasing monotonicity of ψ; see the upshot of the paragraph of (4.19). It is natural to think of extending the argument for other reversible ergodic Markov processes on $\mathbb{N}$; one issue for longer range cases is the positivity of d_n; there should be examples of long range reversible ergodic Markov processes on $\mathbb{N}$ where positivity of d_n may be ascertained by a coupling argument, and we believe we have worked out such an example, but it looked too specific to warrant a more general formulation of our results (and the extra work involved in such an attempt), so again we felt content in presenting our approach in the present setting.

Remark 4.13. Going back to the construction leading to Lemma 4.6, for $0 \leq m \leq n$, let $\breve{\mathsf{L}}_{m,n}$ denote the time ω^m takes to give $n-m$ jumps. Then it follows from the properties of ω, ω^m, $m \geq 0$, as discussed in the paragraphs preceding the statement of Lemma 4.6, that $\{\breve{\mathsf{L}}_{m,n},\, 0 \leq m \leq n < \infty\}$ is a *subadditive* triangular array, and a Law of Large Numbers for τ_n under $\mathbf{P}_{\hat{\nu}}$ would follow, once we establish the ergodicity of $\{\breve{\mathsf{L}}_{nk,(n+1)k}, n \in \mathbb{N}\}$, other conditions for the application of the Subadditive Ergodic Theorem being readily seen to hold. This would require a more substantial argument than for the corresponding result for $\{\mathsf{L}_{nk,(n+1)k}, n \in \mathbb{N}\}$, made briefly above (in the second paragraph below (4.50)), since independence is lost. Perhaps a promising strategy would be one similar to that which we undertake in next section, to the same effect; see Remark 5.2. For this, if for nothing else, we refrained from pursuing this specific point in this paper.

Remark 4.14. The assumption of positivity on φ, made at the beginning of Subsection 4.1, was for simplicity of exposition only. The changes required in the argument if we allow for $\varphi(n) = 0$ for $n \geq n_0$ for some arbitrary $n_0 \geq 1$ have only a notational impact — in this case, we note, the auxiliary process Y introduced in the proof of Lemma 4.5 is a birth-and-death process on $\{0, \ldots, n_0 - 1\}$, thus making much of the steps of the reasoning actually simpler (since we will be dealing with finite sums rather than series).

5 CLT under other initial conditions

We present the proof of Theorem 3.3 in two arguments, spelling out the broad descriptions in Subsection 3.2, in two subsequent subsections, one for $d \leq 2$, and another one for $d \geq 3$. We first state and prove a lemma which enters both arguments, concerning successive coalescence of coupled versions of the environments, one started from $\mathbf{0}$, and the other from $\hat{\mu}_0$, over certain times related to displacements of (x_n).

Consider two coalescing versions of the environment, $\mathring{\omega}$ and ω, the former one starting from $\mathbf{0}$, and the latter starting from $\hat{\mu}_0$ as above, such that $\mathring{\omega}_{\mathbf{x}}(t) \leq \omega_{\mathbf{x}}(t)$ for all $\mathbf{x}$ and t, and for $\mathbf{x} \in \mathbb{Z}^d$, let $\mathsf{T}_\mathbf{x}$ denote the coalescence time of $\mathring{\omega}_\mathbf{x}$ and $\omega_\mathbf{x}$, i.e.,

$$\mathsf{T}_\mathbf{x} = \inf\{s > 0 : \mathring{\omega}_\mathbf{x}(s) = \omega_\mathbf{x}(s)\}. \tag{5.1}$$

Now let $\mathring{X}$ and X be versions of the random walks on $\mathbb{Z}^d$ in the respective environments, both starting from $\mathbf{0}(\in \mathbb{Z}^d)$. Let us suppose, for simplicity, that they have the same embedded chain (x_n). For $n \in \mathbb{N}$, let $\mathcal{B}_n$ denote $\{-2^n, -2^n+1, \dots, 2^n-1, 2^n\}^d$, let $\mathring{\mathcal{H}}_n$ (resp., $\mathcal{H}_n$) denote the hitting time of $\mathbb{Z}^d \setminus \mathcal{B}_n$ by $\mathring{X}$ (resp., X), and consider the event $\mathring{\mathsf{A}}_n$ (resp., A_n) that $\mathsf{T}_\mathbf{x} \le \mathring{\mathcal{H}}_n$ (resp., $\mathsf{T}_\mathbf{x} \le \mathcal{H}_n$) for all $\mathbf{x} \in \mathcal{B}_{n+1}$. Let also h_n denote the hitting time of $\mathbb{Z}^d \setminus \mathcal{B}_n$ by (x_n).

Lemma 5.1.

$$\mathbf{P}_\mathbf{0}(\mathring{\mathsf{A}}_n^c \text{ infinitely often}) = \mathbf{P}_{\hat{\mu}_0}(\mathsf{A}_n^c \text{ infinitely often}) = 0. \tag{5.2}$$

Proof. Under our conditions, the argument is quite elementary, and for this reason we will be rather concise. Let us first point out that both $\mathring{\mathcal{H}}_n$ and $\mathcal{H}_n$ are readily seen to be bounded from below stochastically by $\bar{\mathcal{H}}_n := \sum_{i=1}^{\mathsf{h}_n} \mathcal{E}_i$, where $\mathcal{E}_1, \mathcal{E}_2, \dots$ are iid standard exponential random variables, which are independent of h_n and of $\mathring{\omega}$ and ω.

It follows readily from Kolmogorov's Maximal Inequality that for all $n \in \mathbb{N}$

$$\mathbf{P}(\mathsf{h}_n \le 2^n) = \mathbf{P}\Big(\max_{1 \le i \le 2^n} \|\mathsf{x}_i\| > 2^n\Big) \le \text{const } 2^{-n}, \tag{5.3}$$

and by the above mentioned domination and elementary well-known large deviation estimates, we find that

$$\mathbf{P}_\mathbf{0}(\mathring{\mathcal{H}}_n \le 2^{n-1}) \vee \mathbf{P}_{\hat{\mu}_0}(\mathcal{H}_n \le 2^{n-1}) \le \mathbf{P}(\bar{\mathcal{H}}_n \le 2^{n-1}) \le \text{const } 2^{-n}. \tag{5.4}$$

We henceforth treat only the first probability in (5.2); the argument for the second one is identical.

The probability of the event that $\mathring{\mathcal{H}}_n \le 2^{n-1}$ and $\mathsf{T}_\mathbf{x} > \mathcal{H}_n$ for some $x \in \mathcal{B}_{n+1}$ is bounded above by

$$\text{const } 2^{dn}\, \mathbb{P}(\mathsf{T}_\mathbf{0} > 2^{n-1}). \tag{5.5}$$

It may now be readily checked that $\mathsf{T}_\mathbf{0}$ is stochastically dominated by the hitting time of the origin by a simple random walk on $\mathbb{Z}$ in continuous time with homogeneous jump rates equal to 1, with probability p to jump to the left, initially distributed as $\bar{\mu}$. Thus, given $\delta > 0$

$$\mathbb{P}(\mathsf{T}_\mathbf{0} > 2^{n-1}) \le \bar{\mu}([\delta 2^n, \infty)) + \mathbb{P}\Big(\sum_{i=1}^{\delta 2^n} H_i > 2^{n-1}\Big), \tag{5.6}$$

where H_1, H_2 are iid random variables distributed as the hitting time of the origin by a simple asymmetric random walk on $\mathbb{Z}$ in continuous time with homogeneous jump rates equal to 1, with probability p to jump to the left, starting from 1. H_1 is well known to have a positive exponential moment; it follows from elementary large deviation estimates that we may choose $\delta > 0$ such that the latter term on the right hand side of (5.6) is bounded above by const e^{-b2^n} for some constant $b > 0$ and all n. Using this bound, and substituting (3.6) in (5.6), we find that

$$\mathbb{P}(\mathsf{T}_\mathbf{0} > 2^{n-1}) \le \text{const } e^{-b' 2^n} \tag{5.7}$$

for some $b' > 0$ and all n, and (5.2) upon a suitable use of the Borel-Cantelli Lemma. □

Remark 5.2. As vaguely mentioned in Remark 4.13 at the end of the previous section, a seemingly promising strategy for establishing the ergodicity of $\{\check{\mathsf{L}}_{nk,(n+1)k}, n \in \mathbb{N}\}$ would be to approximate an event of $\mathcal{F}^+_{m'}$, the σ-field generated by $\{\check{\mathsf{L}}_{nk,(n+1)k}, n \ge m'\}$, by one generated by a version of an environment starting from $\mathbf{0}$ at time $\breve{\mathsf{L}}_{0,mk}$, coupled to the original environment in a coalescing way as above, with suitable couplings of the jump times and destinations, with fixed $m \in \mathbb{N}_*$ and $m' \gg m$. Ergodicity would follow by the independence of the latter σ-field and $\mathcal{F}_m^-$, the σ-field generated by $\{\breve{\mathsf{L}}_{(n-1)k,nk}, 1 \leq n \leq m\}$. We have not attempeted to work this idea out in detail; if we did, it looks as though we might face the same issues arising in the extension of the CLT, as treated in the present section, thus possibly not getting a better result than Theorem 3.3.

5.1 Proof of Theorem 3.3 for $d \leq 2$

We start by fixing the coalescing environments $\mathring{\omega}$ and ω, as above, and considering two independent random walks, denoted $\mathring{X}$ and X' in the respective environments $\mathring{\omega}$ and ω. The jump times of $\mathring{X}$ and X' are obtained from $\mathring{\mathcal{M}}$ and $\mathcal{M}'$, as in the original construction of our model, where $\mathring{\mathcal{M}}$ and $\mathcal{M}'$ are independent versions of $\mathcal{M}$.

For the jump destinations of $\mathring{X}$ and X', we will change things a little, and consider independent families $\mathring{\xi} = \{\mathring{\xi}_{\mathsf{z}}, \mathsf{z} \in \mathring{\mathcal{M}}\}$ and $\xi' = \{\xi'_{\mathsf{z}}, \mathsf{z} \in \mathcal{M}'\}$ of independent versions of ξ_1. The jump destination of $\mathring{X}$ at the time corresponding to an a.s. unique point z of $\mathring{\mathcal{M}}$ is then given by $\mathring{\xi}_{\mathsf{z}}$, and correspondingly for X'.

Let $\mathsf{D} = \big(\mathsf{D}(s) := \mathring{X}(s) - X'(s), s \geq 0\big)$, which is clearly a continuous-time jump process, and consider the embedded chain of D, denoted $\mathsf{d} = (\mathsf{d}_n)_{n\in\mathbb{N}}$. We claim that under the conditions of Theorem 3.3 for $d \leq 2$, d is recurrent, that is, it a.s. returns to the origin infinitely often.

Before justifying the claim, let us indicate how to reach the conclusion of the proof of Theorem 3.3 for $d \leq 2$ from this. We consider the sequence of return times of D to the origin, i.e., $\tilde{\sigma}_0 = 0$, and for $n \geq 1$,

$$\tilde{\sigma}_n = \inf\big\{s > \tilde{\sigma}_{n-1} : \mathsf{D}(s) = 0 \text{ and } \mathsf{D}(s-) \neq 0\big\}. \tag{5.8}$$

It may be readily checked, in particular using the recurrence claim, that this is an infinite sequence of a.s. finite stopping times given $\mathring{\omega}, \omega$, such that $\tilde{\sigma}_n \to \infty$ as $n \to \infty$.

Then, for each $n \in \mathbb{N}$, we define a version of X', denoted X_n, coupled to $\mathring{X}$ and X' as follows: $X_n(s) = X'(s)$ for $s \leq \tilde{\sigma}_n$, and for $s > \tilde{\sigma}_n$, the jump times and destinations of X_n are defined from ω as before, except that we replace the Poisson marks of $\mathcal{M}'$ in the half space above $\tilde{\sigma}_n$ by the corresponding marks of $\mathring{\mathcal{M}}$, and we use the corresponding jump destinations of $\mathring{\xi}$. It may be readily checked that X_n is a version of X', and that starting at $\tilde{\sigma}_n$, and as long as X_n and $\mathring{X}$ see the same respective environments, they remain together.

It then follows from Lemma 5.1 that there exists a finite random time N such that $\mathring{X}(t)$ and $X'(t)$ each see only coupled environments for $t > N$, and thus so do $\mathring{X}(t)$ and $X_n(t)$ for $t > \tilde{\sigma}_n > N$. It then follows from the considerations above that given $\mathring{\omega}, \omega$, $n \in \mathbb{N}$ and $x \in \mathbb{R}$

$$\begin{aligned} \Big|P\Big(\frac{X'(t)}{\sqrt{t}} < x\Big) - P\Big(\frac{\mathring{X}(t)}{\sqrt{t}} < x\Big)\Big| &= \Big|P\Big(\frac{X_n(t)}{\sqrt{t}} < x\Big) - P\Big(\frac{\mathring{X}(t)}{\sqrt{t}} < x\Big)\Big| \\ \leq P\big(\{t > \tilde{\sigma}_n > N\}^c\big) &\leq P(\tilde{\sigma}_n \geq t) + P(N \geq \tilde{\sigma}_n), \end{aligned} \tag{5.9}$$

and it follows that the limsup as $t \to \infty$ of the left hand side of (5.9) is bounded above by the latter probability in the same expression. The result (for X') follows since (it does for $\mathring{X}$, by Theorem 3.2, and) n is arbitrary.

In order to check the recurrence claim, notice that if π, the distribution of ξ_1, is symmetric, then d is readily seen to be a discrete time random walk on $\mathbb{Z}^d$ with jump distribution given by π, and the claim follows from well-known facts about mean zero random walks with finite second moments for $d \leq 2$. This completes the argument for Theorem 3.3 for $d = 2$.

For $d=1$ and asymmetric π, d is no longer Markovian, but we may resort to Theorem 1 of [11] to justify the claim as follows. Let us fix a realization of $\mathring{\omega}$, ω, $\mathring{\mathcal{M}}$ and $\mathcal{M}'$ (such that no two marks in $\mathring{\mathcal{M}}\cup\mathcal{M}'$ have the same time coordinate, which is of course an event of full probability). Let us now dress d up as a *controlled random walk (crw)* (conditioned on $\mathring{\omega}$, ω, $\mathring{\mathcal{M}}$ and $\mathcal{M}'$), in the language of [11]; see paragraph before the statement of Theorem 1 therein.

There are two kinds of jump distributions for d ($p=2$, in the notation of [11]): F_1 denotes the distribution of ξ_1, and F_2 denotes the distribution of $-\xi_1$. In order to conform to the set up of [11], we will also introduce two independent families of (jump) iid random variables (which will in the end not be used), namely, $\breve{\xi}=\{\breve{\xi}_{\mathsf{z}},\,\mathsf{z}\in\mathring{\mathcal{M}}\}$ and $\xi''=\{\xi''_{\mathsf{z}},\,\mathsf{z}\in\mathcal{M}'\}$, independent of, but having the same marginal distributions as, $\mathring{\xi}$ (and ξ').

Let us see how the choice between each of the two distributions is made at each step of d. This is done using the indicator functions ψ_n, introduced and termed in [11] the *choice of game at time* $n\geq 1$, inductively, as follows.

Given $\mathring{\omega}$, ω, $\mathring{\mathcal{M}}$ and $\mathcal{M}'$, let ζ_1 denote the earliest point of $\mathring{\mathcal{N}}_{\mathbf{0}}\cup\mathcal{N}'_{\mathbf{0}}$, where $\mathring{\mathcal{N}}_{\mathbf{x}}$, $\mathcal{N}'_{\mathbf{x}}$, $\mathbf{x}\in\mathbb{Z}^d$, are defined from $(\mathring{\omega},\mathring{\mathcal{M}})$ and $(\omega,\mathcal{M}')$, respectively, as $\mathcal{N}_{\mathbf{x}}$ was defined from $(\omega,\mathcal{M})$ at the beginning of Section 2, and let η_1 denote the time coordinate of ζ_1, and set $\psi_1=1+\mathbb{1}\{\zeta_1\in\mathcal{N}'_{\mathbf{0}}\}$, and

$$X_1^i:=\begin{cases}\mathring{\xi}_{\zeta_1}, & \text{if } \psi_1=1 \text{ and } i=1,\\ -\breve{\xi}_{\zeta_1}, & \text{if } \psi_1=1 \text{ and } i=2,\\ \xi''_{\zeta_1}, & \text{if } \psi_1=2 \text{ and } i=1,\\ -\xi'_{\zeta_1}, & \text{if } \psi_1=2 \text{ and } i=2.\end{cases}\tag{5.10}$$

Notice that X_1^1 and X_1^2 are independent and distributed as F_1 and F_2, respectively, and that a.s.

$$\mathring{X}(\eta_1) = X_1^{\psi_1}\mathbb{1}\{\psi_1=1\}+\mathring{X}(0)\mathbb{1}\{\psi_1=2\},\tag{5.11}$$
$$X'(\eta_1) = X_1^{\psi_1}\mathbb{1}\{\psi_1=2\}+X'(0)\mathbb{1}\{\psi_1=1\}.\tag{5.12}$$

For $n\geq 2$, having defined ζ_j, η_j, ψ_j, X_j^i, $j<n$, $i=1,2$, let ζ_n denote the earliest point of $\mathring{\mathcal{N}}_{\mathring{X}(\eta_{n-1})}(\eta_{n-1})\cup\mathcal{N}'_{X'(\eta_{n-1})}(\eta_{n-1})$, where for $\mathbf{x}\in\mathbb{Z}^d$ and $t\geq 0$, $\mathring{\mathcal{N}}_{\mathbf{x}}(t)$, $\mathcal{N}'_{\mathbf{x}}(t)$ denote the points of $\mathring{\mathcal{N}}_{\mathbf{x}}$, $\mathcal{N}'_{\mathbf{x}}$ with time coordinates above t, respectively.

Let now η_n denote the time coordinate of ζ_n, and set $\psi_n=1+\mathbb{1}\{\zeta_n\in\mathcal{N}'_{X'(\eta_{n-1})}(\eta_{n-1})\}$, and

$$X_n^i:=\begin{cases}\mathring{\xi}_{\zeta_n}, & \text{if } \psi_1=1 \text{ and } i=1,\\ -\breve{\xi}_{\zeta_n}, & \text{if } \psi_1=1 \text{ and } i=2,\\ \xi''_{\zeta_n}, & \text{if } \psi_1=2 \text{ and } i=1,\\ -\xi'_{\zeta_n}, & \text{if } \psi_1=2 \text{ and } i=2.\end{cases}\tag{5.13}$$

Notice that $\{X_j^i;\,1\leq j\leq n,\,i=1,2\}$ are independent and X_j^1 and X_j^2 are distributed as F_1 and F_2, respectively, for all j. Moreover, a.s.

$$\mathring{X}(\eta_n) = X_n^{\psi_n}\mathbb{1}\{\psi_n=1\}+\mathring{X}(\eta_{n-1})\mathbb{1}\{\psi_n=2\},\tag{5.14}$$
$$X'(\eta_n) = X_n^{\psi_n}\mathbb{1}\{\psi_n=2\}+X'(\eta_{n-1})\mathbb{1}\{\psi_n=1\}.\tag{5.15}$$

We then have that for $n\geq 1$, $\mathsf{d}_n=\sum_{j=1}^n X_j^{\psi_j}$. One may readily check that (given $\mathring{\omega}$, ω, $\mathring{\mathcal{M}}$ and $\mathcal{M}'$) d is a crw in the setup of Theorem 1 of [11], an application of which readily yields the claim, and the proof of Theorem 3.3 for $d\leq 2$ is complete.

Remark 5.3. We did not find an extension of the above mentioned theorem of [11] to $d=2$, or any other way to show recurrence of (d_n) for general asymmetric π within the conditions of Theorem 3.3.

5.2 Proof of Theorem 3.3 for $d \geq 3$

We now cannot expect to have recurrence of d, quite on the contrary, but transience suggests that we may have enough of a regeneration scheme, and we pursue precisely this idea, in order to implement which, we resort to *cut times* of the trajectory of (x_n), to ensure the existence of infinitely many of which, we need to restrict to boundedly supported π's.

We will be rather sketchy in this subsection, since the ideas are all quite simple and/or have appeared above in a similar guise.

We now discuss a key concept and ingredient of our argument: cut times for $\mathsf{x} = (\mathsf{x}_n)$. First, some notation: for $i, j \in \mathbb{N}$, $i \leq j$, let $\mathsf{x}[i,j] := \bigcup_{k=i}^{j} \{\mathsf{x}_k\}$, and $\mathsf{x}[i,\infty) := \bigcup_{l=1}^{\infty} \mathsf{x}[i,l]$, and set

$$\mathsf{K}_1 = \inf \{n \in \mathbb{N} : \mathsf{x}[0,n] \cap \mathsf{x}[n+1,\infty) = \emptyset\}, \tag{5.16}$$

and, recursively, for $\ell \geq 2$,

$$\mathsf{K}_\ell := \inf \{n > \mathsf{K}_{\ell-1} : \mathsf{x}[0,n] \cap \mathsf{x}[n+1,\infty) = \emptyset\}. \tag{5.17}$$

$(\mathsf{K}_\ell)_{\ell \in \mathbb{N}_*}$ is a sequence of *cut times* for (x_n); under our conditions, it is ensured to be an a.s. well defined infinite sequence of finite entries, according to Theorem 1.2 of [15].

We will have three versions of the environment coupled in a coalescent way, as above, with different initial conditions: $\mathring{\omega}$, starting from $\mathbf{0}$; ω, starting from $\hat{\mu}_0$; and $\tilde{\omega}$, starting from $\hat{\nu}$; in particular, we have that $\mathring{\omega}_{\mathbf{x}}(t) \leq \omega_{\mathbf{x}}(t), \tilde{\omega}_{\mathbf{x}}(t)$ for all $\mathbf{x} \in \mathbb{Z}^d$ and $t \geq 0$. We may suppose that the initial conditions of ω and $\tilde{\omega}$ are independent.

We now consider several coupled versions of versions of our random walk, starting with two: X, in the environment ω, as in the statement of Theorem 3.3; and $\mathring{X}$, in the environment $\mathring{\omega}$. X and $\mathring{X}$ are constructed from the same x and V, following the alternative construction of Subsection 2.1. Let ς_ℓ and $\mathring{\varsigma}_\ell$ be the time X and $\mathring{X}$ take to give K_ℓ jumps, respectively. It may be readily checked, similarly as in Section 3.1 — see (4.29), (4.51) —, from the environmental monotonicity pointed to in the above paragraph and the present construction of X and $\mathring{X}$, that $\mathring{\varsigma}_\ell \leq \varsigma_\ell$ for all $\ell \in \mathbb{N}_*$.

Finally, for each $\ell \in \mathbb{N}_*$, we consider three modifications of $\mathring{X}$ and X, namely, $\mathring{X}_\ell$, X_ℓ and X'_ℓ, defined as follows:

$$\mathring{X}_\ell(t) = \begin{cases} \mathring{X}(t), & \text{for } t \leq \mathring{\varsigma}_\ell, \\ \text{evolves in the environment } \tilde{\omega}, & \text{for } t > \mathring{\varsigma}_\ell; \end{cases} \tag{5.18}$$

$$X_\ell(t) = \begin{cases} X(t), & \text{for } t \leq \varsigma_\ell, \\ \text{evolves in the environment } \tilde{\omega}, & \text{for } t > \varsigma_\ell; \end{cases} \tag{5.19}$$

$$X'_\ell(t) = \begin{cases} X(t), & \text{for } t \leq \varsigma_\ell, \\ \text{evolves in the environment } \tilde{\omega}(\cdot - \varsigma_\ell + \mathring{\varsigma}_\ell), & \text{for } t > \varsigma_\ell. \end{cases} \tag{5.20}$$

Let U denote the first time after which $\mathring{X}$ and X see the same environments $\mathring{\omega}, \omega, \tilde{\omega}$ (from where they stand at each subsequent time). Lemma 5.1 ensures that U is a.s. finite. Let us consider the event $A_{\ell,t} := \{t > \varsigma_\ell > U\}$. It readily follows that in $A_{\ell,t}$

$$\mathring{X}(t) = \mathring{X}_\ell(t) = X'_\ell(t + \varsigma_\ell - \mathring{\varsigma}_\ell) \text{ and } X(t) = X_\ell(t). \tag{5.21}$$

Given $\mathring{\omega}, \omega, \tilde{\omega}$, let $P^{\mathring{\omega},\omega,\tilde{\omega}}$ denote the probability measure underlying our coupled random walks. Since $\hat{\nu}$ is invariant for the environmental BD processes, it follows readily from our construction that $P^{\mathring{\omega},\omega,\tilde{\omega}}(X_\ell \in \cdot)$ and $P^{\mathring{\omega},\omega,\tilde{\omega}}(X'_\ell \in \cdot)$ have the same distribution (as random probability measures).

For $R=(-\infty,r_1)\times\cdots\times(-\infty,r_d)$ a semi-infinite open hyperrectangle of $\mathbb{R}^d$, we have that

$$\begin{aligned}&\big|P^{\mathring{\omega},\omega,\tilde{\omega}}\big(X(t)\in R\sqrt{\gamma t}\big)-P^{\mathring{\omega},\omega,\tilde{\omega}}\big(X_\ell(t)\in R\sqrt{\gamma t}\big)\big|\\ &\leq P^{\mathring{\omega},\omega,\tilde{\omega}}(A^c_{\ell,t})\leq P^{\mathring{\omega},\omega,\tilde{\omega}}(\varsigma_\ell\geq t)+P^{\mathring{\omega},\omega,\tilde{\omega}}(U\geq\varsigma_\ell)\end{aligned}\tag{5.22}$$

— as before, $\gamma=1/\mu$; see statement of Theorem 3.3 —, and it follows that

$$\limsup_{\ell\to\infty}\limsup_{t\to\infty}\big|P^{\mathring{\omega},\omega,\tilde{\omega}}\big(X(t)\in R\sqrt{\gamma t}\big)-P^{\mathring{\omega},\omega,\tilde{\omega}}\big(X_\ell(t)\in R\sqrt{\gamma t}\big)\big|=0\tag{5.23}$$

for a.e. $\mathring{\omega},\omega,\tilde{\omega}$.

Similarly, we find that for a.e. $\mathring{\omega},\omega,\tilde{\omega}$,

$$\limsup_{\ell\to\infty}\limsup_{t\to\infty}\big|P^{\mathring{\omega},\omega,\tilde{\omega}}\big(X'_\ell(t)\in R\sqrt{\gamma t}\big)-P^{\mathring{\omega},\omega,\tilde{\omega}}\big(\mathring{X}((t-\delta_\ell)^+)\in R\sqrt{\gamma t}\big)\big|=0,\tag{5.24}$$

where $\delta_\ell=\varsigma_\ell-\mathring{\varsigma}_\ell$.

Now letting $B_{\ell,t,\epsilon}$ denote the event $\big\{\|\mathring{X}((t-\delta_\ell)^+)-\mathring{X}(t)\|\leq\epsilon\sqrt{\gamma t}\big\}$, where $\epsilon>0$, we have that

$$\begin{aligned}&\big|P^{\mathring{\omega},\omega,\tilde{\omega}}\big(\mathring{X}((t-\delta_\ell)^+)\in R\sqrt{\gamma t}\big)-P^{\mathring{\omega},\omega,\tilde{\omega}}\big(\mathring{X}(t)\in R\sqrt{\gamma t}\big)\big|\\ &\leq P^{\mathring{\omega},\omega,\tilde{\omega}}\big(\mathring{X}(t)\in(R^+_\epsilon\setminus R^-_\epsilon)\sqrt{\gamma t}\big)+P^{\mathring{\omega},\omega,\tilde{\omega}}\big(B^c_{\ell,t,\epsilon}\big),\end{aligned}\tag{5.25}$$

where $R^\pm_\epsilon=(-\infty,r_1\pm\epsilon)\times\cdots\times(-\infty,r_d\pm\epsilon)$.

We now claim that for all $\ell\in\mathbb{N}_*$ and $\epsilon>0$

$$\limsup_{t\to\infty}P^{\mathring{\omega},\omega,\tilde{\omega}}\big(B^c_{\ell,t,\epsilon}\big)=0\tag{5.26}$$

for a.e. $\mathring{\omega},\omega,\tilde{\omega}$.

It then follows from (5.24), (5.25), (5.26) and Theorem 3.2 that for $\epsilon>0$

$$\limsup_{\ell\to\infty}\limsup_{t\to\infty}\big|P^{\mathring{\omega},\omega,\tilde{\omega}}\big(X'_\ell(t)\in R\sqrt{\gamma t}\big)-\Phi(R)\big|\leq\Phi(R^+_\epsilon\setminus R^-_\epsilon)\tag{5.27}$$

for a.e. $\mathring{\omega},\omega,\tilde{\omega}$, where Φ is the d-dimensional centered Gaussian probability measure with covariance matrix Σ. Since ϵ is arbitrary, and the left hand side of (5.27) does not depend on ϵ, we find that it vanishes for a.e. $\mathring{\omega},\omega,\tilde{\omega}$.

From the remark in the paragraph right below (5.21), we have that $P^{\mathring{\omega},\omega,\tilde{\omega}}(X_\ell(t)\in R\sqrt{\gamma t})$ is distributed as $P^{\mathring{\omega},\omega,\tilde{\omega}}\big(X'_\ell(t)\in R\sqrt{\gamma t}\big)$; it follows that

$$\limsup_{\ell\to\infty}\limsup_{t\to\infty}\big|P^{\mathring{\omega},\omega,\tilde{\omega}}\big(X_\ell(t)\in R\sqrt{\gamma t}\big)-\Phi(R)\big|=0\tag{5.28}$$

for a.e. $\mathring{\omega},\omega,\tilde{\omega}$, and it follows from (5.23) that

$$\limsup_{t\to\infty}\big|P^{\mathring{\omega},\omega,\tilde{\omega}}\big(X(t)\in R\sqrt{\gamma t}\big)-\Phi(R)\big|=0\tag{5.29}$$

for a.e. $\mathring{\omega},\omega,\tilde{\omega}$, which is the claim of Theorem 3.3.

In order to complete the proof, it remains to establish (5.26). For that, we first note that

$$\|\mathring{X}((t-\delta_\ell)^+)-\mathring{X}(t)\|=\Big\|\sum_{i=\mathsf{N}_{(t-\delta_\ell)^+}}^{\mathsf{N}_t}\xi_i\Big\|\leq K\big(\mathsf{N}_t-\mathsf{N}_{(t-\delta_\ell)^+}\big),\tag{5.30}$$

where K is the radius of the support of π, and N_t, we recall from Subsection 4.2, counts the jumps of $\mathring{X}$ up to time t. Thus, the probability on the left hand side of (5.26) is bounded above by

$$P^{\mathring{\omega},\omega,\tilde{\omega}}\big(\mathsf{N}_t-\mathsf{N}_{(t-u)^+}>\epsilon K^{-1}t\big)+P^{\mathring{\omega},\omega,\tilde{\omega}}\big(\delta_\ell>u\big),\tag{5.31}$$

where $u > 0$ is arbitrary.

One may readily check from our conditions on φ that $\mathsf{N}_t - \mathsf{N}_{(t-u)^+}$ is stochastically dominated by a Poisson distribution of mean u for each t, and it follows that the first term in (5.31) vanishes as $t \to \infty$ for a.e. $\mathring{\omega}, \omega, \tilde{\omega}$; (5.26) then follows since u is arbitrary and δ_ℓ is finite a.s. □

Remark 5.4. The assumptions of spatial homogeneity in the tail of $\mu_{\mathbf{x}}$ (see (3.6)), as well as of homogeneity of the rates of the underlying birth-and-death process may most possibly be relaxed in the above approach, within a similar reasoning for a CLT as above.

It seems quite clear that we could make things work with (3.6) (for $\mu_{\mathbf{x}}$) replaced by an exponential tail with the rate not vanishing too fast as $\|\mathbf{x}\|$ diverges.

Obtaining a domination similar/ly to that/as in the paragraph of (5.7) would (be probably be a bit trickier and) require that p_n does not approach $1/2$ too fast as n diverges (possibly in a more restrictive way than assumed in Subsection 3.1) —- notice that in this case the second term on the right hand side of (5.6) would involve a sum of *not* identically distributed (independent) random variables possibly with tails heavier than exponential.

A Equivalence of (3.2) and (3.3)

The formulas we will present below may be found in the literature, either explicitly or from other explicit formulas [16]. For this reason, but also in an attempt of self-containment, we will deduce them, rather briefly and sketchily, trusting the reader to be readily able to fill details in (or go to the literature).

For $n \in \mathbb{N}_*$, let $\mathcal{T}_n$ and $\mathcal{S}_n$ denote $\mathbb{E}_n(T_{n-1})$ and $\mathbb{E}_n(T_{n-1}^2)$, respectively. Under $\mathbb{P}_n$, by the Markov property, we have that

$$T_{n-1} = 1 + \mathbb{1}\{\mathsf{w}_1 = n+1\}(T_n' + T_n''), \tag{A.1}$$

where $\mathbb{1}\{\mathsf{w}_1 = n+1\}, T_n', T_n''$ are independent, T_n' has the same distribution as T_{n-1} under $\mathbb{P}_n$ and T_n'' has the same distribution as T_n under $\mathbb{P}_{n+1}$. It follows that for $n \geq 1$

$$\mathcal{T}_n = \frac{1}{q_n} + \rho_n \mathcal{T}_{n+1} \quad \text{and} \quad \mathcal{S}_n = \sigma_n + \rho_n \mathcal{S}_{n+1}, \tag{A.2}$$

where $\sigma_n = s_n/q_n$, and $s_n = 1 + 2p_n(\mathcal{T}_n + \mathcal{T}_{n+1} + \mathcal{T}_n\mathcal{T}_{n+1})$.[5] It then readily follows that

$$\mathcal{T}_n = \frac{1}{R_{n-1}} \sum_{\ell \geq n} \frac{1}{q_\ell} R_{\ell-1} \quad \text{and} \quad \mathcal{S}_1 = \sum_{\ell \geq 1} \sigma_\ell R_{\ell-1}. \tag{A.3}$$

Our conditions on $\mathbf{p}, \mathbf{q}$, to the effect that $\frac{1}{2} \leq q_n \leq 1$ for all n and that p_n is bounded away from 0, imply that $\mathcal{T}_n \asymp \frac{S_{n-1}}{R_{n-1}}$ and $\sigma_n \asymp \mathcal{T}_n\mathcal{T}_{n+1}$. It follows that

$$\mathbb{E}_1(T_0^2) = \mathcal{S}_1 \asymp \sum_{\ell \geq 1} \frac{S_{\ell-1}S_\ell}{R_\ell} \asymp \sum_{\ell \geq 1} \frac{S_\ell^2}{R_\ell}, \tag{A.4}$$

and the second equivalence is established. For the first one, we have that

$$\mathbb{E}_\nu(T_0) = \sum_{n \geq 1} \nu_n \sum_{i=1}^{n} \mathcal{T}_i < \infty, \tag{A.5}$$

[5] The finitude of $\mathcal{T}_n$ follows from the ergodicity of w; that of $\mathcal{S}_n$ may be checked by comparison to versions of w reflected at large states, using a similar reasoning as in the present argument for the reflected versions of $\mathcal{S}_n$, which are more obviously finite.

noting that $\nu_n \asymp R_{n-1}$, and using again our conditions on $\mathbf{p}, \mathbf{q}$ as above, if and only if

$$\infty > \sum_{n\geq 0} R_n \sum_{i=0}^{n} \frac{S_i}{R_i} = \sum_{i\geq 0} \frac{S_i^2}{R_i}, \tag{A.6}$$

completing the argument.

B Relations of (3.2) to moments of ν

B.1 (3.2) implies that ν has a first moment

This follows immediately from the equivalence argued in Appendix A and the fact, also argued above, that the first condition in (3.3) is equivalent to the finiteness of the middle term in (A.6). Notice that the latter condition implies that

$$\sum_{n\geq 0} nR_n < \infty, \tag{B.1}$$

since $S_i \geq R_i$ for all i, which, in turn, using again that $\nu_n \asymp R_{n-1}$ and our conditions on $\mathbf{p}, \mathbf{q}$, implies that $\sum_{n\geq 0} n\nu_n < \infty$.

B.2 A second moment of ν implies (3.2)

Using again that $\nu_n \asymp R_{n-1}$ and our conditions on $\mathbf{p}, \mathbf{q}$, we have that a second moment of ν is equivalent to

$$\sum_{n\geq 0} n^2 R_n < \infty. \tag{B.2}$$

We may write (3.2) as

$$\sum_{n\geq 1} \frac{1}{R_n} \Big(\sum_{\ell\geq n} R_\ell \Big)^2 \asymp \sum_{n\geq 1} \frac{1}{R_n} \sum_{\ell\geq k\geq n} R_\ell R_k = \sum_{\ell\geq k\geq 1} R_\ell R_k \sum_{n\leq k} \frac{1}{R_n} \leq \sum_{\ell\geq 1} R_\ell \sum_{k\leq \ell} k, \tag{B.3}$$

where the latter inequality follows from the fact that the inner sum on the left hand side is bounded above by k/R_k. It is now quite clear that the right hand side of (B.3) is bounded above by constant times the left hand side of (B.2).

It is perhaps worth pointing out that even if the bound above seems quite crude, we were not able to show that (3.2) does not imply a second moment of ν, so it is conceivable that it does (under our conditions).

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