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Uniqueness and non-uniqueness of the Gaussian free field evolution under the two-dimensional Wick ordered cubic wave equation

Tadahiro Oh[1,a], Mamoru Okamoto[2,b] and Nikolay Tzvetkov[3,c]

[1]*School of Mathematics, The University of Edinburgh and The Maxwell Institute for the Mathematical Sciences, James Clerk Maxwell Building, The King's Buildings, Peter Guthrie Tait Road, Edinburgh EH9 3FD, United Kingdom,* [a]*hiro.oh@ed.ac.uk*
[2]*Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan,* [b]*okamoto@math.sci.osaka-u.ac.jp*
[3]*Ecole Normale Supérieure de Lyon, UMPA, UMR CNRS-ENSL 5669, 46, allée d'Italie, 69364-Lyon Cedex 07, France,* [c]*nikolay.tzvetkov@ens-lyon.fr*

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Abstract. We study the nonlinear wave equation (NLW) on the two-dimensional torus $\mathbb{T}^2$ with Gaussian random initial data on $H^s(\mathbb{T}^2) \times H^{s-1}(\mathbb{T}^2)$, $s < 0$, distributed according to the base Gaussian free field μ associated with the invariant Gibbs measure studied by Thomann and the first author (2020). In particular, we investigate the approximation property of the corresponding solution by smooth (random) solutions. Our main results in this paper are two-fold. (i) We show that the solution map for the renormalized cubic NLW defined on the Gaussian free field μ is the unique extension of the solution map defined for smoothed Gaussian initial data obtained by mollification, independent of mollification kernels. (ii) We also show that there is a regularization of the Gaussian initial data so that the corresponding smooth solutions almost surely have no limit in the natural topology. This second result in particular states that one can not use arbitrary smooth approximation for the renormalized cubic NLW dynamics.

As a preliminary step for proving (ii), we establish a (deterministic) norm inflation result at general initial data for the (unrenormalized) cubic NLW on $\mathbb{T}^d$ and $\mathbb{R}^d$ in negative Sobolev spaces, extending the norm inflation result by Christ, Colliander, and Tao (2003).

Résumé. On considère l'équation des ondes (NLW) posée sur le tore de dimension deux $\mathbb{T}^2$ avec une condition initiale aléatoire dans $H^s(\mathbb{T}^2) \times H^{s-1}(\mathbb{T}^2)$, $s < 0$, distribuée selon le champ libre gaussien μ associé à la mesure invariante de Gibbs étudiée par Thomann et le premier auteur (2020). En particulier, nous essayons de comprendre si on peut approximer les solutions avec condition initiale typique par des solutions lisses aléatoires. Nous obtenons deux résultats complémentaires : (i) Nous démontrons que le flot du NLW cubique renormalisé défini sur le champ libre gaussien est l'unique extension du flot défini sur des données gaussiennes régularisées par convolution (et cela indépendamment du noyau de convolution). (ii) Nous démontrons également qu'il existe une régularisation des données initiales gaussiennes telle que les solutions régulières correspondantes n'ont pas de limite presque sûrement dans la topologie naturelle. Par conséquent, nous ne pouvons pas utiliser une approximation arbitraire pour construire la dynamique du NLW cubique renormalisé. Une étape préliminaire dans la preuve de (ii) consiste en une élaboration significative sur un résultat d'inflation de norme dû à Christ, Colliander, et Tao (2003).

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1. Introduction

1.1. *Nonlinear wave equations*

We consider the defocusing nonlinear wave equation (NLW) on $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$:

$$
\begin{cases} \partial_t^2 u + (1 - \Delta)u + u^k = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), \end{cases} \qquad (x, t) \in \mathbb{T}^2 \times \mathbb{R}, \tag{1.1}
$$

where $k \geq 3$ is an odd integer and the unknown function u is real-valued.[1] In particular, we study the Cauchy problem[2] (1.1) with Gaussian random initial data (u_0^ω, u_1^ω) distributed according to the massive Gaussian free field[3] μ on $\mathcal{H}^s(\mathbb{T}^2) \stackrel{\text{def}}{=} H^s(\mathbb{T}^2) \times H^{s-1}(\mathbb{T}^2)$, $s < 0$, with the covariance operator $(\mathrm{Id} - \Delta)^{-1+s}$, whose density is formally given by[4]

$$d\mu = Z^{-1} e^{-\frac{1}{2}\int_{\mathbb{T}^2}(u^2 + |\nabla u|^2)\,dx}\, du \otimes e^{-\frac{1}{2}\int_{\mathbb{T}^2} v^2\,dx}\, dv. \tag{1.2}$$

This problem naturally appears in the study of invariant Gibbs measures for (1.1); see the next subsection. In particular, the (renormalized) NLW on $\mathbb{T}^2$ is known to be almost surely globally well-posed with respect to the massive Gaussian free field μ (see Theorem A below).

Our main goal in this paper is to study the approximation property of the (random) solution to the renormalized NLW with[5] $\mathcal{L}(u_0^\omega, u_1^\omega) = \mu$ (constructed in Theorem A) by smooth (random) solutions. In other words, we are interested in understanding the following question: "In what sense is the solution map: $(u_0^\omega, u_1^\omega) \mapsto (u, \partial_t u)$ to the (renormalized) NLW with $\mathcal{L}(u_0^\omega, u_1^\omega) = \mu$ an extension of the solution map, a priori defined on smooth (random) initial data?" A natural way to study this question is to approximate the rough initial data by regular functions and see whether the obtained sequence of smooth solutions converges to a unique limit (independent of the choice of the regularization). This is the strongest form of uniqueness and it basically holds when the problem is deterministically locally well-posed, allowing us to conclude that *any* approximation would give a good approximating sequence of smooth solutions, tending to the unique limit. It turns out that for our problem at hand with $\mathcal{L}(u_0^\omega, u_1^\omega) = \mu$, this strongest form of uniqueness does not hold because of the low regularity of the initial data. See (ii) below. This gives rise to the "non-uniqueness" part in the title of this paper. On the other hand, if we restrict our attention to regularization by convolution (which is a very particular way of approximating the rough initial data), then the sequence converges to a unique limit, justifying the "uniqueness" part of the title.

In this paper, we will establish the following two claims:

(i) We show that the solution map: $(u_0^\omega, u_1^\omega) \mapsto (u, \partial_t u)$ to the (renormalized) NLW with $\mathcal{L}(u_0^\omega, u_1^\omega) = \mu$ is the unique extension of the solution map defined for smoothed Gaussian initial data obtained by mollification (Theorem 1.6). Here, the uniqueness refers to the fact that the whole sequence of regularized solutions converges. Note that convergence of a subsequence typically follows from weak solution (= compactness) arguments; see [9,10,52]. Moreover, the limiting solution map is independent of mollification kernels. See Theorem 1.6.

(ii) We show that there exists a regularization of the Gaussian initial data (u_0^ω, u_1^ω) with $\mathcal{L}(u_0^\omega, u_1^\omega) = \mu$ such that the corresponding smooth solutions almost surely have no limit in the natural topology; see Theorem 1.7. We prove this second result by establishing *almost sure norm inflation* for the renormalized NLW (Proposition 1.10).

As a preliminary step for (ii), we prove (deterministic) norm inflation for NLW in negative Sobolev spaces (Theorem 1.11) by following the argument in [42]. See Section 1.6.

1.2. *Invariant Gibbs measures*

With $v = \partial_t u$, we can write the equation (1.1) in the following Hamiltonian formulation:

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\partial H}{\partial(u, v)},$$

where $H = H(u, v)$ is the Hamiltonian given by

$$H(u, v) = \frac{1}{2}\int_{\mathbb{T}^2}\big(u^2 + |\nabla u|^2\big)\,dx + \frac{1}{2}\int_{\mathbb{T}^2} v^2\,dx + \frac{1}{k+1}\int_{\mathbb{T}^2} u^{k+1}\,dx. \tag{1.3}$$

By drawing an analogy to the finite dimensional setting, the Hamiltonian structure of the equation and the conservation of the Hamiltonian suggest that the Gibbs measure $P_2^{(k+1)}$ of the form:

$$\text{“}dP_2^{(k+1)} = Z^{-1}\exp\big(-H(u, v)\big)\,du \otimes dv\text{”} \tag{1.4}$$

---

[1] The equation (1.1) is also referred to as the nonlinear Klein–Gordon equation. We, however, simply refer to (1.1) as NLW in the following. Moreover, we only consider real-valued functions in the following. The modifications required to handle the complex-valued case are straightforward. See [52].

[2] More precisely, we study a renormalized version of (1.1). See the Wick ordered NLW (1.18) below.

[3] In fact, μ is a measure on a vector $(u_0.u_1)$, given as the tensor product of the mass Gaussian free fields on the u_0 component and the white noise measure on the u_1 component. For simplicity, however, we refer to μ as the (massive) Gaussian free field in the following.

[4] Henceforth, we use Z to denote various normalization constants so that the corresponding measures are probability measures when appropriate.

[5] Given a random variable X, we use $\mathcal{L}(X)$ to denote the law (= distribution) of X.

is invariant under the dynamics of (1.1). By substituting (1.3) for $H(u,v)$ in the exponent, we can rewrite the formal expression (1.4) as

$$\begin{aligned} dP_2^{(k+1)} &= Z^{-1} e^{-\frac{1}{k+1}\int_{\mathbb{T}^2} u^{k+1}\,dx} e^{-\frac{1}{2}\int_{\mathbb{T}^2}(u^2+|\nabla u|^2)\,dx}\, du \otimes e^{-\frac{1}{2}\int_{\mathbb{T}^2} v^2\,dx}\, dv \\ &\sim e^{-\frac{1}{k+1}\int_{\mathbb{T}^2} u^{k+1}\,dx}\, d\mu, \end{aligned} \tag{1.5}$$

where μ is the massive Gaussian free field defined in (1.2).

Recall that the Gaussian measure μ in (1.2) is the induced probability measure under the map:

$$\omega \in \Omega \longmapsto \big(u_0^\omega, u_1^\omega\big),$$

where (u_0^ω, u_1^ω) is given by the following random Fourier series:[6]

$$\big(u_0^\omega, u_1^\omega\big) = \bigg(\sum_{n\in\mathbb{Z}^2} \frac{g_{0,n}(\omega)}{\langle n\rangle} e^{in\cdot x}, \sum_{n\in\mathbb{Z}^2} g_{1,n}(\omega) e^{in\cdot x}\bigg). \tag{1.6}$$

Here, $\langle n\rangle = \sqrt{1+|n|^2}$ and $\{g_{0,n}, g_{1,n}\}_{n\in\mathbb{Z}^2}$ is a sequence of independent standard complex-valued Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$ conditioned that $g_{j,-n} = \overline{g_{j,n}}$, $n \in \mathbb{Z}^2$, $j = 0, 1$. It is easy to check that (u_0^ω, u_1^ω) belongs to $\mathcal{H}^s(\mathbb{T}^2) \setminus \mathcal{H}^0(\mathbb{T}^2)$, $s < 0$, almost surely. In particular, for an odd integer $k \geq 3$, we have $\int_{\mathbb{T}^2} u^{k+1}\,dx = \infty$ almost surely with respect to μ and thus the density $e^{-\frac{1}{k+1}\int_{\mathbb{T}^2} u^{k+1}\,dx}$ in (1.5) vanishes almost surely. As a result, the expression in (1.5) does not make sense as a probability measure. This forces us to renormalize the potential part of the Hamiltonian, which enables us to define the Gibbs measure $P_2^{(k+1)}$ corresponding to the renormalized Hamiltonian as a probability measure (absolutely continuous with respect to the Gaussian free field μ). See [21,25,52,59] for details. As a consequence, one is led to study the renormalized NLW dynamics (see (1.18) below) associated with the renormalized Hamiltonian.

1.3. *Wick ordered NLW*

In this subsection, we go over a derivation of the renormalized NLW by directly introducing a renormalization at the level of the equation. By writing (1.1) in the Duhamel formulation with the random initial data (u_0^ω, u_1^ω) in (1.6), we have

$$u(t) = S(t)\big(u_0^\omega, u_1^\omega\big) - \int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle} u^k\big(t'\big)\,dt', \tag{1.7}$$

where $\langle\nabla\rangle = \sqrt{1-\Delta}$ and $S(t)$ denotes the linear wave propagator given by

$$S(t)(f,g) = \cos\big(t\langle\nabla\rangle\big) f + \frac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle} g.$$

Let z denote the random linear solution given by

$$z = z^\omega = S(t)\big(u_0^\omega, u_1^\omega\big). \tag{1.8}$$

Recalling that $(u_0^\omega, u_1^\omega) \in \mathcal{H}^s(\mathbb{T}^2) \setminus \mathcal{H}^0(\mathbb{T}^2)$, $s < 0$, almost surely, we see that $z(t)$ is merely a Schwartz distribution. Hence, there is an issue in making sense of the power $z^k(t)$ and thus the full nonlinearity $u^k(t)$ appearing in (1.7). In fact, by following the argument in [44,48], a phenomenon of triviality may be shown for (1.1) without renormalization (at least when $k = 3$). Namely, by considering smooth solutions u_N to (1.1) with regularized random initial data, we may show that, as the regularization is removed, u_N converges to a trivial solution $u \equiv 0$. This shows the necessity of a proper renormalization at the level of the equation.

With (1.8), we easily see that, for any $t \in \mathbb{R}$, the distribution of $z(t)$ is once again given by the massive Gaussian free field μ in (1.2). Namely, μ is invariant under the linear wave dynamics. Indeed, we have

$$\big(z(t), \partial_t z(t)\big) = \bigg(\sum_{n\in\mathbb{Z}^2} \frac{g_{0,n}^t}{\langle n\rangle} e^{in\cdot x}, \sum_{n\in\mathbb{Z}^2} g_{1,n}^t e^{in\cdot x}\bigg), \tag{1.9}$$

[6] We drop the harmless factor 2π in the following.

where

$$g^t_{0,n} \overset{\text{def}}{=} \cos\big(t\langle n\rangle\big)g_{0,n} + \sin\big(t\langle n\rangle\big)g_{1,n},$$
$$g^t_{1,n} \overset{\text{def}}{=} -\sin\big(t\langle n\rangle\big)g_{0,n} + \cos\big(t\langle n\rangle\big)g_{1,n}. \tag{1.10}$$

It is easy to check that $\{g^t_{0,n}, g^t_{1,n}\}_{n\in\mathbb{Z}^2}$ forms a sequence of independent standard complex-valued Gaussian random variables conditioned that

$$g^t_{j,-n} = \overline{g^t_{j,n}} \tag{1.11}$$

for any $n \in \mathbb{Z}^2$ and $j = 0, 1$. This shows that the massive Gaussian free field μ in (1.2) is invariant under the linear wave dynamics.

Let $\mathbf{P}_N$ denote the frequency projection onto the spatial frequencies $\{|n| \le N\}$ and set $z_N = \mathbf{P}_N z$. Then, for each $(x,t) \in \mathbb{T}^2 \times \mathbb{R}$, $z_N(x,t)$ is a mean-zero real-valued Gaussian random variable with variance[7]

$$\sigma_N \overset{\text{def}}{=} \operatorname{Var}\big(z_N(x,t)\big) = \mathbb{E}\big[z_N^2(x,t)\big] = \sum_{|n|\le N} \frac{1}{\langle n\rangle^2} \sim \log N. \tag{1.12}$$

Note that σ_N is independent of $(x,t) \in \mathbb{T}^2 \times \mathbb{R}$, reflecting the translation-invariant nature of the problem. We then define the Wick powers $:\!z_N^\ell\!:$, $\ell \in \mathbb{N}\cup\{0\}$, by setting

$$:\!z_N^\ell(x,t)\!: \overset{\text{def}}{=} H_\ell\big(z_N(x,t); \sigma_N\big) \tag{1.13}$$

in a pointwise manner, where $H_\ell(x;\sigma)$ denotes the Hermite polynomial of degree ℓ with a parameter $\sigma > 0$. See Section 2 for more on the Hermite polynomials. We now recall the following proposition from [26,53].

Proposition 1.1. *Let $\ell \in \mathbb{N}\cup\{0\}$. Then, for any $p < \infty$, $T > 0$, and $\varepsilon > 0$, the sequence $\{:\!z_N^\ell\!:\}_{N\in\mathbb{N}}$ is Cauchy in $L^p(\Omega; C([-T,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2)))$. Denoting the limit by*

$$:\!z^\ell\!: = :\!z_\infty^\ell\!: \overset{\text{def}}{=} \lim_{N\to\infty} :\!z_N^\ell\!:, \tag{1.14}$$

we have $:\!z^\ell\!: \in C([-T,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2))$, almost surely.

In [53], the convergence was shown only in $L^p(\Omega; L^q([-T,T]; W^{-\varepsilon,r}(\mathbb{T}^2)))$ for $q, r < \infty$. By repeating the argument in [26, Proposition 2.1], however, we can easily upgrade this to the claimed regularity result in Proposition 1.1. One may also apply Proposition 2.7 below and directly verify Proposition 1.1. See Section 4.1. See also [27,28,43].

Given $N \in \mathbb{N}$, consider the following truncated NLW:

$$\partial_t^2 u_N + (1-\Delta)u_N + \mathbf{P}_N\big[(\mathbf{P}_N u_N)^k\big] = 0$$

with the random initial data (u_0^ω, u_1^ω) in (1.6). In view of the Duhamel formula, it is natural to decompose u_N as

$$u_N = z + v_N \tag{1.15}$$

with $v_N = \mathbf{P}_N v_N$. Then, by the binomial theorem, we have

$$(\mathbf{P}_N u_N)^k = (z_N + v_N)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} z_N^\ell \cdot v_N^{k-\ell} \tag{1.16}$$

[7] While it may be common to denote the variance by σ_N^2, we chose to use σ_N to denote the variance in (1.12) so that it is consistent with the notation $H(x;\sigma)$ for the Hermite polynomial with a parameter σ, which is used for the Wick renormalization (1.13); see (2.1) and (2.2). See also Kuo's book [36, Chapter 9].

and thus we see that there is an issue in taking a limit as $N \to \infty$, since the limit of z_N^ℓ does not exist. By recalling the following identities for the Hermite polynomials:

$$H_k(x+y) = \sum_{\ell=0}^{k} \binom{k}{\ell} H_\ell(y) \cdot x^{k-\ell} \quad \text{and} \quad H_k(x;\sigma) = \sigma^{\frac{k}{2}} H_k\big(\sigma^{-\frac{1}{2}} x\big),$$

we define the renormalized nonlinearity $:\!(\mathbf{P}_N u_N)^k\!:$ by setting

$$\begin{aligned} :\!(\mathbf{P}_N u_N)^k\!: &= \mathcal{N}^k_{(\mathbf{P}_N u_0^\omega, \mathbf{P}_N u_1^\omega)}(u_N) \\ &\stackrel{\text{def}}{=} H_k(z_N + v_N; \sigma_N) = \sum_{\ell=0}^{k} \binom{k}{\ell} H_\ell(z_N;\sigma_N) \cdot v_N^{k-\ell} \\ &= \sum_{\ell=0}^{k} \binom{k}{\ell} :\!z_N^\ell\!: \cdot\, v_N^{k-\ell}. \end{aligned} \tag{1.17}$$

Namely, we replaced z_N^ℓ in (1.16) by the Wick power $:\!z_N^\ell\!:$. In view of Proposition 1.1, we can take a limit of (1.17) as $N \to \infty$. This leads to the following Wick ordered NLW:

$$\begin{cases} \partial_t^2 u + (1-\Delta)u + :\!u^k\!: = 0, \\ (u, \partial_t u)|_{t=0} = \big(u_0^\omega, u_1^\omega\big), \end{cases} \tag{1.18}$$

where the Wick ordered nonlinearity $:\!u^k\!:$ is defined by

$$:\!u^k\!: = \mathcal{N}^k_{(u_0^\omega, u_1^\omega)}(u) \stackrel{\text{def}}{=} \sum_{\ell=0}^{k} \binom{k}{\ell} :\!z^\ell\!: \cdot\, v^{k-\ell} \tag{1.19}$$

for functions u of the form:

$$u = z + v \tag{1.20}$$

with some sufficiently smooth v such that $v^{k-\ell}$ in (1.19) makes sense. We stress that the Wick ordered nonlinearity $:\!u^k\!:$ is not defined for general functions u but is defined only for functions u of the form (1.20).

In [53], the first author and Thomann studied the Wick ordered NLW (1.18) by considering the following fixed point problem for the residual term $v = u - z$:

$$\begin{cases} \partial_t^2 v + (1-\Delta)v + :\!(v+z)^k\!: = 0, \\ (v, \partial_t v)|_{t=0} = (0,0). \end{cases} \tag{1.21}$$

A result of interest to us reads as follows:

Theorem A ([53]). *The Wick ordered NLW (1.18) is almost surely globally well-posed with respect to the massive Gaussian free field μ in (1.2). Moreover, the solution $(u, \partial_t u)$ to (1.18) almost surely lies in the class*:

$$(u, \partial_t u) \in (z, \partial_t z) + C\big(\mathbb{R}; \mathcal{H}^{1-\varepsilon}(\mathbb{T}^2)\big) \subset C\big(\mathbb{R}; \mathcal{H}^{-\varepsilon}(\mathbb{T}^2)\big). \tag{1.22}$$

for any $\varepsilon > 0$.

Remark 1.2. Consider the following truncated Wick ordered NLW:

$$\begin{cases} \partial_t^2 u_N + (1-\Delta)u_N + \mathbf{P}_N\big[:\!(\mathbf{P}_N u_N)^k\!:\big] = 0, \\ (u_N, \partial_t u_N)|_{t=0} = \big(u_0^\omega, u_1^\omega\big), \end{cases} \tag{1.23}$$

where the truncated Wick ordered nonlinearity is interpreted as in (1.17) for u_N of the form (1.15). Then, it follows from iterating the local theory in [53] that, for given $T > 0$, the solution u_N to (1.23) converges almost surely to the solution u to (1.18) in $C([-T,T]; H^{-\varepsilon}(\mathbb{T}^2))$, $\varepsilon > 0$ (and the residual part $v_N = u_N - z$ converges to $v = u - z$ in $C([-T,T]; H^{1-\varepsilon}(\mathbb{T}^2))$, almost surely).

The proof of almost sure local well-posedness of (1.18) follows from studying the fixed point problem (1.21) for v with Sobolev's inequality[8] and the space-time control on the stochastic terms (Proposition 1.1). The almost sure global well-posedness follows from (i) almost sure global well-posedness of the Wick ordered NLW (1.18) with respect to the Gibbs measure $P_2^{(k+1)}$ (by Bourgain's invariant measure argument [5,6,12]) and (ii) the mutual absolute continuity of the Gibbs measure $P_2^{(k+1)}$ and the massive Gaussian free field μ. Lastly, the second claim (1.22) follows from iterating the local-in-time argument with Proposition 1.1.

Let u be the random solution to the Wick ordered NLW (1.18) with $\mathcal{L}((u, \partial_t u)|_{t=0}) = \mu$ constructed in Theorem A. In the following, we study the approximation property of the random solution u to the renormalized NLW (1.18) by the smooth solutions corresponding to smooth approximating (random) initial data. We point out that the renormalized nonlinearity $:u^k:$ in (1.18) is defined for the specific random initial data (u_0^ω, u_1^ω) in (1.6). In particular, in considering the renormalized dynamics corresponding to smooth random initial data, we need to make it clear what we mean by the renormalized nonlinearity for smooth random initial data. This is the topic of the next subsection.

1.4. *Renormalized NLW with smooth Gaussian initial data*

In this subsection, we consider the renormalized NLW with *smooth* Gaussian random initial data. While there is no need to consider any renormalization in studying (1.1) with smooth random initial data, we introduce a renormalization even for smooth random initial data so that we can study smooth approximations to the Wick ordered NLW (1.18) with $\mathcal{L}(u_0^\omega, u_1^\omega) = \mu$. For this purpose, we introduce the following definition.

Definition 1.3. Let $(\varphi_0^\omega, \varphi_1^\omega)$ be an $\mathcal{H}^s(\mathbb{T}^2)$-valued random variable for some $s \geq 0$. Set

$$\sigma(t) \stackrel{\text{def}}{=} \operatorname{Var}\big(S(t)\big(\varphi_0^\omega, \varphi_1^\omega\big)\big) = \mathbb{E}\big[\big(S(t)\big(\varphi_0^\omega, \varphi_1^\omega\big)\big)^2\big] - \big(\mathbb{E}\big[S(t)\big(\varphi_0^\omega, \varphi_1^\omega\big)\big]\big)^2.$$

Then, we define the renormalized nonlinearity $\mathcal{N}^k_{(\varphi_0^\omega, \varphi_1^\omega)}(v)$ by

$$\mathcal{N}^k_{(\varphi_0^\omega, \varphi_1^\omega)}(v) \stackrel{\text{def}}{=} H_k\big(S(t)\big(\varphi_0^\omega, \varphi_1^\omega\big) + v; \sigma(t)\big).$$

In view of the previous discussion, we aim to study the following problem:

$$\begin{cases} \partial_t^2 v + (1-\Delta)v + \mathcal{N}^k_{(\varphi_0^\omega, \varphi_1^\omega)}(v) = 0, \\ (v, \partial_t v)|_{t=0} = (0,0) \end{cases} \tag{1.24}$$

for a sequence of (smoother) random initial data $(\varphi_0^\omega, \varphi_1^\omega) \in \mathcal{H}^0(\mathbb{T}^2)$ approximating (u_0^ω, u_1^ω) given in (1.6). Our goal is then to try to understand how much the obtained sequence of (smoother) solutions converges to the solution obtained in Theorem A (modulo the free evolution), i.e. the solution $v = u - z = u - S(t)(u_0^\omega, u_1^\omega)$ to (1.21). For this purpose, we will first solve (1.24) for a large class of $(\varphi_0^\omega, \varphi_1^\omega)$ in $\mathcal{H}^s(\mathbb{T}^2)$, $s \geq 0$.

Let us now describe the class of data $(\varphi_0^\omega, \varphi_1^\omega)$ for which we study (1.24). Let $(\phi_0, \phi_1) \in \mathcal{H}^s(\mathbb{T}^2)$, $s \geq 0$, with the Fourier series expansions

$$\phi_j = \sum_{n\in\mathbb{Z}^2} \widehat{\phi}_j(n) e^{in\cdot x} \quad \text{with } \widehat{\phi}_j(-n) = \overline{\widehat{\phi}_j(n)},\ j = 0, 1.$$

We define the randomization $(\phi_0^\omega, \phi_1^\omega)$ of (ϕ_0, ϕ_1) by setting

$$\phi_j^\omega \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}^2} g_{j,n}(\omega)\widehat{\phi}_j(n) e^{in\cdot x}, \tag{1.25}$$

where $\{g_{0,n}, g_{1,n}\}_{n\in\mathbb{Z}^2}$ is as in (1.6). Let $(r_0, r_1) \in \mathcal{H}^{s+1}(\mathbb{T}^2)$. We then study (1.24) with $(\varphi_0^\omega, \varphi_1^\omega)$ given by

$$\big(\varphi_0^\omega, \varphi_1^\omega\big) = \big(\phi_0^\omega, \phi_1^\omega\big) + (r_0, r_1). \tag{1.26}$$

[8] While the argument in [53] used the Strichartz estimates, it is possible to prove the local well-posedness part in Theorem A by Sobolev's inequality. See [28].

Note that

$$\sigma(t) = \mathrm{Var}\big(S(t)\big(\varphi_0^\omega, \varphi_1^\omega\big)\big) = \mathrm{Var}\big(S(t)\big(\phi_0^\omega, \phi_1^\omega\big)\big)$$

$$= \sum_{n\in\mathbb{Z}^2}\bigg(\cos^2\big(t\langle n\rangle\big)\big|\widehat{\phi}_0(n)\big|^2 + \frac{\sin^2(t\langle n\rangle)}{\langle n\rangle^2}\big|\widehat{\phi}_1(n)\big|^2\bigg) \lesssim \|(\phi_0,\phi_1)\|_{\mathcal{H}^0}^2 < \infty, \tag{1.27}$$

which shows that the renormalized nonlinearity $\mathcal{N}^k_{(\varphi_0^\omega,\varphi_1^\omega)}(v)$ in (1.24) is well defined for the random data $(\varphi_0^\omega, \varphi_1^\omega)$ given in (1.26). Compare this with the renormalized nonlinearity in (1.21) which is defined only via a limiting procedure via Proposition 1.1. We have the following proposition on almost sure global existence of unique solutions to the Wick ordered NLW (1.24) with the random data $(\varphi_0^\omega, \varphi_1^\omega)$ given by (1.26).

Proposition 1.4. *Let $k \geq 3$ be an odd integer and let $s \in \mathbb{R}$ satisfy*

$$\text{(i) } s > 0 \text{ when } k = 3 \quad \text{and} \quad \text{(ii) } s \geq 1 \text{ when } k \geq 5.$$

Given $(\phi_0, \phi_1) \in \mathcal{H}^s(\mathbb{T}^2)$ and $(r_0, r_1) \in \mathcal{H}^{s+1}(\mathbb{T}^2)$, let $(\phi_0^\omega, \phi_1^\omega)$ be the randomization of (ϕ_0, ϕ_1) defined in (1.25) and define $(\varphi_0^\omega, \varphi_1^\omega)$ as in (1.26). Then, there exists almost surely a unique global solution $(v, \partial_t v) \in C(\mathbb{R}; \mathcal{H}^1(\mathbb{T}^2))$ to (1.24).

We present the proof of Proposition 1.4 in Section 3. The almost sure local well-posedness for $s \geq 0$ (with any k) follows from a standard fixed point argument with the probabilistic Strichartz estimate (Lemma 2.4). See, for example, [13,58]. As for the almost sure global well-posedness, we proceed with a Gronwall argument as in [13] when $k = 3$. For $k \geq 5$, we also use the integration-by-parts trick, introduced in [47], to control higher order terms with respect to v

Remark 1.5. (i) Observe that if the data in (1.26) is deterministic, i.e. $\phi_j^\omega = 0$, then $\sigma(t) = 0$ and the nonlinearity is of pure power type, namely $\mathcal{N}^k_{(\varphi_0^\omega,\varphi_1^\omega)}(v)$ becomes $(S(t)(r_0, r_1) + v)^k$.

(ii) For simplicity of the presentation, we chose $(r_0, r_1) \in \mathcal{H}^{s+1}(\mathbb{T}^2)$ in the statement of Proposition 1.4 such that the Cameron–Martin theorem [14] allows us to reduce the proof to the case $r_0 = r_1 = 0$ at the beginning of Section 3. In fact, a slight modification of the argument in Section 3 shows that Proposition 1.4 also holds for $(r_0, r_1) \in \mathcal{H}^1(\mathbb{T}^2)$, whether (r_0, r_1) is deterministic or random. Indeed, given $(r_0, r_1) \in \mathcal{H}^1(\mathbb{T}^2)$, by setting $w = v + S(t)(r_0, r_1)$, where v is a solution to (1.24), we see that w satisfies the following Cauchy problem:

$$\begin{cases} \partial_t^2 w + (1-\Delta)w + H_k\big(S(t)\big(\phi_0^\omega, \phi_1^\omega\big) + w(t); \sigma(t)\big) = 0, \\ (w, \partial_t w)|_{t=0} = (r_0, r_1) \in \mathcal{H}^1\big(\mathbb{T}^2\big). \end{cases} \tag{1.28}$$

Then, by noting that Lemma 3.1 on local well-posedness holds for general $\mathcal{H}^1$-initial data, global well-posedness of (1.28) follows from proceeding as in the proof of Proposition 1.4 presented in Section 3, which is about controlling the $\mathcal{H}^1$-norm of a solution; see (3.7). Once we have constructed a unique global-in-time solution w to (1.28), we simply set $v = w - S(t)(r_0, r_1)$, which is a unique global-in-time solution to (1.24).

1.5. *Unique and non-unique extensions of the solution map to the Wick ordered NLW with the Gaussian free field μ as initial data*

In this subsection, we state our main results in this paper. In the following, we restrict our attention to the cubic case ($k = 3$). In the previous subsection, we constructed almost surely well-defined global-in-time dynamics for the Wick ordered NLW (1.24) with smooth random initial data (Proposition 1.4). In particular, there exists a solution map, sending smooth random initial data to smooth random solutions. On the other hand, Theorem A shows that the solution map "extends" to the (rough) Gaussian random initial data (u_0^ω, u_1^ω) of the form (1.6), distributed according to the massive Gaussian free field μ in (1.2). In the following, we investigate in what sense the solution map constructed in Proposition 1.4 extends to that in Theorem A.

We first establish a (partial) positive answer. Namely, we show that the solution map constructed in Theorem A is the unique extension of the solution map defined on a certain class of smooth random initial data. We say that a smooth function $\rho \in L^1(\mathbb{R}^2)$ is a mollification kernel if $\int_{\mathbb{R}^2} \rho(x)\,dx = 1$ and $\operatorname{supp}\rho \subset (-\frac{1}{2}, \frac{1}{2}]^2$. Given a mollification kernel ρ, define ρ_δ by setting

$$\rho_\delta(x) = \delta^{-2}\rho\big(\delta^{-1}x\big)$$

for $0 < \delta \le 1$. Then, $\{\rho_\delta\}_{\delta\in(0,1]}$ is an approximate identity on $\mathbb{R}^2$. By noting that $\operatorname{supp}\rho_\delta \subset (-\frac{1}{2}, \frac{1}{2}]^2 \cong \mathbb{T}^2$ for any $\delta \in (0,1]$, we see that $\{\rho_\delta\}_{\delta\in(0,1]}$ is also an approximate identity on $\mathbb{T}^2$.

The following theorem shows that the solution map constructed in Theorem A is the unique extension of the solution map defined on smooth random initial data, regularized by a mollification. Here, the uniqueness refers to the convergence of the whole sequence and also to the fact that the extension is independent of mollification kernels ρ.

Theorem 1.6. *Let (u_0^ω, u_1^ω) be the Gaussian random initial data defined in (1.6). Given a mollification kernel ρ, define $(u_{0,\delta}^\omega, u_{1,\delta}^\omega) \in C^\infty(\mathbb{T}^2) \times C^\infty(\mathbb{T}^2)$, $0 < \delta \le 1$, via the regularization by mollification:*

$$u_{0,\delta}^\omega = \rho_\delta * u_0^\omega \quad \text{and} \quad u_{1,\delta}^\omega = \rho_\delta * u_1^\omega, \tag{1.29}$$

where ρ_δ is as above (of course, $\lim_{\delta\to 0} \|(u_{0,\delta}^\omega, u_{1,\delta}^\omega) - (u_0^\omega, u_1^\omega)\|_{\mathcal{H}^s} = 0$, almost surely). Denote by $(v_\delta, \partial_t v_\delta)$ the solution to the Wick ordered NLW (1.24) with

$$(\varphi_0^\omega, \varphi_1^\omega) = (u_{0,\delta}^\omega, u_{1,\delta}^\omega),$$

constructed in Proposition 1.4, and set $u_\delta \stackrel{\text{def}}{=} S(t)(u_{0,\delta}^\omega, u_{1,\delta}^\omega) + v_\delta$. Then, given any $T > 0$ and $s < 0$, $(u_\delta, \partial_t u_\delta)$ converges in probability to $(u, \partial_t u)$ in $C([-T,T]; \mathcal{H}^s(\mathbb{T}^2))$, where $(u, \partial_t u)$ is the solution to the Wick ordered NLW (1.18) with the initial data (u_0^ω, u_1^ω) constructed in Theorem A. Namely, $u = z + v$, where z and v are as in (1.8) and (1.21), respectively.

Next, we turn our attention to a negative direction. We prove the following instability result for the Wick ordered NLW (1.18) with the Gaussian free field μ in (1.2) as initial data.

Theorem 1.7. *Let $s < 0$ and (u_0^ω, u_1^ω) be as in (1.6). Then, there exists a set $\Sigma \subset \Omega$ with $P(\Sigma) = 1$ such that given $\omega \in \Sigma$, there exists a sequence $(u_{0,\varepsilon}^\omega, u_{1,\varepsilon}^\omega) \in C^\infty(\mathbb{T}^2) \times C^\infty(\mathbb{T}^2)$, $0 < \varepsilon \le 1$, such that almost surely*

$$\lim_{\varepsilon\to 0} \left\| (u_{0,\varepsilon}^\omega, u_{1,\varepsilon}^\omega) - (u_0^\omega, u_1^\omega) \right\|_{\mathcal{H}^s} = 0$$

but for every $T > 0$, the solutions v_ε to (1.24) with

$$(\varphi_0^\omega, \varphi_1^\omega) = (u_{0,\varepsilon}^\omega, u_{1,\varepsilon}^\omega)$$

defined in Proposition 1.4 satisfy almost surely

$$\lim_{\varepsilon\to 0} \|v_\varepsilon\|_{L^\infty([-T,T];H^s)} = \infty.$$

As a consequence, $u_\varepsilon \stackrel{\text{def}}{=} S(t)(u_{0,\varepsilon}^\omega, u_{1,\varepsilon}^\omega) + v_\varepsilon$ diverges almost surely in $C([-T,T]; H^s(\mathbb{T}^2))$.

Theorems 1.6 and 1.7 together imply that the choice of regularization of the random initial data plays an important role. On the one hand, there is a class of "admissible" regularizations yielding the conclusion of Theorem 1.6. On the other hand, there is also a regularization, leading to a strong instability. This is a sharp contrast with the smoother regime, where (deterministic) local well-posedness theory, in particular continuous dependence, guarantees *any* regularization gives a good approximation. See Theorems 1.33 and 2.7 in [64] for analogous results in the context of the three-dimensional cubic NLW (without the need of renormalization). One main difference between our results (Theorems 1.6 and 1.7) and those in [64] appears in the fact that, in our problem, the effect of the random initial data shows up in the equation through the renormalized nonlinearity, giving further complication to the problem.

Remark 1.8. In a recent work [60], the third author and Sun established a certain pathological behavior for NLW on the three-dimensional torus $\mathbb{T}^3$ with initial data of super-critical (but positive[9]) regularity. They constructed a dense subset S of the Sobolev space of super-critical regularity such that for any $(u_0, u_1) \in S$, the family of global smooth solutions u_δ, generated by the mollified initial data $(\rho_\delta * u_0, \rho_\delta * u_1)$, diverges. While it is a purely deterministic result, this result nicely complements Theorem 1.6, since it shows that a mollification does not in general (and in fact on a dense set) lead to a good approximation in the super-critical regularity.

[9] In particular, there is no need for renormalization in [60].

Before proceeding to the next subsection, we briefly discuss a reduction of the proof of Theorem 1.7 in the following. Our main strategy is as follows. Given $\varepsilon > 0$, let $(u^\omega_{0,\delta}, u^\omega_{1,\delta})$ be the mollified random initial data as in (1.29) for some small $\delta = \delta(\varepsilon) > 0$. We then construct the smooth solution $v_{\delta,\varepsilon} = v^\omega_{\delta,\varepsilon}$ to (1.24):

$$\begin{cases} \partial_t^2 v_{\delta,\varepsilon} + (1-\Delta) v_{\delta,\varepsilon} + \mathcal{N}^3_{(\varphi^\omega_{0,\delta,\varepsilon}, \varphi^\omega_{1,\delta,\varepsilon})}(v_{\delta,\varepsilon}) = 0, \\ (v_{\delta,\varepsilon}, \partial_t v_{\delta,\varepsilon})|_{t=0} = (0,0), \end{cases} \tag{1.30}$$

where

$$\big(\varphi^\omega_{0,\delta,\varepsilon}, \varphi^\omega_{1,\delta,\varepsilon}\big) = \big(u^\omega_{0,\delta}, u^\omega_{1,\delta}\big) + (\phi_{0,\varepsilon}, \phi_{1,\varepsilon})$$

for some suitably chosen *deterministic* functions $(\phi_{0,\varepsilon}, \phi_{1,\varepsilon}) \in C^\infty(\mathbb{T}^2) \times C^\infty(\mathbb{T}^2)$. The first observation is that the conclusion of Theorem 1.6 holds true even if we replace (u^ω_0, u^ω_1) (and $(u^\omega_{0,\delta}, u^\omega_{1,\delta})$, respectively) by $(u^\omega_0, u^\omega_1) + (\phi_{0,\varepsilon}, \phi_{1,\varepsilon})$ (and $(u^\omega_{0,\delta}, u^\omega_{1,\delta}) + (\phi_{0,\varepsilon}, \phi_{1,\varepsilon})$, respectively) for any $(\phi_{0,\varepsilon}, \phi_{1,\varepsilon}) \in C^\infty(\mathbb{T}^2) \times C^\infty(\mathbb{T}^2)$. Namely, the smooth solution $v_{\delta,\varepsilon}$ to (1.30) converges in probability to the solution v_ε to

$$\begin{cases} \partial_t^2 v_{\varepsilon} + (1-\Delta) v_{\varepsilon} + \mathcal{N}^3_{(\varphi^\omega_{0,\varepsilon}, \varphi^\omega_{1,\varepsilon})}(v_{\varepsilon}) = 0, \\ (v_{\varepsilon}, \partial_t v_{\varepsilon})|_{t=0} = (0,0) \end{cases} \tag{1.31}$$

as $\delta \to 0$, where

$$\big(\varphi^\omega_{0,\varepsilon}, \varphi^\omega_{1,\varepsilon}\big) = \big(u^\omega_0, u^\omega_1\big) + (\phi_{0,\varepsilon}, \phi_{1,\varepsilon}). \tag{1.32}$$

See Remark 4.5. Note that, in (1.31), the nonlinearity $\mathcal{N}^3_{(\varphi^\omega_{0,\varepsilon}, \varphi^\omega_{1,\varepsilon})}(v_\varepsilon)$ is interpreted in the limiting sense as $\delta \to 0$. This observation allows us to drop the smoothness assumption on data in Theorem 1.7. More precisely, Theorem 1.7 is a consequence of the following statement.

Proposition 1.9. *Let $s < 0$ and (u^ω_0, u^ω_1) be as in (1.6). Then, there exists a set $\Sigma \subset \Omega$ with $P(\Sigma) = 1$ such that given $\omega \in \Sigma$ and $\varepsilon > 0$, there exist a solution v^ω_ε to (1.31) on $\mathbb{T}^2$ with the random data $(\varphi^\omega_{0,\varepsilon}, \varphi^\omega_{1,\varepsilon})$ in (1.32) and a random time $t_\varepsilon = t_\varepsilon(\omega) \in (0, \varepsilon)$ such that*

$$\|(\phi_{0,\varepsilon}, \phi_{1,\varepsilon})\|_{\mathcal{H}^s} < \varepsilon \quad \textit{but} \quad \|v^\omega_\varepsilon(t_\varepsilon)\|_{H^s} > \varepsilon^{-1}.$$

In our reduction of Theorem 1.7 to Proposition 1.9, we moved from the smooth setting to the rough setting, contrary to the usual reduction, where one approximates rough objects by smooth objects. This reduction, however, helps us since the solutions v to (1.21) and v_ε to (1.31) satisfy the *same* equation, where the renormalization on the nonlinearity is based on (u^ω_0, u^ω_1) defined in (1.6).

We now express (1.31) in terms of $w_\varepsilon = v_\varepsilon + S(t)(\phi_{0,\varepsilon}, \phi_{1,\varepsilon})$. Then, w_ε satisfies the following perturbed NLW:

$$\begin{cases} \partial_t^2 w_{\varepsilon} + (1-\Delta) w_{\varepsilon} + w_\varepsilon^3 + \mathcal{R}(w_\varepsilon, z) = 0, \\ (w_{\varepsilon}, \partial_t w_{\varepsilon})|_{t=0} = (\phi_{0,\varepsilon}, \phi_{1,\varepsilon}), \end{cases} \tag{1.33}$$

where $\mathcal{R}(w, z)$ is given by

$$\begin{aligned} \mathcal{R}(w,z) &= :(z+w)^3: - w^3 \\ &= 3zw^2 + 3:z^2:w + :z^3:. \end{aligned}$$

Then, the proof of Proposition 1.9 is reduced to the following proposition on almost sure norm inflation for the perturbed NLW (1.33).

Proposition 1.10. *Let $s < 0$ and $z = z^\omega$ be as in (1.8). Then, there exists a set $\Sigma \subset \Omega$ with $P(\Sigma) = 1$ such that given $\omega \in \Sigma$ and $\varepsilon > 0$, there exist a solution w^ω_ε to (1.33) on $\mathbb{T}^2$ and a random time $t_\varepsilon = t_\varepsilon(\omega) \in (0, \varepsilon)$ such that*

$$\big\|\big(w^\omega_\varepsilon(0), \partial_t w^\omega_\varepsilon(0)\big)\big\|_{\mathcal{H}^s} < \varepsilon \quad \textit{and} \quad \|w^\omega_\varepsilon(t_\varepsilon)\|_{H^s} > \varepsilon^{-1}.$$

With $\mathcal{R}(w_\varepsilon, z) = 0$, such a norm inflation phenomenon has been studied for the (unrenormalized) NLW (1.1); see [11,18,64,66]. In Proposition 1.10, we establish norm inflation almost surely in the presence of the random perturbation $\mathcal{R}(w_\varepsilon, z)$. We point out that the known result on norm inflation for NLW (1.1) on $\mathbb{T}^d$ or $\mathbb{R}^d$ in negative Sobolev spaces only covers a partial range $s \le -\frac{d}{2}$ in the general setting; see [18]. While there is a norm inflation result for $s < \frac{1}{6}$ by reducing the analysis to the one-dimensional case via the finite speed of propagation (see [18, Corollary 7]), this result is not useful to our problem due to the genuine two-dimensional nature of the random perturbation. Therefore, we first need to extend the deterministic norm inflation result to cover this missing range $(-\frac{d}{2}, 0)$ *without* reducing the analysis to the one-dimensional setting. In fact, this is the goal of the next subsection. More precisely, we consider the (unrenormalized) NLW and prove norm inflation (at general initial data) in negative Sobolev spaces, including the missing range $(-\frac{d}{2}, 0)$. This will be a basic building block for the proof of Proposition 1.10.

Even with norm inflation for the (unrenormalized) NLW (see Theorem 1.11 below), the actual proof of Proposition 1.10 requires a careful analysis. The main strategy for proving Proposition 1.10 is to establish a good approximation argument for the perturbed NLW (1.33) and the cubic NLW (1.1) and then to invoke the norm inflation for the latter equation. For this purpose, we need to have local well-posedness of the perturbed NLW (1.33) for a sufficiently long time. In [53], Thomann and the first author proved almost sure local well-posedness of (1.33) via the Strichartz estimates and Lemma 2.6. Due to the use of the space-time estimates, such an argument provides a rather short local existence time, which is not sufficient for our purpose. In order to observe the desired growth for norm inflation, we need to *maximize* the local existence time by avoiding any use of space-times estimates such as the Strichartz estimates. Unfortunately, the local well-posedness argument based on Sobolev's inequality and the product estimates (Lemma 2.3) within the framework of the L^2-based Sobolev spaces (see [28]) or the Wiener algebra (see Section 5) does not seem to suffice for our purpose. We instead establish local well-posedness of the perturbed NLW (1.33) in a carefully chosen Fourier–Lebesgue space, which provides a sufficiently large time of local existence *and* allows us to implement an approximation argument. See Section 6 for details.

1.6. *Norm inflation for the* (*unrenormalized*) *NLW in negative Sobolev spaces*

In this subsection, we change gears and consider the following (deterministic) NLW:

$$\begin{cases} \partial_t^2 u + (m - \Delta)u + u^k = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), \end{cases} \qquad (x,t) \in \mathcal{M} \times \mathbb{R}, \tag{1.34}$$

where $m \ge 0$ and $\mathcal{M} = \mathbb{T}^d$ or $\mathbb{R}^d$. When $m = 0$, the equation (1.34) on $\mathbb{R}^d$ enjoys the scaling symmetry, which induces the so-called scaling critical Sobolev index: $s_{\text{scaling}} = \frac{d}{2} - \frac{2}{k-1}$. On the other hand, NLW also enjoys the Lorentzian invariance (conformal symmetry), which yields its own critical regularity $s_{\text{conf}} = \frac{d+1}{4} - \frac{1}{k-1}$ (at least in the focusing case); see [37] and [62, Exercise 3.67]. We then define the critical regularity s_{crit} for a given integer $k \ge 2$ by

$$s_{\text{crit}} \stackrel{\text{def}}{=} \max(s_{\text{scaling}}, s_{\text{conf}}, 0) = \max\left(\frac{d}{2} - \frac{2}{k-1}, \frac{d+1}{4} - \frac{1}{k-1}, 0\right). \tag{1.35}$$

The Cauchy problem (1.34) has been studied extensively and it is known that (1.34) is locally well-posed in $\mathcal{H}^s(\mathcal{M})$ for $s \ge s_{\text{crit}}$ in many cases (possibly under an extra condition); see [33,34,37,61].

On the other hand, ill-posedness of (1.34) below the critical regularity s_{crit} has been studied in various papers [11,18, 37,64,66]. In particular, Christ, Colliander, and Tao [18] proved the following norm inflation phenomenon for NLW (1.34) on $\mathbb{R}^d$; given any $\varepsilon > 0$, there exist a solution u_ε to (1.34) on $\mathbb{R}^d$ and $t_\varepsilon \in (0, \varepsilon)$ such that

$$\big\| \big(u_\varepsilon(0), \partial_t u_\varepsilon(0)\big) \big\|_{\mathcal{H}^s(\mathbb{R}^d)} < \varepsilon \quad \text{but} \quad \big\| u_\varepsilon(t_\varepsilon) \big\|_{H^s(\mathbb{R}^d)} > \varepsilon^{-1}, \tag{1.36}$$

provided that one of the following conditions holds:

$$\text{(a) } 0 < s < s_{\text{scaling}} \quad \text{or} \quad s < -\frac{1}{2}, \quad \text{or} \quad \text{(b) } -\frac{1}{2} < s < s_{\text{sob}} \stackrel{\text{def}}{=} \frac{1}{2} - \frac{1}{k}. \tag{1.37}$$

In particular, when $k = 3$, the norm inflation holds except for $s = -\frac{1}{2}$.[10] We point out that, in [18, Corollary 7], the conditions (a) and (b) are obtained first for $d = 1$ ([18, Theorem 6]) and then extended for $d \ge 2$ by reducing the analysis to the one-dimensional case via the finite speed of propagation.

[10] While Theorem 4 in [18] claims a norm inflation for $s = -\frac{1}{2}$ when $d = 1$, their argument uses a scaling and hence seems to break down when $s = s_{\text{crit}} = -\frac{1}{2}$, contrary to their claim.

The norm inflation (1.36) is a stronger form of instability than discontinuity of the solution map (at the trivial function). In [66], Xia proved norm inflation based at general initial data (see (1.38) below) for NLW on $\mathbb{T}^3$ when $0 < s < s_{\text{scaling}}$. See also the lecture note [64] by the third author. We point out that norm inflation at general initial data can not be reduced to the one-dimensional setting and thus the conditions in (1.37) should be disregarded in the following discussion. In fact, without reducing the analysis to the one-dimensional setting, the argument in [18] yields norm inflation for

$$\text{(c) } d \geq 2 : 0 < s < s_{\text{scaling}} \quad \text{or} \quad s \leq -\frac{d}{2}, \quad \text{or} \quad \text{(d) } d = 1 : s < \frac{1}{6} \quad \text{and} \quad s \neq -\frac{1}{2},$$

leaving a gap $-\frac{d}{2} < s \leq 0$ for $d \geq 2$. See [18, Theorems 4 and 6].

In what follows, we only consider the cubic case ($k = 3$). See [23] for the general case, where Forlano and the second author extended our result (Theorem 1.11) to general $k \geq 2$. The next theorem establishes norm inflation at general initial data in negative Sobolev spaces.

Theorem 1.11. *Given $d \in \mathbb{N}$, let $\mathcal{M} = \mathbb{R}^d$ or $\mathbb{T}^d$. Let $k = 3$ and $m \geq 0$. Suppose that $s \in \mathbb{R}$ satisfies either* (i) $s \leq -\frac{1}{2}$ *when $d = 1$ or* (ii) $s < 0$ *when $d \geq 2$. Fix $(u_0, u_1) \in \mathcal{H}^s(\mathcal{M})$. Then, given any $\varepsilon > 0$, there exist a solution u_ε to* (1.34) *on $\mathcal{M}$ and $t_\varepsilon \in (0, \varepsilon)$ such that*

$$\big\| \big(u_\varepsilon(0), \partial_t u_\varepsilon(0)\big) - (u_0, u_1) \big\|_{\mathcal{H}^s(\mathcal{M})} < \varepsilon \quad \textit{but} \quad \big\| u_\varepsilon(t_\varepsilon) \big\|_{H^s(\mathcal{M})} > \varepsilon^{-1}. \tag{1.38}$$

When $(u_0, u_1) = 0$, Theorem 1.11 reduces to the usual norm inflation (based at the zero function) stated in (1.36). It follows from Theorem 1.11 that the solution map $\Phi : (u_0, u_1) \in \mathcal{H}^s(\mathcal{M}) \mapsto (u, \partial_t u) \in C([-T, T]; \mathcal{H}^s(\mathcal{M}))$ to the cubic NLW is discontinuous everywhere in $\mathcal{H}^s(\mathcal{M})$. Theorem 1.11 fills the regularity gap $s \neq -\frac{1}{2}$ left open in [18] for the usual norm inflation in the case of the cubic nonlinearity ($k = 3$). Furthermore, our argument exploits a more robust high-to-low energy transfer mechanism than that in [18] and yields a norm inflation *without* reducing the analysis to the one-dimensional setting, which is crucial for proving norm inflation at general initial data.

The proof of Theorem 1.11 is a basic building block for proving Proposition 1.10 on almost sure norm inflation for the perturbed NLW (1.33). While the argument in [11,18,66] is based on the (dispersionless) ODE approach and an approximation argument, we adapt the Fourier analytic approach employed in [42], where the first author proved an analogous norm inflation at general initial data for the cubic nonlinear Schrödinger equation on $\mathbb{R}^d$ and $\mathbb{T}^d$ in negative Sobolev spaces. The main idea is to exploit high-to-low energy transfer in the Picard second iterate. We refer readers to the previous works [1,16,32,35,57], where a similar approach has been taken. We also mention the work [4,22] which exploits high-to-low energy transfer.

Let us briefly describe the idea of the proof of Theorem 1.11. By a density argument, we may assume that $(u_0, u_1) \in \mathcal{S}(\mathcal{M}) \times \mathcal{S}(\mathcal{M})$, where $\mathcal{S}(\mathcal{M})$ denotes the class of Schwartz functions if $\mathcal{M} = \mathbb{R}^d$ and the class of C^∞-functions if $\mathcal{M} = \mathbb{T}^d$. See Proposition 5.1 below. Then, the main goal is to construct a pair $(\phi_{0,\varepsilon}, \phi_{1,\varepsilon}) \in C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M})$, $\varepsilon > 0$, such that a solution u_ε to (1.34) with initial data $(u_{0,\varepsilon}, u_{1,\varepsilon}) = (u_0, u_1) + (\phi_{0,\varepsilon}, \phi_{1,\varepsilon})$ satisfies the conclusion of Theorem 1.11.

By expressing u_ε in the Duhamel formulation (with $m = 1$), we have

$$u_\varepsilon(t) = S(t)(u_{0,\varepsilon}, u_{1,\varepsilon}) - \int_0^t \frac{\sin((t - t')\langle\nabla\rangle)}{\langle\nabla\rangle} u_\varepsilon^3\big(t'\big)\, dt'.$$

As in [42], the main ingredient is to express a smooth solution u_ε in the following power series expansion:

$$u_\varepsilon = \sum_{j=0}^\infty \Xi_j(u_{0,\varepsilon}, u_{1,\varepsilon}),$$

where $\Xi_j(u_{0,\varepsilon}, u_{1,\varepsilon})$ denotes homogeneous multilinear terms of degree $2j + 1$ (in the linear solution $S(t)(u_{0,\varepsilon}, u_{1,\varepsilon})$). We then construct $(\phi_{0,\varepsilon}, \phi_{1,\varepsilon})$ such that, as $\varepsilon \to 0$,

(i) $(\phi_{0,\varepsilon}, \phi_{1,\varepsilon})$ tends to 0 in $\mathcal{H}^s(\mathcal{M})$,
(ii) the second order term $\Xi_1(u_{0,\varepsilon}, u_{1,\varepsilon})(t_\varepsilon)$ tends to ∞ for some $t_\varepsilon \to 0$,
(iii) the sum of the higher ordered terms $\Xi_j(u_{0,\varepsilon}, u_{1,\varepsilon})(t_\varepsilon)$, $j \geq 2$, is of smaller order than the second order term $\Xi_1(u_{0,\varepsilon}, u_{1,\varepsilon})(t_\varepsilon)$.

This yields the conclusion of Theorem 1.11. We remark that, in [16,32,35,57], Ξ_j was defined in a recursive manner and the (scaled) modulation space $M_{2,1}(\mathcal{M})$ and its algebra property played an important role. In the following, however, we follow a simplified approach presented in [42] and directly define Ξ_j via the power series expansion indexed by trees and use the Wiener algebra $\mathcal{F}L^1(\mathcal{M})$ instead of the modulation space. This latter approach is more suitable for proving norm inflation at general initial data.

1.7. *Remarks and comments*

We conclude this introduction by several remarks.

(i) In the main results (Theorems 1.6 and 1.7), we only considered the cubic case. It is easy to see that Theorem 1.6 is readily extendable to the case $k \geq 5$. Our method for proving Theorem 1.11 on the norm inflation at general initial data is elementary and can be applied to other power-type nonlinearities. Following this paper, the second author and Forlano [23] recently established an analogous norm inflation result for (1.34) with a general power-type nonlinearity u^k. It is likely that Theorem 1.7 can also be extended for $k \geq 5$. We point out, however, that a careful analysis (beyond establishing norm inflation at general initial data) is needed in proving an analogue of Proposition 1.10. See Section 6.

(ii) The defocusing nature of the equation is needed only in obtaining global solutions (Theorems A and Proposition 1.4).

(iii) Consider the cubic nonlinear Schrödinger equation (NLS) on $\mathbb{T}^d$:

$$i\partial_t u - \Delta u + |u|^2 u = 0. \tag{1.39}$$

In this case, we can introduce a renormalization in a deterministic manner:

$$i\partial_t u - \Delta u + \left(|u|^2 - 2\int |u|^2\, dx \right) u = 0 \tag{1.40}$$

to study the dynamics with either random or deterministic initial data of low regularity. See [6,19,29,51,56]. Thanks to the L^2-conservation, the equations (1.39) and (1.40) are equivalent, at least for smooth solutions, via the invertible gauge transform: $u \mapsto e^{2it\int |u|^2 dx} u$. Furthermore, in the case of Gaussian random initial data (under some regularity restriction), the renormalized equation (1.40) is equivalent to the renormalized equation via the Wick renormalization (as in (1.19) but in the complex-valued setting); see [6,51,52]. We point out that a deterministic renormalization as in (1.40) has also been used to study the fractional NLS; see [54,55].

In case of the cubic NLW, it is tempting to consider a deterministic renormalization analogous to (1.40):

$$\partial_t^2 u + (1-\Delta)u + \left(u^2 - 3\int u^2\, dx \right) u = 0. \tag{1.41}$$

Denoting the nonlinearity in (1.41) by $f(u)$, its spatial Fourier transform is written as

$$\widehat{f(u)}(n) = \sum_{\substack{n = n_1+n_2+n_3 \\ (n_1+n_2)(n_2+n_3)(n_3+n_1)\neq 0}} \prod_{j=1}^{3} \widehat{u}(n_j) - 3\big|\widehat{u}(n)\big|^2 \widehat{u}(n) + \mathbf{1}_{n=0}\big(\widehat{u}(0)\big)^3. \tag{1.42}$$

This renormalization cancels certain resonant interactions ($n_j + n_k = 0$ for $j \neq k$), which allows us to make sense of $f(u)$ for u of the form (1.20) with z as in (1.8) and smoother v. Indeed, the problematic terms z^3 and $3z^2 v$ in $(z+v)^3$ are now modified into $z^3 - 3\int z^2\, dx \cdot z$ and $3(z^2 - \int z^2)v$, each of which has a well-defined meaning.

There are, however, two issues in using the renormalized model (1.42). Unlike the cubic NLS, the renormalized model (1.42) is not naturally associated with the unrenormalized model (1.1) with $k = 3$ in the sense that it is not equivalent to the unrenormalized model even for smooth solutions, in particular, due to the lack of the L^2-conservation for NLW. The second point is that the renormalized model (1.42) possesses finite-time blowup solutions,[11] whereas the Wick ordered NLW (1.18) is almost surely globally well-posed; see Theorem A and Proposition 1.4. See also [28].

(iv) The main results of this paper are readily applicable to the two-dimensional stochastic NLW with space-time white noise forcing studied in [26,28,43]. Moreover, our work provides a natural framework for obtaining similar non-uniqueness results for singular stochastic PDEs. For instance, it would be interesting to establish an analogue of Theorem 1.7 in the context of the stochastic wave equations in higher dimensions [7,8,45,46] and the stochastic heat equations [15,20,30,38]. We mention a recent work [31] on the stochastic Navier–Stokes equations.

This remaining part of the paper is organized as follows. In Section 2, we collect some deterministic and stochastic lemmas. In Section 3, we prove Proposition 1.4. In Section 4, we show the convergence and uniqueness of Wick powers and then present the proof of Theorem 1.6. In Section 5, we prove norm inflation at general initial data for the deterministic

[11] For a function u independent of the spatial variable, the defocusing "renormalized" nonlinearity in (1.41) becomes the focusing (unrenormalized) nonlinearity: $(u^2 - 3\int u^2\, dx)u = -2u^3$, showing that there exists a finite time blowup solution $u(t) \sim \sqrt{2}(T_* - t)^{-1}$ in the sense of asymptotic equality as $t - T_*-$.

cubic NLW (1.34) with $k=3$ (Theorem 1.11). In Section 6, we first establish local well-posedness of the perturbed NLW (1.33) in a carefully chosen Fourier–Lebesgue space (Lemma 6.1) and an approximation lemma (Lemma 6.2), which implies Proposition 1.10. In the Appendix, we present the proof of the almost sure convergence of stochastic objects (Proposition 2.7).

2. Deterministic and stochastic lemmas

2.1. *Hermite polynomials and white noise functional*

First, we recall the Hermite polynomials $H_k(x;\sigma)$ defined via the generating function:

$$F(t,x;\sigma) \overset{\text{def}}{=} e^{tx-\frac{1}{2}\sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x;\sigma). \tag{2.1}$$

For simplicity, we set $F(t,x) \overset{\text{def}}{=} F(t,x;1)$ and $H_k(x) \overset{\text{def}}{=} H_k(x;1)$. Note that $H_k(x;\sigma) = \sigma^{\frac{k}{2}} H_k(\sigma^{-\frac{1}{2}}x)$ holds. In the following, we list the first few Hermite polynomials for readers' convenience:

$$H_0(x;\sigma)=1, \qquad H_1(x;\sigma)=x, \qquad H_2(x;\sigma)=x^2-\sigma, \qquad H_3(x;\sigma)=x^3-3\sigma x. \tag{2.2}$$

For the derivative, the following properties hold:

$$\partial_x H_k(x) = kH_{k-1}(x) \quad \text{and} \quad H_k(x) = xH_{k-1}(x) - \partial_x H_{k-1}(x). \tag{2.3}$$

Next, we define the white noise functional. Let $\xi(x;\omega)$ be the (real-valued) mean-zero Gaussian white noise on $\mathbb{T}^2$ defined by

$$\xi(x;\omega) = \sum_{n\in\mathbb{Z}^2} g_n(\omega) e^{in\cdot x},$$

where $\{g_n\}_{n\in\mathbb{Z}^2}$ is a sequence of independent standard complex-valued Gaussian random variables conditioned that $g_{-n} = \overline{g_n}$, $n\in\mathbb{Z}^2$. It is easy to see that $\xi \in \mathcal{H}^s(\mathbb{T}^2)\setminus\mathcal{H}^{-1}(\mathbb{T}^2)$, $s<-1$, almost surely. In particular, ξ is a distribution, acting on smooth functions. In fact, the action of ξ can be defined on $L^2(\mathbb{T}^2)$. We define the white noise functional $W_{(\cdot)} : L^2(\mathbb{T}^2) \to L^2(\Omega)$ by

$$W_f(\omega) = \langle f, \xi(\omega)\rangle_{L^2} = \sum_{n\in\mathbb{Z}^2} \widehat{f}(n)\overline{g_n}(\omega) \tag{2.4}$$

for a real-valued function $f\in L^2(\mathbb{T}^2)$. Note that $W_f = \xi(f)$ is basically the Wiener integral of f. In particular, W_f is a real-valued Gaussian random variable with mean 0 and variance $\|f\|_{L^2}^2$. Moreover, $W_{(\cdot)}$ is unitary:

$$\mathbb{E}[W_f W_h] = \langle f, h\rangle_{L^2} \tag{2.5}$$

for $f,h\in L^2(\mathbb{T}^2)$. In general, we have the following lemma. See [41, Lemma 1.1.1].

Lemma 2.1. (i) *Let g_1 and g_2 be mean-zero real-valued jointly Gaussian random variables with variances σ_1 and σ_2. Then, we have*

$$\mathbb{E}\big[H_k(g_1;\sigma_1)H_m(g_2;\sigma_2)\big] = \delta_{km}k!\big\{\mathbb{E}[g_1g_2]\big\}^k.$$

(ii) *Let $f,h\in L^2(\mathbb{T}^2)$ such that $\|f\|_{L^2} = \|h\|_{L^2} = 1$. Then, for $k,m\in\mathbb{N}\cup\{0\}$, we have*

$$\mathbb{E}\big[H_k(W_f)H_m(W_h)\big] = \delta_{km}k!\big[\langle f,h\rangle_{L^2}\big]^k.$$

Here, δ_{km} denotes the Kronecker's delta function.

Part (i) of Lemma 2.1 easily follows from the definition (2.1) of the generating function:

$$\mathbb{E}\big[F(t,g_1;\sigma_1)F(s,g_2;\sigma_2)\big]=\sum_{k,m=0}^{\infty}\frac{t^k}{k!}\frac{s^m}{m!}\mathbb{E}\big[H_k(g_1;\sigma_1)H_m(g_2;\sigma_2)\big],$$

while Part (ii) is an immediate corollary of Part (i) and (2.5).

As in [53], we also employ the white noise functional adapted to $z(t)$. In view of (1.9), we define the white noise functional $W^t_{(\cdot)}: L^2(\mathbb{T}^2)\to L^2(\Omega)$ with a parameter $t\in\mathbb{R}$ by

$$W^t_f(\omega)=\big\langle f,\xi^t(\omega)\big\rangle_{L^2}=\sum_{n\in\mathbb{Z}^2}\widehat{f}(n)\overline{g^t_{0,n}}(\omega). \tag{2.6}$$

Here, ξ^t denotes (a specific realization of) the white noise on $\mathbb{T}^2$ given by

$$\xi^t(x;\omega)=\sum_{n\in\mathbb{Z}^2}g^t_{0,n}(\omega)e^{in\cdot x},$$

where $g^t_{0,n}$ is defined in (1.10). Since $\{g^t_{0,n}\}_{n\in\mathbb{Z}^2}$ is a sequence of independent standard Gaussian random variables with $g^t_{0,-n}=\overline{g^t_{0,n}}$, the white noise functional $W^t_{(\cdot)}$ defined in (2.6) satisfies the same properties as the standard white noise functional $W_{(\cdot)}$ defined in (2.4). Moreover, we have the following lemma.

Lemma 2.2. *Let $f,h\in L^2(\mathbb{T}^2)$ such that $\|f\|_{L^2}=\|h\|_{L^2}=1$. Then, for $k,m\in\mathbb{N}\cup\{0\}$ and $t_1,t_2\in\mathbb{R}$, we have*

$$\mathbb{E}\big[H_k\big(W^{t_1}_f\big)H_m\big(W^{t_2}_h\big)\big]=\delta_{km}k!\,\big(\mathfrak{I}(f,h)[t_1-t_2]\big)^k, \tag{2.7}$$

where

$$\mathfrak{I}(f,h)[t]=\sum_{n\in\mathbb{Z}^2}\widehat{f}(n)\overline{\widehat{h}(n)}\cos\big(t\langle n\rangle\big).$$

While Lemma 2.2 follows from a similar argument as in the proof of Lemma 3.4 in [50], for readers' convenience, we provide a proof here.

Proof. From (2.6) with (1.10), we have

$$\begin{aligned}W^{t_1}_f(\omega)+W^{t_2}_h(\omega)&=\sum_{n\in\mathbb{Z}^2}\Big\{\big(\widehat{f}(n)\cos\big(t_1\langle n\rangle\big)+\widehat{h}(n)\cos\big(t_2\langle n\rangle\big)\big)\overline{g_{0,n}(\omega)}\\&\qquad+\big(\widehat{f}(n)\sin\big(t_1\langle n\rangle\big)+\widehat{h}(n)\sin\big(t_2\langle n\rangle\big)\big)\overline{g_{1,n}(\omega)}\Big\}\\&=\sum_{n\in\mathbb{Z}^2}\Big\{\operatorname{Re}\big(\widehat{f}(n)\cos\big(t_1\langle n\rangle\big)+\widehat{h}(n)\cos\big(t_2\langle n\rangle\big)\big)\operatorname{Re}g_{0,n}(\omega)\\&\qquad+\operatorname{Im}\big(\widehat{f}(n)\cos\big(t_1\langle n\rangle\big)+\widehat{h}(n)\cos\big(t_2\langle n\rangle\big)\big)\operatorname{Im}g_{0,n}(\omega)\\&\qquad+\operatorname{Re}\big(\widehat{f}(n)\sin\big(t_1\langle n\rangle\big)+\widehat{h}(n)\sin\big(t_2\langle n\rangle\big)\big)\operatorname{Re}g_{1,n}(\omega)\\&\qquad+\operatorname{Im}\big(\widehat{f}(n)\sin\big(t_1\langle n\rangle\big)+\widehat{h}(n)\sin\big(t_2\langle n\rangle\big)\big)\operatorname{Im}g_{1,n}(\omega)\Big\},\end{aligned}$$

where the second equality follows from (1.11) and the fact that f and h are real-valued. Since $\operatorname{Re}g_{j,n}$ and $\operatorname{Im}g_{j,n}$ are independent Gaussian random variables with mean 0 and variance $\frac{1}{2}$ for $n\neq 0$ (1 if $n=0$) and $g_{j,-n}=\overline{g_{j,n}}$, we have

$$\int_\Omega e^{tW^{t_1}_f(\omega)}e^{sW^{t_2}_h(\omega)}\,dP(\omega)=e^{\frac{1}{2}(t^2\|f\|^2_{L^2}+s^2\|h\|^2_{L^2}+2\mathfrak{I}(f,h)[t_1-t_2])}$$

for any $t,s\in\mathbb{R}$, where once again we used the fact that f and h are real-valued.

Let F be as in (2.1). Then, for any $t, s \in \mathbb{R}$ and $f, h \in L^2(\mathbb{T}^2)$ with $\|f\|_{L^2} = \|h\|_{L^2} = 1$, we have

$$\begin{aligned} \int_\Omega F\big(t, W_f^{t_1}(\omega)\big) F\big(s, W_h^{t_2}(\omega)\big)\, dP(\omega) &= e^{-\frac{t^2+s^2}{2}} \int_\Omega e^{tW_f^{t_1}(\omega)+sW_h^{t_2}(\omega)}\, dP(\omega) \\ &= e^{ts\Im(f,h)[t_1-t_2]}. \end{aligned} \tag{2.8}$$

Thus, it follows from (2.1) and (2.8) that

$$e^{ts\Im(f,h)[t_1-t_2]} = \sum_{k,m=0}^{\infty} \frac{t^k s^m}{k!m!} \int_\Omega H_k\big(W_f^{t_1}(\omega)\big) H_m\big(W_h^{t_2}(\omega)\big)\, dP(\omega).$$

By comparing the coefficients of $t^k s^m$, we obtain (2.7). □

2.2. *Product estimates*

We recall the following product estimates. See [26] for the proof.

Lemma 2.3. *Let* $0 \le \alpha \le 1$.

(i) *Suppose that* $1 < p_j, q_j, r < \infty$, $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}$, $j = 1, 2$. *Then, we have*

$$\big\|\langle\nabla\rangle^\alpha (fg)\big\|_{L^r(\mathbb{T}^d)} \lesssim \big(\|f\|_{L^{p_1}(\mathbb{T}^d)} \big\|\langle\nabla\rangle^\alpha g\big\|_{L^{q_1}(\mathbb{T}^d)} + \big\|\langle\nabla\rangle^\alpha f\big\|_{L^{p_2}(\mathbb{T}^d)} \|g\|_{L^{q_2}(\mathbb{T}^d)}\big).$$

(ii) *Suppose that* $1 < p, q, r < \infty$ *satisfy the scaling condition*: $\frac{1}{p} + \frac{1}{q} \le \frac{1}{r} + \frac{\alpha}{d}$. *Then, we have*

$$\big\|\langle\nabla\rangle^{-\alpha} (fg)\big\|_{L^r(\mathbb{T}^d)} \lesssim \big\|\langle\nabla\rangle^{-\alpha} f\big\|_{L^p(\mathbb{T}^d)} \big\|\langle\nabla\rangle^\alpha g\big\|_{L^q(\mathbb{T}^d)}.$$

Note that while Lemma 2.3 (ii) was shown only for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{\alpha}{d}$ in [26], the general case $\frac{1}{p} + \frac{1}{q} \le \frac{1}{r} + \frac{\alpha}{d}$ follows from the inclusion $L^{r_1}(\mathbb{T}^d) \subset L^{r_2}(\mathbb{T}^d)$ for $r_1 \ge r_2$.

2.3. *Tools from stochastic analysis*

We use the short-hand notation $L_T^q L_x^r = L^q([-T, T]; L^r(\mathbb{T}^2))$ for $T > 0$ and $1 \le q, r \le \infty$, etc. Thanks to the randomization of the initial data, the following probabilistic Strichartz estimates hold.

Lemma 2.4. *Given* $(\phi_0, \phi_1) \in \mathcal{H}^0(\mathbb{T}^2)$, *let* $(\phi_0^\omega, \phi_1^\omega)$ *be its randomization defined in* (1.25). (i) *Given* $2 \le q < \infty$ *and* $2 \le r < \infty$, *there exist* $C, c > 0$ *such that*

$$P\big(\big\|S(t)\big(\phi_0^\omega, \phi_1^\omega\big)\big\|_{L_T^q L_x^r} > \lambda\big) \le C \exp\bigg(-c \frac{\lambda^2}{T^{\frac{2}{q}} \|(\phi_0, \phi_1)\|_{\mathcal{H}^0}^2}\bigg)$$

for any $T > 0$ *and* $\lambda > 0$.

(ii) *Let* $s > 0$ *and* $(\phi_0, \phi_1) \in \mathcal{H}^s(\mathbb{T}^2)$. *Then, given* $2 \le r \le \infty$, *there exist* $C, c > 0$ *such that*

$$P\big(\big\|S(t)\big(\phi_0^\omega, \phi_1^\omega\big)\big\|_{L_T^\infty L_x^r} > \lambda\big) \le C(1+T) \exp\bigg(-c \frac{\lambda^2}{\max(1, T^2) \|(\phi_0, \phi_1)\|_{\mathcal{H}^s}^2}\bigg)$$

for any $T > 0$ *and* $\lambda > 0$.

The probabilistic Strichartz estimate in (i) of Lemma 2.4 is proved in [2,13,19]. See [9,47] for (ii) of Lemma 2.4. While the (deterministic) Strichartz estimate holds only for admissible pairs (see [24,34,37,62]), Lemma 2.4 states that the randomization allows us to take a wide range of exponents.

Next, we recall the Wiener chaos estimates. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of independent standard Gaussian random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is the σ-algebra generated by this sequence. Given $k \in \mathbb{N} \cup \{0\}$, we define the homogeneous Wiener chaoses $\mathcal{H}_k$ to be the closure (under $L^2(\Omega)$) of the span of Fourier–Hermite polynomials $\prod_{n=1}^\infty H_{k_n}(g_n)$, where H_j is the Hermite polynomial of degree j and $k = \sum_{n=1}^\infty k_n$.[12] Then, we have the following Ito–Wiener decomposition:

$$L^2(\Omega, \mathcal{F}, P) = \bigoplus_{k=0}^\infty \mathcal{H}_k.$$

See Theorem 1.1.1 in [41]. We also set

$$\mathcal{H}_{\le k} = \bigoplus_{j=0}^k \mathcal{H}_j$$

for $k \in \mathbb{N}$.

Then, as a consequence of the hypercontractivity of the Ornstein–Uhlenbeck semigroup $U(t) = e^{tL}$ due to Nelson [40], we have the following Wiener chaos estimate [59, Theorem I.22]. See also [63, Proposition 2.4].

Lemma 2.5. *Let $k \in \mathbb{N}$. Then, we have*

$$\|X\|_{L^p(\Omega)} \le (p-1)^{\frac{k}{2}} \|X\|_{L^2(\Omega)}$$

for any $p \ge 2$ and any $X \in \mathcal{H}_{\le k}$.

Note that $:z^\ell(t):$ defined in (1.14) belongs to $\mathcal{H}_{\le \ell}$ for $\ell \in \mathbb{N}$. By using the white noise functional defined in (2.6) and Lemma 2.5, Thomann and the first author [53] proved the following estimate on Wick powers.

Lemma 2.6. *Let $\ell \in \mathbb{N} \cup \{0\}$. Then, given $2 \le q, r < \infty$ and $\varepsilon > 0$, there exists $C, c > 0$ such that*

$$P\Big(\big\|\langle\nabla\rangle^{-\varepsilon} :z^\ell:\big\|_{L^q_T L^r_x} > \lambda\Big) \le C \exp\bigg(-c\frac{\lambda^{\frac{2}{\ell}}}{T^{\frac{2}{q\ell}}}\bigg)$$

for any $T > 0$ and $\lambda > 0$.

Note that an analogous estimate also holds even when $q = r = \infty$; see [28].

We conclude this section by stating a proposition useful for studying regularities of stochastic objects. We say that a stochastic process $X : \mathbb{R}_+ \to \mathcal{D}'(\mathbb{T}^d)$ is spatially homogeneous if $\{X(\cdot, t)\}_{t\in\mathbb{R}_+}$ and $\{X(x_0 + \cdot, t)\}_{t\in\mathbb{R}_+}$ have the same law for any $x_0 \in \mathbb{T}^d$. Given $h \in \mathbb{R}$, we define the difference operator δ_h by setting

$$\delta_h X(t) = X(t+h) - X(t). \tag{2.9}$$

Proposition 2.7. *Let $\{X_N\}_{N\in\mathbb{N}}$ and X be spatially homogeneous stochastic processes $: \mathbb{R}_+ \to \mathcal{D}'(\mathbb{T}^d)$. Suppose that there exists $k \in \mathbb{N}$ such that $X_N(t)$ and $X(t)$ belong to $\mathcal{H}_{\le k}$ for each $t \in \mathbb{R}_+$.*

(i) *Let $t \in \mathbb{R}_+$. If there exists $s_0 \in \mathbb{R}$ such that*

$$\mathbb{E}\Big[\big|\widehat{X}(n,t)\big|^2\Big] \lesssim \langle n\rangle^{-d-2s_0} \tag{2.10}$$

for any $n \in \mathbb{Z}^d$, then we have $X(t) \in W^{s,\infty}(\mathbb{T}^d)$, $s < s_0$, almost surely. Furthermore, if there exists $\gamma > 0$ such that

$$\mathbb{E}\Big[\big|\widehat{X}_N(n,t) - \widehat{X}(n,t)\big|^2\Big] \lesssim N^{-\gamma}\langle n\rangle^{-d-2s_0} \tag{2.11}$$

for any $n \in \mathbb{Z}^d$ and $N \ge 1$, then $X_N(t)$ converges to $X(t)$ in $W^{s,\infty}(\mathbb{T}^d)$, $s < s_0$, almost surely.

(ii) *Let $T > 0$ and suppose that* (i) *holds on $[0, T]$. If there exists $\theta \in (0,1)$ such that*

$$\mathbb{E}\Big[\big|\delta_h \widehat{X}(n,t)\big|^2\Big] \lesssim \langle n\rangle^{-d-2s_0+\theta}|h|^\theta, \tag{2.12}$$

[12] This implies that $k_n = 0$ except for finitely many n's.

for any $n \in \mathbb{Z}^d$, $t \in [0, T]$, *and* $h \in [-1, 1]$,[13] *then we have* $X \in C([0, T]; W^{s,\infty}(\mathbb{T}^d))$, $s < s_0 - \frac{\theta}{2}$, *almost surely. Furthermore, if there exists* $\gamma > 0$ *such that*

$$\mathbb{E}\big[\big|\delta_h \widehat{X}_N(n,t) - \delta_h \widehat{X}(n,t)\big|^2\big] \lesssim N^{-\gamma} \langle n\rangle^{-d-2s_0+\theta} |h|^\theta, \tag{2.13}$$

for any $n \in \mathbb{Z}^d$, $t \in [0, T]$, $h \in [-1, 1]$, *and* $N \geq 1$, *then* X_N *converges to* X *in* $C([0, T]; W^{s,\infty}(\mathbb{T}^d))$, $s < s_0 - \frac{\theta}{2}$, *almost surely.*

Proposition 2.7 follows from a straightforward application of the Wiener chaos estimate (Lemma 2.5). For the proof, see Proposition 3.6 in [39] and the Appendix. In particular, for the almost sure convergence claimed in Proposition 2.7, we need to proceed as in a standard proof of Kolmogorov'a continuity criterion; see the Appendix for details. See also Section 3 in [48].

As a corollary, we also have the following (See Remark A.3).

Corollary 2.8. *Let* $\{X_N\}_{N\in\mathbb{N}}$ *be a spatially homogeneous stochastic process* $: \mathbb{R}_+ \to \mathcal{D}'(\mathbb{T}^d)$. *Suppose that there exists* $k \in \mathbb{N}$ *such that* $X_N(t)$ *belongs to* $\mathcal{H}_{\leq k}$ *for each* $t \in \mathbb{R}_+$.

(i) *Let* $t \in \mathbb{R}_+$. *If there exist* $s_0 \in \mathbb{R}$ *and* $\gamma > 0$ *such that*

$$\mathbb{E}\big[\big|\widehat{X}_N(n,t)\big|^2\big] \lesssim \langle n\rangle^{-d-2s_0},$$
$$\mathbb{E}\big[\big|\widehat{X}_N(n,t) - \widehat{X}_M(n,t)\big|^2\big] \lesssim N^{-\gamma} \langle n\rangle^{-d-2s_0}$$

for any $n \in \mathbb{Z}^d$ *and* $M \geq N \geq 1$, *then* $X_N(t)$ *converges in* $W^{s,\infty}(\mathbb{T}^d)$, $s < s_0$, *almost surely.*

(ii) *Let* $T > 0$ *and suppose that* (i) *holds on* $[0, T]$. *If there exist* $\gamma > 0$ *and* $\theta \in (0, 1)$ *such that*

$$\mathbb{E}\big[\big|\delta_h \widehat{X}_N(n,t)\big|^2\big] \lesssim \langle n\rangle^{-d-2s_0+\theta} |h|^\theta,$$
$$\mathbb{E}\big[\big|\delta_h \widehat{X}_N(n,t) - \delta_h \widehat{X}_M(n,t)\big|^2\big] \lesssim N^{-\gamma} \langle n\rangle^{-d-2s_0+\theta} |h|^\theta,$$

for any $n \in \mathbb{Z}^d$, $t \in [0, T]$, $h \in [-1, 1]$, *and* $M \geq N \geq 1$, *then* X_N *converges in* $C([0, T]; W^{s,\infty}(\mathbb{T}^d))$, $s < s_0 - \frac{\theta}{2}$, *almost surely.*

Proposition 2.7 and Corollary 2.8 have been useful widely in the recent study of singular stochastic PDEs; see for example, [27,28,43,45,46].

3. Global existence of smooth solutions for the renormalized NLW

In this section, we present the proof of Proposition 1.4. We point out that, thanks to the Cameron–Martin theorem [14], we can assume that $r_0 = r_1 = 0$. See also [49]. Hence, it suffices to study

$$\partial_t^2 v + (1 - \Delta)v + H_k\big(S(t)\big(\phi_0^\omega, \phi_1^\omega\big) + v(t); \sigma(t)\big) = 0 \tag{3.1}$$

with the zero initial data, where $\sigma(t)$ is defined by (1.27). In particular, it satisfies

$$\sigma(t) \lesssim \big\|(\phi_0, \phi_1)\big\|_{\mathcal{H}^0}^2 \quad \text{and} \quad \big|\partial_t \sigma(t)\big| \lesssim \big\|(\phi_0, \phi_1)\big\|_{\mathcal{H}^{\frac{1}{2}}}^2. \tag{3.2}$$

We first go over local well-posedness of (3.1). For this purpose, we consider the following deterministic perturbed cubic NLW:

$$\partial_t^2 v + (1 - \Delta)v + H_k\big(f(t) + v(t); \sigma(t)\big) = 0, \tag{3.3}$$

where f is a given deterministic function and $\sigma(t)$ satisfies (3.2).

[13] We impose $h \geq -t$ such that $t + h \geq 0$.

Lemma 3.1. *Let $k \geq 3$ be an odd integer, $(\phi_0, \phi_1) \in \mathcal{H}^0(\mathbb{T}^2)$, $(v_0, v_1) \in \mathcal{H}^1(\mathbb{T}^2)$, and $f \in L^k([t_0, t_0+1]; L^\infty(\mathbb{T}^2))$ for some $t_0 \in \mathbb{R}$. Suppose that there exist $R, \theta > 0$ such that*

$$\big\|(v_0, v_1)\big\|_{\mathcal{H}^1} \leq R \quad \text{and} \quad \|f\|_{L^k(I; L^\infty(\mathbb{T}^2))} \leq |I|^\theta \tag{3.4}$$

for any interval $I \subset [t_0, t_0+1]$. Then, there exist $\tau = \tau(R, \theta, \|(\phi_0, \phi_1)\|_{\mathcal{H}^0}) > 0$ and a unique solution $(v, \partial_t v) \in C([t_0, t_0+\tau]; \mathcal{H}^1(\mathbb{T}^2))$ to (3.3) with $(v, \partial_t v)|_{t=t_0} = (v_0, v_1)$.

Remark 3.2. We point out that the second condition in (3.4) can be weakened as follows. Let $\tau = \tau(R, \theta, \|(\phi_0, \phi_1)\|_{\mathcal{H}^0}) > 0$ be as in Lemma 3.1. If we assume

$$\|f\|_{L^k([t_0, t_0+\tau_*]; L^\infty(\mathbb{T}^2))} \leq \tau_*^\theta$$

for some $0 < \tau_* \leq \tau$ instead of the second condition in (3.4), then the conclusion of Lemma 3.1 still holds on $[t_0, t_0+\tau_*]$.

Proof of Lemma 3.1. Without loss of generality, we may assume $t_0 = 0$ and restrict our attention only to positive times. By writing (3.3) in the Duhamel formulation, we have

$$\begin{aligned} v(t) &= \Phi(v)(t) \\ &\overset{\text{def}}{=} S(t)(v_0, v_1) - \int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle} H_k\big(f(t') + v(t'); \sigma(t')\big)\, dt'. \end{aligned}$$

Let $\vec{\Phi}(v) = (\Phi(v), \partial_t \Phi(v))$ and $\vec{v} = (v, \partial_t v)$. Our goal is to show that $\vec{\Phi}$ is a contraction mapping in a suitable functional framework.

Let $0 < T \leq 1$. Then, it follows from (3.2), (3.4), and Sobolev's inequality that

$$\begin{aligned} \big\|H_k(f+v; \sigma)\big\|_{L^1_T L^2_x} &\leq \sum_{\ell=0}^k \binom{k}{\ell} \big\|H_\ell(f;\sigma) v^{k-\ell}\big\|_{L^1_T L^2_x} \\ &\leq \big\|v^k\big\|_{L^1_T L^2_x} + \sum_{\ell=1}^k \binom{k}{\ell} \big\|H_\ell(f;\sigma)\big\|_{L^1_T L^\infty_x} \big\|v^{k-\ell}\big\|_{L^\infty_T L^2_x} \\ &\lesssim T\|v\|^k_{L^\infty_T L^{2k}_x} + \sum_{\ell=1}^k \big(\|f\|^\ell_{L^\ell_T L^\infty_x} + T\|\sigma\|^{\frac{\ell}{2}}_{L^\infty_T}\big)\|v\|^{k-\ell}_{L^\infty_T L^{2(k-\ell)}_x} \\ &\lesssim T\|v\|^k_{L^\infty_T L^{2k}_x} + \sum_{\ell=1}^k \big(T^{\theta\ell} + T\big\|(\phi_0, \phi_1)\big\|^\ell_{\mathcal{H}^0}\big)\|v\|^{k-\ell}_{L^\infty_T L^{2(k-\ell)}_x} \\ &\lesssim T^{\theta'}\big(1 + \big\|(\phi_0, \phi_1)\big\|^k_{\mathcal{H}^0} + \|v\|^k_{L^\infty_T H^1_x}\big), \end{aligned}$$

where $\theta' = \min(\theta, 1) > 0$. Hence, we have

$$\begin{aligned} \big\|\vec{\Phi}(v)\big\|_{L^\infty_T \mathcal{H}^1_x} &\leq \big\|(v_0, v_1)\big\|_{\mathcal{H}^1} + \big\|H_k(f+v;\sigma)\big\|_{L^1_T L^2_x} \\ &\leq R + CT^{\theta'}\big(1 + \big\|(\phi_0, \phi_1)\big\|^k_{\mathcal{H}^0} + \|v\|^k_{L^\infty_T H^1_x}\big). \end{aligned}$$

A similar computation yields the difference estimate:

$$\begin{aligned} &\big\|\vec{\Phi}(v_1) - \vec{\Phi}(v_2)\big\|_{L^\infty_T \mathcal{H}^1_x} \\ &\quad\leq \big\|H_k(f+v_1;\sigma) - H_k(f+v_2;\sigma)\big\|_{L^1_T L^2_x} \\ &\quad\leq CT^{\theta'}\big(1 + \big\|(\phi_0, \phi_1)\big\|^{k-1}_{\mathcal{H}^0} + \|v_1\|^{k-1}_{L^\infty_T H^1_x} + \|v_2\|^{k-1}_{L^\infty_T H^1_x}\big)\|v_1 - v_2\|_{L^\infty_T H^1_x}. \end{aligned}$$

By taking τ as

$$\tau \sim \left(\frac{\min(1, R)}{1 + \|(\phi_0, \phi_1)\|_{\mathcal{H}^0}^k + R^k}\right)^{\frac{1}{\theta'}},$$

we see that $\vec{\Phi}$ is a contraction mapping on the ball $B_{2R} = \{\vec{v} \in C([0, \tau]; H^1(\mathbb{T}^2)) : \|\vec{v}\|_{L^\infty_\tau \mathcal{H}^1_x} \le 2R\}$. Therefore, we obtain a unique[14] local solution $\vec{v} = (v, \partial_t v) \in C([0, \tau]; \mathcal{H}^1(\mathbb{T}^2))$. □

We now present the proof of Proposition 1.4.

Proof of Proposition 1.4. As in [3,19], it suffices to show the following "almost" almost global existence; given any $T, \varepsilon > 0$, there exists a set $\Omega_{T,\varepsilon} \subset \Omega$ such that $P(\Omega^c_{T,\varepsilon}) < \varepsilon$ and for each $\omega \in \Omega_{T,\varepsilon}$, there exists a solution $\vec{v} = (v, \partial_t v)$ to (3.1) on $[-T, T]$.

Let $z(t) = S(t)(\phi_0^\omega, \phi_1^\omega)$. Given $T, \varepsilon > 0$, we set

$$\Omega_{T,\varepsilon} = \big\{\omega \in \Omega : \|z\|_{L^\infty_{T,x}} + \big\|\langle\nabla\rangle^s \widetilde{z}\big\|_{L^{k+1}_{T,x}} \le M\big\},$$

where M is given by

$$M = M\big(T, \varepsilon, \big\|(\phi_0, \phi_1)\big\|_{\mathcal{H}^s}\big) \sim \langle T\rangle \big\|(\phi_0, \phi_1)\big\|_{\mathcal{H}^s} \left(\log \frac{\langle T\rangle}{\varepsilon}\right)^{\frac{1}{2}} \tag{3.5}$$

and $\widetilde{z}$ is defined by

$$\widetilde{z}(t) = -\sin\big(t\langle\nabla\rangle\big)\phi_0^\omega + \frac{\cos(t\langle\nabla\rangle)}{\langle\nabla\rangle}\phi_1^\omega.$$

Note that $\widetilde{z}$ also satisfies Lemma 2.4 and that

$$\partial_t z = \langle\nabla\rangle\widetilde{z}. \tag{3.6}$$

Then, it follows from Lemma 2.4 that

$$P\big(\Omega^c_{T,\varepsilon}\big) < \varepsilon.$$

We point out that the condition $s > 0$ is needed to apply Lemma 2.4 (ii).

As in [13,47], we use the energy $E(\vec{v}) = H(v, \partial_t v)$, where H is as in (1.3). Using the energy $E(\vec{v})$, we show that there exists $R = R(T, \varepsilon, \|(\phi_0, \phi_1)\|_{\mathcal{H}^s}) > 0$ such that

$$\big\|(v, \partial_t v)\big\|_{L^\infty_T \mathcal{H}^1_x} \le R \tag{3.7}$$

for any $\omega \in \Omega_{T,\varepsilon}$.

For now, let us assume (3.7) and conclude "almost" almost sure global existence. Given $\tau > 0$, we write

$$[-T, T] = \bigcup_{j=-[T/\tau]-1}^{[T/\tau]} \big[j\tau, (j+1)\tau\big] \cap [-T, T].$$

By making $\tau = \tau(M) = \tau(T, \varepsilon, \|(\phi_0, \phi_1)\|_{\mathcal{H}^s}) > 0$ small, we have

$$\|z\|_{L^k([j\tau,(j+1)\tau]; L^\infty(\mathbb{T}^2))} \le \tau^{\frac{1}{k}} M \le \tau^{\frac{1}{2k}}$$

for $\omega \in \Omega_{T,\varepsilon}$. By iteratively applying Lemma 3.1 and Remark 3.2, we can construct a solution $\vec{v}$ to (1.24) (with $r_0 = r_1 = 1$) on $[j\tau, (j+1)\tau]$, $j = -[\frac{T}{\tau}] - 1, \dots, [\frac{T}{\tau}]$. This proves the "almost" almost sure global existence.

[14] At this point, the uniqueness holds only in B_{2R} but by a standard continuity argument, we can extend the uniqueness to the entire $C([0, \tau]; \mathcal{H}^1(\mathbb{T}^2))$.

It remains to prove (3.7). We first consider the $k=3$ case. In this case, it follows from (3.1), (2.2), (3.2), and Hölder's and Young's inequalities that

$$\begin{aligned}
E\big(\vec{v}(t)\big) &= \int_0^t \int_{\mathbb{T}^2} \partial_t v \cdot \big(\partial_t^2 v + (1-\Delta)v + v^3\big)\,dx\,dt' \\
&= \int_0^t \int_{\mathbb{T}^2} \partial_t v \cdot \big(-H_3(z+v;\sigma) + v^3\big)\,dx\,dt' \\
&= \int_0^t \int_{\mathbb{T}^2} \partial_t v \cdot \big(-3zv^2 - 3\big(z^2-\sigma\big)v - z^3 + 3\sigma z\big)\,dx\,dt' \\
&\lesssim \int_0^t \big\|\partial_t v\big(t'\big)\big\|_{L^2_x}\Big\{\big\|z\big(t'\big)\big\|_{L^\infty_x}\big\|v\big(t'\big)\big\|^2_{L^4_x} \\
&\quad + \big(\big\|z\big(t'\big)\big\|^2_{L^8_x} + \big\|(\phi_0,\phi_1)\big\|^2_{\mathcal{H}^0}\big)\big\|v\big(t'\big)\big\|_{L^4_x} \\
&\quad + \big\|z\big(t'\big)\big\|^3_{L^6_x} + \big\|(\phi_0,\phi_1)\big\|^2_{\mathcal{H}^0}\big\|z\big(t'\big)\big\|_{L^2_x}\Big\}\,dt' \\
&\lesssim \big(1 + \|z\|_{L^\infty_T L^\infty_x}\big)\int_0^t E\big(\vec{v}\big(t'\big)\big)\,dt' + \|z\|^8_{L^8_{T,x}} + \big\|(\phi_0,\phi_1)\big\|^8_{\mathcal{H}^0}) \\
&\quad + \|z\|^6_{L^6_{T,x}} + \big\|(\phi_0,\phi_1)\big\|^4_{\mathcal{H}^0}\|z\|_{L^2_{T,x}} \\
&\lesssim (1+M)\int_0^t E\big(\vec{v}\big(t'\big)\big)\,dt' + C\big(T, M, \big\|(\phi_0,\phi_1)\big\|_{\mathcal{H}^0}\big) \qquad (3.8)
\end{aligned}$$

for $\omega \in \Omega_{T,\varepsilon}$. Hence, from Gronwall's inequality, we obtain (3.7) for $k=3$ and $s>0$.

Next, we consider the case $k \geq 5$. From (2.3), we have

$$\begin{aligned}
\partial_t H_\ell\big(z(x,t);\sigma(t)\big) &= \ell H_{\ell-1}\big(z(x,t);\sigma(t)\big)\partial_t z(x,t) \\
&\quad - \mathbf{1}_{\ell\geq 2}\cdot\frac{\ell(\ell-1)}{2}H_{\ell-2}\big(z(x,t);\sigma(t)\big)\partial_t\sigma(t). \qquad (3.9)
\end{aligned}$$

Then, from (3.1) and integration by parts with (3.9), we have

$$\begin{aligned}
E\big(\vec{v}(t)\big) &= \int_0^t \int_{\mathbb{T}^2} \partial_t v \cdot \big(\partial_t^2 v + (1-\Delta)v + v^k\big)\,dx\,dt' \\
&= \int_0^t \int_{\mathbb{T}^2} \partial_t v \cdot \big(-H_k(z+v;\sigma) + v^k\big)\,dx\,dt' \\
&= -\sum_{\ell=1}^k \binom{k}{\ell}\int_0^t \int_{\mathbb{T}^2} \partial_t v \cdot H_\ell(z;\sigma)v^{k-\ell}\,dx\,dt' \\
&= -\sum_{\ell=1}^k \binom{k}{\ell}\frac{1}{k-\ell+1}\bigg\{\int_{\mathbb{T}^2} H_\ell(z;\sigma)v^{k-\ell+1}\,dx\Big|_0^t \\
&\quad -\ell\int_0^t\int_{\mathbb{T}^2} H_{\ell-1}(z;\sigma)\partial_t z\cdot v^{k-\ell+1}\,dx\,dt' \\
&\quad + \mathbf{1}_{\ell\geq 2}\cdot\frac{\ell(\ell-1)}{2}\int_0^t\int_{\mathbb{T}^2} H_{\ell-2}(z;\sigma)\partial_t\sigma\cdot v^{k-\ell+1}\,dx\,dt'\bigg\}. \qquad (3.10)
\end{aligned}$$

From Young's inequality and (3.2), we have

$$\begin{aligned}
\bigg|\int_{\mathbb{T}^2} H_\ell(z;\sigma)v^{k-\ell+1}(t)\,dx\bigg| &\leq C(\delta)\big\|H_\ell\big(z(t);\sigma(t)\big)\big\|^{\frac{k+1}{\ell}}_{L_x^{\frac{k+1}{\ell}}} + \delta\big\|v(t)\big\|^{k+1}_{L^{k+1}_x} \\
&\leq C(\delta)\big(\big\|z(t)\big\|^\ell_{L^{k+1}_x} + \big\|(\phi_0,\phi_1)\big\|^{\frac{\ell}{2}}_{\mathcal{H}^0}\big)^{\frac{k+1}{\ell}} + \delta E\big(\vec{v}(t)\big)
\end{aligned}$$

$$\le C(\delta)\big(M^{k+1} + \|(\phi_0,\phi_1)\|_{\mathcal{H}^0}^{\frac{k+1}{2}}\big) + \delta E\big(\vec{v}(t)\big) \tag{3.11}$$

for $\omega \in \Omega_{T,\varepsilon}$ and $1 \le \ell \le k$, where $\delta > 0$ is a small constant to be chosen later. From (3.6) and Young's and Hölder's inequalities with (3.5), we have

$$\begin{aligned}
&\bigg|\int_0^t \int_{\mathbb{T}^2} H_{\ell-1}(z;\sigma)\partial_t z \cdot v^{k-\ell+1}\,dx\,dt'\bigg| = \bigg|\int_0^t \int_{\mathbb{T}^2} H_{\ell-1}(z;\sigma)\langle\nabla\rangle\widetilde{z} \cdot v^{k-\ell+1}\,dx\,dt'\bigg|\\
&\lesssim \int_0^t \big\|H_{\ell-1}\big(z(t');\sigma(t')\big)\langle\nabla\rangle\widetilde{z}(t')\big\|_{L_x^{\frac{k+1}{\ell}}}^{\frac{k+1}{\ell}} + \|v(t')\|_{L_x^{k+1}}^{k+1}\,dt'\\
&\lesssim \Big(\big(\|z(t)\|_{L_x^{k+1}}^{\ell-1} + \|(\phi_0,\phi_1)\|_{\mathcal{H}^0}^{\frac{\ell-1}{2}}\big)\|\langle\nabla\rangle\widetilde{z}(t)\|_{L_x^{k+1}}\Big)^{\frac{k+1}{\ell}} + \int_0^t E\big(\vec{v}(t')\big)\,dt'\\
&\lesssim C\big(T, M, \|(\phi_0,\phi_1)\|_{\mathcal{H}^0}\big) + \int_0^t E\big(\vec{v}(t')\big)\,dt'
\end{aligned} \tag{3.12}$$

for $\omega \in \Omega_{T,\varepsilon}$ and $1 \le \ell \le k$. Lastly, from Young's inequality and (3.2), we have

$$\begin{aligned}
&\bigg|\int_0^t \int_{\mathbb{T}^2} H_{\ell-2}(z;\sigma)\partial_t\sigma \cdot v^{k-\ell+1}\,dx\,dt'\bigg|\\
&\lesssim \|(\phi_0,\phi_1)\|_{\mathcal{H}^{\frac12}}^2 \int_0^t \big\|H_{\ell-2}\big(z(t');\sigma(t')\big)\big\|_{L_x^{\frac{k+1}{\ell}}}^{\frac{k+1}{\ell}} + \|v(t')\|_{L_x^{k+1}}^{k+1}\,dt'\\
&\lesssim \|(\phi_0,\phi_1)\|_{\mathcal{H}^{\frac12}}^2 \int_0^t \big(\|z(t')\|_{L_x^{k+1}}^{\ell-2} + \|(\phi_0,\phi_1)\|_{\mathcal{H}^0}^{\frac{\ell-2}{2}}\big)^{\frac{k+1}{\ell}} + E\big(\vec{v}(t')\big)\,dt'\\
&\lesssim C\big(T, M, \|(\phi_0,\phi_1)\|_{\mathcal{H}^{\frac12}}\big) + \|(\phi_0,\phi_1)\|_{\mathcal{H}^{\frac12}}^2 \int_0^t E\big(\vec{v}(t')\big)\,dt'
\end{aligned} \tag{3.13}$$

for $\omega \in \Omega_{T,\varepsilon}$ and $2 \le \ell \le k$. Hence, by taking $\delta > 0$ small, it follows from (3.10), (3.11), (3.12), and (3.13) that

$$\begin{aligned}
E\big(\vec{v}(t)\big) &= \int_0^t \frac{d}{dt} E\big(\vec{v}(t')\big)\,dt'\\
&\le \frac12 E\big(\vec{v}(t)\big) + C\big(T, M, \|(\phi_0,\phi_1)\|_{\mathcal{H}^{\frac12}}\big) + \|(\phi_0,\phi_1)\|_{\mathcal{H}^{\frac12}}^2 \int_0^t E\big(v(t')\big)\,dt',
\end{aligned}$$

which implies that

$$E\big(\vec{v}(t)\big) \le C\big(T, M, \|(\phi_0,\phi_1)\|_{\mathcal{H}^{\frac12}}\big) + \|(\phi_0,\phi_1)\|_{\mathcal{H}^{\frac12}}^2 \int_0^t E\big(v(t')\big)\,dt'$$

for $\omega \in \Omega_{T,\varepsilon}$. Therefore, from Gronwall's inequality, we obtain (3.7) for $k \ge 5$ and $s \ge 1$. This concludes the proof of Proposition 1.4. □

Remark 3.3. Noting that Lemma 2.4(i) holds for $s \ge 0$, we see that we can handle all the terms in (3.8) for $s = 0$, except for $\int_0^t \int_{\mathbb{T}^2} \partial_t v \cdot zv^2\,dx\,dt'$. As for this term, we can use Yudovich's argument as in [13] and hence Proposition 1.4 with $k = 3$ indeed holds for $s = 0$.

For $k \ge 5$, we used the assumption $s \ge 1$ to control $\|\langle\nabla\rangle\widetilde{z}\|_{L_{T,x}^{k+1}}$ in (3.12). By proceeding as in [47] via the Littlewood–Paley decomposition, we may extend the result to some $s < 1$. However, since the main purpose of Proposition 1.4 is to give a remark on the almost sure global existence with smooth random initial data, we do not pursue this issue further.

4. Unique limit of smooth solutions with mollified data

In this section, we present the proof of Theorem 1.6. We first prove the almost sure convergence of the Wick powers for the Gaussian initial data (1.6) in Section 4.1. We then show convergence in probability of the Wick powers for smooth Gaussian initial data in Section 4.2. Moreover, we prove that the limit is independent of mollification kernels. In Section 4.3, we go over local well-posedness of the perturbed NLW with deterministic perturbations (Lemma 4.4). Finally, in Section 4.4, we iteratively apply Lemma 4.4 for short time intervals to prove Theorem 1.6.

4.1. *Convergence of the Wick powers*

In this subsection, we present a proof of Proposition 1.1. We first estimate the variance of the Fourier coefficients of the truncated Wick powers $:z_N^\ell(t):$ defined in (1.13).

Lemma 4.1. *Let $\ell \in \mathbb{N} \cup \{0\}$. For any $\varepsilon > 0$, $\gamma > 0$, $n \in \mathbb{Z}^2$, $t \in \mathbb{R}$, and $M \ge N \ge 1$, we have*

$$\mathbb{E}\big[\big|\big\langle :z_N^\ell(t):, e_n\big\rangle_{L^2}\big|^2\big] \lesssim \langle n\rangle^{-2+\varepsilon}, \tag{4.1}$$

$$\mathbb{E}\big[\big|\big\langle :z_N^\ell(t): - :z_M^\ell(t):, e_n\big\rangle_{L^2}\big|^2\big] \lesssim N^{-\gamma}\langle n\rangle^{-2+\varepsilon+\gamma}, \tag{4.2}$$

where $e_n(x) = e^{in\cdot x}$. In addition, for any $\varepsilon > 0$, $\gamma > 0$, $\theta \in (0,1)$, $n \in \mathbb{Z}^2$, $t \in \mathbb{R}$, $h \in [-1,1]$, and $M \ge N \ge 1$, we have

$$\mathbb{E}\big[\big|\big\langle \delta_h :z_N^\ell(t):, e_n\big\rangle_{L^2}\big|^2\big] \lesssim \langle n\rangle^{-2+\varepsilon+\theta}|h|^\theta, \tag{4.3}$$

$$\mathbb{E}\big[\big|\big\langle \delta_h :z_N^\ell(t): - \delta_h :z_M^\ell(t):, e_n\big\rangle_{L^2}\big|^2\big] \lesssim N^{-\gamma}\langle n\rangle^{-2+\varepsilon+\gamma+\theta}|h|^\theta, \tag{4.4}$$

where δ_h is as in (2.9).

Once we prove Lemma 4.1, by choosing γ and θ sufficiently small such that $\gamma + \theta < \varepsilon$, Proposition 1.1 follows from Corollary 2.8.

For the proof of Lemma 4.1, we employ the argument used in the proofs of [52, Lemma 2.5] and [53, Proposition 2.3]. Let us first introduce some notations. For *fixed* $x \in \mathbb{T}^2$, we define

$$\eta_N(x)(\cdot) \overset{\text{def}}{=} \frac{1}{\sigma_N^{\frac12}} \sum_{|n|\le N} \frac{\overline{e_n(x)}}{\langle n\rangle} e_n(\cdot), \tag{4.5}$$

where σ_N is as in (1.12). Note that $\eta_N(x)(\cdot)$ is real-valued with $\|\eta_N(x)\|_{L^2(\mathbb{T}^2)} = 1$ for any $x \in \mathbb{T}^2$ and $N \in \mathbb{N}$. Moreover, we have

$$\big\langle \eta_N(x), \eta_M(y)\big\rangle_{L^2} = \frac{1}{\sigma_N^{\frac12}\sigma_M^{\frac12}} \sum_{|n|\le N} \frac{1}{\langle n\rangle^2} e_n(y-x) = \frac{1}{\sigma_N^{\frac12}\sigma_M^{\frac12}} \sum_{|n|\le N} \frac{1}{\langle n\rangle^2} e_n(x-y) \tag{4.6}$$

for any $x, y \in \mathbb{T}^2$ and $M \ge N \ge 1$.

Proof of Lemma 4.1. We only consider (4.2) and (4.4), since (4.1) and (4.3) follow from an analogous (but simpler) argument.

By (2.6) and (4.5) (see also (1.9)), we note that

$$z_N(x,t) = \sigma_N^{\frac12}\frac{z_N(x,t)}{\sigma_N^{\frac12}} = \sigma_N^{\frac12} W^t_{\eta_N(x)}.$$

Then, from (1.13), we have

$$:z_N^\ell(t): = H_\ell\big(z_N(x,t);\sigma_N\big) = \sigma_N^{\frac{\ell}{2}} H_\ell\big(W^t_{\eta_N(x)}\big). \tag{4.7}$$

Given $n \in \mathbb{Z}^2$, define $\Gamma_\ell(n)$ by

$$\Gamma_\ell(n) \overset{\text{def}}{=} \big\{(n_1,\dots,n_\ell) \in \big(\mathbb{Z}^2\big)^\ell : n_1 + \cdots + n_\ell = n\big\}.$$

For $(n_1,\dots,n_\ell) \in \Gamma_\ell(n)$, we have $\max_j |n_j| \gtrsim |n|$. It follows from (4.7), Lemma 2.1, and (4.6) that

$$\begin{aligned}
&\mathbb{E}\big[\big|\big\langle :z_N^\ell(t): - :z_M^\ell(t):, e_n\big\rangle_{L^2}\big|^2\big] \\
&= \int_{\mathbb{T}^2_x\times\mathbb{T}^2_y} \overline{e_n(x)} e_n(y)
\end{aligned}$$

$$\begin{aligned}
&\int_\Omega \Big[\sigma_N^\ell H_\ell\big(W^t_{\eta_N(x)}\big)\overline{H_\ell\big(W^t_{\eta_N(y)}\big)} + \sigma_M^\ell H_\ell\big(W^t_{\eta_M(x)}\big)\overline{H_\ell\big(W^t_{\eta_M(y)}\big)} \\
&- \sigma_N^{\frac{\ell}{2}}\sigma_M^{\frac{\ell}{2}}\big\{H_\ell\big(W^t_{\eta_N(x)}\big)\overline{H_\ell\big(W^t_{\eta_M(y)}\big)} + H_\ell\big(W^t_{\eta_M(x)}\big)\overline{H_\ell\big(W^t_{\eta_N(y)}\big)}\big\}\Big]\,dP\,dx\,dy \\
&= \ell!\Bigg\{\sum_{\substack{\Gamma_\ell(n)\\ |n_j|\le M}}\prod_{j=1}^{\ell}\frac{1}{\langle n_j\rangle^2} - \sum_{\substack{\Gamma_\ell(n)\\ |n_j|\le N}}\prod_{j=1}^{\ell}\frac{1}{\langle n_j\rangle^2}\Bigg\} \\
&\lesssim N^{-\gamma}\langle n\rangle^{-2+\varepsilon+\gamma}. \qquad (4.8)
\end{aligned}$$

for any $M \ge N \ge 1$. This prove (4.2).

Next, we consider (4.4). From (4.7), Lemmas 2.1, and 2.2 with (4.6), we have

$$\begin{aligned}
&\mathbb{E}\Big[\big|\big\langle \delta_h :z_N^\ell(t): - \delta_h :z_M^\ell(t):, e_n\big\rangle_{L^2}\big|^2\Big] \\
&= \int_{\mathbb{T}^2_x\times\mathbb{T}^2_y} \overline{e_n(x)}e_n(y) \\
&\int_\Omega \Big[\sigma_N^\ell\big\{H_\ell\big(W^{t+h}_{\eta_N(x)}\big)\overline{H_\ell\big(W^{t+h}_{\eta_N(y)}\big)} - H_\ell\big(W^{t+h}_{\eta_N(x)}\big)\overline{H_\ell\big(W^{t}_{\eta_N(y)}\big)} \\
&- H_\ell\big(W^{t}_{\eta_N(x)}\big)\overline{H_\ell\big(W^{t+h}_{\eta_N(y)}\big)} + H_\ell\big(W^{t}_{\eta_N(x)}\big)\overline{H_\ell\big(W^{t}_{\eta_N(y)}\big)}\big\} \\
&+ \sigma_M^\ell\big\{H_\ell\big(W^{t+h}_{\eta_M(x)}\big)\overline{H_\ell\big(W^{t+h}_{\eta_M(y)}\big)} - H_\ell\big(W^{t+h}_{\eta_M(x)}\big)\overline{H_\ell\big(W^{t}_{\eta_M(y)}\big)} \\
&- H_\ell\big(W^{t}_{\eta_M(x)}\big)\overline{H_\ell\big(W^{t+h}_{\eta_M(y)}\big)} + H_\ell\big(W^{t}_{\eta_M(x)}\big)\overline{H_\ell\big(W^{t}_{\eta_M(y)}\big)}\big\} \\
&- \sigma_N^{\frac{\ell}{2}}\sigma_M^{\frac{\ell}{2}}\big\{H_\ell\big(W^{t+h}_{\eta_N(x)}\big)\overline{H_\ell\big(W^{t+h}_{\eta_M(y)}\big)} - H_\ell\big(W^{t+h}_{\eta_N(x)}\big)\overline{H_\ell\big(W^{t}_{\eta_M(y)}\big)} \\
&- H_\ell\big(W^{t}_{\eta_N(x)}\big)\overline{H_\ell\big(W^{t+h}_{\eta_M(y)}\big)} + H_\ell\big(W^{t}_{\eta_N(x)}\big)\overline{H_\ell\big(W^{t}_{\eta_M(y)}\big)} \\
&+ H_\ell\big(W^{t+h}_{\eta_M(x)}\big)\overline{H_\ell\big(W^{t+h}_{\eta_N(y)}\big)} - H_\ell\big(W^{t+h}_{\eta_M(x)}\big)\overline{H_\ell\big(W^{t}_{\eta_N(y)}\big)} \\
&- H_\ell\big(W^{t}_{\eta_M(x)}\big)\overline{H_\ell\big(W^{t+h}_{\eta_N(y)}\big)} + H_\ell\big(W^{t}_{\eta_M(x)}\big)\overline{H_\ell\big(W^{t}_{\eta_N(y)}\big)}\big\}\Big]\,dP\,dx\,dy \\
&= 2\ell! \sum_{\substack{\Gamma_\ell(n)\\ N<\max_j |n_j|\le M}} \Bigg\{\prod_{j=1}^{\ell}\frac{1}{\langle n_j\rangle^2} - \prod_{j=1}^{\ell}\frac{\cos(h\langle n_j\rangle)}{\langle n_j\rangle^2}\Bigg\}. \qquad (4.9)
\end{aligned}$$

By writing the last expression in a telescoping sum and applying the mean-value theorem, we have

$$\begin{aligned}
\text{RHS of (4.9)} &\lesssim \sum_{\substack{\Gamma_\ell(n)\\ N<\max_j |n_j|\le M}} \sum_{k=1}^{\ell}|h|^\theta\langle n_k\rangle^\theta\prod_{j=1}^{\ell}\frac{1}{\langle n_j\rangle^2} \\
&\lesssim N^{-\gamma}\langle n\rangle^{-2+\varepsilon+\gamma+\theta}|h|^\theta.
\end{aligned}$$

This proves (4.4). □

4.2. *Uniqueness of the Wick powers*

In this subsection, we study the Wick powers for smooth Gaussian initial data $(u^\omega_{0,\delta}, u^\omega_{1,\delta})$ in (1.29) and show that they converge in probability to the Wick powers $:z^\ell:$ constructed in the previous subsection, which in particular implies that limit is independent of mollification kernels. In order to signify the dependence on a mollification kernel ρ, we write

$$z_{\rho,\delta} = S(t)\big(u^\omega_{0,\delta}, u^\omega_{1,\delta}\big),$$

$$\sigma_{\rho,\delta} = \mathrm{Var}\big(z_{\rho,\delta}(x,t)\big) = \mathbb{E}\big[z_{\rho,\delta}^2(x,t)\big] = \sum_{n\in\mathbb{Z}^2} \frac{|\widehat{\rho}(\delta n)|^2}{\langle n\rangle^2},$$

$$:z_{\rho,\delta}^{\ell}(x,t): = H_\ell\big(z_{\rho,\delta}(x,t);\sigma_{\rho,\delta}\big),$$

where $(u_{0,\delta}^{\omega}, u_{1,\delta}^{\omega})$ is defined in (1.29). Our main goal in this subsection is to prove the following proposition.

Proposition 4.2. *Let $\ell\in\mathbb{N}\cup\{0\}$. Then, for any $T>0$ and $\varepsilon>0$, the mollified Wick powers $:z_{\rho,\delta}^{\ell}:$ converges in probability to $:z^\ell:$ in $C([-T,T];W^{-\varepsilon,\infty}(\mathbb{T}^2))$ as $\delta\to 0$, where $:z^\ell:$ is defined in (1.14).*

We point out that Proposition 4.2 establishes convergence in probability, *not* almost sure convergence. This is due to the fact that we take a limit along a continuous parameter $\delta\to 0$. Indeed, in the second part of the proof of Proposition 4.2, by restricting our attention to a discrete sequence tending to 0 (i.e. $\delta=\frac{1}{N}$, $N\in\mathbb{N}$), we show that the sequence $\{:z_{\rho,\frac{1}{N}}^{\ell}:\}_{N\in\mathbb{N}}$ converges almost surely.[15]

As in the proof of Proposition 1.1, we first estimate the variance of the Fourier coefficients of the mollified Wick powers $:z_{\rho,\delta}^{\ell}(t):$.

Lemma 4.3. *Let $\ell\in\mathbb{N}\cup\{0\}$. For any $\varepsilon>0$, $\gamma\in(0,1)$, $n\in\mathbb{Z}^2$, $t\in\mathbb{R}$, and $\delta,\delta'\in(0,1]$ we have*

$$\mathbb{E}\Big[\big|\big\langle :z_{\rho,\delta}^{\ell}(t):, e_n\big\rangle_{L^2}\big|^2\Big] \lesssim \langle n\rangle^{-2+\varepsilon}, \tag{4.10}$$

$$\mathbb{E}\Big[\big|\big\langle :z_{\rho,\delta}^{\ell}(t): - :z_{\rho,\delta'}^{\ell}(t):, e_n\big\rangle_{L^2}\big|^2\Big] \lesssim \big|\delta-\delta'\big|^{\gamma}\langle n\rangle^{-2+\varepsilon+\gamma}. \tag{4.11}$$

In addition, for any $\varepsilon>0$, $\gamma,\theta\in(0,1)$, $n\in\mathbb{Z}^2$, $t\in\mathbb{R}$, $h\in[-1,1]$, and $\delta,\delta'\in(0,1]$, we have

$$\mathbb{E}\Big[\big|\big\langle \delta_h :z_{\rho,\delta}^{\ell}(t):, e_n\big\rangle_{L^2}\big|^2\Big] \lesssim \langle n\rangle^{-2+\varepsilon+\theta}|h|^{\theta},$$

$$\mathbb{E}\Big[\big|\big\langle \delta_h :z_{\rho,\delta}^{\ell}(t): - \delta_h :z_{\rho,\delta'}^{\ell}(t):, e_n\big\rangle_{L^2}\big|^2\Big] \lesssim \big|\delta-\delta'\big|^{\gamma}\langle n\rangle^{-2+\varepsilon+\gamma+\theta}|h|^{\theta}. \tag{4.12}$$

Proof. Since these estimates follow from a slight modification of the proof of Lemma 4.1, we give a brief explanation of the proof of (4.11) and (4.12). Proceeding as in (4.8), we have

$$\begin{aligned}
&\mathbb{E}\Big[\big|\big\langle :z_{\rho,\delta}^{\ell}(t): - :z_{\rho,\delta'}^{\ell}(t):, e_n\big\rangle_{L^2}\big|^2\Big]\\
&= \ell! \sum_{\Gamma_\ell(n)} \Bigg\{ \prod_{j=1}^{\ell} \frac{|\widehat{\rho}(\delta n_j)|^2}{\langle n_j\rangle^2} + \prod_{j=1}^{\ell} \frac{|\widehat{\rho}(\delta' n_j)|^2}{\langle n_j\rangle^2}\\
&\qquad - \prod_{j=1}^{\ell} \frac{\widehat{\rho}(\delta n_j)\overline{\widehat{\rho}(\delta' n_j)}}{\langle n_j\rangle^2} - \prod_{j=1}^{\ell} \frac{\widehat{\rho}(\delta' n_j)\overline{\widehat{\rho}(\delta n_j)}}{\langle n_j\rangle^2} \Bigg\}\\
&= \ell! \sum_{\Gamma_\ell(n)} \prod_{j=1}^{\ell} \frac{|\widehat{\rho}(\delta n_j) - \widehat{\rho}(\delta' n_j)|^2}{\langle n_j\rangle^2}.
\end{aligned} \tag{4.13}$$

Since $\rho\in L^1(\mathbb{T}^2)$, it follows from the mean value theorem that

$$\big|\widehat{\rho}(\delta n) - \widehat{\rho}\big(\delta' n\big)\big| \le \int_{\mathbb{T}^2} \big|1 - e^{i(\delta-\delta')n\cdot x}\big|\big|\rho(x)\big|\,dx \lesssim \min\big(1, \big|\delta-\delta'\big||n|\big). \tag{4.14}$$

Hence, (4.11) follows from (4.13) and (4.14).

Similarly, proceeding as in (4.9) with (4.14) and the mean value theorem, we have

$$\mathbb{E}\Big[\big|\big\langle \delta_h :z_{\rho,\delta}^{\ell}(t): - \delta_h :z_{\rho,\delta'}^{\ell}(t):, e_n\big\rangle_{L^2}\big|^2\Big]$$

[15] It seems possible to adapt the argument in the proof of Proposition 2.3 in [65] to prove almost sure convergence of $:z_{\rho,\delta}^{\ell}:$ along a continuous parameter $\delta\to 0$. We, however, do not pursue this issue here.

$$
\begin{aligned}
&= 2\ell! \sum_{\Gamma_\ell(n)} \Bigg\{ \prod_{j=1}^{\ell} \frac{|\widehat{\rho}(\delta n_j)|^2}{\langle n_j\rangle^2} - \prod_{j=1}^{\ell} \frac{|\widehat{\rho}(\delta n_j)|^2 \cos(h\langle n_j\rangle)}{\langle n_j\rangle^2} \\
&\quad + \prod_{j=1}^{\ell} \frac{|\widehat{\rho}(\delta' n_j)|^2}{\langle n_j\rangle^2} - \prod_{j=1}^{\ell} \frac{|\widehat{\rho}(\delta' n_j)|^2 \cos(h\langle n_j\rangle)}{\langle n_j\rangle^2} \\
&\quad - \Bigg(\prod_{j=1}^{\ell} \frac{\widehat{\rho}(\delta n_j)\overline{\widehat{\rho}(\delta' n_j)}}{\langle n_j\rangle^2} - \prod_{j=1}^{\ell} \frac{\widehat{\rho}(\delta n_j)\overline{\widehat{\rho}(\delta' n_j)} \cos(h\langle n_j\rangle)}{\langle n_j\rangle^2} \\
&\quad + \prod_{j=1}^{\ell} \frac{\widehat{\rho}(\delta' n_j)\overline{\widehat{\rho}(\delta n_j)}}{\langle n_j\rangle^2} - \prod_{j=1}^{\ell} \frac{\widehat{\rho}(\delta' n_j)\overline{\widehat{\rho}(\delta n_j)} \cos(h\langle n_j\rangle)}{\langle n_j\rangle^2} \Bigg) \Bigg\} \\
&= \ell! \sum_{\Gamma_\ell(n)} \Bigg\{ \prod_{j=1}^{\ell} \frac{|\widehat{\rho}(\delta n_j) - \widehat{\rho}(\delta' n_j)|^2}{\langle n_j\rangle^2} - \prod_{j=1}^{\ell} \frac{|\widehat{\rho}(\delta n_j) - \widehat{\rho}(\delta' n_j)|^2 \cos(h\langle n_j\rangle)}{\langle n_j\rangle^2} \Bigg\} \\
&\lesssim \big|\delta - \delta'\big|^{\gamma} \langle n\rangle^{-2+\varepsilon+\gamma+\theta} |h|^{\theta}, \qquad (4.15)
\end{aligned}
$$

yielding (4.12). □

We are now ready to prove Proposition 4.2.

Proof of Proposition 4.2. Fix small $\gamma, \theta > 0$ such that $\gamma + \theta < \varepsilon$. Fix $t \in \mathbb{R}$. Then, it follows from (4.10), (4.11), and Lemma 2.5 (see also Remark A.3) that, as $\delta \to 0$, $:z^\ell_{\rho,\delta}(t):$ converges to some limit $:z^\ell_\rho(t):$ in $L^p(\Omega; W^{-\varepsilon,\infty}(\mathbb{T}^2))$ for any finite $p \geq 1$.

Let $\{\delta_j\}_{j\in\mathbb{N}}$ be a sequence satisfying $\delta_j \to 0$ as $j \to \infty$. There exists a subsequence $\{\delta_{j(m)}\}_{m\in\mathbb{N}} \subset \{\delta_j\}_{j\in\mathbb{N}}$ such that $\delta_{j(m)} < m^{-1}$ for $m \in \mathbb{N}$. It follows from Corollary 2.8 with Lemma 4.3 that the subsequence $:z^\ell_{\rho,\delta_{j(m)}}:$ converges almost surely (and hence in measure) to $:z^\ell_\rho:$ in $C([-T,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2))$, as $m \to \infty$. Since the limit $:z^\ell_\rho:$ is independent of the choice of a sequence $\{\delta_j\}_{j\in\mathbb{N}}$, we deduce that $:z^\ell_{\rho,\delta}:$ converges in probability to $:z^\ell_\rho:$ in $C([-T,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2))$.

Next, we prove that the limit is independent of mollification kernels. Since $\rho \in L^1(\mathbb{T}^2)$ and $\widehat{\rho}(0) = 1$, it follows from the mean value theorem that

$$
\bigg|1 - \widehat{\rho}\Big(\frac{n}{N}\Big)\bigg| \leq \int_{\mathbb{T}^2} \big|1 - e^{-i\frac{n}{N}\cdot x}\big| \big|\rho(x)\big| \, dx \lesssim \min\Big(1, \frac{|n|}{N}\Big).
$$

Given $h \in [-1,1]$, proceeding as in (4.15), we have

$$
\begin{aligned}
&\mathbb{E}\Big[\big|\big\langle \delta_h :z^\ell_N(t): - \delta_h :z^\ell_{\rho,\frac{1}{N}}(t):, e_n\big\rangle_{L^2}\big|^2\Big] \\
&= 2\ell! \sum_{\Gamma_\ell(n)} \Bigg\{ \prod_{j=1}^{\ell} \frac{\mathbf{1}_{|n_j|\leq N}}{\langle n_j\rangle^2} - \prod_{j=1}^{\ell} \frac{\mathbf{1}_{|n_j|\leq N} \cos(h\langle n_j\rangle)}{\langle n_j\rangle^2} \\
&\quad + \prod_{j=1}^{\ell} \frac{|\widehat{\rho}(\frac{n_j}{N})|^2}{\langle n_j\rangle^2} - \prod_{j=1}^{\ell} \frac{|\widehat{\rho}(\frac{n_j}{N})|^2 \cos(h\langle n_j\rangle)}{\langle n_j\rangle^2} \\
&\quad - \Bigg(\prod_{j=1}^{\ell} \frac{\mathbf{1}_{|n_j|\leq N}\overline{\widehat{\rho}(\frac{n_j}{N})}}{\langle n_j\rangle^2} - \prod_{j=1}^{\ell} \frac{\mathbf{1}_{|n_j|\leq N}\overline{\widehat{\rho}(\frac{n_j}{N})} \cos(h\langle n_j\rangle)}{\langle n_j\rangle^2} \\
&\quad + \prod_{j=1}^{\ell} \frac{\mathbf{1}_{|n_j|\leq N}\widehat{\rho}(\frac{n_j}{N})}{\langle n_j\rangle^2} - \prod_{j=1}^{\ell} \frac{\mathbf{1}_{|n_j|\leq N}\widehat{\rho}(\frac{n_j}{N}) \cos(h\langle n_j\rangle)}{\langle n_j\rangle^2} \Bigg) \Bigg\} \\
&= 2\ell! \sum_{\Gamma_\ell(n)} \Bigg\{ \prod_{j=1}^{\ell} \frac{|\mathbf{1}_{|n_j|\leq N} - \widehat{\rho}(\frac{n_j}{N})|^2}{\langle n_j\rangle^2} - \prod_{j=1}^{\ell} \frac{|\mathbf{1}_{|n_j|\leq N} - \widehat{\rho}(\frac{n_j}{N})|^2 \cos(h\langle n_j\rangle)}{\langle n_j\rangle^2} \Bigg\}. \qquad (4.16)
\end{aligned}
$$

By writing the summand in a telescoping sum and applying the mean value theorem (to $1 - \cos(h\langle n_j\rangle)$) and (4.16), we have

$$\begin{aligned}\text{RHS of (4.16)} &\lesssim \sum_{\Gamma_\ell(n)} \sum_{k=1}^{\ell} |h|^\theta \langle n_k\rangle^\theta \prod_{j=1}^{\ell} \frac{|\mathbf{1}_{|n_j|\le N} - \widehat{\rho}(\frac{n_j}{N})|^2}{\langle n_j\rangle^2} \\ &\lesssim \sum_{\Gamma_\ell(n)} \sum_{k=1}^{\ell} |h|^\theta \langle n_k\rangle^\theta \prod_{j=1}^{\ell} \frac{\mathbf{1}_{|n_j|>N}}{\langle n_j\rangle^2} \\ &\quad + \sum_{\Gamma_\ell(n)} \sum_{k=1}^{\ell} |h|^\theta \langle n_k\rangle^\theta \prod_{j=1}^{\ell} \frac{|\mathbf{1}_{|n_j|\le N} - \widehat{\rho}(\frac{n_j}{N})|^2}{\langle n_j\rangle^2} \\ &\lesssim N^{-\gamma} \langle n\rangle^{-2+\varepsilon+\gamma+\theta} |h|^\theta.\end{aligned}$$

A similar estimate holds for the difference:

$$\mathbb{E}\Big[\big|\big\langle :z_N^\ell(t): - :z_{\rho,\frac{1}{N}}^\ell(t):, e_n\big\rangle_{L^2}\big|^2\Big].$$

Therefore, from the above computation with Lemma 4.1 and Proposition 2.7, we see that, as $N \to \infty$, $:z_{\rho,\frac{1}{N}}^\ell:$ converges almost surely to $:z^\ell:$ (constructed in Proposition 1.1) in $C([-T,T]; W^{-\varepsilon,\infty}(\mathbb{T}^2))$. Together with the convergence in probability of $\{:z_{\rho,\delta}^\ell:\}_{\delta\in(0,1]}$ to $:z_\rho^\ell:$, we conclude that $:z^\ell: = :z_\rho^\ell:$ almost surely. This completes the proof of Proposition 4.2. □

4.3. *Local well-posedness of the perturbed NLW with deterministic perturbation*

In this subsection, we consider the local well-posedness of the following Cauchy problem:

$$\begin{cases} \partial_t^2 v + (1-\Delta)v + v^3 + 3f_1 v^2 + 3f_2 v + f_3 = 0, \\ (v, \partial_t v)|_{t=0} = (v_0, v_1), \end{cases} \tag{4.17}$$

where f_1, f_2, f_3 are given (deterministic) functions. We define the function space

$$X^s(I) = C\big(I; H^s(\mathbb{T}^2)\big) \cap C^1\big(I; H^{s-1}(\mathbb{T}^2)\big)$$

for $s \in \mathbb{R}$ and an interval $I \subset \mathbb{R}$. If $I = [-T,T]$, we write $X_T^s = X^s([-T,T])$.

Lemma 4.4. *Let* $\frac{1}{2} < s < 1$. *There exists* $\varepsilon = \varepsilon(s) > 0$ *such that if* $f_1, f_2, f_3 \in L_{\text{loc}}^{\frac{2}{\varepsilon}}(\mathbb{R}; W^{-\varepsilon,\frac{2}{\varepsilon}}(\mathbb{T}^2))$, *then the Cauchy problem* (4.17) *is locally well-posed in* $\mathcal{H}^s(\mathbb{T}^2)$. *More precisely, given* $(v_0, v_1) \in \mathcal{H}^s(\mathbb{T}^2)$, *there exist* $T > 0$ *and a unique solution* $v \in X_T^s$ *to* (4.17), *depending continuously on the enhanced data set*

$$\Xi = (v_0, v_1, f_1, f_2, f_3) \tag{4.18}$$

in the class:

$$\mathcal{X}_T^{s,\varepsilon} = \mathcal{H}^s(\mathbb{T}^2) \times L^{\frac{2}{\varepsilon}}\big([-T,T]; W^{-\varepsilon,\frac{2}{\varepsilon}}(\mathbb{T}^2)\big)^3.$$

By using the Strichartz estimates as in [26,53], we can indeed prove local well-posedness of (4.17) for $\frac{1}{4} < s < 1$. Note that $s = \frac{1}{4}$ is the critical regularity as in (1.35). For simplicity, however, we only consider the case $\frac{1}{2} < s < 1$, where the local well-posedness follows from a fixed point argument with Sobolev's inequality and the product estimates (Lemma 2.3).

Proof. The proof is essentially contained in Proposition 4.1 in [28] and thus we will be brief here. By writing (4.17) in the Duhamel formulation, we have

$$v(t) = \Phi_\Xi(v)(t)$$

$$\stackrel{\text{def}}{=} S(t)(v_0, v_1) - \int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle}\big(v^3 + 3f_1v^2 + 3f_2v + f_3\big)(t')\,dt'.$$

We will show that Φ_Λ is a contraction mapping on a ball in X_T^s.

By Sobolev's inequality, we have

$$\begin{aligned}\bigg\|\int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle} v^3(t')\,dt'\bigg\|_{X_T^s} &\lesssim T\|v^3\|_{L_T^\infty H_x^{s-1}} \lesssim T\|v^3\|_{L_T^\infty L_x^{\frac{2}{2-s}}}\\ &\lesssim T\|v\|^3_{L_T^\infty L_x^{\frac{6}{2-s}}} \lesssim T\|v\|^3_{L_T^\infty H_x^{\frac{1+s}{3}}}\\ &\lesssim T\|v\|^3_{X_T^s} \end{aligned} \tag{4.19}$$

for $\frac{1}{2} \le s \le 1$. From Lemma 2.3 and Sobolev's inequality, we have

$$\begin{aligned}&\bigg\|\int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle}\big(f_1v^2\big)(t')\,dt'\bigg\|_{X_T^s}\\ &\lesssim \big\|\langle\nabla\rangle^{-\varepsilon}\big(f_1v^2\big)\big\|_{L_T^1L_x^2}\\ &\lesssim T^{1-\frac{\varepsilon}{2}}\big\|\langle\nabla\rangle^{-\varepsilon}f_1\big\|_{L_T^{\frac{2}{\varepsilon}}L_x^{\frac{2}{\varepsilon}}}\big\|\langle\nabla\rangle^{\varepsilon}\big(v^2\big)\big\|_{L_T^\infty L_x^2}\\ &\lesssim T^{1-\frac{\varepsilon}{2}}\big\|\langle\nabla\rangle^{-\varepsilon}f_1\big\|_{L_T^{\frac{2}{\varepsilon}}L_x^{\frac{2}{\varepsilon}}}\big\|\langle\nabla\rangle^{\varepsilon}v\big\|_{L_T^\infty L_x^4}\|v\|_{L_T^\infty L_x^4}\\ &\lesssim T^{1-\frac{\varepsilon}{2}}\big\|\langle\nabla\rangle^{-\varepsilon}f_1\big\|_{L_T^{\frac{2}{\varepsilon}}L_x^{\frac{2}{\varepsilon}}}\|v\|^2_{X_T^s},\end{aligned} \tag{4.20}$$

provided that $\frac{1}{2} < s < 1$ and $\varepsilon = \varepsilon(s) > 0$ is sufficiently small. Similarly, we have

$$\bigg\|\int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle}(f_2v)\,dt'\bigg\|_{X_T^s} \lesssim T^{1-\frac{\varepsilon}{2}}\big\|\langle\nabla\rangle^{-\varepsilon}f_2\big\|_{L_T^{\frac{2}{\varepsilon}}L_x^{\frac{2}{\varepsilon}}}\big\|\langle\nabla\rangle^{\varepsilon}v\big\|_{X_T^s}, \tag{4.21}$$

$$\bigg\|\int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle} f_3(t')\,dt'\bigg\|_{X_T^s} \lesssim T^{1-\frac{\varepsilon}{2}}\big\|\langle\nabla\rangle^{-\varepsilon}f_3\big\|_{L_T^{\frac{2}{\varepsilon}}L_x^{\frac{2}{\varepsilon}}}. \tag{4.22}$$

A standard argument with (4.19)–(4.22) then shows that Φ_Ξ is a contraction on a small ball in X_T^s by choosing $T = T(\|\Xi\|_{\mathcal{X}_1^{s,\varepsilon}}) > 0$ sufficiently small. Moreover, a slight modification of the argument allows us to show continuous dependence of the solution on the enhanced data set Ξ in (4.18). Since the argument is standard, we omit details. □

4.4. *Proof of Theorem* 1.6

We conclude this section by presenting the proof of Theorem 1.6. Set $v = u - z$, where u is the solution constructed in Theorem A and $z = S(t)(u_0^\omega, u_1^\omega)$ is as in (1.8). Let $v_{\rho,\delta}$ be the solution of (1.24) with the mollified initial data $(u_{0,\delta}^\omega, u_{1,\delta}^\omega)$ defined in (1.29) with a mollification kernel ρ. Let $T > 0$ and $\frac{1}{2} < s_0 < 1$. In view of Proposition 4.2, it suffices to show that $v_{\rho,\delta}$ converges in probability to v in $C([-T,T];\mathcal{H}^{s_0}(\mathbb{T}^2))$.

By Theorem A, the global solution $u \in C(\mathbb{R}; H^s(\mathbb{T}^2))$ to (1.18) satisfies $v = u - z \in C(\mathbb{R}; H^{s_0}(\mathbb{T}^2))$, almost surely. In particular, from the construction of the global solution, for any $\eta > 0$, there exists $R = R(T,\eta) \ge 1$ such that $\Omega_1 = \{\omega \in \Omega : \|(v, \partial_t v)\|_{L_T^\infty \mathcal{H}_x^{s_0}} \le R\}$ satisfies

$$P\big(\Omega_1^c\big) < \frac{\eta}{4}. \tag{4.23}$$

We divide the interval $[-T,T]$ into finitely many subintervals:

$$[-T,T] = \bigcup_{j=-[T/\tau]-1}^{[T/\tau]} I_j,\ I_j = \big[j\tau, (j+1)\tau\big] \cap [-T,T],$$

where $\tau > 0$ is to be chosen later. Let $\varepsilon = \varepsilon(s_0) > 0$ be as in Lemma 4.4. We set

$$\Omega_2 = \bigg\{\omega \in \Omega : \big\|:z^\ell:\big\|_{L^{\frac{2}{\varepsilon}}(I_j; W^{-\varepsilon,\frac{2}{\varepsilon}}(\mathbb{T}^2))} \le 1, \ell = 1, 2, 3, j = -\bigg[\frac{T}{\tau}\bigg] - 1, \dots, \bigg[\frac{T}{\tau}\bigg]\bigg\}.$$

By Lemma 2.6 and taking $\tau = \tau(T, \eta) > 0$ small, we have

$$\begin{aligned} P(\Omega_2^c) &\le \sum_{\ell=1}^{3} \sum_{j=-[\frac{T}{\tau}]-1}^{[\frac{T}{\tau}]} P\Big(\big\|:z^\ell:\big\|_{L^{\frac{2}{\varepsilon}}(I_j; W^{-\varepsilon,\frac{2}{\varepsilon}}(\mathbb{T}^2))} > 1\Big) \\ &\lesssim \sum_{\ell=1}^{3} \frac{T}{\tau} \exp\big(-c\tau^{-\frac{\varepsilon}{\ell}}\big) \\ &\lesssim \frac{T}{\tau} \tau \exp\Big(-\frac{c}{2}\tau^{-\frac{\varepsilon}{3}}\Big) \\ &= T \exp\Big(-\frac{c}{2}\tau^{-\frac{\varepsilon}{3}}\Big) < \frac{\eta}{4}. \end{aligned} \tag{4.24}$$

Moreover, we set

$$\Omega_{3,\delta} = \big\{\omega \in \Omega : \| :z^\ell: - :z^\ell_{\rho,\delta}: \|_{L_T^{\frac{2}{\varepsilon}} W_x^{-\varepsilon,\frac{2}{\varepsilon}}} \le 8^{-\frac{T}{\tau}-5}, \ell = 1, 2, 3\big\}.$$

From Proposition 4.2, there exists $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$, we have

$$P\big(\Omega_{3,\delta}^c\big) < \frac{\eta}{4}. \tag{4.25}$$

Then, we define $\Omega_{T,\eta,\delta} = \Omega_1 \cap \Omega_2 \cap \Omega_{3,\delta}$. It follows from (4.23), (4.24), and (4.25) that

$$P\big(\Omega_{T,\eta,\delta}^c\big) < \frac{3}{4}\eta. \tag{4.26}$$

Let $w_{\rho,\delta} = v - v_{\rho,\delta}$. Then, $w_{\rho,\delta}$ satisfies

$$\begin{cases} \partial_t^2 w_{\rho,\delta} + (1-\Delta) w_{\rho,\delta} + \mathcal{N}^3_{(u_0^\omega, u_1^\omega)}(v) - \mathcal{N}^3_{(u_{0,\delta}^\omega, u_{1,\delta}^\omega)}(v_{\rho,\delta}) = 0, \\ (w_{\rho,\delta}, \partial_t w_{\rho,\delta})|_{t=0} = (0,0), \end{cases}$$

where $\mathcal{N}^3_{(u_0^\omega, u_1^\omega)}(v)$ is well defined thanks to Theorem A. From (1.19), we have

$$\begin{aligned} &\mathcal{N}^3_{(u_0^\omega, u_1^\omega)}(v) - \mathcal{N}^3_{(u_{0,\delta}^\omega, u_{1,\delta}^\omega)}(v_{\rho,\delta}) \\ &\quad = v^3 - v_{\rho,\delta}^3 + 3\big(v^2 - v_{\rho,\delta}^2\big) z + 3 v_{\rho,\delta}^2 (z - z_{\rho,\delta}) + 3 w_{\rho,\delta} :z^2: \\ &\qquad + 3 v_{\rho,\delta}\big(:z^2: - :z^2_{\rho,\delta}:\big) + :z^3: - :z^3_{\rho,\delta}: \\ &\quad = -3v^2 w_{\rho,\delta} + 3v(2v - w_{\rho,\delta}) w_{\rho,\delta} + w_{\rho,\delta}^3 + 3(2v - w_{\rho,\delta}) w_{\rho,\delta} z \\ &\qquad + 3\big(v^2 - 2v w_{\rho,\delta} + w_{\rho,\delta}^2\big)(z - z_{\rho,\delta}) + 3 w_{\rho,\delta} :z^2: \\ &\qquad + 3(v - w_{\rho,\delta})\big(:z^2: - :z^2_{\rho,\delta}:\big) + :z^3: - :z^3_{\rho,\delta}: . \end{aligned}$$

By taking $\tau = \tau(R) > 0$ sufficiently small, the local well-posedness argument in the proof of Lemma 4.4 yields

$$\|v\|_{X^{s_0}(I_j)} \le 2R \tag{4.27}$$

for $\omega \in \Omega_{T,\eta,\delta}$ and $j = -[\frac{T}{\tau}] - 1, \dots, [\frac{T}{\tau}]$.

In the following, we restrict our attention to positive times, i.e. we work on I_j for $j = 0, \dots, [\frac{T}{\tau}]$. By applying the estimates (4.19), (4.20), (4.21), and (4.22) with (4.27) and taking $\tau = \tau(R) > 0$ sufficiently small, we have

$$\begin{aligned}\|w_{\rho,\delta}\|_{X^{s_0}(I_j)} &\le \big\|\big(w_{\rho,\delta}(j\tau), \partial_t w_{\rho,\delta}(j\tau)\big)\big\|_{\mathcal{H}^{s_0}} \\ &\quad + C\tau^{1-\frac{\varepsilon}{2}}\Big(\big(R^2 + \|w_{\rho,\delta}\|^2_{X^{s_0}(I_j)}\big)\|w_{\rho,\delta}\|_{X^{s_0}(I_j)} \\ &\quad + \big(R + \|w_{\rho,\delta}\|_{X^{s_0}(I_j)}\big)\|w_{\rho,\delta}\|_{X^{s_0}(I_j)} \\ &\quad + \big(R^2 + \|w_{\rho,\delta}\|^2_{X^{s_0}(I_j)}\big)\|z - z_{\rho,\delta}\|_{L_T^{\frac{2}{\varepsilon}} W_x^{-\varepsilon,\frac{2}{\varepsilon}}} \\ &\quad + \big(R + \|w_{\rho,\delta}\|_{X^{s_0}(I_j)}\big)\| :z^2: - :z^2_{\rho,\delta}: \|_{L_T^{\frac{2}{\varepsilon}} W_x^{-\varepsilon,\frac{2}{\varepsilon}}} \\ &\quad + \| :z^3: - :z^3_{\rho,\delta}: \|_{L_T^{\frac{2}{\varepsilon}} W_x^{-\varepsilon,\frac{2}{\varepsilon}}}\Big) \\ &\le \big\|\big(w_{\rho,\delta}(j\tau), \partial_t w_{\rho,\delta}(j\tau)\big)\big\|_{\mathcal{H}^{s_0}} \\ &\quad + \frac{1}{2}\sum_{\ell=1}^{3}\Big(\|w_{\rho,\delta}\|^\ell_{X^{s_0}(I_j)} + \| :z^\ell: - :z^\ell_{\rho,\delta}: \|_{L_T^{\frac{2}{\varepsilon}} W_x^{-\varepsilon,\frac{2}{\varepsilon}}}\Big)\end{aligned}$$

for any $\omega \in \Omega_{T,\eta,\delta}$ and $j = 0, \dots, [\frac{T}{\tau}]$. By setting

$$A = \sum_{\ell=1}^{3} \| :z^\ell: - :z^\ell_{\rho,\delta}: \|_{L_T^{\frac{2}{\varepsilon}} W_x^{-\varepsilon,\frac{2}{\varepsilon}}},$$

we have

$$\begin{aligned}\|w_{\rho,\delta}\|_{X^{s_0}(I_j)} &\le 2\big\|\big(w_{\rho,\delta}(j\tau), \partial_t w_{\rho,\delta}(j\tau)\big)\big\|_{\mathcal{H}^{s_0}} \\ &\quad + \|w_{\rho,\delta}\|^2_{X^{s_0}(I_j)} + \|w_{\rho,\delta}\|^3_{X^{s_0}(I_j)} + A.\end{aligned} \tag{4.28}$$

When $j = 0$, since $(w_{\rho,\delta}(0), \partial_t w_{\rho,\delta}(0)) = (0, 0)$ and $A < 8^{-3}$, a continuity argument yields

$$\|w_{\rho,\delta}\|_{X^{s_0}(I_0)} \le 2A.$$

In particular, we have $\|(w_{\rho,\delta}(\tau), \partial_t w_{\rho,\delta}(\tau))\|_{\mathcal{H}^{s_0}} \le 2A$. For $j = 1, \dots, [\frac{T}{\tau}]$, since $A < 8^{-j-3}$ for $\omega \in \Omega_{T,\eta,\delta}$, we can repeatedly apply (4.28) and the continuity argument to obtain

$$\|w_{\rho,\delta}\|_{X^{s_0}(I_j)} \le 2 \cdot 8^j A.$$

Hence, we have

$$\|w_{\rho,\delta}\|_{L_T^\infty H_x^{s_0}} \le 2 \cdot 8^{[\frac{T}{\tau}]+1} \sum_{\ell=1}^{3} \| :z^\ell: - :z^\ell_{\rho,\delta}: \|_{L_T^{\frac{2}{\varepsilon}} W_x^{-\varepsilon,\frac{2}{\varepsilon}}}. \tag{4.29}$$

Finally, from (4.29) and Proposition 4.2, we see that for any $\lambda > 0$, there exists $\delta_1 \in (0, \delta_0)$ such that

$$P\big(\big\{\omega \in \Omega_{T,\eta,\delta} : \|w_{\rho,\delta}\|_{L_T^\infty H_x^{s_0}} > \lambda\big\}\big) < \frac{\eta}{4}$$

for $0 < \delta < \delta_1$. Together with (4.26), we conclude that $w_{\rho,\delta}$ converges in probability to 0 in $C([-T, T]; H^{s_0}(\mathbb{T}^2))$. Recalling that $w_{\rho,\delta} = v - v_{\rho,\delta}$, this concludes the proof of Theorem 1.6.

Remark 4.5. Since $(\phi_{0,\varepsilon}, \phi_{1,\varepsilon})$ is smooth, Theorem A with the Cameron–Martin theorem [14] implies almost sure global existence of the solution v_ε to (1.31); see [49]. Moreover, for any $T > 0$ and $\eta > 0$, there exists $\widetilde{R} = \widetilde{R}(T, \eta, \phi_{0,\varepsilon}, \phi_{1,\varepsilon})$ such that

$$P\big(\big\|(v_\varepsilon, \partial_t v_\varepsilon)\big\|_{L_T^\infty \mathcal{H}_x^{s_0}} > \widetilde{R}\big) < \frac{\eta}{4}.$$

Then, we can use this bound instead of (4.23) and repeat the argument presented above to conclude that the solution $v_{\delta,\varepsilon}$ to (1.30) converges in probability to the solution v_ε to (1.31).

5. Norm inflation for the (unrenormalized) NLW in negative Sobolev spaces

In this section, we present the proof of Theorem 1.11, norm inflation for the cubic NLW (1.34) with $k = 3$. In the remaining part of the paper, when we refer to (1.34) (and (1.1)), it is understood that $k = 3$. Furthermore, for simplicity of the presentation, we set $m = 1$, where m denotes the mass $m \geq 0$ in (1.34). Namely, we consider (1.1) with $k = 3$.

We first state the following norm inflation result for smooth initial data.

Proposition 5.1. *Let $d \in \mathbb{N}$. Suppose that $s \in \mathbb{R}$ satisfies either* (i) $s \leq -\frac{1}{2}$ *when $d = 1$ or* (ii) $s < 0$ *when $d \geq 2$. Fix $\vec{u}_0 = (u_0, u_1) \in \mathcal{S}(\mathcal{M}) \times \mathcal{S}(\mathcal{M})$. Then, given any $n \in \mathbb{N}$, there exist a solution u_n to the cubic NLW* (1.1) *with $k = 3$ and $t_n \in (0, \frac{1}{n})$ such that*

$$\big\| \big(u_n(0), \partial_t u_n(0)\big) - (u_0, u_1) \big\|_{\mathcal{H}^s(\mathcal{M})} < \frac{1}{n} \quad \text{and} \quad \big\| u_n(t_n) \big\|_{H^s(\mathcal{M})} > n. \tag{5.1}$$

Once we prove Proposition 5.1, Theorem 1.11 follows from the density of $\mathcal{S}(\mathcal{M}) \times \mathcal{S}(\mathcal{M})$ in $\mathcal{H}^s(\mathcal{M})$ and a diagonal argument. See [42,57,66]. While the basic structure of the argument is the same as that presented in [42], we establish different multilinear estimates by exploiting one degree of smoothing in the Duhamel integral operator $\mathcal{I}$ in (5.4) below. In the following, we fix $\vec{u}_0 \in \mathcal{S}(\mathcal{M}) \times \mathcal{S}(\mathcal{M})$ and may suppress the dependence of various constants on $\vec{u}_0$.

Before proceeding further, we introduce some notations. Given $\mathcal{M} = \mathbb{R}^d$ or $\mathbb{T}^d$, let $\widehat{\mathcal{M}}$ denote the Pontryagin dual of $\mathcal{M}$, i.e.

$$\widehat{\mathcal{M}} = \begin{cases} \mathbb{R}^d & \text{if } \mathcal{M} = \mathbb{R}^d, \\ \mathbb{Z}^d & \text{if } \mathcal{M} = \mathbb{T}^d. \end{cases} \tag{5.2}$$

When $\widehat{\mathcal{M}} = \mathbb{Z}^d$, we endow it with the counting measure. We then define the Fourier–Lebesgue space $\mathcal{F}L^{s,p}(\mathcal{M})$ by the norm:

$$\| f \|_{\mathcal{F}L^{s,p}(\mathcal{M})} = \big\| \langle \xi \rangle^s \widehat{f} \big\|_{L^p(\widehat{\mathcal{M}})}.$$

In particular, $\mathcal{F}L^1(\mathcal{M}) \stackrel{\text{def}}{=} \mathcal{F}L^{0,1}(\mathcal{M})$ corresponds to the Wiener algebra. We also define

$$\overrightarrow{\mathcal{F}L}^{s,p}(\mathcal{M}) \stackrel{\text{def}}{=} \mathcal{F}L^{s,p}(\mathcal{M}) \times \mathcal{F}L^{s-1,p}(\mathcal{M}). \tag{5.3}$$

In Section 5.1, we first go over local well-posedness of (1.1) in the Wiener algebra $\overrightarrow{\mathcal{F}L}^{0,1}(\mathcal{M})$. Then, we express solutions in a power series expansion in terms of initial data, where the summation ranges over all finite ternary trees. We then establish basic nonlinear estimates on the multilinear terms arising in the power series expansion in Section 5.2. In Section 5.3, we present the proof of Proposition 5.1.

5.1. *Power series expansion indexed by trees*

We define the Duhamel integral operator $\mathcal{I}$ by

$$\mathcal{I}[u_1, u_2, u_3](t) \stackrel{\text{def}}{=} -\int_0^t \frac{\sin((t - t')\langle \nabla \rangle)}{\langle \nabla \rangle} [u_1 u_2 u_3]\big(t'\big)\, dt'. \tag{5.4}$$

When all the three arguments u_1, u_2, and u_3 are identical, we use the following shorthand notation:

$$\mathcal{I}^3[u] \stackrel{\text{def}}{=} \mathcal{I}[u, u, u]. \tag{5.5}$$

We say that u is a solution to (1.1) with $(u, \partial_t u)|_{t=0} = (u_0, u_1)$ if u satisfies the following Duhamel formulation:

$$u(t) = S(t)(u_0, u_1) + \mathcal{I}^3[u](t). \tag{5.6}$$

We first state the local well-posedness of (1.1) in $\overrightarrow{\mathcal{F}L}^{0,1}(\mathcal{M})$.

Lemma 5.2. *The cubic NLW* (1.1) *with* $k = 3$ *is locally well-posed in* $\overrightarrow{\mathcal{F}L}^{0,1}(\mathcal{M})$. *More precisely, given* $\vec{u}_0 = (u_0, u_1) \in \overrightarrow{\mathcal{F}L}^{0,1}(\mathcal{M})$, *there exist* $T \sim \|\vec{u}_0\|^{-1}_{\overrightarrow{\mathcal{F}L}^{0,1}} > 0$ *and a unconditionally unique solution* $u \in C([-T, T]; \mathcal{F}L^1(\mathcal{M}))$, *satisfying* (5.6).

The unconditional uniqueness refers to the uniqueness of solutions in the entire $C([-T, T]; \mathcal{F}L^1(\mathcal{M}))$. Unconditional uniqueness is a concept of uniqueness which does not depend on how solutions are constructed.

In view of the boundedness of $S(t)$ in $\overrightarrow{\mathcal{F}L}^{0,1}(\mathcal{M})$ and the algebra property of $\mathcal{F}L^1(\mathcal{M})$ together with the bound:

$$\int_0^t \frac{|\sin((t-t')\langle\xi\rangle)|}{\langle\xi\rangle} dt' \le Ct^2 \tag{5.7}$$

uniformly in $\xi \in \widehat{\mathcal{M}}$ (also see (6.17) below) Lemma 5.2 follows from a standard fixed point argument. We omit details.

Let $\vec{u}_0 \in \overrightarrow{\mathcal{F}L}^{0,1}(\mathcal{M})$. Then, (the proof of) Lemma 5.2 guarantees the convergence of the following Picard iteration scheme:

$$P_0(\vec{\phi}) = S(t)\vec{u}_0 \quad \text{and} \quad P_j(\vec{u}_0) = S(t)\vec{u}_0 + \mathcal{I}^3\big[P_{j-1}(\vec{u}_0)\big], \quad j \in \mathbb{N}, \tag{5.8}$$

at least for short times. It follows from (5.5) and (5.8) that P_j consists of multilinear terms of degrees at most 3^j (in $\vec{u}_0$). In the following, we discuss a more general recursive scheme and express a solution in a power series indexed by trees as in [17,42]. We introduce the following notion of (ternary) trees. Our trees refer to a particular subclass of usual trees with the following properties:

Definition 5.3. (i) Given a partially ordered set $\mathcal{T}$ with partial order $\le$, we say that $b \in \mathcal{T}$ with $b \le a$ and $b \ne a$ is a child of $a \in \mathcal{T}$, if $b \le c \le a$ implies either $c = a$ or $c = b$. If the latter condition holds, we also say that a is the parent of b.

(ii) A tree $\mathcal{T}$ is a finite partially ordered set, satisfying the following properties:

- Let $a_1, a_2, a_3, a_4 \in \mathcal{T}$. If $a_4 \le a_2 \le a_1$ and $a_4 \le a_3 \le a_1$, then we have $a_2 \le a_3$ or $a_3 \le a_2$.
- A node $a \in \mathcal{T}$ is called terminal, if it has no child. A non-terminal node $a \in \mathcal{T}$ is a node with exactly three children.
- There exists a maximal element $r \in \mathcal{T}$ (called the root node) such that $a \le r$ for all $a \in \mathcal{T}$.
- $\mathcal{T}$ consists of the disjoint union of $\mathcal{T}^0$ and $\mathcal{T}^\infty$, where $\mathcal{T}^0$ and $\mathcal{T}^\infty$ denote the collections of non-terminal nodes and terminal nodes, respectively.

Note that the number $|\mathcal{T}|$ of nodes in a tree $\mathcal{T}$ is $3j + 1$ for some $j \in \mathbb{N} \cup \{0\}$, where $|\mathcal{T}^0| = j$ and $|\mathcal{T}^\infty| = 2j + 1$. Let us denote the collection of trees of j generations (i.e. with j parental nodes) by $\mathbf{T}(j)$, i.e.

$$\mathbf{T}(j) \stackrel{\text{def}}{=} \big\{\mathcal{T} : \mathcal{T} \text{ is a tree with } |\mathcal{T}| = 3j + 1\big\}.$$

Recall the following exponential bound on the number $\#\mathbf{T}(j)$ of trees of j generations. See [42] for a proof.

Lemma 5.4. *Let* $\mathbf{T}(j)$ *be as above. Then, there exists* $C > 0$ *such that*

$$\#\mathbf{T}(j) \le C^j$$

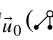
for all $j \in \mathbb{N} \cup \{0\}$.

Next, we express the solution u constructed in Lemma 5.2 in a power series indexed by trees. Fix $\vec{u}_0 \in \overrightarrow{\mathcal{F}L}^{0,1}(\mathcal{M})$. Given a tree $\mathcal{T} \in \mathbf{T}(j)$, $j \in \mathbb{N} \cup \{0\}$, we associate a multilinear operator (in $\vec{u}_0$) by the following rules:

- Replace a non-terminal node "$\circ$" by the Duhamel integral operator $\mathcal{I}$ defined in (5.4) with its three children as arguments u_1, u_2, and u_3,
- Replace a terminal node "$\bullet$" by the linear solution $S(t)\vec{u}_0$.

In the following, we denote this mapping from $\bigcup_{j=0}^\infty \mathbf{T}(j)$ to $\mathcal{D}'(\mathcal{M} \times [-T, T])$ by $\Psi_{\vec{u}_0}$.

For example, $\Psi_{\vec{u}_0}$ maps the trivial tree "$\bullet$", consisting only of the root node to the linear solution $S(t)\vec{u}_0$. Namely, we have $\Psi_{\vec{u}_0}(\bullet) = S(t)\vec{u}_0$. Similarly, we have

$\Psi_{\vec{u}_0}($  $) = \mathcal{I}^3\big[S(t)\vec{u}_0\big],$

$\Psi_{\vec{u}_0}($  $) = \mathcal{I}\big[\mathcal{I}^3\big[S(t)\vec{u}_0\big], S(t)\vec{u}_0, S(t)\vec{u}_0\big],$

where $\mathcal{I}^3$ is as in (5.5). In view of the algebra property of $\mathcal{F}L^1(\mathcal{M})$ along with the continuity and boundedness of $S(t)$, we have $\Psi_{\vec{u}_0}(\mathcal{T}) \in C([-T,T];\mathcal{F}L^1(\mathcal{M}))$ for any tree $\mathcal{T}$, provided $\vec{u}_0 \in \overrightarrow{\mathcal{F}L}^{0,1}(\mathcal{M})$. Note that, if $\mathcal{T} \in \mathbf{T}(j)$, then $\Psi_{\vec{u}_0}(\mathcal{T})$ is $(2j+1)$-linear in $\vec{u}_0$.

Lastly, we define Ξ_j by

$$\Xi_j(\vec{u}_0) \stackrel{\text{def}}{=} \sum_{\mathcal{T}\in\mathbf{T}(j)} \Psi_{\vec{u}_0}(\mathcal{T}). \tag{5.9}$$

When $j=0$ and 1, we have

$$\Xi_0(\vec{u}_0) = S(t)\vec{u}_0 \quad \text{and} \quad \Xi_1(\vec{u}_0) = \mathcal{I}^3\big[S(t)\vec{u}_0\big]. \tag{5.10}$$

Then, from Lemma 5.4, (5.9), the definition of $\Psi_{\vec{u}_0}(\mathcal{T})$, and Young's inequality together with (5.7), we obtain the following lemma. See [42].

Lemma 5.5. *There exists $C>0$ such that*

$$\big\|\Xi_j(\vec{u}_0)(t)\big\|_{\mathcal{F}L^1} \le C^j t^{2j} \|\vec{u}_0\|^{2j+1}_{\overrightarrow{\mathcal{F}L}^{0,1}}.$$

for all $\vec{u}_0 \in \overrightarrow{\mathcal{F}L}^{0,1}(\mathcal{M})$ and all $j\in\mathbb{N}$. In particular, there exist $T \sim \|\vec{u}_0\|^{-1}_{\overrightarrow{\mathcal{F}L}^{0,1}} > 0$ such that the power series expansion:

$$u = \sum_{j=0}^{\infty} \Xi_j(\vec{u}_0) = \sum_{j=0}^{\infty}\sum_{\mathcal{T}\in\mathbf{T}(j)} \Psi_{\vec{u}_0}(\mathcal{T}) \tag{5.11}$$

converges in $C([-T,T];\mathcal{F}L^1(\mathcal{M}))$.

It is easy to check that u defined by the power series (5.11) is indeed a solution to the cubic NLW (1.1). Then, thanks to the unconditional uniqueness of the solution constructed in Lemma 5.2, we conclude that the power series expansion (5.11) must agree with the solution constructed in Lemma 5.2. Note that the time of local existence in Lemma 5.2 and the time of convergence in Lemma 5.5 are of the same order $\sim \|\vec{u}_0\|^{-1}_{\overrightarrow{\mathcal{F}L}^{0,1}} > 0$.

5.2. *Multilinear estimates*

We first go over our choice of initial data for proving Proposition 5.1. Given $n\in\mathbb{N}$, fix $N = N(n) \gg 1$ (to be chosen later). We define $\vec{\phi}_n = (\phi_{0,n}, \phi_{1,n})$ by setting

$$\widehat{\phi}_{0,n}(\xi) = R \sum_{j\in\{-2,-1,1,2\}} \mathbf{1}_{jNe_1+Q_A}(\xi) \quad \text{and} \quad \phi_{1,n} = N\phi_{0,n}, \tag{5.12}$$

where $Q_A = [-\frac{A}{2}, \frac{A}{2})^d$, $e_1 = (1,0,\dots,0)$, $R = R(N) \ge 1$, and $A = A(N) \gg 1$, satisfying

$$RA^{\frac{d}{2}} \gg \|\vec{u}_0\|_{\overrightarrow{\mathcal{F}L}^{0,1}}, \quad \text{and} \quad A \ll N, \tag{5.13}$$

are to be chosen later. Note that we have

$$\|\vec{\phi}_n\|_{\mathcal{H}^s} \sim RA^{\frac{d}{2}}N^s \quad \text{and} \quad \|\vec{\phi}_n\|_{\overrightarrow{\mathcal{F}L}^{0,1}} \sim RA^d, \tag{5.14}$$

for any $s\in\mathbb{R}$. Lastly, given $\vec{u}_0 \in \mathcal{S}(\mathcal{M})\times\mathcal{S}(\mathcal{M})$, set $\vec{u}_{0,n} = (u_{0,n}, u_{1,n})$ by

$$\begin{aligned} \vec{u}_{0,n} = (u_{0,n}, u_{1,n}) &= (u_0, u_1) + (\phi_{0,n}, \phi_{1,n}) \\ &= \vec{u}_0 + \vec{\phi}_n. \end{aligned} \tag{5.15}$$

Let u_n be the corresponding solution to (1.1) with $(u_n, \partial_t u_n)|_{t=0} = \vec{u}_{0,n}$. Lemmas 5.4 and 5.5 with (5.14) guarantee the convergence of the following power series expansion:

$$u_n = \sum_{j=0}^{\infty} \Xi_j(\vec{u}_{0,n}) = \sum_{j=0}^{\infty} \Xi_j(\vec{u}_0 + \vec{\phi}_n), \tag{5.16}$$

on $[-T, T]$, as long as

$$T \lesssim \left(\|\vec{u}_0\|_{\overrightarrow{\mathcal{F}L}^{0,1}} + RA^d\right)^{-1} \sim \left(RA^d\right)^{-1}, \tag{5.17}$$

where the last equivalence follows from (5.13). Our main goal is to show that u_n satisfies (5.1) by estimating each of $\Xi_j(\vec{u}_{0,n})$ in the power series expansion (5.16).

We now state the basic multilinear estimates. Keep in mind that implicit constants in Lemma 5.6 depend on (various norms of) $\vec{u}_0$.

Lemma 5.6. *Let $\vec{u}_{0,n} = (u_{0,n}, u_{1,n})$ and $\vec{\phi}_n = (\phi_{0,n}, \phi_{1,n})$ be as in (5.15) and (5.12). Let $s < 0$. Then, there exists $C > 0$ such that*

$$\|\vec{u}_{0,n} - \vec{u}_0\|_{\mathcal{H}^s} \le CRA^{\frac{d}{2}} N^s, \tag{5.18}$$

$$\left\| \Xi_0(\vec{u}_{0,n})(t) \right\|_{H^s} \le C\left(1 + RA^{\frac{d}{2}} N^s\right), \tag{5.19}$$

$$\left\| \Xi_1(\vec{u}_{0,n})(t) - \Xi_1(\vec{\phi}_n)(t) \right\|_{H^s} \le Ct^2 \|\vec{u}_0\|_{\mathcal{H}^0} R^2 A^{2d}, \tag{5.20}$$

$$\left\| \Xi_1(\vec{\phi}_n)(t) \right\|_{H^s} \le Ct^2 R^3 A^{2d} \cdot f(A), \tag{5.21}$$

$$\left\| \Xi_j(\vec{u}_{0,n})(t) \right\|_{H^s} \le C^j t^{2j} R^{2j+1} A^{2dj} \cdot f(A), \tag{5.22}$$

for any integer $j \ge 2$, where $f(A)$ is given by

$$f(A) = \begin{cases} 1, & \text{if } s < -\frac{d}{2}, \\ (\log A)^{\frac{1}{2}}, & \text{if } s = -\frac{d}{2}, \\ A^{\frac{d}{2}+s}, & \text{if } s > -\frac{d}{2}. \end{cases} \tag{5.23}$$

This lemma in particular shows that the power series (5.16) is convergent in $C([-T, T]; H^s(\mathcal{M}))$, provided that $T^2 R^2 A^{2d} \ll 1$, which is consistent with (5.17).

Proof. Recalling that $\vec{\phi}_n = \vec{u}_{0,n} - \vec{u}_0$, the first two estimates (5.18) and (5.19) follow from (5.14) and the boundedness of $S(t)$ on $\mathcal{H}^s(\mathcal{M})$.

Next, we prove (5.20). In this case, we use the multilinearity of Ξ_1. See (5.4) and (5.10). By the Cauchy–Schwarz inequality, (5.7), and Young's inequality with (5.13), we have

$$\begin{aligned} \left\| \Xi_1(\vec{u}_{0,n})(t) - \Xi_1(\vec{\phi}_n)(t) \right\|_{H^s} &\le \left\| \Xi_1(\vec{u}_{0,n})(t) - \Xi_1(\vec{\phi}_n)(t) \right\|_{L^2} \\ &\lesssim t^2 \|\vec{u}_0\|_{\mathcal{H}^0} \left(\|\vec{u}_0\|^2_{\overrightarrow{\mathcal{F}L}^{0,1}} + \|\vec{\phi}_n\|^2_{\overrightarrow{\mathcal{F}L}^{0,1}} \right) \\ &\lesssim t^2 \|\vec{u}_0\|_{\mathcal{H}^0} \left(1 + R^2 A^{2d}\right) \\ &\lesssim t^2 \|\vec{u}_0\|_{\mathcal{H}^0} R^2 A^{2d}. \end{aligned}$$

Lastly, we consider (5.21) and (5.22). It follows from the definition (5.12) that $\operatorname{supp} \mathcal{F}[S(t)\vec{\phi}_n]$ consists of four disjoint cubes of volume $\sim A^d$. Given $\mathcal{T} \in \mathbf{T}(j)$, $\Psi_{\vec{\phi}_n}(\mathcal{T})$ is basically a $(2j+1)$-fold product of $S(t)\vec{\phi}_n$ under iterated time integrations and spatial smoothing. Hence, the spatial support of $\mathcal{F}[\Psi_{\vec{\phi}_n}(\mathcal{T})]$ consists of (at most) 4^{2j+1} cubes of volume $\sim A^d$. Namely, we have

$$\left|\operatorname{supp} \mathcal{F}\left[\Psi_{\vec{\phi}_n}(\mathcal{T})\right]\right| \le C^j A^d = \left|C_0^j Q_A\right|$$

for some $C, C_0 > 0$. Noting that, for $s < 0$, $\langle \xi \rangle^s$ is a decreasing function in $|\xi|$, we obtain

$$\left\| \langle \xi \rangle^s \right\|_{L^2_\xi(\operatorname{supp} \mathcal{F}[\Psi_{\vec{\phi}_n}(\mathcal{T})])} \le \left\| \langle \xi \rangle^s \right\|_{L^2_\xi(C_0^j Q_A)} \lesssim C^j f(A). \tag{5.24}$$

By (5.7) and Young's inequality, we have

$$\big\|\mathcal{I}[u_1,u_2,u_3](t)\big\|_{\mathcal{F}L^p} \le Ct^2\prod_{j=1}^{3}\|u_j\|_{\mathcal{F}L^{p_j}} \tag{5.25}$$

for $1\le p, p_1, p_2, p_3\le\infty$, satisfying

$$\frac{1}{p}+2=\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}.$$

Then, by first applying (5.24) and then iteratively applying (5.25), we have

$$\begin{aligned}\big\|\Psi_{\vec{\phi}_n}(\mathcal{T})(t)\big\|_{H^s} &\le \big\|\langle\xi\rangle^s\big\|_{L^2_\xi(\operatorname{supp}\mathcal{F}[\Psi_{\vec{\phi}_n}(\mathcal{T})])}\big\|\Psi_{\vec{\phi}_n}(\mathcal{T})(t)\big\|_{\mathcal{F}L^\infty}\\ &\le C^jt^{2j}f(A)\|\vec{\phi}_n\|_{\overrightarrow{\mathcal{F}L}^{0,1}}\|\vec{\phi}_n\|^{2j}_{\overrightarrow{\mathcal{F}L}^{0,q}}\\ &\le C^jt^{2j}RA^d\cdot R^{2j}A^{\frac{2j}{q}d}f(A)\\ &= C^jt^{2j}R^{2j+1}A^{2dj}f(A),\end{aligned} \tag{5.26}$$

where q satisfies $2j=\frac{2j}{q}+1$. Hence, it follows from (5.26) with (5.9) and Lemma 5.4 that

$$\big\|\Xi_j(\vec{\phi}_n)(t)\big\|_{H^s}\le C^jt^{2j}R^{2j+1}A^{2dj}f(A). \tag{5.27}$$

In particular, this proves (5.21).

Next, we estimate the difference $\Xi_j(\vec{u}_0+\vec{\phi}_n)-\Xi_j(\vec{\phi}_n)$. Since we do not know anything about the Fourier support of $\vec{u}_0$, we simply proceed with a loss in the first step:

$$\big\|\Xi_j(\vec{u}_0+\vec{\phi}_n)(t)-\Xi_j(\vec{\phi}_n)(t)\big\|_{H^s}\le\big\|\Xi_j(\vec{u}_0+\vec{\phi}_n)(t)-\Xi_j(\vec{\phi}_n)(t)\big\|_{L^2}. \tag{5.28}$$

Then, with (5.9), Lemma 5.4, the multilinearity of $\Psi_{\vec{\phi}}(\mathcal{T})$, and (5.25), we have

$$\big\|\Xi_j(\vec{u}_0+\vec{\phi}_n)(t)-\Xi_j(\vec{\phi}_n)(t)\big\|_{L^2}\le C^jt^{2j}\|\vec{u}_0\|_{\overrightarrow{\mathcal{F}L}^{0,1}}\big(\|\vec{u}_0\|^{2j}_{\overrightarrow{\mathcal{F}L}^{0,r}}+\|\vec{\phi}_n\|^{2j}_{\overrightarrow{\mathcal{F}L}^{0,r}}\big), \tag{5.29}$$

where r satisfies $2j=\frac{2j}{r}+\frac{1}{2}$. Hence, from (5.28) and (5.29), we obtain

$$\begin{aligned}\big\|\Xi_j(\vec{u}_0+\vec{\phi}_n)(t)-\Xi_j(\vec{\phi}_n)(t)\big\|_{H^s}&\le C^jt^{2j}\|\vec{u}_0\|_{\overrightarrow{\mathcal{F}L}^{0,1}}\big(\|\vec{u}_0\|^{2j}_{\overrightarrow{\mathcal{F}L}^{0,\tilde{q}}}+R^{2j}A^{2dj-\frac{d}{2}}\big)\\ &\le C^jt^{2j}\|\vec{u}_0\|_{\overrightarrow{\mathcal{F}L}^{0,1}}R^{2j}A^{2dj-\frac{d}{2}},\end{aligned} \tag{5.30}$$

where the last step follows from (5.13). Therefore, the desired estimate (5.22) follows from (5.27) and (5.30) with (5.13). □

Next, we state a crucial lemma, establishing a lower bound on $\Xi_1(\vec{\phi}_n)$. As in [16,35,42], the argument exploits the high-to-low energy transfer mechanism in $\Xi_1(\vec{\phi}_n)$.

Lemma 5.7. *Let $\vec{\phi}_n=(\phi_{0,n},\phi_{1,n})$ be as in (5.12) and $s<0$. Then, for $0<t\ll N^{-1}$, we have*

$$\big\|\Xi_1(\phi_n)(t)\big\|_{H^s}\gtrsim t^2R^3A^{2d}\cdot f(A), \tag{5.31}$$

where $f(A)$ is as in (5.23).

Proof. From (5.4), we have

$$\begin{aligned}&\mathcal{F}\big[\Xi_1(\vec{\phi}_n)(t)\big](\xi)\\ &=-\int_{\xi=\xi_1+\xi_2+\xi_3}\int_0^t\frac{\sin((t-t')\langle\xi\rangle)}{\langle\xi\rangle}\Bigg(\prod_{j=1}^{3}\mathcal{F}\big[S(t')\vec{\phi}_n\big](\xi_j)\Bigg)d\xi_1\,d\xi_2\,d\xi_3\,dt'.\end{aligned} \tag{5.32}$$

From the definition (5.12), we have $|\xi_j| \lesssim N$ for $\xi_j \in \operatorname{supp} \widehat{\phi}_{k,n}$, $k = 0, 1$. Then, for $0 < t \ll N^{-1} \ll 1$, we have

$$\begin{aligned} \cos\big(t\langle\xi_j\rangle\big) &= 1 + O\big(t^2\langle\xi_j\rangle^2\big) > \frac{1}{2}, \\ \frac{t}{2} < \frac{\sin(t\langle\xi_j\rangle)}{\langle\xi_j\rangle} &= t + O\big(t^3\langle\xi_j\rangle^2\big) \ll N^{-1}. \end{aligned} \tag{5.33}$$

Moreover, in view of $\xi = \xi_1 + \xi_2 + \xi_3$, we have

$$\frac{\sin((t-t')\langle\xi\rangle)}{\langle\xi\rangle} = t - t' + O\big(\big(t-t'\big)^3\langle\xi\rangle^2\big) > \frac{1}{2}\big(t-t'\big) \tag{5.34}$$

for $0 < t' < t \ll N^{-1} \ll 1$.

Recalling that

$$\mathbf{1}_{a+Q_A} * \mathbf{1}_{b+Q_A}(\xi) \gtrsim A^d \mathbf{1}_{a+b+Q_A}(\xi)$$

for all $a, b, \xi \in \widehat{\mathcal{M}}$ and $A \geq 1$, where $\widehat{\mathcal{M}}$ is as in (5.2), it follows from (5.32), (5.33), and (5.34) with (5.12) that

$$\big|\mathcal{F}\big[\Xi_1(\vec{\phi}_n)(t)\big](\xi)\big| \gtrsim t^2 R^3 A^{2d} \cdot \mathbf{1}_{Q_A}(\xi).$$

Lastly, noting that $\|\langle\xi\rangle^s\|_{L^2_\xi(Q_A)} \sim f(A)$, we obtain (5.31). □

5.3. *Proof of Proposition* 5.1

We conclude this section by briefly discussing the proof of Proposition 5.1. As in [42], it suffices to show that, given $n \in \mathbb{N}$, the following properties hold:

$$\begin{aligned} &\text{(i)} \quad R A^{\frac{d}{2}} N^s \ll \frac{1}{n}, \\ &\text{(ii)} \quad T^2 R^2 A^{2d} \ll 1, \\ &\text{(iii)} \quad T^2 R^3 A^{2d} \cdot f(A) \gg n, \\ &\text{(iv)} \quad T^2 R^3 A^{2d} \cdot f(A) \gg T^4 R^5 A^{4d} \cdot f(A), \\ &\text{(v)} \quad T \ll N^{-1}, \\ &\text{(vi)} \quad R A^{\frac{d}{2}} \gg 1 \end{aligned}$$

for some A, R, T, and N, depending on n. Here, $f(A)$ is as in (5.23). As mentioned before, implicit constants depend on (fixed) $\vec{u}_0 \in \mathcal{S}(\mathcal{M}) \times \mathcal{S}(\mathcal{M})$.

We first show how the conditions (i)–(vi) imply Proposition 5.1. This argument is essentially contained in [42][16] but we include it for readers' convenience. The first condition (i) together with (5.18) in Lemma 5.6 verifies the first estimate in (5.1). The second condition (ii) with (5.17) guarantees local existence of the solution u_n on $[-T, T]$ with $(u_n, \partial_t u_n)|_{t=0} = \vec{u}_{0,n}$ and the convergence of the power series expansion (5.16) in $C([-T, T]; \mathcal{F}L^1(\mathcal{M}))$. Moreover, assuming the conditions (ii)–(vi), it follows from Lemmas 5.6 and 5.7 with the power series expansion (5.16) that

$$\begin{aligned} \big\|u_n(T)\big\|_{H^s} &\geq \big\|\Xi_1(\vec{\phi}_n)(T)\big\|_{H^s} - \big\|\Xi_0(\vec{u}_{0,n})\big\|_{H^s} \\ &\quad - \big\|\Xi_1(\vec{u}_{0,n})(T) - \Xi_1(\vec{\phi}_n)(T)\big\|_{H^s} - \Big\|\sum_{j=2}^{\infty} \Xi_j(\vec{u}_{0,n})(T)\Big\|_{H^s} \\ &\gtrsim T^2 R^3 A^{2d} \cdot f(A) - \big(1 + R A^{\frac{d}{2}} N^s\big) \\ &\quad - T^2 R^2 A^{2d} \|\vec{u}_0\|_{\mathcal{H}^0} - T^4 R^5 A^{4d} \cdot f(A) \\ &\sim T^2 R^3 A^{2d} \cdot f(A) \gg n. \end{aligned}$$

[16] Simply replace $T \ll N^{-2}$ in [42] by $T^2 \ll N^{-2}$ in our setting.

This verifies the second estimate in (5.1) at time $t_n = T$. Lastly, by choosing $N = N(n)$ sufficiently large, the condition (v) guarantees that $t_n \in (0, \frac{1}{n})$. This completes the proof of Proposition 5.1.

Therefore, it remains to verify the conditions (i)–(vi). Note that the conditions (i)–(iv) are identical to those in the Schrödinger case studied in [42] with $T^2(\ll N^{-2})$ replaced $T \ll N^{-2}$. Namely, we simply use the same choices for A and R and the square root for the choice of T from [42].

• **Case 1:** $s < -\frac{d}{2}$. In this case, we set

$$A = N^{\frac{1}{d}(1-\delta)}, \qquad R = N^{2\delta}, \quad \text{and} \quad T = N^{-1-\frac{3}{2}\delta}, \tag{5.35}$$

where $\delta > 0$ is sufficiently small such that $s < -\frac{1}{2} - \frac{3}{2}\delta$.

• **Case 2:** $s = -\frac{d}{2}$. In this case, we set

$$A = \frac{N^{\frac{1}{d}}}{(\log N)^{\frac{1}{16d}}}, \qquad R = 1, \quad \text{and} \quad T = \frac{1}{N(\log N)^{\frac{1}{16}}}.$$

• **Case 3:** $-\frac{d}{2} < s < 0$. Recall that this case is relevant only for $d \geq 2$. We set

$$A = N^{\frac{2}{d}-\delta}, R = N^{-1-s+\frac{d}{2}\delta-\theta}, \quad \text{and} \quad T = N^{-1+s+\frac{1}{2}d\delta+\frac{1}{2}\theta}, \tag{5.36}$$

where $\delta \gg \theta > 0$ are sufficiently small such that

$$-2s > d\delta + \theta \quad \text{and} \quad -s\delta > 2\theta.$$

Then, by repeating the argument in [42], we see that the conditions (i)–(vi) are satisfied in each case.

Remark 5.8. It is easy to check that the choices in (5.35) of Case 1 is also valid for $s < -\frac{1}{2}$. Namely, Cases 1 and 3 are sufficient to prove Proposition 5.1 for $d \geq 2$. In particular, a logarithmic divergence as in Case 2 appears only when $d = 1$, since $s = -\frac{1}{2}$ is the scaling critical regularity.

6. Almost sure norm inflation for the Wick ordered cubic NLW

In this section, we present the proof of Proposition 1.10 on almost sure norm inflation for the Wick ordered cubic NLW on $\mathbb{T}^2$. While the discussion in Section 5 was for a general dimension $d \geq 1$, we restrict our attention to the two-dimensional case in this section.

6.1. *Local well-posedness of the Wick ordered NLW*

In this subsection, we briefly go over local well-posedness of the perturbed NLW (1.33) on $\mathbb{T}^2$. More precisely, we consider

$$\begin{cases} \partial_t^2 v + (1-\Delta)v + v^3 + \mathcal{R}(v,z) = 0, \\ (v, \partial_t v)|_{t=0} = (\phi_0, \phi_1), \end{cases} \tag{6.1}$$

where $\mathcal{R}(v, z)$ is given by

$$\mathcal{R}(v,z) = :(z+v)^3: - v^3 = 3zv^2 + 3:z^2:v + :z^3:.$$

In [53], Thomann and the first author proved almost sure local well-posedness of (6.1) via the Strichartz estimates and Lemma 2.6. Note that, while only the zero initial data[17] for (6.1) is considered in [53], the same proof applies to any $(\phi_0, \phi_1) \in \mathcal{H}^s(\mathbb{T}^2)$, $s > s_{\text{crit}} = \frac{1}{4}$. See also [26]. On the other hand, in proving Proposition 1.10, we need to *maximize* the local existence time. In this respect, the Strichartz estimates are not very efficient. In order to simultaneously handle the Wick powers and make the local existence time longer, we prove local well-posedness of (6.1) in the Fourier–Lebesgue space $\overrightarrow{\mathcal{F}L}^{\alpha,\frac{1}{1-\alpha}}(\mathbb{T}^2)$ for sufficiently small $\alpha > 0$, where $\overrightarrow{\mathcal{F}L}^{\alpha,\frac{1}{1-\alpha}}(\mathbb{T}^2)$ is as in (5.3).

[17] This corresponds to the Wick ordered NLW (1.18) with the random initial data (1.6).

Lemma 6.1. *Let $\alpha > 0$ be sufficiently small. Then, the perturbed NLW (6.1) is almost surely locally well-posed in $\overrightarrow{\mathcal{F}L}^{\alpha,\frac{1}{1-\alpha}}(\mathbb{T}^2)$ on a time interval $[-T,T]$, where*

$$T \gtrsim \Big\{\max\Big(\big\|(\phi_0,\phi_1)\big\|_{\overrightarrow{\mathcal{F}L}^{\alpha,\frac{1}{1-\alpha}}}^{\frac{1}{1-\alpha}}, K_\omega\big(1+\big\|(\phi_0,\phi_1)\big\|_{\overrightarrow{\mathcal{F}L}^{\alpha,\frac{1}{1-\alpha}}}\big)\Big)\Big\}^{-1} \tag{6.2}$$

for some almost surely finite constant $K_\omega > 0$. Moreover, we have

$$\sup_{t\in[-T,T]}\big\|v(t)\big\|_{\mathcal{F}L^{\alpha,\frac{1}{1-\alpha}}} \lesssim \big\|(\phi_0,\phi_1)\big\|_{\overrightarrow{\mathcal{F}L}^{\alpha,\frac{1}{1-\alpha}}}.$$

Proof. Let $\mathcal{I}$ be the Duhamel integral operator defined in (5.4). Then, by the algebra property of $\mathcal{F}L^1(\mathbb{T}^2)$ with (5.7), we have

$$\big\|\mathcal{I}[v_1,v_2,v_3]\big\|_{L_T^\infty \mathcal{F}L_x^1} \lesssim T^2\prod_{j=1}^3 \|v_j\|_{L_T^\infty \mathcal{F}L_x^1}. \tag{6.3}$$

On the other hand, by Sobolev's inequality, we have

$$\begin{aligned}\big\|\mathcal{I}[v_1,v_2,v_3]\big\|_{L_T^\infty H_x^{\frac12}} &\le T\big\|\langle\nabla\rangle^{-\frac12}(u_1u_2u_3)\big\|_{L_T^\infty L_x^2} \lesssim T\|u_1u_2u_3\|_{L_T^\infty L_x^{\frac43}}\\ &\le T\prod_{j=1}^3\|v_j\|_{L_T^\infty L_x^4} \lesssim T\prod_{j=1}^3\|v_j\|_{L_T^\infty H_x^{\frac12}}.\end{aligned} \tag{6.4}$$

By the interpolation of weighted ℓ^p-spaces applied to (6.3) and (6.4) with $\alpha = \theta\cdot\frac12 + (1-\theta)\cdot 0 = \frac{\theta}{2}$, we obtain

$$\big\|\mathcal{I}[v_1,v_2,v_3]\big\|_{L_T^\infty \mathcal{F}L_x^{\alpha,\frac{1}{1-\alpha}}} \lesssim T^{2(1-\alpha)}\prod_{j=1}^3\|v_j\|_{L_T^\infty \mathcal{F}L_x^{\alpha,\frac{1}{1-\alpha}}} \tag{6.5}$$

for $0<\alpha<\frac12$.

Next, we consider the terms

$$\begin{aligned}\mathrm{I} &= \int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle}[v_1v_2z](t')\,dt',\\ \mathrm{II} &= \int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle}\big[v:z^2:\big](t')\,dt',\\ \mathrm{III} &= \int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle}:z^3(t'):\,dt'.\end{aligned} \tag{6.6}$$

By Proposition 1.1, there exists an almost surely finite constant $K_\omega>0$ such that

$$\big\|:z^\ell:\big\|_{L^\infty([-1,1];W_x^{-\frac{\alpha}{2},\infty})} \le K_\omega \tag{6.7}$$

for $\ell=1,2,3$. Let $0<T\le 1$ in the following. Note that Hausdorff–Young's inequality yields that $\mathcal{F}L^{\frac{1}{1-\alpha}}(\mathbb{T}^2)\hookrightarrow L^{\frac{1}{\alpha}}(\mathbb{T}^2)$, in particular, $\mathcal{F}L^{\frac{1}{1-\alpha}}(\mathbb{T}^2)\hookrightarrow L^4(\mathbb{T}^2)$ holds if $0<\alpha\le\frac14$. Then, it follows from Hölder's inequality, Lemma 2.3, and (6.7) with $\alpha<\frac12$ that

$$\begin{aligned}\|\mathrm{I}\|_{L_T^\infty \mathcal{F}L_x^{\alpha,\frac{1}{1-\alpha}}} &\le T\big\|\langle\nabla\rangle^{-1+\alpha}[v_1v_2z]\big\|_{L_T^\infty \mathcal{F}L_x^{0,\frac{1}{1-\alpha}}} \lesssim T\big\|\langle\nabla\rangle^{-\frac{\alpha}{2}}[v_1v_2z]\big\|_{L_T^\infty L_x^2}\\ &\lesssim T\big\|\langle\nabla\rangle^{\frac{\alpha}{2}}[v_1v_2]\big\|_{L_T^\infty L_x^2}\big\|\langle\nabla\rangle^{-\frac{\alpha}{2}}z\big\|_{L_T^\infty L_x^{\frac{4}{\alpha}}}\\ &\lesssim TK_\omega\|v_1\|_{L_T^\infty \mathcal{F}L_x^{\alpha,\frac{1}{1-\alpha}}}\|v_2\|_{L_T^\infty \mathcal{F}L_x^{\alpha,\frac{1}{1-\alpha}}}.\end{aligned} \tag{6.8}$$

Similarly, by Hölder's inequality, Lemma 2.3, and (6.7), we have

$$\begin{aligned}\|\mathrm{II}\|_{L^\infty_T \mathcal{F}L_x^{\alpha,\frac{1}{1-\alpha}}} &\lesssim T\big\|\langle\nabla\rangle^{-\frac{\alpha}{2}}\big[v:z^2:\big]\big\|_{L^\infty_T L^2_x}\\ &\lesssim T\big\|\langle\nabla\rangle^{\frac{\alpha}{2}}v\big\|_{L^\infty_T L^2_x}\big\|\langle\nabla\rangle^{-\frac{\alpha}{2}}:z^2:\big\|_{L^\infty_T L_x^{\frac{4}{\alpha}}}\\ &\lesssim TK_\omega\|v\|_{L^\infty_T \mathcal{F}L_x^{\alpha,\frac{1}{1-\alpha}}} \end{aligned}\tag{6.9}$$

and

$$\|\mathrm{III}\|_{L^\infty_T \mathcal{F}L_x^{\alpha,\frac{1}{1-\alpha}}} \lesssim T\big\|\langle\nabla\rangle^{-\frac{\alpha}{2}}:z^3:\big\|_{L^\infty_T L^2_x} \le TK_\omega. \tag{6.10}$$

By putting (6.5), (6.8), (6.9), and (6.10) together, a standard fixed point argument establishes almost sure local well-posedness of (6.1), provided that $T = T(\omega)$ such sufficiently small such that

$$\begin{aligned} T^{2(1-\alpha)}\big\|(\phi_0,\phi_1)\big\|^2_{\overrightarrow{\mathcal{F}L}^{\alpha,\frac{1}{1-\alpha}}} &\lesssim 1,\\ TK_\omega\big(1+\big\|(\phi_0,\phi_1)\big\|_{\overrightarrow{\mathcal{F}L}^{\alpha,\frac{1}{1-\alpha}}}\big) &\lesssim 1, \end{aligned}$$

yielding the condition (6.2). □

6.2. *Proof of Proposition* 1.10

In this subsection, we present the proof of Proposition 1.10. We prove this almost sure norm inflation result by viewing (1.33) as the (unrenormalized) NLW (1.1) with a random perturbation and invoking the norm inflation result (Proposition 5.1) for the cubic NLW (1.1).

Let $s < 0$. Given $n \in \mathbb{N}$, fix $N = N(n) \gg 1$ to be chosen later. Let $\vec{\phi}_n = (\phi_{0,n}, \phi_{1,n})$ be as in (5.12) with $A = A(N)$, $R = R(N)$, and $T = T(N) > 0$ as in Section 5.3. Then, by taking some small $\alpha > 0$, we have

$$T \lesssim \|\vec{\phi}_n\|^{-\frac{1}{1-\alpha}}_{\overrightarrow{\mathcal{F}L}^{\alpha,\frac{1}{1-\alpha}}} = \big(RA^{2(1-\alpha)}N^\alpha\big)^{-\frac{1}{1-\alpha}} = \big(RN^\alpha\big)^{-\frac{1}{1-\alpha}}A^{-2}. \tag{6.11}$$

In fact, when $s < -\frac{1}{2}$, it follows from (5.35) and Remark 5.8 that

$$T\big(RN^\alpha\big)^{\frac{1}{1-\alpha}}A^2 = N^{-1-\frac{3}{2}\delta}N^{\frac{2\delta+\alpha}{1-\alpha}}N^{1-\delta} = N^{-\frac{\delta-(2+5\delta)\alpha}{2(1-\alpha)}}. \tag{6.12}$$

Hence, (6.11) holds, provided that $\alpha < \frac{\delta}{2+5\delta}$. When $-\frac{1}{2} \le s < 0$, (5.36) yields

$$T\big(RN^\alpha\big)^{\frac{1}{1-\alpha}}A^2 = N^{-1+s+\delta+\frac{\theta}{2}}N^{\frac{-1-s+\delta-\theta+\alpha}{1-\alpha}}N^{2-2\delta} = N^{-\frac{\theta+(2s-2\delta+\theta)\alpha}{2(1-\alpha)}}, \tag{6.13}$$

and hence (6.11) holds, provided that $\alpha < \frac{\theta}{-2s+2\delta-\theta}$.

Let $u = u(n)$ and $v = v(n)$ be the solutions to the unrenormalized NLW (1.1) and the perturbed NLW (1.33) with the initial data $\vec{\phi}_n$, respectively. Then, the above observation guarantees that u and v exist on $[-T, T]$. Moreover, in view of (5.17), the power series expansion (5.16) for u_n (with $\vec{u}_0 = 0$) converges uniformly on $[-T, T]$. Then, it follows from Proposition 5.1 that

$$\big\|u(T)\big\|_{H^s} \gg n \tag{6.14}$$

for suitably chosen $N = N(n, \omega) \gg 1$. Therefore, Proposition 1.10 follows from (6.14) once we prove the following approximation lemma.

Lemma 6.2. *Given $n \in \mathbb{N}$, let $u = u(n)$, $v = v(n)$, and $T = T(n)$ be as above. Namely, they are the solutions to the unrenormalized NLW (1.1) and the perturbed NLW (1.33) with the initial data $\vec{\phi}_n$, respectively. Then, there exists $C_\omega > 0$, almost surely tending to 0 as $n \to 0$ (and hence $N \to \infty$), such that*

$$\sup_{t\in[-T,T]}\big\|u(t)-v(t)\big\|_{L^2_x} \le C_\omega. \tag{6.15}$$

Proof. By our choice of T in (5.35) and (5.36), (6.12), (6.13), and the local well-posedness of (1.1) and (1.33) in $\overrightarrow{\mathcal{F}L}^{0,1}(\mathbb{T}^2)$ and $\overrightarrow{\mathcal{F}L}^{\alpha,\frac{1}{1-\alpha}}(\mathbb{T}^2)$, respectively, there exists $\varepsilon > 0$ such that

$$\begin{aligned} T^2\|u\|^2_{L^\infty_T\mathcal{F}L^1_x} + T^{2(1-\alpha)}\|v\|^2_{L^\infty_T\mathcal{F}L^{\alpha,\frac{1}{1-\alpha}}_x} \\ \lesssim T^2\|\vec{\phi}_n\|^2_{\overrightarrow{\mathcal{F}L}^{0,1}} + T^{2(1-\alpha)}\|\vec{\phi}_n\|^2_{\overrightarrow{\mathcal{F}L}^{\alpha,\frac{1}{1-\alpha}}} \ll T^{2\varepsilon}. \end{aligned} \tag{6.16}$$

By Young's inequality with (5.7), we have

$$\big\|\mathcal{I}[u_1,u_2,u_3]\big\|_{L^\infty_T L^2_x} \le T^2\|u_1\|_{L^\infty_T L^2_x}\prod_{j=2}^{3}\|u_j\|_{L^\infty_T\mathcal{F}L^1_x}. \tag{6.17}$$

By Hölder's and Young's inequalities with a variant of (5.7), we have

$$\begin{aligned} \big\|\mathcal{I}[u_1,u_2,u_3]\big\|_{L^\infty_T L^2_x} &\lesssim \bigg\|\int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle^{1-2\alpha-2\varepsilon}}[u_1u_2u_3](t')\,dt'\bigg\|_{L^\infty_T\mathcal{F}L_x^{\frac{2}{1-2\alpha-\varepsilon}}} \\ &\lesssim T^{2-2\alpha-2\varepsilon}\|u_1\|_{L^\infty_T L^2_x}\prod_{j=2}^{3}\|u_j\|_{L^\infty_T\mathcal{F}L_x^{\frac{4}{4-2\alpha-\varepsilon}}} \\ &\lesssim T^{2-2\alpha-2\varepsilon}\|u_1\|_{L^\infty_T L^2_x}\prod_{j=2}^{3}\|u_j\|_{L^\infty_T\mathcal{F}L_x^{\alpha,\frac{1}{1-\alpha}}}. \end{aligned} \tag{6.18}$$

Similarly, we have

$$\begin{aligned} \big\|\mathcal{I}[u_1,u_2,u_3]\big\|_{L^\infty_T L^2_x} &\lesssim \bigg\|\int_0^t \frac{\sin((t-t')\langle\nabla\rangle)}{\langle\nabla\rangle^{1-\alpha-\varepsilon}}[u_1u_2u_3](t')\,dt'\bigg\|_{L^\infty_T\mathcal{F}L_x^{\frac{4}{2-2\alpha-\varepsilon}}} \\ &\lesssim T^{2-\alpha-\varepsilon}\|u_1\|_{L^\infty_T L^2_x}\|u_2\|_{L^\infty_T\mathcal{F}L^1_x}\|u_3\|_{L^\infty_T\mathcal{F}L_x^{\frac{4}{4-2\alpha-\varepsilon}}} \\ &\lesssim T^{2-\alpha-\varepsilon}\|u_1\|_{L^\infty_T L^2_x}\|u_2\|_{L^\infty_T\mathcal{F}L^1_x}\|u_3\|_{L^\infty_T\mathcal{F}L_x^{\alpha,\frac{1}{1-\alpha}}}. \end{aligned} \tag{6.19}$$

Hence, it follows from (6.17), (6.18), and (6.19) that

$$\begin{aligned} \big\|\mathcal{I}^3[u]-\mathcal{I}^3[v]\big\|_{L^\infty_T L^2_x} &\lesssim T^{-2\varepsilon}\Big\{T^2\|u\|^2_{L^\infty_T\mathcal{F}L^1_x} + T^{2(1-\alpha)}\|v\|^2_{L^\infty_T\mathcal{F}L_x^{\alpha,\frac{1}{1-\alpha}}}\Big\}\|u-v\|_{L^\infty_T L^2_x} \\ &\ll \|u-v\|_{L^\infty_T L^2_x}, \end{aligned} \tag{6.20}$$

where the last inequality follows from (6.16).

Let I, II, and III be as in (6.6). Then, proceeding as in the proof of Lemma 6.1 with a variant of (5.7), we obtain

$$\begin{aligned} \|\mathrm{I}\|_{L^\infty_T L^2_x} &\lesssim T^{2-\frac{\alpha}{2}}\big\|\langle\nabla\rangle^{-\frac{\alpha}{2}}\big[v^2z\big]\big\|_{L^\infty_T L^2_x} \lesssim T^{2-\frac{\alpha}{2}}K_\omega\|v\|^2_{L^\infty_T\mathcal{F}L_x^{\alpha,\frac{1}{1-\alpha}}}, \\ \|\mathrm{II}\|_{L^\infty_T L^2_x} &\le T^{2-\frac{\alpha}{2}}\big\|\langle\nabla\rangle^{-\frac{\alpha}{2}}\big[v:z^2:\big]\big\|_{L^\infty_T L^2_x} \lesssim T^{2-\frac{\alpha}{2}}K_\omega\|v\|_{L^\infty_T\mathcal{F}L_x^{\alpha,\frac{1}{1-\alpha}}}, \\ \|\mathrm{III}\|_{L^\infty_T L^2_x} &\le T^{2-\frac{\alpha}{2}}\big\|\langle\nabla\rangle^{-\frac{\alpha}{2}}:z^3:\big\|_{L^\infty_T L^2_x} \lesssim T^{2-\frac{\alpha}{2}}K_\omega. \end{aligned}$$

Hence, in view of (6.16), we can choose $n \gg 1$ (and hence $N \gg 1$ and $T \ll 1$) depending on ω such that

$$\|\mathrm{I}\|_{L^\infty_T L^2_x} + \|\mathrm{II}\|_{L^\infty_T L^2_x} + \|\mathrm{III}\|_{L^\infty_T L^2_x} \ll 1. \tag{6.21}$$

Finally, noting that $u = \mathcal{I}^3[u]$ and $v = \mathcal{I}^3[v] + \mathrm{I} + \mathrm{II} + \mathrm{III}$, the desired bound (6.15) follows from (6.20) and (6.21). □

Appendix: On almost sure convergence of stochastic objects

We present the proof of Proposition 2.7. First, we show the following lemma which relates the decay in the hypothesis of Proposition 2.7 to the boundedness of the relevant norms.

Lemma A.1. *Let $\{X_N\}$ and X satisfy the assumption in Proposition 2.7.*

(i) *For $p \geq 1$, $s < s_0$, $t \in [0, T]$, and $N \geq 1$, we have*

$$\mathbb{E}\big[\|X(t)\|_{W^{s,\infty}}^p\big] \lesssim p^{\frac{kp}{2}}, \tag{A.1}$$

$$\mathbb{E}\big[\|X_N(t) - X(t)\|_{W^{s,\infty}}^p\big] \lesssim p^{\frac{kp}{2}} N^{-\gamma p}. \tag{A.2}$$

(ii) *For $p \geq 1$, $s < s_0 - \frac{\theta}{2}$, $t \in [0, T]$, $h \in [-1, 1]$, and $N \geq 1$, we have*

$$\mathbb{E}\big[\|\delta_h X(t)\|_{W^{s,\infty}}^p\big] \lesssim |h|^{\theta p}, \tag{A.3}$$

$$\mathbb{E}\big[\|\delta_h X_N(t) - \delta_h X(t)\|_{W^{s,\infty}}^p\big] \lesssim N^{-\gamma p}|h|^{\theta p}. \tag{A.4}$$

Proof. We only consider the proof of (A.1) since the remaining estimates follow from the same argument with (2.11), (2.12), and (2.13).

From $s < s_0$ and Sobolev's inequality, there exists finite $r > 1$ such that $W^{\frac{s_0-s}{2},r}(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$. Then, it follows from Lemma 2.5 that

$$\begin{aligned}\big\|\|X(t)\|_{W^{s,\infty}}\big\|_{L^p(\Omega)} &\lesssim \big\|\|\langle\nabla\rangle^{s+\frac{s_0-s}{2}} X(t)\|_{L^p(\Omega)}\big\|_{L^r} \\ &\lesssim p^{\frac{k}{2}} \big\|\|\langle\nabla\rangle^{s+\frac{s_0-s}{2}} X(t)\|_{L^2(\Omega)}\big\|_{L^r}\end{aligned} \tag{A.5}$$

for $p \geq r$.

Now, note that the spatially homogeneity yields that

$$\mathbb{E}\big[\widehat{X}(n_1, t)\widehat{X}(n_2, t)\big] = 0 \tag{A.6}$$

if $n_1 + n_2 \neq 0$. Indeed, we have

$$\begin{aligned}\mathbb{E}\big[\widehat{X}(n_1, t)\widehat{X}(n_2, t)\big] &= \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \mathbb{E}\big[X(x_1, t)X(x_2, t)\big]e^{-i(n_1\cdot x_1+n_2\cdot x_2)}\,dx_1\,dx_2 \\ &= \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \mathbb{E}\big[X(x_1, t)X(x_2, t)\big]e^{-i(n_1+n_2)\cdot x_1+in_2\cdot(x_1-x_2)}\,dx_1\,dx_2.\end{aligned}$$

The spatially homogeneity implies that $\mathbb{E}[X(x_1, t)X(x_2, t)]$ is a function of $x_1 - x_2$. Then, by a change of variables $y_2 = x_1 - x_2$, we have, for some function F on $\mathbb{T}^d$,

$$\mathbb{E}\big[\widehat{X}(n_1, t)\widehat{X}(n_2, t)\big] = \int_{\mathbb{T}^d} \widehat{F}(n_2)e^{-i(n_1+n_2)\cdot x_1}\,dx_1,$$

which vanishes unless $n_1 + n_2 = 0$. We thus obtain (A.6). Therefore, from (A.6) and (2.10), we have

$$\|\langle\nabla\rangle^{s+\frac{s_0-s}{2}} X(x, t)\|_{L^2(\Omega)}^2 \lesssim \sum_{n\in\mathbb{Z}^d} \langle n\rangle^{s+s_0}\mathbb{E}\big[|\widehat{X}(n, t)|^2\big] \lesssim \sum_{n\in\mathbb{Z}^d} \langle n\rangle^{-d+s-s_0} \lesssim 1.$$

By combining this with (A.5), we obtain (A.1). □

We now present the proof of Proposition 2.7.

Proof of Proposition 2.7. (i) From (A.1), we have $X(t) \in W^{s,\infty}(\mathbb{T}^d)$ almost surely. Given $j \in \mathbb{N}$, it follows from Chebyshev's inequality and (A.2) that

$$\sum_{N=1}^\infty P\bigg(\|X_N(t) - X(t)\|_{W^{s,\infty}} > \frac{1}{j}\bigg) \lesssim \sum_{N=1}^\infty e^{-cN^{\frac{2\gamma}{k}} j^{-\frac{2}{k}}} < \infty.$$

Therefore, we conclude from the Borel–Cantelli lemma that there exists Ω_j with $P(\Omega_j) = 1$ such that for each $\omega \in \Omega_j$, there exists $M = M(\omega) \in \mathbb{N}$ such that $\|X_N(t;\omega) - X(t;\omega)\|_{W^{s,\infty}} < \frac{1}{j}$ for any $N \geq M$. By setting $\Sigma = \bigcap_{j=1}^{\infty} \Omega_j$, we have $P(\Sigma) = 1$. Hence, we conclude that $X_N(t)$ converges almost surely to $X(t)$ in $W^{s,\infty}(\mathbb{T}^d)$. Note that the set of almost sure convergence depends on $t \in [0, T]$ at this point.

(ii) Next, we prove the second part of Proposition 2.7. By (A.3), Kolmogorov's continuity criterion implies that $X \in C([0,T]; W^{s,\infty}(\mathbb{T}^d))$ almost surely. We now modify the proof of Kolmogorov's continuity criterion to prove almost sure convergence of $\{X_N\}_{N\in\mathbb{N}}$ in $C([0,T]; W^{s,\infty}(\mathbb{T}^d))$.

In the following, fix $t \in [0,T]$ and $h \in [-1,1]$ (such that $t + h \in [0,T]$). We choose $p \gg 1$ such that

$$\theta p \geq 1 + \varepsilon > 1 \quad \text{and} \quad \gamma p > 2. \tag{A.7}$$

Let $Y_N = X_N - X$. Then, for any $\alpha > 0$, it follows from Chebyshev's inequality, (A.4), and (A.7) that

$$\begin{aligned}
&P\bigg(\sup_{N\in\mathbb{N}} \max_{j=1,\dots,2^\ell} N^{\frac{\gamma}{2}} \Big\| Y_N\Big(\frac{j}{2^\ell}\Big) - Y_N\Big(\frac{j-1}{2^\ell}\Big)\Big\|_{W^{s,\infty}} \geq 2^{-\alpha\ell}\bigg) \\
&= P\bigg(\bigcup_{N\in\mathbb{N}} \bigcup_{j=1}^{2^\ell} \Big\| Y_N\Big(\frac{j}{2^\ell}\Big) - Y_N\Big(\frac{j-1}{2^\ell}\Big)\Big\|_{W^{s,\infty}} \geq N^{-\frac{\gamma}{2}} 2^{-\alpha\ell}\bigg) \\
&\leq \sum_{N=1}^{\infty} \sum_{j=1}^{2^\ell} P\bigg(\Big\| Y_N\Big(\frac{j}{2^\ell}\Big) - Y_N\Big(\frac{j-1}{2^\ell}\Big)\Big\|_{W^{s,\infty}} \geq N^{-\frac{\gamma}{2}} 2^{-\alpha\ell}\bigg) \\
&\leq \sum_{N=1}^{\infty} \sum_{j=1}^{2^\ell} N^{\frac{\gamma p}{2}} 2^{\alpha p \ell} \mathbb{E}\bigg[\Big\| Y_N\Big(\frac{j}{2^\ell}\Big) - Y_N\Big(\frac{j-1}{2^\ell}\Big)\Big\|_{W^{s,\infty}}^p\bigg] \\
&\lesssim 2^{(\alpha p - \varepsilon)\ell} \sum_{N=1}^{\infty} N^{-\frac{\gamma p}{2}} \lesssim 2^{(\alpha p - \varepsilon)\ell}.
\end{aligned}$$

Now, let $\alpha \in (0, \frac{\varepsilon}{p})$, i.e. $\alpha p - \varepsilon < 0$. Then, summing over $\ell \in \mathbb{N}$, we obtain

$$\sum_{\ell=0}^{\infty} P\bigg(\sup_{N\in\mathbb{N}} \max_{j=1,\dots,2^\ell} N^{\frac{\gamma}{2}} \Big\| Y_N\Big(\frac{j}{2^\ell}\Big) - Y_N\Big(\frac{j-1}{2^\ell}\Big)\Big\|_{W^{s,\infty}} \geq 2^{-\alpha\ell}\bigg) < \infty.$$

Hence, by the Borel–Cantelli lemma, there exists a set $\widetilde{\Sigma} \subset \Omega$ with $P(\widetilde{\Sigma}) = 1$ such that, for each $\omega \in \widetilde{\Sigma}$, we have

$$\sup_{N\in\mathbb{N}} \max_{j=1,\dots,2^\ell} N^{\frac{\gamma}{2}} \Big\| Y_N\Big(\frac{j}{2^\ell};\omega\Big) - Y_N\Big(\frac{j-1}{2^\ell};\omega\Big)\Big\|_{W^{s,\infty}} \leq 2^{-\alpha\ell}$$

for all $\ell \geq L = L(\omega)$. This in particular implies that there exists $C = C(\omega) > 0$ such that

$$\max_{j=1,\dots,2^\ell} \Big\| Y_N\Big(\frac{j}{2^\ell};\omega\Big) - Y_N\Big(\frac{j-1}{2^\ell};\omega\Big)\Big\|_{W^{s,\infty}} \leq C(\omega) N^{-\frac{\gamma}{2}} 2^{-\alpha\ell} \tag{A.8}$$

for any $\ell \geq 0$, uniformly in $N \in \mathbb{N}$.

For simplicity, let $T = 1$ and $t \in [0,1]$. Express t in the following binary expansion:

$$t = \sum_{j=1}^{\infty} \frac{b_j}{2^j}, \tag{A.9}$$

where $b_j \in \{0,1\}$. Let $t_\ell = \sum_{j=1}^{\ell} \frac{b_j}{2^j}$ and $t_0 = 0$. Then, from (A.8), we have

$$\|Y_N(t;\omega)\|_{W^{s,\infty}} \leq \sum_{\ell=1}^{\infty} \|Y_N(t_\ell;\omega) - Y_N(t_{\ell-1};\omega)\|_{W^{s,\infty}} + \|Y_N(0;\omega)\|_{W^{s,\infty}}$$

$$\le C(\omega)N^{-\frac{\gamma}{2}}\sum_{\ell=1}^{\infty}2^{-\alpha\ell}+\big\|Y_N(0;\omega)\big\|_{W^{s,\infty}}$$

$$\le C'(\omega)N^{-\frac{\gamma}{2}}+\big\|Y_N(0;\omega)\big\|_{W^{s,\infty}} \tag{A.10}$$

for $\omega\in\widetilde{\Sigma}$. Note that the right-hand side of (A.10) is independent of $t\in[0,1]$. Hence, by taking a supremum in $t\in[0,1]$, we obtain

$$\big\|X_N(\omega)-X(\omega)\big\|_{C([0,1];W^{s,\infty}(\mathbb{T}^d))}\le C'(\omega)N^{-\frac{\gamma}{2}}+\big\|Y_N(0;\omega)\big\|_{W^{s,\infty}} \longrightarrow 0,$$

as $N\to\infty$. Here, we used Part (i) of Proposition 2.7; $Y_N(0)=X_N(0)-X(0)$ converges to 0 in $W^{s,\infty}(\mathbb{T}^d)$, almost surely. This yields almost sure convergence of $\{X_N\}_{N\in\mathbb{N}}$ in $C([0,1];W^{s,\infty}(\mathbb{T}^d))$, which completes the proof of Proposition 2.7. □

Remark A.2. By slightly modifying the argument, we can also prove that X_N converges almost surely to X in $C^\alpha([0,1];W^{s,\infty}(\mathbb{T}^d))$ for $\alpha<\frac{\varepsilon}{p}$ (and hence $\alpha<\theta$ by taking $p\to\infty$ in view of (A.7)).

Let $t,\tau\in[0,1]$ such that $\frac{1}{2^{j-1}}\le|t-\tau|\le\frac{1}{2^j}$. Express t and τ in the binary expansions (A.9) and

$$\tau=\sum_{j=1}^{\infty}\frac{c_j}{2^j},$$

where $c_j\in\{0,1\}$, and set $\tau_\ell=\sum_{j=1}^{\ell}\frac{c_j}{2^j}$. Then, from (A.8), we have

$$\begin{aligned}\big\|Y_N(t;\omega)-Y_N(\tau;\omega)\big\|_{W^{s,\infty}} &\le \sum_{\ell=j+1}^{\infty}\big\|Y_N(t_\ell;\omega)-Y_N(t_{\ell-1};\omega)\big\|_{W^{s,\infty}}\\ &\quad+\big\|Y_N(t_j;\omega)-Y_N(\tau_j;\omega)\big\|_{W^{s,\infty}}\\ &\quad+\sum_{\ell=j+1}^{\infty}\big\|Y_N(\tau_\ell;\omega)-Y_N(\tau_{\ell-1};\omega)\big\|_{W^{s,\infty}}\\ &\le C(\omega)N^{-\frac{\gamma}{2}}\sum_{\ell=j}^{\infty}2^{-\alpha\ell}\\ &\le C'(\omega)N^{-\frac{\gamma}{2}}2^{-\alpha j}\end{aligned}$$

for $\omega\in\widetilde{\Sigma}$. Then, dividing both sides by $2^{-\alpha j}$ and taking a supremum in $t\ne\tau$, we obtain

$$\|X_N-X\|_{C^\alpha([0,1];W^{s,\infty}(\mathbb{T}^d))}\le C''(\omega)N^{-\frac{\gamma}{2}},$$

which tends to 0 as $N\to\infty$.

Remark A.3. If $\{X_N\}$ satisfies the assumption of Corollary 2.8, then by proceeding as in the proof of Lemma A.1, we have

(i) For $p\ge1$, $s<s_0$, $t\in[0,T]$, and $M\ge N\ge1$, we have

$$\mathbb{E}\Big[\big\|X_N(t)\big\|_{W^{s,\infty}}^p\Big]\lesssim p^{\frac{kp}{2}}, \tag{A.11}$$

$$\mathbb{E}\Big[\big\|X_N(t)-X_M(t)\big\|_{W^{s,\infty}}^p\Big]\lesssim p^{\frac{kp}{2}}N^{-\gamma p}. \tag{A.12}$$

(ii) For $p\ge1$, $s<s_0-\frac{\theta}{2}$, $t\in[0,T]$, $h\in[-1,1]$, and $M\ge N\ge1$, we have

$$\mathbb{E}\Big[\big\|\delta_hX_N(t)\big\|_{W^{s,\infty}}^p\Big]\lesssim_p |h|^{\theta p}, \tag{A.13}$$

$$\mathbb{E}\Big[\big\|\delta_hX_N(t)-\delta_hX_M(t)\big\|_{W^{s,\infty}}^p\Big]\lesssim_p N^{-\gamma p}|h|^{\theta p}. \tag{A.14}$$

It follows from (A.11) and (A.12) that $X_N(t)$ converges to some $X(t)$ in $L^p(\Omega; W^{s,\infty}(\mathbb{T}^d))$ and also in $W^{s,\infty}(\mathbb{T}^d)$, almost surely. Moreover, (A.1) and (A.2) hold. Then, by applying Fatou's lemma applied to (A.13) and (A.14) (in taking $N \to \infty$), we obtain (A.3) and (A.4), which allows us to repeat the proof of Proposition 2.7.

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