Electron. Commun. Probab. **27** (2022), article no. 59, 1–12.
https://doi.org/10.1214/22-ECP503
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Simple halfspace depth*

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Abstract

The halfspace depth is a prominent tool of nonparametric inference for multivariate data. We consider it in the general context of finite Borel measures μ on $\mathbb{R}^d$. The halfspace depth of a point $x \in \mathbb{R}^d$ is defined as the infimum of the μ-masses of halfspaces that contain x. We say that a measure μ has a simple (halfspace) depth if the set of all attained halfspace depth values of μ on $\mathbb{R}^d$ is finite. We give a complete description of measures with simple depths by showing that the halfspace depth of μ is simple if and only if μ is atomic with finitely many atoms. This result completely resolves the halfspace depth characterization problem for the particular situation of simple halfspace depths and datasets. We also discuss the cardinality of the set of the attained halfspace depth values.

Keywords: characterization; flag halfspace; halfspace depth; statistical depth; Tukey depth.
MSC2020 subject classifications: 62H05; 62G35.
Submitted to ECP on April 10, 2022, final version accepted on November 27, 2022.

1 Introduction: simple halfspace depth

In multivariate spaces $\mathbb{R}^d$ with $d > 1$ no canonical notions of quantiles, ranks, or orderings exist. To perform nonparametric analysis of such data, one therefore often considers data-dependent orderings of points based on the so-called statistical depth functions. Depths are supposed to quantify the centrality of any point $x \in \mathbb{R}^d$ with respect to (w.r.t.) a dataset in $\mathbb{R}^d$, or more generally, w.r.t. a given (probability) measure μ on $\mathbb{R}^d$. The higher the depth of x is, the more centrally positioned x is in the mass of μ. Many statistical depth functions have been developed in the past decades [20]. We focus on the classical halfspace depth (also called Tukey depth) [1, 19], which is already for 40 years a subject of active research in statistics and probability [3, 4, 8, 9, 10, 14, 16].

Denote by $\mathcal{M}\left(\mathbb{R}^d\right)$ the collection of all finite Borel measures on $\mathbb{R}^d$, and let $\mathbb{S}^{d-1} = \left\{x \in \mathbb{R}^d \colon \|x\| = 1\right\}$ be the unit sphere in $\mathbb{R}^d$. For $x \in \mathbb{R}^d$, write

$$\mathcal{H}(x) = \left\{\left\{y \in \mathbb{R}^d \colon \langle y, v\rangle \geq \langle x, v\rangle\right\} : v \in \mathbb{S}^{d-1}\right\}$$

for the set of all closed halfspaces that contain x on their boundary. The *halfspace depth* of $x \in \mathbb{R}^d$ w.r.t. $\mu \in \mathcal{M}\left(\mathbb{R}^d\right)$ is defined by

$$D\left(x;\mu\right) = \inf\left\{\mu(H) \colon H \in \mathcal{H}(x)\right\}. \tag{1.1}$$

*P. Laketa was supported by the OP RDE project "International mobility of research, technical and administrative staff at the Charles University", grant CZ.02.2.69/0.0/0.0/18_053/0016976. The work of S. Nagy was supported by Czech Science Foundation (EXPRO project n. 19-28231X).

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The motivation for this work comes from the halfspace depth characterization problem: Given a measure $\mu \in \mathcal{M}\left(\mathbb{R}^d\right)$, is it possible that there exists $\nu \neq \mu$ such that $D\left(x;\mu\right) = D\left(x;\nu\right)$ for all $x \in \mathbb{R}^d$? Measures μ for which the answer is negative are said to be *characterized* by their halfspace depth. The problem of determination of all characterized measures is of importance in multivariate inference, as only for them the halfspace depth provides a valid nonparametric representative. It is known that for $d > 1$ there are measures that are not characterized by halfspace depth [13]. Nevertheless, large collections of measures are known to be characterized. In particular, finitely atomic measures, that is purely atomic measures with finitely many atoms, are determined by their halfspace depth uniquely [6, 18], and efficient algorithms for their reconstruction from depth have been developed [7]. The general problem of identifying all measures with unique halfspace depth is, however, still open [12].

We say that a measure $\mu \in \mathcal{M}\left(\mathbb{R}^d\right)$ has a *simple depth* if the halfspace depth function $x \mapsto D\left(x;\mu\right)$ attains only finitely many different values. Finitely atomic measures have simple depths. The main result of the present note is the converse to this claim — finitely atomic measures are the only ones with simple depths. Combining the present finding with the exact reconstruction procedure for μ from its simple halfspace depth function [7] we therefore completely resolve the halfspace depth characterization problem for simple depths, finitely atomic measures, and datasets.

In Section 2 we gather preliminaries necessary to prove our main result. We introduce a *flag halfspace* — a useful symmetric intermediary between a closed and an open halfspace in $\mathbb{R}^d$ that simplifies proofs. Indeed, a common problem with the theoretical analysis of the halfspace depth is that the infimum in (1.1) does not have to be attained. It is guaranteed to be attained if: (i) μ is *smooth*[1] or (ii) for μ finitely atomic. For that reason, it is common in the literature that theoretical results on the halfspace depth are formulated only for these two special classes of measures (smooth or atomic). As proved in [15], replacing closed halfspaces with their flag counterparts in (1.1), the infimum becomes a minimum. This observation greatly facilitates the analysis for general measures and shortens the proofs of our main results considerably. Section 3 is devoted to the proof of our main result on simple depths. In addition, several observations regarding the cardinality of the set of the attained depth values are given. It is shown that a measure whose support has finitely many connected components cannot attain countably infinite number of different depth values, and several illustrating examples complete the picture.

Notations

In addition to the standard apparatus of probability theory, we use the terminology and results from convex geometry. Our basic reference is [17].

The affine hull $\operatorname{aff}(S)$ of $S \subseteq \mathbb{R}^d$ is the smallest affine subspace (flat) of $\mathbb{R}^d$ containing S. The dimension $\dim(S)$ of S is defined as the dimension of $\operatorname{aff}(S)$. We write $\operatorname{int}(S)$, $\operatorname{cl}(S)$, and $\operatorname{bd}(S)$ for the interior, closure, and boundary of $S \subseteq \mathbb{R}^d$. The relative interior $\operatorname{relint}(S)$, relative closure $\operatorname{relcl}(S)$, and relative boundary $\operatorname{relbd}(S)$ of S is the interior, closure, and boundary of S, respectively, when considered only in the affine space $\operatorname{aff}(S)$. In case $\dim(S) = d$, the interior is the same as the relative interior etc. The relatively open line segment between two different points $x, y \in \mathbb{R}^d$ is denoted by $L(x, y)$.

The collection of all closed halfspaces in $\mathbb{R}^d$ is denoted by $\mathcal{H}$. For a generic halfspace from $\mathcal{H}$ we usually write H; $H_{x,v}$ denotes a halfspace $\left\{y \in \mathbb{R}^d \colon \langle y, v\rangle \geq \langle x, v\rangle\right\}$ whose boundary passes through $x \in \mathbb{R}^d$ and has inner normal $v \in \mathbb{R}^d \setminus \{0\}$. For an affine space $A \subseteq \mathbb{R}^d$ and $x \in A$ we write $\mathcal{H}(x, A)$ for the set of all relatively closed $\dim(A)$-dimensional

[1] We say that $\mu \in \mathcal{M}\left(\mathbb{R}^d\right)$ is smooth if the μ-mass of every hyperplane in $\mathbb{R}^d$ is zero.

halfspaces H in A whose relative boundary contains x. We say that a sequence of closed halfspaces $\{H_{x_n,v_n}\}_{n=1}^{\infty} \subset \mathcal{H}$ converges to $H_{x,v} \in \mathcal{H}$ if $x_n \to x$ and $v_n \to v$. Finally, for any of the symbols $\mathcal{H}$, $\mathcal{H}(x)$, or $\mathcal{H}(x,A)$, a superscript $\circ$ refers to the corresponding relatively open halfspaces, e.g. $\mathcal{H}^{\circ}(x,A) = \{\mathrm{relint}\,(H) : H \in \mathcal{H}(x,A)\}$.

We write $\mathrm{supp}\,(\mu)$ for the support[2] of a measure $\mu \in \mathcal{M}\left(\mathbb{R}^d\right)$. The restriction of $\mu \in \mathcal{M}\left(\mathbb{R}^d\right)$ to a Borel set $S \subseteq \mathbb{R}^d$ is denoted by $\mu|_S \in \mathcal{M}\left(\mathbb{R}^d\right)$ and is defined by $\mu|_S\,(B) = \mu\,(B \cap S)$ for $B \subseteq \mathbb{R}^d$ Borel. The collection of all finitely atomic measures $\mu \in \mathcal{M}\left(\mathbb{R}^d\right)$ is denoted by $\mathcal{A}\left(\mathbb{R}^d\right)$; it is exactly the set of finite Borel measures in $\mathbb{R}^d$ with finite support.

2 Preliminaries: flag halfspaces

We begin by recalling the definition of flag halfspaces together with the main theorem from [15]. The idea of flag halfspaces rests in the fact that even though the infimum in (1.1) does not have to be attained, by a compactness argument there always exists a convergent sequence $\{H_n\}_{n=1}^{\infty}$ of closed halfspaces from $\mathcal{H}(x)$ whose μ-masses converge to $D\,(x;\mu)$. Denote by $\lim_{n\to\infty} H_n = H$ the limiting halfspace. Then $\mu(H) \geq \lim_{n\to\infty} \mu(H_n) = D\,(x;\mu)$ in general. In order to construct a flag halfspace F satisfying $\mu(F) = D\,(x;\mu)$ one first takes the open halfspace $\mathrm{int}\,(H)$. Then one considers the depth of x in the lower dimensional space $\mathrm{bd}\,(H)$ w.r.t. the restriction of μ to $\mathrm{bd}\,(H)$, and proceeds to obtain another open halfspace in the $(d-1)$-dimensional space $\mathrm{bd}\,(H)$ whose μ-mass approximates the depth of x inside that hyperplane. This procedure continues iteratively until one reaches dimension 1, and gets an open halfline that originates in x. The flag halfspace F is then defined as the union of all these relatively open halfspaces of dimensions $1,\dots,d$, respectively, and the (zero-dimensional) point x itself. For details we refer to [15].

Definition 2.1. *A* flag halfspace *at a point $x \in \mathbb{R}^d$ is any set of the form*

$$F = \{x\} \cup \left(\bigcup_{k=1}^{d} G_k \right)$$

where $G_d \in \mathcal{H}^{\circ}(x)$, and $G_k \in \mathcal{H}^{\circ}(x, \mathrm{relbd}\,(G_{k+1}))$ for every $k = 1,\dots,d-1$. We call G_k the k-dimensional face *of F. The inner normal vector $v_k \in \mathbb{S}^{d-1}$ of G_k (when G_k is considered as a set inside $\mathrm{aff}\,(G_k)$) is called the* k-dimensional inner normal *of F, $k = 1,\dots,d$. The collection of all flag halfspaces at x is denoted by $\mathcal{F}(x)$. We write $\mathcal{F}(x,A)$ for the system of all flag halfspaces at $x \in A$ considered in an affine subspace $A \subseteq \mathbb{R}^d$.*

The reason for introducing flag halfspaces is the following theorem. Its proof uses the ideas outlined above; in detail it can be found in [15].

Theorem 2.2. *For any $\mu \in \mathcal{M}\left(\mathbb{R}^d\right)$ and $x \in \mathbb{R}^d$ we have*

$$D\,(x;\mu) = \min\left\{\mu\,(F) : F \in \mathcal{F}(x)\right\}.$$

In particular, there always exists a flag halfspace $F \in \mathcal{F}(x)$ such that $D\,(x;\mu) = \mu(F)$.

We proceed by deriving several auxiliary results on flag halfspaces that will be useful in what follows. Note that a flag halfspace is neither an open nor a closed set. In contrast to a usual closed halfspace, a complement of a flag halfspace $F \in \mathcal{F}(x)$ is, except for its central point x, again a flag halfspace from $\mathcal{F}(x)$, i.e. $(\mathbb{R}^d \setminus F) \cup \{x\} \in \mathcal{F}(x)$. A simple consequence is the following characterization.

Lemma 2.3. *Let $x \in F \subseteq \mathbb{R}^d$. Then $F \in \mathcal{F}(x)$ if and only if both these statements hold:*

[2]The support of $\mu \in \mathcal{M}\left(\mathbb{R}^d\right)$ is defined as the smallest closed set in $\mathbb{R}^d$ of full μ-mass [2, page 227].

(i) F *is convex,*

(ii) *If* $x \in L(y,z)$ *for* $y,z \in \mathbb{R}^d$*, then* $y \in F$ *if and only if* $z \notin F$*.*

Proof. Assume first that $F \in \mathcal{F}(x)$. To prove the convexity of F, take two arbitrary points $y, z \in F = \{x\} \cup \left(\bigcup_{k=1}^d G_k\right)$, $y \neq z$. Denote $G_0 = \{x\}$. Then $y \in G_k$ and $z \in G_l$ for some $k, l \in \{0, \dots, d\}$, without loss of generality (w.l.o.g.) $k \geq l$. Then $z \in \operatorname{relcl}(G_k)$ giving $L(y,z) \subset G_k \subset F$, and the set F is convex. To prove (ii), we first denote by $G_k^- = \operatorname{relint}(\operatorname{aff}(G_k) \setminus G_k)$ the relative interior of the complement of G_k in its affine hull. Note that for all $k \in \{1, \dots, d\}$ we have that $x \in \operatorname{relbd}(G_k)$, and $x \in L(y,z)$ implies that $y \in G_k$ if and only if $z \in G_k^-$. Because $F \cap \left(\bigcup_{k=1}^d G_k^-\right) = \emptyset$ we conclude that $y \in F$ if and only if $z \notin F$, verifying condition (ii).

For the opposite implication, assume that F is a convex set satisfying (ii). W.l.o.g. take x to be the origin and denote by $l_v = \{r\, v \colon r > 0\}$ the halfline from the origin in direction $v \in \mathbb{S}^{d-1}$. Conditions (i) and (ii) imply that for any $v \in \mathbb{S}^{d-1}$ one of the open halflines l_v and l_{-v} belongs to F, while the other one has empty intersection with F. Therefore, F is a convex cone and $F \neq \mathbb{R}^d$, so it has to be contained in a closed halfspace $H_d \in \mathcal{H}(x)$ using the same argument as in [17, Theorem 1.3.9]. Set $G_d = \operatorname{int}(H_d)$ and take any $y \in G_d$. Surely $-y \notin H_d$, so $-y \notin F$ and consequently $y \in F$. Therefore, $G_d \subseteq F \subseteq H_d$. Now take $F_{d-1} = F \cap \operatorname{bd}(G_d)$, that is $F = G_d \cup F_{d-1}$. Obviously, F_{d-1} is a convex set satisfying (ii), so we can proceed in the same manner as before, reducing our construction to the $(d-1)$-dimensional space $\operatorname{bd}(G_d)$. We conclude that there is $G_{d-1} \in \mathcal{H}^\circ(x, \operatorname{bd}(G_d))$ such that $G_{d-1} \subseteq F_{d-1} \subseteq \operatorname{relcl}(G_{d-1})$. Denoting $F_{d-2} = F_{d-1} \cap \operatorname{relbd}(G_{d-1})$, we are able to write $F = G_d \cup G_{d-1} \cup F_{d-2}$. Continuing this procedure, we eventually reach $G_1 \in \mathcal{H}^\circ(x, \operatorname{relbd}(G_2))$ such that $G_1 \subseteq F_1 \subseteq \operatorname{relcl}(G_1) = G_1 \cup \{x\}$. Because $x \in F_1$ and F_1 is one-dimensional, it has to be $F_1 = G_1 \cup \{x\}$. Finally, $F = \{x\} \cup \left(\bigcup_{k=1}^d G_k\right) \in \mathcal{F}(x)$. □

Further consequences of the characterization of flag halfspaces from Lemma 2.3 are summarized in the following lemma.

Lemma 2.4. *Let* $F \in \mathcal{F}(x)$*.*

(i) *For any* $y \in F$ *the flag halfspace* $F_y = F + (y - x) = \{z + (y - x) \colon z \in F\} \in \mathcal{F}(y)$ *is a subset of* F*. If, in addition* $y \neq x$*, then* $F_y \subset F \setminus \{x\}$*.*

(ii) *If* A *is an affine subspace of* $\mathbb{R}^d$ *and* $x \in A$*, then* $F \cap A \in \mathcal{F}(x, A)$*.*

Proof. Part (i) follows directly from the fact that $F - x$ is a convex cone in $\mathbb{R}^d$, as implied by the equivalent characterization from Lemma 2.3. A convex cone is closed under Minkowski addition of its elements [17, Section 1.1] giving directly $F_y \subseteq F$. If $y \neq x$, certainly $L(x,y) \cup \{x\} \subseteq F \setminus F_y$.

For part (ii) note that F is convex and satisfies condition (ii) from Lemma 2.3. Then $A \cap F$ is also a convex set that satisfies condition (ii) of Lemma 2.3, and therefore $F \cap A \in \mathcal{F}(x, A)$ due to Lemma 2.3 again. □

For a compact convex set $C \subseteq \mathbb{R}^d$ and $v \in \mathbb{S}^{d-1}$ there always exists [17, Theorem 1.3.2] a closed halfspace $H \in \mathcal{H}$ with inner normal v that supports C, i.e. $C \subset H$ and $C \cap \operatorname{bd}(H) \neq \emptyset$. We say that the opposite closed halfspace $\operatorname{cl}\left(\mathbb{R}^d \setminus H\right)$ *touches* the set C. The dimension of the touching face $\operatorname{bd}(H) \cap C$ may, however, take any value between 0 and $d-1$. Our final preliminary observation concerns a refinement of this result using flag halfspaces — one can always find a flag halfspace with given inner normal vectors that intersects C at exactly one point.

Lemma 2.5. *For any compact convex set $C \subset \mathbb{R}^d$, $x \in C$ and $F \in \mathcal{F}(x)$ there exists $y \in \operatorname{relbd}(C)$ and $F_y \in \mathcal{F}(y)$ such that (i) $F_y \subseteq F$, (ii) the k-dimensional inner normal of F_y is the same as that of F for all $k = 1, \dots, d$, and (iii) $F_y \cap C = \{y\}$.*

Proof. Denote $F = \{x\} \cup \left(\bigcup_{k=1}^{d} G_k\right)$. There exists a closed halfspace H that touches C at $x_d \in C \cap F$ and has the same inner normal as G_d. Then the flag halfspace $\widetilde{F}_d = F + (x_d - x) \in \mathcal{F}(x_d)$ satisfies $\widetilde{F}_d \subseteq F$ by Lemma 2.4. Also, all the inner normals of $\widetilde{F}_d$ and F coincide, and the d-dimensional face of $\widetilde{F}_d$ is $\widetilde{G}_d = \operatorname{int}(H)$. Denote $A_{d-1} = \operatorname{bd}\left(\widetilde{G}_d\right)$ and $C_{d-1} = C \cap A_{d-1}$. Surely, C_{d-1} is a compact convex set in the subspace A_{d-1}, and we can repeat the previous procedure in A_{d-1}. We iterate the process, decreasing the dimension by one in each step. Eventually, we reach $C_0 = \{y\} \subset C$ and $F_y = F + (y - x) \in \mathcal{F}(y)$ having all the desired properties. □

3 Simple halfspace depth

The apparatus of flag halfspaces developed in Section 2 allows us to state results concerning a measure $\mu \in \mathcal{M}\left(\mathbb{R}^d\right)$, knowing only its halfspace depth function $D(\cdot; \mu)$. In the theory that follows, a crucial concept will be that of the *(halfspace depth) central regions* of μ at level $\alpha \geq 0$ defined by

$$D_\alpha(\mu) = \left\{x \in \mathbb{R}^d \colon D(x; \mu) \geq \alpha\right\}, \quad \text{and} \quad U_\alpha(\mu) = \left\{x \in \mathbb{R}^d \colon D(x; \mu) > \alpha\right\}. \tag{3.1}$$

In multivariate statistics, the central regions $D_\alpha(\mu)$ play the role of the inter-quantile regions of μ. They are closed, convex and nested. For $\alpha > 0$ the region $D_\alpha(\mu)$ is compact. Also the regions $U_\alpha(\mu)$ are nested and convex, but not necessarily closed or open sets. The closure of $U_\alpha(\mu)$ is always contained in $D_\alpha(\mu)$.

We begin with an auxiliary result that is interesting by itself — the halfspace depth of μ cannot be constant on a (relatively) open set of positive μ-mass.

Lemma 3.1. *Let $\mu \in \mathcal{M}\left(\mathbb{R}^d\right)$ and let $K \subset \mathbb{R}^d$ be a relatively open set of points of equal halfspace depth of μ that contains at least two points. Then $\mu(K) = 0$.*

Proof. The following simple observation will be useful.

Lemma 3.2. *For $\mu \in \mathcal{M}\left(\mathbb{R}^d\right)$ and an open set $L \subseteq \mathbb{R}^d$ with $\mu(L) > 0$ there exists an open ball $B \subseteq L$ with $\mu(B) > 0$.*

Proof. Since the space $\mathbb{R}^d$ is separable, any open set L can be written as a countable union of open balls [2, Proposition 2.1.4]. Thus, by countable additivity of μ, there has to exist a ball with positive μ-mass. □

We continue with the proof of Lemma 3.1. Denote by $\alpha \geq 0$ the common depth value of all the points in K, i.e. $K \subseteq D_\alpha(\mu) \setminus U_\alpha(\mu)$. Aiming to derive a contradiction assume that $\mu(K) > 0$. Applying Lemma 3.2 to the space $\operatorname{aff}(K)$ we see that for $\mu(K) > 0$ to be true, K must contain a relatively open ball of positive μ-mass. Since balls are convex, we can also assume, w.l.o.g., that K is convex. Put

$$n = \min\{\dim(M) \colon M \subseteq K, \mu(M) > 0 \text{ and } M \text{ is convex}\}. \tag{3.2}$$

The system on the right hand side of (3.2) is non-empty since it contains K as its element.

If $n = 0$, then $\mu(\{x\}) > 0$ for some $x \in K$. Because $\alpha = D(x; \mu)$, Theorem 2.2 allows us to pick $F \in \mathcal{F}(x)$ such that $\mu(F) = \alpha$. Since K is relatively open, convex, and contains at least two points, there is a relatively closed line segment determined by points $y, z \in K$ such that $x \in L(y, z)$. Lemma 2.3 implies that exactly one of the points y, z belongs to F.

Suppose that $y \in F$ and denote $F' = F + (y - x) \in \mathcal{F}(y)$. From part (i) of Lemma 2.4 it follows that $\mu(F') \leq \mu(F) - \mu(\{x\}) < \alpha$, which contradicts $y \in K \subseteq D_\alpha(\mu)$.

Therefore, it must be $n > 0$, and we can find $M \subseteq K$ convex with $\mu(M) > 0$ and $\dim(M) = n > 0$. Denote $A = \operatorname{aff}(M)$. We may assume that M is relatively open (in A), otherwise, we consider the set $K \cap A \supseteq M$ instead of M. Take any $x \in M$. There is $F_x \in \mathcal{F}(x)$ such that $\mu(F_x) = \alpha$ by Theorem 2.2. We apply Lemma 2.5 to obtain $y \in \operatorname{relbd}(M)$ and $F_y \in \mathcal{F}(y)$ such that $F_y \subseteq F_x$ and $F_y \cap \operatorname{relcl}(M) = \{y\} \subset \operatorname{relbd}(M)$. Because M is relatively open, it follows that

$$F_y \cap M = \emptyset. \tag{3.3}$$

From part (ii) of Lemma 2.4 we know that F'_x defined as $A \cap F_x$ is an element of $\mathcal{F}(x, A)$. Denote by $G_x = \operatorname{relint}(F'_x) \in \mathcal{H}^\circ(x, A)$ the n-dimensional face of F'_x. Because of (3.3) we know that the sets $M \cap G_x$ and F_y are disjoint subsets of F_x, and we can write

$$\mu(F_y) + \mu(M \cap G_x) \leq \mu(F_x) = \alpha. \tag{3.4}$$

Because $y \in \operatorname{relbd}(M)$, and M is a subset of the closed set $D_\alpha(\mu)$, it must be $y \in D_\alpha(\mu)$. By the definition of the halfspace depth and Theorem 2.2 we therefore have $\mu(F_y) \geq \alpha$, which together with (3.4) gives that

$$\mu(M \cap G_x) = 0. \tag{3.5}$$

Define the system of sets $\mathcal{G}$ by $\{G_x : x \in M\}$, and let $S = M \setminus \left(\bigcup \{G : G \in \mathcal{G}\}\right)$. Note that because M is convex and each $G \in \mathcal{G}$ is an open halfspace in A, also the set S is convex. With the intention of reaching a contradiction, suppose that $\dim(S) = \dim(A)$. Because the relative interior of any non-empty convex set in $\mathbb{R}^d$ is non-empty [17, Theorem 1.1.13], there exists $x \in \operatorname{relint}(S)$. Because $x \in M$, from the way we defined $\mathcal{G}$ it follows that there must exist $G_x \in \mathcal{G}$ such that $x \in \operatorname{relbd}(G_x)$. But then, $\dim(S) = \dim(A) = n$, $x \in \operatorname{relint}(S)$, and $G_x \in \mathcal{H}^\circ(x, A)$ implies that necessarily $S \cap G_x \neq \emptyset$, a contradiction with the definition of S. We have shown that $\dim(S) < \dim(A) = \dim(M)$.

We showed in (3.5) that $\mu(M \cap G) = 0$ for all $G \in \mathcal{G}$. Therefore, $\bigcup \{G \cap M : G \in \mathcal{G}\}$ is a union of a system of sets that are relatively open in A, each of μ-mass 0. Lemma 3.2 applied to the space A guarantees that also $\mu\left(\bigcup \{G \cap M : G \in \mathcal{G}\}\right) = 0$, leading to $\mu(S) = \mu(M) > 0$. But this is in contradiction with our choice of n in (3.2) because we found a convex set $S \subseteq K$ of positive μ-mass such that $\dim(S) < \dim(M) = n$. ☐

Our complete characterization of measures with simple halfspace depths is a consequence of Lemma 3.1.

Theorem 3.3. *A measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ has a simple depth if and only if $\mu \in \mathcal{A}(\mathbb{R}^d)$.*

Proof. The halfspace depth of a finitely atomic measure $\mu \in \mathcal{A}(\mathbb{R}^d)$ is known to be simple, with all depth regions (3.1) being convex polytopes [7, Lemma 1]. To prove the non-trivial implication of Theorem 3.3 we proceed in two steps. First, in Lemma 3.4 we show that a simple depth cannot correspond to an atomic measure with infinitely many atoms — we denote such measures by $\mathcal{A}_\infty(\mathbb{R}^d)$. Then, we exclude also measures that are not purely atomic in Lemma 3.5.

Lemma 3.4. *A measure $\mu \in \mathcal{A}_\infty(\mathbb{R}^d)$ does not have a simple depth.*

Proof. Suppose, for a contradiction, that the depth of $\mu \in \mathcal{A}_\infty(\mathbb{R}^d)$ is simple. Then there must exist infinitely many atoms of μ with the same depth. Denote by $\alpha \geq 0$ the smallest value so that $D(x; \mu) = \alpha$ for infinitely many atoms x of μ, and denote the set of such atoms by A. For each $x \in A$ we find a flag halfspace $F_x \in \mathcal{F}(x)$ of μ-mass α

by Theorem 2.2. Note that for any $y \in F_x$, $y \neq x$, we have $F_y = F_x + (y - x) \in \mathcal{F}(y)$ and $D(y;\mu) \leq \mu(F_y) \leq \alpha - \mu(\{x\}) < \alpha$ due to part (i) of Lemma 2.4. Therefore, $F_x \cap D_\alpha(\mu) = \{x\}$.

Since A is countably infinite, we can enumerate its different elements, and write $A = \{x_n : n = 1, 2, \dots\}$. Define $\{y_n\}_{n=1}^\infty \subset \mathbb{R}$ by $y_n = \mu(\{x_n\}) - \mu(\{x_1\})$. We first show that there exists a subsequence $\{z_n\}_{n=1}^\infty$ of $\{y_n\}_{n=1}^\infty$ such that $z_n \neq z_m$ for every $n \neq m$. Indeed, otherwise the sequence $\{y_n\}_{n=1}^\infty$ would attain only finitely many different values, so there would exist an infinite constant subsequence $\{y_{n_k}\}_{k=1}^\infty$ of $\{y_n\}_{n=1}^\infty$. In that case, however, for some $c \in \mathbb{R}$ we would have $y_{n_k} = \mu(\{x_{n_k}\}) - \mu(\{x_1\}) = c$ and $\mu(\{x_{n_k}\}) = \mu(\{x_1\}) + c$ for each $k = 1, 2, \dots$. That is, however, impossible because each $\mu(\{x_{n_k}\})$ would then have to be the same positive constant, which cannot happen as the total mass of μ is finite.

Since $\mu(F_{x_n}) = \alpha$ for each $n = 1, 2, \dots$, we have

$$\mu(F_{x_1} \setminus \{x_1\}) - \mu(F_{x_n} \setminus \{x_n\}) = (\alpha - \mu(\{x_1\})) - (\alpha - \mu(\{x_n\})) = \mu(\{x_n\}) - \mu(\{x_1\}). \quad (3.6)$$

We have established that $F_{x_n} \cap D_\alpha(\mu) = \{x_n\}$ for each x_n. That means that the mass in (3.6) is equal to $\sum_{x \in M} \mu(\{x\}) - \sum_{x \in N} \mu(\{x\})$ where M and N are disjoint subsets of the finite set of all atoms outside $D_\alpha(\mu)$. From the considerations above, we concluded that there are infinitely many different values (3.6), so there have to be infinitely many atoms outside $D_\alpha(\mu)$. Consequently, there would have to exist infinitely many atoms with the same depth β for some $\beta < \alpha$, a contradiction with our choice of α. □

Lemma 3.5. *A measure $\mu \in \mathcal{M}\left(\mathbb{R}^d\right)$ with a non-trivial non-atomic part does not have a simple depth.*

Proof. Denote by $\nu \in \mathcal{M}\left(\mathbb{R}^d\right)$ the non-atomic part of $\mu \in \mathcal{M}\left(\mathbb{R}^d\right)$ and suppose for a contradiction that the depth of μ is simple. Writing $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_m$ for all the depth values attained by μ, we can decompose the sample space into disjoint sets

$$\mathbb{R}^d = \bigcup_{i=1}^m \left(D_{\alpha_i}(\mu) \setminus U_{\alpha_i}(\mu)\right).$$

Because $\nu(\mathbb{R}^d) > 0$, there must exist $i \in \{1, \dots, m\}$ such that $\nu(D_{\alpha_i}(\mu) \setminus U_{\alpha_i}(\mu)) > 0$. Because $U_{\alpha_i}(\mu) = D_{\alpha_{i+1}}(\mu)$ for $i = 1, \dots, m-1$ and $U_{\alpha_m}(\mu) = \emptyset$ we know that each $U_\alpha(\mu)$ is closed. For every $x \in D_{\alpha_i}(\mu) \setminus U_{\alpha_i}(\mu)$ therefore exists an open ball B_x centered at x that has empty intersection with $U_{\alpha_i}(\mu)$ and consequently

$$D_{\alpha_i}(\mu) \setminus U_{\alpha_i}(\mu) \subseteq \bigcup \left\{B_x : x \in D_{\alpha_i}(\mu)\right\}.$$

On the right hand side of the previous display we have a union of open sets. Using Lemma 3.2, it must be that $\nu(B_x \cap D_{\alpha_i}(\mu)) > 0$ for some $x \in D_{\alpha_i}(\mu)$. Consider the restriction $\lambda = \nu|_{B_x \cap D_{\alpha_i}(\mu)} \in \mathcal{M}\left(\mathbb{R}^d\right)$, denote by C the convex hull of $\operatorname{supp}(\lambda)$, and write $A = \operatorname{aff}(C)$. Certainly, C is a non-empty compact convex set with $\operatorname{relint}(C) \subseteq D_{\alpha_i}(\mu) \setminus U_{\alpha_i}(\mu)$. Note that it must be $\dim(A) > 0$ and $\operatorname{relint}(C) \neq \emptyset$ since ν is non-atomic and C is non-empty and convex [17, Theorem 1.1.13]. Pick $z \in \operatorname{relint}(C)$ and find $F \in \mathcal{F}(z)$ from Theorem 2.2 such that $\mu(F) = \alpha$. Then $F \cap A \in \mathcal{F}(z, A)$ by part (ii) of Lemma 2.4, and G defined by $\operatorname{relint}(F \cap A)$ is an element of $\mathcal{H}^\circ(z, A)$. Because, in the subspace A, z lies in the interior of the convex hull of the support of λ, every open halfspace in A with z on its boundary has to be of positive λ-mass. Therefore,

$$\nu(G \cap C) = \lambda(G \cap C) > 0. \quad (3.7)$$

Take $F_y \subset F$ to be the flag halfspace from Lemma 2.5 that touches C with the same collection of inner normals as F. Then $F_y \cap C = \{y\} \subset \mathrm{relbd}\,(C)$, meaning that $F_y \setminus \{y\}$ and $G \cap C$ are disjoint sets. Since $F_y \subset F$ and $G \cap C \subset F$, we obtain that

$$F_y \cup (G \cap C) = (F_y \setminus \{y\}) \cup (G \cap C) \subset F,$$

leading to

$$\nu(F_y) + \nu(G \cap C) = \nu\left(F_y\right) - \nu\left(\{y\}\right) + \nu\left(G \cap C\right) = \nu(F_y \setminus \{y\}) + \nu(G \cap C) \leq \nu(F). \quad (3.8)$$

Because ν is non-atomic we have used $\nu(\{y\}) = 0$. Now, denote by $\tau = \mu - \nu \in \mathcal{M}\left(\mathbb{R}^d\right)$ the (possibly trivial) atomic part of measure μ. Because $F_y \subset F$ we have $\tau(F_y) \leq \tau(F)$, and we can conclude from (3.8) that

$$\mu(F_y) + \nu(G \cap C) = \tau(F_y) + \nu\left(F_y\right) + \nu\left(G \cap C\right) \leq \tau(F) + \nu(F) = \mu(F) = \alpha.$$

Applying (3.7) to the inequality in the last formula we obtain $\mu(F_y) < \alpha$, which is the desired contradiction with Theorem 2.2 since we chose $y \in C \subset D_\alpha(\mu)$. □

Combining Lemmas 3.4 and 3.5 we cover both cases in Theorem 3.3. □

As a direct consequence of Theorem 3.3 we obtain a complete description of the halfspace depth of measures from $\mathcal{A}\left(\mathbb{R}^d\right)$. Indeed, by Theorem 3.3 we know that a given function $f \colon \mathbb{R}^d \to [0, \infty)$ that attains only finitely many different values can be a halfspace depth of a measure μ only if $\mu \in \mathcal{A}\left(\mathbb{R}^d\right)$. For such measures, all the upper level sets (3.1) are convex polytopes [7, Lemma 1]. Not every simple function f with convex polytopal upper level sets is, however, a depth function of a measure. To determine whether f is a depth function, one (i) runs the reconstruction scheme from [7] to obtain a candidate measure $\widetilde{\mu} \in \mathcal{A}\left(\mathbb{R}^d\right)$; (ii) computes the depth $D\left(\cdot; \widetilde{\mu}\right)$; and (iii) compares that depth function with f. The two functions coincide if and only if f is a depth of a measure, and that measure must be $\widetilde{\mu}$.

Depths that attain countably many values

Led by our characterization of measures with simple depths, one may suspect that the depth of $\mu \in \mathcal{M}\left(\mathbb{R}^d\right)$ attains at most countably many values if and only if μ is atomic. An example is any distribution in $\mathbb{R}$ supported in the integers, whose depth attains countably many different values. The situation is, however, not as easy for all $\mu \in \mathcal{A}_\infty\left(\mathbb{R}^d\right)$, as shown in the following example.

Example 3.6. From the definition of the halfspace depth (1.1) in $\mathbb{R}$ it is clear that for $\mu \in \mathcal{M}\left(\mathbb{R}\right)$ and $x \in \mathbb{R}$ smaller than the median of μ in $\mathbb{R}$, the halfspace depth of x equals the value of the cumulative distribution function of μ at x. Consider now $\mu \in \mathcal{A}_\infty\left(\mathbb{R}\right)$ such that $\mu\left(\{x_n\}\right) > 0$ for each $n = 1, 2, \dots$, for an enumeration $\{x_n\}_{n=1}^\infty$ of all rational points in the interval $[0, 1]$. Denoting by m the median of μ and taking any two different points $0 < x < y < m$, there exists a rational number x_n such that $x < x_n < y$. Because x_n is an atom of μ, it must be $D\left(x; \mu\right) < D\left(y; \mu\right)$. Therefore, any two different points in the interval $(0, m)$ have different depth values, and the set of all attained depth values of μ must be uncountably infinite.

It turns out that for a measure $\mu \in \mathcal{M}\left(\mathbb{R}^d\right)$ to attain uncountably many different depth values, it is enough that μ has contiguous support in a non-trivial convex set. The notion of contiguous support of a measure is a standard requirement that ensures regular behavior of the halfspace depth. It goes back to [5] where it was defined that the support of μ is contiguous if the set $\mathrm{supp}\left(\mu\right)$ cannot be separated by a slab, that is a non-empty open set between two parallel hyperplanes in $\mathbb{R}^d$, of null μ-mass. Convex, or connected support is certainly contiguous. In our treatment, we need only a substantially weaker condition. We say that $\mu \in \mathcal{M}\left(\mathbb{R}^d\right)$ has a *contiguous support in a convex set* $C \subseteq \mathbb{R}^d$ if for all $x, y \in \operatorname{relint}(C)$ and closed halfspaces $H \in \mathcal{H}(x, \operatorname{aff}(C))$, $H' \in \mathcal{H}(y, \operatorname{aff}(C))$ such that $H' \subset H$ it follows that $\mu(H') < \mu(H)$. Of course, if the restriction of μ to $\operatorname{aff}(C)$ has contiguous support, then its support is also contiguous in C.

Lemma 3.7. *If $\mu \in \mathcal{M}\left(\mathbb{R}^d\right)$ has contiguous support in a convex set $C \subseteq \mathbb{R}^d$ that contains at least two points, then the set of the attained depth values of μ in the set C is uncountably infinite.*

Proof. Denote $A = \operatorname{aff}(C)$, and write $\operatorname{int}_A(S)$ for the interior of a set $S \subseteq A$ when considered in the affine space A. Every convex set contains a closed convex subset of the same dimension, meaning that we can suppose w.l.o.g. that C is closed. First we show

$$\operatorname{int}_A\left(C \cap \operatorname{cl}\left(D_\alpha(\mu) \setminus U_\alpha(\mu)\right)\right) = \emptyset \quad \text{for all } \alpha \geq 0. \tag{3.9}$$

Denote $K = C \cap \operatorname{cl}\left(D_\alpha(\mu) \setminus U_\alpha(\mu)\right)$ and take, for a contradiction, $x \in \operatorname{int}_A(K)$. There is an open ball $B \subseteq K$ in space A centered at x. Take $F \in \mathcal{F}(x)$ such that $\mu(F) = \alpha$ from Theorem 2.2. Due to Lemma 2.4, part (ii), $F' = A \cap F \in \mathcal{F}(x, A)$. Then $G = \operatorname{relint}(F') \in \mathcal{H}^\circ(x, A)$, and since we assumed $\dim(A) > 0$, we get that there exists $y \in B \cap G$. Note that necessarily $\{x, y\} \subset \operatorname{int}_A(K) \subseteq \operatorname{relint}(C)$. Denote the normal of G in A by $v \in \mathbb{S}^{d-1}$. Then $H_{y,v} \in \mathcal{H}(y, A)$, $H_{(x+y)/2,v} \in \mathcal{H}((x+y)/2, A)$ and $H_{y,v} \subset H_{(x+y)/2,v} \subset F'$, so by the assumption of contiguous support, it must be $\mu\left(H_{y,v}\right) < \mu\left(H_{(x+y)/2,v}\right) \leq \alpha$, which contradicts $y \in B \subset \operatorname{cl}\left(D_\alpha(\mu) \setminus U_\alpha(\mu)\right) \subseteq \operatorname{cl}\left(D_\alpha(\mu)\right) = D_\alpha(\mu)$. We have shown (3.9).

Suppose now for a contradiction that there are only countably many different values $\{\alpha_i\}_{i=1}^\infty$ of the depth $D(\cdot;\mu)$ on C. That means that

$$C = \bigcup_{i=1}^\infty \left(C \cap \operatorname{cl}\left(D_{\alpha_i}(\mu) \setminus U_{\alpha_i}(\mu)\right)\right). \tag{3.10}$$

Due to (3.9) we know that in A, each of the countably many sets on the right hand side of (3.10) is closed with empty interior. By the Baire category theorem [11, Theorem 48.2] it means that C must have empty interior in A, which contradicts with the fact that C is convex and $\dim(C) = \dim(A)$ [17, Theorem 1.1.13]. □

One has to be careful when interpreting the result of Lemma 3.7. Its assumption is a property of the measure $\mu \in \mathcal{M}\left(\mathbb{R}^d\right)$ rather than just a property of its support $\operatorname{supp}(\mu) \subseteq \mathbb{R}^d$. To see this, consider $\mu \in \mathcal{A}_\infty\left(\mathbb{R}^2\right)$ that attaches positive mass to the elements of the countably infinite sequence of atoms

$$A = \left\{(j/n, 1/n) \in [0,1]^2 \colon n = 1, 2, \dots, \text{ and } j = 0, 1, \dots, n\right\}.$$

The support of μ, being a closed set, is the union of A with the relatively closed line segment L between the origin and the point $(1,0) \in \mathbb{R}^2$. Nevertheless, $\mu(L) = 0$ and μ does not satisfy the conditions of Lemma 3.7. On the other hand, the measure $\nu \in \mathcal{M}\left(\mathbb{R}^2\right)$ obtained as a sum of μ and the uniform probability distribution on L does possess contiguous support in the non-trivial convex set L, and Lemma 3.7 applies. At the same time, certainly, $\operatorname{supp}(\mu) = \operatorname{supp}(\nu)$.

Depth and topological connectedness of the support

Contiguous support of μ from Lemma 3.7 is not necessary for the halfspace depth of μ to attain uncountably many values. A beautiful example of a measure with a totally disconnected[3] support in $\mathbb{R}$ whose cumulative distribution function (and therefore also its halfspace depth) attains uncountably many values [2, Section 5.5, Problem 8] is the Cantor measure in the interval $[0,1]$.

More generally, there appears to be only weak relation between the degree of topological connectedness of $\operatorname{supp}(\mu)$ and the cardinality of the set of depth values of μ. Our observation to be made is the following.

Theorem 3.8. *A measure $\mu \in \mathcal{M}\left(\mathbb{R}^d\right)$ whose support is a union of finitely many connected sets has either simple depth, or its depth function $D(\cdot;\mu)$ attains uncountably many values.*

Proof. Denote by $S_1,\dots,S_m \subseteq \mathbb{R}^d$ the connected components of $\operatorname{supp}(\mu)$, i.e. the disjoint closed connected sets such that $\bigcup_{j=1}^m S_j = \operatorname{supp}(\mu)$. If each S_j is a single point set, then $\mu \in \mathcal{A}\left(\mathbb{R}^d\right)$ and the depth of μ is simple by Theorem 3.3. Suppose therefore that the set S_1 contains two different points, and denote by C the convex hull of S_1. We will show that μ satisfies the assumptions of Lemma 3.7 with this choice of C.

We know that S_1 can be disconnected from each S_j in the sense that there exist open sets $U_2,\dots,U_m$ in $\mathbb{R}^d$ such that $S_1 \subset U_j$ and $U_j \cap S_j = \emptyset$, for each $j = 2,\dots,m$. Define $U_1 = \bigcap_{j=2}^m U_j$. This is an open set in $\mathbb{R}^d$ that satisfies $S_1 \subset U_1$ and $U_1 \cap \left(\bigcup_{j=2}^m S_j\right) = \emptyset$. Take any two points $x \neq y$ from $\operatorname{relint}(C)$, $H \in \mathcal{H}(x, \operatorname{aff}(C))$, and $H' \in \mathcal{H}(y, \operatorname{aff}(C))$ such that $H' \subset H$. Since S_1 is connected, there must exist $z \in S_1$ located in the relatively open slab in $A = \operatorname{aff}(C)$ between the boundaries of H and H'. Because $z \in S_1 \subseteq \operatorname{supp}(\mu)$, each open ball B_z in $\mathbb{R}^d$ centered at z is of positive μ-mass. We take such a ball B_z that is contained in U_1, which is possible because $z \in S_1 \subset U_1$ and U_1 is open in $\mathbb{R}^d$. The open set $B_z \setminus A \subseteq U_1 \setminus S_1$ is constructed to be disjoint with $\operatorname{supp}(\mu)$. Hence, $\mu(B_z \setminus A) = 0$ and necessarily $\mu(B_z \cap A) = \mu(B_z) > 0$ for every ball B_z centered at z with a small enough radius. Take B_z so small that the relatively open ball $B_z \cap A$ is contained the slab $H \setminus H'$ in A. Then $\mu(H) = \mu(H \setminus H') + \mu(H') \geq \mu(B_z \cap A) + \mu(H') > \mu(H')$, and we can apply Lemma 3.7, which concludes the proof. □

For measures $\mu \in \mathcal{M}\left(\mathbb{R}^d\right)$ whose support has infinitely many connected components, the depth function $D(\cdot;\mu)$ can attain both countably and uncountably many different values. We conclude with a simple example of an atomic measure with a totally disconnected support that is contiguous in the sense of Lemma 3.7.

Example 3.9. Consider $\mu \in \mathcal{A}_\infty\left(\mathbb{R}^2\right)$ whose support is $S = \bigcup_{n=1}^\infty S_n$ with

$$S_n = \left\{ \left(n \cos\left(\frac{2\,\pi\,j}{n^2}\right), n \sin\left(\frac{2\,\pi\,j}{n^2}\right) \right) \in \mathbb{R}^2 \colon j \in \left\{1,\dots,n^2\right\} \right\}.$$

Each set S_n contains n^2 equidistant points on the circumference of a circle of radius n. The distance of two adjacent points in S_n approaches zero as $n \to \infty$. The set S is totally disconnected. For any $H \in \mathcal{H}$ we have that $S_n \cap H \neq \emptyset$ for all n large enough, and for any $H' \in \mathcal{H}$ such that $H' \subset H$ there exists an element of $S \cap \operatorname{int}(H \setminus H')$. Thus, the condition of Lemma 3.7 is satisfied with $C = \mathbb{R}^2$, and the depth of μ attains uncountably many different values.

Conclusions

It remains to summarize our findings. By Theorem 3.3, only a finitely atomic measure can possess a halfspace depth with finitely many values. By Theorem 3.8 any measure

[3] A set is said to be *(topologically) connected* if it cannot be partitioned into two non-empty subsets that are contained in disjoint open sets. A set is *totally disconnected* if all its connected subsets are one-point sets [11, Chapter 3].

with a non-trivial (that is, containing at least two points) connected component of its support necessarily attains uncountably many values of the halfspace depth. Lemma 3.7, however, asserts that also measures with “sufficiently irregular” totally disconnected supports can attain uncountably many depth values. Altogether, it appears that in general it is a relatively rare situation that a halfspace depth of a measure would attain only countably infinite number of different depth values, and it is not straightforward to characterize such measures. A conclusive answer to the halfspace depth characterization problem for general atomic measures therefore appears elusive.

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