Electron. Commun. Probab. **27** (2022), article no. 58, 1–12.
https://doi.org/10.1214/22-ECP498
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

On the duration of stays of Brownian motion in domains in Euclidean space

Dimitrios Betsakos[*] Maher Boudabra[†] Greg Markowsky[‡]

Abstract

Let T_D denote the first exit time of a Brownian motion from a domain D in $\mathbb{R}^n$. Given domains $U, W \subseteq \mathbb{R}^n$ containing the origin, we investigate the cases in which we are more likely to have fast exits from U than W, meaning $\mathbf{P}(T_U < t) > \mathbf{P}(T_W < t)$ for t small. We show that the primary factor in the probability of fast exits from domains is the proximity of the closest regular part of the boundary to the origin. We also prove a result on the complementary question of longs stays, meaning $\mathbf{P}(T_U > t) > \mathbf{P}(T_W > t)$ for t large. This result, which applies only in two dimensions, shows that the unit disk $\mathbb{D}$ has the lowest probability of long stays amongst all Schlicht domains.

Keywords: Brownian motion; exit time distribution; capacity.
MSC2020 subject classifications: 60J65; 60J45; 31A15; 31B15.
Submitted to ECP on June 6, 2022, final version accepted on November 6, 2022.

1 Introduction

In what follows, a *domain* refers to a connected open set of $\mathbb{R}^n$. The distribution of the exit time of a Brownian motion from a domain in $\mathbb{R}^n$ gives a measure of the size and information about the shape of the domain. Naturally, a small domain will have an exit time which is smaller, in some sense, than that of a large domain. However, making this statement precise is a bit tricky, and this question has been of interest to a number of researchers, especially in two dimensions. For examples, the reader is referred to the recent papers [2, 3, 4, 17, 20], as well as to the older works [11, 13]. This paper is concerned with this question, and particularly with the relationship between the probability that the exit time is small and the proximity of the boundary of the domain to the starting point of the Brownian motion.

Before stating our results, let us fix notation. Let $\mathbf{B}_t$, $t \geq 0$ denote standard Brownian motion moving in $\mathbb{R}^n$, $n \geq 2$. We denote by $\mathbf{P}$ and $\mathbf{E}$ the corresponding probability measure and expectation, respectively, and will use superscripts in $\mathbf{P}^x$ and $\mathbf{E}^x$ to signify that we are using the probability measure associated with Brownian motion starting from x (see e.g. [23]). If ever the starting point is not mentioned, then we assume it is

*Department of Mathematics, Aristotle University, Greece. E-mail: betsakos@math.auth.gr
†Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, KSA. (The work was primarily conducted in the School of Mathematics at Monash University.)
E-mail: maher.boudabra@kfupm.edu.sa
‡School of Mathematical sciences, Monash University, Australia. E-mail: greg.markowsky@monash.edu

the origin. For a domain D containing the origin in $\mathbb{R}^n$, we denote by T_D the first exit time of $\mathbf{B}_t$ from D; that is

$$T_D = \inf\{t \geq 0 : \mathbf{B}_t \notin D\}.$$

We also let $\mathrm{d}(A)$ denote the distance from the origin to the set $A \subset \mathbb{R}^n$; that is,

$$\mathrm{d}(A) = \inf\{\|x\| : x \in A\}.$$

For our first result, we consider the quantity $\mathbf{P}(T_D < t)$ for t small (this is what we mean by “fast exits”) and its relation to $\mathrm{d}(\partial D)$. The authors of this paper have already proved some results in this direction, in [8]. There, we considered this problem only in two dimensions, and assumed also that the domains in question were simply connected. Our main result in that paper was the following.

Theorem A. [8] *Suppose that U, W are two simply connected domains in $\mathbb{R}^2$ both containing the origin, and that $\mathrm{d}(\partial U) < \mathrm{d}(\partial W)$. Then, for all sufficiently small $t > 0$,*

$$\mathbf{P}^0(T_U < t) > \mathbf{P}^0(T_W < t). \tag{1.1}$$

In fact,

$$\lim_{t \to 0^+} \frac{\mathbf{P}^0(T_U < t)}{\mathbf{P}^0(T_W < t)} = \infty.$$

The proof there depended heavily upon simple connectivity and the topology of the plane, and did not seem at the time to generalize to higher dimensions. However, subsequent study has shown that the result does admit a considerable generalization to $\mathbb{R}^n$, and furthermore that the requirement of simple connectivity was unnecessary. Dropping the requirement of simple connectivity, however, does significantly complicate the proof, and compels us to introduce the concept of *regular* boundary points, which we now discuss.

For a closed set $K \subset \mathbb{R}^n$, let $\tau_K = \inf\{t > 0 : B_t \in K\}$. We will call this the *hitting time* of the set K. A point x is called *regular* for K if $\mathbf{P}^x(\tau_K = 0) = 1$ and *irregular* otherwise (see [23] or [26]). Intuitively speaking, a regular point x for K is a point from which the Brownian motion hits $K \backslash \{x\}$ immediately upon starting at x. Note that $\mathbf{P}^x(\tau_K = 0) \in \{0, 1\}$ by Blumenthal's zero-one law; that is, if x is irregular then $\mathbf{P}^x(\tau_K = 0) = 0$. Let K^{r} denote the set of regular points of K. This set contains the interior of K. This leaves only points in ∂K as unknowns, and these may or may not be regular.

An important example is when K is taken to be the complement of a domain. A planar domain with an irregular boundary point is $\mathbb{D} \backslash \{0\}$, and the boundary point 0 is irregular while those on the unit circle are regular. On the other hand, it is well known that all boundary points in any simply connected domain in the plane (strictly smaller than $\mathbb{C}$ itself) are regular (see [24], for instance).

The main result of this paper can now be stated, and is as follows.

Theorem 1.1. *Let U, W be domains in $\mathbb{R}^n$, both containing the origin. Suppose that $\mathrm{d}((\partial U)^r) < \mathrm{d}((\partial W)^r)$. Then*

$$\lim_{t \to 0+} \frac{\mathbf{P}^0(T_U < t)}{\mathbf{P}^0(T_W < t)} = \infty. \tag{1.2}$$

We also consider the complementary problem of “long stays”; that is, the behavior of the quantity $\mathbf{P}(T_D < t)$ for t large. An important quantity related to this problem is the fundamental frequency $\lambda(D)$ of the domain D (i.e. the first eigenvalue of the Dirichlet Laplacian); see Subsection 2.5 for its definition. We will prove the following result.

Theorem 1.2. *Let U, W be domains in $\mathbb{R}^n$, both containing the origin. Suppose that $\lambda(U) > \lambda(W)$. Then*

$$\lim_{t\to\infty} \frac{\mathbf{P}^0(T_W > t)}{\mathbf{P}^0(T_U > t)} = \infty. \tag{1.3}$$

This result is connected to a topic in our previous work [8]. To describe it, we focus only on the two-dimensional case. We first need a definition. For a domain $D \subset \mathbb{R}^2$, let

$$\mathrm{H}(D) = \sup\{p > 0 : \mathbf{E}[(T_D)^p] < \infty\};$$

note that $\mathrm{H}(D)$ is proved in [11] to be exactly equal to half of the Hardy number of D, a purely analytic quantity, as defined in [15], and is therefore calculable for a number of common domains. The domains we consider satisfy a normalization condition: they are *Schlicht*. A planar simply connected domain $D \subsetneq \mathbb{C}$ is Schlicht if it contains the origin and $D = f(\mathbb{D})$, where f is the Riemann map with $f(0) = 0$ and $f'(0) = 1$. It is known that $\mathrm{H}(D) \geq \frac{1}{4}$ as long as $D \neq \mathbb{C}$ is Schlicht ([11]). The following is our prior result.

Theorem B. [8] *Suppose that U, W are Schlicht domains and $\mathrm{H}(U) > \mathrm{H}(W)$. Then*

$$\limsup_{t\to\infty} \frac{\mathbf{P}^0(T_W > t)}{\mathbf{P}^0(T_U > t)} = \infty. \tag{1.4}$$

We conjectured there that this proposition remained true with the $\limsup$ replaced by $\lim$, but we were not able to prove it (except when W is a wedge). Subsequent study has revealed another result in this direction. Before stating the result, we provide a bit of motivation.

In [13], Davis explored the relation of planar Brownian motion to classical complex analysis (for more on this topic, see [18] and the references therein). Among a number of important ideas was Davis' statement that "the distribution of T_D is an intuitively appealing measure of the size of D" (notation changed to match that in this paper). Davis then suggested applying this idea to the set of Schlicht domains. He conjectured that if D is a Schlicht domain, then

$$\mathbf{P}^0(T_D < t) \leq \mathbf{P}^0(T_{\mathbb{D}} < t),$$

for all $t > 0$. However, McConnell disproved this for sufficiently small t and D an infinite strip, in [19]; note that this follows also from our results on fast exits (Theorem A), since $\mathrm{d}(\partial D) < 1$ for any Schlicht domain D other than $\mathbb{D}$ (this is a consequence of Schwarz's lemma: if $\mathbb{D} \subsetneq D$, then Schwarz's lemma implies that $1 > |(f^{-1})'(0)| = 1/|f'(0)|$, a contradiction). We will prove that Davis' conjecture is correct for large t, i.e. for long stays. Our result is as follows.

Theorem 1.3. *Let D be a Schlicht domain other than $\mathbb{D}$. Then there exists $t_o > 0$ such that for every $t > t_o$,*

$$\mathbf{P}^0(T_D < t) < \mathbf{P}^0(T_{\mathbb{D}} < t). \tag{1.5}$$

Theorems 1.2, 1.3 are immediate consequences of known results. The proof of Theorem 1.1 is more involved and uses a number of known deep results. The next section contains the necessary preliminaries, and the subsequent section contains the proofs. A short final section contains some concluding remarks.

2 Preliminaries

In this section, we collect the topics and results we will need for the proofs of our theorems.

2.1 Killed Brownian motion

A Brownian motion running in $\mathbb{R}^n$ admits the transition density

$$p(t,x,y)=\frac{1}{(2\pi t)^{n/2}}e^{\frac{-|x-y|^2}{2t}}.$$

This means that the probability that Brownian motion starting at x is in a Borel set A at time t is equal to $\int_A p(t,x,y)dy$. Note that p satisfies the heat equation: $\partial_t p=\frac{1}{2}\Delta p$. Our primary interest will be in the "stopped transition density" $p_D(t,x,y)$ (see e.g. [23]), which is taken in relation to a domain D in $\mathbb{R}^n$ and applies to Browian motion killed upon leaving D. In particular, if K is outside D, then $\int_K p_D(t,x,y)dy=0$. By the strong Markov property, the formula of $p_D(t,x,y)$ is given by

$$p_D(t,x,y)=p(t,x,y)-\mathbf{E}^x(p(t-T_D,B_{T_D},y)1_{T_D<t})$$

and one can see that $p_D(t,x,y)\le p(t,x,y)$. Note that for every Borel set $K\subset\mathbb{R}^n$, the identity

$$\mathbf{P}^x(B_t\in K,t<T_D)=\int_K p_D(t,x,y)\,m_n(dy)$$

persists; here and below m_n is the n-dimensional Lebesgue measure. Furthermore, p_D also satisfies the heat equation in D with zero boundary values.

Two basic properties of the transition density p_D are the domain monotonicity: if $D\subset\Omega$ then $p_D(t,x,y)\le p_\Omega(t,x,y)$, and the semigroup property:

$$p_D(t+s,x,y)=\int_D p_D(t,x,z)p_D(s,z,y)\,m_n(dz),\quad t,s>0,\ x,y\in D.$$

An additional property of p_D is the principle of "not feeling the boundary". We will need this principle in its simplest form (see [12]): If D is a convex domain in $\mathbb{R}^n$, then for every $x,y\in D$,

$$\lim_{t\to 0+}\frac{p_D(t,x,y)}{p(t,x,y)}=1.$$

When D is a ball, the stopped transition density has a radial monotonicity property. Probably, this property is known to experts, but we haven't found it in the literature, and thus we provide a proof.

Lemma 2.1. *Let D be the ball in $\mathbb{R}^n$ centered at the origin with radius R. Then for every $t>0$, the function $p_D(t,0,\cdot)$ is radially strictly decreasing.*

Proof. By symmetry, $p_D(t,0,x)$ is a radial function. Let $e_1=(1,0,\dots,0)$ be the first coordinate vector and let $0<r<s<R$. Consider the $(n-1)$-dimensional plane P (a line when $n=2$), perpendicular to e_1 and passing from the point $\frac{r+s}{2}e_1$. Let

$$D_+:=\{x=(x_1,x_2,\dots,x_n)\in D:x_1>\frac{r+s}{2}\},$$

$$D_-:=\{x=(x_1,x_2,\dots,x_n)\in D:x_1<\frac{r+s}{2}\}.$$

Note that the reflection of D_+ in P is contained in D_-, and that the reflection of the point se_1 in P is the point re_1.

Now, since p_D satisfies the heat equation, we can use a special case of a polarization result [10, Theorem 9.4], [6, Theorem 4], and conclude that

$$p_D(t,0,se_1)<p_D(t,0,re_1).\tag{2.1}$$

□

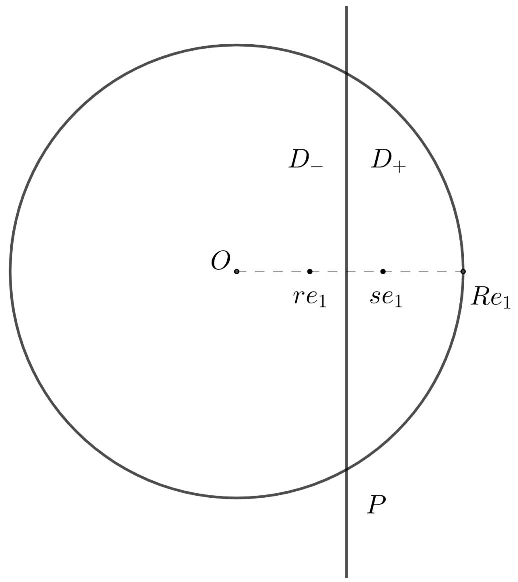

Figure 1: The ball, D_+, and D_-.

An important related object is *Green's function*. We give its probabilistic definition; see e.g. [23], [26]. With p_D as above, we define

$$G_D(x,y) = \int_0^\infty p_D(s,x,y)ds.$$

2.2 Logarithmic and Newtonian capacity

We now discuss the *capacity* of a set in $\mathbb{R}^n$. The definitions of capacity of sets in $\mathbb{R}^n$ are not entirely standardized in the literature; we will follow the definitions used in [23], [24]. For a compact set K in $\mathbb{R}^n$ with $n \geq 2$, define

$$R_n(K) := \inf \int_{K\times K} f(x-y)\mu(dx)\mu(dy),$$

where

$$f(x) = \begin{cases} -\ln|x|, & \text{if } n = 2 \\ |x|^{2-n}, & \text{if } n \geq 3, \end{cases}$$

and where the infinimum is taken over all probability measures μ having their support on K. If $K \subset \mathbb{R}^2$, its *logarithmic capacity* is defined by $c_2(K) = e^{-R_2(K)}$. If $K \subset \mathbb{R}^n$, $n \geq 3$, its *Newtonian capacity* is $c_n(K) = R_n(K)^{-1}$. These capacities have an important connection with Brownian motion in n dimensions, as we now describe.

A compact set K is referred to as *polar* when $\mathbf{P}(\tau_K < \infty) = 0$, i.e when it is not hit by Brownian motion with probability one. Otherwise, it is called *nonpolar*. By a theorem of Kakutani (see e.g. [21, Thm. 8.20]), a compact set $K \subset \mathbb{R}^n$ is nonpolar if and only if $c_n(K) > 0$.

It is worth mentioning that the $\mathbf{P}(\tau_K < \infty)$ can take values other than zero or one if the dimension n is at least 3. To see this, consider the ball $\{|\ x\ | < r\}$. Then the probability to hit that ball starting from $x \notin \{|\ x\ | < r\}$ is given by $(\frac{r}{|x|})^{n-2}$ (see e.g. [23, p.56]), where $n \geq 3$ is the dimension of the space.

2.3 Condenser capacity

We now discuss a related topic, the capacity of a condenser. A *condenser* is a pair (D,K), where D is a region and $K \subset D$ is a compact set, both in $\mathbb{R}^n$. The *capacity* of (D,K) is defined to be the infimum of the Dirichlet integral

$$\operatorname{cap}(D,K) = \inf \int_{D\setminus K} |\nabla u|^2 m_n(dx),$$

where the infimum is taken over all smooth functions u with $u = 1$ on K and $u = 0$ on ∂D. This quantity has a number of nice properties that will be relevant to us. For instance, Dirichlet's principle implies that if a minimizer exists then it must be harmonic. When the minimizer does exist it can be interpreted in terms of Brownian motion by $u(z) = \mathbf{P}^z(B_{T_{D\setminus K}} \in K)$. Furthermore, this type of capacity in two dimensions can be shown to be conformally invariant, in the sense that if f is a conformal map from D onto another domain D', then $\operatorname{cap}(D,K) = \operatorname{cap}(D', f(K))$. The condenser capacity $\operatorname{cap}(D,K)$ is also known as the *Green capacity* of K with respect to D. See [1], [27], [23] for more on this topic.

2.4 Equilibrium measure

Let K be a compact subset of a domain D in $\mathbb{R}^n$. We assume that D possesses a finite Green's function G_D. Then there exists a unique Borel measure $\mu_{K,D}$ on K such that

$$\mathbf{P}^x(\tau_K < T_D) = \int_K G_D(x,y)\,\mu_{K,D}(dy), \quad x \in D. \tag{2.2}$$

This is the *equilibrium measure* of K with respect to D. Its total measure is equal to the capacity of the condenser (D,K): $\mu_{K,D}(K) = \operatorname{cap}(D,K)$; see [23, Chapter 6].

2.5 Fundamental frequency

If D is a planar domain, its *fundamental frequency* is given by

$$\lambda(D) = \inf_{\phi} \frac{\int_D |\nabla \phi|^2 \, dm_n}{\int_D \phi^2 \, dm_n}, \tag{2.3}$$

where the infimum is taken over all smooth nonzero functions ϕ with compact support in D. If the Laplacian has a sequence of Dirichlet eigenvalues on D (e.g. when $m_n(D) < \infty$), then $\lambda(D)$ represents the first eigenvalue. We note, however, that $\lambda(D)$ is defined by (2.3) even when there are no eigenvalues, and that $\lambda(D)$ may be equal to zero.

The next theorem (see e.g. [26, §3.1] or [5, Theorem II 4.13]) gives the connection between the principal Dirichlet eigenvalue and Brownian motion which we will exploit.

Theorem C. *If D is a domain in $\mathbb{C}$, then for every $z \in D$,*

$$\lim_{t\to+\infty} \frac{2}{t} \log \frac{1}{\mathbf{P}^z(T_D > t)} = \lambda(D). \tag{2.4}$$

The following theorem provides a characterization of the disk as an extremal representative of the class of Schlicht domains; recall that these are images of the unit disk under conformal maps f with $f(0) = 0$ and $f'(0) = 1$. For a proof, see [22, §5.8] and [16].

Theorem D. *If D is a Schlicht domain then $\lambda(D) \le \lambda(\mathbb{D})$, with equality if and only if $D = \mathbb{D}$.*

Armed with this large assortment of tools, we can now tackle the proofs of our theorems.

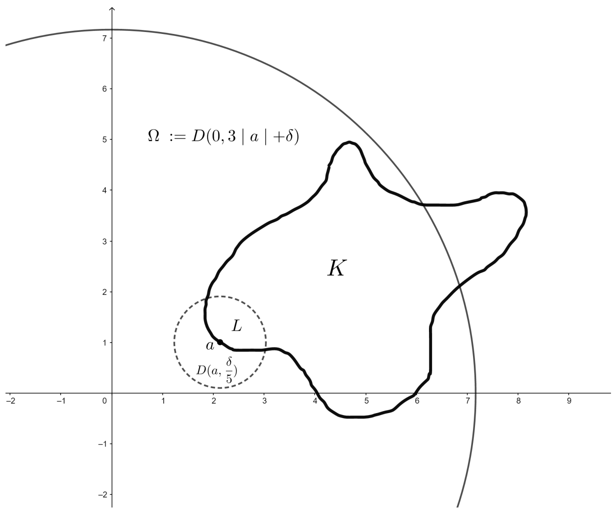

Figure 2: K, L, and Ω.

3 Proofs of Theorems 1.1, 1.2 and 1.3

We will begin with the proof of Theorem 1.1, which differs in details depending on whether the dimension n satisfies $n = 2$ or $n \geq 3$. We will give a complete proof in the case $n = 2$, and then indicate how the proof must be adjusted when $n \geq 3$. In the statement and proof of our results we will often denote by C a generic constant which will be allowed to change from line to line. The heart of the proof is contained in the following lemma.

Lemma 3.1. *Let K be a compact set in the plane with $0 \notin K$ and $\mathrm{c}_2(K) > 0$. Let a be a regular point of K. Let $\delta > 0$. There exist positive constants $C = C(K, a, \delta)$ and $T = T(K, a, \delta)$ such that for every $t \in (0, T)$,*

$$\mathbf{P}^0(\tau_K < t) \geq C\, e^{-(|a|+\delta)^2/(2t)}. \tag{3.1}$$

Proof. We will use a modification of a trick taken from [14, Proof of Lemma 3.6].

For an open ball we will use the notation $B(z_0, r) = \{z : |z - z_0| < r\}$. Set $L := K \cap \overline{B(a, \delta/5)}$ and $\Omega := B(0, 3|a| + \delta)$, i.e the ball centered at the origin and of radius $3|a| + \delta$. We will eventually stop the Brownian motion upon exiting Ω, which allows to use the equilibrium measure. Figure 2 may help the reader understand this setup.

Note that a is a regular point of L and $c_2(L) > 0$. Since $L \subset K$, for every $t > 0$ we have

$$\mathbf{P}^0(\tau_L < t) \leq \mathbf{P}^0(\tau_K < t) \tag{3.2}$$

Observe also that for $t > 0$,

$$\begin{aligned} \mathbf{P}^0(\tau_L < t) &\geq \mathbf{P}^0(\tau_L < t \quad \text{and} \quad \tau_L < T_\Omega) \\ &= \mathbf{P}^0(\tau_L < T_\Omega) - \mathbf{P}^0(\tau_L \geq t \quad \text{and} \quad \tau_L < T_\Omega). \end{aligned} \tag{3.3}$$

By [23, Theorem 5.1, p. 190], for every $z \in \mathbb{C}$,

$$\begin{aligned} \mathbf{P}^z(\tau_L < T_\Omega) &= \int_L G_\Omega(z, x) \mu_{L,\Omega}(dx) \\ &= \int_L \int_0^\infty p_\Omega(s, z, x)\, ds\, \mu_{L,\Omega}(dx), \end{aligned} \tag{3.4}$$

where G_Ω is the Green function for Ω, p_Ω is the transition density for Brownian motion killed upon exiting Ω and $\mu_{L,\Omega}$ is the equilibrium measure of L with respect to Ω.

By the (simple) Markov property, the equation (3.4), Fubini's theorem, the domain monotonicity, the semigroup property of the transition density, and a change of variables,

$$\begin{aligned}
&\mathbf{P}^0(\tau_L \geq t \quad \text{and} \quad \tau_L < T_\Omega) &(3.5)\\
=\;& \int_{\Omega\setminus L} p_{\Omega\setminus L}(t,0,z)\, \mathbf{P}^z(\tau_L < T_\Omega)\, m_n(dz)\\
=\;& \int_{\Omega\setminus L} p_{\Omega\setminus L}(t,0,z) \int_L \int_0^\infty p_\Omega(s,z,x) ds\; \mu_{L,\Omega}(dx)\, m_n(dz)\\
=\;& \int_L \int_0^\infty \int_{\Omega\setminus L} p_{\Omega\setminus L}(t,0,z) p_\Omega(s,z,x) m_n(dz)\; ds\; \mu_{L,\Omega}(dx)\\
\leq\;& \int_L \int_0^\infty \int_\Omega p_\Omega(t,0,z) p_\Omega(s,z,x) m_n(dz)\; ds\; \mu_{L,\Omega}(dx)\\
=\;& \int_L \int_0^\infty p_\Omega(t+s,0,x)\, ds\; \mu_{L,\Omega}(dx)\\
=\;& \int_L \int_t^\infty p_\Omega(s,0,x) ds\; \mu_{L,\Omega}(dx).
\end{aligned}$$

Combining (3.2), (3.3), (3.4), (3.5), we obtain

$$\begin{aligned}
&\mathbf{P}^0(\tau_K < t) &(3.6)\\
\geq\;& \int_L \int_0^\infty p_\Omega(s,0,x) ds\; \mu_{L,\Omega}(dx) - \int_L \int_t^\infty p_\Omega(s,0,x) ds\; \mu_{L,\Omega}(dx)\\
=\;& \int_L \int_0^t p_\Omega(s,0,x) ds\; \mu_{L,\Omega}(dx).
\end{aligned}$$

To obtain a lower bound, we will use the fact that for the disk Ω the transition density $p_\Omega(s,0,x)$ is a radial function of x, i.e. depends only on $|x|$, and is also decreasing in $|x|$ (Lemma 2.1). We We will also use the fact that $\mu_{L,\Omega}(L)$ is equal to the Green capacity (or condenser capacity) $\operatorname{cap}(\Omega, L)$, and obtain:

$$\begin{aligned}
\int_L \int_0^t p_\Omega(s,0,x) ds\; \mu_{L,\Omega}(dx) \;&\geq\; \int_L \int_0^t p_\Omega(s,0,|a|+\delta/5) ds\; \mu_{L,\Omega}(dx)\\
&=\; \operatorname{cap}(\Omega, L) \int_0^t p_\Omega(s,0,|a|+\delta/5)\, ds. &(3.7)
\end{aligned}$$

Now we use "the principle of not feeling the boundary" (see Subsection 2.1) and find a positive $T_1 = T_1(a,\delta)$ such that

$$p_\Omega(s,0,|a|+\delta/5) \geq \frac{1}{2} p(s,0,|a|+\delta/5), \qquad s \in (0, T_1). \tag{3.8}$$

By (3.6), (3.7), and (3.8), for $t \in (0, T_1)$,

$$\begin{aligned}
\mathbf{P}^0(\tau_K < t) \;&\geq\; \frac{\operatorname{cap}(\Omega, L)}{2} \int_0^t p(s,0,|a|+\delta/5) ds &(3.9)\\
&=\; \frac{\operatorname{cap}(\Omega, L)}{2} \int_0^t \frac{1}{2\pi s} e^{-(|a|+\delta/5)^2/(2s)}\, ds.
\end{aligned}$$

By elementary calculus (L'Hôpital's rule), there exists a positive number $T = T(a,\delta) < T_1$ such that for every $t \in (0, T)$,

$$\int_0^t \frac{1}{2\pi s} e^{-(|a|+\delta/5)^2/(2s)}\, ds \geq e^{-(|a|+\delta)^2/(2t)}. \tag{3.10}$$

Also, by [7, Lemma 1], and denoting by L^* the disk $B(0, c_2(L))$, we get

$$\operatorname{cap}(\Omega, L) \geq \operatorname{cap}(\Omega, \overline{L^*}) = 2\pi \left(\log \frac{3|a| + \delta}{\mathrm{c}_2(L)} \right)^{-1}, \tag{3.11}$$

which is a quantity that depends only on K, a and δ.

We combine (3.9), (3.10), (3.11) to conclude that for $t \in (0, T)$, for some constant $C > 0$, we have

$$\mathbf{P}^0(\tau_K < t) \geq C\, e^{-(|a|+\delta)^2/(2t)}. \quad \square \tag{3.12}$$

Proof of Theorem 1.1. We start with some reductions. First, we assume without loss of generality that $\mathrm{d}((\partial W)^r) = 1$. Then for every $t > 0$,

$$\mathbf{P}^0(T_W < t) \leq \mathbf{P}^0(T_{\mathbb{D}} < t), \tag{3.13}$$

and so we may assume that $W = \mathbb{D}$.

Let $a \in (\partial U)^r$ be a point with $|a| < 1$. Choose a disk $B(a, r)$ with $0 < r < 1 - |a|$, and set $K := (\mathbb{C} \setminus U) \cap \overline{B(a, r)}$. Note that K has positive logarithmic capacity and that $U \subset \mathbb{C} \setminus K$. Therefore,

$$\mathbf{P}^0(T_U < t) \geq \mathbf{P}^0(T_{\mathbb{C}\setminus K} < t) = \mathbf{P}^0(\tau_K < t). \tag{3.14}$$

So we may assume that $U = \mathbb{C} \setminus K$, where K is a compact subset of $\mathbb{D}$ with $\mathrm{c}_2(K) > 0$.

By Lemma 3.1, for every $\delta > 0$, there exist positive constants C, T depending on K, a, and δ such that for every $t \in (0, T)$,

$$\mathbf{P}^0(\tau_K < t) \geq C\, e^{-(|a|+\delta)^2/(2t)}, \tag{3.15}$$

where a is a regular point of K. By taking δ small enough, we may assume that $|a| + \delta < 1$. On the other hand, an ingenious argument due to McConnell [19] shows that, for all $t > 0$ and all positive integers $m \geq 3$,

$$\mathbf{P}^{\mathbf{0}}(T_{\mathbb{D}} < t) \leq c(m)\, e^{-\frac{\cos^2(\pi/m)}{2t}}. \tag{3.16}$$

By fixing m large enough, we see that for any $\epsilon > 0$ we can find a constant $C_\epsilon > 0$ such that for all $t > 0$ we have

$$\mathbf{P}^{\mathbf{0}}(T_{\mathbb{D}} \leq t) < C_\epsilon\, e^{-\frac{(1-\epsilon)}{2t}}.$$

Thus Lemma 3.1 gives

$$\begin{aligned} \lim_{t\to 0+} \frac{\mathbf{P}^{\mathbf{0}}(T_U < t)}{\mathbf{P}^{\mathbf{0}}(T_W < t)} \quad &\geq \quad \lim_{t\to 0+} \frac{\mathbf{P}^{\mathbf{0}}(\tau_K < t)}{\mathbf{P}^{\mathbf{0}}(T_{\mathbb{D}} < t)} \\ &\geq \quad \lim_{t\to 0+} \frac{C(K, a, \delta)\ \exp\left[-\frac{(|a|+\delta)^2}{2t}\right]}{C_\epsilon\ \exp\left[-\frac{1-\epsilon}{2t}\right]}. \end{aligned} \tag{3.17}$$

Choose ϵ and δ small enough so that $(|a| + \delta)^2 < 1 - \epsilon$. Then the limit above is equal to ∞ and this completes the proof of the result in two dimensions.

The proof in three or higher dimensions follows along exactly the same lines, with several necessary modifications, which we now indicate. In $n \geq 3$ dimensions, we will use the Newtonian capacity rather than the logarithmic. All other concepts used carry over directly to the higher dimensions with no real change, except that some of the estimates have to be changed. In particular, inequality (3.11) is specific to two dimensions; it is sufficient to replace it with [27, Cor. 1]. Furthermore, McConnell's estimate (3.16) is also specific to two dimensions, but it can be replaced by [25, Cor. 3.4]. The result follows then as before. $\square$

Proof of Theorem 1.2. Suppose that $\lambda(U) > \lambda(W)$. By Theorem C (in Subsection 2.5),

$$\begin{aligned}\lim_{t\to+\infty}\left(\frac{\mathbf{P}^0(T_W > t)}{\mathbf{P}^0(T_U > t)}\right)^{2/t} &= \lim_{t\to+\infty}\exp\left[\frac{2}{t}\log\frac{\mathbf{P}^0(T_W > t)}{\mathbf{P}^0(T_U > t)}\right] \\ &= \exp(\lambda(U) - \lambda(W)) > 1. \end{aligned} \tag{3.18}$$

It follows easily that

$$\lim_{t\to\infty}\frac{\mathbf{P}^0(T_W > t)}{\mathbf{P}^0(T_U > t)} = \infty. \tag{3.19}$$

Proof of Theorem 1.3

Suppose that D is a Schlicht domain other than $\mathbb{D}$. By Theorems C and D (in Subsection 2.5),

$$\begin{aligned}\lim_{t\to+\infty}\left(\frac{\mathbf{P}^0(T_D > t)}{\mathbf{P}^0(T_{\mathbb{D}} > t)}\right)^{2/t} &= \lim_{t\to+\infty}\exp\left[\frac{2}{t}\log\frac{\mathbf{P}^0(T_D > t)}{\mathbf{P}^0(T_{\mathbb{D}} > t)}\right] \\ &= \exp(\lambda(\mathbb{D}) - \lambda(D)) > 1. \end{aligned} \tag{3.20}$$

It follows that there exists $t_o > 0$ such that for every $t > t_o$,

$$\mathbf{P}^0(T_D > t) > \mathbf{P}^0(T_{\mathbb{D}} > t),$$

which is equivalent to (1.5). □

3.1 Concluding remarks

We conclude with the following thoughts, presented informally.

1. It is natural to ask whether anything can be said on the questions addressed in Theorems 1.1 and 1.2 in the equality cases, i.e. when $\mathrm{d}((\partial U)^r) = \mathrm{d}((\partial W)^r)$ or $\lambda(U) = \lambda(W)$. We will give some examples that show that nothing much can be said on this without additional conditions being brought into the picture.

We will work in two dimensions. Let $\mathbb{D}$ be the unit disk as usual, and let $\mathcal{H}$ denote the right half-plane $\{\Re(z) > 1\}$. Note that $\mathrm{d}((\partial\mathbb{D})^r) = \mathrm{d}((\partial\mathcal{H})^r)$. The distribution of $T_{\mathcal{H}}$ is well known and can be obtained by the reflection principle of one dimensional Brownian motion (see [21]), but that of $T_{\mathbb{D}}$ is more difficult and we must resort to asymptotics. Fortunately, such asymptotics are known, in particular if $\rho_{\mathbb{D}}(t)$ denotes the density of $T_{\mathbb{D}}$ then by Corollary 3.4 of [25] we have

$$\rho_{\mathbb{D}}(t) \overset{t\to 0}{\sim} C\frac{1}{t^2}e^{-1/2t}.$$

As indicated above, the distribution of $T_{\mathcal{H}}$ is known from the reflection principle, and the asymptotics are seen to be

$$\rho_{\mathcal{H}}(t) \overset{t\to 0}{\sim} C\frac{1}{t^{3/2}}e^{-1/2t}.$$

We then have, by L'Hôpital's rule,

$$\lim_{t\to 0+}\frac{\mathbf{P}^0(T_{\mathbb{D}} < t)}{\mathbf{P}^0(T_{\mathcal{H}} < t)} = \lim_{t\to 0+}\frac{\rho_{\mathbb{D}}(t)}{\rho_{\mathcal{H}}(t)} = \infty.$$

This clearly precludes any sort of result on domains with boundary equally close to 0 without additional ideas being added.

2. For large t and bounded domains U and W, the ratio

$$\frac{\mathbf{P}^0(T_U > t)}{\mathbf{P}^0(T_W > t)}$$

depends on the first Dirichlet eigenvalues of U and W. In fact, we have the following expansion, valid for large t:

$$\mathbf{P}^0(T_U > t) = u_0(0)e^{-\lambda(U)t} + o(e^{-\lambda(U)t}),$$

where u_0, up to a scaling factor, is the first eigenfunction of the Laplacian operator on U. The above expansion is a straightforward consequence of the Hilbert-Schmidt expansion of the killed transition density [5] (since this density solves the heat equation). Thus, if $\lambda(U) = \lambda(W)$, we get

$$\lim_{t\to+\infty} \frac{\mathbf{P}^0(T_U > t)}{\mathbf{P}^0(T_W > t)} = \frac{u_0(0)}{w_0(0)},$$

where naturally w_0 is the first eigenfunction of W. As the first eigenfunctions u_0 and w_0 vanish continuously at the boundaries, we may translate each domain in order to make $u_0(0)$ or $w_0(0)$ as small as we like, and thus the aforementioned estimate can be made as small or as large as we want. In other words, nothing can be concluded in general regarding the probability of long stays when both domains share the same first Dirichlet eigenvalue, at least not without some additional ideas.

When the domain is unbounded the first Dirichlet eigenvalue is often zero, as in the case of a cone. In such cases, the long term behavior of $\mathbf{P}^0(T_U < t)$ can have rational decay. We are not sure how to handle matters in this case, however we venture a conjecture which is the natural complement of Theorem 1.3. In a number of ways, the Koebe domain $\mathcal{K} := \mathbb{C}\backslash(-\infty, -1/4]$ is the extremal domain in the Schlicht class among domains which are large, just as $\mathbb{D}$ is extremal among domains that are small. The following then seems likely.

Conjecture. *Let D be a Schlicht domain other than a rotation of $\mathcal{K}$. Then there exists $t_o > 0$ such that for every $t > t_o$,*

$$\mathbf{P}^0(T_D > t) < \mathbf{P}^0(T_{\mathcal{K}} > t). \tag{3.21}$$

That is, we conjecture that, among all Schlicht domains, the Koebe domain is the one with the highest probability of long stays. This is almost, but not quite, a consequence of Proposition 5 of [8].

3. We have given an analytic proof of Lemma 2.1, but it is also possible to prove it in a more probabilistic manner. Here is a sketch of one such proof. The Brownian transition density is symmetric in the space variables (see [5, Thm. II.4.4]), so, employing the notation used in proof of the lemma, it suffices to prove $p_D(t, se_1, 0) < p_D(t, re_1, 0)$. However, this can be obtained as a straightforward application of the method of coupling expounded in [9], see in particular Theorem 2 in that paper. Other proofs are also available.

References

[1] G.D. Anderson, M.K. Vamanamurthy, *The Newtonian capacity of a space condenser*. Indiana Univ. Math. J. 34 (1985), no. 4, 753–776. MR0808824

[2] R. Bañuelos, T. Carroll, *Brownian motion and the fundamental frequency of a drum*, Duke Mathematical Journal, 75(1994), 575–602. MR1291697

[3] R. Bañuelos, T. Carroll, *The maximal expected lifetime of Brownian motion*, Mathematical Proceedings of the Royal Irish Academy, 2011, 1–11. MR2840260

[4] R. Bañuelos, P. Mariano, J. Wang, *Bounds for exit times of Brownian motion and the first Dirichlet eigenvalue for the Laplacian*, arXiv:2003.06867, 2020.

[5] R. Bass, *Probabilistic Techniques in Analysis*. Springer Science & Business Media, 1994. MR1329542

[6] D. Betsakos, *Equality cases in the symmetrization inequalities for Brownian transition functions and Dirichlet heat kernels*. Ann. Acad. Sci. Fenn. Math. 33 (2008), no. 2, 413–427. MR2431373

[7] D. Betsakos, *Geometric versions of Schwarz's lemma for quasiregular mappings*. Proc. Amer. Math. Soc. 139 (2011), no. 4, 1397–1407. MR2748432

[8] D. Betsakos, M. Boudabra, G. Markowsky, *On the probability of fast exits and long stays of a planar Brownian motion in simply connected domains*. J. Math. Anal. Appl. 493 (2021). MR4153849

[9] M. Boudabra, G. Markowsky, *Maximizing the p-th moment of exit time of planar Brownian motion from a given domain*. J. Appl. Probab. 54 (2020), no. 4, 1135–1149. MR4179602

[10] F. Brock, A.Yu. Solynin, *An approach to symmetrization via polarization*. Trans. Amer. Math. Soc. 352 (2000), 1759–1796. MR1695019

[11] D. Burkholder, *Exit times of Brownian motion, harmonic majorization, and Hardy spaces*. Advances in Math. 26 (1977), no. 2, 182–205. MR0474525

[12] Z. Ciesielski, *Heat conduction and the principle of not feeling the boundary.* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 14 (1966), 435–440. MR0206530

[13] B. Davis, *Brownian motion and analytic functions*. Ann. Probab. 7 (1979), 913–932. MR0548889

[14] A. Grigor'yan, L. Saloff-Coste, *Hitting probabilities for Brownian motion on Riemannian manifolds*. J. Math. Pures Appl. (9) 81 (2002), no. 2, 115–142. MR1994606

[15] L. Hansen, *Hardy classes and ranges of functions*. Michigan Math. J. 17 (1970), 235–248. MR0262512

[16] J. Hersch, *On symmetric membranes and conformal radius: Some complements to Pólya's and Szegö's inequalitites*. Arch. Rational Mech. Anal. 20 (1965), 378–390. MR0186929

[17] D. Kim, *Quantitative inequalities for the expected lifetime of Brownian motion*, Michigan Math. J., (to appear). MR4302556

[18] G. Markowsky, *Planar Brownian Motion and Complex Analysis*, arXiv:2012.08574, 2020. MR0845659

[19] T.R. McConnell, *The size of an analytic function as measured by Lévy's time change*. Ann. Probab. 13 (1985), 1003–1005. MR0799435

[20] J. P. Mendez-Hernández, *Brascamp-Lieb-Luttinger inequalities for convex domains of finite inradius*, Duke Mathematical Journal, 113 (2002), 75, 93–131. MR1905393

[21] P. Mörters, Y. Peres, *Brownian Motion*. Cambridge University Press, 2010. MR2604525

[22] G. Pólya, G. Szegö, *Isoperimetric Inequalities in Mathematical Physics*. Annals of Mathematics Studies, no. 27, Princeton University Press, Princeton, N. J., 1951. MR0043486

[23] S.C. Port, C.J. Stone, *Brownian Motion and Classical Potential Theory.* Academic Press 1978. MR0492329

[24] T. Ransford, *Potential Theory in the Complex Plane.* Cambridge Univ. Press, 1995. MR1334766

[25] G. Serafin, *Exit times densities of the Bessel process*. Proc. Am. Math. Soc. 145(7) (2017), 3165–3178. MR3637962

[26] A.S. Sznitman, *Brownian Motion, Obstacles and Random Media*. Springer Science & Business Media, 1998. MR1717054

[27] N. Zorii, *Precise estimate of the 2-capacity of a condenser*. Ukrainian Math. J. 42(2) (1990), 224–228. MR1053428

Acknowledgments. The authors are grateful to two anonymous referees for helpful comments.