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The weak functional representation of historical martingales*

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Abstract

A weak extension of the Dupire derivative is derived, which turns out to be the adjoint operator of the integral with respect to the martingale measure associated with the historical Brownian motion a benchmark example of a measure valued process. This extension yields the explicit form of the martingale representation of historical functionals, which we compare to a classical result on the representation of historical functionals derived in [7].

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1 Introduction

Dupire's landmark work [5] on the functional Itō-formula gave rise to a completely new approach to numerous questions in the field of stochastic calculus. One of the applications of the so-called functional or Dupire derivative developed by Cont and Fournié is presented in [3] as well as [1], where they extend the derivative to a weak one for square-integrable Brownian functionals and use it to derive their martingale representation.

The classical versions of the functional Itō-formula are derived for $\mathbb{R}^d$-valued processes (see [5], [2], [3], [10]) and there are only a few extensions to infinite-dimensional processes. For functionals of Dawson-Watanabe superprocesses, this was done in [11]. In the present work, we transfer the approach in [3] and [1] to an infinite-dimensional setting and derive the martingale representation formula of the following form. If $H = (H(t))_{t\in[\tau,T]}$ is a historical Brownian motion and Y a square-integrable martingale with respect to the filtration generated by H then, by Theorem 3.8, Y allows the representation

$$Y(t) = Y(0) + \int_0^t \nabla_M Y(s,y) M_H(ds,dy), \quad \text{for all } t \in [\tau,T]$$

where the operator ∇_M is an extension of the Dupire derivative and M_H is the martingale measure associated with H (in the sense of [17]).

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A historical Brownian motion is an enriched version of the super-Brownian motion that also contains information on genealogy. Generally speaking, if $y \in C([0,T],\mathbb{R}^d)$ and $y^t(s) = y(t \wedge s)$, $Y^t(y) = y^t$ is a time-inhomogeneous Borel strong Markov process on an appropriate space, sometimes called the path process, and the historical Brownian motion is the Dawson-Watanabe superprocess associated with Y^t ([15], [16]). Historical Brownian motions have been studied extensively by Perkins and his co-authors (e.g. [4], [12], [14], [7] and [15]), whose notation we follow in this work.

The predictable representation property for Dawson-Watanabe superprocesses and other measure-valued processes was first studied in [6] and [13], which are based on the fundamental insight of Jacod [9]. In the paper [7] Evans and Perkins derived the explicit form of the integrand in the representation of historical functionals. Their approach is based on the cluster representation of the historical Brownian motion.

In contrast, we base our approach on properties derived from the martingale problem of the historical Brownian motion. To highlight the differences, we provide a brief introduction to the approach in [7] as well as a comparison of the resulting representations in Section 4.

Before that, we start by introducing the underlying concepts in Section 2 including a brief introduction to historical Brownian motions as well as path processes. In Section 3, the weak extension of the functional derivative is derived and the martingale representation formula obtained.

Finally, it is worth mentioning that, if one works with a super-Brownian motion X instead of a historical Brownian motion, a result similar to the one in Section 3 can be obtained. For more details we refer to the remark after Theorem 3.8. In the special case that the martingale of interest, Y, can be expressed as $Y(t) = F(t, X^t)$ for a sufficiently nice F, one can directly apply the functional Itō-formula introduced in [11] to obtain the martingale representation.

2 The setting

Let $T \in (0,\infty)$ be fixed and set $C = C([0,T],\mathbb{R}^d)$, the space of continuous functions mapping $[0,T]$ to $\mathbb{R}^d$, equipped with the compact-open topology. Denote by $\mathcal{C}$ its Borel-σ-algebra and by $\mathcal{C}_t = \sigma(y(s), s \le t)$ its canonical filtration.

Next, for any y, $w \in C$ and $s \in [0,T]$, define $y^s(t) = y(s \wedge t)$ and the function $y/s/w$ glued together at s

$$(y/s/w)(t) = \begin{cases} y(t), & \text{if } t < s, \\ w(t-s), & \text{if } t \ge s. \end{cases}.$$

A function $Z : [0,T] \times C \to \mathbb{R}$ is $(\mathcal{C}_t)$-predictable if and only if it is Borel measurable and it holds $Z(t,y) = Z(t,y^t)$ for all $t \in [0,T]$ (see Section V.2 in [16]).

Denote by $M_F(C)$ the space of finite measures on C equipped with the topology of weak convergence and define for $t \in [0,T]$

$$M_F(C)^t = \{m \in M_F(C) : y = y^t \text{ for } m\text{-a.a. } y\}.$$

For $m \in M_F(C)$ and $\phi : C \to \mathbb{R}$, we set $\langle m, \phi\rangle = \int_C \phi(y) m(dy)$.

If P_x denotes the Wiener measure on $(C,\mathcal{C})$ starting at x, $\tau \in [0,T]$ and $m \in M_F(C)^\tau$, define the measure $P_{\tau,m} \in M_F(C)$ by

$$P_{\tau,m}(A) = \int_C P_{y(\tau)}(\{w : y/\tau/w \in A\}) dm(y).$$

Define the space Ω_H by

$$\Omega_H = \{H \in C([\tau,T], M_F(C)) : H(t) \in M_F(C)^t \ \forall t \in [\tau,T]\}.$$

To introduce the historical Brownian motion, let

$$\tilde{S} = \{(\tau, m) : \tau \in [0, T],\, m \in M_F(C)^\tau\}$$

and define for $(\tau, m) \in \tilde{S}$

$$\begin{aligned} F_{\tau,m} = \{\phi : [\tau, T] \times C \to \mathbb{R} : \phi \text{ is } (\mathcal{C}_t)\text{-predictable, } P_{\tau,m}\text{-a.s. right-continuous and} \\ \sup_{s\in[\tau,T]} |\phi(s, y)| \le K \text{ holds } P_{\tau,m}\text{-a.s. for some } K\} \end{aligned}$$

as well as

$$\begin{aligned} D(A_{\tau,m}) = \{\phi \in F_{\tau,m} : \text{there exists a } A_{\tau,m}\Phi \in F_{\tau,m} \text{ such that} \\ \phi(t, y) - \phi(\tau, y) - \int_\tau^t A_{\tau,m}\phi(s, y)ds \\ \text{is a } (\mathcal{C}_t)_{t\in[\tau,T]}\text{-martingale under } P_{\tau,m}\}. \end{aligned}$$

Given these notations, we can define the historical Brownian motion via its martingale problem. A predictable process $(H(t), t \in [\tau, T])$ on $\bar{\Omega} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[\tau,T]}, \mathbb{P})$ with sample paths a.s. in Ω_H is a historical Brownian motion on $\bar{\Omega}$, starting at $(\tau, m) \in \tilde{S}$ and with branching rate $\gamma > 0$, if and only if its law $\mathbb{P}_{\tau,m}$ solves the martingale problem (see [15])

$$\begin{aligned} Z(t)(\phi) = \langle H(t), \phi(t, \cdot)\rangle - \langle m, \phi(\tau, \cdot)\rangle - \int_\tau^t \langle H(s), A_{\tau,m}\phi(s, \cdot)\rangle ds, \quad t \in [\tau, T], \\ \text{is a continuous } (\mathcal{F}_t)\text{-martingale under } \mathbb{P}_{\tau,m} \text{ for all } \phi \in D(A_{\tau,m}) \\ \text{such that } Z(\tau)(\phi) = 0 \text{ and with the quadratic variation given by} \\ [Z(\phi)]_t = \int_\tau^t \langle H(s), \gamma\phi(s, \cdot)^2\rangle ds \quad \text{for all } t \in [\tau, T]. \end{aligned} \tag{MP$_{HBM}$}$$

The process $Z(t)(\phi)$ in (MP$_{HBM}$) gives rise to a martingale measure in the sense of [17], which we denote by M_H (see [15]). This allows us to write

$$Z(t)(\phi) = \int_{(\tau,t)\times C} \phi(s, y) M_H(ds, dy).$$

From [15], we know that the historical Brownian motion can also be defined by a more explicit martingale problem. To introduce this alternative martingale problem, denote by $C_0^\infty(\mathbb{R}^d)$ the space of infinitely continuously differentiable functions with compact support mapping $\mathbb{R}^d$ to $\mathbb{R}$ and define

$$D_0 = \{\phi : C \to \mathbb{R} : \ \phi(y) = \psi(y(t_1), \dots, y(t_n)), 0 \le t_1 \le \dots \le t_n \le T,\, \psi \in C_0^\infty(\mathbb{R}^{nd})\}$$

as well as

$$\tilde{D}_0 = \{\phi : [0, T] \times C \to \mathbb{R} : \phi(t, y) = \psi(y^t) \text{ for some } \psi \in D_0\}.$$

Further, set for $\phi \in \tilde{D}_0$

$$\begin{aligned} \phi_i(t, y) &= \sum_{j=1}^n 1_{t\le t_j} \psi_{(j-1)d+i}(y(t_1 \wedge t), \dots, y(t_n \wedge t)), \\ \phi_{i,j}(t, y) &= \sum_{k=1}^n \sum_{\ell=1}^n 1_{t\le t_k\wedge t_\ell} \psi_{(k-1)d+i,(\ell-1)+j}(y(t_1 \wedge t), \dots, y(t_n \wedge t)), \\ \bar{\Delta}\phi(t, y) &= \sum_{i=1}^d \phi_{i,i}(t, y), \end{aligned}$$

with ψ_i and ψ_{ij} being the first and second order partial derivatives of ψ. Using these functions, Itō's lemma yields that for all $\phi \in \tilde{D}_0$ and $(\tau, m) \in \tilde{S}$

$$A_{\tau,m}\phi(t,y) = \frac{\bar{\Delta}}{2}\phi(t,y)$$

holds, which is needed for the following result.

Theorem 2.1 ([15]). *A $(\mathcal{C}_t)_t$-predictable process $H(t)$, $t \in [\tau, T]$ on $\bar{\Omega}$ is a historical Brownian motion starting at $(\tau, m) \in \tilde{S}$ and with branching rate $\gamma > 0$ if and only if $H(t) \in M_F(C)^t$ for all $t \in [\tau, T]$ and the law $\mathbb{P}_{\tau,m}$ of H is a solution to the following martingale problem*

$$\begin{gathered} Z(t)(\phi) = \langle H(t), \phi\rangle - \langle m, \phi\rangle - \int_\tau^t \langle H(s), \tfrac{\bar{\Delta}}{2}\phi(s,\cdot)\rangle ds, \quad t \in [\tau, T], \\ \textit{is a continuous } (\mathcal{F}_t)\textit{-martingale under } \mathbb{P}_{\tau,m} \textit{ for all } \phi \in D_0 \\ \textit{such that } Z(\tau)(\phi) = 0 \textit{ and with the quadratic variation given by} \\ [Z(\phi)]_t = \int_\tau^t \langle H(s), \gamma\phi^2\rangle ds \quad \textit{for all } t \in [\tau, T]. \end{gathered} \tag{MP$_0$}$$

From now on we consider the historical Brownian motion on its canonical path space $(\Omega_H, \mathcal{H}[\tau, T], (\mathcal{H}_t)_{t\in[\tau,T]}, \mathbb{P}_{\tau,m})$ with the coordinate process $H(t)(\omega) = \omega(t)$. The σ-algebra $\mathcal{H}[\tau, T]$ and the filtration $(\mathcal{H}_t)_{t\in[\tau,T]}$, are defined as the $\mathbb{P}_{\tau,m}$-completions of the corresponding σ-algebra and filtration generated by the coordinate process H.

Now we can introduce the concepts necessary to develop the result in Section 3, starting with a metric on the space of measure-valued càdlàg functions and differentiation of (non-anticipating) functionals on that space.

Denote by $D([\tau, T], M_F(C))$ the space of right continuous functions from $[\tau, T]$ to $M_F(C)$ with left limits and equip the space with the supremum metric $\tilde{d}(\omega, \omega') = \sup_{u\in[\tau,T]} d_P(\omega(u), \omega'(u))$ for all ω, $\omega' \in D([\tau, T], M_F(C))$, where d_P is the Prokhorov metric on $M_F(C)$. This allows us to define an equivalence relation on the space $[\tau, T] \times D([\tau, T], M_F(C))$ by

$$(t, \omega) \sim (t', \omega') \quad \Leftrightarrow \quad t = t' \text{ and } \omega^t = {\omega'}^{t'},$$

which gives rise to the quotient space

$$\Lambda_T = \{(t, \omega^t) : (t, \omega) \in [\tau, T] \times D([\tau, T], M_F(C))\} = [\tau, T] \times D([\tau, T], M_F(C))/\sim,$$

which we equip with the metric (for all (t, ω), $(t', \omega') \in \Lambda_T$)

$$d_\infty((t, \omega), (t', \omega')) = \sup_{u\in[\tau,T]} d_P(\omega(t \wedge u), \omega'(t' \wedge u)) + |t - t'|.$$

A functional F on $[\tau, T] \times D([\tau, T], M_F(C))$ mapping to $\mathbb{R}$ is called non-anticipating if it is a measurable map on the space of stopped paths, i.e. $F : \Lambda_T \to \mathbb{R}$. In other words, F is non-anticipating if $F(t, \omega) = F(t, \omega^t)$ holds for all $\omega \in D([\tau, T], M_F(C))$.

A functional $F : \Lambda_T \to \mathbb{R}$ is continuous with respect to d_∞ if for all $(t, \omega) \in \Lambda_T$ there exists for every $\varepsilon > 0$ an $\eta > 0$ such that for all $(t', \omega') \in \Lambda_T$ with $d_\infty((t, \omega), (t', \omega')) < \eta$ we have $|F(t, \omega) - F(t', \omega')| < \varepsilon$ (joint continuity in t and ω).

For a non-anticipating functional $F : \Lambda_T \to \mathbb{R}$ and $y \in C$, the functional derivative of F in direction y is given for all $(t, \omega) \in \Lambda_T$ by

$$\mathcal{D}_y F(t, \omega) = \lim_{\varepsilon\to 0} \frac{F(t, \omega + \varepsilon\delta_y 1_{[t,T]}) - F(t, \omega)}{\varepsilon}$$

if the limit exists.

We conclude this introductory section by defining the space of simple functions, which plays an important role in the development of the stochastic integral with respect to a martingale measure (see [17]). The space of simple functions, $\mathcal{S}$ is defined as the space of functions from $\Omega_H \times [\tau, T] \times C$ to $\mathbb{R}$ that are linear combinations of functions of form

$$\Phi_{\Gamma,B,a}(\omega, t, y) = \Gamma(\omega) 1_B(y) 1_{(a,T]}(t)$$

with Γ being bounded and $\mathcal{H}_a$-measurable, $B \in \mathcal{C}$ and $a \in [\tau, T]$. The σ-algebra generated by $\mathcal{S}$ is the product of the Borel σ-algebra on C and the predictable Borel σ-algebra on $[\tau, T] \times \Omega$ and called the predictable σ-algebra. Functions measurable with respect to this σ-algebra are called predictable.

3 Result

Let $\mathcal{L}^2(M_H)$ be the space of predictable functions $\Phi : \Omega_H \times [0, T] \times C \to \mathbb{R}$ satisfying

$$\|\Phi\|^2_{\mathcal{L}^2(M_H)} := \mathbb{E}_{\tau,m}\left[\gamma \int_{[\tau,T]\times C} \Phi(s, y)^2 H(s)(dy)ds\right] < \infty.$$

For functions in this space, the integral with respect to the martingale measure M_H exists, i.e.

$$I_{M_H}(\Phi) := \int_{[\tau,\cdot]\times C} \Phi(s, y) M_H(ds, dy) < \infty \tag{3.1}$$

for $\Phi \in \mathcal{L}^2(M_H)$.

Denote by $\mathcal{M}^2$ the space of square-integrable $(\mathcal{H}_t)_t$-martingales with initial value zero and with norm

$$\|Y\|^2_{\mathcal{M}^2} := \mathbb{E}_{\tau,m}\left[Y(T)^2\right].$$

Further, let $\mathcal{U}$ be the linear span of functions of form

$$\Phi_{\Gamma,\psi,a}(\omega, t, y) = \Gamma(\omega)\psi(y) 1_{(a,T]}(t),$$

where Γ is a $\mathcal{H}_a$-measurable, bounded random variable, $\tau \le a \le T$ and $\psi \in D_0$. As the pointwise limit of of $\mathcal{U}$ and $\mathcal{S}$ are equal, all functions in $\mathcal{U}$ are predictable. Further, if the L^∞ bounds of Γ^2 and ψ^2 are given by K_{Γ^2} and K_{ψ^2}, respectively, we have

$$\begin{aligned}\|\Phi_{\Gamma,\psi,a}\|^2_{\mathcal{L}^2(M_H)} &= \mathbb{E}\left[\int_{[\tau,T]\times C} (\Gamma\psi(s, y) 1_{(a,T]}(s))^2 H(s)(dy)ds\right] \\ &\le K_{\Gamma^2} K_{\psi^2} \mathbb{E}\left[\int_{[a,T]\times C} H(s)(dy)ds\right] < \infty,\end{aligned}$$

where we get the finiteness of the last term by the integrability of the total mass (cf. e.g. Corollary 2.2 in [15]). Therefore, $\mathcal{U} \subset \mathcal{L}^2(M_H)$ and, since $\Psi \in D_0$ can be approximated by step functions, this yields that $\mathcal{U}$ is dense in $\mathcal{L}^2(M_H)$.

In a first step, we analyze the predictable representation property and the form of the integrand on martingales build from the functions in $\mathcal{U}$, more precisely on the space

$$I_{M_H}(\mathcal{U}) = \left\{Y : Y(t) = \int_{[\tau,t]\times C} \Phi(s, y) M_H(ds, dy), \Phi \in \mathcal{U},\ t \in [\tau, T]\right\}.$$

Proposition 3.1. *Let $\Phi \in \mathcal{U}$, with $\Phi(\omega) = \Phi_{\Gamma,\psi,a}(\omega, s, y) = \Gamma(\omega)\psi(y)\mathbf{1}_{(a,T)}(s)$. Then:*

1. *It holds*

$$I_{M_H}(\Phi)(t) = \Gamma(\omega_a)\left(\langle H^t, \psi\rangle - \langle \omega(a), \psi\rangle - \int_a^t \langle \omega(s), \tfrac{\bar{\Delta}}{2}\psi\rangle ds\right) 1_{t>a} = F(t, H^t(\omega))$$

with $F = F_{\Phi_{\Gamma,\psi,a}}$ *defined by*

$$\begin{aligned} F : [\tau, T] \times D([\tau, T], M_F(C)) &\to \mathbb{R} \\ (t, \omega) &\mapsto \Gamma(\omega)\Bigg(\langle \omega(t), \psi\rangle - \langle \omega(a), \psi\rangle \\ &\qquad - \int_a^t \langle \omega(s), \tfrac{\bar{\Delta}}{2}\psi\rangle ds\Bigg) 1_{t>a}. \end{aligned}$$

2. *For F defined as in the first part and $(t, \omega) \in \Lambda_T$, it holds*

$$\mathcal{D}_y F(t, \omega) = \Gamma(\omega)\psi(y) 1_{(a,T]}(t) = \Phi_{\Gamma,\psi,a}(t, \omega). \tag{3.2}$$

3. *Set $Y(t) = F(t, H^t)$. As $(t, H^t(\omega)) \in \Lambda_T$, we can define the operator ∇_M on $I_{M_H}(\mathcal{U})$ by*

$$\nabla_M : I_{M_H}(\mathcal{U}) \ni Y \mapsto \nabla_M Y \in \mathcal{L}^2(M_H), \tag{3.3}$$

where

$$\nabla_M Y : (\omega, t, y) \mapsto \nabla_M Y(\omega, t, y) := \mathcal{D}_y F(t, H^t(\omega)).$$

Then, the representation

$$Y(t) = \int_{[\tau,t]\times C} \nabla_M Y(s, y) M_H(ds, dy) \tag{3.4}$$

holds for all $Y \in I_{M_H}(\mathcal{U})$.

Proof. We only have to prove (3.2). To do so, we first acknowledge that, for any $\omega \in D([\tau, T], M_F(C))$,

$$\begin{aligned} &\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\Bigg(\langle (\omega + \varepsilon\delta_y 1_{[t,T]})(t), \psi\rangle - \langle (\omega + \varepsilon\delta_y 1_{[t,T]})(a), \psi\rangle - \int_a^t \langle (\omega + \varepsilon\delta_y 1_{[t,T]})(s), \tfrac{\bar{\Delta}}{2}\psi\rangle ds \\ &\qquad - \langle \omega(t), \psi\rangle + \langle \omega(a), \psi\rangle + \int_a^t \langle \omega(s), \tfrac{\bar{\Delta}}{2}\psi\rangle ds\Bigg) \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\varepsilon\psi(y) - \varepsilon\int_a^t \langle \delta_y 1_{[t,T]}, \tfrac{\bar{\Delta}}{2}\psi\rangle ds\right) = \psi(y) \end{aligned}$$

holds. Further,

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\Gamma(\omega + \varepsilon\delta_y 1_{[t,T]}) 1_{t>a} - \Gamma(\omega) 1_{t>a}\right) = 0$$

holds since Γ is $\mathcal{H}_a$ measurable. In combination with the product rule of differentiation, this completes the proof of (3.2). □

Definition 3.2. *A linear operator Π mapping from its domain $D(\Pi)$ into a Hilbert space $\mathcal{H}$ is called an* extension *of the linear operator $\tilde{\Pi} : D(\tilde{\Pi}) \to \mathcal{H}$ if $D(\tilde{\Pi}) \subset D(\Pi)$ and $\tilde{\Pi}v = \Pi v$ for all $v \in D(\tilde{\Pi})$.*

To derive the martingale representation for elements in $\mathcal{M}^2$, we extend the operator ∇, which is defined in (3.3) and based on the functional derivative $\mathcal{D}$, to an operator ∇_M on $\mathcal{M}$. A crucial step towards extending the operator is proving that $I_{M_H}(\mathcal{U})$ is a dense subspace of $\mathcal{M}^2$. For this, we need the following result.

Proposition 3.3. *The mapping*

$$I_{M_H} : \mathcal{L}^2(M_H) \ni \Phi \mapsto \int_{[\tau,\cdot]\times C} \Phi(s,y) M_H(ds,dy) \in \mathcal{M}^2$$

is an isometry.

Proof. As M_H is an orthogonal martingale measure with covariation $\nu(ds,dy) = \gamma H(s)(dy)ds$, we obtain (Theorem 2.5 in [17]) for all Φ and Ψ in $\mathcal{L}^2(M_H)$

$$\begin{aligned}
&\mathbb{E}\left[\int_{[\tau,t]\times C} \Phi(s,y) M_H(ds,dy) \int_{[\tau,t]\times C} \Psi(s,y) M_H(ds,dy)\right] \\
&= \mathbb{E}\left[\gamma \int_{[\tau,t]\times C} \Phi(s,y)\Psi(s,y) H(s)(dy)ds\right]
\end{aligned} \tag{3.5}$$

for all $t \in [\tau, T]$. Setting $\Psi = \Phi$ in (3.5) yields

$$\begin{aligned}
\|I_{M_H}(\Phi)\|^2_{\mathcal{M}^2} &= \mathbb{E}\left[\left(\int_{[\tau,T]\times C} \Phi(s,y) M_H(ds,dy)\right)^2\right] \\
&= \mathbb{E}\left[\gamma \int_{[\tau,T]\times C} \Phi(s,y)^2 H(s)(dy)ds\right] = \|\Phi\|^2_{\mathcal{L}^2(M_H)}.
\end{aligned}$$

□

Next, to verify that $I_{M_H}(\mathcal{U})$ is indeed a subspace of $\mathcal{M}^2$, consider any $\Phi \in \mathcal{U}$. We then get

$$\begin{aligned}
I_{M_H}(\Phi)(t) &= \int_{[\tau,t]\times C} \Gamma\psi(y) 1_{(a,T]}(s) M_H(ds,dy) = \Gamma \int_{[a,t]\times C} \psi(y) M_H(ds,dy) 1_{t>a} \\
&= \Gamma(M(t)(\psi) - M(a)(\psi))1_{t>a} = \Gamma\left(\langle H(t),\psi\rangle - \langle H(a),\psi\rangle - \int_a^t \langle H(s), \tfrac{\bar\Delta}{2}\psi\rangle ds\right) 1_{t>a}.
\end{aligned}$$

As $\Psi \in D_0$ and Γ is $\mathcal{H}_a$-measurable, we get from the martingale problem (MP_0) that elements in $I_{M_H}(\mathcal{U})$ are martingales with $Y(\tau) = 0$. In addition, as $\mathcal{U} \subset \mathcal{L}^2(M_H)$, we get for all $\Phi \in \mathcal{U}$

$$\begin{aligned}
\mathbb{E}[(I_{M_H}(\Phi)(t))^2] &= \mathbb{E}\left[\left(\int_{[\tau,t]\times C} \Phi(s,y) M_H(ds,dy)\right)^2\right] \\
&= \mathbb{E}\left[\gamma \int_{[\tau,t]\times C} \Phi(s,y)^2 H(s)(dy)ds\right] \le \mathbb{E}\left[\gamma \int_{[\tau,T]\times C} \Phi(s,y)^2 H(s)(dy)ds\right] < \infty.
\end{aligned}$$

Consequently, $I_{M_H}(\Phi)$ is square-integrable and thus $I_{M_H}(\mathcal{U}) \subset \mathcal{M}^2$.

From Theorem 4.7 in [7] and Example 3.1 in [13] we get the existence of a unique $\rho \in \mathcal{L}^2(M_H)$ such that

$$Y(t) = \int_{[\tau,t]\times C} \rho(s,y) M_H(ds,dy) \quad \text{for all } t \in [\tau,T] \tag{3.6}$$

holds $\mathbb{P}_{\tau,m}$-a.s.. Consequently, the representation (3.4) is unique and by Proposition 3.3 the mapping I_{M_H} is a bijective isometry, which allows us to prove the following.

Proposition 3.4. *The space $\{\nabla_M Y : Y \in I_{M_H}(\mathcal{U})\}$ is dense in $\mathcal{L}^2(M_H)$ and the space $I_{M_H}(\mathcal{U})$ is dense in $\mathcal{M}^2$.*

Proof. From (3.2) and the definition of ∇_M, we obtain that $\{\nabla_M Y : Y \in I_{M_H}(\mathcal{U})\} = \mathcal{U}$ holds. As $\mathcal{U}$ is dense in $\mathcal{L}^2(M_H)$, this yields the density of $\{\nabla_M Y : Y \in I_{M_H}(\mathcal{U})\}$ in $\mathcal{L}^2(M_H)$. Further, as I_{M_H} is a bijective isometry, we get the density of $I_{M_H}(\mathcal{U})$ in $\mathcal{M}^2$. □

Taking this density result into account, we can prove the following proposition, which can be interpreted as an integration by parts formula.

Proposition 3.5. *If $Y \in I_{M_H}(\mathcal{U})$, $\nabla_M Y$ is the unique element in $\mathcal{L}^2(M_H)$ such that*

$$\mathbb{E}[Y(T)Z(T)] = \mathbb{E}\left[\gamma \int_{[\tau,T]\times C} \nabla_M Y(s,y) \nabla_M Z(s,y) H(s)(dy) ds\right] \tag{3.7}$$

holds for all $Z \in I_{M_H}(\mathcal{U})$.

Proof. From (3.4) and (3.5) we get

$$\begin{aligned}\mathbb{E}[Y(T)Z(T)] &= \mathbb{E}\left[\int_{[\tau,T]\times C} \nabla_M Y(s,y) M_H(ds,dy) \int_{[\tau,T]\times C} \nabla_M Z(s,y) M_H(ds,dy)\right] \\ &= \mathbb{E}\left[\gamma \int_{[\tau,T]\times C} \nabla_M Y(s,y) \nabla_M Z(s,y) H(s)(dy) ds\right].\end{aligned}$$

The uniqueness is obtained from the following: Assume $\Phi \in \mathcal{L}^2(M_H)$ also satisfies

$$\mathbb{E}[Y(T)Z(T)] = \mathbb{E}\left[\gamma \int_{[\tau,T]\times C} \Phi(s,y) \nabla_M Z(s,y) H(s)(dy) ds\right].$$

Then,

$$0 = \mathbb{E}\left[\gamma \int_{[\tau,T]\times C} (\Phi(s,y) - \nabla_M Y(s,y)) \nabla_M Z(s,y) H(s)(dy) ds\right]$$

holds for all $Z \in I_{M_H}(\mathcal{U})$. As $\{\nabla_M Z : Z \in I_{M_H}(\mathcal{U})\}$ is dense in $\mathcal{L}^2(M_H)$, this yields the equality of Φ and $\nabla_M Y$ in $\mathcal{L}^2(M_H)$ and thus the uniqueness. □

The interpretation as an integration by parts formula becomes clear if we write (3.7) in the following alternative form, which holds for all $\Phi \in \mathcal{L}^2(M_H)$:

$$\mathbb{E}\left[Y(T) \int_{[\tau,T]\times C} \Phi(s,y) M_H(ds,dy)\right] = \mathbb{E}\left[\gamma \int_{[\tau,T]\times C} \nabla_M Y(s,y) \Phi(s,y) H(s)(dy) ds\right].$$

By using the uniqueness of $\nabla_M Y$ in (3.7), we can extend the operator ∇_M from the subspace $I_{M_H}(\mathcal{U})$ to all of $\mathcal{M}^2$.

Theorem 3.6. *The operator ∇_M defined on $I_{M_H}(\mathcal{U})$ can be uniquely extend to a bounded operator*

$$\begin{array}{rccc}\nabla_M : & \mathcal{M}^2 & \to & \mathcal{L}^2(M_H) \\ & Y & \mapsto & \nabla_M Y.\end{array}$$

This operator is a bijection and the unique continuous extension is given by the following: For a given $Y \in \mathcal{M}^2$, $\nabla_M Y$ is the unique element in $\mathcal{L}^2(M_H)$ such that

$$\mathbb{E}[Y(T)Z(T)] = \mathbb{E}\left[\gamma \int_{[\tau,T]\times C} \nabla_M Y(s,y) \nabla_M Z(s,y) H(s)(dy) ds\right] \tag{3.8}$$

holds for all $Z \in I_{M_H}(\mathcal{U})$.

Proof. As $\nabla_M : I_{M_H}(\mathcal{U}) \to \mathcal{L}^2(M_H)$ is a bounded linear operator, $\mathcal{L}^2(M_H)$ is a Hilbert space and $I_{M_H}(\mathcal{U})$ is dense in $\mathcal{M}^2$, the BLT theorem (bounded linear transformation theorem; see e.g. Theorem 5.19 in [8]) yields the existence of a unique continuous bounded extension of ∇_M to $\mathcal{M}^2$. As the restriction of the operator defined by (3.8) to $I_{M_H}(\mathcal{U})$ coincides with the initial operator by Proposition 3.5, (3.8) uniquely defines the continuous extension.

As for every $Y \in \mathcal{M}^2$ there exists a unique ρ such that (3.6) holds, we can combine (3.6) with (3.5) and (3.4) to get for all $Z \in I_{M_H}(\mathcal{U})$

$$\begin{aligned}\mathbb{E}[Y(T)Z(T)] &= \mathbb{E}\left[\int_{[\tau,T]\times C} \rho(s,y) M_H(ds,dy) Z(T)\right] \\ &= \mathbb{E}\left[\int_{[\tau,T]\times C} \rho(s,y) \nabla_M Z(s,y) H(s)(dy) ds\right].\end{aligned}$$

Thus, ρ and $\nabla_M Y$ have to coincide in $\mathcal{L}^2(M_H)$ because of the uniqueness of the integrand in (3.6).

Using this, we can prove that the operator is bijective. To do so, let Y, $Y' \in \mathcal{M}^2$ with $\nabla_M Y = \nabla_M Y'$ and $\nabla_M Y$, $\nabla_M Y' \in \mathcal{L}^2(M_H)$. Then, as $I_{M_H}(\mathcal{U})$ is dense in $\mathcal{M}^2$, we get from

$$\begin{aligned}0 &= \mathbb{E}\left[\int_{[\tau,T]\times C} (\nabla_M Y(s,y) - \nabla_M Y'(s,y)) \nabla_M Z(s,y) H(s)(dy) ds\right] \\ &= \mathbb{E}\left[\left(\int_{[\tau,T]\times C} \nabla_M Y(s,y) M_H(ds,dy) - \int_{[\tau,T]\times C} \nabla_M Y'(s,y) M_H(ds,dy)\right) Z(T)\right] \\ &= \mathbb{E}[(Y(T) - Y'(T)) Z(T)]\end{aligned}$$

for all $Z \in I_{M_H}(\mathcal{U})$ that $Y = Y'$ in $\mathcal{M}^2$ holds. Consequently, the operator ∇_M is injective and as for every $\Phi \in \mathcal{L}^2(M_H)$ the process given by

$$Y = \int_{[\tau,\cdot]\times C} \Phi(s,y) M_H(ds,dy)$$

is in $\mathcal{M}^2$ and satisfies $\nabla_M Y = \Phi$, the operator is also surjective. Therefore, the operator is bijective. □

The operator ∇_M defined on $\mathcal{M}^2$ has the following nice properties which, while not of particular interest for the martingale representation formula, are worth mentioning.

Proposition 3.7. *The operator ∇_M defined on $\mathcal{M}^2$ is an isometry and the adjoint operator of I_{M_H}, the stochastic integral with respect to the martingale measure M_H.*

Proof. Let $Y \in \mathcal{M}^2$. As in the previous case, we get the isometry property from

$$\begin{aligned}\|\nabla_M Y\|^2_{\mathcal{L}^2(M_X)} &= \mathbb{E}\left[\gamma \int_{[\tau,T]\times C} (\nabla_M Y(s,y))^2 H(s)(dy) ds\right] \\ &= \mathbb{E}\left[\left(\int_{[\tau,T]\times C} \nabla_M Y(s,y) M_H(ds,dy)\right)^2\right] \\ &= \left\|\int_{[\tau,\cdot]\times C} \nabla_M Y(s,y) M_H(ds,dy)\right\|^2_{\mathcal{M}^2} \\ &= \|Y\|^2_{\mathcal{M}^2}.\end{aligned}$$

To show that ∇_M is the adjoint operator of I_{M_H}, let $\Phi \in \mathcal{L}^2(M_H)$. It then follows

$$\begin{aligned}\langle I_{M_H}(\Phi), Y\rangle_{\mathcal{M}^2} &= \mathbb{E}\left[\int_{[\tau,T]\times C} \Phi(s,y) M_H(ds,dy) Y(T)\right] \\ &= \mathbb{E}\left[\int_{[\tau,T]\times C} \Phi(s,y) M_H(ds,dy) \int_{[\tau,T]\times C} \nabla_M Y(s,y) M_H(ds,dy)\right] \\ &= \mathbb{E}\left[\gamma \int_{[\tau,T]\times C} \Phi(s,y) \nabla_M Y(s,y) H(s)(dy) ds\right] \\ &= \langle \Phi, \nabla_M Y\rangle_{\mathcal{L}^2(M_H)},\end{aligned}$$

which proves that ∇_M is the adjoint operator. □

From the above, we obtain the following martingale representation formula that extends (3.4) to all square-integrable $(\mathcal{H}_t)_t$-martingales.

Theorem 3.8. *For any square integrable $(\mathcal{H}_t)_t$-martingale Y and every $t \in [\tau, T]$ it holds*

$$Y(t) = Y(0) + \int_{[\tau,t]\times C} \nabla_M Y(s,y) M_H(ds,dy) \quad \mathbb{P}_{\tau,m} - a.s.. \tag{3.9}$$

Proof. First, assume that $Y \in \mathcal{M}^2$. From the proof of Theorem 3.6 we know that the unique integrand ρ in (3.6) is given by $\nabla_M Y$. Therefore, for $Y \in \mathcal{M}^2$ and $t \in [\tau, T]$, it holds

$$Y(t) = \int_{[\tau,t]\times C} \nabla_M Y(s,y) M_H(ds,dy)$$

$\mathbb{P}_{\tau,m}$-almost surely.

To obtain the result for all square-integrable $(\mathcal{H}_t)_t$-martingales Y, we can once again get a process $\tilde{Y} \in \mathcal{M}^2$ by setting $\tilde{Y} = Y - Y(0)$. Then, applying the above to $\tilde{Y}$ and adding $Y(0)$ to both sides yields (3.9). □

Remark 3.9. By following the same steps, one can obtain a version of Theorem 3.8 based on the super-Brownian motion instead of the historical Brownian motion. While we leave the details to the reader, we do note that the space D_0 in the definition of $\mathcal{U}$ is replaced by the Schwartz space and the uniqueness of integrand in (3.4) is proved in [6].

4 Comparison to the Result by Evans and Perkins

We start this section with a brief summary of the results in [6] and [7] before we compare our result to the existing one. Note that the following results from the literature are slightly adjusted to match our setting and that we skip over most of the details for the sake of brevity.

Let X be a Dawson-Watanabe superprocess and M_X the associated martingale measure. If a square-integrable functional F is applied to X, then the result of interest in [6] states that there exists an unique suitable integrand ϕ^F such that $F(X^T)$ can be written as

$$F(X^T) = \mathbb{E}[F(X^T)] + \int_{[0,T]\times E} \phi^F(s,x) M_X(ds,dx). \tag{4.1}$$

As an immediate consequence, every square-integrable martingale Y can be written as

$$Y(t) = \mathbb{E}[Y(0)] + \int_{[0,t]\times E} f(s,x) M_X(ds,dx)$$

for some square-integrable integrand f.

In [7] a result similar to (4.1) is proven in a scenario with X replaced by a historical Brownian motion H. In addition to that, if the function F satisfies some regularity conditions, the authors obtained the explicit form of the integrand ϕ^F and thus the explicit form of the representation, which is given by

$$F(H^T) = \mathbb{E}[F(H^T)] + \int_{[\tau,T]\times C} \mathcal{J}_{s;y} F(H^T) M_H(ds, dy). \tag{4.2}$$

The integrand $\mathcal{J}_{s;y} F(H^T)$ is given by a specific predictable projection of the process

$$J_{s;y} F(H^T) = \int_{C([\tau,T],M_F(C))} F(H^T + h) - F(H^T) \mathbb{Q}^{s;y^{s-}}(dh),$$

with $\mathbb{Q}^{s;y^{s-}}$ playing the role of the canonical measure in the Poisson cluster representation of the path of H from s to T and the operator $J_{s;y}$ resembling the Malliavin derivative for diffusions. The proof of (4.2) is motivated by Bismut's perturbation argument in the context of Brownian motion. For details on these concepts, we refer to the original work [7].

To compare the result in [7] to the one presented in Section 3, note that, if we set $t = T$, (3.9) becomes

$$Y(T) = \mathbb{E}[Y(T)] + \int_{[\tau,T]\times C} \nabla_M Y(s, y) M_H(ds, dy).$$

As $Y(T)$ is $\mathcal{H}_T$-measurable, there exists a function G such that $Y(T) = G(H^T)$, from which we get a representation of form

$$G(H^T) = \mathbb{E}[G(H^T)] + \int_{[\tau,T]\times C} \nabla_M G(H^T)(s, y) M_H(ds, dy).$$

If the function G satisfies the regularity conditions on F in [7], the uniqueness of the representation yields that the two integrands have to coincide with respect to $\|\cdot\|_{\mathcal{L}^2(M_H)}$, i.e.

$$\nabla_M G(H^T)(s, y) = \mathcal{J}_{s;y} G(H^T) \quad \text{with respect to } \|\cdot\|_{\mathcal{L}^2(M_H)}.$$

In particular, if F in (4.2) is such that $F(H^T) \in I_{M_H}(\mathcal{U})$, we can use the definition of ∇_M on $I_{M_H}(\mathcal{U})$ as well as (3.2) to compute the integrand using the Dupire derivative so that

$$\mathcal{J}_{s;y} F(H^T) = \mathcal{D}_y F(s, H^s) \quad \text{with respect to } \|\cdot\|_{\mathcal{L}^2(M_H)}.$$

In conclusion, the approach by Evans and Perkins is based on a two step derivation. In a first step, $J_{s;y} F(H^T)$ is computed. As $J_{s;y} F(H^T)$ is not predictable, computing the predictable projection $\mathcal{J}_{s;y} F(H^T)$ is necessary to obtain the integrand in the representation. This is in line with the derivation of the classical derivation of the Clark-Ocone formula based on Malliavin calculus.

In contrast to that, our approach is based on a single step as the Dupire derivative $\mathcal{D}$ as well as the operator ∇_M are already predictable. However, if one starts with a function G, it is necessary to first derive the martingale Y from the functional G for this single step procedure. Of course, this also constitutes a predictable projection. Hence, one can conclude that the operations *predicatable projection* and *differentiation* commute. A detailed comparison of the the approach based on Malliavin calculus and the approach based on functional calculus for real-valued diffusions is presented in Chapter 7.3 in [1], where this commutativity is also discussed.

Finally, while the approach in [7] requires a deeper understanding of the cluster representation of the historical Brownian motion and its paths, our approach is solely based on properties obtained from the martingale problem defining the historical Brownian motion.

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