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Regularization by random translation of potentials for the continuous PAM and related models in arbitrary dimension*

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Abstract

We study a regularization by noise phenomenon for the continuous parabolic Anderson model with a potential shifted along paths of fractional Brownian motion. We demonstrate that provided the Hurst parameter is chosen sufficiently small, this shift allows to establish well-posedness and stability to the corresponding problem – without the need of renormalization – in any dimension. We moreover provide a robustified Feynman-Kac type formula for the unique solution to the regularized problem building upon regularity estimates for the local time of fractional Brownian motion as well as non-linear Young integration.

Keywords: regularization by noise; parabolic Anderson model; non-linear Young integral; multiplicative noise; stochastic partial differential equations; Feynman-Kac formula.
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1 Introduction

Consider the problem

$$\partial_t u = \frac{1}{2}\Delta u - Vu, \qquad u(0) = f. \tag{1.1}$$

As it is well known, provided f and V are sufficiently smooth, we obtain the unique solution to the above problem via the Feynman-Kac formula

$$u(t,x) = \mathbb{E}^x\left[f(W_t)\exp\left(-\int_0^t V(s, W_{t-s})ds\right)\right],$$

where $(W_t)_t$ is a standard Brownian motion in $\mathbb{R}^d$ on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_t)$ and $\mathbb{E}^x[(\cdot)]$ denotes the expectation conditional on the Brownian motion starting in $x \in \mathbb{R}^d$. In the case of V enjoying only distributional regularity, this reasoning is no longer applicable. A famous example in this context is the continuous parabolic Anderson model (PAM)

$$\partial_t u = \frac{1}{2}\Delta u - \xi u, \qquad u(0) = f, \tag{1.2}$$

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where ξ is spatial white noise on $\mathbb{T}^d$ or $\mathbb{R}^d$. While in the case $d = 1$ (1.2) is still well posed due to the regularizing effect of the Laplacian, already $d = 2, 3$ require renormalization and the implementation of advanced tools from the theory of singular SPDEs such as regularity structures or paracontrolled distributions [28], [17], [18], [16], [1], [26, Example 1.21]. These approaches break down for $d \geq 4$.

In the following, we intend to study evolution problems of the above form with a potential shifted along paths of fractional Brownian motion w^H of Hurst parameter H, i.e.

$$\partial_t u = \frac{1}{2}\Delta u - \widetilde{V} u, \qquad u(0) = f, \tag{1.3}$$

where we denote $\widetilde{V}(t, x) := V(x - w_t^H)$ in case V enjoys sufficient regularity to admit pointwise evaluations or respectivelty $\widetilde{V}(t, \varphi) = \langle V, \varphi(\cdot + w_t^H)\rangle$ for any smooth test function φ in case V only enjoys some distributional regularity. The study of (1.3) is motivated by its formal connection to transport noise perturbations of (1.1). Indeed, provided v solves

$$\partial_t v = \frac{1}{2}\Delta v - V v + \nabla v \cdot \dot{w}_t^H, \qquad u(0) = f, \tag{1.4}$$

then formally, $u(t, x) := v(t, x - w_t^H)$ is a solution to (1.3). Conversely, given a solution u to (1.3), $v(t, x) := u(t, x + w_t^H)$ formally solves (1.4)[1].

We show that by shifting the reference frame of the potential in such a way, we can restore well-posedness to (1.3) and obtain stability results even in the setting of distributional valued V. As an application, we establish well-posedness of a shifted parabolic Anderson model

$$\partial_t u = \frac{1}{2}\Delta u - \widetilde{\xi} u, \qquad u(0) = f \tag{1.5}$$

in arbitrary dimension, provided the Hurst parameter H is chosen sufficiently small.

Towards this end, we exploit a pathwise regularization phenomenon on the level of the Feynman-Kac formula building mainly on [20]. More concretely, we establish in Lemma 2.2 below that for smooth potentials V^ϵ, we may rewrite the Feynman-Kac formula for (1.3) as

$$u^\epsilon(t, x) = \mathbb{E}^x\left[f(W_t)\exp\left(-(\mathcal{I}A^{t,\epsilon})_t\right)\right], \tag{1.6}$$

where $\mathcal{I}$ denotes Gubinelli's Sewing operator[2], $A^{t,\epsilon}$ the germ

$$A_{s,r}^{t,\epsilon} = (V^\epsilon * L_{s,r})(W_{t-s}),$$

and L the local time associated with the fractional Brownian motion w^H. Due to Young's inequality in Besov spaces, (1.6) might be well defined even for distributional potentials V, provided the local time L enjoys sufficient spatial regularity[3]. We therefore refer to (1.6) as a robustified Feynman-Kac formula. This main observation allows us to subsequently pass by a mollification argument: Given a distributional V, we consider first V^ϵ smooth such that $\|V^\epsilon - V\|_{H^{-\eta}} \to 0$. For such V^ϵ, we may express the unique solution u^ϵ to the associated PDE as in (1.6) by Lemma 2.2. Next, in Lemma 2.4 we establish that due to the regularizing effect of the local time L, $(u^\epsilon)_\epsilon$ will converge in appropriate topologies to

$$u(t, x) = \mathbb{E}^x\left[f(W_t)\exp\left(-(\mathcal{I}A^t)_t\right)\right], \tag{1.7}$$

[1] We refer to [4], [31] where a similar transformation was employed to study regularization by transport noise for transport equations. Note that to make this connection rigorous, the main challenge consists in giving meaning to $\nabla v \cdot \dot{w}_t^H$, which is done in [4], [31] using higher order rough path theory (refer also to [8, Section 2.4], [3]).

[2] Refer to [15] and Lemma 3.3 in the Appendix.

[3] This is precisely what [20, Theorem 3.1], also cited explicitly below, provides us with.

that then becomes a candidate for a solution to (1.3). Notice that the expression $\widetilde{V}u$ appearing in the equation is however a priori ill-defined. Yet, exploiting our explicit robustified Feynman-Kac representation (1.7), we can easily establish higher spatial regularity of u. In particular, we demonstrate that provided the initial condition f is chosen sufficiently smooth and H sufficiently small, u enjoys sufficient regularity for the product

$$V(\cdot)u(t, \cdot + w_t^H)$$

to be well defined in the sense of Lemma 1.5 for any $t \in [0, T]$. Under these more restrictive conditions, we are thus able to identify (1.7) as a weak solution to (1.3).

1.1 Formulation and discussion of the main result

Theorem 1.1. *For $\eta \geq 0$ and $d \in \mathbb{N}$, let $V \in H^{-\eta}(\mathbb{R}^d)$ and $f \in C^1(\mathbb{R}^d)$. Let w^H be a d-dimensional fractional Brownian motion on $(\Omega^H, \mathcal{F}^H, \mathbb{P}^H)$ whose Hurst parameter satisfies $H < \frac{1}{2}(1 + \eta + d/2)^{-1}$. Then for any smooth V^ϵ such that $\|V^\epsilon - V\|_{H^{-\eta}} \to 0$ and $\widetilde{V^\epsilon}(t, x) := V^\epsilon(x - w_t^H)$, the sequence $(u^\epsilon)_\epsilon$ of unique solutions to the the problem*

$$\partial_t u^\epsilon = \frac{1}{2}\Delta u^\epsilon - \widetilde{V^\epsilon} u^\epsilon, \qquad u(0) = f \tag{1.8}$$

is Cauchy in $C([0, T] \times \mathbb{R}^d)$ equipped with the topology of uniform convergence, $\mathbb{P}^H$-almost surely. For $A_{s,u}^t = (V * L_{s,u})(W_{t-s})$, the limit $u \in C([0, T] \times \mathbb{R}^d)$ satisfies the Feynman-Kac formula

$$u(t, x) = \mathbb{E}^x[f(W_t) \exp -(\mathcal{I}A^t)_t], \tag{1.9}$$

independent of the sequence $(V^\epsilon)_\epsilon$ chosen to approximate $V \in H^{-\eta}$. Moreover, provided further $\eta \notin \mathbb{N}$ and $H < \frac{1}{2}(1 + \eta + \lceil \eta \rceil + d/2)^{-1}$ and $f \in C^{\lceil \eta \rceil}$, we have $u(t, \cdot) \in C^{\lceil \eta \rceil}$ uniformly in $[0, T]$, $\mathbb{P}^H$-almost surely. In particular the product $V(\cdot)u(t, \cdot + w_t^H)$ is well defined for any $t \in [0, T]$ in the sense of Lemma 1.5 and u is a weak solution to

$$\partial_t u = \frac{1}{2}\Delta u - \widetilde{V}u, \qquad u(0) = f \tag{1.10}$$

i.e. for any $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $t \in [0, T]$ we have

$$\langle u_t - f, \varphi \rangle = \int_0^t \langle u_s, \frac{1}{2}\Delta\varphi \rangle ds + \int_0^t \langle V(\cdot)u(\cdot + w_s^H), \varphi(\cdot + w_s^H) \rangle ds.$$

Corollary 1.2 (Regularized PAM)**.** *Consider spatial white noise on the d-dimensional torus $\mathbb{T}^d$. Its realizations are known to lie in $H^{-(d/2+\epsilon)}$ for any $\epsilon > 0$ almost surely [34]. Hence, imposing $H < \frac{1}{2}(1 + d)^{-1}$, we can apply the first part of our main result Theorem 1.1. Demanding further $H < \frac{1}{2}(1 + d + \lceil d/2 + 1/4 \rceil)^{-1}$, we may employ the second part, yielding a weak solution.*

Remark 1.3. Note that due to the robustness of our approach, several canonical extensions to the above statements follow readily mutatis mutandis: Instead of considering the Laplacian, more general non-degenerate diffusion operators can be considered in (1.10). Moreover, instead of shifting the potential V along paths of fractional Brownian motion, shifts along any path that admits a sufficiently regular local time are conceivable.

Remark 1.4 (On uniqueness)**.** By Theorem 1.1 the function u of (1.9) is independent of the sequence $(V^\epsilon)_\epsilon$ chosen to approximate the problem. We therefore obtain uniqueness of solutions to (1.10) in the class of functions that are limits of solutions to mollified problems of the form of (1.8). For a genuine uniqueness statement without this restriction, one could attempt to show that any solution to (1.10) admits the Feynman-Kac formula (1.9) along the lines of [25, Theorem 5.7.6] for example. Note however that since such results require some minimal regularity on $\widetilde{V}$ (typically continuity), we would require again a mollification step,thus obtaining again only uniqueness in the class of function that are limits of solutions to the mollified problem.

1.2 Short overview of existing literature

The idea of employing a robustified Feynman-Kac formula in the study of heat equations with some form of multiplicative noise can be traced back to at least [30], [23], [22] for various types of space-time fractional Brownian motions. [21] combines these considerations with non-linear Young theory similar in spirit to the setting presented in this article. Remark however that our qualitatively different approach of considering random translations of the potential allows us to treat considerably more singular potentials. Furthermore, robustifications of the Feynman-Kac formula have been employed in the study of rough stochastic PDEs for example in [8, Chapter 12] or [7]. Regularization by additive noise for the multiplicative stochastic heat equation was recently established by [6] building upon ideas on pathwise regularization by noise in the spirit of [5], [10], [20]. Let us mention that this approach to regularization by noise has recently seen numerous applications for example to interacting particle systems [19], distribution dependent SDEs [12], [13] and multiplicative SDEs [11], [2].

1.3 Notation

We employ the standard notation necessary to formulate and apply the Sewing Lemma for which we refer to Appendix 3.B. Let $\mathcal{S}'$ denote the space of tempered distributions. For $\eta \in \mathbb{R}$, we denote by H^η the inhomogeneous Bessel potential space of order η, i.e.

$$H^\eta := \left\{ f \in \mathcal{S}' | \ \|f\|_{H^\eta} = \left\| (1 + |(\cdot)|)^\eta \hat{f} \right\|_{L^2} < \infty \right\}.$$

Moreover, for $\alpha > 0$ and $\alpha \notin \mathbb{N}$ we denote by $\mathcal{C}^\alpha$ the Hölder space

$$\mathcal{C}^\alpha := \left\{ f \in \mathcal{S}' | \ \|f\|_{\mathcal{C}^\alpha} = \|f\|_{C^{\lfloor \alpha \rfloor}} + \sup_{x \neq y} \frac{|(D^k f)(x) - (D^k f)(y)|}{|x-y|^{\alpha - \lfloor \alpha \rfloor}} < \infty \right\},$$

where for $n \in \mathbb{N}_0$

$$\|f\|_{C^n} = \sum_{k=0}^{n} \left\| D^k f \right\|_\infty . \tag{1.11}$$

We denote by C^n the space of n-times continuously differentiable functions such that (1.11) is finite. Let us remark that the first two spaces above are related to more general Besov spaces in the sense that $H^\alpha = B^\alpha_{2,2}$ for any $\alpha \in \mathbb{R}$ and $\mathcal{C}^\alpha = B^\alpha_{\infty,\infty}$ for $\alpha > 0$ and $\alpha \notin \mathbb{N}$. In particular, note that by Young's inequality in Besov spaces [27], we have

$$\|f * g\|_{\mathcal{C}^{\alpha-\eta}} \lesssim \|f\|_{H^\alpha} \|g\|_{H^{-\eta}} \tag{1.12}$$

for any $\alpha - \eta$ such that $\alpha - \eta > 0$ and $\alpha - \eta \notin \mathbb{N}$. Let us also recall the multiplication theorem for Besov spaces (see e.g. [29, Corollary 2.1.35], [33, Theorem 19.7]) adapted to our setting:

Lemma 1.5. *Let $\alpha > 0$ such that $\alpha - \eta > 0$. Then for any $\epsilon > 0$ and $u, v \in \mathcal{S}'$ we have*

$$\|u \cdot V\|_{H^{-\eta-\epsilon}} \lesssim \|u\|_{\mathcal{C}^\alpha} \|V\|_{H^{-\eta}} ,$$

i.e. the multiplication operator extends to a continuous bilinear map $\cdot : \mathcal{C}^\alpha \times H^{-\eta} \to H^{-\eta-\epsilon}$.

2 Proof of Theorem 1.1

For the readers convenience we begin by stating the following result on the regularity of local times associated with fractional Brownian motion that we will use throughout. We refer to the Appendix 3.A the basic definitions of occupation measures, local times and the occupation times formula.

Lemma 2.1 ([20, Theorem 3.1])**.** *Let w^H be a d-dimensional fractional Brownian motion of Hurst parameter $H < 1/d$ on $(\Omega^H, \mathcal{F}^H, \mathbb{P}^H)$. Then there exists a null set $\mathcal{N}$ such that for all $\omega \in \mathcal{N}^c$, the path $w^H(\omega)$ has a local time $L(\omega)$ and for $\lambda < \frac{1}{2H} - \frac{d}{2}$ and $\gamma \in [0, 1 - (\lambda + \frac{d}{2})H)$ we have*

$$\|L_{s,t}(\omega)\|_{H^\lambda} \leq C_T(\omega)|t-s|^\gamma. \tag{2.1}$$

for any $s,t \in [0,T]$, where $L_{s,t} = L_t - L_s$.

Throughout the remainder of the paper and for $H < 1/d$ satisfying additionally the conditions demanded in the statements below, we shall fix a realization of fractional Brownian motion w^H on $(\Omega^H, \mathcal{F}^H, \mathbb{P}^H)$ that admits a local time L and for which Lemma 2.1 can be applied in the corresponding regularity regime.

2.1 Solutions to the mollified equation converge

For $\eta \geq 0$ and $V \in H^{-\eta}$, let V^ϵ be a mollification, i.e. V^ϵ smooth such that $\|V^\epsilon - V\|_{H^{-\eta}} \to 0$. Then for $\widetilde{V^\epsilon}(t,x) := V^\epsilon(x - w_t^H)$ we know that the unique solution to the problem

$$\partial_t u^\epsilon = \frac{1}{2}\Delta u^\epsilon - (\widetilde{V^\epsilon})u^\epsilon, \qquad u^\epsilon(0) = f, \tag{2.2}$$

is given by

$$u^\epsilon(t,x) = \mathbb{E}^x\left[f(W_t)\exp\left(-\int_0^t V^\epsilon(W_{t-s} - w_s^H)ds\right)\right],$$

where $(W_t)_t$ is a standard Brownian motion in $\mathbb{R}^d$ on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_t)$ and $\mathbb{E}^x[(\cdot)]$ denotes the expectation conditional on the Brownian motion starting in $x \in \mathbb{R}^d$. We first establish that we may replace the above Lebesgue integral in time by an appropriate sewing that is capable of leveraging the regularizing effect due to the highly oscillating fractional Brownian motion. Towards this end, we exploit the Sewing Lemma 3.3.

Lemma 2.2 (Identification of Riemann integral as Sewing)**.** *For $\delta > 0$, let $V \in H^{(1\vee d/2)+\delta}(\mathbb{R}^d)$. Let W be a standard Brownian motion on $\mathbb{R}^d$ on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_t)$. Then for almost every $\omega \in \Omega$ the germ*

$$A_{s,r}^t := (V * L_{s,r})(W_{t-s}(\omega)) = \int_s^r V(W_{t-s}(\omega) - w_v^H)dv$$

admits a Sewing $(\mathcal{I}A^t)$ on $[0,t]$ and moreover for any $t \in [0,T]$ we have

$$\int_0^t V(W_{t-s} - w_s^H)ds = (\mathcal{I}A^t)_t.$$

Proof. Remark first that by Lemma 2.1 we have for some $\epsilon > 0$ and any $(s,t) \in \Delta_2([0,T])$:

$$\|L_{s,t}\|_{L^2} \leq C_T|t-s|^{1/2+\epsilon}.$$

Moreover, by Young's inequality in Besov spaces (1.12), we have that

$$\|V * L_{s,t}\|_{\mathcal{C}^{1+\delta}} \lesssim \|V\|_{H^{1+\delta}} \|L_{s,t}\|_{L^2}.$$

We also have for almost every $\omega \in \Omega$ that $W \in C^{1/2-\epsilon/2}$. We therefore obtain for $(s,u,r,t) \in \Delta_4([0,T])$ that

$$\begin{aligned} |(\delta A^t)_{s,u,r}| &= |V * L_{u,r}(W_{t-s}) - V * L_{u,r}(W_{t-u})| \\ &\lesssim \|V\|_{H^{1+\delta}} \|L_{s,t}\|_{L^2} |r-u|^{1/2+\epsilon} |u-s|^{1/2-\epsilon/2}. \end{aligned}$$

We conclude that A^t does indeed admit a Sewing on $[0,t]$ and for $(s,u,t) \in \Delta_3([0,T])$ we have

$$|A^t_{s,u} - (\mathcal{I}A^t)_{s,u}| = O(|u-s|^{1+\epsilon/2}).$$

Next, observe that the germ

$$\widetilde{A}^t_{s,u} := \int_s^u V(W_{t-v} - w^H_v)dv$$

trivially admits a Sewing, as $\delta\widetilde{A}^t = 0$ wherefore we have $(\mathcal{I}\widetilde{A}^t) = \widetilde{A}^t$. Moreover note that because of $V \in H^{d/2+\delta} \hookrightarrow \mathcal{C}^\delta$ we have for $(s,u,t) \in \Delta_3$

$$\left|\widetilde{A}^t_{s,u} - A^t_{s,u}\right| \lesssim \int_s^u |W_{t-s} - W_{t-v}|^\delta dv \lesssim |u-s|^{1+\delta(1-\epsilon/2)},$$

allowing to conclude

$$\left|\int_s^u V(W_{t-v} - w^H_v)dv - (\mathcal{I}A^t)_{s,u}\right| \le |\widetilde{A}^t_{s,u} - A^t_{s,u}| + |A^t_{s,u} - (\mathcal{I}A^t)_{s,u}| \lesssim |u-s|^{1+\delta(1-\epsilon/2)}.$$

Hence the function

$$s \in [0,t] \to \int_0^s V(W_{t-r} - w^H_r)dr - (\mathcal{I}A^t)_s$$

is constant. Since it moreover starts in zero, this establishes the claim. □

Remark 2.3. Remark that in the above statement, we did not exploit any regularization from the local time, but instead demanded regularity of the potential V. As a consequence, the only constraint on the Hurst parameter at this stage is $H < 1/d$, which simply assures the existence of a local time. In the following, we will impose further restrictions on the Hurst parameter, allowing to pass to less regular potentials V.

By the previous Lemma 2.2, we have that indeed

$$\begin{aligned} u^\epsilon(t,x) &= \mathbb{E}^x\left[f(W_t)\exp\left(-\int_0^t V^\epsilon(W_{t-s} - w^H_s)ds\right)\right] \\ &= \mathbb{E}^x\left[f(W_t)\exp\left(-(\mathcal{I}A^{t,\epsilon})_t\right)\right], \end{aligned}$$

where

$$A^{t,\epsilon}_{s,u} := (V^\epsilon * L_{s,u})(W_{t-s}).$$

In the next Lemma we address the question: Under which condition on V and H is it possible to pass to a limit $\epsilon \to 0$?

Lemma 2.4 (Convergence of mollifications)**.** *For $\eta \ge 0$, $d \in \mathbb{N}$, let $f \in C^1(\mathbb{R}^d)$, $V \in H^{-\eta}(\mathbb{R}^d)$ and $H < \frac{1}{2}(1+\eta+d/2)^{-1}$. Let u^ϵ be the unique solution to the mollified problem* (2.2) *and set*

$$u(t,x) := \mathbb{E}^x\left[f(W_t)\exp\left(-(\mathcal{I}A^t)_t\right)\right] \tag{2.3}$$

where

$$A^t_{s,u} := (V * L_{s,u})(W_{t-s}).$$

Then u^ϵ converges uniformly to u on $[0,T] \times \mathbb{R}^d$.

Proof. Let us start by establishing that u is well defined under the conditions stated in the Lemma. By Lemma 2.1, we have for some $\delta > 0$

$$\|L_{s,t}\|_{H^{1+\eta+\delta}} \lesssim |t-s|^{1/2+\delta}.$$

Note that again, by Young's inequality in Besov spaces, we have

$$\|V * L_{s,t}\|_{\mathcal{C}^{1+\delta}} \leq \|V\|_{H^{-\eta}} \|L_{s,t}\|_{H^{1+\eta+\delta}}$$

meaning again in particular that $V * L_{s,t}$ lies in C^1. Similar to the previous Lemma 2.2, we can conclude that indeed, A^t admits a sewing, since

$$\begin{aligned} |(\delta A^t)_{s,u,r}| &= |V * L_{u,r}(W_{t-s}) - V * L_{u,r}(W_{t-u})| \\ &\lesssim \underbrace{\left(\sup_{x \neq y \in [0,T]} \frac{|W_x(\omega) - W_y(\omega)|}{|x-y|^{1/2-\delta/2}} \right)}_{=: c_\delta(\omega)} |r-u|^{1/2+\delta} |u-s|^{1/2-\delta/2}. \end{aligned}$$

The above ensures that for almost every $\omega \in \Omega$ the expression $(\mathcal{I}A^t)_t$ is well defined. We further demonstrate that $(\mathcal{I}A^t)_t$ admits exponential moments with respect to the measure $\mathbb{P}$, allowing to establish well posedness of (2.3). By the Sewing Lemma 3.3, we have for $t \in [0,T]$ the a priori bound

$$\begin{aligned} |(\mathcal{I}A^t)_t| &\leq |A^t_{0,t}| + |(\mathcal{I}A^t)_{0,t} - A^t_{0,t}| \\ &\leq \|V * L_{0,t}\|_\infty + \left\|\delta A^t\right\|_{1+\delta/2} T^{1+\delta/2} \\ &\lesssim 1 + c_\delta(\omega). \end{aligned} \tag{2.4}$$

Hence, we conclude that for some $a > 0$, we have

$$|\mathbb{E}^x \left[f(W_t) \exp\left(-(\mathcal{I}A^t)_t\right) \right]| \lesssim \|f\|_\infty \mathbb{E}^x[\exp\left(a c_\delta(\omega)\right)] < \infty$$

by Lemma 3.5. This shows that the function u in (2.3) is well defined as a function in $C^0([0,T] \times \mathbb{R}^d)$. Towards establishing convergence, let us first remark that similar to (2.4), we have

$$|(\mathcal{I}A^{t,\epsilon})_t| \lesssim 1 + c_\delta(\omega)$$

uniformly in $\epsilon > 0$. This permits the following bound

$$\begin{aligned} |u^\epsilon_t(x) - u_t(x)| &\leq \left| \mathbb{E}^x \left[f(W_t) (e^{-(\mathcal{I}A^t)_t} - e^{-(\mathcal{I}A^{t,\epsilon})_t}) \right] \right| \\ &\leq \mathbb{E}^x \left[|f(W_t)| e^{a c_\delta} |(\mathcal{I}A^t)_t - (\mathcal{I}A^{t,\epsilon})_t|) \right]. \end{aligned}$$

Next, note that due to the linearity of the Sewing operator $\mathcal{I}$

$$|(\mathcal{I}A^t)_t - (\mathcal{I}A^{t,\epsilon})_t| \leq \left\|A^t - A^{t,\epsilon}\right\|_{1/2} T^{1/2} + \left\|\delta(A^t - A^{t,\epsilon})\right\|_{1+\delta/2} T^{1+\delta/2}.$$

We have moreover that

$$|(A^t - A^{t,\epsilon})_{s,r}| \lesssim \|V - V^\epsilon\|_{H^{-\eta}} \|L_{s,r}\|_{H^{\eta+\delta}} \lesssim \|V - V^\epsilon\|_{H^{-\eta}} |r-s|^{1/2},$$

as well as similar to the above calculations

$$\begin{aligned} |(\delta(A^t - A^{t,\epsilon}))_{s,u,r}| &= |(V - V^\epsilon) * L_{u,r}(W_{t-s}) - (V - V^\epsilon) * L_{u,r}(W_{t-u})| \\ &\lesssim c_\delta(\omega) \|V - V^\epsilon\|_{H^{-\eta}} |r-u|^{1/2+\delta} |u-s|^{1/2-\delta/2}. \end{aligned}$$

Overall, this permits to conclude that

$$|u^\epsilon_t(x) - u_t(x)| \lesssim \|V - V^\epsilon\|_{H^{-\eta}} \|f\|_\infty \mathbb{E}^x[c_\delta \exp\left(a(1 + c_\delta)\right)] \lesssim \|V - V^\epsilon\|_{H^{-\eta}} \|f\|_\infty$$

exploiting Lemma 3.5. Hence we have established the claim. □

2.2 Weak solutions

In the following section, we undertake to establish in what sense and under what conditions the function u obtained in the previous section solves our original problem. We recall again that the main obstacle to address lies in the appearance of the product $\widetilde{V}u$. This obstacle will be overcome by establishing higher spatial regularity of u building upon our robustified Feynman-Kac representation for u in Lemma 2.4 provided we demand more regularity in the initial condition f as well as the local time L. In this way, Lemma 1.5 allows us to give a meaning to the product $\widetilde{V}u$, thereby allowing to conclude that u satisfies the original problem in the weak sense of Theorem 1.1.

Lemma 2.5 (Spatial regularity of u)**.** *For $n \in \mathbb{N}$ suppose $H < \frac{1}{2}(1+\eta+n+d/2)^{-1}$, $f \in C^n$ and $V \in H^{-\eta}$. Then for every $t \in [0,T]$, the function*

$$u(t,x) := \mathbb{E}^x \left[f(W_t) \exp\left(-(\mathcal{I}A^t)_{0,t}\right)\right], \tag{2.5}$$

where

$$A^t_{s,r} = (V * L_{s,r})(W_{t-s})$$

lies in C^n. Moreover, $u^\epsilon \to u$ in C^n, uniformly in $t \in [0,T]$.

Proof. Fix $t \in [0,T]$. We show that the function

$$x \to (\mathcal{I}A^t(x))_t,$$

where

$$A^t_{s,r}(x) = (V * L_{s,r})(W_{t-s} + x)$$

is n times differentiable and that moreover, all derivatives up to order n are uniformly bounded in space, integrable with respect to $\mathbb{P}$. Towards this end, let us note that

$$D^k_x(\mathcal{I}A^t(x))_t = (\mathcal{I}(D^k_x A^t)(x))_t. \tag{2.6}$$

This can be established by using Lemma 3.4. For the sake of conciseness, we restrict ourselves to the case $d=1$ and one derivative. Let us define

$$A^{t,n}_{s,r} = n\left((V * L_{s,r})(W_{t-s} + x + 1/n) - (V * L_{s,r})(W_{t-s} + x)\right).$$

Then it can be seen easily that $\|A^{t,n} - D_x A^t\|_{1/2} \to 0$ uniformly in $t \in [0,T]$. Moreover,

$$\begin{aligned}
\delta A^{t,n}_{s,u,r} &= n\left((V * L_{u,r})(W_{t-s} + x + 1/n) - (V * L_{u,r})(W_{t-s} + x)\right) \\
&\quad - n\left((V * L_{u,r})(W_{t-u} + x + 1/n) + (V * L_{u,r})(W_{t-u} + x)\right) \\
&= (V * D_x L_{u,r})(W_{t-s} + x) - (V * D_x L_{u,r})(W_{t-u} + x) + O(1/n)|u-s|^{1+2\delta} \\
&\lesssim c_\delta(\omega)|r-u|^{1/2+\delta}|u-s|^{1/2-\delta/2} + O(1/n)|u-s|^{1+2\delta},
\end{aligned}$$

meaning that indeed $\sup_n \|\delta A^{t,n}\|_{1+\delta/2} < \infty$, allowing to conclude (2.6) by Lemma 3.4. By the Faà di Bruno formula, we have

$$\begin{aligned}
&\frac{d^n}{dx^n}\left(f(W_t + x)\exp\left(-(\mathcal{I}A^t(x))_t\right)\right) \\
&= \exp\left(-(\mathcal{I}A^t(x))_t\right) \sum_{k=0}^{n} \binom{n}{k} \left(D^{n-k}_x f(W_t + x)\right) B_k(-(\mathcal{I}D_x A^t(x))_t, \cdots - (\mathcal{I}D^k_x A^t(x))_t),
\end{aligned}$$

where B_k denotes the k-th complete Bell polynomial with the convention $B_0 = 1$. Note in particular that $(\mathcal{I}D^k_x A^t(x))_t$ is uniformly bounded in space due to the regularity of the local time in this setting. Moreover, we have for any $k \le n$ the a priori bound

$$|(\mathcal{I}D^k_x A^t(x))_t| \lesssim (1 + c_\delta(\omega)).$$

We therefore have for any $a > 0$

$$|B_k(-D(\mathcal{I}A^t)_{0,t}, \dots D^k(\mathcal{I}A^t)_{0,t})| \lesssim \exp\left(ac_\gamma(\omega)\right)$$

wherefore

$$\frac{d^n}{dx^n}\left(f(B_t + x)\exp\left(-(\mathcal{I}A^t)_{0,t}\right)\right)$$

is well defined, uniformly bounded in $x \in \mathbb{R}^d$ and integrable with respect to $\mathbb{P}$. Overall, this allows to conclude that indeed $u(t, \cdot) \in C^n$ for any $t \in [0, T]$. Finally, going through similar considerations for u^ϵ and remarking that

$$(\mathcal{I}D_x^k(A^t - A^{t,\epsilon}(x)))_t \to 0$$

uniformly in $t, x \in [0, T] \times \mathbb{R}^d$, we infer that $u^\epsilon(t, \cdot) \to u(t, \cdot)$ in C^n uniformly in $t \in [0, T]$. □

Invoking Lemma 1.5, we can now conclude the proof of Theorem 1.1 by observing the following:

Corollary 2.6. *Suppose $H < \frac{1}{2}(1 + \eta + \lceil\eta\rceil + d/2)^{-1}$, $f \in C^{\lceil\eta\rceil}$ and $V \in H^{-\eta}$. Then for any $t \in [0, T]$, the product*

$$\langle(\widetilde{V}u)_t, \varphi\rangle := \langle V(\cdot)u(t, \cdot + w_t^H), \varphi(\cdot + w_t^H)\rangle$$

is well defined in the sense of Lemma 1.5. In particular, we infer that u is a weak solution to the original problem.

3 Appendix

3.A Local times and occupation times formula

We recall for the reader the basic concepts of occupation measures, local times and the occupation times formula. A comprehensive review paper on these topics is [14].

Definition 3.1. *Let $w : [0, T] \to \mathbb{R}^d$ be a measurable path. Then the occupation measure at time $t \in [0, T]$, written μ_t^w is the Borel measure on $\mathbb{R}^d$ defined by*

$$\mu_t^w(A) := \lambda(\{s \in [0, t] : \ w_s \in A\}), \quad A \in \mathcal{B}(\mathbb{R}^d),$$

where λ denotes the standard Lebesgue measure.

The occupation measure thus measures how much time the process w spends in certain Borel sets. Provided for any $t \in [0, T]$, the measure is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$, we call the corresponding Radon-Nikodym derivative local time of the process w:

Definition 3.2. *Let $w : [0, T] \to \mathbb{R}^d$ be a measurable path. Assume that there exists a measurable function $L^w : [0, T] \times \mathbb{R}^d \to \mathbb{R}_+$ such that*

$$\mu_t^w(A) = \int_A L_t^w(z)dz,$$

for any $A \in \mathcal{B}(\mathbb{R}^d)$ and $t \in [0, T]$. Then we call L^w local time of w.

Note that by the definition of the occupation measure, we have for any bounded measurable function $f : \mathbb{R}^d \to \mathbb{R}$ that

$$\int_0^t f(w_s)ds = \int_{\mathbb{R}^d} f(z)\mu_t^w(dz). \tag{3.1}$$

The above equation (3.1) is called occupation times formula. Remark that in particular, provided w admits a local time, we also have for any $x \in \mathbb{R}^d$

$$\int_0^t f(x - w_s)ds = \int_{\mathbb{R}^d} f(x - z)\mu_t^w(dz) = \int_{\mathbb{R}^d} f(x - z)L_t^w(z)dz = (f * L_t^w)(x). \tag{3.2}$$

3.B The sewing lemma

We recall the Sewing Lemma due to [15] (see also [8, Lemma 4.2]). Let E be a Banach space, $[0,T]$ a given interval. Let Δ_n denote the n-th simplex of $[0,T]$, i.e. $\Delta_n := \{(t_1,\dots,t_n)|0 \le t_1 \cdots \le t_n \le T\}$. For a function $A : \Delta_2 \to E$ define the mapping $\delta A : \Delta_3 \to E$ via

$$(\delta A)_{s,u,t} := A_{s,t} - A_{s,u} - A_{u,t}.$$

Provided $A_{t,t} = 0$ we say that for $\alpha, \beta > 0$ we have $A \in C_2^{\alpha,\beta}(E)$ if $\|A\|_{\alpha,\beta} < \infty$, where

$$\|A\|_\alpha := \sup_{(s,t)\in\Delta_2} \frac{\|A_{s,t}\|_E}{|t-s|^\alpha}, \qquad \|\delta A\|_\beta := \sup_{(s,u,t)\in\Delta_3} \frac{\|(\delta A)_{s,u,t}\|_E}{|t-s|^\beta}$$

and $\|A\|_{\alpha,\beta} := \|A\|_\alpha + \|\delta A\|_\beta$. For a function $f : [0,T] \to E$, we denote $f_{s,t} := f_t - f_s$. Moreover, if for any sequence $(\mathcal{P}^n([s,t]))_n$ of partitions of $[s,t]$ whose mesh size goes to zero, the quantity

$$\lim_{n\to\infty} \sum_{[u,v]\in\mathcal{P}^n([s,t])} A_{u,v}$$

converges to the same limit, we note

$$(\mathcal{I}A)_{s,t} := \lim_{n\to\infty} \sum_{[u,v]\in\mathcal{P}^n([s,t])} A_{u,v}.$$

Lemma 3.3 (Sewing, [15])**.** *Let $0 < \alpha \le 1 < \beta$. Then for any $A \in C_2^{\alpha,\beta}(E)$, $(\mathcal{I}A)$ is well defined (we say that A admits the sewing $(\mathcal{I}A)$). Moreover, denoting $(\mathcal{I}A)_t := (\mathcal{I}A)_{0,t}$, we have $(\mathcal{I}A) \in C^\alpha([0,T],E)$ and $(\mathcal{I}A)_0 = 0$ and for some constant $c > 0$ depending only on β we have*

$$\|(\mathcal{I}A)_t - (\mathcal{I}A)_s - A_{s,t}\|_E \le c\,\|\delta A\|_\beta\,|t-s|^\beta.$$

Lemma 3.4 (Lemma A.2 [9])**.** *For $0 < \alpha \le 1 < \beta$ and E a Banach space, let $A \in C_2^{\alpha,\beta}(E)$ and $(A^n)_n \subset C_2^{\alpha,\beta}(E)$ such that for some $R > 0$ $\sup_{n\in\mathbb{N}} \|\delta A^n\|_\beta \le R$ and such that $\|A^n - A\|_\alpha \to 0$. Then*

$$\|\mathcal{I}(A - A^n)\|_\alpha \to 0.$$

3.C Exponential moments for the Hölder modulus of continuity of Brownian motion

For the sake of completeness, we provide a sketch of the proof that the γ-Hölder modulus of continuity of Brownian motion is exponentially integrable for $\gamma \in (0,1/2)$. Refer also to [24] for more refined integrability statements.

Lemma 3.5. *Let B be a standard Brownian motion. Then for any $\gamma < 1/2$ and $a > 0$, we have*

$$\mathbb{E}\left[\exp\left(a \sup_{s\neq t\in[0,T]} \frac{|B_t - B_s|}{|t-s|^\gamma}\right)\right] < \infty.$$

Proof. Without loss of generality, set $T = 1$. Remark first that we have for $k \in \mathbb{N}$

$$\mathbb{E}[|B_t - B_s|^k] \le |t-s|^{k/2}(k-1)!!$$

We follow the classical proof of Kolmogorov's continuity theorem (refer for example to [32, Theorem 10.1]). For $D_m := 2^{-m}\mathbb{N}_0 \cap [0,1)$ and $D = \cup_m D_m$ set

$$\Delta_m = \{(s,t) \in D_m \times D_m : \ |t-s| \le 2^{-m}\}.$$

We then have for $\sigma_j := \sup_{(s,t)\in\Delta_j} |B(t) - B(s)|$ the bound

$$\mathbb{E}[\sigma_j^k] \leq \sum_{(s,t)\in\Delta_j} \mathbb{E}[|B_t - B_s|^k] \leq 2 \cdot 2^{j(1-k/2)})(k-1)!!$$

Following further the proof of Kolmogorov's continuity theorem as in [32, Theorem 10.1], we obtain

$$\begin{aligned}\mathbb{E}\left[\left(\sup_{s\neq t\in D} \frac{|B_t - B_s|}{|t-s|^\gamma}\right)^k\right]^{1/k} &\leq 2^{1+\gamma} \sum_{j=0}^\infty 2^{j\gamma} \mathbb{E}[\sigma_j^k]^{1/k} \\ &\leq 2^{1/k} \cdot 2^{1+\gamma}((k-1)!!)^{1/k} \sum_{j=0}^\infty 2^{j\gamma}(2^{-j(1/2-1/k)}).\end{aligned}$$

Now let k_0 be the smallest natural such that $\gamma < 1/2 - 1/k_0$. We then obtain for any $k \geq k_0$

$$\begin{aligned}\mathbb{E}\left[\left(\sup_{s\neq t\in D} \frac{|B_t - B_s|}{|t-s|^\gamma}\right)^k\right]^{1/k} &\leq 2^{1/k} \cdot 2^{1+\gamma}((k-1)!!)^{1/k} \sum_{j=0}^\infty 2^{j(\gamma+1/k-1/2)} \\ &\leq 2^{1/k} \cdot 2^{1+\gamma}((k-1)!!)^{1/k} \sum_{j=0}^\infty 2^{j(\gamma+1/k_0-1/2)} \\ &\leq C2^{1/k} \cdot 2^{1+\gamma}((k-1)!!)^{1/k}.\end{aligned}$$

Therefore, we have

$$\sum_{k=k_0}^\infty \frac{a^k}{k!} \mathbb{E}\left[\left(\sup_{s\neq t\in[0,1]} \frac{|B_t - B_s|}{|t-s|^\gamma}\right)^k\right] \leq 2\sum_{k=k_0}^\infty \frac{(2^{1+\gamma}Ca)^k}{k!!} < \infty,$$

which proves the claim. □

References

[1] R. Allez and K. Chouk, *The continuous Anderson hamiltonian in dimension two*, 2015, arXiv:1511.02718.

[2] F. Bechtold and M. Hofmanová, *Weak solutions for singular multiplicative SDEs via regularization by noise*, 2022, arXiv:2203.13745.

[3] C. Bellingeri, A. Djurdjevac, P. K. Friz, and N. Tapia, *Transport and continuity equations with (very) rough noise*, Partial Differential Equations and Applications **2** (2021), no. 4. MR4338294

[4] R. Catellier, *Rough linear transport equation with an irregular drift*, Stochastics and Partial Differential Equations: Analysis and Computations **4** (2016), no. 3, 477–534. MR3538009

[5] R. Catellier and M. Gubinelli, *Averaging along irregular curves and regularisation of ODEs*, Stochastic Processes and their Applications **126** (2016), no. 8, 2323–2366. MR3505229

[6] R. Catellier and F. Harang, *Pathwise regularization of the stochastic heat equation with multiplicative noise through irregular perturbation*, 2021, arXiv:2101.00915.

[7] J. Diehl, P. Friz, and W. Stannat, *Stochastic partial differential equations: a rough paths view on weak solutions via Feynman–Kac*, Annales de la Faculté des sciences de Toulouse: Mathématiques **Ser. 6, 26** (2017), no. 4, 911–947 (en). MR3746646

[8] P. Friz and M. Hairer, *A course on rough paths*, Universitext, Springer, Cham, 2014. MR3289027

[9] L. Galeati, *Nonlinear young differential equations: A review*, Journal of Dynamics and Differential Equations (2021).

[10] L. Galeati and M. Gubinelli, *Noiseless regularisation by noise*, Revista Matemática Iberoamericana **38** (2021), no. 2, 433–502. MR4404773

[11] L. Galeati and F. Harang, *Regularization of multiplicative sdes through additive noise*, 2020, arXiv:2008.02335.

[12] L. Galeati, F. A. Harang, and A. Mayorcas. Distribution dependent SDEs driven by additive fractional brownian motion. *Probability Theory and Related Fields*, May 2022.

[13] L. Galeati, F. Harang, and A. Mayorcas, *Distribution dependent SDEs driven by additive continuous noise*, Electronic Journal of Probability **27** (2022), 1–38. MR4388460

[14] D. Geman and J. Horowitz, *Occupation densities*, The Annals of Probability **8** (1980), no. 1, 1–67. MR0556414

[15] M Gubinelli, *Controlling rough paths*, J. Func. Anal. **216** (2004), no. 1, 86–140. MR2091358

[16] M. Gubinelli, P. Imkeller, and N. Perkowski, *Paracontrolled distributions and singular PDEs*, Forum of Mathematics, Pi **3e6** (2015), 1–75. MR3406823

[17] M. Hairer and C. Labbé, *A simple construction of the continuum parabolic Anderson model on* $\mathbf{R}^2$, Electronic Communications in Probability **20** (2015), no. none, 1–11. MR3358965

[18] M.Hairer and C. Labbé, *Multiplicative stochastic heat equations on the whole space*, Journal of the European Mathematical Society **20** (2018), no. 4, 1005–1054. MR3779690

[19] F. Harang and A. Mayorcas, *Pathwise regularisation of singular interacting particle systems and their mean field limits*, 2020, arXiv:2010.15517.

[20] F. Harang and N. Perkowski, *C∞-regularization of ODEs perturbed by noise*, Stochastics and Dynamics (2021), 2140010. MR4342752

[21] Y. Hu and K. Lê, *Nonlinear young integrals and differential systems in Hölder media*, Transactions of the American Mathematical Society **369** (2016), no. 3, 1935–2002. MR3581224

[22] Y. Hu, F. Lu, and D. Nualart, *Feynman–Kac formula for the heat equation driven by fractional noise with Hurst parameter* $H < 1/2$, The Annals of Probability **40** (2012), no. 3, 1041–1068. MR2962086

[23] Y. Hu, D. Nualart, and J. Song, *Feynman–Kac formula for heat equation driven by fractional white noise*, The Annals of Probability **39** (2011), no. 1, 291–326. MR2778803

[24] T. P. Hytönen and M. C. Veraar, *On Besov regularity of Brownian motions in infinite dimensions*, Probability and Mathematical Statistics **28** (2008), Fasc. 1, 143–162. MR2445509

[25] I. Karatzas and I.K.S. Shreve, *Brownian motion and stochastic calculus*, Graduate Texts in Mathematics (113) (Book 113), Springer New York, 1991. MR1121940

[26] W. König, *The parabolic Anderson model*, Springer International Publishing, 2016. MR3526112

[27] F. Kühn and R. Schilling, *Convolution inequalities for Besov and Triebel–Lizorkin spaces, and applications to convolution semigroups*, 2021, arXiv:2101.03886. MR4339468

[28] C. Labbé, *The continuous Anderson hamiltonian in* $d \leq 3$, Journal of Functional Analysis **277** (2019), no. 9, 3187–3235. MR3997634

[29] J. Martin, *Refinements of the solution theory for singular SPDEs*, Ph.D. thesis, Humboldt-Universität zu Berlin, 2018.

[30] O. Mocioalca and F. Viens, *Skorohod integration and stochastic calculus beyond the fractional brownian scale*, Journal of Functional Analysis **222** (2005), no. 2, 385–434. MR2132395

[31] T. Nilssen, *Rough linear PDE's with discontinuous coefficients – existence of solutions via regularization by fractional Brownian motion*, Electronic Journal of Probability **25** (2020), no. none, 1–33. MR4089784

[32] R. Schilling and L. Partzsch, *Brownian motion: An introduction to stochastic processes*, De Gruyter, 2012. MR2962168

[33] W. van Zuijlen, *Theory of function spaces*, Lecture notes (2020), http://www.wias-berlin.de/people/vanzuijlen/LN_theory_of_function_spaces.pdf.

[34] M. Veraar, *Regularity of gaussian white noise on the d-dimensional torus*, Banach Center Publications **95** (2011), 385–398. MR2918350

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