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Limiting behavior for the excursion area of band-limited spherical random fields*

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Abstract

In this paper we investigate some geometric functionals for band-limited Gaussian and isotropic spherical random fields in dimension 2. In particular, we focus on the area of excursion sets, providing its behavior in the high energy limit. Our results are based on Wiener chaos expansion for non linear transform of Gaussian fields and on an explicit derivation on the high-frequency limit of the covariance function of the field. As a simple corollary we establish also the Central Limit Theorem for the excursion area.

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1 Introduction and background

Let $\{T_\ell(x), x \in \mathbb{S}^2\}$ denote the spherical harmonics, which are solutions of the Helmholtz equation:

$$\Delta_{\mathbb{S}^2} T_\ell(x) + \ell(\ell+1)T_\ell(x) = 0,\ \ell = 1, 2, \dots;$$

where $\Delta_{\mathbb{S}^2}$ is the spherical Laplacian. We can put on these eigenfunctions a random structure such that $\{T_\ell(x), x \in \mathbb{S}^2\}$ are isotropic, centred Gaussian, with covariance function given by

$$\mathbb{E}[T_\ell(x)T_\ell(y)] = \frac{2\ell+1}{4\pi} P_\ell(\cos d(x,y)),$$

where P_ℓ is the Legendre polynomial and $d(x,y)$ is the spherical geodesic distance between x and y, $d(x,y) = \arccos(\langle x,y\rangle)$. After choosing a standard basis $\{Y_{\ell m}(x)\}$ of $L^2(\mathbb{S}^2)$, the random fields $\{T_\ell(x)\}$ can be expressed by

$$T_\ell(x) = \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x), \tag{1.1}$$

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where $\{a_{\ell m}\}$ is the array of random spherical harmonic coefficients, which are independent, safe for the condition $\bar{a}_{\ell m} = (-1)^m a_{\ell,-m}$; for $m \neq 0$ they are standard complex-valued Gaussian variables, while $a_{\ell 0}$ is a standard real-valued Gaussian variable. They satisfy

$$\mathbb{E}[a_{\ell m}\bar{a}_{\ell' m'}] = \delta_{\ell}^{\ell'}\delta_m^{m'}.$$

The excursion set of T_ℓ is defined as

$$A_u(T_\ell) := \{x \in \mathbb{S}^2 : T_\ell(x) \geq u\},$$

where $u \in \mathbb{R}$. The geometry of the excursion sets of random eigenfunctions can be described by the so called Lipschitz Killing Curvatures (LKCs), which in two dimensions are the excursion area, half of the boundary length and the Euler-Poincaré characteristic. These functionals have been studied in many different papers, among them [9], [10], [11] focused on the area; [20], [8], [21] on the nodal length (boundary length at level $u = 0$); [4], [3] on the Euler-Poincaré characteristic. In these works the authors established asymptotic variances and limiting distributions (in the high frequency domain) of these geometric functionals. Moreover, the local behavior of the excursion area and of the nodal length has been investigated in [18], [19] respectively. Indeed random eigenfunctions restricted to subdomains of $\mathbb{S}^2$ are studied and differences and analogies with the case of the whole sphere have been highlighted. We also refer to [7] and [15] for results in the d-dimensional sphere $\mathbb{S}^d$. In this framework we aim to extend these results to the case of band-limited functions. They have recently received attention for example in [2], [13], where their nodal domains have been studied; in [12], which considers the connected components of zero sets Gaussian random fields and in [16], where the topology of nodal sets is analyzed.

Our model is described here below.

Let us consider the sequence $\alpha_{n,\beta}$ given by

$$\alpha_{n,\beta} := \sqrt{1 - \frac{1}{n^\beta}}$$

with $0 \leq \beta \leq 1, \beta \in \mathbb{R}$, $n \in \mathbb{N}$ and $\{T_\ell\}$ defined as in (1.1). The band-limited functions here are random fields $\{\bar{T}_{\alpha_{n,\beta}}(x), x \in \mathbb{S}^2\}$ defined by

$$\bar{T}_{\alpha_{n,\beta}}(x) = \sqrt{C_{n,\beta}} \sum_{\ell=\alpha_{n,\beta} n}^{n} T_\ell(x), \tag{1.2}$$

where

$$C_{n,\beta} := \frac{4\pi}{n^2(1-\alpha_{n,\beta}^2) + 2n + 1} = \frac{4\pi}{n^{2-\beta} + 2n + 1}.$$

$\{\bar{T}_{\alpha_{n,\beta}}(x)\}$ are centred Gaussian with $\mathbb{E}[\bar{T}_{\alpha_{n,\beta}}(x)^2] = 1$ and covariance function given by

$$\bar{\Gamma}_{\alpha_{n,\beta}}(x,y) = C_{n,\beta}\left(\sum_{\ell=\alpha_{n,\beta} n}^{n} \mathbb{E}[T_\ell(x)T_\ell(y)]\right) = C_{n,\beta} \sum_{\ell=\alpha_{n,\beta} n}^{n} \frac{2\ell+1}{4\pi} P_\ell(\cos d(x,y)). \tag{1.3}$$

We consider the excursion sets

$$A_u(\bar{T}_{\alpha_{n,\beta}}) := \{x \in \mathbb{S}^2 : \bar{T}_{\alpha_{n,\beta}}(x) \geq u\},$$

with $u \in \mathbb{R}$, $u \neq 0$; in this paper we focus on the area of these regions, which we denote by $S_{\alpha_{n,\beta}}(u)$, when $\beta \in (0,1)$. Along the lines of [10], we can see that $S_{\alpha_{n,\beta}}(u)$ can be written as a function of the random field itself in the following way

$$S_{\alpha_{n,\beta}}(u) = \int_{\mathbb{S}^2} 1_{\{\bar{T}_{\alpha_{n,\beta}}(x) > u\}}\, dx,$$

where $1(\cdot)$ is, as usual, the characteristic function which takes value one if the condition in the argument is satisfied, zero otherwise. This expression allows to project the area into the orthonormal system generated by Hermite polynomials (Wiener chaoses projection) $H_k(u)$, $k \in \mathbb{N}$, that is

$$H_0(u) = 1, H_1(u) = u, H_2(u) = u^2 - 1, \ldots, H_k(t) = tH_{k-1}(t) - H'_{k-1}(t), k \geq 1.$$

Indeed, since $1(\cdot) \in L^2(\mathbb{S}^2)$, it can be expanded as

$$1_{\{\bar{T}_{\alpha_{n,\beta}}(x)>u\}} = \sum_{q=0}^{\infty} \frac{J_q(u)}{q!} H_q(\bar{T}_{\alpha_{n,\beta}}(x)),$$

in $L^2(\Omega)$. The coefficients $\{J_q(\cdot)\}$ have the analytic expressions $J_0(u) = \Phi(u)$, $J_1(u) = -\phi(u)$, $J_2(u) = -u\phi(u)$, $J_3(u) = (1-u^2)\phi(u)$ and in general

$$J_q(u) = -H_{q-1}(u)\phi(u),$$

(see [10] and [14]) where $\phi(\cdot)$ and $\Phi(\cdot)$ are the density function and the distribution function of a standard Gaussian variable. It follows that

$$S_{\alpha_{n,\beta}}(u) = \sum_{q=0}^{\infty} \frac{J_q(u)}{q!} \int_{\mathbb{S}^2} H_q(\bar{T}_{\alpha_{n,\beta}}(x))\, dx. \tag{1.4}$$

Note that if we take $\beta = 1$, the random field in (1.2) is the eigenfunction T_ℓ and the behavior of the LKCs in this case has already been investigated. Indeed in [10] the authors have been proved that the projection on the first component vanishes identically and the correspondent series in (1.4) is dominated simply by the second chaotic component. More explicitly, the variance of this single term is asymptotically equivalent to the variance of the full series, and its asymptotic distribution (Gaussian) gives also the limiting behavior of the excursion area. On the contrary, when $\beta = 0$, the expansion in (1.4) does not have any leading component, namely, each chaotic component has the same asymptotic behavior (as it happens for the defect case, defined as the difference between positive and negative regions, when only one eigenfunction is considered, see [11] and [15]). It could then be suspected that the limiting behavior may depend on the value of β, but this turns out not to be the case. Indeed, we will prove that for any $0 < \beta < 1$ the second chaotic component is still the leading term of the series expansion in (1.4) and so no phase transition with respect to β arises.

2 Main result

Let us consider the expansion of the excursion area given in (1.4), we can write

$$\begin{aligned} S_{\alpha_{n,\beta}}(u) &= (1-\Phi(u)) \int_{\mathbb{S}^2} dx + \phi(u) \int_{\mathbb{S}^2} H_1(\bar{T}_{\alpha_{n,\beta}}(x))\, dx \\ &\quad + u\phi(u) \frac{1}{2} \int_{\mathbb{S}^2} H_2(\bar{T}_{\alpha_{n,\beta}}(x))\, dx + \sum_{q=3}^{\infty} \frac{J_q(u)}{q!} \int_{\mathbb{S}^2} H_q(\bar{T}_{\alpha_{n,\beta}}(x))\, dx, \end{aligned} \tag{2.1}$$

in the $L^2(\Omega)$-convergence sense. Denoting

$$h_{\beta;q} := \int_{\mathbb{S}^2} H_q(\bar{T}_{\alpha_{n,\beta}}(x))\, dx \qquad q = 1, 2, \ldots,$$

we have that

$$S_{\alpha_{n,\beta}}(u) = \sum_{q=0}^{\infty} \frac{J_q(u)}{q!} h_{\beta;q}.$$

Remark 2.1. Note that $\int_{\mathbb{S}^2} H_1(\bar{T}_{\alpha_{n,\beta}}(x))\,dx = 0$, indeed

$$\int_{\mathbb{S}^2} H_1(\bar{T}_{\alpha_{n,\beta}}(x))\,dx = \int_{\mathbb{S}^2} \bar{T}_{\alpha_{n,\beta}}(x)\,dx = \sqrt{C_{n,\beta}} \sum_{\ell=\alpha_{n,\beta}n}^{n} \int_{\mathbb{S}^2} T_\ell(x)\,dx = 0$$

thanks to the orthogonal property of Spherical Harmonics (see [6], page 66).

Remark 2.2. The choice of $C_{n,\beta}$ in (1.3) is such that $\mathrm{Var}[\bar{T}_{\alpha_{n,\beta}}(x)] = 1$. Indeed,

$$\begin{aligned}\mathrm{Var}[\bar{T}_{\alpha_{n,\beta}}(x)] &= \frac{4\pi}{n^2(1-\alpha_{n,\beta}^2)+2n+1} \sum_{\ell=\alpha_{n,\beta}n}^{n} [\mathrm{Var}\, T_\ell(x)] \\ &= \frac{4\pi}{n^2(1-\alpha_{n,\beta}^2)+2n+1} \sum_{\ell=\alpha_{n,\beta}n}^{n} \frac{2\ell+1}{4\pi} = 1.\end{aligned}$$

The main result of this paper gives the high energy behavior of the variances of $h_{\beta;q}$ to the vary of $q \geq 2$.

Theorem 2.3. *For* $0<\beta<1$, *as* $n\to\infty$,

$$\begin{aligned}&\mathrm{Var}(h_{\beta;2}) = \frac{32\pi^2}{n^{2-\beta}} + O\left(\frac{1}{n^{3-2\beta}}\right) && \text{for } q=2,\\ &\mathrm{Var}(h_{\beta;4}) = O\left(\frac{\log n}{n^2}\right) && \text{for } q=4,\\ &\mathrm{Var}(h_{\beta;q}) = O\left(\frac{1}{n^2}\right) && \text{for } q=3 \text{ and } q\geq 5.\end{aligned}$$

Remark 2.4. From Theorem 2.3 it follows that the second chaos is the leading term for all $\beta\in(0,1)$. The same holds when only one eigenfunction is considered (namely if $\beta=1$). Indeed in that case it has been proved, in [10], that the variance of $h_{1;2}$ is $32\pi^2\frac{2}{2\ell+1}$ for all ℓ.

For the continuity of the norm and the orthogonality of the Hermite polynomials, the following expansion holds in the $L^2(\Omega)$ sense:

$$\begin{aligned}\mathrm{Var}[S_{\alpha_{n,\beta}}(u)] = 0 + 0 + \frac{u^2\phi(u)^2}{4}\mathrm{Var}\left[\int_{\mathbb{S}^2} H_2(\bar{T}_{\alpha_{n,\beta}}(x))\,dx\right] \\ + \sum_{q=3}^{\infty}\frac{J_q(u)^2}{q!^2}\mathrm{Var}\left[\int_{\mathbb{S}^2} H_q(\bar{T}_{\alpha_{n,\beta}}(x))\,dx\right].\end{aligned}$$

Then, as a corollary, we get

Corollary 2.5. *For* $0<\beta<1$, *as* $n\to\infty$,

$$\mathrm{Var}(S_{\alpha_{n,\beta}}(u)) = \frac{u^2\phi(u)^2}{4}\,\mathrm{Var}(h_{\beta;2}) + o\left(\mathrm{Var}(h_{\beta;2})\right) = 32\pi^2\frac{u^2\phi(u)^2}{4}\frac{1}{n^{2-\beta}} + o\left(\frac{1}{n^{2-\beta}}\right).$$

The key role in the proof of Theorem 2.3 is played by the derivation of the asymptotic behavior of the covariance function defined in (1.3) and it is given here below in Lemma 2.6.

Denoting by N the North Pole, we fix $x=N$ and, in view of the isotropy, we can write $\bar{\Gamma}_{\alpha_{n,\beta}}(x,y) = \bar{\Gamma}_{\alpha_{n,\beta}}(\cos\theta)$ with $\theta\in[0,\pi)$. Changing variable $\theta = \frac{\psi}{\alpha_{n,\beta}n}$, the following lemma gives the asymptotic behavior in the high frequency limit of the covariance function for $1<\psi\leq\alpha_{n,\beta}n(\pi-\epsilon)$, for any $\epsilon>0$.

Lemma 2.6. *Given $\bar{\Gamma}_{\alpha_{n,\beta}}(x,y)$ as in (1.3), for $0<\beta<1, \beta\in\mathbb{R}$, we have that, for $1<\psi<\alpha_{n,\beta}n\pi$, as $n\to\infty$,*

$$\bar{\Gamma}_{\alpha_{n,\beta}}\left(\cos\frac{\psi}{\alpha_{n,\beta}n}\right)=\frac{C_{n,\beta}}{4\pi}\left(\sin\frac{\psi}{2\alpha_{n,\beta}n}\right)^{-1}\left(\frac{\psi/(n\alpha_{n,\beta})}{\sin(\psi/(\alpha_{n,\beta}n))}\right)^{1/2}\frac{\alpha_{n,\beta}n}{\sqrt{\psi}}$$
$$\times\left[\sqrt{\frac{2}{\pi}}(-2)\sin\left(\frac{\psi}{2}-\frac{3\pi}{4}+\frac{(n+1)}{2\alpha_{n,\beta}n}\psi\right)\sin\left(\frac{n+1}{n\alpha_{n,\beta}}\psi-\psi\right)+O\left(\frac{1}{\psi}\right)\right]. \tag{2.2}$$

Remark 2.7. Note that, writing $\frac{(n+1)\psi}{\alpha_{n,\beta}}=\psi+\frac{\psi}{2n^{\beta}}+O(\frac{\psi}{n})+O(\frac{\psi}{n^{2\beta}})$ and calling $\zeta=\frac{\psi}{2n^{\beta}}+O(\frac{\psi}{n^{2\beta}})+O(\frac{\psi}{n})$,

$$\sin\left(\frac{\psi}{2}-\frac{3\pi}{4}+\frac{(n+1)}{2\alpha_{n,\beta}n}\psi\right)\sin\left(\frac{n+1}{n\alpha_{n,\beta}}\psi-\psi\right)=\left(\sin\left(\psi-\frac{3\pi}{4}\right)\cos\zeta+\cos\left(\psi-\frac{3}{4}\pi\right)\sin\zeta\right)\sin\zeta$$

and then (2.2) becomes

$$\bar{\Gamma}_{\alpha_{n,\beta}}\left(\cos\frac{\psi}{\alpha_{n,\beta}n}\right)=\frac{C_{n,\beta}}{4\pi}\left(\sin\left(\frac{\psi}{2\alpha_{n,\beta}n}\right)\right)^{-1}\left(\frac{\psi/\alpha_{n,\beta}n}{\sin(\psi/\alpha_{n,\beta}n)}\right)^{1/2}\frac{(\alpha_{n,\beta}n)}{\sqrt{\psi}}$$
$$\times\sqrt{\frac{2}{\pi}}(-2)\sin\zeta\left(-\sin\left(\psi-\frac{3\pi}{4}\right)\cos\zeta+\cos\left(\psi-\frac{3\pi}{4}\right)\sin\zeta+O\left(\frac{1}{\psi\zeta}\right)\right). \tag{2.3}$$

Finally, we observe that, for Parseval's identity,

$$\int_{\mathbb{S}^2}H_2(\bar{T}_{\alpha_{n,\beta}}(x))\,dx=\int_{\mathbb{S}^2}(\bar{T}_{\alpha_{n,\beta}}(x)^2-1)\,dx=C_{n,\beta}\sum_{\ell}\sum_{\ell'}\int_{\mathbb{S}^2}T_\ell(x)T_{\ell'}(x)\,dx-4\pi$$
$$=C_{n,\beta}\sum_{\ell=\alpha_{n,\beta}n}^{n}\sum_{m=-\ell}^{\ell}|a_{\ell m}|^2-4\pi, \tag{2.4}$$

which are sums of independent Gaussian random variables. The mean of (2.4) is zero and then, from Theorem 2.3, the Central Limit Theorem follows for $h_{\beta;2}$. As a consequence, along the same lines as in [7], we can establish the validity of the Central Limit Theorem for $S_{\alpha_{n,\beta}}$. Note that from (2.1) it is easy to see that $\mathbb{E}[S_{\alpha_{n,\beta}}(u)]=(1-\Phi(u))4\pi$.

Corollary 2.8. *For all $0<\beta<1$, as $n\to\infty$,*

$$\frac{S_{\alpha_{n,\beta}}(u)-\mathbb{E}[S_{\alpha_{n,\beta}}(u)]}{\sqrt{\mathrm{Var}[S_{\alpha_{n,\beta}}(u)]}}\to_d Z,$$

where $Z\sim N(0,1)$ and $\to_d$ denotes convergence in distribution.

Remark 2.9. When $\beta=0$ all the chaotic components have the same asymptotic behavior. We are not going to discuss in details this case but the idea is that the covariance function behaves like

$$\bar{\Gamma}_{\alpha_{n,0}}\left(\cos\frac{\psi}{n}\right)\sim C_{n,0}\frac{1}{\sin(\psi/n)}\left(\frac{\psi/n}{\sin(\psi/n)}\right)^{1/2}(n+1)\sqrt{\frac{2}{(n+1)(\psi/n)}}\cos\left(\psi-\frac{3\pi}{4}\right) \tag{2.5}$$

and then for all q

$$\mathrm{Var}(h_{0,q})\sim\frac{1}{n^2}\int_1^{\pi n}\frac{1}{\psi^{\frac{3}{2}q-1}}\cos\left(\psi-\frac{3\pi}{4}\right)^q d\psi\sim\frac{1}{n^2}$$

since the integral converges.

3 Proof of the main result (Theorem 2.3)

Proof of Theorem 2.3 assuming Lemma 2.6. First of all we remind the following property (see for instance [6], page 98): let Z_1, Z_2 be jointly Gaussian; then, for all $q_1, q_2 \geq 0$

$$\mathbb{E}[H_{q_1}(Z_1)H_{q_2}(Z_2)] = q_1!\delta_{q_1}^{q_2}\mathbb{E}[Z_1 Z_2]. \tag{3.1}$$

Now we start computing the variance of $h_{\beta;2}$; hence,

$$\begin{aligned}\mathrm{Var}(h_{\beta;2}) &= \mathrm{Var}\left[\int_{\mathbb{S}^2} H_2(\bar{T}_{\alpha_{n,\beta}}(x))\,dx\right] = \mathbb{E}\left[\int_{\mathbb{S}^2} H_2(\bar{T}_{\alpha_{n,\beta}}(x))\,dx\right]^2 \\ &= \mathbb{E}\left[\int_{\mathbb{S}^2\times\mathbb{S}^2} H_2(\bar{T}_{\alpha_{n,\beta}}(x))H_2(\bar{T}_{\alpha_{n,\beta}}(y))\,dxdy\right] = \int_{\mathbb{S}^2\times\mathbb{S}^2}\mathbb{E}[H_2(\bar{T}_{\alpha_{n,\beta}}(x))H_2(\bar{T}_{\alpha_{n,\beta}}(y))]\,dxdy,\end{aligned} \tag{3.2}$$

which is, in view of (3.1), equal to

$$2\int_{\mathbb{S}^2\times\mathbb{S}^2}\bar{\Gamma}_{\alpha_{n,\beta}}(x,y)^2\,dxdy.$$

Using (1.3) we get

$$\mathrm{Var}(h_{\beta;2}) = 2C_{n,\beta}^2\int_{\mathbb{S}^2\times\mathbb{S}^2}\sum_{\ell=\alpha_{n,\beta}n}^{n}\sum_{\ell'=\alpha_{n,\beta}n}^{n}\frac{2\ell+1}{4\pi}P_\ell(\langle x,y\rangle)\frac{2\ell'+1}{4\pi}P_{\ell'}(\langle x,y\rangle)\,dxdy \tag{3.3}$$

and exchaging integrals and sums, and applying the duplication property (see [6], Ch. 3), that is,

$$\int_{\mathbb{S}^2}\frac{2\ell+1}{4\pi}P_\ell(\langle x,y\rangle)\frac{2\ell'+1}{4\pi}P_{\ell'}(\langle y,z\rangle)\,dy = \frac{2\ell+1}{4\pi}P_\ell(\langle x,z\rangle)\delta_\ell^{\ell'},$$

(3.3) is equal to

$$2C_{n,\beta}^2\sum_\ell\frac{2\ell+1}{4\pi}\int_{\mathbb{S}^2}P_\ell(\langle x,x\rangle)\,dx.$$

Since $P_\ell(0)=1\ \ \forall\ell$, we conclude that

$$\begin{aligned}\mathrm{Var}(h_{\beta;2}) &= 2C_{n,\beta}^2\sum_{\ell=\alpha_{n,\beta}n}^{n}\frac{2\ell+1}{4\pi}\int_{\mathbb{S}^2}dx = 2C_{n,\beta}^2\sum_{\ell=\alpha_{n,\beta}n}^{n}\frac{2\ell+1}{4\pi}4\pi \\ &= 2C_{n,\beta}^2[(n(n+1)-\alpha_{n,\beta}n(\alpha_{n,\beta}n-1))+n+1-\alpha_{n,\beta}n] \\ &= 2C_{n,\beta}^2(n^2(1-\alpha_{n,\beta}^2)+2n+1) = \frac{2(4\pi)^2}{n^2(1-\alpha_n^2)+2n+1} = \frac{2(4\pi)^2}{n^{2-\beta}+2n+1}.\end{aligned}$$

Finally, we observe that

$$\mathrm{Var}(h_{\beta;2}) = \frac{32\pi^2}{n^{2-\beta}}\left(1-\frac{2}{n^{1-\beta}}+o\left(\frac{1}{n^{1-\beta}}\right)\right) = \frac{32\pi^2}{n^{2-\beta}}-\frac{64\pi^2}{n^{3-2\beta}}+o\left(\frac{1}{n^{3-2\beta}}\right). \tag{3.4}$$

Let us focus now on the variance of the chaotic component $h_{\beta;q}$, for each $q>2$. Hence, same computations as in (3.2) lead to

$$\mathrm{Var}(h_{\beta;q}) = \mathrm{Var}\left(\int_{\mathbb{S}^2}H_q(\bar{T}_{\alpha_{n,\beta}}(x))\,dx\right) = q!\int_{\mathbb{S}^2\times\mathbb{S}^2}\bar{\Gamma}_{\alpha_{n,\beta}}(x,y)^q\,dxdy$$

and because of isotropy it is

$$= 2\pi|\mathbb{S}^2|q!\int_0^\pi\bar{\Gamma}_{\alpha_{n,\beta}}(\cos\theta)^q\sin\theta d\theta. \tag{3.5}$$

Note that, as it happens for example in [9], for symmetry properties of the Legendre polynomials, we can write

$$\int_0^{\pi} \bar{\Gamma}_{\alpha_{n,\beta}}(\cos\theta)^q \sin\theta d\theta = 2\int_0^{\pi/2} \bar{\Gamma}_{\alpha_{n,\beta}}(\cos\theta)^q \sin\theta d\theta. \tag{3.6}$$

This can be seen recalling definition (1.3) and using the fact that $P_\ell(\cdot)$ is an odd function if ℓ is odd while it is even if ℓ is even. Then the only case where (3.6) does not hold is when the integral in (3.5) is equal to zero. Then we have

$$\begin{aligned}\mathrm{Var}(h_{\beta;q}) &= 2\pi|\mathbb{S}^2|q!2\int_0^{\pi/2} \bar{\Gamma}_{\alpha_{n,\beta}}(\cos\theta)^q \sin\theta d\theta = 4\pi|\mathbb{S}^2|q!\int_0^{1/(\alpha_{n,\beta}n)} \bar{\Gamma}_{\alpha_{n,\beta}}(\cos\theta)^q \sin\theta d\theta \\ &+ 4\pi|\mathbb{S}^2|q!\int_{1/(\alpha_{n,\beta}n)}^{\pi/2} \bar{\Gamma}_{\alpha_{n,\beta}}(\cos\theta)^q \sin\theta d\theta. \end{aligned}\tag{3.7}$$

Note that for $\theta \in [0, 1/(\alpha_{n,\beta}n))$, since $|\bar{\Gamma}_{\alpha_{n,\beta}}(x,y)| \le 1$, changing variable $\theta = \dfrac{\psi}{\alpha_{n,\beta}n}$, we have that

$$\begin{aligned}\int_0^{1/(\alpha_{n,\beta}n)} \bar{\Gamma}_{\alpha_{n,\beta}}(\cos\theta)^q \sin\theta\, d\theta &= O\left(\int_0^{1/(\alpha_{n,\beta}n)} |\sin\theta|\, d\theta\right) \\ &= O\left(\frac{1}{(\alpha_{n,\beta}n)}\int_0^1 \frac{\psi}{(\alpha_{n,\beta}n)}\, d\psi\right) = O\left(\frac{1}{n^2}\right). \end{aligned}\tag{3.8}$$

For the second integral in (3.7), changing the variable $\theta = \frac{\psi}{\alpha_{n,\beta}n}$, we obtain

$$\begin{aligned}\int_{1/(\alpha_{n,\beta}n)}^{\pi/2} \bar{\Gamma}_{\alpha_{n,\beta}}(\cos\theta)^q \sin\theta d\theta &= \frac{1}{\alpha_{n,\beta}n}\int_1^{\pi\alpha_{n,\beta}n/2} \bar{\Gamma}_{\alpha_{n,\beta}}\left(\cos\frac{\psi}{\alpha_{n,\beta}n}\right)^q \sin\left(\frac{\psi}{\alpha_{n,\beta}n}\right) d\psi \\ &= \frac{1}{\alpha_{n,\beta}n}\int_1^{n^\beta} \bar{\Gamma}_{\alpha_{n,\beta}}\left(\cos\frac{\psi}{\alpha_{n,\beta}n}\right)^q \sin\left(\frac{\psi}{\alpha_{n,\beta}n}\right) d\psi \\ &+ \frac{1}{\alpha_{n,\beta}n}\int_{n^\beta}^{\pi\alpha_{n,\beta}n/2} \bar{\Gamma}_{\alpha_{n,\beta}}\left(\cos\frac{\psi}{\alpha_{n,\beta}n}\right)^q \sin\left(\frac{\psi}{\alpha_{n,\beta}n}\right) d\psi. \end{aligned}$$

We denote

$$I_{1,q} := \frac{1}{\alpha_{n,\beta}n}\int_1^{n^\beta} \bar{\Gamma}_{\alpha_{n,\beta}}\left(\cos\frac{\psi}{\alpha_{n,\beta}n}\right)^q \sin\left(\frac{\psi}{\alpha_{n,\beta}n}\right) d\psi$$

and

$$I_{2,q} := \frac{1}{\alpha_{n,\beta}n}\int_{n^\beta}^{\pi\alpha_{n,\beta}n/2} \bar{\Gamma}_{\alpha_{n,\beta}}\left(\cos\frac{\psi}{\alpha_{n,\beta}n}\right)^q \sin\left(\frac{\psi}{\alpha_{n,\beta}n}\right) d\psi.$$

Let us focus on $I_{2,q}$. From Lemma 2.6, taking the q-power, we get that

$$\begin{aligned}\bar{\Gamma}_{\alpha_{n,\beta}}\left(\cos\frac{\psi}{\alpha_{n,\beta}n}\right)^q &= \left\{\frac{C_{n,\beta}}{4\pi}\left(\sin\frac{\psi}{2\alpha_{n,\beta}n}\right)^{-1}\left(\frac{\psi/(n\alpha_{n,\beta})}{\sin(\psi/(\alpha_{n,\beta}n))}\right)^{1/2}\frac{\alpha_{n,\beta}n}{\sqrt{\psi}}\right. \\ &\times\left.\left[\sqrt{\frac{2}{\pi}}(-2)\sin\left(\frac{\psi}{2}-\frac{3\pi}{4}+\frac{(n+1)}{2\alpha_{n,\beta}n}\psi\right)\sin\left(\frac{n+1}{n\alpha_{n,\beta}}\psi-\psi\right)+O\left(\frac{1}{\psi}\right)\right]\right\}^q. \end{aligned}\tag{3.9}$$

Exploiting that

$$\sin(\psi/(\alpha_{n,\beta}n)) = O\left(\frac{\psi}{n\alpha_{n,\beta}}\right) \tag{3.10}$$

and bounding by a constant the term in the square brackets in (3.9), we can say

$$I_{2,q} = O\bigg((\alpha_{n,\beta} n)^{q-1} C_{n,\beta}^q \int_{n^\beta}^{n\alpha_{n,\beta}\pi/2} \frac{1}{\psi^{q/2}} \left(\frac{\psi}{\alpha_{n,\beta} n} \right)^{1-q} d\psi \bigg)$$
$$= O\bigg((\alpha_{n,\beta} n)^{2q-2} C_{n,\beta}^q \int_{n^\beta}^{n\alpha_{n,\beta}\pi/2} \frac{1}{\psi^{3q/2-1}}\, d\psi \bigg).$$

We consider now $(\alpha_{n,\beta} n)^{2q-2} C_{n,\beta}^q = (\alpha_{n,\beta} n)^{2q-2} \frac{1}{(n^{2-\beta}+2n+1)^q}$. We can write it as

$$\frac{n^{2q-2}}{(n^{2-\beta})^q} \frac{1}{\left(1+\frac{2n+1}{n^{2-\beta}}\right)^q} \left(1 - \frac{1}{n^\beta}\right)^{q-1} = \frac{n^{-2}}{n^{-\beta q}} \left(1 + O\left(\frac{2n+1}{n^{2-\beta}}\right)\right)\left(1 + O\left(\frac{1}{n^\beta}\right)\right)$$
$$= \frac{n^{-2}}{n^{-\beta q}} \left(1 + O\left(\frac{1}{n^\beta}\right) + O\left(\frac{1}{n^{1-\beta}}\right)\right). \tag{3.11}$$

Hence for all $q \geq 4$, we have that

$$I_{2,q} = O\left(\frac{n^{q\beta}}{n^2} \left(\frac{1}{(\alpha_{n,\beta} n)^{3/2q-2}} - \frac{1}{n^{\beta(3q/2-2)}} \right) \right) = O\left(\frac{1}{n^2}\right). \tag{3.12}$$

For $q = 3$, from (3.9), we get

$$I_{2,q=3} \sim (K_3)^3 \frac{n^{3\beta}}{n^2} \int_{n^\beta}^{\alpha_{n,\beta} n\pi/2} \frac{1}{\psi^{9/2-1}} \times \sin\left(\psi - \frac{3\pi}{4} + \frac{(n+1)}{2\alpha_{n,\beta} n}\psi\right)^3 \sin\left(\frac{n+1}{2\alpha_{n,\beta} n}\psi - \psi\right)^3 d\psi,$$

where $K_3 = \frac{1}{4\pi}\sqrt{\frac{2}{\pi}}(-2)$. In view of the following formula

$$(\sin(a))^3(\sin(b))^3 = \frac{1}{32}\Big[-3\cos(a-3b) + \cos(3a-3b) + 9\cos(a-b) - 3\cos(3a-b)$$
$$- 9\cos(a+b) + 3\cos(3a+b) + 3\cos(a+3b) - \cos(3a+3b)\Big],$$

integration by parts applied to each summands gives

$$I_{2,q=3} = O\left(\frac{1}{n^2}\right). \tag{3.13}$$

To be clear, let us study for example the first term, it is equal to

$$(K_3)^3 \frac{-3}{32} \frac{n^{3\beta}}{n^2} \int_{n^\beta}^{\alpha_{n,\beta} n\pi/2} \frac{\cos(4\psi - \frac{3\pi}{4} - \frac{n+1}{n\alpha_{n,\beta}}\psi)}{\psi^{9/2-1}}\, d\psi = \frac{-3(K_3)^3}{32(4 - \frac{n+1}{n\alpha_{n,\beta}})} \frac{n^{3\beta}}{n^2}$$
$$\times \left[\frac{\sin(4\psi - \frac{3\pi}{4} - \frac{n+1}{n\alpha_{n,\beta}}\psi)}{\psi^{7/2}} \bigg|_{n^\beta}^{\alpha_{n,\beta} n\pi/2} + \frac{7}{2} \int_{n^\beta}^{\alpha_{n,\beta} n\pi/2} \frac{\sin(4\psi - \frac{3\pi}{4} - \frac{n+1}{n\alpha_{n,\beta}}\psi)}{\psi^{9/2}}\, d\psi \right]$$
$$= O\left(\frac{1}{n^{2+\beta/2}}\right).$$

The other integrals can be done in the same way.

As far as $I_{1,q}$ is concerned, taking the q-power of (2.3) and exploting again (3.10) for $\sin\psi/(\alpha_{n,\beta} n)$ and $\sin\zeta$, we get

$$I_{1,q} = \frac{1}{\alpha_{n,\beta} n} \int_1^{n^\beta} \left[\frac{C_{n,\beta}}{4\pi} \left(\sin\frac{\psi}{2\alpha_{n,\beta} n}\right)^{-1} \left(\frac{\psi/\alpha_{n,\beta} n}{\sin(\psi/\alpha_{n,\beta} n)}\right)^{1/2} \frac{(\alpha_{n,\beta} n)}{\sqrt{\psi}} \right.$$
$$\left. \times \sqrt{\frac{2}{\pi}}(-2)\sin\zeta \left(-\sin\left(\psi - \frac{3\pi}{4}\right)\cos\zeta + \cos\left(\psi - \frac{3\pi}{4}\right)\sin\zeta + O\left(\frac{1}{\psi}\right)\right)\right]^q \sin\frac{\psi}{\alpha_{n,\beta}}\, d\psi$$
$$= O\left((C_{n,\beta})^q (\alpha_{n,\beta})^{2q-2} \int_1^{n^\beta} \psi^{1-q/2} \frac{1}{\psi^q} \zeta^q \, d\psi \right) = O\left(\frac{(C_{n,\beta})^q (\alpha_{n,\beta})^{2q-2}}{n^{\beta q}} \int_1^{n^\beta} \psi^{1-q/2}\, d\psi \right). \tag{3.14}$$

For $q>4$ the integral in (3.14) converges and in view of (3.11) we conclude that

$$I_{1,q}=\begin{cases}O\left(\frac{1}{n^2}\right) & \text{if } q>4\\ O\left(\frac{\log n}{n^2}\right) & \text{if } q=4.\end{cases} \tag{3.15}$$

It remains to study $I_{1,q=3}$. Developing the third power in (3.14), we obtain

$$\begin{aligned}I_{1,q=3}&\sim\frac{C^3_{n,\beta}(\alpha_{n,\beta}n)^2}{(4\pi)^3}\int_1^{n^\beta}\frac{1}{\sin(\psi/2\alpha_{n,\beta})^3}\left(\frac{\psi/(\alpha_{n,\beta})}{\sin(\psi/\alpha_{n,\beta})}\right)^{3/2}\frac{1}{\psi^{3/2}}\sin^3\zeta\\&\times\left(\sin\left(\psi-\frac{3\pi}{4}\right)^3\cos^3\zeta+\cos^3\left(\psi-\frac{3\pi}{4}\right)\sin^3\zeta-3\sin\left(\psi-\frac{3\pi}{4}\right)\cos\zeta\right.\\&\left.\times\cos^2\left(\psi-\frac{3\pi}{4}\right)\sin^2\zeta+3\sin^2\left(\psi-\frac{3\pi}{4}\right)\cos^2\zeta\cos\left(\psi-\frac{3\pi}{4}\right)\sin\zeta\right)\sin(\psi/\alpha_{n,\beta})\,d\psi.\end{aligned}$$

Replacing once again $\sin(\psi/(\alpha_{n,\beta}n))\sim\psi/(\alpha_{n,\beta}n)$, $\cos\zeta\sim 1$ and $\sin\zeta\sim\zeta$ we get

$$\begin{aligned}I_{1,q=3}&\sim\frac{C^3_{n,\beta}(\alpha_{n,\beta}n)^2}{(4\pi)^3}2^3\int_1^{n^\beta}\frac{(\alpha_{n,\beta})^2}{\psi^2}\frac{1}{\psi^{3/2}}\frac{\psi^3}{n^{3\beta}}\\&\times\left(\sin\left(\psi-\frac{3\pi}{4}\right)^3+\cos^3\left(\psi-\frac{3\pi}{4}\right)\frac{\psi^3}{n^{3\beta}}-3\sin\left(\psi-\frac{3\pi}{4}\right)\cos^2\left(\psi-\frac{3\pi}{4}\right)\frac{\psi^2}{n^{2\beta}}\right.\\&\left.+3\sin^2\left(\psi-\frac{3\pi}{4}\right)\cos\left(\psi-\frac{3\pi}{4}\right)\frac{\psi}{n^\beta}\right)d\psi\end{aligned}$$

and in view of (3.11) we have

$$\begin{aligned}I_{1,q=3}\sim\frac{n^{3\beta}}{n^2(4\pi)^3n^{3\beta}}4\int_1^{n^\beta}\psi^{-1/2}&\left(\sin\left(\psi-\frac{3\pi}{4}\right)^3+\cos^3\left(\psi-\frac{3\pi}{4}\right)\frac{\psi^3}{n^{3\beta}}\right.\\&\left.-3\sin\left(\psi-\frac{3\pi}{4}\right)\cos^2\left(\psi-\frac{3\pi}{4}\right)\frac{\psi^2}{n^{2\beta}}+3\sin^2\left(\psi-\frac{3\pi}{4}\right)\cos\left(\psi-\frac{3\pi}{4}\right)\frac{\psi}{n^\beta}\right)d\psi.\end{aligned}$$

Integrating by parts it can be seen that the integral converges and then that

$$I_{1,q=3}=O\left(\frac{1}{n^2}\right). \tag{3.16}$$

Indeed, let us consider, for instance, the first summand, we have that

$$\begin{aligned}\int_1^{n^\beta}\frac{\sin\left(\psi-\frac{3\pi}{4}\right)^3}{\psi^{1/2}}d\psi&=\int_1^{n^\beta}\frac{\sin\left(\psi-\frac{3\pi}{4}\right)}{\psi^{1/2}}-\frac{\cos\left(\psi-\frac{3\pi}{4}\right)^2\sin\left(\psi-\frac{3\pi}{4}\right)}{\psi^{1/2}}d\psi\\&=-\left[\frac{\cos\left(\psi-\frac{3\pi}{4}\right)}{\psi^{1/2}}\Bigg|_1^{n^\beta}+\frac{1}{2}\int_1^{\pi n^\beta}\frac{\cos\left(\psi-\frac{3\pi}{4}\right)}{\psi^{3/2}}d\psi\right]\\&\quad+\frac{1}{3}\left[\frac{\cos\left(\psi-\frac{3\pi}{4}\right)^3}{\psi^{1/2}}\Bigg|_1^{n^\beta}-\int_1^{\pi n^\beta}\frac{\cos\left(\psi-\frac{3\pi}{4}\right)^3}{\psi^{3/2}}d\psi\right]<\infty.\end{aligned}$$

Putting together (3.15), (3.16), (3.12), (3.13) and (3.8) in (3.7), and remembering the result in (3.4) the thesis of Theorem 2.3 follows. □

4 Proof of Lemma 2.6

Before proving Lemma 2.6, using the same notation as in [5], we recall the Hilb's asymptotic formula (see [17], Theorem 8.21.12):

$$P_n^{(1,0)}(\cos\theta)=\left(\sin\frac{\theta}{2}\right)^{-1}\left\{\left(\frac{\theta}{\sin\theta}\right)^{1/2}J_1((n+1)\theta)+R_{1,n}(\theta)\right\}, \tag{4.1}$$

where $P_n^{(1,0)}(\cdot)$ is a Jacobi Polynomial, which in general is defined by

$$P_n^{(\alpha,\beta)}(x) = \sum_{s=0}^{n} \binom{n+\alpha}{s}\binom{n+\beta}{n-s}\left(\frac{x-1}{2}\right)^{n-s}\left(\frac{x+1}{2}\right)^{s},$$

$$R_{1,n}(\theta) = \begin{cases} \theta^3 O(n), & 0 \le \theta \le c/n \\ \theta^{1/2} O(n^{-3/2}), & c/n \le \theta \le \pi - \epsilon \end{cases} \tag{4.2}$$

and J_1 is the Bessel function of order 1.

Proof of Lemma 2.6. Looking at the covariance function in (1.3) we can write $\bar{\Gamma}_{\alpha_{n,\beta}}(x,y)$ as

$$\bar{\Gamma}_{\alpha_{n,\beta}}(x,y) = C_{n,\beta}\Big(\sum_{\ell=0}^{n} \frac{2\ell+1}{4\pi} P_\ell(\langle x,y\rangle) - \sum_{\ell=0}^{n\alpha_{n,\beta}-1} \frac{2\ell+1}{4\pi} P_\ell(\langle x,y\rangle)\Big).$$

Thanks to the following formula ([5], page 6), derived by the Christoffel-Darboux formula (see [1]),

$$\sum_{\ell=0}^{n}\sum_{m=-\ell}^{\ell} Y_{\ell,m}(x)Y_{\ell,m}(y) = \frac{n+1}{4\pi} P_n^{(0,1)}(\cos\theta(x,y)),$$

and to the addition formula ([6] page 66):

$$\sum_{m=-\ell}^{\ell} Y_{\ell,m}(x)Y_{\ell,m}(y) = \frac{2\ell+1}{4\pi} P_\ell(\cos\theta(x,y)),$$

we obtain that

$$\bar{\Gamma}_{\alpha_{n,\beta}}(\cos\theta) = C_{n,\beta}\left[\frac{n+1}{4\pi} P_n^{(1,0)}(\cos\theta(x,y)) - \frac{n\alpha_{n,\beta}}{4\pi} P_{n\alpha_{n,\beta}-1}^{(1,0)}(\cos\theta(x,y))\right].$$

Applying the Hilb's asymptotics formula given in (4.1) we get

$$\begin{aligned}\bar{\Gamma}_{\alpha_{n,\beta}}(\cos\theta) = \frac{C_{n,\beta}}{4\pi}\left(\sin\frac{\theta}{2}\right)^{-1}\Bigg[(n+1)\left(\frac{\theta}{\sin\theta}\right)^{1/2} J_1((n+1)\theta) + (n+1)R_{1,n}(\theta) + \\ - n\alpha_{n,\beta}\left(\frac{\theta}{\sin\theta}\right)^{1/2} J_1(n\alpha_{n,\beta}\theta) - n\alpha_{n,\beta} R_{1,n\alpha_{n,\beta}}(\theta)\Bigg].\end{aligned}$$

In view of (4.2) the error term $(n+1)R_{1,n}(\theta) - n\alpha_{n,\beta}R_{1,n\alpha_{n,\beta}}(\theta)$, for $\frac{c}{n\alpha_{n,\beta}} \le \theta \le \pi - \epsilon$, is equal to

$$\theta^{\frac12} O\left(n^{-\frac32}\right)(n+1) + n\alpha_{n,\beta}\theta^{\frac12} O\left(n^{-\frac32}\alpha_{n,\beta}^{-\frac32}\right) = \theta^{\frac12} O\left(n^{-\frac12} + n^{-\frac12}\alpha_{n,\beta}^{-\frac12}\right) = O\left(\frac{1}{\sqrt{n}}\right).$$

Now, changing variable $\theta = \frac{\psi}{\alpha_{n,\beta} n}$ and exploiting the expansion of the Bessel functions (see [17], page 15-16):

$$J_1(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{3\pi}{4}\right) - \frac{3}{4\sqrt{2\pi}x^{3/2}} \sin\left(x - \frac{3\pi}{4}\right) + O(x^{-5/2}), \text{ as } x \to \infty,$$

we find

$$\begin{aligned}\bar{\Gamma}_{\alpha_{n,\beta}}\left(\cos\frac{\psi}{\alpha_{n,\beta} n}\right) = \frac{C_{n,\beta}}{4\pi}\left(\sin\left(\frac{\psi}{2\alpha_{n,\beta} n}\right)\right)^{-1}\Bigg\{\left(\frac{\psi/(n\alpha_{n,\beta})}{\sin(\psi/(\alpha_{n,\beta} n))}\right)^{1/2} \\ \times\Bigg[(n+1)\Bigg(\left(\frac{2\alpha_{n,\beta} n}{\pi(n+1)\psi}\right)^{1/2} \times \cos\Big(\frac{\psi(n+1)}{\alpha_{n,\beta} n} - \frac{3\pi}{4}\Big) - \frac{3}{4\sqrt{2\pi}}\left(\frac{1}{\psi}\right)^{3/2}\left(\frac{\alpha_{n,\beta} n}{n+1}\right)^{3/2}\end{aligned}$$

$$\times \sin\left(\frac{n+1}{\alpha_{n,\beta}n}\psi - \frac{3\pi}{4}\right) + O\left(\left(\frac{\alpha_{n,\beta}n}{(n+1)\psi}\right)^{5/2}\right) - n\alpha_{n,\beta}\left(\sqrt{\frac{2}{\pi\psi}}\cos\left(\psi - \frac{3\pi}{4}\right)\right.$$
$$\left.\left. - \frac{3}{4\sqrt{2\pi}}\left(\frac{1}{\psi^{3/2}}\right) \times \sin\left(\psi - \frac{3\pi}{4}\right) + O\left(\frac{1}{\psi^{5/2}}\right)\right)\right] + O\left(\frac{1}{\sqrt{n}}\right)\Bigg\}$$

which leads to

$$\bar{\Gamma}_{\alpha_{n,\beta}}\left(\cos\frac{\psi}{\alpha_{n,\beta}n}\right) = \frac{C_{n,\beta}}{4\pi}\left(\sin\frac{\psi}{2\alpha_{n,\beta}n}\right)^{-1}\Bigg\{\left(\frac{\psi/(n\alpha_{n,\beta})}{\sin(\psi/(\alpha_{n,\beta}n))}\right)^{1/2}\alpha_{n,\beta}n$$
$$\times\left[\sqrt{\frac{2}{\pi\psi}}\left(\left(\frac{n+1}{\alpha_{n,\beta}n}\right)^{1/2} \times \cos\left((n+1)\frac{\psi}{\alpha_{n,\beta}n} - \frac{3\pi}{4}\right) - \cos\left(\psi - \frac{3\pi}{4}\right)\right)\right.$$
$$\left. + \frac{3}{4\sqrt{2\pi}}\left(\frac{1}{\psi}\right)^{3/2}\left(\sin\left(\psi - \frac{3\pi}{4}\right) - \left(\frac{\alpha_{n,\beta}n}{n+1}\right)^{1/2}\sin\left(\frac{n+1}{\alpha_{n,\beta}n}\psi - \frac{3\pi}{4}\right)\right) + O\left(\frac{1}{\psi^{5/2}n^{\beta}}\right)\right]\Bigg\}.$$

Using the Taylor expansion $(1+x)^{\gamma} = 1 + \gamma x + \frac{\gamma(\gamma-1)}{2}x^2 + o(x^2) \quad (x \to 0)$ applied to

$$\left(\frac{n+1}{\alpha_{n,\beta}n}\right)^{1/2} = \left(1+\frac{1}{n}\right)^{1/2}\left(1-\frac{1}{n^{\beta}}\right)^{-1/4} = 1 + \frac{1}{4n^{\beta}} + \frac{5}{32n^{2\beta}} + \frac{1}{2n} + o\left(\frac{1}{n^{2\beta}}\right) + o\left(\frac{1}{n}\right)$$

and

$$\left(\frac{\alpha_{n,\beta}n}{n+1}\right)^{1/2} = \left(1+\frac{1}{n}\right)^{-1/2}\left(1-\frac{1}{n^{\beta}}\right)^{1/4} = 1 - \frac{1}{4n^{\beta}} - \frac{3}{32n^{2\beta}} - \frac{1}{2n} + o\left(\frac{1}{n^{2\beta}}\right) + o\left(\frac{1}{n}\right),$$

we obtain

$$\bar{\Gamma}_{\alpha_{n,\beta}}\left(\cos\frac{\psi}{\alpha_{n,\beta}n}\right) = \frac{C_{n,\beta}}{4\pi}\left(\sin\frac{\psi}{2\alpha_{n,\beta}n}\right)^{-1}\Bigg\{\left(\frac{\psi/(n\alpha_{n,\beta})}{\sin(\psi/(\alpha_{n,\beta}n))}\right)^{1/2}\alpha_{n,\beta}n\left[\sqrt{\frac{2}{\pi\psi}}\right.$$
$$\times\left(\cos\left(\frac{\psi(n+1)}{\alpha_{n,\beta}n} - \frac{3\pi}{4}\right) - \cos\left(\psi - \frac{3\pi}{4}\right)\right) + \sqrt{\frac{2}{\pi\psi}}\frac{1}{4n^{\beta}}\cos\left((n+1)\frac{\psi}{\alpha_{n,\beta}n} - \frac{3\pi}{4}\right)$$
$$+ \frac{3}{4\sqrt{2\pi}}\left(\frac{1}{\psi}\right)^{3/2}\left(\sin\left(\psi - \frac{3\pi}{4}\right) - \sin\left(\frac{n+1}{\alpha_{n,\beta}n}\psi - \frac{3\pi}{4}\right)\right)$$
$$\left.+ \frac{3}{4\sqrt{2\pi}}\left(\frac{1}{\psi}\right)^{3/2}\frac{1}{4n^{\beta}}\sin\left(\psi\frac{n+1}{n\alpha_{n,\beta}} - \frac{3\pi}{4}\right) + O\left(\frac{1}{\sqrt{\psi}n^{2\beta}} + \frac{1}{\psi^{3/2}n^{2\beta}} + \frac{1}{\psi^{5/2}n^{\beta}}\right)\right]\Bigg\}.$$

Exploiting the addition formulas of sine and cosine

$$\cos\left(\frac{\psi(n+1)}{\alpha_{n,\beta}n} - \frac{3\pi}{4}\right) - \cos\left(\psi - \frac{3\pi}{4}\right) = -2\sin\left(\frac{\psi}{2} - \frac{3\pi}{4} + \frac{(n+1)}{2\alpha_{n,\beta}n}\psi\right)\sin\left(\frac{n+1}{n\alpha_{n,\beta}}\psi - \psi\right),$$
$$\sin\left(\psi - \frac{3\pi}{4}\right) - \sin\left(\frac{\psi(n+1)}{\alpha_{n,\beta}n} - \frac{3\pi}{4}\right) = 2\cos\left(\frac{\psi}{2} - \frac{3\pi}{4} + \frac{(n+1)}{2\alpha_{n,\beta}n}\psi\right)\sin\left(\frac{\psi}{2} - \frac{n+1}{2n\alpha_{n,\beta}}\psi\right),$$

we conclude that

$$\bar{\Gamma}_{\alpha_{n,\beta}}\left(\cos\frac{\psi}{\alpha_{n,\beta}n}\right) = \frac{C_{n,\beta}}{4\pi}\left(\sin\frac{\psi}{2\alpha_{n,\beta}n}\right)^{-1}\left(\frac{\psi/(n\alpha_{n,\beta})}{\sin(\psi/(\alpha_{n,\beta}n))}\right)^{1/2}\frac{\alpha_{n,\beta}n}{\sqrt{\psi}}$$
$$\times\left[\sqrt{\frac{2}{\pi}}(-2)\sin\left(\frac{\psi}{2} - \frac{3\pi}{4} + \frac{(n+1)}{2\alpha_{n,\beta}n}\psi\right)\sin\left(\frac{n+1}{n\alpha_{n,\beta}}\psi - \psi\right) + O\left(\frac{1}{\psi}\right)\right]. \qquad \square$$

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