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The logarithmic anti-derivative of the Baik-Rains distribution satisfies the KP equation

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Abstract

It has been discovered that the Kadomtsev-Petviashvili (KP) equation governs the distribution of the fluctuation of many random growth models. In particular, the Tracy-Widom distributions appear as special self-similar solutions of the KP equation. We prove that the anti-derivative of the Baik-Rains distribution, which governs the fluctuation of the models in the KPZ universality class starting with stationary initial data, satisfies the KP equation. The result is first derived formally by taking a limit of the generating function of the KPZ equation, which satisfies the KP equation. Then we prove it directly using the explicit Painlevé II formulation of the Baik-Rains distribution.

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1 Introduction

The fluctuations for the KPZ universality class depend on the initial data. Let $h(x,t)$ be the solution of the Kardar-Parisi-Zhang equation,

$$\partial_t h(x,t) = \frac{1}{2}(\partial_x h(x,t))^2 + \frac{1}{2}\partial_x^2 h(x,t) + \xi(x,t). \tag{1.1}$$

Here $\xi(x,t)$ is space-time Gaussian white noise,

$$\mathbb{E}(\xi(x,t)\xi(y,s)) = \delta(x-y)\delta(t-s). \tag{1.2}$$

The equation is ill-posed because the quadratic non-linear term does not make sense for a realization of a solution. A typical solution $h(x,t)$ looks like a Brownian motion in x variable. One way to make sense of the equation is through the Hopf-Cole transformation. The Hopf-Cole solution of the KPZ equation is defined to be: $h(t,x) = -\log z(t,x)$, where $z(t,x)$ is the solution of the stochastic heat equation with multiplicative white noise,

$$\partial_t z(t,x) = \frac{1}{2}\partial_x^2 z(t,x) + z(t,x)\xi(t,x). \tag{1.3}$$

This is well-posed interpreted as an Itô integral equation.

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It has been independently proved recently in both [QS20] and [Vir20] that the KPZ equation converges to the KPZ fixed point. Thus all the universal fluctuation behaviors can be observed on large space and time scales under the KPZ scaling as $\epsilon \to 0$,

$$h_\epsilon(t,x) = \epsilon^{1/2} h(\epsilon^{-3/2} t, \epsilon^{-1} x). \tag{1.4}$$

The fluctuations depend on the initial condition [MQR20]. If the initial condition is $z(0,x) = \delta_0$, which corresponds to $h(0,x) = -\infty$ if $x \neq 0$ and $h(0,x) = 0$ if $x = 0$, i.e. the KPZ equation starting from the narrow wedge initial condition, we observe that when $t \to \infty$,

$$-2^{1/3} t^{-1/3} (h(t, 2^{1/3} t^{2/3} x) - 2^{-1/3} t^{1/3} x^2 - \frac{t}{24} - \log \sqrt{2\pi t}) \to \mathcal{A}_2(x). \tag{1.5}$$

If the initial condition is $z(0,x) = 1$, we observe

$$-2^{1/3} t^{-1/3} (h(t, 2^{1/3} t^{2/3} x) - \frac{t}{24}) \to \mathcal{A}_1(x). \tag{1.6}$$

If the initial condition is $z(0,x) = e^{B(x)}$, where $B(x)$ is a two-sided Brownian motion with $B(0) = 0$, we observe

$$-2^{1/3} t^{-1/3} (h(t, 2^{1/3} t^{2/3} x) - \frac{t}{24}) \to \mathcal{A}_{\text{stat}}(x). \tag{1.7}$$

Here $\mathcal{A}_1(x), \mathcal{A}_2(x), \mathcal{A}_{\text{stat}}(x)$ are stochastic processes whose finite dimensional distributions are given by Fredholm determinants [PS01] [Joh03] [IS04]. These are the conjectured processes which govern the long-time fluctuations of models which belong to the KPZ universality class. $\mathcal{A}_1(x)$ is a stationary process, whose one-point distribution is the Tracy-Widom GOE distribution. The one point marginal of $\mathcal{A}_2(x)$ is given by the Tracy-Widom GUE distribution. The one point marginal of $\mathcal{A}_{\text{stat}}(x)$ is given by the Baik-Rains distribution.

In the paper [QR20], it was stated that the GUE and GOE Tracy-Widom distributions are seen to arise as special similarity solutions of the scalar Kadomtsev-Petviashvili (KP) equation,

$$\partial_t \phi + \phi \partial_r \phi + \frac{1}{12} \partial_r^3 \phi + \frac{1}{4} \partial_r^{-1} \partial_x^2 \phi = 0. \tag{1.8}$$

In this paper, we explain how the Baik-Rains distribution can be seen as a similarity solution of the KP equation. We first explain how the GUE and GOE distributions arise as similarity solutions of the KP equation, then give the definition of the Baik-Rains distribution and then state our main results.

Example 1.1. Tracy-Widom GUE distribution [QR20]: We consider a self-similar solution of (1.8) in the following form,

$$\phi(t,x,r) = t^{-2/3} \psi(t^{-1/3} r + t^{-4/3} x^2). \tag{1.9}$$

This turns (1.8) to

$$\psi''' + 12\psi\psi' - 4r\psi' - 2\psi = 0. \tag{1.10}$$

If we look for solutions of the form $\psi = -q^2$, then the above equation becomes the Painlevé II equation,

$$q'' = rq + 2q^3. \tag{1.11}$$

In this way, we recover the GUE distribution, since $F_{\text{GUE}}(r) = \exp\{-\int_r^{-\infty} du(u-r)q^2(u)\}$.

Example 1.2. Tracy-Widom GOE distribution[QR20]: We consider a self-similar solution of (1.8) in the following form,

$$\phi(t,r) = (t/4)^{-2/3}\psi((t/4)^{-1/3}r). \tag{1.12}$$

This turns (1.8) to

$$\psi''' + 12\psi'\psi - r\psi' - 2\psi = 0. \tag{1.13}$$

If we look for solutions of the form $\psi = \frac{1}{2}(q' - q^2)$, we get the Painlevé II equation again, thus we recover the GOE distribution,

$$F_{\mathrm{GOE}}(r) = \exp\{-\frac{1}{2}\int_r^\infty q(u)du\}F_{\mathrm{GUE}}(r)^{1/2}. \tag{1.14}$$

Now let us look at the definition of the Baik-Rains distribution. Here we present two definitions which turn out to be equivalent. One definition appears in [BCFV15], as the large time fluctuation of the stationary KPZ equation; the other definition will be the main tool to prove our results.

Definition 1.3. *[FS06] For $w, s \in \mathbb{R}$, we define the following functions,*

$$\begin{aligned} \hat{\Phi}_{w,s}(x) &= \int_{\mathbb{R}_-} dze^{wz}K_{\mathbf{Ai},s}(z,x)e^{ws}, \\ \hat{\Psi}_{w,s}(y) &= \int_{\mathbb{R}_-} dze^{wz}\mathbf{Ai}(y+z+s), \\ \rho_s(x,y) &= (I - P_0K_{\mathbf{Ai},s}P_0)^{-1}(x,y), \end{aligned} \tag{1.15}$$

where $P_0(x) = I_{x\geq 0}$ is the projection operator, $\mathrm{Ai}(x)$ is the Airy function, and $K_{\mathrm{Ai},s}$ is the shifted Airy kernel,

$$K_{\mathbf{Ai},s}(x,y) = \int_0^\infty d\lambda \mathbf{Ai}(x+\lambda+s)\mathbf{Ai}(y+\lambda+s). \tag{1.16}$$

The Tracy-Widom GUE distribution can be written as

$$F_{GUE}(s) = \det(I - P_0K_{\mathbf{Ai},s}P_0) \tag{1.17}$$

Then we define the function $g(s,w)$ which appears as a component in the Baik-Rains distribution,

$$g(s,w) = e^{-\frac{1}{3}w^3}[\int_{\mathbb{R}_-^2} dxdye^{w(x+y)}\mathbf{Ai}(x+y+s) + \int_{\mathbb{R}_+^2} dxdy\hat{\Phi}_{w,s}(x)\rho_s(x,y)\hat{\Psi}_{w,s}(y)]. \tag{1.18}$$

Finally, the Baik-Rains distribution is defined to be

$$F_\tau(r) = \frac{\partial}{\partial r}(g(r+\tau^2,\tau)F_{GUE}(r+\tau^2)). \tag{1.19}$$

The function $g(s,w)$ can also be derived by solving the PNG model using the Riemann-Hilbert technique. We also present this equivalent definition here [BR00]. Let $u(x)$ be the solution of the Painlevé II equation,

$$u_{xx} = 2u^3 + xu, \tag{1.20}$$

with the boundary condition,

$$u(x) \sim -\mathrm{Ai}(x) \text{ as } x \to +\infty. \tag{1.21}$$

$v(x)$ is defined to be

$$v(x) = \int_{\infty}^{x} (u(s))^2 ds. \tag{1.22}$$

Then the Tracy-Widom distributions can be defined in terms of u and v. Set

$$\begin{aligned} F(x) &= \exp(\frac{1}{2}\int_x^{\infty} v(x)ds) = \exp(-\frac{1}{2}\int_x^{\infty}(s-x)(u(s))^2 ds), \\ E(x) &= \exp(\frac{1}{2}\int_x^{\infty} u(s)ds). \end{aligned} \tag{1.23}$$

Then we can define the GUE and GOE distributions as

$$\begin{aligned} F_{\mathrm{GUE}}(x) &= F(x)^2 = \exp(\int_x^{\infty}(s-x)(u(s))^2 ds), \\ F_{\mathrm{GOE}}(x) &= F(x)E(x). \end{aligned} \tag{1.24}$$

Then define $F_\tau(r) = H(r+\tau^2; \tau/2, -\tau/2)$, where

$$H(x; w, -w) = \{y'(x,w) - y(x,w)v(x))\} F_{\mathrm{GUE}}(x) = \partial_x (y(x,w) F_{\mathrm{GUE}}(x)), \tag{1.25}$$

where

$$\begin{aligned} y(x,w) = (2u^2 + x - 4w^2)a(x;w)a(x;-w) - (u' + 2wu)b(x;w)a(x;-w) \\ -(u' - 2wu)a(x;w)b(x;-w). \end{aligned} \tag{1.26}$$

Here functions $a(x;w)$ and $b(x;w)$ arise in the Painlevé II Riemann-Hilbert problem. In this paper, we do not need the exact definition of $a(x;w), b(x;w)$, thus we skip the formulation of the Riemann-Hilbert problem here. What we need are the following identities [BR00],

$$\begin{aligned} \partial_x a(x,w) &= u(x)b(x,w), \\ \partial_x b(x,w) &= u(x)a(x,w) - 2wb(x,w), \\ \partial_w a(x,w) &= 2u(x)^2 a(x,w) - (4wu(x) + 2u'(x))b(x,w), \\ \partial_w b(x,w) &= (-4wu(x) + 2u'(x))a(x,w) + (8w^2 - 2x - 2u(x)^2)b(x,w), \\ a(x,w) &= -b(x,-w)e^{\frac{8}{3}w^3 - 2wx}, \\ b(x,w) &= -a(x,-w)e^{\frac{8}{3}w^3 - 2wx}. \end{aligned} \tag{1.27}$$

Remark 1.4. It is proven in the appendix A of [FS06] that $y(s + {w'}^2, w'/2) = g(s + w^2, w)$ when $w' = 2w$, which establishes the equivalence of the two definitions presented above.

The main result we discovered about $F_\tau(r)$ is the following,

Theorem 1.5. *Recall that $F_\tau(r)$ is defined to be the partial derivative of $y(r+\tau^2, \tau/2)$ $F_{GUE}(r+\tau^2)$ in r. If we consider certain scaling form of this anti-derivative of $F_\tau(r)$,*

$$(\partial_r^{-1} F_{t^{-2/3}x})(t^{-1/3}r) = F_{GUE}(t^{-1/3}r + t^{-4/3}x^2) y(t^{-1/3}r + t^{-4/3}x^2, t^{-2/3}x/2), \tag{1.28}$$

its logarithmic derivative $\phi_{br}(x,t,r) = \partial_r^2 \log(\partial_r^{-1} F_{t^{-2/3}x})(t^{-1/3}r)$ satisfies the KP equation,

$$\partial_t \phi_{br} + \phi_{br}\partial_r \phi_{br} + \frac{1}{12}\partial_r^3 \phi_{br} + \frac{1}{4}\partial_r^{-1}\partial_x^2 \phi_{br} = 0. \tag{1.29}$$

Remark 1.6. We know (1.28) and $\partial_r^2 \log F_{\mathrm{GUE}}(t^{-1/3}r + t^{-4/3}x^2)$ both satisfy the KP equation. If we define $\phi_{\mathrm{gue}} = \partial_r^2 \log F_{\mathrm{GUE}}, \psi(x,t,r) = \partial_r^2 \log y(t^{-1/3}r + t^{-4/3}x^2, t^{-2/3}x/2)$, then Theorem 1.5 is equivalent to $\psi(x,t,r)$ satisfying the following equation,

$$\partial_t \psi + \psi \partial_r \psi + \frac{1}{12}\partial_r^3 \psi + \frac{1}{4}\partial_r^{-1}\partial_x^2 \psi + \phi_{\mathrm{gue}}\partial_r \psi + \psi \partial_r \phi_{\mathrm{gue}} = 0. \tag{1.30}$$

In this paper, we will prove that ψ satisfies (1.30), which implies Theorem 1.5.

We first explain where the result comes from in Section 2. The Baik-Rains distribution governs the fluctuation of the stationary KPZ equation in the large time limit. The generating function of the KPZ equation starting with two sided Brownian motion with drifts satisfies the KP equation. One way of seeing this is by checking the cumulants [Dou20], which is formally derived using Bethe ansatz method; another way is by checking its determinantal formula [QR20]. Then we conjecture that the same equation should still be satisfied when the drift goes to zero and time goes to infinity, which gives Theorem 1.5. This is only formal since we assume that if a sequence of functions satisfy an equation, its limit also satisfies the same equation. In section 3, we will prove Theorem 1.5 by directly differentiating $y(t^{-1/3}r + t^{-4/3}x^2, t^{-2/3}x/2)$. Using the identities in (1.27), we find that equation (1.30) holds.

2 Motivation for the result

A key fact we will use is the following theorem,

Theorem 2.1. *[QR20] Suppose a function can be written in the Fredholm determinant form, i.e.* $F(x,t,r) = \det(I-K)_{L^2[0,\infty)}$*, and the integral kernel* $K(u,v,x,t,r)$ *satisfies the following relations,*

$$\begin{aligned} \partial_r K &= (\partial_u + \partial_v)K, \\ \partial_t K &= -\frac{1}{3}(\partial_u^3 + \partial_v^3)K, \\ \partial_x K &= (\partial_u^2 - \partial_v^2)K. \end{aligned} \tag{2.1}$$

Suppose in addition that $\det(I-K) > 0$ *for all finite* t, x, r*, and* K *is real analytic in* t, x *and* r*, and that the trace norm* $\|K\|_1 < 1$ *for* r *in some open real interval. Then in that interval,* $\phi(x,t,r) = \partial_r^2 \log F(x,t,r)$ *satisfies the scalar KP equation,*

$$\partial_t \phi + \phi \partial_r \phi + \frac{1}{12}\partial_r^3 \phi + \frac{1}{4}\partial_r^{-1}\partial_x^2 \phi = 0. \tag{2.2}$$

Remark 2.2. The fact that equations (2.1) lead to the KP equation was discovered several times in [ZS74] [Pop89], but its appearance in the context of random fluctuation interfaces was discovered in [QR20].

It is checked in [QR20] [Dou20] that if $h(x,t)$ is the solution of the KPZ equation with half-Brownian initial data or narrow wedge initial data, the generating function $G(t,x,r) = \mathbb{E}[\exp\{-e^{h(t,x)+\frac{t}{12}-r}\}]$ can be written in the Fredholm determinant form, having a kernel satisfying equations (2.1). It is also checked in [Dou20] that if $h(x,t)$ is the solution of the KPZ equation with the drifted Brownian motion initial condition, its generating function also satisfies the KP equation, by studying the moments of $e^{h(x,t)}$. Here we check it using a different method, which agrees with the results in [Dou20]. We begin with the following theorem.

Theorem 2.3. *[BCFV15] Let* $z_{b,\beta}(t,x)$ *denote the solution to the stochastic heat equation with initial data* $z(0,x) = \exp(B_{b,\beta}(x))$*, where* $B_{b,\beta}(x)$ *is a two-sided Brownian motion with drift* β *to the left of 0 and drift* b *to the right of 0, with* $\beta > b$*, that is,* $B_{b,\beta} = \mathbf{1}_{x\le 0}(B^l(x) + \beta x) + \mathbf{1}_{x>0}(B^r(x) + bx)$*, where* $B^l : (-\infty, 0] \to \mathbb{R}$ *is a Brownian motion without drift pinned at* $B^l(0) = 0$ *and* $B^r : [0,\infty) \to \mathbb{R}$ *is an independent Brownian motion pinned at* $B^r(0) = 0$*. Then for* $S > 0$*,*

$$\mathbb{E}[2(Se^{\frac{x^2}{2t}+\frac{t}{24}}z_{b,\beta}(t,x))^{\frac{\beta-b}{2}}K_{-(\beta-b)}(2\sqrt{Se^{\frac{x^2}{2t}+\frac{t}{24}}z_{b,\beta}(t,x)})] = \Gamma(\beta-b)\det(I-K_{b+\frac{x}{t},\beta+\frac{x}{t}})_{L^2(\mathbb{R}_+)} \tag{2.3}$$

where $K_\nu(z)$ is the modified Bessel function of order ν and the kernel on the right-hand side is given by

$$K_{b,\beta}(u,v) = \frac{1}{(2\pi i)^2}\int dw \int dz \frac{\sigma\pi S^{\sigma(z-w)}}{\sin(\sigma\pi(z-w))}\frac{e^{z^3/3-zv}}{e^{w^3/3-wu}}\frac{\Gamma(\beta-\sigma z)}{\Gamma(\sigma z-b)}\frac{\Gamma(\sigma w-b)}{\Gamma(\beta-\sigma w)}, \tag{2.4}$$

where

$$\sigma = (2/t)^{1/3}. \tag{2.5}$$

The integration contour for w is from $-\frac{1}{4\sigma}-i\infty$ to $-\frac{1}{4\sigma}+i\infty$ and crosses the real axis between b and β. The other contour for z goes from $\frac{1}{4\sigma}-i\infty$ to $\frac{1}{4\sigma}+i\infty$, it also crosses the real axis between b and β and it does not intersect the contour for w.

In order to get the formula for stationary initial data, we need to take the limit as $\beta \to b$ and set $b=0$. To do so, we need to rewrite the kernel $K_{b,\beta}$. Two contours in the integral kernel of $K_{b,\beta}$ intersect the real axis between the pole at b/σ and β/σ, so when $\beta\to b$, two contours will collide. We move the contour of w cross the pole at b/σ and move the contour of z cross the pole at β/σ. Using the residue theorem, we have [BCFV15],

$$\begin{aligned} K_{b,\beta}(u,v) &= \bar{K}_{b,\beta}(u,v) + q_{b,\beta}(u) r_\beta(v)\frac{1}{\sigma\Gamma(\beta-b)} + r_{-b}(u) q_{-\beta,-b}(v)\frac{1}{\sigma\Gamma(\beta-b)} \\ &\quad + \frac{\sigma\pi S^{\beta-b}}{\sin(\pi(\beta-b))} r_{-b}(u) r_\beta(v)\frac{1}{\sigma^2\Gamma(\beta-b)^2}, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} \bar{K}_{b,\beta}(u,v) &= \frac{1}{(2\pi i)^2}\int_{-\frac{1}{4\sigma}+i\mathbb{R}} dw \int_{\frac{1}{4\sigma}+i\mathbb{R}} dz \frac{\sigma\pi S^{\sigma(z-w)}}{\sin(\sigma\pi(z-w))}\frac{e^{z^3/3-zv}}{e^{w^3/3-wu}}\frac{\Gamma(\beta-\sigma z)}{\Gamma(\sigma z-b)}\frac{\Gamma(\sigma w-b)}{\Gamma(\beta-\sigma w)}, \\ q_{b,\beta}(u) &= \frac{1}{2\pi i}\int_{-\frac{1}{4\sigma}+i\mathbb{R}} dw \frac{\sigma\pi S^{\beta-\sigma w}}{\sin(\pi(\beta-\sigma w))} e^{-w^3/3+wu}\frac{\Gamma(\sigma w-b)}{\Gamma(\beta-\sigma w)}, \\ r_b(u) &= e^{b^3/(3\sigma^3)-bu/\sigma}. \end{aligned} \tag{2.7}$$

Notice that the only difference between $K_{b,\beta}$ and $\bar{K}_{b,\beta}$ is that they have different contours. We can write

$$K_{b,\beta}(u,v) = \bar{K}_{b,\beta}(u,v) + \sum_{i=1}^{3} f_i(u) g_i(v), \tag{2.8}$$

with suitable f_i, g_i. Then for the Fredholm determinant, we have the following formula,

$$\det(I-K_{b,\beta}) = \det(I-\bar{K}_{b,\beta})\det[\delta_{i,j} - \langle (I-\bar{K}_{b,\beta})^{-1} f_i, g_j\rangle]_{i,j=1}^3. \tag{2.9}$$

Both kernels $K_{b,\beta}(x,y)$ and $\bar{K}_{b,\beta}(x,y)$ depend on S. For $S=e^{\tau^2+\sigma r}$, where τ is related to x,b,t as $x = bt + \frac{2\tau}{\sigma^2}$ and when $b=0$, we observe that both $K_{b+\frac{x}{t},\beta+\frac{x}{t}}(u,v), \bar{K}_{b+\frac{x}{t},\beta+\frac{x}{t}}(u,v)$ satisfy equations (2.1). The reason that we make this specific choice of S and τ will be clear from the later context; this is the scaling that gives the Baik-Rains distribution. We will do some transformations on the integral kernel so that equations (2.1) become obvious while the determinant of the operator remains unchanged. When $S=e^{\tau^2+\sigma r}$,

$$\begin{aligned} &K_{b+\frac{x}{t},\beta+\frac{x}{t}}(u,v) \\ &\quad = \frac{1}{(2\pi i)^2}\int dw\int dz \frac{\sigma\pi e^{-(z-w)(\tau^2+\sigma r)}}{\sin(\sigma\pi(z-w))}\frac{e^{z^3/3-zv}}{e^{w^3/3-wu}}\frac{\Gamma(\beta+\frac{x}{t}-\sigma z)}{\Gamma(\sigma z-b-\frac{x}{t})}\frac{\Gamma(\sigma w-b-\frac{x}{t})}{\Gamma(\beta+\frac{x}{t}-\sigma w)}. \end{aligned} \tag{2.10}$$

Now let $z \to \sigma z - \frac{x}{t}, w \to \sigma w - \frac{x}{t}$, then

$$K_{b+\frac{x}{t},\beta+\frac{x}{t}}(u,v) = \frac{1}{(2\pi i)^2}\int dw \int dz \frac{\pi e^{-(z-w)(\tau^2+\sigma r)/\sigma}}{\sin(\pi(z-w))} \frac{e^{t(z^3+3z^2x/t+3zx^2/t^2)/6-\frac{1}{\sigma}(z+\frac{x}{t})v}}{e^{t(w^3+3w^2x/t+3wx^2/t^2)/6-\frac{1}{\sigma}(w+\frac{x}{t})u}} \frac{\Gamma(\beta-z)}{\Gamma(z-b)} \frac{\Gamma(w-b)}{\Gamma(\beta-w)}. \tag{2.11}$$

Let $u \to \frac{1}{\sigma}u, v \to \frac{1}{\sigma}v$ and conjugate the kernel by $e^{ux/t}$ (which is equivalent to multiplying the kernel by $e^{(u-v)x/t}$). It becomes

$$K_{b+\frac{x}{t},\beta+\frac{x}{t}}(u,v) = \frac{1}{(2\pi i)^2}\int dw \int dz \frac{\pi e^{-(z-w)(b^2t/2+bx)}}{\sin(\pi(z-w))} \frac{e^{(tz^3+3z^2x)/6-z(v+r)}}{e^{(tw^3+3w^2x)/6-w(u+r)}} \frac{\Gamma(\beta-z)}{\Gamma(z-b)} \frac{\Gamma(w-b)}{\Gamma(\beta-w)}. \tag{2.12}$$

When $b=0$, the only term contains x,t,r,u,v is $\frac{e^{(tz^3+3z^2x)/6-z(v+r)}}{e^{(tw^3+3w^2x)/6-w(u+r)}}$, which clearly satisfies equations (2.1). When $b \neq 0$, it will have extra terms when differentiating t,x coming from $e^{-(z-w)(b^2t/2+bx)}$, which fail to satisfy equations (2.1). The reason we can directly take derivatives under the integral sign and the kernel being analytic in t,x,r come from the following lemma,

Lemma 2.4. *[BCFV15]Let $f(z,\zeta)$ be a complex function in two variables and suppose that*

1. *f is defined on $(z,\zeta) \in A \times C$ where A is an open set and C is a contour,*
2. *For each $z \in A$, define the contour $\gamma_z = \{z + re^{it} : 0 \le t \le 2\pi\}$ with a sufficiently small r such that also the disc around z with radius r lies in A. Suppose that for each $z \in A$,*
$$\int_C \int_{\gamma_z} |f(u,\zeta)||du||d\zeta| < \infty, \tag{2.13}$$
3. *For each $\zeta \in C, z \to f(z,\zeta)$ is analytic in A,*
4. *For each $z \in A, \zeta \to f(z,\zeta)$ is continuous on C.*

Then
$$F(z) = \int_C f(z,\zeta)d\zeta \tag{2.14}$$
is analytic in A with $F'(z) = \int_C \frac{\partial}{\partial z} f(z,\zeta)d\zeta$.

It can be easily seen from the form of the kernel that conditions (1), (3), (4) are satisfied. Since $e^{z^3/3}$ decays along C_z as $e^{-c|\mathrm{Im}(z)|^2}$, $e^{w^3/3}$ decays along C_w as $e^{-c|\mathrm{Im}(w)|^2}$, using the gamma ratio formula, as $|z| \to \infty$, we have [BCFV15]

$$\left|\frac{\Gamma(\beta-\sigma z)}{\Gamma(\sigma z - b)}\right| \simeq |z|^{\beta+b-2\sigma \mathrm{Re}(z)}. \tag{2.15}$$

Similarly we have the same bound for large w. Thus if we integrate β on some finite contour, we have the same polynomial bounds. Thus the whole integrand decays exponentially on the contour, so condition (2) is satisfied.

Here $\bar{K}_{b,\beta}$ also satisfies equations (2.1) if $S = e^{\tau^2+\sigma r}$, because the only difference between $\bar{K}_{b,\beta}$ and $K_{b,\beta}$ is that they have different contours, for which Lemma 2.4 still applies. We denote $\phi_{b,\beta} = \partial_r^2 \log\det(I - K_{b,\beta}), \bar{\phi}_{b,\beta} = \partial_r^2 \log\det(I - \bar{K}_{b,\beta})$ and $\alpha = \partial_r^2 \log\det[\delta_{i,j} - \langle (I-\bar{K}_{b,\beta})^{-1} f_i, g_j\rangle]_{i,j=1}^3$, we have $\phi_{b,\beta} = \bar{\phi}_{b,\beta} + \alpha$ and both $\phi_{b,\beta}, \bar{\phi}_{b,\beta}$ satisfy the KP equation,

$$\begin{aligned} &\partial_t \bar{\phi}_{b,\beta} + \bar{\phi}_{b,\beta}\partial_r \bar{\phi}_{b,\beta} + \frac{1}{12}\partial_r^3 \bar{\phi}_{b,\beta} + \frac{1}{4}\partial_r^{-1}\partial_x^2 \bar{\phi}_{b,\beta} = 0, \\ &\partial_t(\bar{\phi}_{b,\beta}+\alpha) + (\bar{\phi}_{b,\beta}+\alpha)\partial_r(\bar{\phi}_{b,\beta}+\alpha) + \frac{1}{12}\partial_r^3(\bar{\phi}_{b,\beta}+\alpha) + \frac{1}{4}\partial_r^{-1}\partial_x^2(\bar{\phi}_{b,\beta}+\alpha) = 0. \end{aligned} \tag{2.16}$$

Combining these two equations, we obtain an equation for α involving $\bar{\phi}_{b,\beta}$,

$$\partial_t \alpha + \alpha \partial_r \alpha + \frac{1}{12}\partial_r^3 \alpha + \frac{1}{4}\partial_r^{-1}\partial_x^2 \alpha + \alpha \partial_r \bar{\phi}_{b,\beta} + \bar{\phi}_{b,\beta}\partial_r \alpha = 0. \tag{2.17}$$

This is a similar equation to (1.30), except that $\bar{\phi}_{b,\beta}$ is not $\partial_r^2 \log F_{\text{GUE}}$. Now we want to see how equation (2.9) leads us to the Baik-Rains distribution.

In the limit of $\beta \to b$, we have the following results,

Theorem 2.5. *[BCFV15] Let $b + \frac{x}{t} \in (-\frac{1}{4}, \frac{1}{4})$ be fixed. For the kernel $K_{b,\beta}$, we have*

$$\lim_{\beta \to b} \frac{1}{\beta - b} \det(I - K_{b+\frac{x}{t}, \beta + \frac{x}{t}}) = \frac{1}{\sigma}\Xi(S, b + \frac{x}{t}, \sigma), \tag{2.18}$$

where

$$\begin{aligned} \Xi(S, b, \sigma) = &- \det(I - \bar{K}_b)[\frac{b^2}{\sigma^2} + \sigma(2\gamma_E + \ln S) \\ &+ \langle (I - \bar{K}_b)^{-1}(\bar{K}_b r_{-b} + q_b), r_b\rangle + \langle (I - \bar{K}_b)^{-1}(r_{-b} + q_b), q_{-b}\rangle]. \end{aligned} \tag{2.19}$$

Here γ_E is the Euler-Mascheroni constant, and for $b + \frac{x}{t} \in (-\frac{1}{4}, \frac{1}{4})$, $\bar{K}_b = \bar{K}_{b,b}, q_b = q_{b,b}$.

Looking at the function $\Xi(S, b, \sigma)$ and equation (2.9), we can see that

$$\begin{aligned} &\det(I - \bar{K}_{b+\frac{x}{t}, \beta+\frac{x}{t}}) \to \det(I - \bar{K}_{b+\frac{x}{t}}), \\ &\frac{1}{\beta - b}\det[\delta_{i,j} - \langle (I - \bar{K}_{b+\frac{x}{t}, \beta+\frac{x}{t}})^{-1} f_i, g_j\rangle]_{i,j=1}^3 \to [\frac{(b+\frac{x}{t})^2}{\sigma^2} + \sigma(2\gamma_E + \ln S) \\ &+ \frac{1}{\sigma}\langle (I - \bar{K}_{b+\frac{x}{t}})^{-1}(\bar{K}_{b+\frac{x}{t}} r_{-(b+\frac{x}{t})} + q_{b+\frac{x}{t}}), r_{b+\frac{x}{t}}\rangle \\ &+ \langle (I - \bar{K}_{b+\frac{x}{t}})^{-1}(r_{-(b+\frac{x}{t})} + q_{b+\frac{x}{t}}), q_{-(b+\frac{x}{t})}\rangle]. \end{aligned} \tag{2.20}$$

We define the log derivative of the objects after limits as follows,

$$\begin{aligned} &\phi(x,t,r) = \partial_r^2 \log\det(I - \bar{K}_{b+\frac{x}{t}}), \\ &\psi(x,t,r) = \partial_r^2 \log[\frac{(b+\frac{x}{t})^2}{\sigma^2} + \sigma(2\gamma_E + \ln S) \\ &+ \langle (I - \bar{K}_{b+\frac{x}{t}})^{-1}(\bar{K}_{b+\frac{x}{t}} r_{-(b+\frac{x}{t})} + q_{b+\frac{x}{t}}), r_{b+\frac{x}{t}}\rangle + \langle (I - \bar{K}_{b+\frac{x}{t}})^{-1}(r_{-(b+\frac{x}{t})} + q_{b+\frac{x}{t}}), q_{-(b+\frac{x}{t})}\rangle]. \end{aligned} \tag{2.21}$$

We take limit $\beta \to b$ in equation (2.16), then $\phi_{b,\beta}(x,t,r) \to \phi(x,t,r), \alpha(x,t,r) \to \psi(x,t,r)$. In a purely formal way, we assume all the partial derivatives converge to the derivatives of the limit, thus we obtain that $\phi(x,t,r), \psi(x,t,r)$ also satisfy equations (2.16).

In the large time limit, we have the following results [BCFV15],

$$\lim_{t\to\infty} \Xi(e^{-\frac{\tau^2+\sigma r}{\sigma}}, \tau\sigma, \sigma) = g(\sigma r + \tau^2, \tau) F_{\text{GUE}}(\sigma r + \tau^2). \tag{2.22}$$

Remark 2.6. The reader might confuse why it is not the exact result in [BCFV15]. It is because the function g in [BCFV15], which is cited from [BFP10], is not the same g in [FS06]. Their equivalence is claimed in [BFP10], Remark 1.3.

Here the relation of τ to x, t is

$$x = -bt + \frac{2\tau}{\sigma^2}, \sigma = (\frac{2}{t})^{1/3}. \tag{2.23}$$

Under this scaling, we have the following result.

Theorem 2.7. *[BCFV15] Let* $b \in (-\frac{1}{4}, \frac{1}{4})$ *be fixed and consider any* $\tau \in \mathbb{R}$. *Define* $\sigma = (2/t)^{1/3}$ *and consider the scaling* $x = -bt + \frac{2\tau}{\sigma^2}$*. Then, for any* $r \in \mathbb{R}$*,*

$$\lim_{t\to\infty} \mathbb{P}(\frac{h_b(t,x) + \frac{t}{24}(1+12b^2) - 2^{1/3}b\tau t^{2/3}}{(t/2)^{1/3}} \le r) = F_\tau(r). \tag{2.24}$$

When $b = 0$, we have $\sigma^2 x = 2\tau$. We can see that as $t \to \infty$, x must go to infinity in the speed $x \sim t^{2/3}$ so that τ can be a meaningful number. For this reason, we write the variable in the scaling form $h_\epsilon(t,x) = \epsilon^{1/2} h(\epsilon^{-1}x, \epsilon^{-3/2}t)$. To obtain the corresponding formula, we plug $\epsilon^{-1}x \to x, \epsilon^{-3/2}t \to t, \epsilon^{-1/2}r \to r$ into $\Xi(e^{-\frac{\tau^2+\sigma r}{\sigma}}, \tau\sigma, \sigma)$, which we denoted as Ξ_ϵ. Then taking $t \to \infty$ in (2.22) is equivalent to taking $\epsilon \to 0$. We have

$$\begin{aligned}&\lim_{\epsilon\to 0} \Xi_\epsilon(e^{-\frac{\tau^2+r}{\sigma}}, \tau\sigma, \sigma) = \\ &g((2^{1/3}t^{-1/3}r + 2^{-2/3}t^{-4/3}x^2, 2^{-1/3}t^{-2/3}x)F_{\text{GUE}}(2^{1/3}t^{-1/3}r + 2^{-2/3}t^{-4/3}x^2).\end{aligned} \tag{2.25}$$

Now we also write the two components of Ξ in the scaling form. Let $\phi^\epsilon(x,t,r) = \phi(\epsilon^{-1}x,, \epsilon^{-\frac{3}{2}}t, \epsilon^{-1/2}r), \psi^\epsilon(x,t,r) = \psi(\epsilon^{-1}x,, \epsilon^{-\frac{3}{2}}t, \epsilon^{-1/2}r)$, where ϕ, ψ are defined in (2.21). Then equations (2.16) re-scale as follows,

$$\begin{aligned}&\epsilon^{5/2}\partial_t\phi^\epsilon + \epsilon^{5/2}\phi^\epsilon\partial_r\phi^\epsilon + \frac{1}{12}\epsilon^{5/2}\partial_r^3\phi^\epsilon + \epsilon^{5/2}\frac{1}{4}\partial_r^{-1}\partial_x^2\phi^\epsilon = 0, \\ &\epsilon^{5/2}\partial_t\psi^\epsilon + \epsilon^{5/2}\psi^\epsilon\partial_r\psi^\epsilon + \epsilon^{5/2}\frac{1}{12}\partial_r^3\psi^\epsilon + \epsilon^{5/2}\frac{1}{4}\partial_r^{-1}\partial_x^2\psi^\epsilon + \epsilon^{5/2}\psi^\epsilon\partial_r\phi^\epsilon + \epsilon^{5/2}\phi^\epsilon\partial_r\psi^\epsilon = 0.\end{aligned} \tag{2.26}$$

Thus $\phi^\epsilon, \psi^\epsilon$ also satisfy equation (2.16). In the limit as $\epsilon \to 0$, we have

$$\begin{aligned}&\phi^\epsilon(x,t,r) \to \partial_r^2 \log F_{\text{GUE}}(2^{1/3}t^7 - 1/3r + 2^{-2/3}t^{-4/3}x^2), \\ &\psi^\epsilon(x,t,r) \to \partial_r^2 \log g(2^{1/3}t^{-1/3}r + 2^{-2/3}t^{-4/3}x^2, 2^{-1/3}t^{-2/3}x).\end{aligned} \tag{2.27}$$

By formally assuming all the derivatives also converging to the derivatives of the limit, we obtain that $\partial_r^2 \log F_{\text{GUE}}, \partial_r^2 \log g$ satisfy equations (2.16). Since the fact that $\partial_r^2 \log F_{\text{GUE}}$ satisfies the KP equation is a known result, what is left to prove is $\partial_r^2 \log g$ satisfies (2.17).

3 Proof by direct verification

In this section, we are going to prove that $\partial_r^2 \log g$ satisfies (2.17) by directly differentiating the function. First we want to change the function g to its equivalent function y. Recall from Remark 1.4, $y(s + w'^2, w'/2) = g(s + w^2, w)$ when $w' = 2w$. In (2.25), we can see that $w = 2^{-1/3}t^{-2/3}x, s = 2^{1/3}t^{-1/3}r$, which means the correct scaling for y is $y(2^{1/3}t^{-1/3}r + 2^{4/3}t^{-4/3}x^2, 2^{2/3}t^{-2/3}x)$. Then we do another constant scaling on t: $2t \to t$. Now we will show that $B = \partial_r^2 \log y(t^{-1/3}r + t^{-4/3}x^2, \frac{1}{2}t^{-2/3}x)$ satisfies (2.17). We use y' to denote $\partial_1 y$ which is the partial derivative of y with respect to the first variable, to simplify the notation. The ϕ in (2.17) is the log derivative of the GUE distribution: $\phi(t,x,r) = \partial_r^2 \log F_{\text{GUE}}(t^{-1/3}r + t^{-4/3}x^2) = -t^{-2/3}u^2(t^{-1/3}r + t^{-4/3}x^2)$, u is defined in

(1.21). We compute all the terms appearing in (2.17),

$$\begin{aligned}
\partial_r B &= t^{-3/3}(\tfrac{y'''}{y} - \tfrac{3y'y''}{y^2} + \tfrac{2y'^3}{y^3}),\\
B\partial_r B &= t^{-5/3}(\tfrac{y'''y''}{y^2} - \tfrac{3y'y''^2}{y^3} + \tfrac{5y'^3y''}{y^4} - \tfrac{y'''y'^2}{y^3} - \tfrac{2y'^5}{y^5}),\\
\phi\partial_r B &= t^{-5/3}(-u^2)(\tfrac{y'''}{y} - \tfrac{3y'y''}{y^2} + \tfrac{2y'^3}{y^3}),\\
B\partial_r\phi &= t^{-5/3}(-2uu')(\tfrac{y''}{y} - \tfrac{y'^2}{y^2}),\\
\partial_r^2 B &= t^{-4/3}(\tfrac{y^{(4)}}{y} - \tfrac{3y''y''}{y^2} - \tfrac{4y'y'''}{y^2} + \tfrac{12y'^2y''}{y^3} - \tfrac{6y'^4}{y^4}),\\
\partial_r^3 B &= t^{-5/3}(\tfrac{y^{(5)}}{y} - \tfrac{10y''y'''}{y^2} - \tfrac{5y'y^{(4)}}{y^2} + \tfrac{20y'^2y'''}{y^3} + \tfrac{30y'y''^2}{y^3} - \tfrac{60y'^3y''}{y^4} + \tfrac{24y'^5}{y^5}),\\
\partial_t B &= -\tfrac{2}{3}t^{-5/3}(\tfrac{y''}{y} - (\tfrac{y'}{y})^2) - t^{-2/3}(\tfrac{y'''}{y} - \tfrac{3y'y''}{y^2} + \tfrac{2y'^3}{y^3})(\tfrac{1}{3}t^{-4/3}r + \tfrac{4}{3}t^{-7/3}x^2)\\
&\quad - t^{-2/3}(\tfrac{\partial_2 y''}{y} - \tfrac{y''\partial_2 y}{y^2} - \tfrac{2y'\partial_2 y'}{y^2} + \tfrac{2y'^2\partial_2 y}{y^3})\tfrac{1}{3}t^{-5/3}x,\\
\partial_r^{-1}\partial_x^2 B &= 2t^{-5/3}(\tfrac{y''}{y} - \tfrac{y'^2}{y^2}) + 4x^2t^{-9/3}(\tfrac{y'''}{y} - \tfrac{3y'y''}{y^2} + \tfrac{2y'^3}{y^3}) + \tfrac{1}{4}t^{-5/3}\partial_2^2(\tfrac{y'}{y})\\
&\quad +2xt^{-7/3}(\tfrac{\partial_2 y''}{y} - \tfrac{y''\partial_2 y}{y^2} - \tfrac{2y'\partial_2 y'}{y^2} + \tfrac{2y'^2\partial_2 y}{y^3}).
\end{aligned} \tag{3.1}$$

Now plug in every term into equation (1.30). It becomes

$$\begin{aligned}
&\partial_t B + B\partial_r B + \tfrac{1}{12}\partial_r^3 B + \tfrac{1}{4}\partial_r^{-1}\partial_x^2 B + \phi\partial_r B + B\partial_r\phi\\
&= t^{-5/3}(\tfrac{1}{12}(\tfrac{y^{(5)}}{y} - \tfrac{5y'y^{(4)}}{y^2} + \tfrac{2y''y'''}{y^2} - \tfrac{6y'y''^2}{y^3} + \tfrac{8y'^2y'''}{y^3}) - \tfrac{1}{6}(\tfrac{y''}{y} - \tfrac{y'^2}{y^2})\\
&+ \tfrac{1}{16}(\tfrac{\partial_2^2 y'}{y} - \tfrac{2\partial_2 y'\partial_2 y}{y^2} - \tfrac{y'\partial_2^2 y}{y^2} + \tfrac{2y'(\partial_2 y)^2}{y^3}) + \tfrac{1}{3}w(\tfrac{\partial_2 y''}{y} - \tfrac{y''\partial_2 y}{y^2} - \tfrac{2y'\partial_2 y'}{y^2} + \tfrac{2y'^2\partial_2 y}{y^3})\\
&- \tfrac{1}{3}x(\tfrac{y'''}{y} - \tfrac{3y'y''}{y^2} + 2\tfrac{y'^3}{y^3}) - u^2(\tfrac{y'''}{y} - \tfrac{3y'y''}{y^2} + 2\tfrac{y'^3}{y^3}) - 2uu'(\tfrac{y''}{y} - \tfrac{y'^2}{y^2})).
\end{aligned} \tag{3.2}$$

Multiplied by $y^3t^{5/3}$, the right-hand side becomes

$$\begin{aligned}
&\tfrac{1}{12}(y^{(5)}y^2 - 5y'y^{(4)}y + 2y''y'''y - 6y'y''^2 + 8y'^2y''') - \tfrac{1}{6}(y''y^2 - y'^2y)\\
&+ \tfrac{1}{16}(\partial_2^2 y'y^2 - 2\partial_2 y'\partial_2 yy - y'\partial_2^2 yy + 2y'(\partial_2 y)^2) + \tfrac{1}{3}w(\partial_2 y''y^2 - y''\partial_2 yy - 2y'\partial_2 y'y + 2y'^2\partial_2 y)\\
&- \tfrac{1}{3}x(y'''y^2 - 3y'y''y + 2y'^3) - u^2(y'''y^2 - 3y'y''y + 2y'^3) - 2uu'(y''y^2 - y'^2y).
\end{aligned} \tag{3.3}$$

Here $w = t^{-1/3}r + t^{-4/3}x^2$, u comes from the F_{GUE} part. Now we expand the derivatives of y in terms of a, b, u, w. Recall that functions $a(x;w), b(x;w)$ arise in the Riemann-Hilbert problem for the Painlevé II equation. In the following expressions, if we omit the variables, then it means that is the variable in the definition, i.e. a stands for $a(x,w)$, b stands for $b(x,w)$. We use $a(-w), b(-w)$ to represent $a(x,-w), b(x,-w)$,

$$\begin{aligned}
y &= (2u^2 + x - 4w^2)aa(-w) - (u' + 2wu)ba(-w) - (u' - 2wu)ab(-w),\\
y' &= aa(-w),\\
y'' &= uba(x,-w) + uab(-w),\\
y''' &= (u' - 2wu)ba(-w) + 4u^2aa(-w) + (u' + 2wu)ab(-w),\\
y^{(4)} &= 12uu'aa(-w) + (4u^3 + u'' + 4wu' + 4w^2u)ab(-w)\\
&\quad + (u'' + 4u^3 - 4wu' + 4w^2u)ba(-w),\\
y^{(5)} &= (12u'^2 + 16uu'' + 16u^4 + 16w^2u^2)aa(-w),\\
&\quad + (24u^2u' + u''' + 6wu'' + 12w^2u' + 8wu^3 + 8w^3u)ab(-w)\\
&\quad + (24u^2u' + u''' - 6wu'' - 8wu^3 + 12w^2u' - 8w^3u)ba(-w),\\
\partial_2 y &= -8waa(-w) + 2uab(-w) + (-2u)ba(-w),\\
\partial_2^2 y &= (-8 - 16uu')aa(-w) + (16w^2u - 16wu' + 8u^3 + 4ux)ab(-w)\\
&\quad + (16w^2u + 16wu' + 8u^3 + 4ux)ba(-w).
\end{aligned} \tag{3.4}$$

It is not a complete list. We are also required to compute $\partial_2 y', \partial_2^2 y', \partial_2 y''$. Thanks to the identities (1.27) derived in [BR00], the derivatives of $a(x,w), b(x,w)$ behave similarly to $\sin, \cos$, in the sense that the derivatives of $a(x,w), b(x,w)$ are certain combinations of $a(x,w), b(x,w)$ themselves. All the partials of y are in the form of $c_1 aa(-w) + c_2 ab(-w) + c_3 ba(-w)$, where c_1, c_2, c_3 are coefficients consisting of u, w and derivatives of u (there is no $bb(-w)$ term because $bb(-w) = aa(-w)$ by (1.27)). Finally, if we plug in all the partials into equation (3.3), all the terms are canceled and we get 0. The way it cancels is the following: each term in (3.3) is of "degree 3" in y, in the sense that every term is a product of 3 derivatives of y. Since every type of the derivative is in the form of $c_1 aa(-w) + c_2 ab(-w) + c_3 ba(-w)$, their multiplication is of the form $c_1' a^3 a^3(-w) + c_2' a^3 a^2(-w) b(-w) + c_3' a^3 a(-w) b^2(-w) + c_4' a^3 b^3(-w) + c_5' b^3 a^3(-w)$. Notice that many other types can be transformed to one of these by $bb(-w) = aa(-w)$. It turns out that all $c_1', \cdots, c_5'$ are 0, using

$$u_{xx} = 2u^3 + xu. \tag{3.5}$$

From (3.4), we can see that there exist derivatives of u with order 2 and higher. Whenever we encounter this, we use identity (3.5) to reduce the order of derivatives, so that in the coefficients of the final formula, the only variables left are u, u', w, x. Here are two examples of the coefficients for $a^3 a^3(-w)$ and $a^3 a^2(-w) b(-w)$,

$$\begin{aligned} c_1' &= \tfrac{1}{3}(-4w^2 + x + 2u^2)(2 - 32w^4u^2 + 8w^2xu^2 + 16w^2u^4 + 2uu' \\ &\quad - 16w^3uu' + 4wxuu' + 8wu^3u' - 12w^2u'^2 + 3xu'^2 + 6u^2u'^2) \\ &\quad + \tfrac{1}{3}(4w^2 + x + 2u^2)(-2 + 32w^4u^2 - 8w^2xu^2 - 16w^2u^4 - 2uu' \\ &\quad + 16w^3uu' - 4wxuu' - 8wu^3u' + 12w^2u'^2 - 3xu'^2 - 6u^2u'^2) \\ &= 0, \\ c_2' &= \tfrac{5}{12}u(4w^2 + x + 6u^2)(4w^2 - x - 2u^3) \\ &\quad - \tfrac{5}{12}u(4w^2 + x + 6u^2)(4w^2 - x - 2u^2) \\ &= 0. \end{aligned} \tag{3.6}$$

The rest, $c_3', \cdots, c_5'$, vanish in a similar, purely algebraic manner. So, by directly plugging in all the derivatives of the function y, we checked that $\partial_r^2 \log y(t^{-1/3}r + t^{-4/3}x^2, \frac{1}{2}t^{-2/3}x)$ satisfied (1.30), which implies that that $\partial_r^2 \log(y(t^{-1/3}r + t^{-4/3}x^2, \frac{1}{2}t^{-2/3}x) F_{\mathrm{GUE}}(t^{-1/3}r + t^{-4/3}x^2))$ satisfies the KP equation.

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