Electron. Commun. Probab. **27** (2022), article no. 12, 1–8.
https://doi.org/10.1214/22-ECP452
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

A note on first eigenvalue estimates by coupling methods in Kähler and quaternion Kähler manifolds*

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Abstract

In this note, using the Kendall-Cranston coupling, we study on Kähler (resp. quaternion Kähler) manifolds first eigenvalue estimates in terms of dimension, diameter, and lower bounds on the holomorphic (resp. quaternionic) sectional curvature.

Keywords: Kendall-Cranston coupling; Kähler and quaternion Kähler manifolds; first eigenvalue estimates.
MSC2020 subject classifications: Primary 58J65, Secondary 58J50; 60J60; 58J50.
Submitted to ECP on May 30, 2021, final version accepted on January 29, 2022.

1 Introduction

Using probabilistic coupling arguments M.F. Chen and F.Y. Wang [3, 4] proved the following remarkable theorem on Riemannian manifolds:

Theorem 1.1 (Chen-Wang, [3, 4])**.** *Let* $(\mathbb{M}, g)$ *be a compact Riemannian manifold with dimension* n*, diameter* D *and first eigenvalue* λ_1*. Assume that* $\mathrm{Ric} \geq (n-1)k$ *with* $k \in \mathbb{R}$*. Then, for any* C^2 *function* $g : [0, D) \to \mathbb{R}$ *such that* $g(0) = 0$ *and* $g' > 0$ *on* $[0, D)$ *one has*

$$\lambda_1 \geq - \sup_{r \in (0,D)} \frac{\mathcal{A}_k g(r)}{g(r)},$$

where

$$\mathcal{A}_k = 4\frac{\partial^2}{\partial r^2} + (n-1)G(k,r)\frac{\partial}{\partial r},$$

and $G(k, r)$ *is the comparison function* (3.1)*.*

For instance, if $k = 0$, the choice $g(r) = \sin\left(\frac{\pi}{2D}r\right)$ in Theorem 1.1 yields the celebrated Zhong-Yang [16] estimate $\lambda_1 \geq \frac{\pi^2}{D^2}$.

In this note we apply in the context of Kähler and quaternion Kähler manifolds a similar coupling method to improve this first eigenvalue estimate on Kähler and quaternion Kähler manifolds. We rely on the decomposition of Ricci curvature on

*Fabrice Baudoin and Guang Yang were supported in part by National Science Foundation grant DMS-1901315.

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Kähler manifolds into orthogonal Ricci curvature and holomorphic sectional curvature as follows.

$$Ric(X,\overline{X}) = Ric^{\perp}(X,\overline{X}) + R(X,\overline{X},X,\overline{X})/|X|^2,$$

where X is a $(1,0)$-tangent vector of the holomorphic tangent bundle on a Kähler manifold M^n. A similar decomposition holds on a quaternion Kähler geometry. It seems worthy to mention that to obtain Laplacian and Index comparison theorems with model spaces in Kähler (and quaternion Kähler) geometry, it is necessary that the lower bounds of orthogonal Ricci curvature and holomorphic (quaternionic) sectional curvature should be assumed simultaneously, not only a Ricci curvature lower bound (also, see [5]). In this paper, the main geometric ingredients are new estimates of the index form in those settings that build on the previous recent work [1].

Acknowledgement: Shortly after this work was posted on arXiv, the authors were made aware of the references [9, 10] which independently obtained comparable results under similar assumptions but by using analytic methods. The authors thank the anonymous referees for a careful reading and remarks that improved the presentation of the paper.

2 Preliminaries: Kähler and quaternion Kähler manifolds

In this section, for the sake of completeness, we give the definitions we will be using in this paper. We refer to [1] for more details. Throughout the paper, let $(\mathbb{M}, g)$ be a smooth complete Riemannian manifold. Denote by ∇ the Levi-Civita connection on $\mathbb{M}$.

2.1 Kähler manifolds

Definition 2.1. *The manifold $(\mathbb{M}, g)$ is called a Kähler manifold, if there exists a smooth $(1,1)$ tensor J on $\mathbb{M}$ that satisfies:*

- *For every $x \in \mathbb{M}$, and $X, Y \in T_x\mathbb{M}$, $g_x(J_xX, Y) = -g_x(X, J_xY)$;*
- *For every $x \in \mathbb{M}$, $J_x^2 = -\mathbf{Id}_{T_x\mathbb{M}}$;*
- *$\nabla J = 0$.*

The map J is called a complex structure.

It is well-known that Kähler manifolds can be seen as the complex manifolds for which the Chern connection coincide with the Levi-Civita connection, see [11], however the complex viewpoint will not be necessary to state and prove our results, so we will always only consider the real structure on a Kähler manifold and work with the definition above.

We will be considering the holomorphic sectional curvature and orthogonal Ricci curvature which are defined as below. A reason to consider those curvature quantities instead of the more usual Ricci curvature is that in the classical Riemannian comparison theorems involving a Ricci curvature lower bound, the spaces with respect to which the comparison is made are usually spheres, Euclidean spaces and hyperbolic spaces. To develop comparison theorems in the category of Kähler manifolds, it is then necessary to consider more subtle curvature invariants which are adapted to the additional Kähler structure. We refer to [12] and [13] for further details about the geometric interpretation of the holomorphic sectional curvature and orthogonal Ricci curvature.

Let

$$R(X,Y,Z,W) = g((\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z, W)$$

be the Riemannian curvature tensor of $(\mathbb{M}, g)$. The holomorphic sectional curvature of the Kähler manifold $(\mathbb{M}, g, J)$ is defined as

$$H(X) = \frac{R(X,JX,JX,X)}{g(X,X)^2}.$$

The orthogonal Ricci curvature of the Kähler manifold $(\mathbb{M}, g, J)$ is defined for a vector field X such that $g(X,X) = 1$ by

$$\mathrm{Ric}^{\perp}(X,X) = \mathrm{Ric}(X,X) - H(X),$$

where Ric is the usual Riemannian Ricci tensor of $(\mathbb{M}, g)$. Unlike the Ricci tensor, $\mathrm{Ric}^{\perp}$ does not admit polarization, so we never consider $\mathrm{Ric}^{\perp}(u,v)$ for $u \neq v$.

The table below shows the curvature of the Kähler model spaces $\mathbb{C}^m$, $\mathbb{C}P^m$ and $\mathbb{C}H^m$, see [1].

Table 1: Curvatures of Kähler model spaces.

$\mathbb{M}$	H	$\mathrm{Ric}^{\perp}$
$\mathbb{C}^m$	0	0
$\mathbb{C}P^m$	4	$2m-2$
$\mathbb{C}H^m$	-4	$-(2m-2)$

2.2 Quaternion Kähler manifolds

Definition 2.2. *The manifold $(\mathbb{M}, g)$ is called a quaternion Kähler manifold, if there exists a covering of $\mathbb{M}$ by open sets U_i and, for each i, 3 smooth $(1,1)$ tensors I, J, K on U_i such that:*

- *For every $x \in U_i$, and $X, Y \in T_x\mathbb{M}$, $g_x(I_xX, Y) = -g_x(X, I_xY)$, $g_x(J_xX, Y) = -g_x(X, J_xY)$, $g_x(K_xX, Y) = -g_x(X, K_xY)$;*
- *For every $x \in U_i$, $I_x^2 = J_x^2 = K_x^2 = I_xJ_xK_x = -\mathbf{Id}_{T_x\mathbb{M}}$;*
- *For every $x \in U_i$, and $X \in T_x\mathbb{M}$, $\nabla_X I, \nabla_X J, \nabla_X K \in \mathbf{span}\{I, J, K\}$;*
- *For every $x \in U_i \cap U_j$, the vector space of endomorphisms of $T_x\mathbb{M}$ generated by I_x, J_x, K_x is the same for i and j.*

On quaternion Kähler manifolds, we will be considering the following curvatures. Let

$$R(X,Y,Z,W) = g((\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z, W)$$

be the Riemannian curvature tensor of $(\mathbb{M}, g)$. We define the quaternionic sectional curvature of the quaternionic Kähler manifold $(\mathbb{M}, g, J)$ as

$$Q(X) = \frac{R(X,IX,IX,X) + R(X,JX,JX,X) + R(X,KX,KX,X)}{g(X,X)^2}.$$

We define the orthogonal Ricci curvature of the quaternionic Kähler manifold $(\mathbb{M}, g, I, J, K)$ for a vector field X such that $g(X,X) = 1$ by

$$\mathrm{Ric}^{\perp}(X,X) = \mathrm{Ric}(X,X) - Q(X),$$

where Ric is the usual Riemannian Ricci tensor of $(\mathbb{M}, g)$. The table below shows the curvature of the quaternion-Kähler model spaces $\mathbb{H}^m$, $\mathbb{H}P^m$ and $\mathbb{H}H^m$, see [1].

Table 2: Curvatures of the quaternion Kähler model spaces.

$\mathbb{M}$	Q	$\mathrm{Ric}^{\perp}$
$\mathbb{H}^m$	0	0
$\mathbb{H}P^m$	12	$4m-4$
$\mathbb{H}H^m$	-12	$-(4m-4)$

3 Index form estimates on Kähler and quaternion Kähler manifolds

Let $(\mathbb{M}, g)$ be a complete Riemannian manifold with real dimension n and denote by d the Riemannian distance on $\mathbb{M}$. The index form of a vector field X (with not necessarily vanishing endpoints) along a geodesic γ is defined by

$$I(\gamma, X, X) := \int_0^T \left(\langle \nabla_{\gamma'} X, \nabla_{\gamma'} X \rangle - \langle R(\gamma', X)X, \gamma' \rangle \right) dt,$$

where ∇ is the Levi-Civita connection and R is the Riemann curvature tensor of $\mathbb{M}$.

We will denote by

$$\mathrm{Cut}(\mathbb{M}) = \{(x, y) \in \mathbb{M} \times \mathbb{M},\, x \in \mathrm{Cut}(y)\}.$$

where $\mathrm{Cut}(y)$ denotes the cut-locus of y. For $(x, y) \notin \mathrm{Cut}(\mathbb{M})$ we denote

$$\mathcal{I}(x, y) = \sum_{i=1}^{n-1} I(\gamma, Y_i, Y_i)$$

where γ is the unique length parametrized geodesic from x to y and $\{Y_1, \cdots, Y_{n-1}\}$ are Jacobi fields such that at both x and y, $\{\gamma', Y_1, \cdots, Y_{n-1}\}$ is an orthonormal frame.

Throughout the paper, we consider the comparison function:

$$G(k, r) = \begin{cases} -2\sqrt{k} \tan \frac{\sqrt{k}r}{2} & \text{if } k > 0, \\ 0 & \text{if } k = 0, \\ 2\sqrt{|k|} \tanh \frac{\sqrt{|k|}r}{2} & \text{if } k < 0. \end{cases} \tag{3.1}$$

3.1 Kähler case

Let $(\mathbb{M}, g, J)$ be a complete Kähler with complex dimension m (i.e. the real dimension is $2m$). As above, the holomorphic sectional curvature of $\mathbb{M}$ will be denoted by H and the orthogonal Ricci curvature by $\mathrm{Ric}^\perp$.

Theorem 3.1. *Let $k_1, k_2 \in \mathbb{R}$. Assume that $H \geq 4k_1$ and that $\mathrm{Ric}^\perp \geq (2m-2)k_2$. For every $(x, y) \notin \mathrm{Cut}(\mathbb{M})$, one has*

$$\mathcal{I}(x, y) \leq (2m-2)G(k_2, d(x, y)) + 2G(k_1, 2d(x, y)).$$

Remark 3.2. In particular, if $k_1 > 0$ then $\mathbb{M}$ is compact with a diameter $\leq \frac{\pi}{2\sqrt{k_1}}$ and if $k_2 > 0$ then $\mathbb{M}$ is compact with a diameter $\leq \frac{\pi}{\sqrt{k_2}}$.

Proof. Let $(x, y) \notin \mathrm{Cut}(\mathbb{M})$ and denote by $\gamma : [0, r] \to \mathbb{M}$ where $r = d(x, y)$ the unique length parametrized geodesic from $x = \gamma(0)$ to $y = \gamma(r)$. At x, we consider an orthonormal frame $\{X_1(x), \cdots, X_{2m}(x)\}$ such that

$$X_1(x) = \gamma'(0),\ X_2(x) = J\gamma'(0).$$

We denote by $X_1, \cdots, X_{2m}$ the vector fields obtained by parallel transport of $X_1(x), \cdots, X_{2m}(x)$ along γ. Note that $X_1 = \gamma'$ and that $X_2 = J\gamma'$ because $\nabla J = 0$.

We introduce the function

$$\mathfrak{j}(k, t) = \mathfrak{c}(k, t) + \frac{1 - \mathfrak{c}(k, r)}{\mathfrak{s}(k, r)} \mathfrak{s}(k, t)$$

where

$$\mathfrak{s}(k,t)=\begin{cases}\sin\sqrt{k}t & \text{if } k>0,\\ t & \text{if } k=0,\\ \sinh\sqrt{|k|}t & \text{if } k<0,\end{cases}$$

and

$$\mathfrak{c}(k,t)=\begin{cases}\cos\sqrt{k}t & \text{if } k>0,\\ 1 & \text{if } k=0,\\ \cosh\sqrt{|k|}t & \text{if } k<0.\end{cases}$$

We now consider the vector fields defined along γ by $\tilde{X}_2(\gamma(t))=\mathfrak{j}(4k_1,t)X_2$ and for $i=3,\cdots,2m$ by $\tilde{X}_i(\gamma(t))=\mathfrak{j}(k_2,t)X_i$. From the index lemma (Lemma 1.21 in [2]) one has $\mathcal{I}(x,y)\le\sum_{i=2}^{2m}I(\gamma,\tilde{X}_i,\tilde{X}_i)$. We first estimate

$$\begin{aligned}I(\gamma,\tilde{X}_2,\tilde{X}_2)&=\int_0^r\left(\langle\nabla_{\gamma'}\tilde{X}_2,\nabla_{\gamma'}\tilde{X}_2\rangle-\langle R(\gamma',\tilde{X}_2)\tilde{X}_2,\gamma'\rangle\right)dt\\ &\le\int_0^r(\mathfrak{j}'(4k,t)^2+4k\mathfrak{j}(4k,t)^2)dt=2G(k_1,2r).\end{aligned}$$

Then, by a similar computation one has $\sum_{i=3}^{2m}I(\gamma,\tilde{X}_i,\tilde{X}_i)\le(2m-2)G(k_2,r)$. □

3.2 Quaternion Kähler case

Let now $(\mathbb{M},g,I,J,K)$ be a complete quaternion Kähler manifold with quaternionic dimension m (i.e. the real dimension is $4m$). As above, the quaternionic sectional curvature of $\mathbb{M}$ will be denoted by H and the orthogonal Ricci curvature by $\mathrm{Ric}^{\perp}$.

Theorem 3.3. *Let $k_1,k_2\in\mathbb{R}$. Assume that $Q\ge 12k_1$ and that $\mathrm{Ric}^{\perp}\ge(4m-4)k_2$. For every $(x,y)\notin\mathrm{Cut}(\mathbb{M})$,*

$$\mathcal{I}(x,y)\le(4m-4)G(k_2,d(x,y))+6G(k_1,2d(x,y)).$$

Remark 3.4. In particular, if $k_1>0$ then $\mathbb{M}$ is compact with a diameter $\le\frac{\pi}{2\sqrt{k_1}}$ and if $k_2>0$ then $\mathbb{M}$ is compact with a diameter $\le\frac{\pi}{\sqrt{k_2}}$.

Proof. The proof is almost similar to the Kähler case. We can assume $m\ge 2$ since for $m=1$ theorem 3.3 reduces to theorem 1.1. Let $(x,y)\notin\mathrm{Cut}(\mathbb{M})$ and denote by $\gamma:[0,r]\to\mathbb{M}$ where $r=d(x,y)$ the unique length parametrized geodesic from $x=\gamma(0)$ to $y=\gamma(r)$. At x, we consider an orthonormal frame $\{X_1(x),\cdots,X_{4m}(x)\}$ such that

$$X_1(x)=\gamma'(0),\ X_2(x)=I\gamma'(0),\ X_3(x)=J\gamma'(0),\ X_4(x)=K\gamma'(0)$$

We denote by $X_1,\cdots,X_{4m}$ the vector fields obtained by parallel transport of $X_1(x),\cdots,X_{4m}(x)$ along γ and consider the vector fields defined along γ by

$$\tilde{X}_2(\gamma(t))=\mathfrak{j}(4k_1,t)X_2,\ \tilde{X}_3(\gamma(t))=\mathfrak{j}(4k_1,t)X_3,\tilde{X}_4(\gamma(t))=\mathfrak{j}(4k_1,t)X_4$$

and for $i=5,\cdots,4m$ by

$$\tilde{X}_i(\gamma(t))=\mathfrak{j}(k_2,t)X_i.$$

Since along γ one has

$$\nabla_{\gamma'}I,\nabla_{\gamma'}J,\nabla_{\gamma'}K\in\mathbf{span}\{I,J,K\}$$

we deduce that along γ one has

$$\mathbf{span}\{X_2,X_3,X_4\}=\mathbf{span}\{I\gamma',J\gamma',K\gamma'\}.$$

Moreover $\{X_2, X_3, X_4\}$ and $\{I\gamma', J\gamma', K\gamma'\}$ are both orthonormal along γ. One deduces

$$\begin{aligned}&R(\gamma', X_2, X_2, \gamma') + R(\gamma', X_3, X_3, \gamma') + R(\gamma', X_4, X_4, \gamma')\\ =&R(\gamma', I\gamma', I\gamma', \gamma') + R(\gamma', J\gamma', J\gamma', \gamma') + R(\gamma', K\gamma', K\gamma', \gamma')\\ =&Q(\gamma').\end{aligned}$$

Using then the index lemma as in the Kähler case and similar arguments as in the proof of theorem 3.1 yields the conclusion. □

4 First eigenvalue estimates

With the index form estimates of the previous section in hands, we can use the reflection coupling method by M.F. Chen and F.Y. Wang [3, 4] (see also [6, Section 6.7]) to get curvature diameter estimates of the first eigenvalue in Kähler and quaternion Kähler manifolds.

4.1 Kähler case

Let $(\mathbb{M}, g, J)$ be a compact Kähler with complex dimension m. We denote by D the diameter of $\mathbb{M}$ and by λ_1 the first eigenvalue of $\mathbb{M}$.

Theorem 4.1. *Let $k_1, k_2 \in \mathbb{R}$. Assume that $H \geq 4k_1$ and that $\mathrm{Ric}^\perp \geq (2m-2)k_2$. For any C^2 function $g : [0, D] \to \mathbb{R}$ such that $g(0) = 0$ and $g' > 0$ on $[0, D)$ one has*

$$\lambda_1 \geq -\sup_{r \in (0,D)} \frac{\mathcal{L}_{k_1,k_2} g(r)}{g(r)},$$

where

$$\mathcal{L}_{k_1,k_2} = 4\frac{\partial^2}{\partial r^2} + ((2m-2)G(k_2, r) + 2G(k_1, 2r))\,\frac{\partial}{\partial r}.$$

Proof. Let $g : [0, D] \to \mathbb{R}$ be a C^2 function such that $g(0) = 0$ and $g' > 0$ on $[0, D)$. As in [3, 4] we will use the coupling by reflection introduced by Kendall [7].

The argument is easy to explain in the absence of cut-locus. Indeed, for the sole sake of the explanation, assume first that $\mathrm{Cut}(\mathbb{M}) = \emptyset$. From Theorem 2.3.2 in [15] and Theorem 3.1, if $(X_t, Y_t)_{t\geq 0}$ is the Kendall's mirror coupling started from (x, y) then, by denoting $\rho_t = d(X_t, Y_t)$ we have that:

$$d\rho_t \leq 2d\beta_t + ((2m-2)G(k_2, \rho_t) + 2G(k_1, 2\rho_t))\,dt,$$

where $(\beta_t)_{t\geq 0}$ is a Brownian motion on $\mathbb{R}$ with $\langle \beta \rangle_t = 2t$. Using then Itô's formula and the fact that g is non decreasing, we obtain

$$\begin{aligned}g(\rho_t) \leq &\, g(\rho_0) + 2\int_0^t g'(\rho_s)d\beta_s + 4\int_0^t g''(\rho_s)ds\\ &+ \int_0^t ((2m-2)G(k_2, \rho_s) + 2G(k_1, 2\rho_s))\, g'(\rho_s)ds\\ &\leq g(\rho_0) + 2\int_0^t g'(\rho_s)d\beta_s + \int_0^t \mathcal{L}_{k_1,k_2} g(\rho_s)ds\\ &\leq g(\rho_0) + 2\int_0^t g'(\rho_s)d\beta_s - \delta\int_0^t g(\rho_s)ds,\end{aligned}$$

where $\delta = -\sup_{r\in(0,D)} \frac{\mathcal{L}_{k_1,k_2} g(r)}{g(r)}$. By taking expectations, we obtain therefore

$$\mathbb{E}(g(\rho_t)) \leq \mathbb{E}(g(\rho_0)) - \delta\int_0^t \mathbb{E}(g(\rho_s))ds.$$

Thanks to Gronwall's inequality this yields

$$\mathbb{E}(g(\rho_t)) \leq \mathbb{E}(g(\rho_0))e^{-\delta t}.$$

On the other hand, let us now consider an eigenfunction Φ associated to the eigenvalue λ_1, i.e. $\Delta\Phi = -\lambda_1\Phi$. One has

$$\begin{aligned} e^{-\lambda_1 t}|\Phi(x) - \Phi(y)| &= |\mathbb{E}(\Phi(X_t) - \Phi(Y_t))| \\ &\leq \mathbb{E}(|\Phi(X_t) - \Phi(Y_t)|) \\ &\leq \|\nabla\Phi\|_\infty \mathbb{E}(d(X_t, Y_t)) = \|\nabla\Phi\|_\infty \mathbb{E}(\rho_t). \end{aligned}$$

Thanks to our assumptions on g, there exists a constant $C > 0$ so that for every $r \in [0, D]$, $g(r) \geq \frac{1}{C}r$. We obtain therefore

$$e^{-\lambda_1 t}|\Phi(x) - \Phi(y)| \leq C\|\nabla\Phi\|_\infty \mathbb{E}(g(\rho_t)) \leq C\|\nabla\Phi\|_\infty g(d(x, y))e^{-\delta t}.$$

Since it is true for every $t \geq 0$ one concludes that $\lambda_1 \geq \delta$.

In the absence of cut locus issues the Kendall coupling is easily constructed using stochastic differential equations and the above argument is complete. To handle cut-locus issues, it is possible to instead construct the Kendall coupling (X_t, Y_t) as a limit of coupled random walks, see [8] and [14]. In particular, a similar argument as in [8, Lemma 11] yields as above

$$\mathbb{E}(g(\rho_t)) \leq \mathbb{E}(g(\rho_0)) - \delta \int_0^t \mathbb{E}(g(\rho_s))ds.$$

and the conclusion follows then as before. □

Remark 4.2. Theorem 4.1 can be used to improve the lower bound of Theorem 1.1 in some situations like for example the complex projective space $\mathbb{C}P^m$ in Table (1). Indeed $H \geq 4k_1$ and $\mathrm{Ric}^\perp \geq (2m-2)k_2$ imply the Ricci lower bound $\mathrm{Ric} \geq 4k_1 + (2m-2)k_2$ and we always have

$$(2m-2)G(k_2, r) + 2G(k_1, 2r) \leq (2m-1)G\left(\frac{4k_1 + (2m-2)k_2}{2m-1}, r\right).$$

by concavity of $k \to G(k, r)$.

4.2 Quaternion Kähler case

Let now $(\mathbb{M}, g, I, J, K)$ be a complete quaternion Kähler with quaternionic dimension m. As before, we denote by D the diameter of $\mathbb{M}$ and by λ_1 the first eigenvalue of $\mathbb{M}$. Using the coupling by reflection as in the previous section, we obtain the following result.

Theorem 4.3. *Let $k_1, k_2 \in \mathbb{R}$. Assume that $Q \geq 12k_1$ and that $\mathrm{Ric}^\perp \geq (4m-4)k_2$. For any C^2 function $g : [0, D) \to \mathbb{R}$ such that $g(0) = 0$ and $g' > 0$ on $[0, D)$ one has*

$$\lambda_1 \geq -\sup_{r\in(0,D)} \frac{\tilde{\mathcal{L}}_{k_1,k_2} g(r)}{g(r)},$$

where

$$\tilde{\mathcal{L}}_{k_1,k_2} = 4\frac{\partial^2}{\partial r^2} + ((4m-4)G(k_2, r) + 6G(k_1, 2r))\frac{\partial}{\partial r}.$$

Remark 4.4. As in the Kähler case, Theorem 4.3 can be used to improve Theorem 1.1 in some situations like the quaternionic projective space $\mathbb{H}P^m$ in Table (2). Indeed $Q \geq 12k_1$ and $\mathrm{Ric}^\perp \geq (4m-4)k_2$ imply the Ricci lower bound $\mathrm{Ric} \geq 12k_1 + (4m-4)k_2$ and we have

$$(4m-4)G(k_2, r) + 6G(k_1, 2r) \leq (2m-1)G\left(\frac{12k_1 + (4m-4)k_2}{4m-1}, r\right).$$

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