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On the limiting law of line ensembles of Brownian polymers with geometric area tilts

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Dedicated to the memory of Dima Ioffe

Abstract. We study the line ensembles of non-crossing Brownian bridges above a hard wall, each tilted by the area of the region below it with geometrically growing pre-factors. This model, which mimics the level lines of the $(2+1)$D SOS model above a hard wall, was studied in two works from 2019 by Caputo, Ioffe and Wachtel. In those works, the tightness of the law of the top k paths, for any fixed k, was established under either zero or free boundary conditions, which in the former setting implied the existence of a limit via a monotonicity argument. Here we address the open problem of existence of a limit under free boundary conditions: we prove that as the interval length, followed by the number of paths, go to ∞, the top k paths converge to the same limit as in the zero boundary case, as conjectured by Caputo, Ioffe and Wachtel.

Résumé. Nous étudions l'ensemble de lignes déterminé par des mouvements Browniens non-intersectant au-dessus d'un mur solide. Ce modèle, qui imite les lignes de niveaux du modèle $(2+1)$D SOS au-dessus d'un mur, a été étudié en 2019 par Caputo, Ioffe et Wachtel. Dans ces travaux, la tension de la loi des k lignes hautes, pour chaque k fixe, a été obtenue sous des conditions nulles au bord ou des conditions libres au bord. Dans le premier cas, ca implique l'existence d'une limite par un argument de monotonicité. Nous abordons ici le problème ouvert d'existence d'une limite sous des conditions libres au bord : nous démontrons que quand la longueur de l'intervalle, suivi par le nombre de lignes, tend vers l'infinie, les k lignes hautes convergent vers la même limite que dans le cas de conditions nulles au bord, comme conjecturé par Caputo, Ioffe et Wachtel.

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1. Introduction

Entropic repulsion in low temperature $(2+1)$D crystals above a hard wall has been the subject of extensive study in statistical physics. Whereas in the absence of a wall, the surface of the crystal would typically be rigid at height 0, in the presence of a wall, the surface is propelled in order to increase its entropy (i.e., to allow thermal fluctuations going downward), and becomes rigid at some height level which diverges with the side length L of the box.

A rigorous study of this phenomenon in the $(2+1)$D Solid-On-Solid (SOS) model – a low temperature approximation of the 3D Ising model – dates back to Bricmont, El Mellouki and Fröhlich [1] in 1986, where it was shown that, in the presence of a hard wall at height 0, the typical height of a site in the bulk is propelled to order $\log L$. Thereafter, a detailed description of the shape of this random surface was obtained by Caputo et al. [4–6], showing that it typically becomes rigid at a height which is one of two consecutive (explicit) integers, through a sequence of nested level lines each encompassing a $(1-\varepsilon)$-fraction of the sites (analogous behavior was later established [16] for the more general family of $|\nabla\phi|^p$-random surface model, where the SOS model is the case $p=1$). The level lines near the center sides of the box behave as random walks – a ubiquitous feature of interfaces in low temperature spin systems – albeit with cube-root fluctuations, as their laws are tilted by the entropic repulsion effect. The lower the level line, the higher the reward is for generating spikes going downward, and as such, the tilting effect of the level lines increases exponentially as the height decreases.

Whereas the 2D Ising model with a pinning potential is known [11] to have an interface converging to a Ferrari–Spohn diffusion, the behavior in the SOS model – where there are $H \asymp \log L$ interacting level lines, each constrained not to cross its neighbors and inducing a tilt which is a function of the area it encompasses and its height – is far from being understood (see the review [13] for more information).

In this work, we investigate the limiting law of a line ensemble that was studied by Caputo, Ioffe and Wachtel [2,3] to model the level lines of the SOS model in the presence of a hard wall: each level line, $X_1, X_2, \ldots$, where X_1 is the top one, is tilted by the area below it, with the coefficients of these area tilts increasing geometrically.

For more perspective on this model in the context of other models of Brownian polymers constrained above a barrier, starting from the influential work of Ferrari and Spohn [8] (the model there being equivalent by Girsanov's transformation – cf. [17] – to a Brownian excursion with an area tilt), see, e.g., [2,12,14] and the references therein.

Define

$$\mathbb{A}_n^+ = \{\underline{x} \in \mathbb{R}^n : x_1 > x_2 > \cdots > x_n > 0\},$$

its closure $\bar{\mathbb{A}}_n^+$ and, for a designated interval

$$I = [\ell, r] \quad (\ell < r \in \mathbb{R}),$$

let

$$\Omega_n^I = \{X \in \mathsf{C}(I; \mathbb{R}^n) : X(t) \in \mathbb{A}_n^+ \text{ for all } t \in I\}.$$

(Here, for $T \subset \mathbb{R}$ and $\mathcal{X}$ a topological space, we denote by $\mathsf{C}(T; \mathcal{X})$ the space of continuous functions from T to $\mathcal{X}$, equipped with the topology of uniform convergence on compact subsets of $\mathbb{R}$.) Further define the area tilt of $Y \in \mathsf{C}(I; \mathbb{R})$ to be

$$\mathcal{A}_I(Y) = \int_I Y(t)\,\mathrm{d}t,$$

and, for given tilt parameters $\mathfrak{a} > 0$ and $\mathfrak{b} > 1$ and endpoints $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{A}_n^+$ and $\underline{y} = (y_1, \ldots, y_n) \in \mathbb{A}_n^+$, the partition function

$$Z_n^{\underline{x},\underline{y},I} = \mathbf{E}_n^{\underline{x},\underline{y},I}\big[\mathbb{1}_{\Omega_n^I} e^{-\mathfrak{a}\sum_{i=1}^n \mathfrak{b}^{i-1}\mathcal{A}_I(X_i(\cdot))}\big], \tag{1.1}$$

in which $\mathbf{E}_n^{\underline{x},\underline{y},I} = \bigotimes_{i=1}^n \mathbf{E}_1^{x_i,y_i,I}$ and the expectation $\mathbf{E}_1^{x,y,I}$ for $I = [\ell, r]$ is w.r.t. the (unnormalized) path measures of the Brownian bridge which starts at x at time ℓ and ends at y at time r; that is, the total mass of $\mathbf{E}_1^{x_i,y_i,I}$ is $\phi_{r-\ell}(y_i - x_i)$, where hereafter $\|\cdot\|$ stands for the relevant Euclidean norm, with

$$\phi_v(\underline{x}) := (2\pi v)^{-k/2} e^{-\|\underline{x}\|^2/(2v)} \tag{1.2}$$

denoting the density of a centered Gaussian vector of independent coordinates of variance v, whose dimension k is implicitly given by the argument we use. (At no point in our analysis will we need to adjust the tilt parameters $(\mathfrak{a}, \mathfrak{b})$, and as such we do not include them in the notation for brevity.)

Let $\mathcal{B}_n = \mathcal{B}_{n,I}$ be the Borel σ-field on $\mathsf{C}(I, \mathbb{R}^n)$. (We omit I from the notation when no confusion occurs.) For $\Gamma \in \mathcal{B}_{n,I}$ define

$$\mathbb{P}_n^{\underline{x},\underline{y},I}(\Gamma) := \frac{1}{Z_n^{\underline{x},\underline{y},I}} \mathbf{E}_n^{\underline{x},\underline{y},I}\big[\mathbb{1}_\Gamma \mathbb{1}_{\Omega_n^I} e^{-\mathfrak{a}\sum_{i=1}^n \mathfrak{b}^{i-1}\mathcal{A}_I(X_i(\cdot))}\big]. \tag{1.3}$$

Consider $I_T = [-T, T]$. Two classes of boundary conditions that are of interest are:

(a) *Zero boundary conditions*: fixing both $\underline{x}$ and $\underline{y}$ to be zero:

$$\mu_{n,T}^{\circ} = \mathbb{P}_n^{\underline{0},\underline{0},I_T}.$$

(More precisely, this is the limit of $\mathbb{P}_n^{\varepsilon\underline{x},\varepsilon\underline{y},I_T}$ as $\varepsilon \downarrow 0$, which by stochastic domination exists and is independent of the fixed $\underline{x}$, $\underline{y}$ in $\mathbb{A}_n^+$ which one uses.)

(b) *Free boundary conditions* with respect to a σ-finite measure: averaging $\mathbf{E}_n^{\underline{x},\underline{y},I_T}[\cdot]$ over $\underline{x}$, $\underline{y}$ according to a specified σ-finite measure Θ_n on $\mathbb{R}^n$:

$$\mu^{\mathfrak{f}}_{n,T}(\Gamma) = \frac{1}{\mathcal{Z}^{\mathfrak{f}}_{n,T}} \int_{\mathbb{A}_n^+} \int_{\mathbb{A}_n^+} \mathbf{E}_n^{\underline{x},\underline{y},I_T}\big[\mathbb{1}_\Gamma \mathbb{1}_{\Omega_n^{I_T}} e^{-\mathfrak{a}\sum_{i=1}^n \mathfrak{b}^{i-1}\mathcal{A}_{I_T}(X_i(\cdot))}\big]\Theta_n(\mathrm{d}\underline{x})\Theta_n(\mathrm{d}\underline{y}),$$

where

$$\mathcal{Z}^{\mathfrak{f}}_{n,T} := \int_{\mathbb{A}_n^+} \int_{\mathbb{A}_n^+} \mathbf{E}_n^{\underline{x},\underline{y},I_T}\big[\mathbb{1}_{\Omega_n^{I_T}} e^{-\mathfrak{a}\sum_{i=1}^n \mathfrak{b}^{i-1}\mathcal{A}_{I_T}(X_i(\cdot))}\big]\Theta_n(\mathrm{d}\underline{x})\Theta_n(\mathrm{d}\underline{y}).$$

We refer to such $\mu^{\mathfrak{f}}_{n,T}$ as Θ_n-free boundary conditions, reserving Leb-free for the special case of Lebesgue Θ_n, considered in [2,3]. Caputo, Ioffe and Wachtel show in [2,3] that $\mu^{\circ}_{n,T}$ converges to a limit μ°_{∞} as $n, T \to \infty$ (moreover, they proved that for any fixed n, the measures $\mu^{\circ}_{n,T}$ converge as $T \to \infty$ to a limit μ°_n, which then converges to μ°_{∞} as $n \to \infty$), and that for any $c > 0$, the family of Leb-free distributions $\{\mu^{\mathfrak{f}}_{n,T}\}_{n\geq 1, T>c}$ is tight. In this and subsequent statements, the measure μ°_{∞} is defined on $\mathsf{C}(\mathbb{R}, \mathbb{R}^{\mathbb{N}})$, and the convergence is in the sense that for any compact set $\mathcal{K} \subset \mathbb{R}$, integer $k \in \mathbb{N}$ and fixed bounded, continuous function $f : \mathsf{C}(\mathcal{K}; \mathbb{R}^k) \mapsto \mathbb{R}$, we have that

$$\lim_{n,T\to\infty} \int f(X_1, \dots, X_k)\mu^{\circ}_{n,T}(\mathrm{d}X) = \int f(X_1, \dots, X_k)\mu^{\circ}_{\infty}(\mathrm{d}X). \tag{1.4}$$

For Leb-free boundary conditions, Caputo, Ioffe and Wachtel conjectured that $\mu^{\mathfrak{f}}_{n,T}$ converges as well, and to the same limit μ°_{∞} as when $n, T \to \infty$. Our main result confirms that when $T \to \infty$ followed by $n \to \infty$, this holds more generally, whenever for $n \geq 1$,

$$\mathfrak{c}_n := \limsup_{r\to\infty} r^{-1} \log \Theta_n\big(\mathbb{A}_n^+ \cap \{x_1 \leq r\}\big) < \infty \tag{1.5}$$

(in particular, note that $\mathfrak{c}_n = 0$ when Θ_n is Lebesgue measure).

Theorem 1.1. *Assuming* (1.5), *for any fixed tilt parameters* $\mathfrak{a} > 0$ *and* $\mathfrak{b} > 1$ *and any fixed integer* n, *the measures* $\mu^{\mathfrak{f}}_{n,T}$ *and* $\mu^{\circ}_{n,T}$ *have the same weak limit as* $T \to \infty$. *In particular, if we denote by* μ°_{∞} *the limit of* $\mu^{\circ}_{n,T}$ *as* $n, T \to \infty$, *then*

$$\exists \lim_{n\to\infty} \lim_{T\to\infty} \mu^{\mathfrak{f}}_{n,T} = \mu^{\circ}_{\infty}.$$

Remark 1.2. Our proof easily extends to allow in $\mu^{\mathfrak{f}}_{n,T}$ different measures for $\underline{x}$ and for $\underline{y}$ (as long as both satisfy (1.5)). Note that Theorem 1.1 is optimal in terms of Θ_n, as merely having $\mathcal{Z}^{\mathfrak{f}}_{n,T}$ finite, requires that $\mathfrak{a}T \geq \mathfrak{c}_n$ (see (2.2) and (2.7) at $s = T$). Further, $\sup_n\{\mathfrak{c}_n\}$ must be finite if aiming to exchange the order of limits in n and T.

Our proof of Theorem 1.1 employs the Markovian structure of the problem. In a first step we introduce a (sub-) Markovian Kernel K_t, see (2.1). The key part of the proof is Lemma 2.1, where we prove that K_1 is compact in the appropriate L^2 space; the proof of the lemma involves probabilistic arguments. With the lemma, standard contraction arguments, detailed in Section 2.1, yield the exponential decay (in T) of the dependence in the boundary conditions. We note that some care is needed here due to the non-compactness of the set of possible boundary conditions, but that non-compactness was already handled in [2].

Many interesting open questions remain, chief among which, perhaps, is describing the limiting process $X_\infty(\cdot)$ (on, say, the interval $[0, 1]$). We refer to [2,3] for a list of such problems and note in passing that from (2.13) and PDE theory, one can verify that the limit of $\mu^{\mathfrak{f}}_{n,T}$ when $T \to \infty$, is the stationary solution of the Langevin SDE for invariant probability density φ_1^2 on $\mathbb{A}_n^+$ (where the Perron-Frobenius eigenvector φ_1 of K_1 is the positive C^2-solution of the elliptic PDE $\frac{1}{2}\Delta u = (c + \mathfrak{a}\langle \underline{\mathfrak{b}}, \underline{x}\rangle)u$ with Dirichlet boundary conditions at $\partial\mathbb{A}_n^+$ and the largest possible $c < 0$).

2. Proof of main result

Fix the tilt parameters $\mathfrak{a} > 0$ and $\mathfrak{b} > 1$, and let $n \geq 1$ be an integer. Throughout this proof, for $X \in \Omega_n^I$, we use the abbreviated notation

$$\mathcal{A}_I\big(X(\cdot)\big) := \mathfrak{a}\sum_{i=1}^n \mathfrak{b}^{i-1}\mathcal{A}_I\big(X_i(\cdot)\big).$$

Let $\Gamma \in \mathcal{B}_{n,[0,1]}$, and define

$$K_1^\Gamma(\underline{x},\underline{y}) = \mathbf{E}_n^{\underline{x},\underline{y},[0,1]}\big[\mathbb{1}_\Gamma \mathbb{1}_{\Omega_n^{[0,1]}} e^{-\mathcal{A}_{[0,1]}(X(\cdot))}\big],$$

which we view as a linear operator on $L^2(\mathbb{A}_n^+) = L^2(\mathbb{A}_n^+, \mathrm{Leb})$:

$$\big(K_1^\Gamma f\big)(\underline{x}) = \int_{\mathbb{A}_n^+} K_1^\Gamma(\underline{x},\underline{y}) f(\underline{y})\,\mathrm{d}\underline{y}.$$

With a slight abuse of notation, we continue to write K_1^Γ also when $\Gamma \in \mathcal{B}_{n,\mathbb{R}}$, in which case we understand that Γ was replaced by its restriction to the interval $[0, 1]$. With this convention in mind, we will further be interested in the semigroup

$$K_t^\Gamma(\underline{x},\underline{y}) = \mathbf{E}_n^{\underline{x},\underline{y},[0,t]}\big[\mathbb{1}_\Gamma \mathbb{1}_{\Omega_n^{[0,t]}} e^{-\mathcal{A}_{[0,t]}(X(\cdot))}\big]. \tag{2.1}$$

When referring to the case $\Gamma = \Omega_n^{\mathbb{R}}$ (i.e., the indicator $\mathbb{1}_\Gamma$ within the expectation in the definition of K_1^Γ is omitted), we simply write K_t (with no superscript) in lieu of $K_t^{\Omega_n^{\mathbb{R}}}$, noting that $K_t(\underline{x},\underline{y})$ is precisely the partition function $Z_n^{\underline{x},\underline{y},[0,t]}$ from (1.1).

Observe that K_t is symmetric, in that $K_t(\underline{x},\underline{y}) = K_t(\underline{y},\underline{x})$, as well as positivity preserving:

$$(K_t f)(\underline{x}) = \int K_t(\underline{x},\underline{y}) f(\underline{y})\,\mathrm{d}\underline{y} \geq 0 \quad \text{whenever } f \geq 0.$$

As K_t is symmetric, and given by a continuous time Markov process with killing, it is positive definite (this follows, e.g., by [9, Theorems 1.3.1, Lemma 1.3.2 and Theorem 6.1.1], all applied to Example 1.2.3 there). A key ingredient in the proof will be that K_1 is furthermore relatively compact:

Lemma 2.1. *For every fixed n, the range of the symmetric positive definite operator*

$$(K_1 f)(\underline{x}) = \int_{\mathbb{A}_n^+} \mathbf{E}_n^{\underline{x},\underline{y},[0,1]}\big[\mathbb{1}_{\Omega_n^{[0,1]}} e^{-\mathcal{A}_{[0,1]}(X(\cdot))}\big] f(\underline{y})\,\mathrm{d}\underline{y}$$

consists of continuous functions on $\mathbb{A}_n^+$, and the operator K_1 is compact w.r.t. $L^2(\mathbb{A}_n^+)$.

2.1. *Proof of Theorem 1.1 modulo Lemma 2.1*

We consider throughout the convergence over the interval $[0, 1]$, the changes needed for considering other compact sets (as the set $\mathcal{K}$ in (1.4)) are minimal. Aiming to express the measures $\mu_{n,T}^{\circ}$ and $\mu_{n,T}^{\mathfrak{f}}$ in terms of the operator K_t, we define for $s > 0$

$$\psi_s(\underline{u}) = \int K_s(\underline{u},\underline{x})\Theta_n(\mathrm{d}\underline{x}).$$

Setting $s > 0$ large enough so that $\psi_s \in L^2(\mathbb{A}_n^+)$, in view of the symmetry of K_s and the semigroup property, our goal is then to show that for every $\Gamma \in \mathcal{B}_{n,[0,1]}$, the limit of

$$\mu_{n,T}^{\mathfrak{f}}(\Gamma) = \frac{\iint \psi_s(\underline{x}) K_{T-s}(\underline{x},\underline{u}) K_1^\Gamma(\underline{u},\underline{v}) K_{T-1-s}(\underline{v},\underline{y}) \psi_s(\underline{y})\,\mathrm{d}\underline{u}\,\mathrm{d}\underline{v}\,\mathrm{d}\underline{x}\,\mathrm{d}\underline{y}}{\iint \psi_s(\underline{x}) K_{2T-2s}(\underline{x},\underline{y}) \psi_s(\underline{y})\,\mathrm{d}\underline{x}\,\mathrm{d}\underline{y}} \tag{2.2}$$

as $T \to \infty$ exists and coincides with that of

$$\mu_{n,T}^{\circ}(\Gamma) = \lim_{\varepsilon \downarrow 0} \frac{\iint K_T(\varepsilon\underline{x},\underline{u}) K_1^\Gamma(\underline{u},\underline{v}) K_{T-1}(\varepsilon\underline{y},\underline{v})\,\mathrm{d}\underline{u}\,\mathrm{d}\underline{v}}{K_{2T}(\varepsilon\underline{x},\varepsilon\underline{y})}, \tag{2.3}$$

where by [3, Lemma 2.2], the limit (2.3) exists and is independent of $\underline{x}$, $\underline{y}$ in $\mathbb{A}_n^+$.

By the spectral decomposition theorem, the compact positive definite operator K_1 has a discrete spectrum (except for a possible accumulation point at 0), with positive eigenvalues $\{\lambda_i\}$ and eigenvectors $\{\varphi_i\}$ that form a complete orthonormal basis of $L^2(\mathbb{A}_n^+)$ (see, e.g., [18, Thms. VI.15 and VI.16]). In particular,

$$K_1(\underline{x},\underline{y}) = \sum_{i=1}^\infty \lambda_i \varphi_i(\underline{x})\varphi_i(\underline{y}) \quad \text{for a complete basis } \{\varphi_i\}_{i\geq 1} \text{ with } \langle\varphi_i,\varphi_j\rangle_{L^2(\mathbb{A}_n^+)} = \delta_{ij}.$$

With $K_1(\underline{x}, \underline{y}) > 0$ throughout $\mathbb{A}_n^+ \times \mathbb{A}_n^+$ (e.g., due to parabolic regularity), by the generalized Perron-Frobenius Theorem (see, e.g., the version of the Krein-Rutman Theorem given in [9, Thm. XIII.43]), the top eigenvalue λ_1 has a one-dimensional eigen-space and we may choose the continuous function φ_1 to be strictly positive on $\mathbb{A}_n^+$. That is,

$$\varphi_1 > 0 \quad \text{and} \quad \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq 0.$$

Further, for any $r \geq 1$ and $\underline{x}, \underline{y} \in \mathbb{A}_n^+$,

$$\begin{aligned} K_r(\underline{x}, \underline{y}) &= \mathbf{E}_n^{\underline{x},\underline{y},[0,r]}\big[\mathbb{1}_{\Omega_n^{[0,r]}} e^{-\mathcal{A}_{[0,r]}(X(\cdot))}\big] \leq \mathbf{E}_n^{\underline{x},\underline{y},[0,r]}\big[\mathbb{1}_{\Omega_n^{[0,r]}} e^{-\mathfrak{a}\mathcal{A}_{[0,r]}(X_1(\cdot))}\big] \\ &\leq \mathbf{E}_1^{x_1,y_1,[0,r]}\big[e^{-\mathfrak{a}\mathcal{A}_{[0,r]}(X_1(\cdot))}\big] \\ &= \phi_r(y_1 - x_1) e^{-\mathfrak{a}r\frac{x_1+y_1}{2}} \mathbb{E}\big[e^{-\mathfrak{a}\int_0^r B_s\,\mathrm{d}s}\big] \\ &\leq e^{C_r} e^{-\mathfrak{a}r\frac{x_1+y_1}{2}}, \end{aligned} \tag{2.4}$$

where $\{B_s, s \in [0, r]\}$ is the standard Brownian bridge over $[0, r]$ starting and ending at 0 and $C_r = \frac{\mathfrak{a}^2}{2}\mathbb{E}(\int_0^r B_s\,\mathrm{d}s)^2$ (using in the second line that the total mass of $\mathbf{E}_1^{x_i,y_i,[0,r]}$, $i \geq 2$, is at most one, while for the third line recall that a Brownian bridge between fixed points has the law of the standard bridge plus a straight line connecting these points). Since $K_r(\underline{x}, \underline{y})$ vanishes if either $\underline{x} \notin \mathbb{A}_n^+$ or $\underline{y} \notin \mathbb{A}_n^+$, it follows that

$$\iint K_r(\underline{x}, \underline{y})\,\mathrm{d}\underline{x}\,\mathrm{d}\underline{y} \leq e^{C_r}\bigg[\int_0^\infty x_1^{n-1} e^{-\mathfrak{a}rx_1/2}\,\mathrm{d}x_1\bigg]^2 < \infty, \tag{2.5}$$

and

$$\int K_r(\underline{x}, \underline{x})\,\mathrm{d}\underline{x} \leq e^{C_r}\int_0^\infty x_1^{n-1} e^{-\mathfrak{a}rx_1}\,\mathrm{d}x_1 < \infty. \tag{2.6}$$

Similarly, by the symmetry of K_s, the semigroup property and (2.4),

$$\int \psi_s(\underline{u})^2\,\mathrm{d}\underline{u} = \iint K_{2s}(\underline{x}, \underline{y})\Theta_n(\mathrm{d}\underline{x})\Theta_n(\mathrm{d}\underline{y}) \leq e^{C_{2s}}\bigg[\int_{\mathbb{A}_n^+} e^{-\mathfrak{a}sx_1}\Theta_n(\mathrm{d}\underline{x})\bigg]^2 < \infty, \tag{2.7}$$

provided that $\mathfrak{a}s > \mathfrak{c}_n$ of (1.5), in which case we can decompose

$$\psi_s = \sum_{i=1}^\infty \alpha_{i,s}\varphi_i \quad \text{where } \alpha_{i,s} := \langle \psi_s, \varphi_i\rangle_{L^2(\mathbb{A}_n^+)}, \quad \sum_{i=1}^\infty \alpha_{i,s}^2 = \|\psi_s\|_2^2. \tag{2.8}$$

(Hereafter $\|\cdot\|_2$ denotes the $L^2(\mathbb{R}^n, \mathrm{Leb})$-norm, using $\|\cdot\|_{L^2(\mathbb{A}_n^+)}$ when restricting the domain to $\mathbb{A}_n^+$.) Fixing an integer $\ell > \mathfrak{c}_n/\mathfrak{a}$, we have from (2.7) that $\{\psi_s\}_{s\in[\ell,\ell+1]}$ is bounded in $L^2(\mathbb{A}_n^+)$, and we split any $T \geq \ell + 1$ as $T = t + s$ for $s \in [\ell, \ell + 1)$ and integer $t \geq 1$, to get from the decomposition (2.8) that

$$\int K_{t-1}(\underline{u}, \underline{y})\psi_s(\underline{y})\,\mathrm{d}\underline{y} = \sum_{i=1}^\infty \lambda_i^{t-1}\alpha_{i,s}\varphi_i(\underline{u}).$$

Similarly, for $t \geq 1$,

$$\iint K_t(\underline{x}, \underline{y})\psi_s(\underline{x})\psi_s(\underline{y})\,\mathrm{d}\underline{x}\,\mathrm{d}\underline{y} = \sum_{i=1}^\infty \lambda_i^t \alpha_{i,s}^2 := c_{t,s}.$$

Hence, (2.2) translates for $s = \ell + \{T\}$ and $t = T - s$, into

$$\mu_{n,T}^{\mathfrak{f}}(\Gamma) = \frac{1}{c_{2t,s}}\iint \sum_{i=1}^\infty\sum_{j=1}^\infty \alpha_{i,s}\alpha_{j,s}\lambda_i^t\lambda_j^{t-1}\varphi_i(\underline{u})K_1^\Gamma(\underline{u}, \underline{v})\varphi_j(\underline{v})\,\mathrm{d}\underline{u}\,\mathrm{d}\underline{v}. \tag{2.9}$$

Looking at K_1^Γ and arguing as we did for K_1, we see that for any $\underline{u} \in \mathbb{R}^n$,

$$\int K_1^\Gamma(\underline{u}, \underline{v})^2 \, \mathrm{d}\underline{v} \le \int K_1(\underline{u}, \underline{v})^2 \, \mathrm{d}\underline{v} = K_2(\underline{u}, \underline{u}) < \infty,$$

where the equality holds by the symmetry of K_1 and the definition of K_t and the last inequality by (2.4). In other words, $K_1^\Gamma(\underline{u}, \cdot) \in L^2(\mathbb{A}_n^+)$ for every $\underline{u} \in \mathbb{R}^n$. Moreover, by (2.6) we have that

$$\iint K_1^\Gamma(\underline{u}, \underline{v})^2 \, \mathrm{d}\underline{u} \, \mathrm{d}\underline{v} \le \iint K_1(\underline{u}, \underline{v})^2 \, \mathrm{d}\underline{u} \, \mathrm{d}\underline{v} = \int K_2(\underline{u}, \underline{u}) \, \mathrm{d}\underline{u} < \infty$$

and it follows that

$$K_1^\Gamma \in L^2\big(\mathbb{A}_n^+ \times \mathbb{A}_n^+\big).$$

A complete orthonormal system $\{\varphi_i\}$ w.r.t. $L^2(\mathbb{A}_n^+)$ induces a complete orthonormal system $\{\varphi_i \otimes \varphi_j\}_{i,j\ge 1}$ w.r.t. $L^2(\mathbb{A}_n^+ \times \mathbb{A}_n^+)$; hence, we may decompose K_1^Γ into

$$K_1^\Gamma(\underline{u}, \underline{v}) = \sum_{i,j} \gamma_{i,j} \varphi_i(\underline{u}) \varphi_j(\underline{v})$$

where

$$\gamma_{i,j} := \iint K_1^\Gamma(\underline{u}, \underline{v}) \varphi_i(\underline{u}) \varphi_j(\underline{v}) \, \mathrm{d}\underline{u} \, \mathrm{d}\underline{v}, \qquad \sum_{i,j\ge 1} \gamma_{i,j}^2 = \big\| K_1^\Gamma \big\|^2_{L^2(\mathbb{A}_n^+ \times \mathbb{A}_n^+)} < \infty.$$

This reduces (2.9) into $\mu^{\mathfrak{f}}_{n,T}(\Gamma) = \Xi^{(1)}_{n,T} / \Xi^{(2)}_{n,T}$ where

$$\Xi^{(1)}_{n,T} := \sum_{i,j\ge 1} \gamma_{i,j} \alpha_{i,s} \alpha_{j,s} \widehat{\lambda}_i^t \widehat{\lambda}_j^{t-1}, \qquad \Xi^{(2)}_{n,T} := \lambda_1 \sum_{i=1}^\infty \widehat{\lambda}_i^{2t} \alpha_{i,s}^2, \tag{2.10}$$

and the rescaled eigenvalues $\widehat{\lambda}_i := \lambda_i / \lambda_1 \in [0, 1]$ $(i = 1, 2, \ldots)$ satisfy

$$\widehat{\lambda}_i = 1 \quad \text{and} \quad \sup_{i>1} \widehat{\lambda}_i \le 1 - \delta \quad \text{for } \delta = (\lambda_1 - \lambda_2)/\lambda_1 > 0.$$

We immediately see that $\Xi^{(2)}_{n,T}$ of (2.10) satisfies

$$\lambda_1 \alpha_{1,s}^2 \le \Xi^{(2)}_{n,T} \le \lambda_1 \alpha_{1,s}^2 + \lambda_1 (1-\delta)^{2t} \|\psi_s\|_2^2. \tag{2.11}$$

Further, with $K_s \varphi_1 = \lambda_1^s \varphi_1$ continuous and positive on $\mathbb{A}_n^+$, for any non-zero Θ_n,

$$\alpha_{1,s} = \int_{\mathbb{A}_n^+} (K_s \varphi_1)(\underline{x}) \Theta_n(\mathrm{d}\underline{x}) = \lambda_1^s \int_{\mathbb{A}_n^+} \varphi_1(\underline{x}) \Theta_n(\mathrm{d}\underline{x})$$

is bounded away from zero, uniformly over $s \le \ell + 1$. Consequently,

$$\lim_{T\to\infty} \frac{\Xi^{(2)}_{n,T}}{\alpha_{1,s}^2} = \lambda_1.$$

To treat $\Xi^{(1)}_{n,T}$ of (2.10), note that by Cauchy–Schwarz and having $\sup_{i\ge 2} |\widehat{\lambda}_i| \le 1 - \delta$,

$$\begin{aligned} \Bigg| \sum_{\substack{i,j\ge 1 \\ i+j>2}} \gamma_{i,j} \alpha_{i,s} \alpha_{j,s} \widehat{\lambda}_i^t \widehat{\lambda}_j^{t-1} \Bigg| &\le (1-\delta)^{t-1} \sum_{i,j} |\gamma_{i,j} \alpha_{i,s} \alpha_{j,s}| \\ &\le (1-\delta)^{t-1} \sqrt{\sum_{i,j} \gamma_{i,j}^2} \sqrt{\sum_{i,j} \alpha_{i,s}^2 \alpha_{j,s}^2} \\ &= (1-\delta)^{t-1} \big\| K_1^\Gamma \big\|_{L^2(\mathbb{A}_n^+ \times \mathbb{A}_n^+)} \|\psi_s\|_2^2. \end{aligned} \tag{2.12}$$

Taking $T \to \infty$, we see that

$$\lim_{T\to\infty} \frac{\Xi^{(1)}_{n,T}}{\alpha^2_{1,s}} = \gamma_{1,1}.$$

Altogether, we have thus established that

$$\lim_{T\to\infty} \mu^{\mathfrak{f}}_{n,T}(\Gamma) = \frac{\gamma_{1,1}}{\lambda_1}. \tag{2.13}$$

We now repeat the same analysis for $\mu^{\circ}_{n,T}$, where since the limit as $T \to \infty$ exists, we assume hereafter that T is integer (and set $s = \ell = 1$). Further, for simplicity we opt to take $\underline{y} = \underline{x}$ and let $\psi^{(\varepsilon)}(\underline{u}) := K_1(\varepsilon\underline{x}, \underline{u})$. Inferring that $\psi^{(\varepsilon)} \in L^2(\mathbb{A}^+_n)$ (because $K_2(\varepsilon\underline{x}, \varepsilon\underline{x}) < \infty$), we can write

$$\psi^{(\varepsilon)} = \sum_{i=1}^{\infty} \alpha^{(\varepsilon)}_i \varphi_i,$$

where

$$\alpha^{(\varepsilon)}_i := \langle \psi^{(\varepsilon)}, \varphi_i \rangle_{L^2(\mathbb{A}^+_n)}, \qquad \|\psi^{(\varepsilon)}\|^2_{L^2(\mathbb{A}^+_n)} = K_2(\varepsilon\underline{x}, \varepsilon\underline{x}) = \sum_{i=1}^{\infty} \big(\alpha^{(\varepsilon)}_i\big)^2 < \infty.$$

The exact same argument then shows that $\mu^{\circ}_{n,T}(\Gamma)$ is the limit at $\varepsilon \to 0$ of $\Xi^{(1,\varepsilon)}_{n,T} / \Xi^{(2,\varepsilon)}_{n,T}$ where

$$\Xi^{(1,\varepsilon)}_{n,T} := \sum_{i,j\ge 1} \gamma_{i,j} \alpha^{(\varepsilon)}_i \alpha^{(\varepsilon)}_j \widehat{\lambda}_i^{T-1} \widehat{\lambda}_j^{T-2}, \qquad \Xi^{(2,\varepsilon)}_{n,T} := \lambda_1 \sum_{i=1}^{\infty} \widehat{\lambda}_i^{2T-2} \big(\alpha^{(\varepsilon)}_i\big)^2. \tag{2.14}$$

With $\psi^{(\varepsilon)} > 0$ and $\varphi_1 > 0$, we have as before that $\alpha^{(\varepsilon)}_1 > 0$. Moreover, setting

$$\kappa_\varepsilon := \frac{\|\psi^{(\varepsilon)}\|^2_{L^2(\mathbb{A}^+_n)}}{(\alpha^{(\varepsilon)}_1)^2},$$

we have analogously to (2.11) and (2.12) that

$$0 \le \frac{\Xi^{(2,\varepsilon)}_{n,T}}{(\alpha^{(\varepsilon)}_1)^2} - \lambda_1 \le \lambda_1 (1-\delta)^{2T-2} \kappa_\varepsilon,$$

$$\left| \frac{\Xi^{(1,\varepsilon)}_{n,T}}{(\alpha^{(\varepsilon)}_1)^2} - \gamma_{1,1} \right| \le (1-\delta)^{T-2} \big\| K^{\Gamma}_1 \big\|_{L^2(\mathbb{A}^+_n \times \mathbb{A}^+_n)} \kappa_\varepsilon.$$

We shall employ the following asymptotic as $\varepsilon \to 0$, the proof of which we defer to Section 2.3.

Lemma 2.2. *Setting* $\underline{n} := (2n-1, 2n-3, \dots, 1)$, *we have that*

$$\limsup_{\varepsilon\to 0} \frac{K_2(\varepsilon\underline{n}, \varepsilon\underline{n})}{(\int_{u_1\le 1} K_1(\varepsilon\underline{n}, \underline{u})\varphi_1(\underline{u})\, d\underline{u})^2} < \infty. \tag{2.15}$$

Since K_1 and φ_1 are both positive, (2.15) applies also without the restriction to $u_1 \le 1$, with Lemma 2.2 yielding that κ_ε is uniformly bounded (as $\varepsilon \to 0$), when $\underline{x} = \underline{n}$. Hence, thanks to our freedom to choose the boundary, we have that

$$\lim_{T\to\infty} \mu^{\circ}_{n,T}(\Gamma) = \frac{\gamma_{1,1}}{\lambda_1}, \tag{2.16}$$

which in light of (2.13) concludes our proof of Theorem 1.1, modulo the proofs of Lemmas 2.1 and 2.2.

2.2. *Proof of Lemma* 2.1

Letting

$$\mathfrak{B}_0 = \{ f : \|f\|_{L^2(\mathbb{A}_n^+)} \le 1 \} \quad \text{and}$$
$$\mathfrak{B}_1 = \{ (K_1 f) : f \in \mathfrak{B}_0 \},$$

we will establish compactness by verifying the Fréchet–Kolmogorov criteria (see [20, p. 275], as well as [19]).

First, with $\mathbb{P}$ denoting the law of Brownian motion $\{W(t)\}_{t\in[0,1]}$ in $\mathbb{R}^n$ started at the origin and $\mathbb{E}$ its corresponding expectation, note that

$$(K_1 f)(\underline{x}) = \mathbb{E}\big[\mathbb{1}_{\Omega_n^{[0,1]}}\big(\underline{x} + W(\cdot)\big) e^{-\mathcal{A}_{[0,1]}(\underline{x}+W(\cdot))} f\big(\underline{x} + W(1)\big)\big]. \tag{2.17}$$

Now, setting for f supported on $\mathbb{A}_n^+$,

$$M(f) := \sup_{\underline{x}\in\mathbb{A}_n^+} \mathbb{E}\big[\big|f\big(\underline{x} + W(1)\big)\big|\big], \tag{2.18}$$

note that by Cauchy–Schwarz,

$$M(f)^2 \le \sup_{\underline{x}\in\mathbb{A}_n^+} \mathbb{E}\big[f\big(\underline{x} + W(1)\big)^2\big] \le \|f\|^2_{L^2(\mathbb{A}_n^+)} \sup_{\underline{x},\underline{y}\in\mathbb{A}_n^+} \big\{\phi_1(\underline{y} - \underline{x})\big\} \le 1, \tag{2.19}$$

where $\phi_v(\cdot)$ denotes the density in (1.2) and the last inequality holds for all $f \in \mathfrak{B}_0$.

This readily implies the following uniform bound on $g = K_1 f \in \mathfrak{B}_1$, where by a computation similar to the third line of (2.4), for any $\underline{x} \in \mathbb{A}_n^+$,

$$\big|g(\underline{x})\big| \le \int_{\mathbb{A}_n^+} \mathbf{E}_n^{\underline{x},\underline{y},[0,1]}\big[e^{-\mathfrak{a}\int_0^1 X_1(s)\,ds}\big]\big|f(\underline{y})\big| \mathrm{d}\underline{y} \le ce^{-\frac{\mathfrak{a}}{2}x_1} M(f) \le ce^{-\frac{\mathfrak{a}}{2}x_1}, \tag{2.20}$$

for some finite $c = c(\mathfrak{a})$, independent of $\underline{x}$ and $f \in \mathfrak{B}_0$. We deduce in particular that

$$\limsup_{R\to\infty} \sup_{g\in\mathfrak{B}_1} \int_{\substack{\underline{x}\in\mathbb{A}_n^+\\ x_1>R}} \big|g(\underline{x})\big|^2\, d\underline{x} = 0, \tag{2.21}$$

establishing equitightness (and, due to (2.20), also uniform boundedness, although it is not needed in view of [19]).

It remains to establish equicontinuity for $\mathfrak{B}_1$, where in view of (2.21) and the compactness of $\bar{\mathbb{A}}_n^+ \cap \{x_1 \le R\}$ it suffices to bound, in terms of $\|\underline{h}\|$, the value of

$$\sup_{g\in\mathfrak{B}_1,\underline{x}\in\mathbb{A}_n^+,\underline{x}+\underline{h}\in\mathbb{A}_n^+} \big\{\big|g(\underline{x}+\underline{h}) - g(\underline{x})\big|\big\}.$$

Using the representation (2.17) for $g = K_1 f$, we start by reducing to $\tilde{g}(\cdot)$ in which we extracted out the explicit dependence of the area tilt on $\underline{x}$. Specifically, let

$$\tilde{g}(\underline{x}) := \mathbb{E}\big[\mathbb{1}_{\Omega_n^{[0,1]}}\big(\underline{x} + W(\cdot)\big) e^{-\mathcal{A}_{[0,1]}(W(\cdot))} f\big(\underline{x} + W(1)\big)\big].$$

By a slight abuse of notation, letting $\mathcal{A}_{[0,1]}(\underline{x})$ denote $\mathcal{A}_{[0,1]}(X(\cdot))$ for $X \equiv \underline{x}$, which is nothing but $\mathfrak{a}\langle\underline{\mathfrak{b}}, \underline{x}\rangle$ for $\underline{\mathfrak{b}} := (1, \mathfrak{b}, \ldots, \mathfrak{b}^{n-1})$, we see that

$$g(\underline{x}) = e^{-\mathcal{A}_{[0,1]}(\underline{x})}\tilde{g}(\underline{x}),$$

and therefore,

$$\begin{aligned}\big|g(\underline{x}) - g(\underline{x}+\underline{h})\big| &= \big|e^{-\mathcal{A}_{[0,1]}(\underline{x})}\big(\tilde{g}(\underline{x}) - e^{-\mathcal{A}_{[0,1]}(\underline{h})}\tilde{g}(\underline{x}+\underline{h})\big)\big| \\ &\le \big|e^{\mathcal{A}_{[0,1]}(\underline{h})} - 1\big|\big|g(\underline{x}+\underline{h})\big| + e^{-\mathcal{A}_{[0,1]}(\underline{x})}\big|\tilde{g}(\underline{x}) - \tilde{g}(\underline{x}+\underline{h})\big|.\end{aligned}$$

For the first term note that $|\mathcal{A}_{[0,1]}(\underline{h})| = |\mathfrak{a}\langle\underline{\mathfrak{b}}, \underline{h}\rangle| \le \mathfrak{a}\|\underline{\mathfrak{b}}\|\|\underline{h}\|$ and though $\underline{h}$ may be outside $\mathbb{A}_n^+$, by Taylor expansion and (2.20) we have that for any $\|\underline{h}\| \le 1$,

$$\sup_{g\in\mathfrak{B}_1, \underline{x}+\underline{h}\in\mathbb{A}_n^+} \left|e^{\mathcal{A}_{[0,1]}(\underline{h})} - 1\right|\left|g(\underline{x}+\underline{h})\right| \le C(\mathfrak{a}, \mathfrak{b}, n)\|\underline{h}\|.$$

Further, with $\mathcal{A}_{[0,1]}(\underline{x}) \ge 0$ for all $\underline{x} \in \mathbb{A}_n^+$, it remains only to bound $|\tilde{g}(\underline{x}) - \tilde{g}(\underline{y})|$ uniformly over $g \in \mathfrak{B}_1$, $\underline{x} \in \mathbb{A}_n^+$ and $\underline{y} \in \mathbb{A}_n^+$ such that $\|\underline{y} - \underline{x}\| \le \delta$. To this end, let

$$\tau_{\underline{x}} := \inf\left\{t \ge 0 : \underline{x} + W(t) \notin \mathbb{A}_n^+\right\}, \quad \text{so that} \quad \mathbb{1}_{\Omega_n^{[0,1]}}\big(\underline{x} + W(\cdot)\big) = \mathbb{1}_{\{\tau_{\underline{x}}>1\}}.$$

We then have in terms of

$$\Delta(\underline{x}, \underline{y}) := \mathbb{1}_{\{\tau_{\underline{y}}>1\}} f\big(W(1) + \underline{y}\big) - \mathbb{1}_{\{\tau_{\underline{x}}>1\}} f\big(W(1) + \underline{x}\big)$$

and $\eta \in (0, 1)$, that

$$\left|\tilde{g}(\underline{y}) - \tilde{g}(\underline{x})\right| = \left|\mathbb{E}\left[e^{-\mathcal{A}_{[0,1]}(W(\cdot))}\Delta(\underline{x}, \underline{y})\right]\right| \le \mathbb{E}\left[|\Psi_1|\right] + \left|\mathbb{E}[\Psi_2]\right|,$$

where

$$\Psi_1 := e^{-\mathcal{A}^*_{[0,1-\eta]}(W(\cdot))}\left(e^{-\mathcal{A}_{[1-\eta,1]}(W(\cdot)-W(1-\eta))} - 1\right)\Delta(\underline{x}, \underline{y}),$$
$$\Psi_2 := e^{-\mathcal{A}^*_{[0,1-\eta]}(W(\cdot))}\Delta(\underline{x}, \underline{y}),$$

and

$$\mathcal{A}^*_{[0,1-\eta]}\big(W(\cdot)\big) = \mathcal{A}_{[0,1-\eta]}\big(W(\cdot)\big) + \mathcal{A}_{[1-\eta,1]}\big(W(1-\eta)\big).$$

To bound $\mathbb{E}|\Psi_1|$, use the fact that $|\Delta(\underline{x}, \underline{y})| \le |f(W(1) + \underline{y})| + |f(W(1) + \underline{x})|$ together with Hölder's inequality to infer that $\mathbb{E}|\Psi_1|$ is at most

$$\mathbb{E}\left[e^{-4\mathcal{A}^*_{[0,1-\eta]}(W(\cdot))}\right]^{\frac{1}{4}}\mathbb{E}\left[\left|e^{-\mathcal{A}_{[1-\eta,1]}(W(\cdot)-W(1-\eta))} - 1\right|^4\right]^{\frac{1}{4}}\Big(2\sup_{\underline{x}\in\mathbb{A}_n^+}\mathbb{E}\left[f\big(W(1)+\underline{x}\big)^2\right]\Big)^{\frac{1}{2}}.$$

Noting that the variance of the centered Gaussian $\mathcal{A}^*_{[0,1-\eta]}(W(\cdot))$ is at most some $v = v(\mathfrak{a}, \mathfrak{b}, n)$ finite, the first expectation above is uniformly bounded (namely, by e^{8v}). Similarly, by (2.19), the third term is at most $\sqrt{2}$, uniformly over $f \in \mathfrak{B}_0$. Finally, with $\mathcal{A}_{[1-\eta,1]}(W(\cdot) - W(1-\eta))$ a centered Gaussian of variance $c(\mathfrak{a}, \mathfrak{b}, n)\eta^2$ for some finite $c(\mathfrak{a}, \mathfrak{b}, n)$, the expectation in the second term is at most $\varepsilon_0(\eta) \to 0$ as $\eta \to 0$. Overall, we conclude that

$$\mathbb{E}|\Psi_1| \le \varepsilon_1(\eta) \downarrow 0 \quad \text{as} \quad \eta \downarrow 0, \quad \text{uniformly over} \quad g \in \mathfrak{B}_1, \underline{x} \in \mathbb{A}_n^+. \tag{2.22}$$

Turning to $\Psi_2 = e^{-\mathcal{A}^*_{[0,1-\eta]}(W(\cdot))}\Delta(\underline{x}, \underline{y})$, utilizing the identity

$$\mathbb{1}_{\{\tau_{\underline{y}}>1\}} = 1 - \mathbb{1}_{\{\tau_{\underline{y}}\le 1-\eta, \tau_{\underline{x}}\le 1-\eta\}} - \mathbb{1}_{\{1-\eta<\tau_{\underline{y}}\le 1\}} - \mathbb{1}_{\{\tau_{\underline{y}}\le 1-\eta, 1-\eta<\tau_{\underline{x}}\le 1\}} - \mathbb{1}_{\{\tau_{\underline{y}}\le 1-\eta, \tau_{\underline{x}}>1\}},$$

and its dual where the roles of $\tau_{\underline{y}}$ and $\tau_{\underline{x}}$ have been exchanged, yields the decomposition

$$\Delta(\underline{x}, \underline{y}) = \Upsilon_1 - \Upsilon_2(\underline{y}) - \Upsilon_3(\underline{y}, \underline{x}) - \Upsilon_4(\underline{y}, \underline{x}) + \Upsilon_2(\underline{x}) + \Upsilon_3(\underline{x}, \underline{y}) + \Upsilon_4(\underline{x}, \underline{y}),$$

where

$$\Upsilon_1 := \left[f\big(W(1) + \underline{y}\big) - f\big(W(1) + \underline{x}\big)\right](1 - \mathbb{1}_{\{\tau_{\underline{x}}\le 1-\eta, \tau_{\underline{y}}\le 1-\eta\}}),$$
$$\Upsilon_2(\underline{y}) := f\big(W(1) + \underline{y}\big)\mathbb{1}_{\{1-\eta<\tau_{\underline{y}}\le 1\}},$$
$$\Upsilon_3(\underline{y}, \underline{x}) := f\big(W(1) + \underline{y}\big)\mathbb{1}_{\{\tau_{\underline{y}}\le 1-\eta, 1-\eta<\tau_{\underline{x}}\le 1\}},$$
$$\Upsilon_4(\underline{y}, \underline{x}) := f\big(W(1) + \underline{y}\big)\mathbb{1}_{\{\tau_{\underline{y}}\le 1-\eta, \tau_{\underline{x}}>1\}}.$$

For the contribution to $|\mathbb{E}[\Psi_2]|$ due to Υ_1, condition on $\mathcal{F}_{1-\eta} = \sigma(\{W(s)\}_{s\le 1-\eta})$, on which the indicator in Υ_1 is measurable, to get

$$\begin{aligned}\big|\mathbb{E}\big[\Upsilon_1 e^{-\mathcal{A}^*_{[0,1-\eta]}(W(\cdot))}\big]\big| \le \mathbb{E}\big[e^{-\mathcal{A}^*_{[0,1-\eta]}W(\cdot))}\big]& \\ \cdot \sup_{\underline{z}}\big|\mathbb{E}\big[f\big(W(1)+\underline{y}\big) - f\big(W(1)+\underline{x}\big) | W(1-\eta) = \underline{z}\big]\big|.&\end{aligned}$$

While treating $\mathbb{E}|\Psi_1|$, we saw that the first term on the right-hand is some finite $C(\mathfrak{a},\mathfrak{b},n)$, independently of $\underline{x}$, $\underline{h}$. For the second term, extending $f\in\mathfrak{B}_0$ from $\mathbb{A}_n^+$ to $\mathbb{R}^n$ via $f(\underline{x})=0$ for $\underline{x}\notin\mathbb{A}_n^+$, yields that $\|f\|_2 = \|f\|_{L^2(\mathbb{A}_n^+)}\le 1$. Thus, performing a change of variable $\underline{v} := W(1)+\underline{y}$ in $\mathbb{E}[f(W(1)+\underline{y}) \mid W(1-\eta)=\underline{z}]$ and $\underline{v} := W(1)+\underline{x}$ in $\mathbb{E}[f(W(1)+\underline{x}) \mid W(1-\eta)=\underline{z}]$, we get that the absolute difference between these expectations is

$$\begin{aligned}&\left|\int\big[\phi_\eta(\underline{v}-\underline{y}-\underline{z}) - \phi_\eta(\underline{v}-\underline{x}-\underline{z})\big]f(\underline{v})\,\mathrm{d}\underline{v}\right| \\ &\quad\le \|f\|_{L^2(\mathbb{A}_n^+)}\eta^{-n/4}\big\|\phi_1\big(\underline{w}-\eta^{-1/2}(\underline{y}-\underline{x})\big)-\phi_1(\underline{w})\big\|_2 \le C(n)\eta^{-n/4-1/2}\delta,\end{aligned}$$

where the first inequality is obtained by Cauchy–Schwarz and an additional change of variable $\underline{w}=\eta^{-1/2}(\underline{v}-\underline{z}-\underline{x})$, and the second inequality by an easy computation (utilizing that $1-e^{-r}\le r$). Thus, choosing

$$\delta \le \eta^{n/4+1}, \tag{2.23}$$

makes the contribution of Υ_1 negligible.

To deal with the contribution of $\Upsilon_2(\underline{y})$ to $|\mathbb{E}[\Psi_2]|$, observe that by Hölder's inequality,

$$\begin{aligned}&\mathbb{E}\big[e^{-\mathcal{A}^*_{[0,1-\eta]}(W(\cdot))}\mathbb{1}_{\{1-\eta<\tau_{\underline{y}}\le 1\}}\big|f\big(W(1)+\underline{y}\big)\big|\big] \\ &\quad\le \mathbb{E}\big[e^{-4\mathcal{A}^*_{[0,1-\eta]}(W(\cdot))}\big]^{\frac14}\mathbb{P}(1-\eta<\tau_{\underline{y}}\le 1)^{\frac14}\Big(\sup_{\underline{y}\in\mathbb{A}_n^+}\mathbb{E}\big[f\big(W(1)+\underline{y}\big)^2\big]\Big)^{\frac12}.\end{aligned}\tag{2.24}$$

While bounding $\mathbb{E}|\Psi_1|$ we have seen that the first and third terms are at most some $c(\mathfrak{a},\mathfrak{b},n)$ finite, uniformly over $\mathfrak{B}_0$, so it suffices to show that

$$\varepsilon_2(\eta) := \sup_{\underline{y}\in\mathbb{A}_n^+}\big\{\mathbb{P}(1-\eta<\tau_{\underline{y}}\le 1)\big\}\to 0 \quad \text{as } \eta\to 0. \tag{2.25}$$

Indeed, taking a union bound over the n different boundaries of $\mathbb{A}_n^+$ that are considered in $\tau_{\underline{y}}$, reduces, up to the factor n, to the bound in case $n=1$, namely for the first hitting time T_b of level $-b<0$ by a standard Brownian motion B_t. The corresponding probability density $f_{T_b}(t) = be^{-b^2/(2t)}/\sqrt{2\pi t^3}$ is bounded, uniformly over b and $t\ge 1/2$, thereby yielding (2.25).

The same analysis applies to the contributions from the Υ_3 terms.

Analogously to (2.24) the contribution of $\Upsilon_4(\underline{y},\underline{x})$ to $|\mathbb{E}[\Psi_2]|$ is bounded above by

$$\begin{aligned}&\mathbb{E}\big[e^{-\mathcal{A}^*_{[0,1-\eta]}(W(\cdot))}\mathbb{1}_{\{\tau_{\underline{y}}\le 1-\eta,\tau_{\underline{x}}>1\}}\big|f\big(W(1)+\underline{y}\big)\big|\big] \\ &\quad\le C\|f\|_{L^2(\mathbb{A}_n^+)}\mathbb{P}(\tau_{\underline{y}}\le 1-\eta,\tau_{\underline{x}}>1)^{1/4}\le C\varepsilon_3(\delta,\eta)^{1/4},\end{aligned}$$

for some $C=C(\mathfrak{a},\mathfrak{b},n)$, any $f\in\mathfrak{B}_0$ and

$$\varepsilon_3(\delta,\eta) := \sup_{\underline{x},\underline{y}\in\mathbb{A}_n^+,\|\underline{x}-\underline{y}\|\le\delta}\mathbb{P}(\tau_{\underline{y}}\le 1-\eta,\tau_{\underline{x}}>1).$$

With the same bound applying for $\Upsilon_4(\underline{x},\underline{y})$, it remains only to show that $\varepsilon_3(\delta,\eta)\to 0$ as $\delta\to 0$ (for any fixed $\eta>0$). To this end, by a union bound over the n different boundaries of $\mathbb{A}_n^+$, as done for proving (2.25), the probability in question is at most n times the probability that standard Brownian motion $B(t) := (W_i(t)-W_{i+1}(t))/\sqrt{2}$ reaches level $-b$ by time

$1-\eta$ (here $b=(y_i-y_{i+1})/\sqrt{2}$), while remaining above $-(b+\delta)$ up till time 1. With Brownian motion a strong Markov process of independent increments, we thus deduce by the reflection principle that

$$n^{-1}\varepsilon_3(\delta,\eta)\le \mathbb{P}\Big(\inf_{s\le\eta}\{B(s)\}>-\delta\Big)=1-2\mathbb{P}\big(B(\eta)\ge\delta\big)=\mathbb{P}\big(\big|B(\eta)\big|<\delta\big),$$

which goes to zero as $\delta\to 0$ (for any fixed $\eta>0$). This completes the proof of Lemma 2.1.

2.3. *Proof of Lemma* 2.2

Setting $\widehat{K}_t$ for the operator K_t in the case $\mathfrak{a}=0$ (no area tilt), we first establish (2.15) for $\widehat{K}_t$. Namely, we show that,

$$\limsup_{\varepsilon\to 0}\frac{\widehat{K}_2(\varepsilon\underline{n},\varepsilon\underline{n})}{(\int_{u_1\le 1}\widehat{K}_1(\varepsilon\underline{n},\underline{u})\varphi_1(\underline{u})\,d\underline{u})^2}<\infty. \tag{2.26}$$

Our starting point for (2.26) is the following explicit formula, valid for any $\underline{y}\in\mathbb{A}_n^+$ and any $t,\varepsilon>0$,

$$\widehat{K}_t(\varepsilon\underline{n},\underline{y})=2^{n^2}\phi_t(\underline{y})e^{-\varepsilon^2\|\underline{n}\|^2/(2t)}\prod_i\sinh\left(\frac{\varepsilon y_i}{t}\right)\prod_{j<k}\left[\sinh^2\left(\frac{\varepsilon y_j}{t}\right)-\sinh^2\left(\frac{\varepsilon y_k}{t}\right)\right]. \tag{2.27}$$

Indeed, for $\varepsilon=1$ this is the explicit evaluation in [10, Display below (24)] of the Karlin–McGregor determinantal formula [15] for the transition kernel,

$$q_t(x,y)=\phi_t(y-x)-\phi_t(y+x)=2\phi_t(y)e^{-x^2/(2t)}\sinh(xy/t),$$

of a scalar Brownian motion absorbed at level zero, when starting at the distinguished point $\underline{n}$. We thus get (2.27) by noting that the non-trivial factors $\sinh(x_iy_j/t)$ are invariant to changing from $(\varepsilon\underline{n},\underline{y})$ to $(\underline{n},\varepsilon\underline{y})$.

In particular, with $g(x):=\sinh(x/2)$ being zero at $x=0$ and globally Lipschitz(L) on $[0,2n]$, we get from (2.27) that for some c_n, C_n finite and any $\varepsilon\in[0,1]$,

$$\begin{aligned}\widehat{K}_2(\varepsilon\underline{n},\varepsilon\underline{n})&\le c_n\prod_i g\big(\varepsilon^2n_i\big)\prod_{j<k}\big[g^2\big(\varepsilon^2n_j\big)-g^2\big(\varepsilon^2n_k\big)\big]\\&\le c_nL^{n^2}\prod_i\big(\varepsilon^2n_i\big)\prod_{j<k}\big[\big(\varepsilon^2n_j\big)^2-\big(\varepsilon^2n_k\big)^2\big]=C_n\varepsilon^{2n^2}.\end{aligned} \tag{2.28}$$

Next, noting that on $\mathbb{R}_+$ both $\sinh(x)\ge x$ and $\sinh^2(x)-x^2$ are non-decreasing, we deduce from (2.27) that for any $\underline{u}\in\mathbb{A}_n^+$ and $\varepsilon\in[0,1]$,

$$\widehat{K}_1(\varepsilon\underline{n},\underline{u})\ge 2^{n^2}e^{-\|\underline{n}\|^2/2}\varepsilon^{n^2}\hat{\phi}(\underline{u}),\quad\text{where }\hat{\phi}(\underline{u}):=\phi_1(\underline{u})\prod_i u_i\prod_{j<k}\big(u_j^2-u_k^2\big).$$

With $\hat{\phi}(\cdot)$ and $\varphi_1(\cdot)$ positive on $\mathbb{A}_n^+$, we get from the latter bound that

$$\inf_{\varepsilon\in[0,1]}\varepsilon^{-n^2}\int_{u_1\le 1}\widehat{K}_1(\varepsilon\underline{n},\underline{u})\varphi_1(\underline{u})\,d\underline{u}>0,$$

which in combination with (2.28) establishes (2.26).

Next, recall that $K_t(\underline{x},\underline{y})$ is point-wise decreasing in $\mathfrak{a}$ and in particular bounded from above by $\widehat{K}_t(\underline{x},\underline{y})$; thus, the sought bound (2.15) for K_t follows from (2.26) once we show that for some finite $C=C(\mathfrak{a},\mathfrak{b},n)$ and any $\underline{u}\in\mathbb{A}_n^+$ with $u_1\le 1$,

$$\sup_{\varepsilon\in(0,1]}\left\{\frac{\widehat{K}_1(\varepsilon\underline{n},\underline{u})}{K_1(\varepsilon\underline{n},\underline{u})}\right\}\le C. \tag{2.29}$$

Turning to the latter bound, we define for finite M the event

$$\Gamma_M:=\Big\{\max_{t\in[0,1]}\{X_1(t)\}\le M\Big\},$$

noting that for $c := \mathfrak{a}\langle \underline{\mathfrak{b}}, \underline{1}\rangle$, any $u_1 \le 1$ and $\varepsilon \le 1$,

$$K_1(\varepsilon\underline{n}, \underline{u}) \ge e^{-cM} \mathbf{E}_n^{\varepsilon\underline{n},\underline{u},[0,1]}[\mathbb{1}_{\Gamma_M} \mathbb{1}_{\Omega_n^{[0,1]}}] = e^{-cM} \widehat{K}_1(\varepsilon\underline{n}, \underline{u}) \widehat{\mathbb{P}}_n^{\varepsilon\underline{n},\underline{u},[0,1]}(\Gamma_M)$$
$$\ge e^{-cM} \widehat{K}_1(\varepsilon\underline{n}, \underline{u}) \widehat{\mathbb{P}}_n^{\underline{n},\underline{n},[0,1]}(\Gamma_M),$$

where $\widehat{\mathbb{P}}_n^{\underline{x},\underline{y},[0,1]}$ is the measure $\mathbb{P}_n^{\underline{x},\underline{y},[0,1]}$ from (1.3) corresponding to $\mathfrak{a} = 0$, and with the second inequality due to [7, Lemma 2.7] (taking there $A = [0, 1]$, $f \equiv 0$, noting that $\underline{n} > \underline{u}$ and $\underline{n} > \varepsilon\underline{n}$ whenever $u_1 \le 1$ and $\varepsilon \le 1$ and that the event Γ_M is decreasing).

Finally, moving to the unconditional space of n independent bridges rooted at $\underline{n}$, $\underline{n}$ via a multiplicative cost of at most $1/\widehat{K}_1(\underline{n}, \underline{n})$, we see that $\widehat{\mathbb{P}}_n^{\underline{n},\underline{n},[0,1]}(\Gamma_M^c)$ is at most $\mathbb{P}(\sup_{s\in[0,1]}\{B(s)\} > M - 2n)/\widehat{K}_1(\underline{n}, \underline{n})$ for a one dimensional Brownian bridge from $(0, 1)$ to $(1, 1)$. By the tightness of the maximum of the latter bridge (and recalling that $\widehat{K}_1(\underline{n}, \underline{n}) > 0$), one thus has for M large, depending only on n, that

$$\widehat{\mathbb{P}}_n^{\underline{n},\underline{n},[0,1]}(\Gamma_M) \ge \frac{1}{2}.$$

Combining the last two displays yields (2.29), thereby completing the proof of Lemma 2.2.

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