Electronic Journal of Statistics
Vol. 15 (2021) 3249–3286
ISSN: 1935-7524
https://doi.org/10.1214/21-EJS1860

An inverse norm weight spatial sign test for high-dimensional directional data*

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Abstract: In this paper, we focus on the high-dimensional location testing problem of directional data under the assumption of rotationally symmetric distributions, where the data dimension is potentially much larger than the sample size. We study the family of directional weighted spatial sign tests for this testing problem and establish the asymptotic null distributions and local power properties of this family. In particular, we find that the test based on the inverse norm weight, named as the inverse norm weight spatial sign test, has the maximum asymptotic power in this family. As demonstrated by extensive numerical results, the inverse norm weight spatial sign test has advantages in empirical power compared with some other members in the family as well as some existing tests.

MSC2020 subject classifications: 62G10.
Keywords and phrases: Directional data, high dimension, location test, rotationally symmetric distribution, weighted spatial sign.

Received December 2020.

1. Introduction

Directional data have been widely studied in meteorology [7], astronomy [2], earth science [22] and biology [10]. These fields naturally yield a large number of directional data, such as the wind direction data and the earth scale spatial

*This work was supported by NSFC grants 11501092, 11571068, 11671073, 11671-178, the Fundamental Research Funds for the Central Universities grant 241201-7BJ002, the Key Laboratory of Applied Statistics of MOE (KLAS) grants 130026507 and 130028612.

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data, which are commonly considered as an implementation of a random vector on a hypersphere. Besides, some data from other fields are essential only for their relative size, hence being projected onto a unit hypersphere, where the projection removes the overall scale factor associated with the size. For analyzing directional data, the family of rotationally symmetric distributions is one of the most commonly used distributions, which has played a central role in the application of directional statistics [15]. One prominent member of the family of rotationally symmetric distributions is the Fisher-von Mises-Langevin (FvML) distribution, which plays a critical role in directional statistics, on par with that of the Gaussian distribution in the classic multivariate setting.

The literature on the inference of location parameters under the assumption of rotationally symmetric distributions is abundant. For example, [23], [3], [11], [13] and [19] considered the location inference problems in low-dimensional situations. [21] and [5] tackled the problem for spherical regression, while [1] considered the location testing problem under axial frames. [8] considered the problem of testing rotational symmetry on hyperspheres, and introduced two locally asymptotically maximin tests against two classes of directional distributions in the Le Cam sense.

In this paper, we focus on the high-dimensional location testing problem of directional data under the assumption of rotationally symmetric distributions, where the null hypothesis is that the location parameter vector is equal to a given vector on the unit hypersphere. To deal with such a testing problem, [23] proposed a traditional Watson test based on the mean of the spherical sample. [17] proposed a class of rank tests and discussed the Le Cam optimality of the tests. Then, [12] proposed a high-dimensional Watson test by standardizing the traditional Watson statistic.

To develop some alternative high-dimensional tools for testing location under rotationally symmetric distributions, in this paper we study the family of directional weighted spatial sign tests for testing location of directional data under rotationally symmetric distributions. In particular, we find that the test based on the inverse norm weight, named as the inverse norm weight spatial sign test, has the maximum asymptotic power in this family. Indeed, the proposed inverse norm weight spatial sign test is an extension of the inverse norm sign test devised for general data. We then present its asymptotic properties under the unified framework of directional weighted spatial sign tests, and demonstrate its empirical power advantages via extensive numerical results.

The rest of the paper is organized as follows. In Section 2, we introduce the family of directional weighted spatial sign tests for high-dimensional directional data under rotationally symmetric distributions. Besides, we establish the corresponding theoretical results, including the limiting null distributions, the asymptotic power under the local alternative and the asymptotic relative efficiency results in Section 3. Then, we investigate the numerical performance of the proposed test in comparison with its main competitors in Section 4. Finally, we conclude the paper with some discussions in Section 5, and relegate some additional numerical results as well as the technical proofs to the appendix.

2. Test statistics

Let $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n \in \mathcal{S}^{p-1} \doteq \{\mathbf{x} \in \mathbb{R}^p : \|\mathbf{x}\| = 1\}$ be a sequence of p-dimensional independent and identically distributed (iid) observations from a rotationally symmetric distribution with spherical location $\boldsymbol{\theta} \in \mathcal{S}^{p-1}$. We recall that a random vector $\boldsymbol{X}_i$ is said to be rotationally symmetric about some location $\boldsymbol{\theta} \in \mathcal{S}^{p-1}$ if its distribution is invariant under rotations about $\boldsymbol{\theta}$, i.e. if $\boldsymbol{O}\boldsymbol{X}_i$ has the same distribution as $\boldsymbol{X}_i$ for any orthogonal $p \times p$ matrix $\boldsymbol{O}$ satisfying $\boldsymbol{O}\boldsymbol{\theta} = \boldsymbol{\theta}$. In particular, $\boldsymbol{X}_i$ has the density of $c_{f,p} f(\mathbf{x}^{\mathrm{T}}\boldsymbol{\theta})$ with $\mathbf{x} \in \mathcal{S}^{p-1}$, where $f : [-1, 1] \to \mathbb{R}^+$ is an absolutely continuous function, called the angular function, and $c_{f,p}$ is the standardization constant. For example, the FvML distribution, one of the most popular members of the rotationally symmetric distributions, is obtained by taking the angular function $t \to \exp(\kappa t)$, where $t \in [-1, 1]$ and $\kappa \geq 0$ is the concentration parameter.

Our interest is to test the location hypotheses

$$H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0 \quad \text{versus} \quad H_1 : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0, \tag{2.1}$$

for some given location $\boldsymbol{\theta}_0 \in \mathcal{S}^{p-1}$. To test (2.1), we study the family of directional weighted spatial sign tests, where the test statistics are constructed as

$$\begin{aligned} T_n(\omega) \doteq & \frac{2}{n(n-1)} \sum\sum_{i<j} \omega(\|(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\|)\omega(\|(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_j\|) \\ & \times \mathbf{U}\{(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\}^{\mathrm{T}}\mathbf{U}\{(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_j\}. \end{aligned} \tag{2.2}$$

Here, $\omega(\cdot)$ is a nonnegative and continuous weight function on $\mathbb{R}^+$, $\boldsymbol{I}_p$ denotes the $p \times p$ identity matrix and $\mathbf{U}(\cdot)$ denotes the spatial sign function with $\mathbf{U}(\boldsymbol{a}) \doteq \boldsymbol{a}/\|\boldsymbol{a}\|$ if $\boldsymbol{a} \neq \mathbf{0}$ and $\mathbf{U}(\boldsymbol{a}) \doteq \mathbf{0}$ if $\boldsymbol{a} = \mathbf{0}$. Define

$$\widehat{\sigma}_n^2(\omega) \doteq 2n^{-4}p^{-1} \sum\sum_{i \neq j} \omega(\|(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\|)^2 \omega(\|(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_j\|)^2. \tag{2.3}$$

According to Theorems 3.1 and 3.2 presented in the next section, for each ω, H_0 will be rejected when $T_n(\omega)/\sqrt{\widehat{\sigma}_n^2(\omega)} > z_\alpha$, where z_α is the upper α-quantile of the standard normal distribution $\mathcal{N}(0, 1)$ with significance level α. We will call this test the $T_n(\omega)$-based test.

In fact, some members of this family are closely related to existing tests for (2.1). For example, taking $\omega(t) = \omega_{\mathrm{N}}(t) \doteq t$ for $t \geq 0$, i.e. using the norm (N) weight function ω_{N}, the test statistic is

$$\begin{aligned} T_n(\omega_{\mathrm{N}}) = & \frac{2}{n(n-1)} \sum\sum_{i<j} \|(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\| \|(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_j\| \\ & \times \mathbf{U}\{(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\}^{\mathrm{T}}\mathbf{U}\{(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_j\}. \end{aligned} \tag{2.4}$$

The main part of $T_n(\omega_{\mathrm{N}})$ is the same as that of the high-dimensional Watson test statistic proposed by [12]:

$$\widetilde{W}_n \doteq \frac{\sqrt{2(p-1)}}{\sum_{i=1}^n v_{i0}^2} \sum\sum_{i<j} \|(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\| \|(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_j\|$$

$$\times\, \mathbf{U}\{(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\}^{\mathrm{T}}\mathbf{U}\{(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_j\},$$

where $v_{i0} \doteq \|(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\|$.

The relationship between the two statistics is discussed in the following proposition. To present this relationship, we need to impose the following condition.

(C0) $\mathbb{E}(v_i^4)/\mathbb{E}^2(v_i^2) = O(1)$, $\mathbb{E}(v_i^2) \neq 0$ and for each $i \in \{1\cdots,n\}$, $\mathbb{E}\{v_i^{-4}\}$ exists for sufficiently large p.

Here, $v_i \doteq \|(\mathbf{I}_p - \boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}})\boldsymbol{X}_i\| = (1-u_i^2)^{1/2}$, where $u_i \doteq \boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{\theta}$ and $\boldsymbol{X}_i$ is rotationally symmetric about $\boldsymbol{\theta}$ with concentration parameter κ. The probability density function of u_i is

$$c_{p,f,\kappa}\left(1-u^2\right)^{(p-3)/2} f_\kappa(u),$$

where $u \in [-1,1]$ and $c_{p,f,\kappa} \doteq 1/\int_{-1}^{1}\left(1-u^2\right)^{(p-3)/2} f_\kappa(u)\,du$. It can be seen that the distributions of u_i and v_i do not depend on $\boldsymbol{\theta}$.

Proposition 2.1. *As n, $p \to \infty$, under condition (C0) and H_0,*

$$\widetilde{W}_n / \frac{T_n(\omega_{\mathrm{N}})}{\sqrt{\widehat{\sigma}_n^2(\omega_N)}} \to 1 \text{ in probability.}$$

This proposition indicates that the $T_n(\omega_{\mathrm{N}})$-based and $\widetilde{W}_n$-based tests are asymptotically equivalent, where the $\widetilde{W}_n$-based test rejects H_0 when $\widetilde{W}_n > z_\alpha$. We recall that all proofs are discussed in the appendix.

In fact, $\widetilde{W}_n$ is a standardized version of the traditional Watson test statistic [23, 20]

$$W_n \doteq \frac{n(p-1)\bar{\boldsymbol{X}}_n^{\mathrm{T}}(\boldsymbol{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\bar{\boldsymbol{X}}_n}{1-\frac{1}{n}\sum_{i=1}^n(\boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{\theta}_0)^2},$$

with asymptotic mean $p-1$ and asymptotic variance $2(p-1)$ under the null hypothesis, where $\bar{\boldsymbol{X}}_n \doteq \frac{1}{n}\sum_{i=1}^n \boldsymbol{X}_i$. The W_n-based test rejects H_0 when W_n is larger than the upper α-quantile of χ^2_{p-1}.

Taking $\omega(t) = \omega_{\mathrm{C}}(t) \doteq 1$ for $t \geq 0$, i.e. using the constant (C) weight function ω_{C}, the test statistic is

$$T_n(\omega_{\mathrm{C}}) = \frac{2}{n(n-1)}\sum\sum_{i<j}\mathbf{U}\{(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\}^{\mathrm{T}}\mathbf{U}\{(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_j\}, \tag{2.5}$$

which is very similar to the test that was mentioned in [18].

In addition to these existing ones, we also consider some other members of this family. Taking $\omega(t) = \omega_{\mathrm{IN}}(t) \doteq t^{-1}$ for $t \geq 0$, i.e. using the inverse norm (IN) weight function ω_{IN}, the test statistic is

$$\begin{aligned} T_n(\omega_{\mathrm{IN}}) = &\frac{2}{n(n-1)}\sum\sum_{i<j}\|(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\|^{-1}\|(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_j\|^{-1} \\ &\times \mathbf{U}\{(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\}^{\mathrm{T}}\mathbf{U}\{(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_j\}, \end{aligned} \tag{2.6}$$

which can be regarded as an extension of the inverse norm sign test (INST) proposed by [6] on directional data. In this paper, we name the $T_n(\omega_{\mathrm{IN}})$-based test as the inverse norm weight spatial sign test.

Taking $\omega(t) = \omega_{\mathrm{S}}(t) \doteq t^2$ and $\omega(t) = \omega_{\mathrm{R}}(t) \doteq t^{1/2}$ respectively, the test statistics are

$$\begin{aligned} T_n(\omega_{\mathrm{S}}) = & \frac{2}{n(n-1)} \sum\sum_{i<j} \|(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\|^2 \|(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_j\|^2 \\ & \times \mathbf{U}\{(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\}^{\mathrm{T}} \mathbf{U}\{(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_j\}, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} T_n(\omega_{\mathrm{R}}) = & \frac{2}{n(n-1)} \sum\sum_{i<j} \|(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\|^{1/2} \|(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_j\|^{1/2} \\ & \times \mathbf{U}\{(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\}^{\mathrm{T}} \mathbf{U}\{(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_j\}. \end{aligned} \tag{2.8}$$

Now, we have introduced the family of directional weighted spatial sign tests. In the following section, we will establish the theoretical results of the whole family, which imply that the $T_n(\omega_{\mathrm{IN}})$-based test has the maximum asymptotic power among this family. Then, extensive numerical results will demonstrate the empirical power advantages of the $T_n(\omega_{\mathrm{IN}})$-based test compared with some other members of this family as well as some existing tests.

3. Theoretical results

In this section, we will establish the theoretical results of the whole family of directional weighted spatial sign tests with the general form of weight function $\omega(\cdot)$.

3.1. Null distribution

(C1) $b_4(\omega) = O\{b_2(\omega)^2\}$, $b_2(\omega) \neq 0$ and for each $i \in \{1 \cdots, n\}$, $\mathbb{E}\{v_i^{-4}\}$ exists for sufficiently large p.

Here, $b_k(\omega) \doteq \mathbb{E}\{\omega^k(v_i)\}$ for any positive integer k. Note that by choosing $\omega = \omega_{\mathrm{N}}$, condition (C1) becomes condition (C0), which leads to $\mathbb{E}(v_i^4)/\mathbb{E}^2(v_i^2) = o(n)$, previously used in Theorem 3.1 (iv) of [12] and Theorem 3.1 (b) of [20].

Theorem 3.1. *Under condition (C1) and H_0, as n, $p \to \infty$, $T_n(\omega)/\sigma_n(\omega) \to \mathcal{N}(0,1)$ in distribution, where $\sigma_n^2(\omega) \doteq 2n^{-2}p^{-1}b_2^2(\omega)$.*

We find a ratio-consistent estimation of $\sigma_n^2(\omega)$ in the following theorem.

Theorem 3.2. *Under condition (C1) and H_0, as n, $p \to \infty$, $\widehat{\sigma}_n^2(\omega)/\sigma_n^2(\omega) \to 1$ in probability, where $\widehat{\sigma}_n^2(\omega)$ is defined in (2.3).*

3.2. Asymptotic power

Then, we investigate the asymptotic distribution of $T_n(\omega)$ under the following local alternative.

(C2) $\|\boldsymbol{\theta}-\boldsymbol{\theta}_0\|^2\sqrt{\mathbb{E}(v_i^{-4})} = O(n^{-1}p^{-1/2})$ and $\mathrm{var}(u_i) = o(1)$ for each $i \in \{1,\cdots,n\}$, where $u_i = \boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{\theta}$.

Note that $\mathrm{var}(u_i)=o(1)$ is a weaker condition than condition (c), i.e. $\sqrt{p}\mathbb{E}(u_i^2) = o(1)$, of Theorem 3.1 in [20] due to $\mathrm{var}(u_i)\le \mathbb{E}(u_i^2)$, which was used to derive the local power of $\widetilde{W}_n$. Condition (C2) ensures that the difference between $\boldsymbol{\theta}$ and $\boldsymbol{\theta}_0$ is not too large, so that the variance of $T_n(\omega)$ can be asymptotically described by $\sigma_n^2(\omega)$.

Theorem 3.3. *Under conditions (C1)–(C2), as n, $p\to\infty$,*

$$[T_n(\omega)-c_0(\omega)^2\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}]/\sigma_n(\omega)\to\mathcal{N}(0,1) \text{ in distribution,}$$

where $c_0(\omega)\doteq\mathbb{E}(u_i)\mathbb{E}\{\omega(v_i)v_i^{-1}\}$.

According to Theorems 3.1 and 3.3, the local power of the $T_n(\omega)$-based test against an alternative $\boldsymbol{\theta}$ that satisfies condition (C2) is

$$\begin{aligned}
&\beta_{n,p}(\omega)\\
\doteq&\mathrm{P}_{\boldsymbol{\theta}}\left[T_n(\omega)>\sigma_n(\omega)z_\alpha\right]\\
=&\mathrm{P}_{\boldsymbol{\theta}}\left[\frac{T_n(\omega)-c_0(\omega)^2\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}}{\sigma_n(\omega)}>z_\alpha-\frac{c_0(\omega)^2\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}}{\sigma_n(\omega)}\right].
\end{aligned}$$

Then, we obtain the asymptotic power of the $T_n(\omega)$-based test:

$$\begin{aligned}
&\beta(\omega)\\
\doteq&\lim_{n,p\to\infty}\beta_{n,p}(\omega)\\
=&\lim_{n,p\to\infty}\Phi\left[-z_\alpha+\frac{\mathbb{E}^2(u_i)\mathbb{E}^2\{\omega(v_i)v_i^{-1}\}}{\mathbb{E}\{\omega^2(v_i)\}}\frac{p^{1/2}n(\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta})}{\sqrt{2}}\right]. \qquad (3.1)
\end{aligned}$$

To find a weight function ω that can make $\beta(\omega)$ reach the maximum value, we only need to find the maximum value of $\mathbb{E}^2(u_i)\mathbb{E}^2\{\omega(v_i)v_i^{-1}\}/\mathbb{E}\{\omega^2(v_i)\}$, because $\Phi\left[-z_\alpha+\frac{\mathbb{E}^2(u_i)\mathbb{E}^2\{\omega(v_i)v_i^{-1}\}}{\mathbb{E}\{\omega^2(v_i)\}}\frac{p^{1/2}n(\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta})}{\sqrt{2}}\right]$ in (3.1) is an increasing function of $\mathbb{E}^2(u_i)\mathbb{E}^2\{\omega(v_i)v_i^{-1}\}/\mathbb{E}\{\omega^2(v_i)\}$. By proving $\mathbb{E}^2\{\omega(v_i)v_i^{-1}\}/\mathbb{E}\{\omega^2(v_i)\}\le \mathbb{E}({v_i}^{-2})$, we will obtain that taking $\omega=\omega_{\mathrm{IN}}$ makes $\beta(\omega)$ reach the maximum value.

Theorem 3.4. *For all the weight functions satisfying conditions (C1)–(C2), as n, $p\to\infty$, the maximum value of the asymptotic power $\beta(\omega)$ is*

$$\beta(\omega_{\mathrm{IN}})=\lim_{n,p\to\infty}\Phi\left[-z_\alpha+\mathbb{E}^2(u_i)\mathbb{E}({v_i}^{-2})\frac{p^{1/2}n\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}}{\sqrt{2}}\right].$$

In contrast, the asymptotic power of the $T_n(\omega_{\mathrm{C}})$-based, $T_n(\omega_{\mathrm{N}})$-based, $T_n(\omega_{\mathrm{S}})$-based and $T_n(\omega_{\mathrm{R}})$-based tests are

$$\beta(\omega_{\mathrm{N}}) = \lim_{n,p\to\infty} \Phi\left[-z_\alpha + \mathbb{E}^2(u_i)\mathbb{E}^{-1}(v_i^2)\frac{p^{1/2}n\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}}{\sqrt{2}}\right],$$
$$\beta(\omega_{\mathrm{C}}) = \lim_{n,p\to\infty} \Phi\left[-z_\alpha + \mathbb{E}^2(u_i)\mathbb{E}^{2}(v_i^{-1})\frac{p^{1/2}n\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}}{\sqrt{2}}\right],$$
$$\beta(\omega_{\mathrm{S}}) = \lim_{n,p\to\infty} \Phi\left[-z_\alpha + \mathbb{E}^2(u_i)\frac{\mathbb{E}^2(v_i)}{\mathbb{E}(v_i^4)}\frac{p^{1/2}n\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}}{\sqrt{2}}\right],$$
$$\beta(\omega_{\mathrm{R}}) = \lim_{n,p\to\infty} \Phi\left[-z_\alpha + \mathbb{E}^2(u_i)\frac{\mathbb{E}^2(v_i^{-1/2})}{\mathbb{E}(v_i)}\frac{p^{1/2}n\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}}{\sqrt{2}}\right],$$

respectively.

Next, we consider the asymptotic power of the $T_n(\omega)$-based test in situation of the FvML distribution.

Corollary 3.1. *Suppose $\boldsymbol{X}_i$ follows a FvML distribution. Under conditions (C1) and (C2), as $n, p \to \infty$, the asymptotic power of the $T_n(\omega)$-based test is*

$$\beta = \lim_{n,p\to\infty} \Phi\left[-z_\alpha + \mathbb{E}^2(u_i)\mathbb{E}(v_i^{-2})\frac{p^{1/2}n\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}}{\sqrt{2}}\right],$$

which does not depend on the choice of the weight function ω.

This corollary indicates that if $\boldsymbol{X}_i$ follows a FvML distribution, all the directional weighted spatial sign tests have the same asymptotic power.

3.3. Asymptotic relative efficiency

Then, we derive the asymptotic relative efficiencies (AREs) between the $T_n(\omega_{\mathrm{IN}})$-based test and the $T_n(\omega_{\mathrm{C}})$-based, $T_n(\omega_{\mathrm{N}})$-based, $T_n(\omega_{\mathrm{S}})$-based, $T_n(\omega_{\mathrm{R}})$-based tests, respectively. Specifically,

$$\mathrm{ARE}_{\mathrm{IN,N}} = \mathbb{E}(v_i^{-2})\mathbb{E}(v_i^2) \geq 1,$$
$$\mathrm{ARE}_{\mathrm{IN,C}} = \mathbb{E}(v_i^{-2})\mathbb{E}^{-2}(v_i^{-1}) = 1 + \mathrm{var}(v_i^{-1})\mathbb{E}^{-2}(v_i^{-1}) \geq 1,$$
$$\mathrm{ARE}_{\mathrm{IN,S}} = \mathbb{E}(v_i^{-2})\mathbb{E}(v_i^4)\mathbb{E}^{-2}(v_i) \geq 1,$$
$$\mathrm{ARE}_{\mathrm{IN,R}} = \mathbb{E}(v_i^{-2})\mathbb{E}(v_i)\mathbb{E}^{-2}(v_i^{-1/2}) \geq 1.$$

For rotationally symmetric distributions, there are many choices of angular functions. Below, we present the formulas for calculating the AREs under some commonly used angular functions. First, when $\boldsymbol{X}_i$ follows a rotationally symmetric distribution, the density function of $\boldsymbol{X}_i$ is $c_{p,f,\kappa} f_\kappa(\mathbf{x}^{\mathrm{T}}\theta)$, where $c_{p,f,\kappa} = 1/\int_{-1}^{1}(1-t^2)^{(p-3)/2} f_\kappa(t)dt$. Hence,

$$\mathrm{ARE}_{\mathrm{IN,N}} = \frac{c_{p,f,\kappa}^2}{c_{p-2,f,\kappa}c_{p+2,f,\kappa}},$$

$$\mathrm{ARE}_{\mathrm{IN,C}} = \frac{c^2_{p-1,f,\kappa}}{c_{p-2,f,\kappa}c_{p,f,\kappa}},$$

$$\mathrm{ARE}_{\mathrm{IN,S}} = \frac{c^2_{p+1,f,\kappa}}{c_{p-2,f,\kappa}c_{p+4,f,\kappa}},$$

$$\mathrm{ARE}_{\mathrm{IN,R}} = \frac{c^2_{p-\frac{1}{2},f,\kappa}}{c_{p-2,f,\kappa_1}c_{p+1,f,\kappa_1}}.$$

Next, when $\boldsymbol{X}_i$ follows the mixture of two rotationally symmetric distributions with the same location parameter, the density function of $\boldsymbol{X}_i$ can be denoted as

$$\lambda c_{p,f,\kappa_1} f_{\kappa_1}(\mathbf{x}^{\mathrm{T}}\boldsymbol{\theta}) + (1-\lambda) c_{p,f,\kappa_2} f_{\kappa_2}(\mathbf{x}^{\mathrm{T}}\boldsymbol{\theta}), \tag{3.2}$$

where $c_{p,f,\kappa} = 1/\int_{-1}^{1}(1-t^2)^{(p-3)/2} f_\kappa(t)dt$. Then, we have

$$\mathrm{ARE}_{\mathrm{IN,N}} = \left[\lambda\frac{c_{p,f,\kappa_1}}{c_{p-2,f,\kappa_1}} + (1-\lambda)\frac{c_{p,f,\kappa_2}}{c_{p-2,f,\kappa_2}}\right] \times \left[\lambda\frac{c_{p,f,\kappa_1}}{c_{p+2,f,\kappa_1}} + (1-\lambda)\frac{c_{p,f,\kappa_2}}{c_{p+2,f,\kappa_2}}\right],$$

$$\mathrm{ARE}_{\mathrm{IN,C}} = \left[\lambda\frac{c_{p,f,\kappa_1}}{c_{p-2,f,\kappa_1}} + (1-\lambda)\frac{c_{p,f,\kappa_2}}{c_{p-2,f,\kappa_2}}\right] \Big/ \left[\lambda\frac{c_{p,f,\kappa_1}}{c_{p-1,f,\kappa_1}} + (1-\lambda)\frac{c_{p,f,\kappa_2}}{c_{p-1,f,\kappa_2}}\right]^2,$$

$$\mathrm{ARE}_{\mathrm{IN,S}} = \left[\lambda\frac{c_{p,f,\kappa_1}}{c_{p-2,f,\kappa_1}} + (1-\lambda)\frac{c_{p,f,\kappa_2}}{c_{p-2,f,\kappa_2}}\right] \times \left[\lambda\frac{c_{p,f,\kappa_1}}{c_{p+4,f,\kappa_1}} + (1-\lambda)\frac{c_{p,f,\kappa_2}}{c_{p+4,f,\kappa_2}}\right] \Big/ \left[\lambda\frac{c_{p,f,\kappa_1}}{c_{p+1,f,\kappa_1}} + (1-\lambda)\frac{c_{p,f,\kappa_2}}{c_{p+1,f,\kappa_2}}\right]^2,$$

$$\mathrm{ARE}_{\mathrm{IN,R}} = \left[\lambda\frac{c_{p,f,\kappa_1}}{c_{p-2,f,\kappa_1}} + (1-\lambda)\frac{c_{p,f,\kappa_2}}{c_{p-2,f,\kappa_2}}\right] \times \left[\lambda\frac{c_{p,f,\kappa_1}}{c_{p+1,f,\kappa_1}} + (1-\lambda)\frac{c_{p,f,\kappa_2}}{c_{p+1,f,\kappa_2}}\right] \Big/ \left[\lambda\frac{c_{p,f,\kappa_1}}{c_{p-\frac{1}{2},f,\kappa_1}} + (1-\lambda)\frac{c_{p,f,\kappa_2}}{c_{p-\frac{1}{2},f,\kappa_2}}\right]^2.$$

In the following, we investigate the ARE results numerically under the non-mixed and mixed rotationally symmetric distributions respectively. Here, we consider the following three types of angular functions.

(F1) The FvML angular function $\exp(\kappa t)$ with $t \in [-1, 1]$.
(F2) The angular function $4^{-\kappa\cdot\arccos(t)}$ with $t \in [-1, 1]$.
(F3) The angular function $6^{-\kappa\cdot\arcsin(t)}$ with $t \in [-1, 1]$.

Table 1 presents the corresponding ARE results of the non-mixed rotationally symmetric distributions, where some different choices of κ and p are considered. Then, Tables 2, 3 and 4 present the corresponding ARE results of the mixed distributions for (F1)–(F3) respectively, where two settings are considered: (1) $\lambda = 0.9$ with $\kappa_1 = \kappa/10$ and $\kappa_2 = 10\kappa$; (2) $\lambda = 0.6$ with $\kappa_1 = \kappa/5$ and $\kappa_2 = 5\kappa$. Although the $T_n(\omega_{\mathrm{IN}})$-based test does not show obvious advantages in Table 1, it has obvious advantages as shown in Tables 2, 3 and 4, which indicates that the proposed $T_n(\omega_{\mathrm{IN}})$-based test performs better than the other four directional weighted spatial sign tests in situation of mixed distributions.

TABLE 1
The ARE results of the non-mixed rotationally symmetric distributions for the three types of angular functions.

p	κ	$\text{ARE}_{\text{IN,N}}$	$\text{ARE}_{\text{IN,C}}$	$\text{ARE}_{\text{IN,S}}$	$\text{ARE}_{\text{IN,R}}$
		(F1)			
100	p	1.011	1.003	1.026	1.006
200	p	1.006	1.001	1.013	1.003
300	p	1.004	1.001	1.008	1.002
400	p	1.003	1.001	1.006	1.002
100	p^2	1.021	1.005	1.046	1.012
200	p^2	1.010	1.003	1.023	1.006
300	p^2	1.007	1.002	1.015	1.004
400	p^2	1.005	1.001	1.011	1.003
		(F2)			
100	$2p$	1.037	1.009	1.083	1.021
150	$2p$	1.024	1.006	1.055	1.014
200	$2p$	1.018	1.005	1.041	1.010
250	$2p$	1.014	1.004	1.032	1.008
300	$2p$	1.012	1.003	1.027	1.007
100	$3p$	1.039	1.010	1.089	1.022
150	$3p$	1.026	1.006	1.059	1.014
200	$3p$	1.019	1.005	1.044	1.011
250	$3p$	1.015	1.004	1.035	1.009
300	$3p$	1.013	1.003	1.029	1.007
		(F3)			
100	$2p$	1.038	1.010	1.088	1.022
150	$2p$	1.025	1.006	1.057	1.014
200	$2p$	1.019	1.005	1.043	1.011
250	$2p$	1.015	1.004	1.034	1.008
300	$2p$	1.013	1.003	1.028	1.007
100	$3p$	1.040	1.010	1.091	1.022
150	$3p$	1.026	1.007	1.060	1.015
200	$3p$	1.020	1.005	1.045	1.011
250	$3p$	1.016	1.004	1.035	1.009
300	$3p$	1.013	1.003	1.029	1.007

3.4. Two rank-based tests

To make a more extensive comparison in the following simulation studies, in this subsection we consider two rank-based weights mentioned in [16]. The statistics are

$$R_W = \frac{2}{n(n-1)} \sum\sum_{i<j} R_i R_j \mathbf{U}\{(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\}^{\mathrm{T}} \mathbf{U}\{(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_j\}$$

and

$$R_S = \frac{2}{n(n-1)} \sum\sum_{i<j} R_i^2 R_j^2 \mathbf{U}\{(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\}^{\mathrm{T}} \mathbf{U}\{(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_j\}$$

respectively. Here, R_i is the rank of v_i among $\{v_1, v_2, \cdots, v_n\}$,

Table 2
The ARE results of the mixed rotationally symmetric distributions for angular function (F1) with two different choices of λ.

λ	p	κ	$\text{ARE}_{\text{IN,N}}$	$\text{ARE}_{\text{IN,C}}$	$\text{ARE}_{\text{IN,S}}$	$\text{ARE}_{\text{IN,R}}$
0.9	100	p^2	9.583	2.978	11.831	6.531
	200	p^2	9.699	2.998	11.840	6.616
	300	p^2	9.739	3.005	11.842	6.645
	400	p^2	9.759	3.009	11.843	6.660
	100	p	1.784	1.307	2.041	1.565
	200	p	1.775	1.303	2.029	1.557
	300	p	1.772	1.301	2.025	1.555
	400	p	1.770	1.300	2.023	1.554
0.6	100	p^2	6.520	1.570	14.042	3.220
	200	p^2	6.524	1.569	13.905	3.223
	300	p^2	6.526	1.569	13.859	3.225
	400	p^2	6.527	1.568	13.837	3.225
	100	p	1.869	1.184	2.851	1.463
	200	p	1.855	1.180	2.825	1.455
	300	p	1.851	1.179	2.817	1.452
	400	p	1.849	1.178	2.813	1.451

Table 3
The ARE results of the mixed rotationally symmetric distributions for angular function (F2) with two different choices of λ.

λ	p	κ	$\text{ARE}_{\text{IN,N}}$	$\text{ARE}_{\text{IN,C}}$	$\text{ARE}_{\text{IN,S}}$	$\text{ARE}_{\text{IN,R}}$
0.9	100	$2p$	67.816	5.764	83.333	33.717
	150	$2p$	66.906	5.731	82.109	33.328
	200	$2p$	66.459	5.715	81.509	33.136
	250	$2p$	66.194	5.706	81.152	33.022
	300	$2p$	66.018	5.699	80.916	32.947
	100	$3p$	139.115	6.703	172.029	59.975
	150	$3p$	137.236	6.669	169.279	59.306
	200	$3p$	136.313	6.653	167.930	58.977
	250	$3p$	135.765	6.643	167.129	58.781
	300	$3p$	135.401	6.636	166.599	58.651
0.6	100	$2p$	37.454	2.026	94.224	9.777
	150	$2p$	36.951	2.018	92.583	9.682
	200	$2p$	36.705	2.013	91.778	9.635
	250	$2p$	36.558	2.011	91.300	9.607
	300	$2p$	36.461	2.009	90.984	9.589
	100	$3p$	64.558	2.127	168.273	13.679
	150	$3p$	63.686	2.118	164.869	13.552
	200	$3p$	63.257	2.114	163.200	13.489
	250	$3p$	63.003	2.112	162.209	13.452
	300	$3p$	62.834	2.110	161.552	13.428

$$
\begin{aligned}
\sigma_n^2(R_W) \doteq & 2n^{-4}p^{-1}\sum\sum_{i\neq j} i^2 j^2 = 2n^{-4}p^{-1}\left\{\left(\sum_i i^2\right)^2 - \sum_i i^4\right\} \\
= & \frac{(n^2-1)(4n^2-1)(5n+6)}{90n^3p}
\end{aligned}
$$

Table 4
The ARE results of the mixed rotationally symmetric distributions for angular function (F3) with two different choices of λ.

λ	p	κ	$\mathrm{ARE}_{\mathrm{IN,N}}$	$\mathrm{ARE}_{\mathrm{IN,C}}$	$\mathrm{ARE}_{\mathrm{IN,S}}$	$\mathrm{ARE}_{\mathrm{IN,R}}$
0.9	100	$2p$	107.571	6.380	132.694	48.888
	150	$2p$	106.121	6.347	130.647	48.334
	200	$2p$	105.409	6.330	129.643	48.062
	250	$2p$	104.985	6.321	129.047	47.900
	300	$2p$	104.705	6.314	128.653	47.792
	100	$3p$	211.010	7.188	262.208	83.162
	150	$3p$	208.153	7.155	257.689	82.265
	200	$3p$	206.750	7.138	255.473	81.824
	250	$3p$	205.917	7.128	254.156	81.561
	300	$3p$	205.364	7.122	253.284	81.387
0.6	100	$2p$	53.704	2.095	138.357	12.217
	150	$2p$	52.980	2.086	135.713	12.102
	200	$2p$	52.624	2.082	134.417	12.045
	250	$2p$	52.413	2.080	133.647	12.011
	300	$2p$	52.273	2.078	133.137	11.989
	100	$3p$	84.398	2.170	223.751	16.106
	150	$3p$	83.256	2.161	218.766	15.963
	200	$3p$	82.695	2.157	216.324	15.892
	250	$3p$	82.361	2.155	214.874	15.850
	300	$3p$	82.140	2.153	213.914	15.822

and

$$\sigma_n^2(R_S) \doteq 2n^{-4}p^{-1}\sum\sum_{i\neq j} i^4 j^4 = 2n^{-4}p^{-1}\left\{\left(\sum_i i^4\right)^2 - \sum_i i^8\right\}$$
$$= \frac{(n^2-1)(4n^2-1)(9n^5+20n^4-15n^3-50n^2+n+30)}{450n^3p}.$$

Proposition 3.1. *Under H_0, both $R_W/\sqrt{\sigma_n^2(R_W)}$ and $R_S/\sqrt{\sigma_n^2(R_S)}$ are asymptotically standard normal.*

The R_W-based and R_S-based tests rejects H_0 when $R_W/\sqrt{\sigma_n^2(R_W)} > z_\alpha$ and $R_S/\sqrt{\sigma_n^2(R_S)} > z_\alpha$, respectively.

4. Simulation results

In this section, we present some simulation results to investigate the performance of the W_n-based, $\widetilde{W}_n$-based, R_W-based and R_S-based, $T_n(\omega_{\mathrm{N}})$-based, $T_n(\omega_{\mathrm{C}})$-based, $T_n(\omega_{\mathrm{IN}})$-based, $T_n(\omega_{\mathrm{S}})$-based and $T_n(\omega_{\mathrm{R}})$-based tests, abbreviated as W_n, $\widetilde{W}_n$, R_W, R_S, T_{N}, T_{C}, T_{IN}, T_{S} and T_{R} respectively. We let $\boldsymbol{\theta}_0 = (1, 0, \cdots, 0)^{\mathrm{T}} \in \mathcal{S}^{p-1}$,

$$\boldsymbol{\theta} = \boldsymbol{\theta}_0 + \delta(-2\delta, 2\sqrt{1-\delta^2}, 0, \cdots, 0)^{\mathrm{T}} \in \mathcal{S}^{p-1}. \tag{4.1}$$

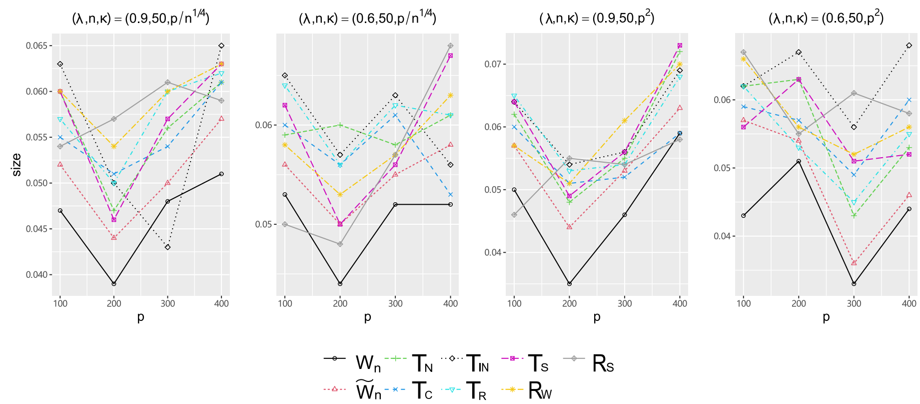

Fig 1. *The empirical size results of the nine tests under the mixed rotationally symmetric distributions at 5% level with angular function (F1) under settings* I and II.

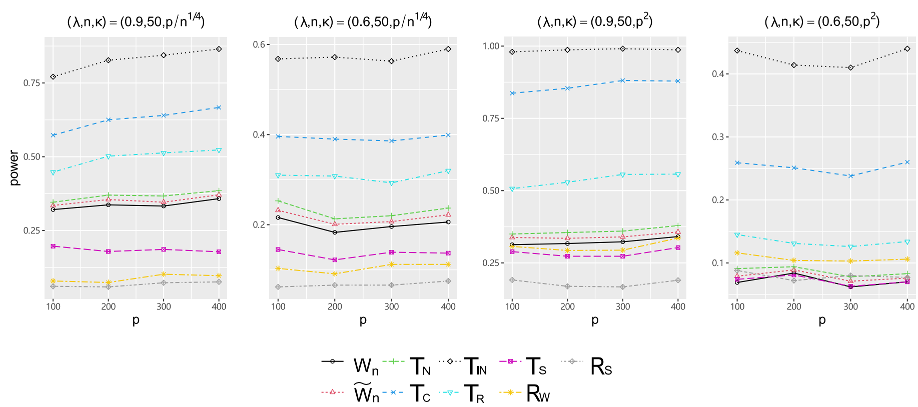

Fig 2. *The empirical power results of the nine tests under the mixed rotationally symmetric distributions at 5% level with angular function (F1) under settings* I and II.

We consider two settings of mixed distributions in (3.2), where for setting I, $(\lambda, \kappa_1, \kappa_2, \delta) = (0.9, \kappa/10, 10\kappa, 2\Delta)$, and for setting II, $(\lambda, \kappa_1, \kappa_2, \delta) = (0.6, \kappa/5, 5\kappa, \Delta/2)$. κ and Δ will be set according to the angular function type in the following text.

For angular function (F1), we let $n = 50$, $p \in \{100, 200, 300\}$ and consider two cases:

(1) $\kappa = p/n^{1/4}$, $\Delta = p^{3/4}/(\sqrt{n}\kappa)$,
(2) $\kappa = p^2$, $\Delta = p^{1/4}/\sqrt{n\kappa}$.

Figures 1 and 2 summarize the empirical size and power results of the nine tests by using angular function (F1), where all the simulation results are based on 1,000 replications. The results in Figure 1 suggest that all these nine tests can

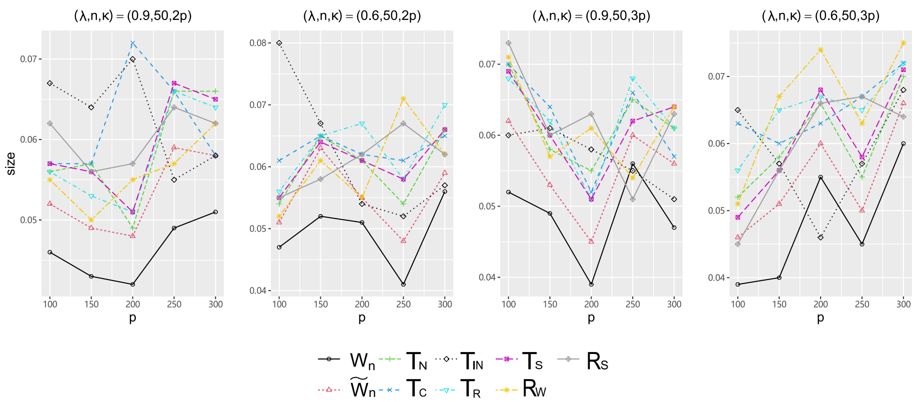

FIG 3. *The empirical size results of the nine tests under the mixed rotationally symmetric distributions at 5% level with angular function (F2) under settings* I and II.

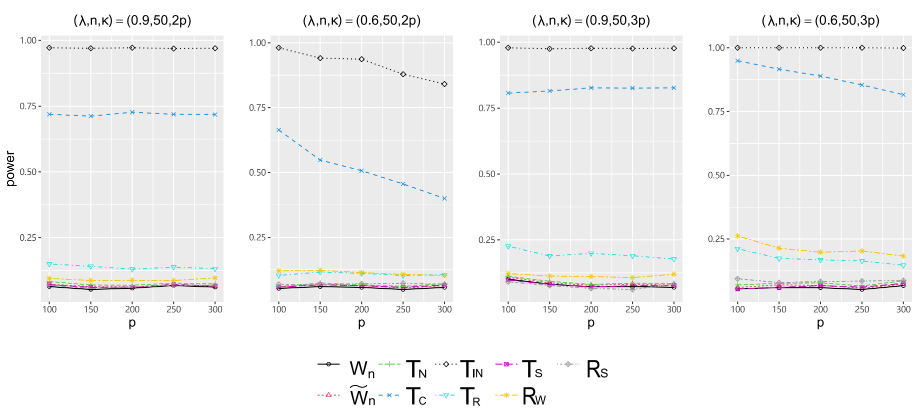

FIG 4. *The empirical power results of the nine tests under the mixed rotationally symmetric distributions at 5% level with angular function (F2) under settings* I and II.

properly control the empirical size. For power performance, the power results in Figure 2 suggest that T_{IN} can outperform the remaining tests.

For angular functions (F2) and (F3), we let $\kappa \in \{2p, 3p\}$, $\Delta = 1/(n^{1/2}p^{1/2})$, $n = 50$ and $p \in \{100, 150, 200, 250, 300\}$. Figures 3 and 5 summarize the empirical size of the nine tests by using angular functions (F2) and (F3), respectively. Similarly, all the simulation results are based on 1,000 replications. The size results in Figures 3 and 5 are similar to that for angular function (F1). For power performance, Figure 4 presents the power curves for angular function (F2), while Figure 6 presents the power curves for angular function (F3). From Figures 4 and 6, we see that T_{IN} and T_{C} can outperform the remaining tests in power comparison, while T_{IN} performs much better than T_{C}. In addition, we place some additional simulation results for the non-mixed distributions in Appendix.

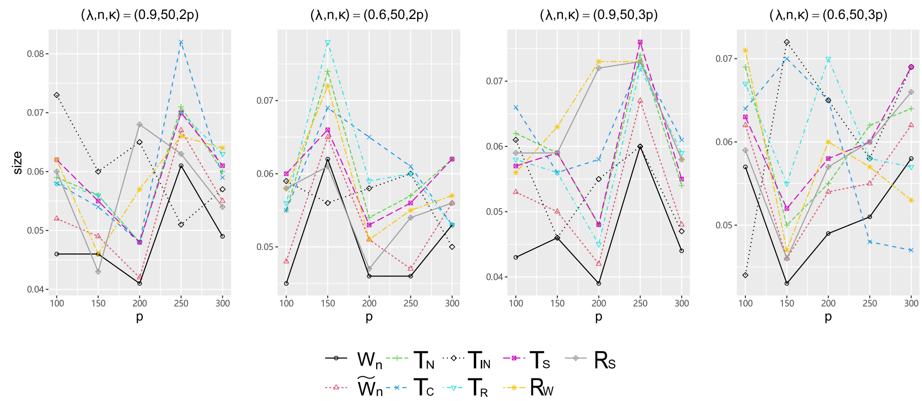

Fig 5. *The empirical size results of the nine tests under the mixed rotationally symmetric distributions at 5% level with angular function (F3) under settings* I and II.

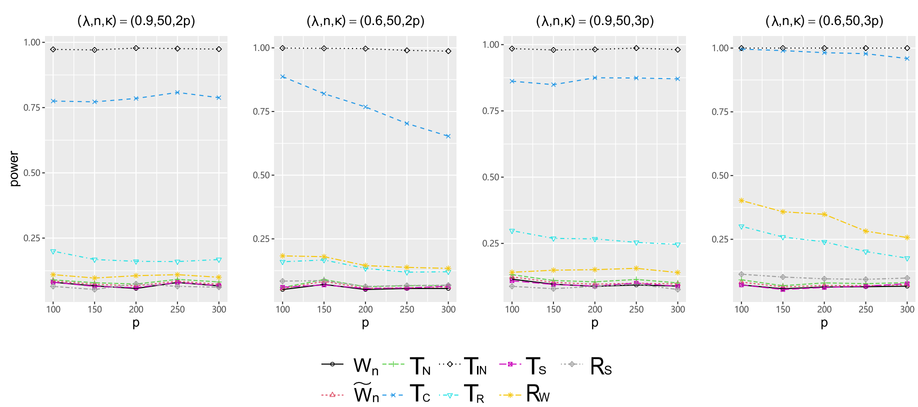

Fig 6. *The empirical power results of the nine tests under the mixed rotationally symmetric distributions at 5% level with angular function (F3) under settings* I and II.

5. Discussion

We have studied the general framework of directional weighted spatial sign tests for testing location of directional data under the commonly used rotationally symmetric distributions, which includes many members closely related to the existing tests. The asymptotic properties of the family of tests have been established under the rotationally symmetric distributions. Within this framework of asymptotic properties, we find that the $T_n(\omega_{\mathrm{IN}})$-based test has the maximum asymptotic power in this family. Then, the power advantages of the $T_n(\omega_{\mathrm{IN}})$-based test have been fully demonstrated by the theoretical and numerical results.

Appendix

Some additional simulation results

In this subsection, we present some additional simulation results of the nine tests, as X_i follows the non-mixed rotationally symmetric distributions.

For angular function (F1), we perform simulations under the following three settings:

(a) $\kappa = p^2$, $\delta = p^{1/4}/\sqrt{n\kappa}$,
(b) $\kappa = p$, $\delta = \sqrt{0.5 + \sqrt{1 + 0.25}}p^{3/4}/(\sqrt{n}\kappa)$,
(c) $\kappa = p/n^{1/4}$, $\delta = p^{3/4}/(\sqrt{n}\kappa)$,

which are the same as settings (i)–(iii) used in [20], with concentration κ and location

$$\boldsymbol{\theta} = (1, 0, \cdots, 0)^{\mathrm{T}} + \delta(-2\delta, 2\sqrt{1 - \delta^2}, 0, \cdots, 0)^{\mathrm{T}} \in \mathcal{S}^{p-1}. \tag{A.1}$$

Let $(n, p) \in \{(100, 400), (200, 800)\}$.

Table 5 summarizes the empirical size and power results of the nine tests by using the angular function (F1). The size results in Table 5 suggest that all these nine tests can properly control the empirical size. For power performance, the power results in Table 5 suggest that the power of the $T_n(\omega)$-based weighted spatial sign tests are quite similar to that of the W_n-based and $\widetilde{W}_n$-based tests. The power of the R_W-based and R_S-based tests are significantly smaller than those of the remaining tests.

Next, for angular functions (F2) and (F3), we let $p \in \{100, 150, 200, 250, 300\}$, $n = 40$ and $\kappa \in \{2p, 3p\}$, respectively. The setting of $\boldsymbol{\theta}$ is the same as (A.1) for angular function (F1). We fix $\delta = \sqrt{0.65}/[n^{1/2}p^{1/4}\{\mathbb{E}(v_i^{-4})\}^{1/4}]$ for angular function (F2), and fix $\delta = \sqrt{0.70}/[n^{1/2}p^{1/4}\{\mathbb{E}(v_i^{-4})\}^{1/4}]$ for angular function (F3).

Table 6 summarizes the empirical size of the nine tests for angular functions (F2) and (F3), which suggests that all these tests generally control the size. In particular, we find that the empirical size of the $T_n(\omega)$-based weighted spatial sign tests are slightly larger than that of the W_n-based and $\widetilde{W}_n$-based tests. These may indicate a slight anti-conservatism of the proposed tests.

For power comparison, Figure 7 shows the power curves of $\kappa = 2p$ and $3p$ respectively for angular function (F2), while Figure 8 shows the power curves of $\kappa = 2p$ and $3p$ respectively for angular function (F3). From Figures 7 and 8, it can be seen that some of the proposed directional weighted spatial sign tests are more powerful, but not significantly more powerful, than the W_n-based and $\widetilde{W}_n$-based tests, considering the slight anti-conservatism of the proposed tests. In addition, the empirical power of the R_S-based and R_W-based tests are much lower than that of the remaining tests. In short, in the case of non-mixed rotationally symmetric distributions, the power of all tests except the R_S-based and R_W-based tests are similar.

Table 5
The empirical size and power results of the nine tests under the non-mixed rotationally symmetric distributions at 5% level with angular function (F1).

n	p	κ	W_n	$\widetilde{W}_n$	T_{N}	T_{C}	T_{IN}	T_{S}	T_{R}	R_W	R_S
						size					
100	400	p^2	0.054	0.058	0.062	0.059	0.060	0.061	0.059	0.082	0.076
100	400	p	0.051	0.054	0.056	0.059	0.061	0.052	0.057	0.055	0.057
100	400	$p/n^{1/4}$	0.054	0.057	0.062	0.062	0.060	0.064	0.062	0.066	0.058
200	800	p^2	0.048	0.050	0.050	0.051	0.052	0.047	0.051	0.050	0.051
200	800	p	0.050	0.051	0.053	0.052	0.052	0.054	0.053	0.052	0.047
200	800	$p/n^{1/4}$	0.060	0.062	0.064	0.064	0.063	0.064	0.064	0.047	0.050
						power					
100	400	p^2	0.814	0.822	0.831	0.830	0.824	0.827	0.832	0.693	0.514
100	400	p	0.826	0.834	0.840	0.840	0.845	0.837	0.841	0.677	0.495
100	400	$p/n^{1/4}$	0.777	0.785	0.793	0.791	0.791	0.793	0.793	0.631	0.451
200	800	p^2	0.837	0.841	0.845	0.845	0.846	0.844	0.846	0.654	0.463
200	800	p	0.838	0.842	0.854	0.847	0.845	0.852	0.851	0.690	0.490
200	800	$p/n^{1/4}$	0.794	0.805	0.810	0.808	0.807	0.809	0.808	0.601	0.411

Table 6
The empirical size results of the nine tests under the non-mixed rotationally symmetric distributions at 5% level with angular functions (F2) and (F3).

n	p	κ	W_n	$\widetilde{W}_n$	T_{N}	T_{C}	T_{IN}	T_{S}	T_{R}	R_W	R_S
						(F2)					
40	100	200	0.049	0.053	0.060	0.062	0.062	0.061	0.059	0.068	0.069
40	100	300	0.043	0.046	0.052	0.056	0.054	0.056	0.056	0.056	0.055
40	150	300	0.050	0.057	0.068	0.061	0.065	0.070	0.064	0.071	0.072
40	150	450	0.044	0.052	0.060	0.059	0.061	0.056	0.058	0.060	0.060
40	200	400	0.045	0.050	0.059	0.055	0.055	0.058	0.056	0.069	0.071
40	200	600	0.044	0.051	0.058	0.056	0.051	0.058	0.055	0.060	0.060
40	250	500	0.046	0.052	0.057	0.063	0.065	0.053	0.062	0.057	0.052
40	250	750	0.041	0.047	0.057	0.058	0.057	0.056	0.058s	0.057	0.057
40	300	600	0.055	0.058	0.062	0.061	0.064	0.063	0.060	0.057	0.053
40	300	900	0.041	0.045	0.052	0.055	0.053	0.050	0.053	0.052	0.063
						(F3)					
40	100	200	0.042	0.046	0.058	0.058	0.057	0.052	0.058	0.058	0.057
40	100	300	0.044	0.053	0.062	0.060	0.063	0.061	0.058	0.059	0.064
40	150	300	0.040	0.048	0.052	0.055	0.057	0.062	0.053	0.067	0.059
40	150	450	0.046	0.052	0.057	0.057	0.056	0.062	0.056	0.060	0.061
40	200	400	0.044	0.046	0.052	0.057	0.056	0.052	0.057	0.062	0.068
40	200	600	0.045	0.050	0.053	0.051	0.050	0.053	0.054	0.058	0.058
40	250	500	0.048	0.054	0.055	0.056	0.054	0.054	0.055	0.056	0.068
40	250	750	0.045	0.050	0.057	0.062	0.061	0.052	0.060	0.048	0.046
40	300	600	0.056	0.062	0.071	0.072	0.073	0.072	0.074	0.056	0.050
40	300	900	0.052	0.058	0.071	0.066	0.067	0.067	0.066	0.047	0.040

Technical details

If $\boldsymbol{X}_i$ are rotationally symmetric on $\mathcal{S}^{p-1}$ about $\boldsymbol{\theta}$, then we have the tangent-normal decomposition of $\boldsymbol{X}_i$, $\boldsymbol{X}_i = u_i\boldsymbol{\theta} + v_i\boldsymbol{S}_i$, where $u_i = \boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{\theta}$, $v_i^2 = 1 - u_i^2 = \|(\mathbf{I}_p - \boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}})\boldsymbol{X}_i\|^2$, $\boldsymbol{S}_i = \mathbf{U}\{(\mathbf{I}_p - \boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}})\boldsymbol{X}_i\}$. In addition, $\boldsymbol{S}_i$ are independent with u_i, and $\boldsymbol{S}_i$ is uniformly distributed on $\mathcal{S}^{p-1}(\boldsymbol{\theta}^{\perp}) : \{\boldsymbol{S} \in \mathbb{R}^p : \|\boldsymbol{S}\| = 1, \boldsymbol{S}^{\mathrm{T}}\boldsymbol{\theta} = 0\}$. Before starting to prove, we recall Lemma B.1. in [4].

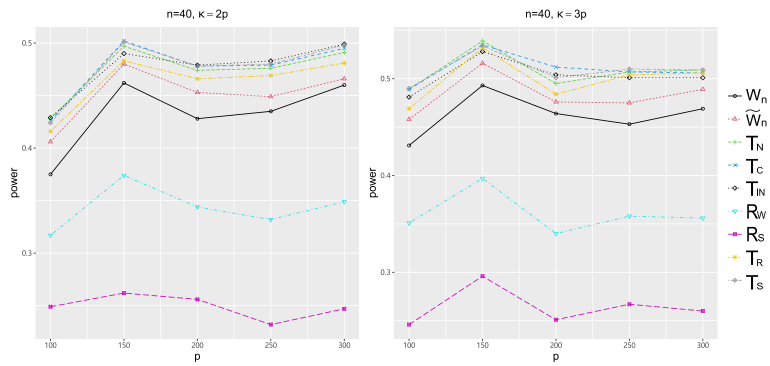

FIG 7. *The power curves of nine tests under the non-mixed rotationally symmetric distributions for angular function (F2) with* $\kappa = 2p$ *and* $3p$.

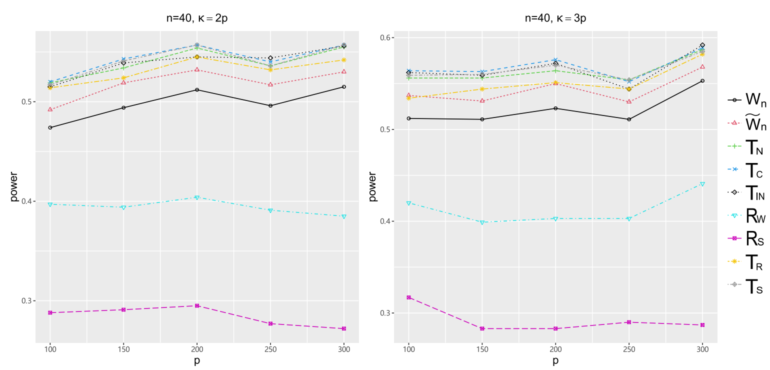

FIG 8. *The power curves of nine tests under the non-mixed rotationally symmetric distributions for angular function (F3) with* $\kappa = 2p$ *and* $3p$.

Lemma A.1. *Let* $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n$ *be a sequence of independent and identically distributed observations with a rotationally symmetric distribution, due to* $\boldsymbol{X}_i = u_i\boldsymbol{\theta} + v_i\boldsymbol{S}_i$, *we have*

$$(i)\ \mathbb{E}(\boldsymbol{S}_i\boldsymbol{S}_i^{\mathrm{T}}) = \frac{1}{p-1}(\mathbf{I}_p - \boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}})\ for\ any\ i;$$

$$(ii)\ \mathbb{E}\left\{(\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{S}_j)^2\right\} = \frac{1}{p-1}\ for\ any\ i \neq j;$$

$$(iii)\ \mathbb{E}\left\{(\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{S}_j)^4\right\} = \frac{3}{p^2-1}\ for\ any\ i \neq j.$$

Proof of Theorem 3.1

First, it is noteworthy that under H_0, $\boldsymbol{X}_i$ are rotationally symmetric with $\boldsymbol{\theta}_0$, hence $\boldsymbol{S}_i$ and u_i are independent. We have $\mathbb{E}\{T_n(\omega)\} = 0$ under H_0 due to

$\mathbb{E}(\boldsymbol{S}_i) = \mathbf{0}$. Then, the variance of $T_n(\omega)$ is

$$\begin{aligned}\operatorname{var}\{T_n(\omega)\} =& \frac{2}{n(n-1)}\mathbb{E}\left\{\omega^2(v_i)\omega^2(v_j)(\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{S}_j)^2\right\}\\ =& \frac{2}{n(n-1)}\left[\mathbb{E}\left\{\omega^2(v_i)\right\}\right]^2\mathbb{E}(\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{S}_j)^2\\ =& \frac{2}{n(n-1)(p-1)}b_2^2(\omega)\\ =& \sigma_n^2(\omega)\{1+o(1)\}.\end{aligned}$$

The normality of $T_n(\omega)$ has yet to be proven. Define $W_{nk} = \sum_{i=2}^k Z_{ni}$ where $Z_{ni} = 2/\{n(n-1)\}\sum_{j=1}^{i-1}\boldsymbol{V}_i^{\mathrm{T}}\boldsymbol{V}_j$, $\boldsymbol{V}_i = \omega(v_i)\boldsymbol{S}_i$. Let $\mathbf{A} = \mathbb{E}(\boldsymbol{V}_i\boldsymbol{V}_i^{\mathrm{T}})$. Note that $\mathbf{A} = b_2(\omega)(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})/(p-1)$ due to Lemma A.1. Let $\mathcal{F}_{n,i} \doteq \sigma\{\boldsymbol{V}_1,\cdots,\boldsymbol{V}_i\}$ be the σ-field generated by $\{\boldsymbol{V}_j, j\le i\}$. Obviously, $\mathbb{E}(Z_{ni} \mid \mathcal{F}_{n,i-1}) = 0$ and it follows that $\{W_{nk}, \mathcal{F}_{n,k}; 2\le k\le n\}$ is a zero mean martingale. According to the Martingale central limit theorem in [9], we only need to show

$$\frac{\sum_{j=2}^n \mathbb{E}(Z_{nj}^2 \mid \mathcal{F}_{n,j-1})}{\sigma_n^2(\omega)} \xrightarrow{p} 1, \tag{A.2}$$

and

$$\mathbb{E}\left\{\sum_{j=2}^n \mathbb{E}(Z_{nj}^4|\mathcal{F}_{n,j-1})\right\} = o\{\sigma_n^4(\omega)\}. \tag{A.3}$$

It can be shown that

$$\begin{aligned}\sum_{j=2}^n \mathbb{E}(Z_{nj}^2|\mathcal{F}_{n,j-1}) =& \sum_{j=2}^n \mathbb{E}\left[\left\{\frac{2}{n(n-1)}\sum_{i=1}^{j-1}\boldsymbol{V}_i^{\mathrm{T}}\boldsymbol{V}_j\right\}^2|\mathcal{F}_{n,j-1}\right]\\ =& \sum_{j=2}^n \frac{4}{n^2(n-1)^2}\mathbb{E}\left\{\left(\sum_{i_1=1}^{j-1}\sum_{i_2=1}^{j-1}\boldsymbol{V}_{i_1}^{\mathrm{T}}\boldsymbol{V}_j\boldsymbol{V}_{i_2}^{\mathrm{T}}\boldsymbol{V}_j\right)|\mathcal{F}_{n,j-1}\right\}\\ =& \sum_{j=2}^n \frac{4}{n^2(n-1)^2}\sum_{i_1=1}^{j-1}\sum_{i_2=1}^{j-1}\boldsymbol{V}_{i_1}^{\mathrm{T}}\mathbb{E}(\boldsymbol{V}_j\boldsymbol{V}_j^{\mathrm{T}})\boldsymbol{V}_{i_2}\\ =& C_{n1} + C_{n2},\end{aligned}$$

where

$$C_{n1} = \frac{4}{n^2(n-1)^2}\sum_{j=2}^n\sum_{i=1}^{j-1}\boldsymbol{V}_i^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_i \text{ and } C_{n2} = \frac{8}{n^2(n-1)^2}\sum_{j=2}^n\sum_{i_1<i_2}^{j-1}\sum^{j-1}\boldsymbol{V}_{i_1}^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_{i_2}.$$

We first consider C_{n1}. Due to $\mathbf{A} = \mathbb{E}(\boldsymbol{V}_i\boldsymbol{V}_i^{\mathrm{T}})$, we have

$$\mathbb{E}(C_{n1}) = \frac{2}{n(n-1)}\mathbb{E}(\boldsymbol{V}_i^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_i)$$

$$
\begin{aligned}
&= \frac{2}{n(n-1)} \operatorname{tr}\{\mathbb{E}(\mathbf{A}\boldsymbol{V}_i\boldsymbol{V}_i^{\mathrm{T}})\} \\
&= \frac{2}{n(n-1)} \operatorname{tr}(\mathbf{A}^2) \\
&= \sigma_n^2(\omega)\{1+o(1)\},
\end{aligned}
$$

where $\mathbf{A} = b_2(\omega)(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})/(p-1)$ and $\boldsymbol{S}_i = (\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i/v_i$. We also obtain

$$
\begin{aligned}
\boldsymbol{S}_i^{\mathrm{T}}\mathbf{A}\boldsymbol{S}_i &= \frac{1}{v_i^2(p-1)} b_2(\omega)\boldsymbol{X}_i^{\mathrm{T}}(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}))(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}))\boldsymbol{X}_i \\
&= \frac{1}{p-1} b_2(\omega).
\end{aligned}
$$

Thus,

$$
(\boldsymbol{S}_i^{\mathrm{T}}\mathbf{A}\boldsymbol{S}_i)^2 = \frac{b_2^2(\omega)}{(p-1)^2}. \tag{A.4}
$$

In addition,

$$
\begin{aligned}
\operatorname{var}(C_{n1}) &= \operatorname{var}\left(\frac{4}{n^2(n-1)^2}\sum_{j=2}^{n}\sum_{i=1}^{j-1}\boldsymbol{V}_i^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_i\right) \\
&= \operatorname{var}\left\{\frac{4}{n^2(n-1)^2}\sum_{j=2}^{n}\sum_{i=1}^{j-1}\frac{1}{p-1}b_2(\omega)\omega^2(v_i)\right\} \\
&= \operatorname{var}\left\{\frac{4}{n^2(n-1)^2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{1}{p-1}b_2(\omega)\omega^2(v_i)\right\} \\
&= \operatorname{var}\left\{\frac{4}{n^2(n-1)^2}\sum_{i=1}^{n}\frac{n-i}{p-1}b_2(\omega)\omega^2(v_i)\right\} \\
&\le O\{n^{-5}(p-1)^{-2}\}b_4(\omega)b_2^2(\omega) \\
&= o\{\sigma_n^4(\omega)\},
\end{aligned}
$$

where we used Condition (C1) in the last equality. Then, we have

$$
C_{n1}/\sigma_n^2(\omega) \xrightarrow{p} 1.
$$

Similarly,

$$
\begin{aligned}
\mathbb{E}(C_{n2}^2) &= \mathbb{E}\left\{\frac{8}{n^2(n-1)^2}\sum_{j=2}^{n}\sum_{i_1<i_2}^{j-1}\sum^{j-1}\boldsymbol{V}_{i_1}^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_{i_2}\right\}^2 \\
&= O(n^{-8})\mathbb{E}\left(\sum_{j_1=2}^{n}\sum_{j_2=2}^{n}\sum_{i_1<i_2}^{j_1-1}\sum^{j_1-1}\sum_{i_3<i_4}^{j_2-1}\sum^{j_2-1}\boldsymbol{V}_{i_1}^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_{i_2}\boldsymbol{V}_{i_3}^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_{i_4}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= O(n^{-8})\mathbb{E}\left\{\sum_{j_1<j_2}^{n}\sum^{n}\sum_{i_1<i_2}^{j_1-1}\sum^{j_1-1}(\boldsymbol{V}_{i_1}^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_{i_2})^2\right\}\\
&= O(n^{-8})\mathbb{E}\left\{\sum_{j_2=2}^{n}\sum_{j_1=1}^{j_2-1}\sum_{i_1<i_2}^{j_1-1}\sum^{j_1-1}(\boldsymbol{V}_{i_1}^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_{i_2})^2\right\}\\
&= O(n^{-4})\mathbb{E}\left\{(\boldsymbol{V}_i^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_j)^2\right\}\\
&= O(n^{-4})\mathbb{E}\left\{\omega^2(v_i)\omega^2(v_j)(\boldsymbol{S}_i^{\mathrm{T}}\mathbf{A}\boldsymbol{S}_j)^2\right\}\\
&= O(n^{-4})b_2^2(\omega)\mathrm{tr}\left\{\mathbb{E}(\boldsymbol{S}_i^{\mathrm{T}}\mathbf{A}\boldsymbol{S}_j\boldsymbol{S}_j^{\mathrm{T}}\mathbf{A}\boldsymbol{S}_i)\right\}\\
&= O(n^{-4}p^{-3})b_2^4(\omega) = o\{\sigma_n^4(\omega)\},
\end{aligned}
$$

where $i \neq j$ in the fourth equality and we used $\mathbf{A} = (p-1)^{-1}b_2(\omega)(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})$ and Lemma A.1 (ii) in the seventh equality. Then (A.2) holds.

Next, we only need to show

$$
\mathbb{E}\left\{\sum_{j=2}^{n}\mathbb{E}(Z_{nj}^4|\mathcal{F}_{n,j-1})\right\} = o\{\sigma_n^4(\omega)\}.
$$

Note that

$$
\mathbb{E}\left\{\sum_{j=2}^{n}\mathbb{E}(Z_{nj}^4|\mathcal{F}_{n,j-1})\right\} = \sum_{j=2}^{n}\mathbb{E}(Z_{nj}^4) = O(n^{-8})\sum_{j=2}^{n}\mathbb{E}\left(\sum_{i=1}^{j-1}\boldsymbol{V}_j^{\mathrm{T}}\boldsymbol{V}_i\right)^4
$$

can be decomposed as $Q+P$, where

$$
\begin{aligned}
Q =& O(n^{-8})\sum_{j=2}^{n}\sum_{i_1<i_2}^{j-1}\sum^{j-1}\mathbb{E}\left(\boldsymbol{V}_j^{\mathrm{T}}\boldsymbol{V}_{i_1}\boldsymbol{V}_{i_1}^{\mathrm{T}}\boldsymbol{V}_j\boldsymbol{V}_j^{\mathrm{T}}\boldsymbol{V}_{i_2}\boldsymbol{V}_{i_2}^{\mathrm{T}}\boldsymbol{V}_j\right),\\
P =& O(n^{-8})\sum_{j=2}^{n}\sum_{i=1}^{j-1}\mathbb{E}\left\{(\boldsymbol{V}_j^{\mathrm{T}}\boldsymbol{V}_i)^4\right\}.
\end{aligned}
$$

Because

$$
\begin{aligned}
Q =& O(n^{-5})\mathbb{E}\left(\boldsymbol{V}_j^{\mathrm{T}}\boldsymbol{V}_{i_1}\boldsymbol{V}_{i_1}^{\mathrm{T}}\boldsymbol{V}_j\boldsymbol{V}_j^{\mathrm{T}}\boldsymbol{V}_{i_2}\boldsymbol{V}_{i_2}^{\mathrm{T}}\boldsymbol{V}_j\right)\\
=& O(n^{-5})\mathrm{tr}\left\{\mathbb{E}(\boldsymbol{V}_{i_1}\boldsymbol{V}_{i_1}^{\mathrm{T}}\boldsymbol{V}_j\boldsymbol{V}_j^{\mathrm{T}}\boldsymbol{V}_{i_2}\boldsymbol{V}_{i_2}^{\mathrm{T}}\boldsymbol{V}_j\boldsymbol{V}_j^{\mathrm{T}})\right\}\\
=& O(n^{-5})\mathrm{tr}\left[\mathbb{E}\left\{\mathbb{E}\left(\boldsymbol{V}_{i_1}\boldsymbol{V}_{i_1}^{\mathrm{T}}\boldsymbol{V}_j\boldsymbol{V}_j^{\mathrm{T}}\boldsymbol{V}_{i_2}\boldsymbol{V}_{i_2}^{\mathrm{T}}\boldsymbol{V}_j\boldsymbol{V}_j^{\mathrm{T}}|\boldsymbol{V}_j\right)\right\}\right]\\
=& O(n^{-5})\mathrm{tr}\left\{\mathbb{E}(\mathbf{A}\boldsymbol{V}_j\boldsymbol{V}_j^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_j\boldsymbol{V}_j^{\mathrm{T}})\right\}\\
=& O(n^{-5})\mathbb{E}\left\{(\boldsymbol{V}_j^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_j)^2\right\},
\end{aligned}
$$

hence, $Q = O\{n^{-5}(p-1)^{-2}\}b_4(\omega)b_2^2(\omega) = o\{\sigma_n^4(\omega)\}$ due to (A.4) and Condition (C1). Similarly, due to Lemma A.1 (iii), we can show that

$$
P \quad = \quad O(n^{-6})\mathbb{E}\left\{(\boldsymbol{V}_j^{\mathrm{T}}\boldsymbol{V}_i)^4\right\}
$$

$$\begin{aligned}
&= O(n^{-6})\mathbb{E}\left\{\omega^4(v_i)\omega^4(v_j)(\boldsymbol{S}_j^{\mathrm{T}}\boldsymbol{S}_i)^4\right\} \\
&= O(n^{-6})b_4^2(\omega)\mathbb{E}\left\{(\boldsymbol{S}_j^{\mathrm{T}}\boldsymbol{S}_i)^4\right\} \\
&= O(n^{-6}p^{-2})b_2^4(\omega) \\
&= O\{n^{-2}\sigma_n^4(\omega)\}. \qquad \text{(A.5)}
\end{aligned}$$

Then, we complete the proof. □

Proof of Theorem 3.2

We have

$$\begin{aligned}
\widehat{\sigma}_n^2(\omega) =& 2n^{-4}p^{-1}\sum\sum_{i\neq j}\omega(\|(\mathbf{I}_p-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\|)^2\omega(\|(\mathbf{I}_p-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_j\|)^2 \\
=& 2n^{-4}p^{-1}\sum\sum_{i\neq j}\omega(v_i)^2\omega(v_j)^2.
\end{aligned}$$

First of all,

$$\mathbb{E}(\widehat{\sigma}_n^2(\omega)) = 2n^{-2}p^{-1}b_2^2(\omega) = \sigma_n^2(\omega)\{1+o(1)\}.$$

Next, we have

$$\begin{aligned}
\mathrm{var}\{\widehat{\sigma}_n^2(\omega)\} =& \mathbb{E}\left[\left\{2n^{-4}p^{-1}\sum\sum_{i\neq j}\omega(v_i)^2\omega(v_j)^2\right\}^2\right] \\
& - \left[\mathbb{E}\left\{2n^{-4}p^{-1}\sum\sum_{i\neq j}\omega(v_i)^2\omega(v_j)^2\right\}\right]^2 \\
=& O(n^{-6}p^{-2})b_4^2(\omega) + O(n^{-5}p^{-2})b_2^2(\omega)b_4(\omega) \\
=& o\{\sigma_n^4(\omega)\},
\end{aligned}$$

where the last equality is due to Condition (C1), which leads to

$$\widehat{\sigma}_n^2(\omega)/\sigma_n^2(\omega) \xrightarrow{p} 1. \qquad \square$$

Proof of Theorem 3.3

We have

$$\begin{aligned}
U\{\,(\mathbf{I}_p-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\} =& U\{\,(\mathbf{I}_p-\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}})\boldsymbol{X}_i + (\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\,\} \\
=& \{(\mathbf{I}_p-\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}})\boldsymbol{X}_i + (\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\,\} \\
& \times \left\{v_i^2 + \boldsymbol{X}_i^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\right\}^{-1/2} \\
=& \{\boldsymbol{S}_i + v_i^{-1}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\}
\end{aligned}$$

$$\times \left\{1 + v_i^{-2} \boldsymbol{X}_i^{\mathrm{T}} (\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}) \boldsymbol{X}_i\right\}^{-1/2}$$
$$= \{\boldsymbol{S}_i + v_i^{-1} (\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}) \boldsymbol{X}_i\} \{1 + \alpha_i\}^{-1/2},$$

where $\alpha_i = v_i^{-2} \boldsymbol{X}_i^{\mathrm{T}} (\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}) \boldsymbol{X}_i$. Note that

$$\|(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}) \boldsymbol{X}_i\| = v_i (1 + \alpha_i)^{1/2}.$$

Thus,

$$\begin{aligned}
&T_n(\omega) \\
=& \frac{2}{n(n-1)} \sum\sum_{i<j} \omega\{v_i(1+\alpha_i)^{1/2}\} \omega\{v_j(1+\alpha_j)^{1/2}\} U((\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}) \boldsymbol{X}_i)^{\mathrm{T}} \\
&\times U((\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}) \boldsymbol{X}_j) \\
=& \frac{2}{n(n-1)} \sum\sum_{i<j} \omega\{v_i(1+\alpha_i)^{1/2}\} \omega\{v_j(1+\alpha_j)^{1/2}\} \\
&\times \{\boldsymbol{S}_i + v_i^{-1} (\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}) \boldsymbol{X}_i\}^{\mathrm{T}} \{\boldsymbol{S}_j + v_j^{-1} (\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}) \boldsymbol{X}_j\} \\
&\times (1+\alpha_i)^{-1/2} (1+\alpha_j)^{-1/2} \\
=& \frac{2}{n(n-1)} \sum\sum_{i<j} \boldsymbol{V}_i^{\mathrm{T}} \boldsymbol{V}_j + A_1 + A_2 + A_3 + A_4 + A_5 + A_6,
\end{aligned}$$

where

$$\begin{aligned}
A_1 =& \frac{2}{n(n-1)} \sum\sum_{i<j} \left[\omega\{v_i(1+\alpha_i)^{1/2}\} \omega\{v_j(1+\alpha_j)^{1/2}\} - \omega(v_i)\omega(v_j)\right] \boldsymbol{S}_i^{\mathrm{T}} \boldsymbol{S}_j, \\
A_2 =& \frac{2}{n(n-1)} \sum\sum_{i<j} \omega\{v_i(1+\alpha_i)^{1/2}\} \omega\{v_j(1+\alpha_j)^{1/2}\} \\
&\times \{v_i^{-1} \boldsymbol{X}_i^{\mathrm{T}} (\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}) \boldsymbol{S}_j + v_j^{-1} \boldsymbol{S}_i^{\mathrm{T}} (\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}) \boldsymbol{X}_j\}, \\
A_3 =& \frac{2}{n(n-1)} \sum\sum_{i<j} \omega\{v_i(1+\alpha_i)^{1/2}\} \omega\{v_j(1+\alpha_j)^{1/2}\} \\
&\times \{v_i^{-1} v_j^{-1} \boldsymbol{X}_i^{\mathrm{T}} (\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} + \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}} - \boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}) \boldsymbol{X}_j\}, \\
A_4 =& \frac{2}{n(n-1)} \sum\sum_{i<j} \omega\{v_i(1+\alpha_i)^{1/2}\} \omega\{v_j(1+\alpha_j)^{1/2}\} \boldsymbol{S}_i^{\mathrm{T}} \boldsymbol{S}_j \\
&\times \{(1+\alpha_i)^{-1/2} (1+\alpha_j)^{-1/2} - 1\}, \\
A_5 =& \frac{2}{n(n-1)} \sum\sum_{i<j} \omega\{v_i(1+\alpha_i)^{1/2}\} \omega\{v_j(1+\alpha_j)^{1/2}\} \\
&\times \{v_i^{-1} \boldsymbol{X}_i^{\mathrm{T}} (\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}) \boldsymbol{S}_j + v_j^{-1} \boldsymbol{S}_i^{\mathrm{T}} (\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}) \boldsymbol{X}_j\} \\
&\times \{(1+\alpha_i)^{-1/2} (1+\alpha_j)^{-1/2} - 1\}, \\
A_6 =& \frac{2}{n(n-1)} \sum\sum_{i<j} \omega\{v_i(1+\alpha_i)^{-1/2}\} \omega\{v_j(1+\alpha_j)^{-1/2}\} \{v_i^{-1} v_j^{-1} \boldsymbol{X}_i^{\mathrm{T}}
\end{aligned}$$

$$\times (\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} + \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}} - \boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}})\boldsymbol{X}_j\} \times \{(1+\alpha_i)^{-1/2}(1+\alpha_j)^{-1/2} - 1\}.$$

Because of the tangent-normal decomposition, we have

$$\alpha_i = v_i^{-2}u_i^2 - v_i^{-2}u_i^2\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{\theta} - 2v_i^{-1}u_i\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{\theta} - (\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{\theta}_0)^2,$$

then,

$$\begin{aligned}
\mathbb{E}(\alpha_i) =&\mathbb{E}(v_i^{-2}u_i^2)\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\} - \mathbb{E}\{(\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{\theta}_0)^2\}\\
=&\mathbb{E}(v_i^{-2}u_i^2)\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\} - \mathbb{E}[\{\boldsymbol{S}_i^{\mathrm{T}}(\boldsymbol{\theta}_0 - \boldsymbol{\theta})\}^2]\\
\le&\mathbb{E}(v_i^{-2}u_i^2)\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\} - \|\boldsymbol{\theta}_0 - \boldsymbol{\theta}\|^2\\
\le&\sqrt{\mathbb{E}(v_i^{-4}u_i^4)}\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}.
\end{aligned}$$

According to $\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta} = 1 - (\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_0)^2 = \|\boldsymbol{\theta} - \boldsymbol{\theta}_0\|^2 - \|\boldsymbol{\theta} - \boldsymbol{\theta}_0\|^4/4$ and Condition (C2), we have $\mathbb{E}(\alpha_i) = o(1)$, then we concentrate on the variance of α_i, where

$$\begin{aligned}
\mathrm{var}(\alpha_i) \le&\mathbb{E}(\alpha_i^2)\\
\le&4\mathbb{E}\left(v_i^{-4}u_i^4\right)\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}^2 + 4\mathbb{E}(\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{\theta}_0)^4\\
&+ 8\mathbb{E}\left(u_iv_i^{-1}\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{\theta}\right)^2\\
=&4\mathbb{E}\left(v_i^{-4}u_i^4\right)\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}^2 + 4\mathbb{E}(\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{\theta}_0 - \boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{\theta})^4\\
&+8\mathbb{E}\left(u_i^2v_i^{-2}\right)\mathrm{tr}\{\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\mathbb{E}(\boldsymbol{S}_i\boldsymbol{S}_i^{\mathrm{T}})\}\\
=&o(1) + 4\mathbb{E}\{\boldsymbol{S}_i^{\mathrm{T}}(\boldsymbol{\theta}_0 - \boldsymbol{\theta})\}^4\\
\le&4\|\boldsymbol{\theta} - \boldsymbol{\theta}_0\|^4 + o(1).
\end{aligned}$$

Due to Condition (C2), we have $\mathrm{var}(\alpha_i) = o(1)$ and then $\alpha_i = o_p(1)$. Thus,

$$\begin{aligned}
&\mathbb{E}(A_1^2)\\
=&\frac{4}{n^2(n-1)^2}\mathbb{E}\left(\sum\sum_{i<j}\left[\omega\{v_i(1+\alpha_i)^{1/2}\}\omega\{v_j(1+\alpha_j)^{1/2}\} - \omega(v_i)\omega(v_j)\right]\right.\\
&\left.\times\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{S}_j\right)^2\\
=&\frac{4}{n^2(n-1)^2}\mathbb{E}\left\{\sum\sum_{i<j}\left(\left[\omega(v_i)^{-1}\omega\{v_i(1+\alpha_i)^{1/2}\}\omega(v_j)^{-1}\omega\{v_j(1+\alpha_j)^{1/2}\}\right.\right.\right.\\
&\left.\left.\left.-1\right]\times\boldsymbol{V}_i^{\mathrm{T}}\boldsymbol{V}_j\right)^2\right\}\\
=&O(n^{-2})\mathbb{E}\left(\left[\omega(v_i)^{-1}\omega\{v_i(1+\alpha_i)^{1/2}\}\omega(v_j)^{-1}\omega\{v_j(1+\alpha_j)^{1/2}\} - 1\right]\right.\\
&\left.\times\boldsymbol{V}_i^{\mathrm{T}}\boldsymbol{V}_j\right)^2
\end{aligned}$$

$$\le O(n^{-2})\sqrt{\mathbb{E}\left(\left[\omega(v_i)^{-1}\omega\{v_i(1+\alpha_i)^{1/2}\}\omega(v_j)^{-1}\omega\{v_j(1+\alpha_j)^{1/2}\}-1\right]^4\right)}$$
$$\times\sqrt{\mathbb{E}\{(\boldsymbol{V}_i^{\mathrm{T}}\boldsymbol{V}_j)^4\}}$$
$$=o\{n^{-2}b_4(\omega)\}\sqrt{\mathbb{E}\left\{(\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{S}_j)^4\right\}}$$
$$=o\{\sigma_n^2(\omega)\},$$

where the second last equality is due to $\alpha_i \to 0$ and Lemma A.1 (iii). This leads to $A_1 = o_p\{\sigma_n(\omega)\}$. Next, for the first term in A_2, we have

$$\mathbb{E}(A_{21}^2) = \mathbb{E}\left(\left[\frac{2}{n(n-1)}\sum\sum_{i<j}\omega\{v_i(1+\alpha_i)^{1/2}\}\omega\{v_j(1+\alpha_j)^{1/2}\}v_i^{-1}\right.\right.$$
$$\left.\left.\times\boldsymbol{X}_i^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{S}_j\right]^2\right)$$
$$=O(n^{-4})\mathbb{E}\left(\left[\sum\sum_{i<j}\omega\{v_i(1+\alpha_i)^{1/2}\}\omega\{v_j(1+\alpha_j)^{1/2}\}v_i^{-1}\right.\right.$$
$$\left.\left.\times\boldsymbol{X}_i^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{S}_j\right]^2\right)$$
$$=O(n^{-4})\mathbb{E}\left(\left[\sum\sum_{i<j}\omega\{v_i(1+\alpha_i)^{1/2}\}\omega\{v_j(1+\alpha_j)^{1/2}\}v_i^{-1}\right.\right.$$
$$\left.\left.\times\boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{S}_j\right]^2\right)$$
$$\doteq O(n^{-2})\mathbb{E}\left\{\left[\omega(v_i)\omega(v_j)v_i^{-1}(u_i\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{S}_j+v_i\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{S}_j)\right]^2\right\}$$
$$+O(n^{-1})\mathbb{E}\left\{\omega(v_{i_1})\omega(v_{i_2})\omega(v_j)^2v_{i_1}^{-1}v_{i_2}^{-1}u_{i_1}u_{i_2}\right\}$$
$$\times\mathbb{E}(\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{S}_j\boldsymbol{S}_j^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{\theta}),$$

where $i_1 \neq i_2 < j$. We have that

$$O(n^{-2})\mathbb{E}[\{\omega(v_i)\omega(v_j)v_i^{-1}(u_i\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{S}_j+v_i\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{S}_j)\}^2]$$
$$\le O(n^{-2})\mathbb{E}\left\{\omega(v_i)\omega(v_j)v_i^{-1}u_i\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{S}_j\right\}^2+O(n^{-2})\mathbb{E}\left\{\omega(v_i)\omega(v_j)\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{S}_j\right\}^2$$
$$=O(n^{-2})\mathbb{E}\{\omega^2(v_i)\omega^2(v_j)v_i^{-2}u_i^2\}\mathbb{E}\{(\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{S}_j)^2\}$$
$$+O\{n^{-2}b_2^2(\omega)\}\mathbb{E}\{(\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{S}_j)^2\}$$
$$\le O(n^{-2})\mathbb{E}\{\omega^2(v_j)\}\sqrt{\mathbb{E}(v_i^{-4}u_i^4)\mathbb{E}\{\omega^4(v_i)\}}\mathbb{E}(\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{S}_j\boldsymbol{S}_j^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{\theta})$$
$$+O\{n^{-2}b_2^2(\omega)\}\mathbb{E}(\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{S}_j\boldsymbol{S}_j^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{S}_i)$$
$$=o\{\sigma_n^2(\omega)\},$$

where the second last equality is due to Lemma A.1 (i) and Condition (C2).

Then, due to $i_1 \neq i_2 < j$, we have that

$$O(n^{-1})\mathbb{E}\left\{\omega(v_{i_1})\omega(v_{i_2})\omega(v_j)^2v_{i_1}^{-1}v_{i_2}^{-1}u_{i_1}u_{i_2}\right\}\mathbb{E}(\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{S}_j\boldsymbol{S}_j^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{\theta})$$

$$
\begin{aligned}
&=O(n^{-1})b_2(\omega)[\mathbb{E}\{\omega(v_i)v_i^{-1}u_i\}]^2\mathbb{E}(\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{S}_j\boldsymbol{S}_j^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{\theta})\\
&\leq O\{n^{-1}(p-1)^{-1}\}b_2(\omega)\mathbb{E}\{\omega^2(v_i)v_i^{-2}u_i^2\}\left\{\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{\theta}-(\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{\theta})^2\right\}\\
&\leq O\{n^{-1}(p-1)^{-1}b_2(\omega)\}\sqrt{\mathbb{E}\{\omega^4(v_i)\}\mathbb{E}(v_i^{-4}u_i^4)}\left\{\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{\theta}-(\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{\theta})^2\right\}\\
&=O\{n^{-1}(p-1)^{-1}b_2^2(\omega)\}\sqrt{\mathbb{E}(v_i^{-4}u_i^4)}\left\{\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{\theta}-(\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{\theta})^2\right\}\\
&=o\{\sigma_n^2(\omega)\},
\end{aligned}
$$

where $i \neq j$ in the first equality, and the second last equality is due to Condition (C1). In addition,

$$
\begin{aligned}
&\left\{\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{\theta}-(\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{\theta})^2\right\}\\
=&(\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{\theta}_0)^2\left(\|\boldsymbol{\theta}-\boldsymbol{\theta}_0\|^2-\|\boldsymbol{\theta}-\boldsymbol{\theta}_0\|^4/4\right),
\end{aligned}
$$

hence the last equation holds and $A_{21} = o_p\{\sigma_n(\omega)\}$. Similarly, $A_{22} = o_p\{\sigma_n(\omega)\}$, hence $A_2 = o_p\{\sigma_n(\omega)\}$. Next, define $\beta_i \doteq \omega\{v_i(1+\alpha_i)^{1/2}\}v_i^{-1}$, and $\beta_0 \doteq \mathbb{E}(\beta_i)$. Then $\beta_0 = c_0(\omega)\{1+o(1)\}$.

$$
\begin{aligned}
\mathbb{E}(\beta_i^2) =&\mathbb{E}\{\omega^2(v_i)v_i^{-2}\}\{1+o(1)\}\\
\leq&[\mathbb{E}\{\omega^4(v_i)\}\mathbb{E}(v_i^{-4})]^{1/2}\\
\leq&O\{b_2(\omega)\}\sqrt{\mathbb{E}(v_i^{-4})}.
\end{aligned}
$$

Note that $\beta_0^2 \leq \mathbb{E}(\beta_i^2)$. We let

$$
\begin{aligned}
A_3' =&\frac{2}{n(n-1)}\sum\sum_{i<j}\omega\{v_i(1+\alpha_i)^{1/2}\}\omega\{v_j(1+\alpha_j)^{1/2}\}v_i^{-1}v_j^{-1}\\
&\{\mathbb{E}(u_iu_j)\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}+v_iv_j\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{S}_j\}.
\end{aligned}
$$

Next, we will show $A_3 = A_3' + o_p\{\sigma_n(\omega)\}$ i.e. $A_3 - A_3' = o_p\{\sigma_n(\omega)\}$.

$$
\begin{aligned}
\mathbb{E}(A_3-A_3') =&\mathbb{E}[\frac{2}{n(n-1)}\sum\sum_{i<j}\omega\{v_i(1+\alpha_i)^{1/2}\}v_i^{-1}\omega\{v_j(1+\alpha_j)^{1/2}\}v_j^{-1}\\
&\times\{u_iu_j-\mathbb{E}^2(u_i)\}\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}]\\
=&\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\mathbb{E}[\omega\{v_i(1+\alpha_i)^{1/2}\}v_i^{-1}\omega\{v_j(1+\alpha_j)^{1/2}\}\\
&\times v_j^{-1}\{u_iu_j-\mathbb{E}^2(u_i)\}]\\
=&\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\sqrt{\mathbb{E}(\beta_i^2\beta_j^2)\mathbb{E}(u_iu_j-\mathbb{E}^2(u_i))^2}\\
\leq&\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\mathbb{E}(\beta_i^2)\sqrt{\mathbb{E}^2(u_i^2)-\mathbb{E}^4(u_i)}\\
\leq&\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}}-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\sqrt{b_4(\omega)\mathbb{E}(v_i^{-4})}\sqrt{\mathbb{E}^2(u_i^2)-\mathbb{E}^4(u_i)}\\
=&o\{\sigma_n(\omega)\},
\end{aligned}
$$

where the last equality is due to Condition (C2). Hence we have

$$\mathbb{E}^2(u_i^2) - \mathbb{E}^4(u_i) = o(1).$$

So, we define $u_i u_j - \mathbb{E}^2(u_i) = \gamma_{ij}$, then

$$\begin{aligned}
\text{var}(A_3 - A_3') =&\mathbb{E}\{\frac{2}{n(n-1)}\sum\sum_{i<j}\beta_i\beta_j\gamma_{ij}\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}^2 \\
&- \mathbb{E}^2\{\frac{2}{n(n-1)}\beta_i\beta_j\gamma_{ij}\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\} \\
=&\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}^2\{O(n^{-1})\mathbb{E}(\beta_i^2\beta_j\beta_l\gamma_{ij}\gamma_{il}) \\
&+ O(n^{-2})\mathbb{E}(\beta_i^2\beta_j^2\gamma_{ij}^2)\} \\
=&D_1 + D_2,
\end{aligned}$$

where $i < j \neq l$ in the second inequality. At first, we focus on D_1. Due to $i < j \neq l$,

$$\begin{aligned}
D_1 =&\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}^2 O(n^{-1})\mathbb{E}(\beta_i^2\beta_j\beta_l\gamma_{ij}\gamma_{il}) \\
=&\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}^2 O(n^{-1})[\mathbb{E}(\beta_i^2\beta_j\beta_l u_i^2 u_j u_l) - \mathbb{E}\{\beta_i^2\beta_j\beta_l\mathbb{E}^2(u_i)u_i u_j\} \\
&- \mathbb{E}\{\beta_i^2\beta_j\beta_l\mathbb{E}^2(u_i)u_i u_l\} + \mathbb{E}\{\beta_i^2\beta_j\beta_l\mathbb{E}^4(u_i)\}] \\
\leq&\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}^2 O(n^{-1})b_4(\omega)\mathbb{E}(v_i^{-4}) \\
=&O\{n^{-1}b_2^2(\omega)\}\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}^2\mathbb{E}(v_i^{-4}) \\
=&o\{\sigma_n^2(\omega)\},
\end{aligned}$$

where the first inequality is because of the Cauchy inequality and $|u_i| \leq 1$. The Condition (C2) brings about the last equality. Next, similar to the above process, we have

$$\begin{aligned}
D_2 &= \{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}^2 O(n^{-2})\mathbb{E}[\beta_i^2\beta_j^2\{u_i^2u_j^2 + \mathbb{E}^4(u_i) - 2u_iu_j\mathbb{E}^2(u_i)\}] \\
&\leq O(n^{-2})\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}^2 b_4(\omega)\mathbb{E}(v_i^{-4}) \\
&\leq O\{n^{-2}b_2^2(\omega)\}\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}^2\mathbb{E}(v_i^{-4}) \\
&= o\{\sigma_n^2(\omega)\}.
\end{aligned}$$

Therefore, we can conclude that $A_3 = A_3' + o_p\{\sigma_n(\omega)\}$. Then, for A_3', we have

$$\begin{aligned}
\mathbb{E}(A_3') =&\beta_0^2\mathbb{E}^2(u_i)\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta} \\
=&c_0^2(\omega)\mathbb{E}^2(u_i)\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\{1 + o(1)\}.
\end{aligned}$$

Moreover,

$$\text{var}(A_3') =\mathbb{E}\left(\frac{2}{n(n-1)}\sum\sum_{i<j}\left[\beta_i\beta_j\mathbb{E}^2(u_i)\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}\right.\right.$$

$$
\begin{aligned}
&+\omega(v_i)\omega(v_j)\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{S}_j]\Big)^2 - \beta_0^4\mathbb{E}^4(u_i)\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}^2 \\
=&\mathbb{E}\{\frac{2}{n(n-1)}\sum\sum_{i<j}\beta_i\beta_j\mathbb{E}^2(u_i)\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}^2 \\
&+ \mathbb{E}\{\frac{2}{n(n-1)}\sum\sum_{i<j}\omega(v_i)(1+\alpha_i)^{1/2}\omega(v_j)(1+\alpha_j)^{1/2}\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{S}_j\}^2 \\
&- \beta_0^4\mathbb{E}^4(u_i)\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}^2 \\
=&O\{n^{-4}(\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta})^2\}\mathbb{E}\{\sum\sum_{i<j}\sum\sum_{k<l}(\beta_i\beta_j\beta_k\beta_l - \beta_0^4)\} \\
&+ O\{n^{-2}b_2^2(\omega)\}\mathbb{E}(\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{S}_j\boldsymbol{S}_j^{\mathrm{T}}\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}\boldsymbol{S}_i) \\
=&(\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta})^2\{O(n^{-2})\mathbb{E}(\beta_i^2\beta_j^2 - \beta_0^4) \\
&+ O(n^{-1})\beta_0^2\mathbb{E}(\beta_i^2 - \beta_0^2)\} + o(n^{-2}b_2^2(\omega)p^{-2}) \\
\le&(\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta})^2[O(n^{-2})\{\mathbb{E}(\beta_i^2)\}^2 + O(n^{-1})\{\mathbb{E}(\beta_i^2)\}^2] \\
&+ o\{n^{-2}b_2^2(\omega)p^{-2}\} \\
=&(\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta})^2\{O(n^{-1})b_2^2(\omega)\mathbb{E}(v_i^{-4})\} + o\{\sigma_n^2(\omega)\} \\
=&o\{\sigma_n^2(\omega)\},
\end{aligned}
$$

where the last equality is due to Condition (C2), which leads to

$$
A_3 = c_0(\omega)^2\mathbb{E}^2(u_i)\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\} + o_p\{\sigma_n(\omega)\}.
$$

For A_4, by the Cauchy inequality, we have

$$
\begin{aligned}
\mathbb{E}(A_4^2) =&\frac{4}{n^2(n-1)^2}\sum\sum_{i<j}\mathbb{E}\left[\boldsymbol{V}_i^{\mathrm{T}}\boldsymbol{V}_j\{(1+\alpha_i)^{-1/2}(1+\alpha_j)^{-1/2} - 1\}\right]^2 \\
&\times\{1+o(1)\} \\
=&O(n^{-2})\mathbb{E}\{(\boldsymbol{V}_i^{\mathrm{T}}\boldsymbol{V}_j)^2\}\mathbb{E}\left[\{(1+\alpha_i)^{-1/2}(1+\alpha_j)^{-1/2} - 1\}^2\right] \\
&\times\{1+o(1)\} \\
=&o\{n^{-2}b_2^2(\omega)(p-1)^{-1}\} \\
=&o\{n^{-2}p^{-1}b_2^2(\omega)\} \\
=&o\{\sigma_n^2(\omega)\},
\end{aligned}
$$

hence $A_4 = o_p\{\sigma_n(\omega)\}$. Similarly, we can obtain that $\mathbb{E}(A_5^2) = o\{\mathbb{E}(A_2^2)\}$ and $\mathbb{E}(A_6^2) = o\{\mathbb{E}(A_3^2)\}$, which lead to $A_5 = o_p\{\sigma_n(\omega)\}$, $A_6 = o_p\{\sigma_n(\omega)\}$, respectively.

Combining the above results, we get

$$
T_n(\omega) = \frac{2}{n(n-1)}\sum\sum_{i<j}\boldsymbol{V}_i^{\mathrm{T}}\boldsymbol{V}_j + c_0(\omega)^2\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\} + o_p\{\sigma_n(\omega)\},
$$

using the same procedure as in the proof of Theorem 1. Then, we obtain the conclusion. □

Proof of theorem 3.4

Under conditions (C1)–(C2), given $\boldsymbol{\theta}$ and $\boldsymbol{\theta}_0$, we find that the right side of (3.1) is an increasing function of $\mathbb{E}^2\{\omega(v_i)v_i^{-1}\}/\mathbb{E}\{\omega^2(v_i)\}$. Interestingly, according to the Cauchy inequality, it can be seen that

$$\frac{\mathbb{E}^2\{\omega(v_i)v_i^{-1}\}}{\mathbb{E}\{\omega^2(v_i)\}} \leq \frac{\mathbb{E}\{\omega^2(v_i)\}\mathbb{E}({v_i}^{-2})}{\mathbb{E}\{\omega^2(v_i)\}} = \mathbb{E}(v_i^{-2}). \tag{A.6}$$

The above inequality holds for any nonnegative continuous weight function satisfying the condition (C1). So, the maximum value of the asymptotic power in the whole family is

$$\lim_{n,p\to\infty} \Phi\left[-z_\alpha + \mathbb{E}^2(u_i)\mathbb{E}({v_i}^{-2})\frac{p^{1/2}n\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}}{\sqrt{2}}\right].$$

Besides, when taking $\omega = \omega_{\mathrm{IN}}$, the asymptotic power is

$$\beta(\omega_{\mathrm{IN}}) = \lim_{n,p\to\infty} \Phi\left[-z_\alpha + \mathbb{E}^2(u_i)\mathbb{E}({v_i}^{-2})\frac{p^{1/2}n\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}}{\sqrt{2}}\right],$$

which equals to the maximum value of the asymptotic power in the whole family.

In addition, we will verify that under the FvML and mixed FvML distributions, the weight ω_{IN} satisfies condition (C1).

When X_i follows a FvML distribution with density function $c_{p,\kappa}\exp(\kappa\mathbf{x}^{\mathrm{T}}\boldsymbol{\theta})$. Then, the density function of u_i is $c_{p,\kappa}(1-u^2)^{\frac{p-3}{2}}\exp(\kappa u)$, where

$$c_{p,\kappa} = 1/\int_{-1}^{1}(1-u^2)^{\frac{p-3}{2}}\exp(\kappa u)du.$$

So, we have

$$\mathbb{E}(v_i^{-4}) = c_{p,\kappa}\int_{-1}^{1}(1-u^2)^{\frac{p-7}{2}}\exp(\kappa u)du$$

and

$$\mathbb{E}(v_i^{-2}) = c_{p,\kappa}\int_{-1}^{1}(1-u^2)^{\frac{p-5}{2}}\exp(\kappa u)du.$$

Then, according to (S.2.7) in the supplement of [4], we have

$$\mathbb{E}(v_i^{-4}) = \frac{(\kappa/2)^2\Gamma\left(\frac{p-5}{2}\right)\mathcal{I}_{\frac{p}{2}-3}(\kappa)}{\Gamma\left(\frac{p-1}{2}\right)\mathcal{I}_{\frac{p}{2}-1}(\kappa)}$$

and

$$\mathbb{E}(v_i^{-2}) = \frac{(\kappa/2)\Gamma\left(\frac{p-3}{2}\right)\mathcal{I}_{\frac{p}{2}-2}(\kappa)}{\Gamma\left(\frac{p-1}{2}\right)\mathcal{I}_{\frac{p}{2}-1}(\kappa)},$$

where

$$\mathcal{I}_\nu(\kappa) \doteq \frac{(\kappa/2)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_{-1}^{1}\left(1-s^2\right)^{\nu-\frac{1}{2}}\exp(\kappa s)ds$$

is the modified Bessel function of the first kind and of order v. Then, we have

$$\frac{\mathbb{E}(v_i^{-4})}{\mathbb{E}^2(v_i^{-2})} = \frac{\Gamma\left(\frac{p-5}{2}\right)\mathcal{I}_{\frac{p}{2}-3}(\kappa)\Gamma\left(\frac{p-1}{2}\right)\mathcal{I}_{\frac{p}{2}-1}(\kappa)}{\Gamma\left(\frac{p-3}{2}\right)\mathcal{I}_{\frac{p}{2}-2}(\kappa)\Gamma\left(\frac{p-3}{2}\right)\mathcal{I}_{\frac{p}{2}-2}(\kappa)} = \frac{(p-3)\mathcal{I}_{\frac{p}{2}-3}(\kappa)\mathcal{I}_{\frac{p}{2}-1}(\kappa)}{(p-5)\mathcal{I}_{\frac{p}{2}-2}(\kappa)\mathcal{I}_{\frac{p}{2}-2}(\kappa)}.$$

Then, according to (S.2.6) and Lemma S.2.2(i) in the supplement of [4], we can obtain

$$\begin{aligned}\frac{\mathbb{E}(v_i^{-4})}{\mathbb{E}^2(v_i^{-2})} &= \frac{(p-3)\mathcal{I}_{\frac{p}{2}-3}(\kappa)\mathcal{I}_{\frac{p}{2}-1}(\kappa)}{(p-5)\mathcal{I}_{\frac{p}{2}-2}(\kappa)\mathcal{I}_{\frac{p}{2}-2}(\kappa)}\\ &= \frac{p-3}{p-5}\left(A_{p-2}^2(\kappa) + \frac{p-4}{\kappa}A_{p-2}(\kappa)\right)\\ &\le \frac{p-3}{p-5}\left[1 + \frac{p-4}{(p-4)/2 + \sqrt{\kappa^2 + (p-4)^2/4}}\right]\\ &\le \frac{p-3}{p-5}\left[1 + \frac{p-4}{p-4}\right]\\ &= O(1),\end{aligned}$$

for all $p > 5$, where $A_p(\kappa) \doteq I_{p/2}(\kappa)/I_{p/2-1}(\kappa)$ and

$$A_{p-2}(\kappa) \le \frac{\kappa}{(p-4)/2 + \sqrt{\kappa^2 + (p-4)^2/4}}.$$

When $\boldsymbol{X}_i$ follows the mixture of two FvML distributions with the same location parameter, the density function of $\boldsymbol{X}_i$ can be denoted as

$$\lambda c_{p,\kappa_1}\exp(\kappa_1\mathbf{x}^{\mathrm{T}}\boldsymbol{\theta}) + (1-\lambda)c_{p,\kappa_2}\exp(\kappa_2\mathbf{x}^{\mathrm{T}}\boldsymbol{\theta}). \tag{A.7}$$

the density function of u_i is

$$\lambda c_{p,\kappa_1}(1-u^2)^{\frac{p-3}{2}}\exp(\kappa_1 u) + (1-\lambda)c_{p,\kappa_2}(1-u^2)^{\frac{p-3}{2}}\exp(\kappa_2 u).$$

So, we have

$$\begin{aligned}\mathbb{E}(v_i^{-4}) =& \lambda c_{p,\kappa_1}\int_{-1}^{1}(1-u^2)^{\frac{p-7}{2}}\exp(\kappa_1 u)du\\ &+ (1-\lambda)c_{p,\kappa_2}\int_{-1}^{1}(1-u^2)^{\frac{p-7}{2}}\exp(\kappa_2 u)du\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}(v_i^{-2}) =& \lambda c_{p,\kappa_1}\int_{-1}^{1}(1-u^2)^{\frac{p-5}{2}}\exp(\kappa_1 u)du\\ &+ (1-\lambda)c_{p,\kappa_2}\int_{-1}^{1}(1-u^2)^{\frac{p-5}{2}}\exp(\kappa_2 u)du.\end{aligned}$$

Then, according to (S.2.7) in the supplement of [4], we have

$$\begin{aligned}
\mathbb{E}(v_i^{-4}) &= \lambda \frac{(\kappa_1/2)^2 \Gamma\left(\frac{p-5}{2}\right) \mathcal{I}_{\frac{p}{2}-3}(\kappa_1)}{\Gamma\left(\frac{p-1}{2}\right) \mathcal{I}_{\frac{p}{2}-1}(\kappa_1)} + (1-\lambda) \frac{(\kappa_2/2)^2 \Gamma\left(\frac{p-5}{2}\right) \mathcal{I}_{\frac{p}{2}-3}(\kappa_2)}{\Gamma\left(\frac{p-1}{2}\right) \mathcal{I}_{\frac{p}{2}-1}(\kappa_2)} \\
&= \lambda \frac{\kappa_1^2 \mathcal{I}_{\frac{p}{2}-3}(\kappa_1)}{(p-3)(p-5)\mathcal{I}_{\frac{p}{2}-1}(\kappa_1)} + (1-\lambda) \frac{\kappa_2^2 \mathcal{I}_{\frac{p}{2}-3}(\kappa_2)}{(p-3)(p-5)\mathcal{I}_{\frac{p}{2}-1}(\kappa_2)} \\
&= \frac{\lambda \kappa_1^2}{(p-3)(p-5)} \left(1 + \frac{p-4}{\kappa_1 A_{p-2}(\kappa_1)}\right) \\
&\quad + \frac{(1-\lambda)\kappa_2^2}{(p-3)(p-5)} \left(1 + \frac{p-4}{\kappa_2 A_{p-2}(\kappa_2)}\right) \\
&\leq \frac{\lambda \kappa_1^2}{(p-3)(p-5)} \left(1 + \frac{p-4}{\kappa_1 A_{p-2}(\kappa_1)}\right) \\
&\quad + \frac{(1-\lambda)\kappa_2^2}{(p-3)(p-5)} \left(1 + \frac{p-4}{\kappa_2 A_{p-2}(\kappa_2)}\right),
\end{aligned}$$

where $\mathcal{I}_{\frac{p}{2}-3}(\kappa) = \mathcal{I}_{\frac{p}{2}-1}(\kappa) + \frac{p-4}{\kappa}\mathcal{I}_{\frac{p}{2}-2}(\kappa)$ due to (S.2.6) in supplement of [4].

$$\begin{aligned}
\mathbb{E}(v_i^{-2}) &= \lambda \frac{(\kappa_1/2) \Gamma\left(\frac{p-3}{2}\right) \mathcal{I}_{\frac{p}{2}-2}(\kappa_1)}{\Gamma\left(\frac{p-1}{2}\right) \mathcal{I}_{\frac{p}{2}-1}(\kappa_1)} + (1-\lambda) \frac{(\kappa_2/2) \Gamma\left(\frac{p-3}{2}\right) \mathcal{I}_{\frac{p}{2}-2}(\kappa_2)}{\Gamma\left(\frac{p-1}{2}\right) \mathcal{I}_{\frac{p}{2}-1}(\kappa_2)} \\
&= \lambda \frac{\kappa_1 \mathcal{I}_{\frac{p}{2}-2}(\kappa_1)}{(p-3)\mathcal{I}_{\frac{p}{2}-1}(\kappa_1)} + (1-\lambda) \frac{\kappa_2 \mathcal{I}_{\frac{p}{2}-2}(\kappa_2)}{(p-3)\mathcal{I}_{\frac{p}{2}-1}(\kappa_2)} \\
&= \frac{\lambda \kappa_1}{(p-3)A_{p-2}(\kappa_1)} + \frac{(1-\lambda)\kappa_2}{(p-3)A_{p-2}(\kappa_2)}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\frac{\mathbb{E}(v_i^{-4})}{\mathbb{E}^2(v_i^{-2})} &= \frac{\frac{\lambda \kappa_1^2}{(p-3)(p-5)} \left(1 + \frac{p-4}{\kappa_1 A_{p-2}(\kappa_1)}\right) + \frac{(1-\lambda)\kappa_2^2}{(p-3)(p-5)} \left(1 + \frac{p-4}{\kappa_2 A_{p-2}(\kappa_2)}\right)}{\left[\frac{\lambda \kappa_1}{(p-3)A_{p-2}(\kappa_1)} + \frac{(1-\lambda)\kappa_2}{(p-3)A_{p-2}(\kappa_2)}\right]^2} \\
&\leq \frac{\frac{\lambda \kappa_1^2}{(p-3)(p-5)} \left(1 + \frac{p-4}{\kappa_1 A_{p-2}(\kappa_1)}\right) + \frac{(1-\lambda)\kappa_2^2}{(p-3)(p-5)} \left(1 + \frac{p-4}{\kappa_2 A_{p-2}(\kappa_2)}\right)}{\left[\frac{\lambda \kappa_1}{(p-3)A_{p-2}(\kappa_1)}\right]^2 + \left[\frac{(1-\lambda)\kappa_2}{(p-3)A_{p-2}(\kappa_2)}\right]^2}.
\end{aligned}$$

Because

$$\frac{\frac{\kappa_1^2}{(p-3)(p-5)} \left(1 + \frac{p-4}{\kappa_1 A_{p-2}(\kappa_1)}\right)}{\left[\frac{\kappa_1}{(p-3)A_{p-2}(\kappa_1)}\right]^2} = O(1)$$

and

$$\frac{\frac{\kappa_2^2}{(p-3)(p-5)} \left(1 + \frac{p-4}{\kappa_1 A_{p-2}(\kappa_2)}\right)}{\left[\frac{\kappa_2}{(p-3)A_{p-2}(\kappa_2)}\right]^2} = O(1),$$

Then, we have

$$\frac{\mathbb{E}(v_i^{-4})}{\mathbb{E}^2(v_i^{-2})} = O(1).$$

Then, we obtain the conclusion that under the FvML and mixed FvML distributions, the weight ω_{IN} satisfies condition (C1). Similarly, it is easy to verify that under the FvML and mixed FvML distributions, the weight $\omega_{\mathrm{N}}, \omega_{\mathrm{C}}, \omega_{\mathrm{R}}, \omega_{\mathrm{S}}$ satisfies condition (C1).

Proof of Proposition 2.1

Under H_0, we have that

$$\widetilde{W}_n / \frac{T_n(\omega_{\mathrm{N}})}{\sqrt{\sigma_n^2(\omega_N)}} = \sqrt{\frac{p-1}{p}} \frac{n-1}{n} \frac{b_2(\omega_N)}{n^{-1}\sum_{i=1}^n v_{i0}^2}.$$

Therefore, $\widetilde{W}_n / \frac{T_n(\omega_{\mathrm{N}})}{\sqrt{\sigma_n^2(\omega_N)}} \to^p 1$ as n, $p \to \infty$, because $n^{-1}\sum_{i=1}^n v_{i0}^2 \to^p \mathbb{E}(v_{i0}^2) = b_2(\omega_N)$. Then, under Condition (C0) and H_0, this leads to $\widetilde{W}_n / \frac{T_n(\omega_{\mathrm{N}})}{\sqrt{\hat{\sigma}_n^2(\omega_N)}} \to^p 1$ as n, $p \to \infty$, due to Theorem 3.2. □

Proof of Corollary 3.1

$\boldsymbol{X}_i$ follows the FvML distribution, hence we have that $f(\cdot) = \exp(\cdot)$.

First, we will prove that v_i converges to a constant with probability one. To this end, we just need to show that, under the FvML distribution, u_i converges to $\mathbb{E}(u_i)$ with probability one. Specifically, due to the equation on the last line of the fourth page of [14], we have

$$\operatorname{var}(u_i) = 1 - \frac{p-1}{\kappa} A_p(\kappa) - \{A_p(\kappa)\}^2, \tag{A.8}$$

where $A_p(\kappa) \doteq I_{p/2}(\kappa)/I_{p/2-1}(\kappa)$, and

$$\mathcal{I}_\nu(\kappa) \doteq \frac{(\kappa/2)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)} \int_{-1}^1 \left(1-s^2\right)^{\nu-\frac{1}{2}} \exp(\kappa s) ds$$

is the modified Bessel function of the first kind and of order v. Due to Lemma S.2.2(i) in the supplement of [4], we have

$$A_p(\kappa) \geq G_{\frac{p}{2},\frac{p}{2}}(\kappa) \doteq \frac{\kappa}{\frac{p}{2} + \sqrt{\kappa^2 + \left(\frac{p}{2}\right)^2}}. \tag{A.9}$$

Hence,

$$\operatorname{var}(u_i) = 1 - \frac{p-1}{\kappa} A_p(\kappa) - \{A_p(\kappa)\}^2$$

$$
\begin{aligned}
&\leq 1-\frac{p-1}{\kappa}\frac{\kappa}{\frac{p}{2}+\sqrt{\kappa^2+\left(\frac{p}{2}\right)^2}}-\frac{\kappa^2}{\left(\frac{p}{2}+\sqrt{\kappa^2+\left(\frac{p}{2}\right)^2}\right)^2}\\
&=1-\frac{p-1}{\frac{p}{2}+\sqrt{\kappa^2+\left(\frac{p}{2}\right)^2}}-\frac{\kappa^2}{\left(\frac{p}{2}+\sqrt{\kappa^2+\left(\frac{p}{2}\right)^2}\right)^2}\\
&\leq 1-\frac{p-1}{\frac{p}{2}+\sqrt{\kappa^2+\left(\frac{p}{2}\right)^2}}-\frac{\kappa^2}{\left(\frac{p}{2}+\sqrt{\kappa^2+\left(\frac{p}{2}\right)^2}\right)^2}\\
&\leq \frac{\sqrt{\kappa^2+\left(\frac{p}{2}\right)^2}-\frac{p}{2}+1}{\frac{p}{2}+\sqrt{\kappa^2+\left(\frac{p}{2}\right)^2}}-\frac{\kappa^2}{\left(\frac{p}{2}+\sqrt{\kappa^2+\left(\frac{p}{2}\right)^2}\right)^2}\\
&\leq \frac{1}{\frac{p}{2}+\sqrt{\kappa^2+\left(\frac{p}{2}\right)^2}}+\frac{\sqrt{\kappa^2+\left(\frac{p}{2}\right)^2}-\frac{p}{2}}{\frac{p}{2}+\sqrt{\kappa^2+\left(\frac{p}{2}\right)^2}}-\frac{\kappa^2}{\left(\frac{p}{2}+\sqrt{\kappa^2+\left(\frac{p}{2}\right)^2}\right)^2}\\
&\leq \frac{1}{p}+\frac{(\sqrt{\kappa^2+\left(\frac{p}{2}\right)^2}-\frac{p}{2})\left(\frac{p}{2}+\sqrt{\kappa^2+\left(\frac{p}{2}\right)^2}\right)}{\left(\frac{p}{2}+\sqrt{\kappa^2+\left(\frac{p}{2}\right)^2}\right)^2}\\
&\quad-\frac{\kappa^2}{\left(\frac{p}{2}+\sqrt{\kappa^2+\left(\frac{p}{2}\right)^2}\right)^2}\leq\frac{1}{p}. \qquad \text{(A.10)}
\end{aligned}
$$

Besides, there exist a positive integer p_0 and a real constant c such that

$$
\mathbb{E}\left\{u_i-\mathbb{E}(u_i)\right\}^4\leq c\mathbb{E}\left\{u_i-\mathbb{E}(u_i)\right\}^2
$$

for any $p\geq p_0$ and any $\kappa>0$ (see Lemma S.2.1(ii) in the supplement of [4]). Hence, $\mathbb{E}\left\{u_i-\mathbb{E}(u_i)\right\}^4\leq c/p^2$ due to (A.10). Thus, we have $\mathbb{E}\left\{u_i-\mathbb{E}(u_i)\right\}^4=O(p^{-2})$ and

$$
\sum_p^\infty \mathrm{P}\left(|u_i-\mathbb{E}(u_i)|\geq\epsilon\right)<\sum_p^\infty\frac{\mathbb{E}\{u_i-\mathbb{E}(u_i)\}^4}{\epsilon^4}<\infty,\ \forall\epsilon>0,
$$

that is, $u_i\to\mathbb{E}(u_i)$ with probability one.

Then, we have that v_i converges to a constant with probability one due to $v_i=\sqrt{1-u_i^2}$, which leads to

$$
\frac{\mathbb{E}^2(u_i)\mathbb{E}^2\{\omega(v_i)v_i^{-1}\}}{\mathbb{E}\{\omega^2(v_i)\}}\to\mathbb{E}^2(u_i)\mathbb{E}(v_i^{-2}).
$$

Hence, the asymptotic power of all the $T_n(\omega)$-based tests is

$$\beta = \lim_{n,p\to\infty} \Phi\left[-z_\alpha + \mathbb{E}^2(u_i)\mathbb{E}(v_i^{-2})\frac{p^{1/2}n\{\boldsymbol{\theta}^{\mathrm{T}}(\boldsymbol{\theta}\boldsymbol{\theta}^{\mathrm{T}} - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{\theta}\}}{\sqrt{2}}\right]. \qquad \square$$

Proof of Proposition 3.1

Under H_0, since $\boldsymbol{S}_i = \mathbf{U}\{(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i\}$ and v_i are independent, so $\boldsymbol{S}_i$ and R_i are also independent. We have

$$\mathbb{E}\left(\frac{2}{n(n-1)}\sum\sum_{i<j} R_i R_j \boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{S}_j\right) = 0.$$

Then, the variance of R_W is

$$\begin{aligned}
&\mathrm{var}\left(\frac{2}{n(n-1)}\sum\sum_{i<j} R_i R_j \boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{S}_j\right)\\
&= \frac{2}{n(n-1)}\mathbb{E}\left\{R_i^2 R_j^2 (\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{S}_j)^2\right\}\\
&= \frac{2}{n(n-1)}\mathbb{E}(R_i^2 R_j^2)\mathbb{E}(\boldsymbol{S}_i^{\mathrm{T}}\boldsymbol{S}_j)^2\\
&= \frac{2}{n(n-1)(p-1)}\mathbb{E}(R_i^2 R_j^2)\\
&= 2n^{-4}p^{-1}\sum\sum_{i\neq j} i^2 j^2\{1+o(1)\}\\
&= \sigma_n^2(R_W)\{1+o(1)\},
\end{aligned}$$

where $\sigma_n^2(R_W) = 2n^{-4}p^{-1}\sum\sum_{i\neq j} i^2 j^2 = O(n^2 p^{-1})$. Similar to the proof of Theorem 3.1, to prove the normality of $R_W/\sqrt{\sigma_n^2(R_W)}$, we only need to show

$$\frac{\sum_{j=2}^n \mathbb{E}(Z_{nj}^2 \mid \mathcal{F}_{n,j-1})}{\sigma_n^2(R_W)} \xrightarrow{p} 1, \tag{A.11}$$

and

$$\mathbb{E}\left\{\sum_{j=2}^n \mathbb{E}(Z_{nj}^4|\mathcal{F}_{n,j-1})\right\} = o\{\sigma_n^4(R_W)\}, \tag{A.12}$$

where $Z_{ni} = 2/\{n(n-1)\}\sum_{j=1}^{i-1}\boldsymbol{V}_i^{\mathrm{T}}\boldsymbol{V}_j$, $\boldsymbol{V}_i = R_i\boldsymbol{S}_i$. Let $\mathbf{A} = \mathbb{E}(\boldsymbol{V}_i\boldsymbol{V}_i^{\mathrm{T}})$, it can be shown that

$$\sum_{j=2}^n \mathbb{E}(Z_{nj}^2|\mathcal{F}_{n,j-1}) = \sum_{j=2}^n \mathbb{E}\left[\left\{\frac{2}{n(n-1)}\sum_{i=1}^{j-1}\boldsymbol{V}_i^{\mathrm{T}}\boldsymbol{V}_j\right\}^2 |\mathcal{F}_{n,j-1}\right]$$

$$
\begin{aligned}
=&\sum_{j=2}^{n}\frac{4}{n^2(n-1)^2}\mathbb{E}\left\{\left(\sum_{i_1=1}^{j-1}\sum_{i_2=1}^{j-1}\boldsymbol{V}_{i_1}^{\mathrm{T}}\boldsymbol{V}_j\boldsymbol{V}_{i_2}^{\mathrm{T}}\boldsymbol{V}_j\right)|\mathcal{F}_{n,j-1}\right\}\\
=&\sum_{j=2}^{n}\frac{4}{n^2(n-1)^2}\sum_{i_1=1}^{j-1}\sum_{i_2=1}^{j-1}\boldsymbol{V}_{i_1}^{\mathrm{T}}\mathbb{E}(\boldsymbol{V}_j\boldsymbol{V}_j^{\mathrm{T}})\boldsymbol{V}_{i_2}\\
=&C_{n1}+C_{n2},
\end{aligned}
$$

where

$$
C_{n1}=\frac{4}{n^2(n-1)^2}\sum_{j=2}^{n}\sum_{i=1}^{j-1}\boldsymbol{V}_i^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_i \quad\text{and}\quad C_{n2}=\frac{8}{n^2(n-1)^2}\sum_{j=2}^{n}\sum_{i_1<i_2}^{j-1}\sum^{j-1}\boldsymbol{V}_{i_1}^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_{i_2}.
$$

Next, we consider C_{n1} and note that $\mathbf{A}=\mathbb{E}(R_i^2)(\mathbf{I}_p-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})/(p-1)$ due to Lemma A.1. Furthermore, we have

$$
\begin{aligned}
\mathbb{E}(C_{n1})=&\frac{2}{n(n-1)}\mathbb{E}(\boldsymbol{V}_i^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_i)\\
=&\frac{2}{n(n-1)}\mathrm{tr}\{\mathbb{E}(\mathbf{A}\boldsymbol{V}_i\boldsymbol{V}_i^{\mathrm{T}})\}\\
=&\frac{2}{n(n-1)}\mathrm{tr}(\mathbf{A}^2)\\
=&\sigma_n^2(R_W)\{1+o(1)\},
\end{aligned}
$$

where $\boldsymbol{S}_i=(\mathbf{I}_p-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})\boldsymbol{X}_i/v_i$ and the last equality holds due to $\mathbb{E}(R_i^2R_j^2)=\mathbb{E}^2(R_i^2)\{1-o(1)\}$. We also obtain

$$
\begin{aligned}
\boldsymbol{S}_i^{\mathrm{T}}\mathbf{A}\boldsymbol{S}_i=&\frac{\mathbb{E}(R_i^2)}{v_i^2(p-1)}\boldsymbol{X}_i^{\mathrm{T}}(\mathbf{I}_p-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})(\mathbf{I}_p-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}))(\mathbf{I}_p-\boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}}))\boldsymbol{X}_i\\
=&\frac{\mathbb{E}(R_i^2)}{(p-1)}.
\end{aligned}
$$

Thus,

$$
(\boldsymbol{S}_i^{\mathrm{T}}\mathbf{A}\boldsymbol{S}_i)^2=\frac{\mathbb{E}^2(R_i^2)}{(p-1)^2}. \tag{A.13}
$$

So, due to $\mathbb{E}(R_i^4)=O(n^4)$ and $\mathbb{E}(R_i^2)=O(n^2)$,

$$
\begin{aligned}
\mathrm{var}(C_{n1})=&\mathrm{var}\left(\frac{4}{n^2(n-1)^2}\sum_{j=2}^{n}\sum_{i=1}^{j-1}\boldsymbol{V}_i^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_i\right)\\
=&\mathrm{var}\left\{\frac{4}{n^2(n-1)^2}\sum_{j=2}^{n}\sum_{i=1}^{j-1}\frac{\mathbb{E}(R_i^2)}{(p-1)}R_i^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{var}\left\{\frac{4}{n^2(n-1)^2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{\mathbb{E}(R_i^2)}{(p-1)}R_i^2\right\}\\
&= \mathrm{var}\left\{\frac{4}{n^2(n-1)^2}\sum_{i=1}^{n}\frac{\mathbb{E}(R_i^2)(n-i)}{(p-1)}R_i^2\right\}\\
&\le O\{n^{-5}(p-1)^{-2}\}\mathbb{E}(R_i^4)\mathbb{E}^2(R_i^2)\\
&= o\{\sigma_n^4(R_W)\}.
\end{aligned}
$$

Then, we have $C_{n1}/\sigma_n^2(R_W) \to^p 1$. Similarly,

$$
\begin{aligned}
\mathbb{E}(C_{n2}^2) &= \mathbb{E}\left\{\frac{8}{n^2(n-1)^2}\sum_{j=2}^{n}\sum_{i_1<i_2}^{j-1}\sum^{j-1}\boldsymbol{V}_{i_1}^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_{i_2}\right\}^2\\
&= O(n^{-8})\mathbb{E}\left(\sum_{j_1=2}^{n}\sum_{j_2=2}^{n}\sum_{i_1<i_2}^{j_1-1}\sum^{j_1-1}\sum_{i_3<i_4}^{j_2-1}\sum^{j_2-1}\boldsymbol{V}_{i_1}^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_{i_2}\boldsymbol{V}_{i_3}^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_{i_4}\right)\\
&= O(n^{-8})\mathbb{E}\left\{\sum_{j_1<j_2}^{n}\sum\sum_{i_1<i_2}^{j_1-1}\sum^{j_1-1}(\boldsymbol{V}_{i_1}^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_{i_2})^2\right\}\\
&= O(n^{-8})\mathbb{E}\left\{\sum_{j_2=2}^{n}\sum_{j_1=1}^{j_2-1}\sum_{i_1<i_2}^{j_1-1}\sum^{j_1-1}(\boldsymbol{V}_{i_1}^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_{i_2})^2\right\}\\
&= O(n^{-4})\mathbb{E}\left\{(\boldsymbol{V}_i^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_j)^2\right\}\\
&= O(n^{-4})\mathbb{E}\left\{R_i^2R_j^2(\boldsymbol{S}_i^{\mathrm{T}}\mathbf{A}\boldsymbol{S}_j)^2\right\}\\
&= O(n^{-4})\mathbb{E}(R_i^2R_j^2)\mathrm{tr}\left\{\mathbb{E}(\boldsymbol{S}_i^{\mathrm{T}}\mathbf{A}\boldsymbol{S}_j\boldsymbol{S}_j^{\mathrm{T}}\mathbf{A}\boldsymbol{S}_i)\right\}\\
&= O(n^{-4}p^{-3})\mathbb{E}(R_i^2R_j^2)\mathbb{E}^2(R_i^2) = o\{\sigma_n^4(R_W)\},
\end{aligned}
$$

where $i \neq j$ in the fourth equality and we used $\mathbf{A} = (p-1)^{-1}\mathbb{E}(R_i^2)(\mathbf{I}_p - \boldsymbol{\theta}_0\boldsymbol{\theta}_0^{\mathrm{T}})$ and Lemma A.1 (ii) in the seventh equality. Then (A.11) holds.

Next, we only need to show

$$
\mathbb{E}\left\{\sum_{j=2}^{n}\mathbb{E}(Z_{nj}^4|\mathcal{F}_{n,j-1})\right\} = o\{\sigma_n^4(R_W)\}.
$$

Note that

$$
\begin{aligned}
\mathbb{E}\left\{\sum_{j=2}^{n}\mathbb{E}(Z_{nj}^4|\mathcal{F}_{n,j-1})\right\} &= \sum_{j=2}^{n}\mathbb{E}\left\{\mathbb{E}(Z_{nj}^4|\mathcal{F}_{n,j-1})\right\}\\
&= \sum_{j=2}^{n}\mathbb{E}(Z_{nj}^4)
\end{aligned}
$$

$$
\begin{aligned}
&=\sum_{j=2}^{n}\mathbb{E}\left(\left(\frac{2}{n(n-1)}\sum_{j=1}^{i-1}\boldsymbol{V}_i^{\mathrm{T}}\boldsymbol{V}_j\right)^4\right)\\
&=O(n^{-8})\sum_{j=2}^{n}\mathbb{E}\left(\sum_{i=1}^{j-1}\boldsymbol{V}_j^{\mathrm{T}}\boldsymbol{V}_i\right)^4 \qquad \text{(A.14)}
\end{aligned}
$$

can be decomposed as $Q+P$, where

$$
\begin{aligned}
Q &=O(n^{-8})\sum_{j=2}^{n}\sum_{i_1<i_2}^{j-1}\sum^{j-1}\mathbb{E}\left(\boldsymbol{V}_j^{\mathrm{T}}\boldsymbol{V}_{i_1}\boldsymbol{V}_{i_1}^{\mathrm{T}}\boldsymbol{V}_j\boldsymbol{V}_j^{\mathrm{T}}\boldsymbol{V}_{i_2}\boldsymbol{V}_{i_2}^{\mathrm{T}}\boldsymbol{V}_j\right),\\
P &=O(n^{-8})\sum_{j=2}^{n}\sum_{i=1}^{j-1}\mathbb{E}\left\{(\boldsymbol{V}_j^{\mathrm{T}}\boldsymbol{V}_i)^4\right\}.
\end{aligned}
$$

Because

$$
\begin{aligned}
Q &=O(n^{-5})\mathbb{E}\left(\boldsymbol{V}_j^{\mathrm{T}}\boldsymbol{V}_{i_1}\boldsymbol{V}_{i_1}^{\mathrm{T}}\boldsymbol{V}_j\boldsymbol{V}_j^{\mathrm{T}}\boldsymbol{V}_{i_2}\boldsymbol{V}_{i_2}^{\mathrm{T}}\boldsymbol{V}_j\right)\\
&=O(n^{-5})\mathrm{tr}\left\{\mathbb{E}(\boldsymbol{V}_{i_1}\boldsymbol{V}_{i_1}^{\mathrm{T}}\boldsymbol{V}_j\boldsymbol{V}_j^{\mathrm{T}}\boldsymbol{V}_{i_2}\boldsymbol{V}_{i_2}^{\mathrm{T}}\boldsymbol{V}_j\boldsymbol{V}_j^{\mathrm{T}})\right\}\\
&=O(n^{-5})\mathrm{tr}\left[\mathbb{E}\left\{\mathbb{E}\left(\boldsymbol{V}_{i_1}\boldsymbol{V}_{i_1}^{\mathrm{T}}\boldsymbol{V}_j\boldsymbol{V}_j^{\mathrm{T}}\boldsymbol{V}_{i_2}\boldsymbol{V}_{i_2}^{\mathrm{T}}\boldsymbol{V}_j\boldsymbol{V}_j^{\mathrm{T}}|\boldsymbol{V}_j\right)\right\}\right]\\
&=O(n^{-5})\mathrm{tr}\left\{\mathbb{E}(\mathbf{A}\boldsymbol{V}_j\boldsymbol{V}_j^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_j\boldsymbol{V}_j^{\mathrm{T}})\right\}\\
&=O(n^{-5})\mathbb{E}\left\{(\boldsymbol{V}_j^{\mathrm{T}}\mathbf{A}\boldsymbol{V}_j)^2\right\},
\end{aligned}
$$

$Q=O\{n^{-5}(p-1)^{-2}\}\mathbb{E}(R_j^4)\mathbb{E}^2(R_i^2)=o\{\sigma_n^4(R_W)\}$ due to (A.13). Similarly, due to Lemma A.1 (iii), we can show that

$$
\begin{aligned}
P &= O(n^{-6})\mathbb{E}\left\{(\boldsymbol{V}_j^{\mathrm{T}}\boldsymbol{V}_i)^4\right\}\\
&= O(n^{-6})\mathbb{E}\left\{(R_i^4R_j^4)(\boldsymbol{S}_j^{\mathrm{T}}\boldsymbol{S}_i)^4\right\}\\
&= O(n^{-6})\mathbb{E}(R_i^4R_j^4)\mathbb{E}\left\{(\boldsymbol{S}_j^{\mathrm{T}}\boldsymbol{S}_i)^4\right\}\\
&= O(n^{-6}p^{-2})\mathbb{E}(R_i^4R_j^4)\\
&= o\{\sigma_n^4(R_W)\}. \qquad \text{(A.15)}
\end{aligned}
$$

So, we have complete the proof of $R_W/\sqrt{\sigma_n^2(R_W)}$. Similarly, $R_S/\sqrt{\sigma_n^2(R_S)}$ is also asymptotically standard normal. Finally, we obtain the conclusion. □

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