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Hopf bifurcation in a Mean-Field model of spiking neurons*

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Abstract

We study a family of non-linear McKean-Vlasov SDEs driven by a Poisson measure, modelling the mean-field asymptotic of a network of generalized Integrate-and-Fire neurons. We give sufficient conditions to have periodic solutions through a Hopf bifurcation. Our spectral conditions involve the location of the roots of an explicit holomorphic function. The proof relies on two main ingredients. First, we introduce a discrete time Markov Chain modeling the phases of the successive spikes of a neuron. The invariant measure of this Markov Chain is related to the shape of the periodic solutions. Secondly, we use the Lyapunov-Schmidt method to obtain self-consistent oscillations. We illustrate the result with a toy model for which all the spectral conditions can be analytically checked.

Keywords: McKean-Vlasov SDE; long time behavior; Hopf bifurcation; mean-field interaction; Volterra integral equation; piecewise deterministic Markov process.
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1 Introduction

We consider a mean-field model of spiking neurons. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$, $b : \mathbb{R}_+ \to \mathbb{R}$ and let $\mathbf{N}(du, dz)$ be a Poisson measure on $\mathbb{R}_+^2$ with intensity the Lebesgue measure $dudz$. Consider the following McKean-Vlasov SDE

$$X_t = X_0 + \int_0^t b(X_u)du + J\int_0^t \mathbb{E}\, f(X_u)du - \int_0^t \int_{\mathbb{R}_+} X_{u-}\mathbb{1}_{\{z\leq f(X_{u-})\}}\mathbf{N}(du, dz). \tag{1.1}$$

Here, $J \geq 0$ is a deterministic constant (it models the strength of the interactions) and the initial condition X_0 is independent of the Poisson measure.

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Informally, the SDE (1.1) can be understood in the following sense: between the jumps, X_t solves the scalar ODE $\dot{X}_t = b(X_t) + J\,\mathbb{E}\,f(X_t)$ and X_t jumps to 0 at rate $f(X_t)$. We assume that $b(0) \geq 0$ and that $X_0 \geq 0$, such that the dynamics lives on $\mathbb{R}_+$. This SDE is non-linear in the sense of McKean-Vlasov, because of the interaction term $\mathbb{E}\,f(X_t)$ which depends on the law of X_t. Let $\nu(t, dx) := \mathcal{L}(X_t)$ be the law of X_t. It solves the following non-linear Fokker-Planck equation, in the sense of measures:

$$\begin{aligned} &\partial_t \nu(t, dx) + \partial_x \left[(b(x) + Jr_t)\nu(t, dx)\right] + f(x)\nu(t, dx) = r_t \delta_0(dx) \\ &\nu(0, dx) = \mathcal{L}(X_0), \quad r_t = \int_{\mathbb{R}_+} f(x)\nu(t, dx). \end{aligned} \tag{1.2}$$

Here δ_0 is the Dirac measure in 0. If furthermore $\mathcal{L}(X_t)$ has a density for all t, that is $\mathcal{L}(X_t) = \nu(t, x)dx$ then $\nu(t, x)$ solves the following strong form of (1.2)

$$\begin{aligned} &\partial_t \nu(t, x) + \partial_x \left[(b(x) + Jr_t)\nu(t, x)\right] + f(x)\nu(t, x) = 0, \\ &\nu(0, x)dx = \mathcal{L}(X_0), \quad r_t = \int_{\mathbb{R}_+} f(x)\nu(t, x)dx, \end{aligned}$$

with the boundary condition

$$\forall t > 0, \quad (b(0) + Jr_t)\,\nu(t, 0) = r_t.$$

We study the existence of periodic solution of this non-linear Fokker-Planck equation. We give sufficient conditions for the existence of a Hopf bifurcation around a stationary solution of (1.2).

Associated particle system

Equations (1.1) and (1.2) appeared (see e.g. [7]) as the limit of the following networks of neurons. For each $N \geq 1$, consider *i.i.d.* initial potentials $(X_0^{i,N})_{i \in \{1, \cdots, N\}}$ with law $\mathcal{L}(X_0)$. The *càdlàg* process $(X_t^{i,N})_{i \in \{1, \cdots, N\}} \in \mathbb{R}^N$ is a PDMP: between the jumps each $X_t^{i,N}$ solves the ODE $\dot{X}_t^{i,N} = b(X_t^{i,N})$ and "spikes" with rate $f(X_t^{i,N})$. When a spike occurs, say neuron i spikes at (random) time τ, its potential is reset to 0 while the others receive a "kick" of size $\frac{J}{N}$:

$$X_{\tau_+}^{i,N} = 0, \quad \text{and} \quad \forall k \neq i, \quad X_{\tau_+}^{k,N} = X_{\tau_-}^{k,N} + \frac{J}{N}.$$

This completely defines the particle system. As N goes to infinity, a phenomenon of *propagation of chaos* occurs. In particular, each neuron, say $(X_t^{1,N})_{t \geq 0}$, converges in law to the solution of (1.1). We refer to [13] for a proof of such convergence result under stronger assumptions. There is a qualitative difference between the particle system and the solution of the limit equation (1.1): for a fixed value of N, the particle system is Harris ergodic (see [11], where this result is proved under stronger assumptions on b and f) and so it admits a unique, globally attractive, invariant measure. Thus, there are no stable oscillations when the number of particles is finite. For the limit equation however, the long time behavior is richer: for fixed values of the parameters there can be multiple invariant measures (see [5] and [6] for some explicit examples) and, as shown here, there can exist periodic solutions (see Figure 1).

Literature

From a mathematical point of view, this model has been first introduced by [7], after many considerations by physicists (see for instance [24], [14] and [4] and references therein). In [13], the existence of solution of (1.1), path-wise uniqueness and convergence of the particle system are addressed. The long time behavior of the solution to (1.1) is studied in [5] in the case of weak interactions: b and f being fixed, the authors prove that there exists a constant $\bar{J}$ (depending on b and f) such that for all $J < \bar{J}$, (1.1) admits a unique globally attractive invariant measure. Finally in [6], the local stability of an invariant measure is studied with no further assumptions on the size of the interactions J. It is proved that the stability of an invariant measure is given by the location of the roots of some holomorphic function. In [22], the authors study a "metastable" behavior of the particle system. They give examples of drifts b and rate functions f where the particle system follows the long time behavior of the mean-field model for an exponential large time, before finally converging to its (unique) invariant probability measure.

The model studied in the current paper belongs to the class of generalized integrate-and-fire neurons, whose most celebrated example is the "fixed threshold" model (see for instance [2], [8] and the references therein). Many of the techniques developed here also apply to this variant. However, it would require additional work to overcome the specific difficulties due to the fixed threshold setting. In particular, there are no simple explicit expressions of the kernels introduced in the current paper.

In [10], numerical evidences are given for the existence of a Hopf bifurcation in a close setting: the dynamics between the jumps is (as in [7]) given by

$$\dot{X}_t = -(X_t - \mathbb{E}\, X_t) + J\, \mathbb{E}\, f(X_t).$$

In particular the potentials of each neuron are attracted to their common mean. This models "electrical synapses", while $J\, \mathbb{E}\, f(X_t)$ models the chemical synapses. Oscillations with both electrical and chemical synapses is also studied in a different model in [23]. In this work, the mean-field equation is a 2D-ODE and so the analysis of the Hopf bifurcation is standard. Finally, oscillations with multi-populations such as with both excitatory and inhibitory neurons have been extensively studied in neuroscience. For instance in [9], it is shown that multi-populations of mean-field Hawkes processes can oscillate. Again, the dynamics is reduced to a finite dimension ODE.

It is well-known that the long time behavior of McKean-Vlasov SDEs can be significantly different from markovian SDEs. In [25] and [26], the author gives simple examples of such non-linear SDEs which oscillate. Again, in these examples, the dynamics can be reduced to an ordinary differential equation. To go beyond ODEs, the framework of Delay differential equation is often used: see for instance [27] for the study of Hopf bifurcations for such equations, based on the Lyapunov-Schmidt method. In [20, 21] the authors study periodic solutions of a McKean-Vlasov SDE using a slow-fast approach. Another approach is to use the center manifold theory to reduce infinite dimensional problem to manifold of finite dimension: we refer to [18] (see also [15] for an application to some McKean-Vlasov SDE). Finally, in [19] an abstract framework is presented to study Hopf bifurcations for some classes of regular PDEs. Even though our proof is not based on the PDE (1.2) (but on the Volterra integral equation described below), we follow the methodology of [19] to obtain our main result.

Regularity of the drift and of the jump function.

We make the following regularity assumptions on b and f.

Assumption 1.1. *The drift* $b : \mathbb{R}_+ \to \mathbb{R}$ *is* $\mathcal{C}^2$, *with* $b(0) \geq 0$ *and* $\sup_{x \geq 0} |b'(x)| + |b''(x)| < \infty$.

Assumption 1.2. *The function* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *is* $\mathcal{C}^2$, *strictly increasing, with* $\sup_{x \geq 1} \frac{|f''(x)|}{f(x)} < \infty$ *and there exists a constant* C_f *such that*

1.2(a) for all $x, y \geq 0$, $f(xy) \leq C_f(1 + f(x))(1 + f(y))$.

1.2(b) for all $A > 0$, $\sup_{x \geq 0} A f'(x) - f(x) < \infty$.

1.2(c) for all $x \geq 0$, $|b(x)| \leq C_f(1 + f(x))$.

Remark 1.3. If a non-decreasing function f satisfies Assumption 1.2(a), there exists another constant C_f such that for all $x, y \geq 0$, $f(x+y) \leq C_f(1 + f(x) + f(y))$. Moreover, it also implies that f grows at most at a polynomial rate: there exists a constant $p > 0$ such that

$$\sup_{x \geq 1} f(x)/x^p < \infty.$$

Note that for instance, for all $p \geq 1$, the function $f : x \mapsto x^p$ satisfies Assumption 1.2. More generally, any continuous function such that $f(x) \sim_{x\to\infty} x^p$ for some $p \geq 0$ satisfies Assumption 1.2(a).

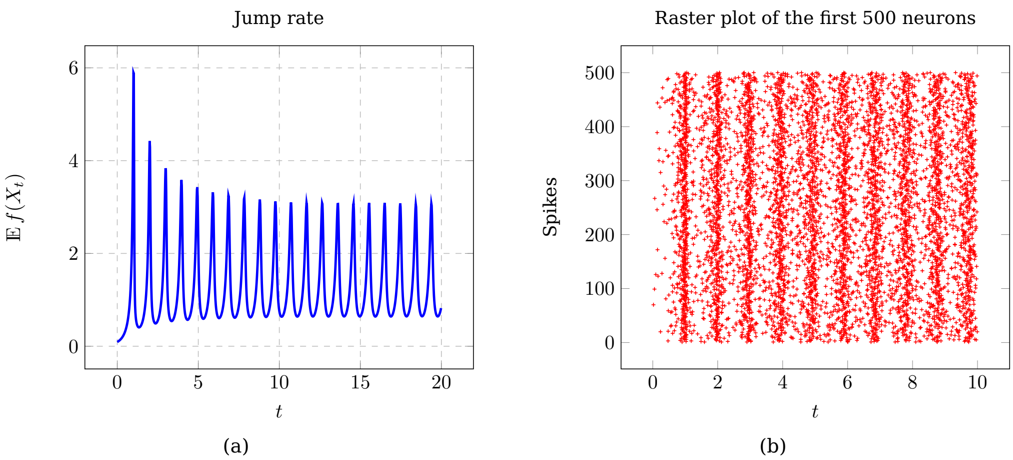

Figure 1: Consider the following example where for all $x \geq 0$, $f(x) = x^{10}$, $b(x) = 2 - 2x$ and $J = 0.8$. Using a Monte-Carlo method, we simulate the particle systems with $N = 8 \cdot 10^5$ neurons, starting at $t = 0$ with i.i.d. uniformly distributed random variables on $[0, 1]$. Stable oscillations appear. (a) Empirical mean number of spikes per unit of time. (b) Each red cross corresponds to a spike of one of the first 500 neurons (spike raster plot).

The Volterra integral equation

As in [5, 6], we study the long time behavior of the solution of (1.1) through its "linearized" version: given a non-negative scalar function $\boldsymbol{a} \in L^\infty(\mathbb{R}_+; \mathbb{R}_+)$, consider the non-homogeneous linear SDE:

$$\forall t \geq s, \quad Y^{\boldsymbol{a},\nu}_{t,s} = Y_s + \int_s^t [b(Y^{\boldsymbol{a},\nu}_{u,s}) + a_u] du - \int_s^t \int_{\mathbb{R}_+} Y^{\boldsymbol{a},\nu}_{u-,s} \mathbb{1}_{\{z \leq f(Y^{\boldsymbol{a},\nu}_{u-,s})\}} \mathbf{N}(du, dz), \quad (1.3)$$

starting with law ν at time s. That is, equation (1.3) is (1.1) where the interactions $J\,\mathbb{E}\, f(X_u)$ have been replaced by the "external current" a_u. For all $t \geq s$ and for all $\boldsymbol{a} \in L^\infty(\mathbb{R}_+; \mathbb{R}_+)$, consider $\tau^{\boldsymbol{a},\nu}_s$ the time of the first jump of $Y^{\boldsymbol{a},\nu}$ after s

$$\tau^{\boldsymbol{a},\nu}_s := \inf\{t \geq s : Y^{\boldsymbol{a},\nu}_{t,s} \neq Y^{\boldsymbol{a},\nu}_{t-,s}\}. \quad (1.4)$$

We introduce the spiking rate $r^\nu_{\boldsymbol{a}}(t, s)$, the survival function $H^\nu_{\boldsymbol{a}}(t, s)$ and the density of the first jump $K^\nu_{\boldsymbol{a}}(t, s)$ to be

$$r^\nu_{\boldsymbol{a}}(t, s) := \mathbb{E}\, f(Y^{\boldsymbol{a},\nu}_{t,s}), \quad H^\nu_{\boldsymbol{a}}(t, s) := \mathbb{P}(\tau^{\boldsymbol{a},\nu}_s > t), \quad K^\nu_{\boldsymbol{a}}(t, s) := -\frac{d}{dt}\mathbb{P}(\tau^{\boldsymbol{a},\nu}_s > t). \quad (1.5)$$

Notation 1.4. *We detail our conventions and notations.*

1. *We use bold letters $\boldsymbol{a}$ for time dependent currents and regular greek letters α for constant currents.*
2. *When $\nu = \delta_x$, we write $r^x_{\boldsymbol{a}}(t,s) := r^{\delta_x}_{\boldsymbol{a}}(t,s)$.*
3. *When $\nu = \delta_0$, we remove the x superscript and write $r_{\boldsymbol{a}}(t,s) := r^{\delta_0}_{\boldsymbol{a}}(t,s)$.*
4. *When $\boldsymbol{a}$ is constant and equal to $\alpha \geq 0$, it holds that $r^\nu_\alpha(t,s) = r^\nu_\alpha(t-s,0)$ and we simply note $r^\nu_\alpha(t) := r^\nu_\alpha(t,0)$.*
5. *Finally, we extend the function $r^\nu_{\boldsymbol{a}}$, for $s > t$ by setting $\forall s > t, \quad r^\nu_{\boldsymbol{a}}(t,s) := 0$.*

We use the same conventions for $H^\nu_{\boldsymbol{a}}$ and $K^\nu_{\boldsymbol{a}}$.

It is known from [5, Prop. 19] (see also [6, Prop. 6] for a shorter proof) that $r^\nu_{\boldsymbol{a}}$ is the solution of the following Volterra integral equation

$$r^\nu_{\boldsymbol{a}}(t,s) = K^\nu_{\boldsymbol{a}}(t,s) + \int_s^t K_{\boldsymbol{a}}(t,u) r^\nu_{\boldsymbol{a}}(u,s) du. \tag{1.6}$$

Moreover, by [5, Lem. 17], one has

$$1 = H^\nu_{\boldsymbol{a}}(t,s) + \int_s^t H_{\boldsymbol{a}}(t,u) r^\nu_{\boldsymbol{a}}(u,s) du. \tag{1.7}$$

Following [17], given $c_1, c_2 : \mathbb{R}^2 \to \mathbb{R}$ two measurable functions, it is convenient to use the notation

$$(c_1 * c_2)(t,s) = \int_s^t c_1(t,u) c_2(u,s) du,$$

such that (1.6) and (1.7) simply write

$$r^\nu_{\boldsymbol{a}} = K^\nu_{\boldsymbol{a}} + K_{\boldsymbol{a}} * r^\nu_{\boldsymbol{a}} \quad \text{and} \quad 1 = H^\nu_{\boldsymbol{a}} + H_{\boldsymbol{a}} * r^\nu_{\boldsymbol{a}}.$$

The invariant measures of (1.1)**.**

Let $\alpha > 0$, define

$$\sigma_\alpha := \inf\{x \geq 0, \ b(x) + \alpha = 0\}, \tag{1.8}$$

and

$$\nu^\infty_\alpha(x) := \frac{\gamma(\alpha)}{b(x)+\alpha} \exp\left(-\int_0^x \frac{f(y)}{b(y)+\alpha} dy\right) \mathbb{1}_{[0,\sigma_\alpha)}(x), \tag{1.9}$$

where $\gamma(\alpha)$ is the normalizing factor, such that $\int_{\mathbb{R}_+} \nu^\infty_\alpha(x)dx = 1$. Note that $\gamma(\alpha)$ is the jump rate under ν^∞_α:

$$\gamma(\alpha) = \nu^\infty_\alpha(f). \tag{1.10}$$

By [5, Prop. 26], ν^∞_α is the unique invariant measure of the linear SDE (1.3) driven by the constant "external current" $\boldsymbol{a} \equiv \alpha$. We define here a central quantity in our work

$$J(\alpha) := \frac{\alpha}{\gamma(\alpha)}. \tag{1.11}$$

It is readily seen that ν^∞_α is an invariant measure of the non-linear equation (1.1) with $J = J(\alpha)$. Reciprocally, for a fixed value of J, the number of invariant measures of (1.1) is the number of solutions $\alpha \geq 0$ to the scalar equation

$$\alpha = J\gamma(\alpha). \tag{1.12}$$

Any such invariant measure is characterized by its corresponding value of α.

Stability of an invariant measure

Fix $J \geq 0$ and consider $\alpha > 0$ a solution of (1.12). So ν_α^∞ is an invariant probability measure of (1.1). A sufficient condition for ν_α^∞ to be locally stable is given in [6].

First, consider $H_\alpha(t)$, defined by (1.5) (with $\nu = \delta_0$ and $\boldsymbol{a} \equiv \alpha$). For $z \in \mathbb{C}$, we denote by $\Re(z)$ and $\Im(z)$ its real and imaginary parts. The Laplace transform of $H_\alpha(t)$ is defined for z with $\Re(z) > -f(\sigma_\alpha)$ (σ_α is given by (1.8)):

$$\widehat{H}_\alpha(z) := \int_0^\infty e^{-zt} H_\alpha(t) dt.$$

We assumed that $b(0) \geq 0$ (see Assumption 1.1) and that $\alpha > 0$. So $\sigma_\alpha > 0$. Because f is strictly increasing (see Assumption 1.2), we deduce that $f(\sigma_\alpha) > 0$. Let

$$\lambda_\alpha^* := -\sup\{\Re(z) |\ \Re(z) > -f(\sigma_\alpha),\ \widehat{H}_\alpha(z) = 0\}. \tag{1.13}$$

By [5, Lem. 34, 36], it holds that $\lambda_\alpha^* > 0$. The constant λ_α^* is related to the rate of convergence of $(Y_{t,0}^{\alpha,\delta_0})$ to its invariant measure ν_α^∞. In particular we have

$$\forall \lambda < \lambda_\alpha^*, \quad \sup_{t \geq 0} |r_\alpha(t) - \gamma(\alpha)| e^{\lambda t} < \infty. \tag{1.14}$$

This describes the long time behavior of an isolated neuron subject to a constant current $\alpha > 0$.

Assumption 1.5. *Assume that the deterministic flow is not degenerate at σ_α:*

$$\sigma_\alpha < \infty \quad \textit{and} \quad b'(\sigma_\alpha) < 0 \tag{1.15}$$

$$\textbf{\textit{or}} \qquad \sigma_\alpha = \infty \quad \textit{and} \quad \inf_{x \geq 0} b(x) + \alpha > 0. \tag{1.16}$$

Recall that $r_\alpha^x(t)$ is given by (1.5) (with $\boldsymbol{a} \equiv \alpha, \nu = \delta_x$ and $s = 0$). Following [6], define

$$\forall t \geq 0, \quad \Theta_\alpha(t) := \int_0^\infty \frac{d}{dx} r_\alpha^x(t) \nu_\alpha^\infty(dx). \tag{1.17}$$

By [6, Prop. 19], $x \mapsto r_\alpha^x(t)$ is $\mathcal{C}^1$ and integrable with respect to ν_α^∞. Moreover, we have

Theorem 1.6 ([6])**.** *Grant Assumptions 1.1, 1.2 and 1.5. It holds that for all $\lambda < \lambda_\alpha^*$ one has $t \mapsto e^{\lambda t}\Theta_\alpha(t) \in L^1(\mathbb{R}_+)$, so that $z \mapsto \widehat{\Theta}_\alpha(z)$ is holomorphic on $\Re(z) > \lambda_\alpha^*$. Assume that*

$$\sup\{\Re(z) \mid z \in \mathbb{C},\ \Re(z) > -\lambda_\alpha^*,\ J(\alpha)\widehat{\Theta}_\alpha(z) = 1\} < 0, \tag{1.18}$$

then the invariant measure ν_α^∞ is locally stable.

We refer to [6, Def. 16] for definition of local stability, in particular for the choice of the distance between two probability measures.

Assume that there exists $\alpha > 0$ such that (1.18) holds, and so ν_α^∞ is locally stable. There are two “canonical” ways to break (1.18) at some *bifurcation point* α_0: either there exists some $\tau_0 > 0$ such that $J(\alpha_0)\widehat{\Theta}_{\alpha_0}(\pm\frac{i}{\tau_0}) = 1$ or $J(\alpha_0)\widehat{\Theta}_{\alpha_0}(0) = 1$. The first case is the subject of this paper: we give explicit conditions to have a Hopf bifurcation.

In the second case, the following lemma shows that $J'(\alpha_0) = 0$. So, at least in the non-degenerate case where $J''(\alpha_0) \neq 0$, the function $\alpha \mapsto J(\alpha)$ is not strictly monotonic in the neighborhoods of α_0: this is a fold bifurcation which typically leads to *bistability* (or *multistability*, etc.).

Lemma 1.7. *Grant Assumptions 1.1 and 1.2, and consider $\alpha_1 > 0$ such that Assumption 1.5 holds in α_1. Then the function $\alpha \mapsto J(\alpha)$ is $\mathcal{C}^2$ in a neighborhood of α_1 with*

$$J'(\alpha) = \frac{1 - J(\alpha)\widehat{\Theta}_\alpha(0)}{\gamma(\alpha)}.$$

Proof. First, recall that $J(\alpha) = \frac{\alpha}{\gamma(\alpha)}$. So it suffices to show that $\alpha \mapsto \gamma(\alpha)$ is $\mathcal{C}^2$ and that $\gamma'(\alpha) = \widehat{\Theta}_\alpha(0)$. Note that if $\alpha_1 > 0$ satisfies Assumption 1.5, then there exists a neighborhood of α_1 such that for any α in this neighborhood, α also satisfies Assumption 1.5. We will see later by Proposition 3.11 that $\alpha \mapsto \gamma(\alpha)$ is $\mathcal{C}^2$ in the neighborhood for α_1. This can also be checked directly: by [5, eq. (31)], it holds that $\gamma(\alpha)^{-1} = \widehat{H}_\alpha(0)$ and we can conclude using the explicit expression satisfied by $H_\alpha(t)$ (see (3.3) below). To end the proof, using again that $\gamma(\alpha)^{-1} = \widehat{H}_\alpha(0)$, we have to deduce that

$$\frac{d}{d\alpha}\widehat{H}_\alpha(0) = -\frac{\widehat{\Theta}_\alpha(0)}{[\gamma(\alpha)]^2}.$$

Following [6], let:

$$\Psi_\alpha(t) := -\int_0^{\sigma_\alpha} \frac{d}{dx} H^x_\alpha(t)\nu^\infty_\alpha(x)dx. \tag{1.19}$$

It holds that (see [6])

$$\Psi_\alpha(t) = \gamma(\alpha)\int_0^\infty H_\alpha(t+u)\frac{f(\varphi^\alpha_{t+u}(0)) - f(\varphi^\alpha_u(0))}{b(\varphi^\alpha_u(0)) + \alpha}du. \tag{1.20}$$

Moreover, let

$$\forall t \geq 0, \quad \Xi_\alpha(t) := \frac{d}{dt}\Psi_\alpha(t). \tag{1.21}$$

We have

$$\forall t \geq 0, \quad \Xi_\alpha(t) = \int_0^{\sigma_\alpha} \frac{d}{dx} K^x_\alpha(t)\nu^\infty_\alpha(x)dx. \tag{1.22}$$

So, using (1.6) (with $\nu = \delta_x$ and $\boldsymbol{a} \equiv \alpha$) we deduce that (see [6, eq. (43)] for more details):

$$\Theta_\alpha = \Xi_\alpha + r_\alpha * \Xi_\alpha. \tag{1.23}$$

Note that $\Psi_\alpha(0) = 0$, $\lim_{t\to\infty} \Psi_\alpha(t) = 0$ and so $\widehat{\Xi}_\alpha(0) = 0$. Let

$$\xi_\alpha(t) := r_\alpha(t) - \gamma(\alpha).$$

Using (1.14) (with $\nu = \delta_0$), we deduce that $\xi_\alpha \in L^1(\mathbb{R}_+)$. So (1.23) yields

$$\Theta_\alpha = \Xi_\alpha + \gamma(\alpha)\Psi_\alpha + \xi_\alpha * \Xi_\alpha. \tag{1.24}$$

We deduce that $\widehat{\Theta}_\alpha(0) = \gamma(\alpha)\widehat{\Psi}_\alpha(0)$. Finally, we have

$$\begin{aligned}
\frac{d}{d\alpha}\widehat{H}_\alpha(0) &= \int_0^\infty \frac{d}{d\alpha}H_\alpha(t)dt \\
&= -\int_0^\infty H_\alpha(t)\int_0^t \frac{f(\varphi^\alpha_t(0)) - f(\varphi^\alpha_\theta(0))}{b(\varphi^\alpha_\theta(0)) + \alpha}d\theta dt \\
&= -\int_0^\infty \int_0^\infty H_\alpha(u+\theta)\frac{f(\varphi^\alpha_{u+\theta}(0)) - f(\varphi^\alpha_\theta(0))}{b(\varphi^\alpha_\theta(0)) + \alpha}d\theta du \quad \text{(using Fubini} \\
&\qquad \text{and the change of variables } u = t - \theta). \\
&= -\frac{\widehat{\Psi}_\alpha(0)}{\gamma(\alpha)}.
\end{aligned}$$

This ends the proof. □

The paper is structured as follows: in Section 2, we state the spectral assumptions and the main result, Theorem 2.8. We give a layout of its proof at the end of Section 2. In Section 3, we give the proof of Theorem 2.8. Finally, in Section 4, we give an explicit example of a drift b and a rate function f for which such Hopf bifurcations occur and the spectral assumptions can be analytically checked.

2 Assumptions and main result

Following [5, 6], we assume that the law of the initial condition belongs to

$$\mathcal{M}(f^2) := \{\nu \in \mathcal{P}(\mathbb{R}_+) : \quad \int_{\mathbb{R}_+} f^2(x)\nu(dx) < \infty\}.$$

For such initial condition, under Assumptions 1.1 and 1.2, the non-linear SDE (1.1) has a unique path-wise solution (see [6, Th. 9]).

Definition 2.1. *A family of probability measures $(\nu(t))_{t\in[0,T]}$ is said to be a T-periodic solution of* (1.1) *if*

1. $\nu(0) \in \mathcal{M}(f^2)$.
2. *For all $t \in [0,T]$, $\nu(t) = \mathcal{L}(X_t)$ where $(X_t)_{t\in[0,T]}$ is the solution of* (1.1) *starting from $X_0 \sim \nu(0)$.*
3. *It holds that $\nu(T) = \nu(0)$.*

In this case, we can obviously extend $(\nu(t))$ for $t \in \mathbb{R}$ by periodicity. Considering now the solution $(X_t)_{t\geq 0}$ of (1.1) defined for $t \geq 0$, it remains true that $\nu(t) = \mathcal{L}(X_t)$ for any $t \geq 0$.

We study the existence of periodic solutions $t \mapsto \mathcal{L}(X_t)$ where (X_t) is the solution of (1.1), near a non-stable invariant measure $\nu^\infty_{\alpha_0}$. We assume that the stability criterion (1.18) is not satisfied for α_0:

Assumption 2.2. *Assume that there exists $\alpha_0 > 0$ and $\tau_0 > 0$ such that*

$$J(\alpha_0)\widehat{\Theta}_{\alpha_0}(\tfrac{i}{\tau_0}) = 1 \quad \textit{and} \quad \frac{d}{dz}\widehat{\Theta}_{\alpha_0}(\tfrac{i}{\tau_0}) \neq 0.$$

Assumption 2.3 (Non-resonance condition)**.** *Assume that for all $n \in \mathbb{Z}\backslash\{-1,1\}$,*

$$J(\alpha_0)\widehat{\Theta}_{\alpha_0}(\tfrac{in}{\tau_0}) \neq 1.$$

Remark 2.4 (Local uniqueness of the invariant measure in the neighborhood of α_0)**.** Under Assumption 2.3, we have $J(\alpha_0)\widehat{\Theta}_\alpha(0) \neq 1$ and so, by Lemma 1.7, it holds that $J'(\alpha_0) \neq 0$. Fix J in the neighborhood of $J(\alpha_0)$. Recall that the values of α such that ν^∞_α is an invariant measure of (1.1) are precisely the solutions of $J(\alpha) = J$. So, in the neighborhood of $\alpha = \alpha_0$, the invariant measure of (1.1) is unique.

Lemma 2.5. *Under Assumption 2.2, there exists $\eta_0, \varrho_0 > 0$ and a function $\mathfrak{Z}_0 \in \mathcal{C}^1((\alpha_0 - \eta_0, \alpha_0 + \eta_0); \mathbb{C})$ with $\mathfrak{Z}_0(\alpha_0) = \frac{i}{\tau_0}$ such that for all $z \in \mathbb{C}$ with $|z - \frac{i}{\tau_0}| < \varrho_0$ and for all $\alpha > 0$ with $|\alpha - \alpha_0| < \eta_0$ we have*

$$J(\alpha)\widehat{\Theta}_\alpha(z) = 1 \iff z = \mathfrak{Z}_0(\alpha). \tag{2.1}$$

Proof. It suffices to apply the implicit function theorem to $(\alpha, z) \mapsto J(\alpha)\widehat{\Theta}_\alpha(z) - 1$. $\square$

Assumption 2.6. *Assume that $\alpha \mapsto \mathfrak{Z}_0(\alpha)$ crosses the imaginary part with non-vanishing speed, that is*

$$\Re\mathfrak{Z}_0'(\alpha_0) \neq 0, \quad \textit{where} \quad \mathfrak{Z}_0'(\alpha) = \frac{d}{d\alpha}\mathfrak{Z}_0(\alpha).$$

Remark 2.7. Using (2.1), Assumption 2.6 is equivalent to

$$\Re\left(\frac{\frac{\partial}{\partial\alpha}\left(J(\alpha)\widehat{\Theta}_\alpha\right)\Big|_{\alpha=\alpha_0}(\frac{i}{\tau_0})}{J(\alpha_0)\frac{\partial}{\partial z}\widehat{\Theta}_{\alpha_0}(\frac{i}{\tau_0})}\right) \neq 0.$$

Our main result is the following.

Theorem 2.8. *Consider b, f satisfying Assumptions 1.1 and 1.2. Let $\alpha_0, \tau_0 > 0$ be such that Assumptions 1.5, 2.2, 2.3 and 2.6 hold. Then, there exists a family of $2\pi\tau_v$-periodic solutions of (1.1), parametrized by $v \in (-v_0, v_0)$, for some $v_0 > 0$. More precisely, there exists a continuous curve $\{(\nu_v(\cdot), \alpha_v, \tau_v),\ v \in (-v_0, v_0)\}$ such that*

1. *For all $v \in (-v_0, v_0)$, $(\nu_v(t))_{t\in\mathbb{R}}$ is a $2\pi\tau_v$-periodic solution of (1.1) with $J = J(\alpha_v)$.*
2. *The curve passes through $(\nu^\infty_{\alpha_0}, \alpha_0, \tau_0)$ at $v = 0$. In particular we have for all $t \in \mathbb{R}$, $\nu_0(t) \equiv \nu^\infty_{\alpha_0}$.*
3. *The "periodic current" $\boldsymbol{a}_v$, defined by*
$$t \mapsto a_v(t) := J(\alpha_v) \int_{\mathbb{R}_+} f(x)\nu_v(t, dx), \tag{2.2}$$
is continuous and $2\pi\tau_v$-periodic. Moreover, its mean over one period is α_v:
$$\frac{1}{2\pi\tau_v}\int_0^{2\pi\tau_v} a_v(u)du = \alpha_v.$$
4. *Furthermore, v is the amplitude of the first harmonic of $\boldsymbol{a_v}$, that is for all $v \in (-v_0, v_0)$*
$$\frac{1}{2\pi\tau_v}\int_0^{2\pi\tau_v} a_v(u)\cos(u/\tau_v)du = v \quad \text{and} \quad \frac{1}{2\pi\tau_v}\int_0^{2\pi\tau_v} a_v(u)\sin(u/\tau_v)du = 0.$$

Every other periodic solution in a neighborhood of $\nu^\infty_{\alpha_0}$ is obtained from a phase-shift of one such ν_v. More precisely, there exists small enough constants $\epsilon_0, \epsilon_1 > 0$ (only depending on b, f, α_0 and τ_0) such that if $(\nu(t))_{t\in\mathbb{R}}$ is any $2\pi\tau$-periodic solution of (1.1) for some value of $J > 0$ such that
$$|\tau - \tau_0| < \epsilon_0 \quad \text{and} \quad \sup_{t\in[0,2\pi\tau]}\left| J\int_{\mathbb{R}_+} f(x)\nu(t, dx) - \alpha_0\right| < \epsilon_1,$$
then there exists a shift $\theta \in [0, 2\pi\tau)$ and $v \in (-v_0, v_0)$ such that $J = J(\alpha_v)$ and
$$\forall t \in \mathbb{R}, \quad \nu(t) = \nu_v(t + \theta).$$

Remark 2.9. Given the "periodic current" $\boldsymbol{a}_v$ defined by (2.2), the shape of the solution is known explicitly: for all $v \in (-v_0, v_0)$, it holds that
$$\nu_v = \tilde{\nu}_{\boldsymbol{a}_v},$$
where $\tilde{\nu}_{\boldsymbol{a}_v}$, defined by (3.24) below, is known explicitly in terms of b, f and $\boldsymbol{a}_v$.

Notation 2.10. *For $T > 0$, we denote by C^0_T the space of continuous and T-periodic functions from $\mathbb{R}$ to $\mathbb{R}$ and by $C^{0,0}_T$ the subspace of centered functions*
$$C^{0,0}_T := \{h \in C^0_T, \quad \int_0^T h(t)dt = 0\}.$$

We now give an outline of the proof of Theorem 2.8. The proof is divided in two main parts.

The first part is devoted to the study of an isolated neuron subject to a periodic external current. That is, given $\tau > 0$ and $\boldsymbol{a} \in C^0_{2\pi\tau}$, we study the jump rate of an isolated neuron driven by $\boldsymbol{a}$. We give in Section 3.1 estimates on the kernels $K_{\boldsymbol{a}}$ and $H_{\boldsymbol{a}}$. We want to characterize the "asymptotic" jump rate of a neuron driven by this external periodic current. That is, informally

$$\forall t \in \mathbb{R}, \quad \rho_{\boldsymbol{a}}(t) = \lim_{k \in \mathbb{N},\ k \to \infty} r_{\boldsymbol{a}}(t, -2\pi k \tau).$$

In order to characterize such limit $\rho_{\boldsymbol{a}}$, we introduce in Section 3.2 a discrete-time Markov Chain corresponding to the phases of the successive spikes of the neuron driven by $\boldsymbol{a}$. We prove that this Markov Chain has a unique invariant measure, which is proportional to $\rho_{\boldsymbol{a}}$. This serves as a definition of $\rho_{\boldsymbol{a}}$. Given this periodic jump rate $\rho_{\boldsymbol{a}} \in C^0_{2\pi\tau}$, we give in Section 3.3 an explicit description of the associated time-periodic probability densities, that we denote $(\tilde{\nu}_{\boldsymbol{a}}(t))_{t \in [0, 2\pi\tau]}$. Consequently, to find a $2\pi\tau$-periodic solution of (1.1), it is equivalent to find $\boldsymbol{a} \in C^0_{2\pi\tau}$ such that

$$\boldsymbol{a} = J \rho_{\boldsymbol{a}}. \tag{2.3}$$

One classical difficulty with Hopf bifurcation is that the period $2\pi\tau$ itself is unknown: τ varies when the interaction parameter J varies. To address this problem, we make in Section 3.4 a change of time to only consider 2π-periodic functions. We define for all $t \in \mathbb{R}$

$$\forall \boldsymbol{d} \in C^0_{2\pi}, \forall \tau > 0, \quad \rho_{\boldsymbol{d},\tau}(t) := \rho_{\boldsymbol{a}}(\tau t), \quad \text{ where } \quad a(t) := d(t/\tau). \tag{2.4}$$

We shall see that this change of time has a simple probabilistic interpretation by scaling b, f and $\boldsymbol{d}$ appropriately. In Section 3.5, we prove that the function $C^0_{2\pi} \times \mathbb{R}^*_+ \ni (\boldsymbol{d}, \tau) \mapsto \rho_{\boldsymbol{d},\tau} \in C^0_{2\pi}$ is $\mathcal{C}^2$-Fréchet differentiable. Furthermore, consider $\boldsymbol{d} = \alpha + \boldsymbol{h}$ with $\boldsymbol{h} \in C^{0,0}_{2\pi}$ and $\alpha > 0$ is the mean of $\boldsymbol{d}$ over one period. We prove that the mean number of spikes over one period only depends on α. The common value is obtained with the particular case $\boldsymbol{h} \equiv 0$ and (1.10). Thus, we prove

$$\forall \boldsymbol{h} \in C^{0,0}_{2\pi}, \forall \tau > 0, \quad \frac{1}{2\pi} \int_0^{2\pi} \rho_{\alpha + \boldsymbol{h},\tau}(u) du = \gamma(\alpha). \tag{2.5}$$

In the second part of the proof, we find self-consistent periodic solutions using the Lyapunov-Schmidt method. We introduce in Section 3.6 the following functional

$$C^{0,0}_{2\pi} \times \mathbb{R}^*_+ \times \mathbb{R}^*_+ \ni (\boldsymbol{h}, \alpha, \tau) \mapsto G(\boldsymbol{h}, \alpha, \tau) := (\alpha + \boldsymbol{h}) - J(\alpha) \rho_{\alpha + \boldsymbol{h},\tau}.$$

Using (2.5), this functional takes values in $C^{0,0}_{2\pi}$. The roots of G, described by Proposition 3.17, match with the periodic solutions of (1.1). For instance if $G(\boldsymbol{h}, \alpha, \tau) = 0$, we set $a(t) := \alpha + h(t/\tau)$. This current $\boldsymbol{a}$ solves (2.3) with $J = J(\alpha)$ and so it can be used to define a periodic solution of (1.1). Conversely, to any periodic solution of (1.1), we can associate a root of G. So Theorem 2.8 is equivalent to Proposition 3.17. Sections 3.7, 3.8, 3.9 and 3.10 are then devoted to the proof of Proposition 3.17. In Section 3.7, we prove that the linear operator $D_{\boldsymbol{h}} G(0, \alpha, \tau)$ can be written using a convolution involving Θ_α, given by (1.17). We then follow the method of [19, Ch. I.8]. In Section 3.8, we study the range and the kernel of $D_{\boldsymbol{h}} G(0, \alpha_0, \tau_0)$: we prove that under the spectral Assumptions 2.2 and 2.3, $D_{\boldsymbol{h}} G(0, \alpha_0, \tau_0)$ is a Fredholm operator of index zero, with a kernel of dimension two. The problem of finding the roots of G is a priori of infinite dimension ($\boldsymbol{h}$ belongs to $C^{0,0}_{2\pi}$). In Section 3.9 we apply the Lyapunov-Schmidt method to obtain an equivalent problem of dimension two. Finally in Section 3.10 we study the reduced 2D-problem.

3 Proof of Theorem 2.8

Without risk of confusion, we alleviate the notation in the proofs: we no more use bold letters for small perturbations h of a constant current α_0.

3.1 Preliminaries

Let $T > 0$, $s \in \mathbb{R}$ and $\boldsymbol{a} \in C^0_T$ such that

$$\inf_{t \in [0,T]} a_t > -b(0). \tag{3.1}$$

For $x \geq 0$, we consider $\varphi^{\boldsymbol{a}}_{t,s}(x)$ the solution of the ODE

$$\begin{aligned} \frac{d}{dt}\varphi^{\boldsymbol{a}}_{t,s}(x) &= b(\varphi^{\boldsymbol{a}}_{t,s}(x)) + a_t \\ \varphi^{\boldsymbol{a}}_{s,s}(x) &= x. \end{aligned} \tag{3.2}$$

By Assumption 1.1, this ODE has a unique solution. Moreover, the kernels $H^\nu_{\boldsymbol{a}}(t,s)$ and $K^\nu_{\boldsymbol{a}}(t,s)$, defined by (1.5), have explicit expressions in term of the flow

$$H^\nu_{\boldsymbol{a}}(t,s) = \int_{\mathbb{R}_+} \exp\left(-\int_s^t f(\varphi^{\boldsymbol{a}}_{u,s}(x))du\right)\nu(dx), \tag{3.3}$$

$$K^\nu_{\boldsymbol{a}}(t,s) = \int_{\mathbb{R}_+} f(\varphi^{\boldsymbol{a}}_{t,s}(x)) \exp\left(-\int_s^t f(\varphi^{\boldsymbol{a}}_{u,s}(x))du\right)\nu(dx). \tag{3.4}$$

The function $s \mapsto \varphi^{\boldsymbol{a}}_{t,s}(0)$ belongs to $\mathcal{C}^1((-\infty,t];\mathbb{R}_+)$ and

$$\frac{d}{ds}\varphi^{\boldsymbol{a}}_{t,s}(0) = -\left[b(0) + a_s\right] \exp\left(\int_s^t b'(\varphi^{\boldsymbol{a}}_{\theta,s}(0))d\theta\right). \tag{3.5}$$

In particular, under the assumption (3.1), $s \mapsto \varphi^{\boldsymbol{a}}_{t,s}(0)$ is strictly decreasing on $(-\infty,t]$, for all t. Define then

$$\sigma_{\boldsymbol{a}}(t) := \lim_{s \to -\infty} \varphi^{\boldsymbol{a}}_{t,s}(0) \in \mathbb{R}^*_+ \cup \{+\infty\}. \tag{3.6}$$

Given $\boldsymbol{d} \in C^0_T$ and $\eta > 0$, we consider the following open balls of C^0_T:

$$B^T_\eta(\boldsymbol{d}) := \{\boldsymbol{a} \in C^0_T, \quad \sup_{t \in [0,T]} |a_t - d_t| < \eta\}. \tag{3.7}$$

Lemma 3.1. *Let $T > 0$ and $b : \mathbb{R}_+ \to \mathbb{R}$ such that Assumption 1.1 holds. Let $\alpha_0 > 0$ satisfying Assumption 1.5. There exists $\eta_0 > 0$ such that for all $\boldsymbol{a} \in B^T_{\eta_0}(\alpha_0)$, it holds that*

1. *If $\sigma_{\alpha_0} = \infty$, then for all $t \in [0,T]$, $\sigma_{\boldsymbol{a}}(t) = +\infty$.*
2. *If $\sigma_{\alpha_0} < \infty$, then the function $t \mapsto \sigma_{\boldsymbol{a}}(t)$ belongs to C^0_T and*

$$\inf_{\boldsymbol{a} \in B^T_{\eta_0}(\alpha_0)} \inf_{t \in [0,T]} \sigma_{\boldsymbol{a}}(t) > 0.$$

Proof. Assume first that $\sigma_{\alpha_0} = \infty$, and let $\eta_0 := \frac{1}{2}\left[\inf_{x \geq 0} b(x) + \alpha_0\right]$, which is strictly positive by assumption. Then it holds that $\inf_{t \geq 0} \inf_{x \geq 0} b(x) + a_t \geq \eta_0$ and so

$$\varphi^{\boldsymbol{a}}_{t,s}(0) \geq \eta_0(t-s). \tag{3.8}$$

Letting s tend to $-\infty$, we deduce that $\sigma_{\boldsymbol{a}}(t) = +\infty$.
Assume now that $\sigma_{\alpha_0} < \infty$. Using (1.15), we apply the implicit function theorem to

$$(x,\eta) \mapsto b(x) + \alpha_0 + \eta$$

at $(\sigma_{\alpha_0}, 0)$: there exists $\eta_0 > 0$ and a function $\eta \mapsto \sigma_{\alpha_0+\eta} \in \mathcal{C}^1([0,\eta_0];\mathbb{R}^*_+)$ such that

$$\forall \eta \in [0,\eta_0], \quad \sigma_{\alpha_0} \leq \sigma_{\alpha_0+\eta} < \infty \quad \text{and} \quad \sigma_{\alpha_0+\eta} = \inf_{x \geq 0}\{b(x) + \alpha_0 + \eta = 0\}.$$

In addition, we choose η_0 such that $\eta_0 < b(0) + \alpha_0$. Let $\boldsymbol{a} \in C^0_T$ such that $\sup_{t \in [0,T]} |a_t - \alpha_0| \leq \eta_0$. It holds that

$$\forall t \geq s, \quad \varphi^{\boldsymbol{a}}_{t,s}(0) \leq \varphi^{\alpha_0+\eta_0}_{t,s}(0) \leq \sigma_{\alpha_0+\eta_0}. \tag{3.9}$$

In particular $\sigma_{\boldsymbol{a}}(t) < \infty$. We prove that this function is right-continuous in t. We fix $t \geq s$ and $\eta \in [0, \eta_0]$, we have

$$\varphi^{\boldsymbol{a}}_{t+\eta,s}(0) - \varphi^{\boldsymbol{a}}_{t,s}(0) = \int_t^{t+\eta} b(\varphi^{\boldsymbol{a}}_{t+u,s}(0))du + \int_t^{t+\eta} a_u du.$$

Let $A_0 := \sup_{x \in [0, \sigma_{\alpha_0+\eta_0}]} |b(x)| < \infty$ so $|\varphi^{\boldsymbol{a}}_{t+\eta,s}(0) - \varphi^{\boldsymbol{a}}_{t,s}(0)| \leq (A_0 + ||\boldsymbol{a}||_\infty)\eta$. Letting s tend to $-\infty$ we deduce that $t \mapsto \sigma_{\boldsymbol{a}}(t)$ is right-continuous. Left-continuity is proved similarly. Using $\varphi^{\boldsymbol{a}}_{t+T,s+T}(0) = \varphi^{\boldsymbol{a}}_{t,s}(0)$, it holds that $t \mapsto \sigma_{\boldsymbol{a}}(t)$ is T-periodic. Finally, because $s \mapsto \varphi^{\boldsymbol{a}}_{t,s}(0)$ is strictly decreasing, and takes value 0 when $s = t$, we deduce that $\sigma_{\boldsymbol{a}}(t) > 0$. More precisely, let

$$m_0 := - \min_{x \in [0, \sigma_{\alpha_0+\eta_0}]} b'(x).$$

By (1.15), it holds that $m_0 > 0$. Moreover, using (3.5), we deduce that

$$\frac{d}{ds}\varphi^{\boldsymbol{a}}_{t,s}(0) \leq -(b(0) + \alpha_0 - \eta_0)e^{-m_0(t-s)},$$

and so

$$\forall s \leq t, \quad \varphi^{\boldsymbol{a}}_{t,s}(0) \geq (b(0) + \alpha_0 - \eta_0)\frac{1 - e^{-m_0(t-s)}}{m_0}. \tag{3.10}$$

It ends the proof. □

Lemma 3.2. *Grant Assumptions 1.1 and 1.2. Let $\alpha_0 > 0$ such that Assumption 1.5 holds. There exists $\lambda_0, \eta_0, s_0 > 0$ (only depending on α_0 and b) such that for all $T > 0$, for all $\boldsymbol{a} \in B^T_{\eta_0}(\alpha_0)$, it holds that*

$$\forall t, s, \quad t - s \geq s_0 \implies \varphi^{\boldsymbol{a}}_{t,s}(0) \geq \lambda_0. \tag{3.11}$$

Moreover, if $\sigma_{\alpha_0} = \infty$, λ_0 can be chosen arbitrarily large. Finally, it holds that

$$\sup_{T>0} \sup_{\boldsymbol{a} \in B^T_{\eta_0}(\alpha_0)} \sup_{t \geq s} H_{\boldsymbol{a}}(t,s)e^{f(\lambda_0)(t-s)} < \infty, \tag{3.12}$$

$$\sup_{T>0} \sup_{\boldsymbol{a} \in B^T_{\eta_0}(\alpha_0)} \sup_{t \geq s} K_{\boldsymbol{a}}(t,s)e^{f(\lambda_0)(t-s)} < \infty. \tag{3.13}$$

Proof. The inequality (3.11) is a direct consequence of (3.10) if $\sigma_{\alpha_0} < \infty$ and of (3.8) if $\sigma_{\alpha_0} = \infty$. Then, (3.12) follows from the explicit expression (3.3) of $H_{\boldsymbol{a}}$ and (3.11).

If $\sigma_{\alpha_0} < \infty$ or f is bounded on $\mathbb{R}_+$, (3.13) follows from (3.12) by (3.4).

It remains to prove (3.13) if $\sigma_{\alpha_0} = \infty$ and $\lim_{x \to \infty} f(x) = \infty$. Denote by L the Lipschitz constant of b. The Grönwall's lemma gives the existence of a constant C_1 (only depending on b, α_0, η_0) such that $\varphi^{\boldsymbol{a}}_{t,s}(0) \leq C_1 e^{L(t-s)}$. By Remark 1.3, we know that $f(x) \leq C_2 x^p$. Overall, $f(\varphi^{\boldsymbol{a}}_{t,s}(0)) \leq Ce^{Lp(t-s)}$. Fix $\lambda_0 \geq 0$. There exists λ_1 such that $f(\lambda_1) \geq f(\lambda_0) + Lp$ and (3.11) holds with λ_1. It ends the proof of (3.13). □

3.2 Study of the non-homogeneous linear equation

In this section, we study the asymptotic jump rate of an "isolated" neuron driven by a periodic continuous function. Grant Assumptions 1.1, 1.2 and let $\alpha_0 > 0$ such that Assumption 1.5 holds. Let $\lambda_0, \eta_0 > 0$ be given by Lemma 3.2 and $T > 0$. Consider $\boldsymbol{a} \in B^T_{\eta_0}(\alpha_0)$. Following the terminology of [5], we say that $\boldsymbol{a}$ is the "external current". Let $r_{\boldsymbol{a}}$ be the solution of the Volterra equation $r_{\boldsymbol{a}} = K_{\boldsymbol{a}} + K_{\boldsymbol{a}} * r_{\boldsymbol{a}}$. We consider the following limit

$$\forall t \in [0,T], \quad \rho_{\boldsymbol{a}}(t) = \lim_{k\in\mathbb{N},\ k\to\infty} r_{\boldsymbol{a}}(t, -kT).$$

The goal of this section is to show that $\rho_{\boldsymbol{a}}$ is well defined and to study some of its properties. First, (1.6) and (1.7) write

$$\forall t \in \mathbb{R}, \quad r_{\boldsymbol{a}}(t,-kT) = K_{\boldsymbol{a}}(t,-kT) + \int_{-kT}^{t} K_{\boldsymbol{a}}(t,s) r_{\boldsymbol{a}}(s,-kT)ds,$$
$$1 = H_{\boldsymbol{a}}(t,-kT) + \int_{-kT}^{t} H_{\boldsymbol{a}}(t,s) r_{\boldsymbol{a}}(s,-kT)ds.$$

Letting $k \to \infty$, we find that $\rho_{\boldsymbol{a}}$ has to solve

$$\forall t \in \mathbb{R}, \quad \rho_{\boldsymbol{a}}(t) = \int_{-\infty}^{t} K_{\boldsymbol{a}}(t,s)\rho_{\boldsymbol{a}}(s)ds. \tag{3.14}$$

$$1 = \int_{-\infty}^{t} H_{\boldsymbol{a}}(t,s)\rho_{\boldsymbol{a}}(s)ds. \tag{3.15}$$

Note that if $\rho_{\boldsymbol{a}}$ is a solution of (3.14), then it automatically holds that the function $t \mapsto \int_{-\infty}^{t} H_{\boldsymbol{a}}(t,s)\rho_{\boldsymbol{a}}(s)ds$ is constant (its derivative is null). In Lemma 3.4 below, we prove that the solutions of equation (3.14) form a linear space of dimension 1. Consequently (3.14) together with (3.15) have a unique solution: this will serve as the definition of $\rho_{\boldsymbol{a}}$.

A probabilistic interpretation of (3.14) and (3.15)

Let x be a T-periodic solution of (3.14). We have for all $t \in [0,T]$

$$\begin{aligned} x(t) &= \int_{-\infty}^{T} K_{\boldsymbol{a}}(t,s)x(s)ds, \quad \text{(with the convention } K_{\boldsymbol{a}}(t,s) = 0 \text{ for } s > t) \\ &= \sum_{k\geq 0} \int_{-kT}^{T-kT} K_{\boldsymbol{a}}(t,s)x(s)ds \\ &= \sum_{k\geq 0} \int_{0}^{T} K_{\boldsymbol{a}}(t,u-kT)x(u)du \quad \text{(by the change of variable } u = s+kT). \end{aligned}$$

Define for all $t,s \in [0,T]$

$$K^T_{\boldsymbol{a}}(t,s) := \sum_{k\geq 0} K_{\boldsymbol{a}}(t, s-kT).$$

Note that by Lemma 3.2 we have normal convergence since

$$\forall t,s \in [0,T], \quad K_{\boldsymbol{a}}(t,s-kT) \leq Ce^{-f(\lambda_0)kT},$$

for some constant C only depending on b, f, α_0, η_0 and λ_0. We deduce that x solves

$$x(t) = \int_0^T K^T_{\boldsymbol{a}}(t,s)x(s)ds. \tag{3.16}$$

Using that $\boldsymbol{a}$ is T-periodic, we have

$$\forall t \geq s, \quad K_{\boldsymbol{a}}(t+T, s+T) = K_{\boldsymbol{a}}(t, s). \tag{3.17}$$

Moreover, $K_{\boldsymbol{a}}$ is a probability density so

$$\forall s \in \mathbb{R}, \quad \int_s^\infty K_{\boldsymbol{a}}(t, s) dt = 1. \tag{3.18}$$

From (3.17) and (3.18), we deduce that

$$\forall s \in [0, T], \quad \int_0^T K^T_{\boldsymbol{a}}(t, s) dt = 1. \tag{3.19}$$

In view of (3.19), $K^T_{\boldsymbol{a}}(\cdot, s)$ can be seen as the transition probability kernel of a Markov Chain acting on the continuous space $[0, T]$. The interpretation of this Markov Chain is the following. Let $(Y^{\boldsymbol{a},\nu}_t)_{t\geq 0}$ be the solution of (1.3), starting at time 0 with law ν and driven by the T-periodic current $\boldsymbol{a}$. Define $(\tau_i)_{i\geq 1}$ the times of the successive jumps of $(Y^{\boldsymbol{a},\nu}_t)_{t\geq 0}$. Let $\phi_i \in [0, T)$ and $\Delta_i \in \mathbb{N}$ be defined by:

$$\phi_i := \tau_i - \left\lfloor \frac{\tau_i}{T} \right\rfloor T, \quad \tau_{i+1} - \tau_i =: \Delta_{i+1} T + \phi_{i+1} - \phi_i. \tag{3.20}$$

That is, ϕ_i is the *phase* of the i-ith jump, while Δ_i is the number of “revolutions” between τ_{i-1} and τ_i:

$$\Delta_i = \#\{k \in \mathbb{N}, \quad kT \in [\tau_{i-1}, \tau_i)\}.$$

In other words, if one considers that a period is a “lap”, Δ_i is the number of times we cross the start line of the lap between two spikes.

Then, $(\phi_i, \Delta_i)_{i\geq 0}$ is Markov, with a transition probability given by

$$\forall A \in \mathcal{B}([0, T]),\ \forall n \in \mathbb{N}, \quad \mathbb{P}(\phi_{i+1} \in A, \Delta_{i+1} = n | \phi_i) = \int_A K_{\boldsymbol{a}}(t + nT, \phi_i) dt.$$

In particular, $(\phi_i)_{i\geq 0}$ is Markov, with a transition probability given by $K^T_{\boldsymbol{a}}$. With some slight abuse of notations, we also write $K^T_{\boldsymbol{a}}$ for the linear operator which maps $y \in L^1([0, T])$ to

$$K^T_{\boldsymbol{a}}(y) := t \mapsto \int_0^T K^T_{\boldsymbol{a}}(t, s) y(s) ds \in L^1([0, T]). \tag{3.21}$$

Lemma 3.3. *Let $\boldsymbol{a} \in C^0_T$. The linear operator $K^T_{\boldsymbol{a}} : L^1([0, T]) \to L^1([0, T])$ is a compact operator. Moreover, if $y \in L^1([0, T])$, then $K^T_{\boldsymbol{a}}(y) \in C^0_T$.*

Proof. First, the function $[0, T]^2 \ni (t, s) \mapsto K^T_{\boldsymbol{a}}(t, s)$ is (uniformly) continuous. Let $\epsilon > 0$, there exists $\eta > 0$ such that

$$|t - t'| + |s - s'| \leq \eta \implies |K^T_{\boldsymbol{a}}(t, s) - K^T_{\boldsymbol{a}}(t', s')| \leq \epsilon.$$

It follows that

$$\left|K^T_{\boldsymbol{a}}(y)(t) - K^T_{\boldsymbol{a}}(y)(t')\right| \leq \int_0^T \left|K^T_{\boldsymbol{a}}(t, s) - K^T_{\boldsymbol{a}}(t', s)\right| |y(s)|\, ds \leq \epsilon ||y||_1,$$

and so the function $K^T_{\boldsymbol{a}}(y)$ is continuous. Note that

$$\forall s \in [0, T], \quad K^T_{\boldsymbol{a}}(T, s) = K^T_{\boldsymbol{a}}(0, s),$$

and so $K^T_{\boldsymbol{a}}(y)$ can be extended to a T-periodic function. Altogether, $K^T_{\boldsymbol{a}}(y) \in C^0_T$. To prove that $K^T_{\boldsymbol{a}}$ is a compact operator, we use the Weierstrass approximation Theorem: there exists a sequence of polynomial functions $(t,s) \mapsto P_n(t,s)$ such that $\sup_{t,s\in[0,T]} \left|P_n(t,s) - K^T_{\boldsymbol{a}}(t,s)\right| \to_n 0$ as $n \to \infty$. For each $n \in \mathbb{N}$, the linear operator $L^1([0,T]) \ni y \mapsto P_n(y) := t \mapsto \int_0^T P_n(t,s)y(s)ds$ is of finite-rank. Moreover, the sequence P_n converges to $K^T_{\boldsymbol{a}}$ for the norm operator, and so $K^T_{\boldsymbol{a}}$ is a compact operator (as the limit of finite-rank operators, see [1, Ch. 6]). ☐

Lemma 3.4. *Let $\boldsymbol{a} \in C^0_T$. The Markov Chain $(\phi_i)_{i\geq 0}$ with transition probability kernel $K^T_{\boldsymbol{a}}$ has a unique invariant probability measure $\pi_{\boldsymbol{a}} \in C^0_T$. Moreover, the solutions of* (3.16) *in $L^1([0,T])$ span a vector space of dimension 1.*

Proof. Step 1: any solution of (3.16) *has a constant sign.* Let $x \in L^1([0,T])$ be a solution of (3.16). Because the kernel $K^T_{\boldsymbol{a}}$ is strictly positive and continuous on $[0,T]^2$, it holds that

$$\delta := \inf_{t,s\in[0,T]} K^T_{\boldsymbol{a}}(t,s) > 0. \tag{3.22}$$

We write x_+ for the positive part of x, x_- for its negative part and define $\beta := \min(||x_+||_1, ||x_-||_1)$. We have $K^T_{\boldsymbol{a}}(x_+)(t) \geq \delta\beta$ and $K^T_{\boldsymbol{a}}(x_-)(t) \geq \delta\beta$. Consequently

$$||K^T_{\boldsymbol{a}}(x)||_1 = ||K^T_{\boldsymbol{a}}(x_+) - K^T_{\boldsymbol{a}}(x_-)||_1 \leq ||K^T_{\boldsymbol{a}}(x_+) - \delta\beta||_1 + ||K^T_{\boldsymbol{a}}(x_-) - \delta\beta||_1 = ||x||_1 - 2\delta\beta.$$

To obtain the right-most equality, we used that for all $y \in L^1([0,T])$, $y \geq 0$ yields $||K^T_{\boldsymbol{a}} y||_1 = ||y||_1$. But the identity $K^T_{\boldsymbol{a}}(x) = x$ implies that $\beta = 0$ and so either x_+ or x_- is a.e. null.

Step 2: existence and uniqueness of the invariant probability measure. Let $(K^T_{\boldsymbol{a}})' : L^\infty([0,T]) \to L^\infty([0,T])$ be the dual operator of $K^T_{\boldsymbol{a}}$. We have:

$$\forall v \in L^\infty([0,T]), \quad (K^T_{\boldsymbol{a}})'(v) = s \mapsto \int_0^T K^T_{\boldsymbol{a}}(t,s)v(t)dt.$$

From (3.19), we deduce that 1 is an eigenvalue of $(K^T_{\boldsymbol{a}})'$ (its associated eigenvector is $\mathbf{1}$, the constant function equal to 1). Denoting by N the null space, the Fredholm alternative [1, Th. 6.6] yields dim $N(I - K^T_{\boldsymbol{a}}) =$ dim $N(I - (K^T_{\boldsymbol{a}})')$. So there exists $\pi_{\boldsymbol{a}} \in L^1([0,T])$ such that:

$$\pi_{\boldsymbol{a}} = K^T_{\boldsymbol{a}}(\pi_{\boldsymbol{a}}),\ ||\pi_{\boldsymbol{a}}||_1 = 1.$$

By Step 1, $\pi_{\boldsymbol{a}}$ can be chosen positive, and by Lemma 3.3, $\pi_{\boldsymbol{a}} \in C^0_T$. Uniqueness follows directly from Step 1: if π_1, π_2 are two invariant probability measures, then $x = \pi_1 - \pi_2$ solves (3.16) and so it has a constant sign. Because the mass of x is null, we deduce that $x = 0$.

By Steps 1 and 2, we deduce that the solutions of (3.16) in $L^1([0,T])$ are the $\{\lambda\pi_{\boldsymbol{a}},\ \lambda \in \mathbb{R}\}$. ☐

Remark 3.5. The estimate (3.22) is a strong version of the Doeblin's condition. It holds that

$$\inf_{s\in[0,T]} K^T_{\boldsymbol{a}}(\cdot,s) \geq \delta T \text{Unif}(\cdot),$$

where Unif is the uniform distribution on $[0,T]$. A classical coupling argument shows that for all $i \geq 1$, $||\mathcal{L}(\phi_i) - \pi_{\boldsymbol{a}}||_{\text{TV}} \leq (1-\delta T)^i$, where (ϕ_i) is the Markov Chain defined by (3.20) and $||\cdot||_{\text{TV}}$ denotes the total variation distance between probability measures. This argument provides an alternative proof of the existence and uniqueness of $\pi_{\boldsymbol{a}}$.

We define for all $\theta \in \mathbb{R}$ the following shift operator

$$S_\theta : \begin{array}{ccc} C^0_T & \to & C^0_T \\ x & \mapsto & (x(t+\theta))_t. \end{array}$$

Corollary 3.6. *Given $\boldsymbol{a} \in C^0_T$, equations* (3.14) *and* (3.15) *have a unique solution $\rho_{\boldsymbol{a}} \in C^0_T$. Moreover, it holds that for all $\theta \in \mathbb{R}$,*

$$\rho_{S_\theta(\boldsymbol{a})} = S_\theta(\rho_{\boldsymbol{a}}). \tag{3.23}$$

Proof. Note that there is a one to one mapping between periodic solutions of (3.14) and solutions of (3.16). So by Lemma 3.4, the solution $\rho_{\boldsymbol{a}}$ of equations (3.14) and (3.15) is $\rho_{\boldsymbol{a}} = \frac{\pi_{\boldsymbol{a}}}{c_{\boldsymbol{a}}}$, where $\pi_{\boldsymbol{a}}$ is the invariant measure (on $[0,T]$) of the Markov Chain with transition probability kernel $K^T_{\boldsymbol{a}}$ and $c_{\boldsymbol{a}}$ is given by

$$c_{\boldsymbol{a}} := \int_{-\infty}^t H_{\boldsymbol{a}}(t,s)\pi_{\boldsymbol{a}}(s)ds.$$

Note that $c_{\boldsymbol{a}}$ is constant in time. Define for all $t,s \in [0,T]$:

$$H^T_{\boldsymbol{a}}(t,s) := \sum_{k\geq 0} H_{\boldsymbol{a}}(t, s-kT).$$

Using the same notation that in (3.21), we have $c_{\boldsymbol{a}} = H^T_{\boldsymbol{a}}(\pi_{\boldsymbol{a}})$. Moreover, we have

$$\forall t,s,\theta \in \mathbb{R}, \quad \varphi^{S_\theta(\boldsymbol{a})}_{t,s}(0) = \varphi^{\boldsymbol{a}}_{t+\theta,s+\theta}(0),$$

because both sides satisfy the same ODE with the same initial condition at $t = s$. We deduce from (3.3) and (3.4) that

$$H_{S_\theta(\boldsymbol{a})}(t,s) = H_{\boldsymbol{a}}(t+\theta, s+\theta) \quad \text{and} \quad K_{S_\theta(\boldsymbol{a})}(t,s) = K_{\boldsymbol{a}}(t+\theta,s+\theta).$$

So $S_\theta(\rho_{\boldsymbol{a}})$ solves (3.14) and (3.15), where the kernels are replaced by $K_{S_\theta(\boldsymbol{a})}$ and $H_{S_\theta(\boldsymbol{a})}$. By uniqueness it follows that $\rho_{S_\theta(\boldsymbol{a})} = S_\theta(\rho_{\boldsymbol{a}})$. □

Remark 3.7. Using that $\int_0^T \pi_{\boldsymbol{a}}(s)ds = 1$, we find that the average number of spikes over one period $[0,T]$ is given by

$$\frac{1}{T}\int_0^T \rho_{\boldsymbol{a}}(s)ds = \frac{1}{c_{\boldsymbol{a}}T}.$$

The probabilistic interpretation of $c_{\boldsymbol{a}}$ is the following: remembering the Markov chain defined by (3.20), we have

$$\mathbb{P}(\Delta_{i+1} > k|\phi_i) = H_{\boldsymbol{a}}((k+1)T, \phi_i),$$

and so, if $\mathcal{L}(\phi_i) = \pi_{\boldsymbol{a}}$, we deduce that

$$\mathbb{E}\,\Delta_{i+1} = \mathbb{E}\,\mathbb{E}(\Delta_{i+1}|\phi_i) = \mathbb{E}\left[\sum_{k\geq 0}\mathbb{P}(\Delta_{i+1} > k|\phi_i)\right] = H^T_{\boldsymbol{a}}(\pi_{\boldsymbol{a}}) = c_{\boldsymbol{a}}.$$

In other words, $c_{\boldsymbol{a}}$ is the expected number of “revolutions” between two successive spikes, assuming the phase of each spike follows its invariant measure $\pi_{\boldsymbol{a}}$. We shall see in Proposition 3.16 that $c_{\boldsymbol{a}}$ only depends on the *mean* of $\boldsymbol{a}$. Furthermore, it holds that for $\boldsymbol{a} \equiv \alpha > 0$

$$c_\alpha = H^T_\alpha(1/T) = \frac{1}{T}\int_0^\infty H_\alpha(t)dt = \frac{1}{T\gamma(\alpha)},$$

and so for all t, $\rho_\alpha(t) = \gamma(\alpha)$.

3.3 Shape of the solutions

Let $\boldsymbol{a} \in C^0_T$ such that (3.1) holds. Let $\sigma_{\boldsymbol{a}}(t)$ be defined by (3.6), such that $s \mapsto \varphi^{\boldsymbol{a}}_{t,s}(0)$ is a bijection from $(-\infty, t]$ to $[0, \sigma_{\boldsymbol{a}}(t))$. We denote by $x \mapsto \beta^{\boldsymbol{a}}_t(x)$ its inverse. Note that $t \mapsto \sigma_{\boldsymbol{a}}(t)$ is T-periodic and

$$\forall t \in \mathbb{R}, \forall x \in [0, \sigma_{\boldsymbol{a}}(t)), \quad \beta^{\boldsymbol{a}}_{t+T}(x) = \beta^{\boldsymbol{a}}_t(x) + T.$$

Using that $\varphi^{\boldsymbol{a}}_{t,t}(0) = 0$, we have $\beta^{\boldsymbol{a}}_t(0) = t$.

Notation 3.8. *Given $\boldsymbol{a} \in C^0_T$, we define for all $t \in \mathbb{R}$*

$$\tilde{\nu}_{\boldsymbol{a}}(t,x) := \frac{\rho_{\boldsymbol{a}}(\beta^{\boldsymbol{a}}_t(x))}{b(0) + a(\beta^{\boldsymbol{a}}_t(x))} \exp\left(- \int_{\beta^{\boldsymbol{a}}_t(x)}^t (f + b')(\varphi^{\boldsymbol{a}}_{\theta, \beta^{\boldsymbol{a}}_t(x)}(0)) d\theta \right) \mathbb{1}_{[0, \sigma_{\boldsymbol{a}}(t))}(x), \tag{3.24}$$

where $\rho_{\boldsymbol{a}}$ is the unique solution of the equations (3.14) *and* (3.15)*.*

By the change of variables $u = \beta^{\boldsymbol{a}}_t(x)$, one obtains that for any non-negative measurable test function g

$$\int_0^\infty g(x) \tilde{\nu}_{\boldsymbol{a}}(t,x) dx = \int_{-\infty}^t g(\varphi^{\boldsymbol{a}}_{t,u}(0)) \rho_{\boldsymbol{a}}(u) H_{\boldsymbol{a}}(t,u) du. \tag{3.25}$$

Note moreover that when $\boldsymbol{a}$ is constant and equal to $\alpha > 0$ ($\boldsymbol{a} \equiv \alpha$), (3.24) matches with the definition of the invariant measure ν^∞_α given by (1.9):

$$\forall t \in \mathbb{R}, \quad \sigma_\alpha(t) = \sigma_\alpha \quad \text{and} \quad \tilde{\nu}_\alpha(t) = \nu^\infty_\alpha.$$

The main result of this section is

Proposition 3.9. *Let $\boldsymbol{a} \in C^0_T$ such that $\inf_{t \in \mathbb{R}} a_t > -b(0)$. It holds that $(\tilde{\nu}_{\boldsymbol{a}}(t, \cdot))_t$ is the unique T-periodic solution of* (1.3)*.*

Proof. **Existence**. We first prove that $\tilde{\nu}_{\boldsymbol{a}}(t, \cdot)$ is indeed a T-periodic solution. We follow the same strategy that [5, Prop. 26]. First note that, by (3.25), one has

$$\int_0^\infty f(x) \tilde{\nu}_{\boldsymbol{a}}(t,x) dx = \int_{-\infty}^t K_{\boldsymbol{a}}(t,u) \rho_{\boldsymbol{a}}(u) du = \rho_{\boldsymbol{a}}(t).$$

Consider the solution $(Y^{\boldsymbol{a}, \tilde{\nu}_{\boldsymbol{a}}(0)}_{t,0})$ of (1.3) starting with law $\tilde{\nu}_{\boldsymbol{a}}(0)$ at time $t = 0$ and let $r^{\tilde{\nu}_{\boldsymbol{a}}(0)}_{\boldsymbol{a}}(t) = \mathbb{E}\, f(Y^{\boldsymbol{a}, \tilde{\nu}_{\boldsymbol{a}}(0)}_{t,0})$.

Claim: It holds that for all $t \geq 0$, $r^{\tilde{\nu}_{\boldsymbol{a}}(0)}_{\boldsymbol{a}}(t) = \rho_{\boldsymbol{a}}(t)$.

Proof of the Claim. Recall that $r^{\tilde{\nu}_{\boldsymbol{a}}(0)}_{\boldsymbol{a}}(t)$ is the unique solution of the Volterra equation

$$r^{\tilde{\nu}_{\boldsymbol{a}}(0)}_{\boldsymbol{a}} = K^{\tilde{\nu}_{\boldsymbol{a}}(0)}_{\boldsymbol{a}} + K_{\boldsymbol{a}} * r^{\tilde{\nu}_{\boldsymbol{a}}(0)}_{\boldsymbol{a}}.$$

So, to prove the claim it suffices to show that $\rho_{\boldsymbol{a}}$ also solves this equation. For all $u \leq 0 \leq t$, one has

$$K^{\varphi^{\boldsymbol{a}}_{0,u}(0)}_{\boldsymbol{a}}(t,0) H_{\boldsymbol{a}}(0,u) = K_{\boldsymbol{a}}(t,u).$$

Consequently, we deduce from (3.25) that

$$K^{\tilde{\nu}_{\boldsymbol{a}}(0)}_{\boldsymbol{a}}(t,0) = \int_{-\infty}^0 K_{\boldsymbol{a}}(t,u) \rho_{\boldsymbol{a}}(u) du.$$

So

$$\rho_{\boldsymbol{a}}(t) = \int_{-\infty}^t K_{\boldsymbol{a}}(t,u) \rho_{\boldsymbol{a}}(u) du = K^{\tilde{\nu}_{\boldsymbol{a}}(0)}_{\boldsymbol{a}}(t,0) + \int_0^t K_{\boldsymbol{a}}(t,u) \rho_{\boldsymbol{a}}(u) du,$$

and the conclusion follows. □

Finally, using [5, Prop. 19] and the claim, we deduce that for any non-negative measurable function g

$$\mathbb{E}\, g(Y^{\boldsymbol{a},\tilde{\nu}_{\boldsymbol{a}}(0)}_{t,0}) = \int_0^t g(\varphi^{\boldsymbol{a}}_{t,u}(0)) H_{\boldsymbol{a}}(t,u)\rho_{\boldsymbol{a}}(u)du + \int_0^\infty g(\varphi^{\boldsymbol{a}}_{t,0}(x)) H^x_{\boldsymbol{a}}(t,0)\tilde{\nu}_{\boldsymbol{a}}(0,x)dx.$$

By (3.25) (with $t = 0$ and g replaced by $x \mapsto g(\varphi^{\boldsymbol{a}}_{t,0}(x))H^x_{\boldsymbol{a}}(t,0)$), the second term is equal to

$$\int_{-\infty}^0 g(\varphi^{\boldsymbol{a}}_{t,u}(0)) H_{\boldsymbol{a}}(t,u)\rho_{\boldsymbol{a}}(u)du,$$

and so

$$\forall t \geq 0, \quad \mathbb{E}\, g(Y^{\boldsymbol{a},\tilde{\nu}_{\boldsymbol{a}}(0)}_{t,0}) = \int_{-\infty}^t g(\varphi^{\boldsymbol{a}}_{t,u}(0)) H_{\boldsymbol{a}}(t,u)\rho_{\boldsymbol{a}}(u)du \overset{(3.25)}{=} \int_0^\infty g(x)\tilde{\nu}_{\boldsymbol{a}}(t,x)dx.$$

This ends the proof of the existence.

Uniqueness. Consider $(\nu(t))_{t\in[0,T]}$ a T-periodic solution of (1.3) and define $\rho(t) = \mathbb{E}\, f(Y^{\boldsymbol{a},\nu(0)}_{t,0})$. The function ρ is T-periodic. Moreover, it holds that for all $k \geq 0$, $\rho(t) = \mathbb{E}\, f(Y^{\boldsymbol{a},\nu(0)}_{t,-kT})$ and so (1.6) and (1.7) yields

$$\rho(t) = K^{\nu(0)}_{\boldsymbol{a}}(t,-kT) + \int_{-kT}^t K_{\boldsymbol{a}}(t,u)\rho(u)du$$
$$1 = H^{\nu(0)}_{\boldsymbol{a}}(t,-kT) + \int_{-kT}^t H_{\boldsymbol{a}}(t,u)\rho(u)du.$$

Letting k goes to infinity, we deduce that ρ solves (3.14) and (3.15). By uniqueness, we deduce that for all t, $\rho(t) = \rho_{\boldsymbol{a}}(t)$ (and so ρ is continuous). Finally define τ_t the time of the last spike of $Y^{\boldsymbol{a},\nu(0)}_{t,-kT}$ before t (with the convention that $\tau_t = -kT$ if there is no spike between $-kT$ and t). The law of τ_t is

$$\mathcal{L}(\tau_t)(du) = \delta_{-kT}(du)H^{\nu(0)}_{\boldsymbol{a}}(t,-kT) + \rho_{\boldsymbol{a}}(u)H_{\boldsymbol{a}}(t,u)du.$$

Consequently, for any non-negative test function g:

$$\begin{aligned}\mathbb{E}\, g(Y^{\boldsymbol{a},\nu(0)}_{t,-kT}) &= \mathbb{E}\, g(Y^{\boldsymbol{a},\nu(0)}_{t,-kT}\mathbb{1}_{\tau_t=-kT}) + \mathbb{E}\, g(\varphi^{\boldsymbol{a}}_{t,\tau_t}(0))\mathbb{1}_{\tau_t\in(-kT,t]} \\ &= \int_0^\infty g(\varphi^{\boldsymbol{a}}_{t,-kT}(x))H^x_{\boldsymbol{a}}(t,-kT)\nu(0)(dx) + \int_{-kT}^t g(\varphi^{\boldsymbol{a}}_{t,u}(0))\rho_{\boldsymbol{a}}(u)H_{\boldsymbol{a}}(t,u)du.\end{aligned}$$

Using that $\mathbb{E}\, g(Y^{\boldsymbol{a},\nu(0)}_{t,-kT}) = \mathbb{E}\, g(Y^{\boldsymbol{a},\nu(0)}_{t,0})$ and letting again k to infinity we deduce that

$$\mathbb{E}\, g(Y^{\boldsymbol{a},\nu(0)}_{t,0}) = \int_{-\infty}^t g(\varphi^{\boldsymbol{a}}_{t,u}(0))\rho_{\boldsymbol{a}}(u)H_{\boldsymbol{a}}(t,u)du.$$

So for all t, $\nu(t) \equiv \tilde{\nu}_{\boldsymbol{a}}(t)$. □

3.4 Reduction to 2π-periodic functions

Convention: For now on, we prefer to work with the *reduced period* τ, such that

$$T =: 2\pi\tau, \quad \tau > 0.$$

Consider $\boldsymbol{d} \in C^0_{2\pi\tau}$ and let $\boldsymbol{a}$ be the 2π-periodic function defined by:

$$\forall t \in \mathbb{R}, \quad a(t) := d(\tau t).$$

We define

$$\forall t \in \mathbb{R}, \quad \rho_{\boldsymbol{a},\tau}(t) := \rho_{\boldsymbol{d}}(\tau t),$$

where $\rho_{\boldsymbol{d}}$ is the unique solution of (3.14) and (3.15) (with kernels $K_{\boldsymbol{d}}$ and $H_{\boldsymbol{d}}$). Because

$\rho_{\boldsymbol{d}}$ is $2\pi\tau$-periodic, $\rho_{\boldsymbol{a},\tau}$ is 2π-periodic. Note that when $\boldsymbol{a} \equiv \alpha$ is constant we have

$$\forall \tau > 0, \forall t \in \mathbb{R}, \quad \rho_{\alpha,\tau}(t) = \gamma(\alpha). \tag{3.26}$$

To better understand how $\rho_{\boldsymbol{a},\tau}$ depends on τ, consider $(Y^{\boldsymbol{d},\nu}_{t,s})$ the solution of (1.3), starting with law ν and driven by $\boldsymbol{d}$. Note that for all $t \geq s$

$$\begin{aligned} Y^{\boldsymbol{d},\nu}_{\tau t,\tau s} &= Y^{\boldsymbol{d},\nu}_{\tau s,\tau s} + \int_{\tau s}^{\tau t} b(Y^{\boldsymbol{d},\nu}_{u,\tau s})du + \int_{\tau s}^{\tau t} d_u du - \int_{\tau s}^{\tau t}\int_{\mathbb{R}_+} Y^{\boldsymbol{d},\nu}_{u-,\tau s}\mathbb{1}_{\{\tau z \leq \tau f(Y^{\boldsymbol{d},\nu}_{u-,\tau s})\}}\mathbf{N}(du,dz) \\ &= Y^{\boldsymbol{d},\nu}_{\tau s,\tau s} + \int_{s}^{t} \tau b(Y^{\boldsymbol{d},\nu}_{\tau u,\tau s})du + \int_{s}^{t} \tau a_u du - \int_{s}^{t}\int_{\mathbb{R}_+} Y^{\boldsymbol{d},\nu}_{\tau u-,\tau s}\mathbb{1}_{\{z \leq \tau f(Y^{\boldsymbol{d},\nu}_{\tau u-,\tau s})\}}\tilde{\mathbf{N}}(du,dz). \end{aligned}$$

Here, $\tilde{\mathbf{N}} := g_*\mathbf{N}$ is the push-forward measure of $\mathbf{N}$ by the function

$$g(t,z) := (\tau t, z/\tau).$$

Note that $\tilde{\mathbf{N}}(du,dz)$ is again a Poisson measure of intensity $dudz$, and so $(Y^{\boldsymbol{d},\nu}_{\tau t,\tau s})$ is a (weak) solution of (1.3) for $\tilde{f} := \tau f$, $\tilde{b} := \tau b$ and $\tilde{\boldsymbol{a}} := \tau \boldsymbol{a}$. So, in particular (taking $\nu = \delta_0$), if we define:

$$\begin{aligned} &\frac{d}{dt}\varphi^{\boldsymbol{a},\tau}_{t,s}(0) = \tau b(\varphi^{\boldsymbol{a},\tau}_{t,s}(0)) + \tau a(t); \quad \varphi^{\boldsymbol{a},\tau}_{s,s}(0) = 0, \\ &H_{\boldsymbol{a},\tau}(t,s) := \exp\left(-\int_s^t \tau f(\varphi^{\boldsymbol{a},\tau}_{u,s}(0))du\right), \\ &K_{\boldsymbol{a},\tau}(t,s) := \tau f(\varphi^{\boldsymbol{a},\tau}_{t,s}(0))\exp\left(-\int_s^t \tau f(\varphi^{\boldsymbol{a},\tau}_{u,s}(0))du\right), \end{aligned} \tag{3.27}$$

we have

Lemma 3.10. *Let $\tau > 0$ and $\boldsymbol{a} \in C^0_{2\pi}$. Set, for all $t \in \mathbb{R}$, $d(t) := a(\frac{t}{\tau})$. Then it holds that*

$$\forall t \geq s, \quad H_{\boldsymbol{a},\tau}(t,s) = H_{\boldsymbol{d}}(\tau t, \tau s) \quad \text{and} \quad K_{\boldsymbol{a},\tau}(t,s) = \tau K_{\boldsymbol{d}}(\tau t,\tau s).$$

In view of this result, we deduce that $\rho_{\boldsymbol{a},\tau}$ solves

$$\rho_{\boldsymbol{a},\tau}(t) = \int_{-\infty}^t K_{\boldsymbol{a},\tau}(t,s)\rho_{\boldsymbol{a},\tau}(s)ds, \quad 1 = \tau\int_{-\infty}^t H_{\boldsymbol{a},\tau}(t,s)\rho_{\boldsymbol{a},\tau}(s)ds, \tag{3.28}$$

or equivalently, setting

$$\forall t,s \in [0,2\pi], \quad K^{2\pi}_{\boldsymbol{a},\tau}(t,s) := \sum_{k\geq 0} K_{\boldsymbol{a},\tau}(t,s-2\pi k) \quad \text{and} \quad H^{2\pi}_{\boldsymbol{a},\tau}(t,s) := \sum_{k\geq 0} H_{\boldsymbol{a},\tau}(t,s-2\pi k), \tag{3.29}$$

one has, using the same operator notation as in (3.21)

$$\rho_{\boldsymbol{a},\tau} = K^{2\pi}_{\boldsymbol{a},\tau}(\rho_{\boldsymbol{a},\tau}), \quad 1 = \tau H^{2\pi}_{\boldsymbol{a},\tau}(\rho_{\boldsymbol{a},\tau}).$$

Note that $\rho_{\cdot,\tau}$ and $\rho_{\cdot}$ are linked by (2.4). Consequently equations (3.28) define a unique 2π-periodic continuous function

$$\rho_{\boldsymbol{a},\tau} = \frac{\pi_{\boldsymbol{a},\tau}}{c_{\boldsymbol{a},\tau}}, \tag{3.30}$$

where $\pi_{\boldsymbol{a},\tau}$ is the unique invariant measure of the Markov Chain with transition probability kernel $K^{2\pi}_{\boldsymbol{a},\tau}$ and $c_{\boldsymbol{a},\tau}$ is the constant given by

$$c_{\boldsymbol{a},\tau} := \tau H^{2\pi}_{\boldsymbol{a},\tau}(\pi_{\boldsymbol{a},\tau}).$$

3.5 Regularity of ρ

The goal of this section is to study the regularity of $\rho_{\boldsymbol{a},\tau}$ with respect to $\boldsymbol{a}$ and τ. For $\eta_0 > 0$, recall that $B^{2\pi}_{\eta_0}$ is the open ball of $C^0_{2\pi}$ defined by (3.7). The main result of this section is

Proposition 3.11. *Grant Assumptions 1.1, 1.2 and let $\alpha_0 > 0$ such that Assumption 1.5 holds. Let $\tau_0 > 0$. There exists $\epsilon_0, \eta_0 > 0$ small enough (only depending on b, f, α_0 and τ_0) such that the function*

$$\begin{array}{rcl} B^{2\pi}_{\eta_0}(\alpha_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0) & \to & C^0_{2\pi} \\ (\boldsymbol{a}, \tau) & \mapsto & \rho_{\boldsymbol{a},\tau} \end{array}$$

is $\mathcal{C}^2$ Fréchet differentiable.

The proof of Proposition 3.11 relies on (3.30) and on Lemma 3.14 below, which states that the function $(\boldsymbol{a}, \tau) \mapsto \pi_{\boldsymbol{a},\tau}$ is $\mathcal{C}^2$. Recall Notation 2.10

$$C^{0,0}_{2\pi} := \{u \in C^0_{2\pi} | \int_0^{2\pi} u(s)ds = 0\}.$$

Let $\boldsymbol{a} \in B^{2\pi}_{\eta_0}$ and $\tau > 0$. Because $\int_0^{2\pi} \pi_{\boldsymbol{a},\tau}(u)du = 1$, the space $C^0_{2\pi}$ can be decomposed in the following way

$$C^0_{2\pi} = \text{Span}(\pi_{\boldsymbol{a},\tau}) \oplus C^{0,0}_{2\pi}.$$

We denote by $K^{2\pi}_{\boldsymbol{a},\tau}\big|_{C^0_{2\pi}}$ the restriction of $K^{2\pi}_{\boldsymbol{a},\tau}$ to $C^0_{2\pi}$ (recall that the linear operator $h \mapsto K^{2\pi}_{\boldsymbol{a},\tau} h$ is defined for all $h \in L^1([0, 2\pi])$). Similarly, we denote by $I|_{C^0_{2\pi}}$ the identity operator on $C^0_{2\pi}$. Given a linear operator L, we denote by $N(L)$ its kernel (null-space) and by $R(L)$ its range.

Lemma 3.12. *Grant Assumptions 1.1 and 1.2, let $\alpha_0 > 0$ such that Assumption 1.5 holds and let $\boldsymbol{a} \in B^{2\pi}_{\eta_0}(\alpha_0)$, where $\eta_0 > 0$ is given by Lemma 3.2. It holds that*

$$N(I|_{C^0_{2\pi}} - K^{2\pi}_{\boldsymbol{a},\tau}\big|_{C^0_{2\pi}}) = \text{Span}(\pi_{\boldsymbol{a},\tau}) \quad \textit{and} \quad R(I|_{C^0_{2\pi}} - K^{2\pi}_{\boldsymbol{a},\tau}\big|_{C^0_{2\pi}}) = C^{0,0}_{2\pi}.$$

Proof. We proved in Lemma 3.4 that $N(I - K^{2\pi}_{\boldsymbol{a},\tau}) = \text{Span}(\pi_{\boldsymbol{a},\tau})$. It remains to show that $R(I|_{C^0_{2\pi}} - K^{2\pi}_{\boldsymbol{a},\tau}\big|_{C^0_{2\pi}}) = C^{0,0}_{2\pi}$. The Fredholm alternative [1, Th. 6.6] yields

$$R(I - K^{2\pi}_{\boldsymbol{a},\tau}) = N(I - (K^{2\pi}_{\boldsymbol{a},\tau})')^{\perp},$$

where $(K^{2\pi}_{\boldsymbol{a},\tau})' \in \mathcal{L}\left(L^\infty([0, 2\pi]); L^\infty([0, 2\pi])\right)$ is the dual operator of $K^{2\pi}_{\boldsymbol{a},\tau} \in \mathcal{L}\left(L^1([0, 2\pi]); L^1([0, 2\pi])\right)$. In the proof of Lemma 3.4, it is shown that

$$\mathbf{1} \in N(I - (K^{2\pi}_{\boldsymbol{a},\tau})'),$$

where $\mathbf{1}$ denotes the constant function equal to 1 on $[0, 2\pi]$. The Fredholm alternative yields

$$\dim N(I - (K^{2\pi}_{\boldsymbol{a},\tau})') = \dim N(I - K^{2\pi}_{\boldsymbol{a},\tau}) = 1.$$

So

$$N(I - (K^{2\pi}_{\boldsymbol{a},\tau})') = \text{Span}(\mathbf{1}).$$

It follows that

$$R(I - K^{2\pi}_{\boldsymbol{a},\tau}) = \text{Span}(\mathbf{1})^{\perp} = \{u \in L^1([0, 2\pi]) | \int_0^{2\pi} u(s)ds = 0\}.$$

Finally, using that for $h \in L^1([0, 2\pi])$, one has $K^{2\pi}_{\boldsymbol{a},\tau} h \in C^0_{2\pi}$, one obtains the result for the restrictions to $C^0_{2\pi}$. □

As a consequence, the linear operator $I - K^{2\pi}_{\boldsymbol{a},\tau} : C^{0,0}_{2\pi} \to C^{0,0}_{2\pi}$ is invertible, with a continuous inverse.

Lemma 3.13. *Grant Assumptions 1.1, 1.2 and let $\alpha_0 > 0$ such that Assumption 1.5 holds. Let $\tau_0 > 0$. There exists $\eta_0, \epsilon_0 > 0$ small enough (only depending on b, f, α_0 and τ_0) such* *that the following function is $\mathcal{C}^2$ Fréchet differentiable*

$$\begin{array}{rcl} B^{2\pi}_{\eta_0}(\alpha_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0) & \to & \mathcal{L}(C^0_{2\pi}; C^0_{2\pi}) \\ (\boldsymbol{a}, \tau) & \mapsto & H^{2\pi}_{\boldsymbol{a},\tau}. \end{array}$$

The same result holds for $K^{2\pi}_{\boldsymbol{a},\tau}$.

Proof. We only prove the result for H, the proof for K being similar. Let $\epsilon_0 > 0$ be chosen arbitrary such that $\epsilon_0 < \tau_0$.
Step 1. We introduce relevant Banach spaces: E denotes the set of continuous functions

$$E := \mathcal{C}([0, 2\pi]^2; \mathbb{R}), \quad \text{equipped with } ||w||_E := \sup_{t,s} |w(t, s)|$$

$$E_0 := \{w \in E,\ \forall s \in [0, 2\pi],\ w(2\pi, s) = w(0, s)\}, \quad \text{equipped with } || \cdot ||_E.$$

We define the following application Φ,

$$\begin{array}{rcl} E_0 & \to & \mathcal{L}(C^0_{2\pi}; C^0_{2\pi}) \\ w & \mapsto & \Phi(w) := \left[h \mapsto \left(\int_0^{2\pi} w(t, s)h(s)ds \right)_{t \in [0,2\pi]} \right]. \end{array}$$

Note that Φ is linear and continuous, so in particular $\mathcal{C}^2$. So, to prove the result, it suffices to show that

$$\begin{array}{rcl} B^{2\pi}_{\eta_0}(\alpha_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0) & \to & E_0 \\ (\boldsymbol{a}, \tau) & \mapsto & (H^{2\pi}_{\boldsymbol{a},\tau}(t, s))_{t,s \in [0,2\pi]^2} \end{array}$$

is $\mathcal{C}^2$, where $H^{2\pi}_{\boldsymbol{a},\tau}(t, s)$ is explicitly given by the series (3.29).
Step 2. Let $k \in \mathbb{N}$ be fixed. We prove that the function

$$\begin{array}{rcl} B^{2\pi}_{\eta_0}(\alpha_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0) & \to & E \\ (\boldsymbol{a}, \tau) & \mapsto & (H_{\boldsymbol{a},\tau}(t, s - 2\pi k))_{t,s \in [0,2\pi]^2} \end{array}$$

is $\mathcal{C}^2$. To proceed, we use the explicit expression of $H_{\boldsymbol{a},\tau}(t, s)$, given by (3.27). Note that we have first to show that the function $(\boldsymbol{a}, \tau) \mapsto \varphi^{\boldsymbol{a},\tau}_{t,s}(0) \in \mathbb{R}$ is $\mathcal{C}^2$. This follows (see [12, Th. 3.10.2]) from the fact that $b : \mathbb{R}_+ \to \mathbb{R}$ is $\mathcal{C}^2$ and so the solution of the ODE (3.27) is $\mathcal{C}^2$ with respect to $\boldsymbol{a}$ and τ. Moreover, we have for all $h \in C^0_{2\pi}$,

$$D_{\boldsymbol{a}} \varphi^{\boldsymbol{a},\tau}_{t,s}(0) \cdot h = \int_s^t \tau h(u) \exp \left(\tau \int_u^t b'(\varphi^{\boldsymbol{a},\tau}_{\theta,s}(0)) d\theta \right) du.$$

A similar expression holds for $\frac{d}{d\tau} \varphi^{\boldsymbol{a},\tau}_{t,s}(0)$. Using that f is $\mathcal{C}^2$, we deduce that the function

$$(\boldsymbol{a}, \tau) \mapsto (H_{\boldsymbol{a},\tau}(t, s - 2\pi k))_{t,s} \in E$$

is $\mathcal{C}^2$. Furthermore, we have for instance

$$D_{\boldsymbol{a}} H_{\boldsymbol{a},\tau}(t, s) \cdot h = -H_{\boldsymbol{a},\tau}(t, s) \int_s^t \tau f'(\varphi^{\boldsymbol{a},\tau}_{u,s}(0)) \left[D_{\boldsymbol{a}} \varphi^{\boldsymbol{a},\tau}_{u,s} \cdot h \right] du.$$

So, proceeding as in the proof of Lemma 3.2, we deduce the existence of $\eta_0, \lambda_0, A_0 > 0$ (only depending on b, f, α_0, τ_0 and ϵ_0) such that for all $h \in C^0_{2\pi}$ and for all $\tau \in (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0)$, it holds that

$$\sup_{t,s \in [0,2\pi]^2} \sup_{\boldsymbol{a} \in B^{2\pi}_{\eta_0}(\alpha_0)} |D_{\boldsymbol{a}} H_{\boldsymbol{a},\tau}(t, s - 2\pi k) \cdot h| \leq A_0 ||h||_\infty e^{-2\pi k \lambda_0}.$$

Similar estimates hold for the second derivative with respect to $\boldsymbol{a}$ and for the first and second derivatives with respect to τ.

Step 3. We have

$$\sum_{k\geq 0} \sup_{t,s\in[0,2\pi]^2} \sup_{\boldsymbol{a}\in B^{2\pi}_{\eta_0}(\alpha_0)} \sup_{h\in C^0_{2\pi}, ||h||_\infty\leq 1} |D_{\boldsymbol{a}} H_{\boldsymbol{a},\tau}(t, s-2\pi k)\cdot h| \leq \sum_{k\geq 0} A_0 e^{-2\pi k\lambda_0} < \infty.$$

Using [3, Th. 3.6.1], we deduce that $\boldsymbol{a} \mapsto (H^{2\pi}_{\boldsymbol{a},\tau}(t,s))_{t,s} \in E$ is Fréchet differentiable, with for all $h \in C^0_{2\pi}$

$$D_{\boldsymbol{a}} H^{2\pi}_{\boldsymbol{a},\tau}(t,s)\cdot h = \sum_{k\geq 0} D_{\boldsymbol{a}} H_{\boldsymbol{a},\tau}(t, s-2\pi k)\cdot h.$$

Note that this last series converges again normally, and so $\boldsymbol{a} \mapsto (H^{2\pi}_{\boldsymbol{a},\tau}(t,s))_{t,s}$ is in fact $\mathcal{C}^1$. Applying again [3, Th. 3.6.1], we prove similarly that $\boldsymbol{a} \mapsto H^{2\pi}_{\boldsymbol{a},\tau}(t,s)$ is $\mathcal{C}^2$. The same arguments shows that $\tau \mapsto H^{2\pi}_{\boldsymbol{a},\tau}(t,s)$ is $\mathcal{C}^2$.

Step 4. It remains to prove that $(\boldsymbol{a},\tau) \mapsto (H^{2\pi}_{\boldsymbol{a},\tau}(t,s))_{t,s} \in E_0$ is $\mathcal{C}^2$ (we have proved the result for E, not E_0, in the previous step). Let $t, s \in [0, 2\pi]$ be fixed, define

$$w \in E, \quad \mathcal{E}^t_s(w) := w(t,s) \in \mathbb{R}.$$

The application $\mathcal{E}^t_s$ is linear and continuous. Moreover, we have seen that $H^{2\pi}_{\boldsymbol{a},\tau} \in E_0$, so

$$\forall s \in [0,2\pi], \quad \mathcal{E}^{2\pi}_s(H^{2\pi}_{\boldsymbol{a},\tau}) = \mathcal{E}^0_s(H^{2\pi}_{\boldsymbol{a},\tau}).$$

Differentiating with respect to $\boldsymbol{a}$, we deduce that for all $h \in C^0_{2\pi}$,

$$\forall s \in [0,2\pi], \quad \mathcal{E}^{2\pi}_s(D_{\boldsymbol{a}} H^{2\pi}_{\boldsymbol{a},\tau}\cdot h) = \mathcal{E}^0_s(D_{\boldsymbol{a}} H^{2\pi}_{\boldsymbol{a},\tau}\cdot h),$$

and so $D_{\boldsymbol{a}} H^{2\pi}_{\boldsymbol{a},\tau} \in \mathcal{L}(C^0_{2\pi}, E_0)$. The same results holds for the second derivative with respect to $\boldsymbol{a}$ and the two derivatives with respect to τ. This ends the proof. ☐

Lemma 3.14. *Grant Assumptions 1.1, 1.2 and let $\alpha_0 > 0$ such that Assumption 1.5 holds. Let $\tau_0 > 0$. There exists $\epsilon_0, \eta_0 > 0$ small enough (only depending on b, f, α_0 and τ_0) such that the function*

$$\begin{array}{rcl} B^{2\pi}_{\eta_0}(\alpha_0) \times (\tau_0-\epsilon_0, \tau_0+\epsilon_0) & \to & C^0_{2\pi} \\ (\boldsymbol{a},\tau) & \mapsto & \pi_{\boldsymbol{a},\tau} \end{array}$$

is $\mathcal{C}^2$ Fréchet differentiable.

Remark 3.15. Recall that $\pi_{\boldsymbol{a},\tau}$ is the unique invariant measure of the Markov Chain having $K^{2\pi}_{\boldsymbol{a},\tau}$ has kernel transition probability. So, we study the smoothness of the invariant measure with respect to the parameters $(\boldsymbol{a},\tau)$, knowing the smoothness of the transition probability kernel $(\boldsymbol{a},\tau) \mapsto K^{2\pi}_{\boldsymbol{a},\tau}$. We refer to [16] for such sensibility result in the setting of finite discrete-time Markov Chains. Our approach is different and based on the implicit function theorem. In this proof, we consider independent functions $\boldsymbol{a}$ and h (that is we do not have $\boldsymbol{a} = \alpha_0 + h$).

Proof. Let α_0 and τ_0 be fixed. Let $\delta_0, \epsilon_0 > 0$ be given by Lemma 3.13. Consider the following $\mathcal{C}^2$-Fréchet differentiable function:

$$\begin{array}{rrcl} F: & C^{0,0}_{2\pi} \times B^{2\pi}_{\eta_0}(\alpha_0) \times (\tau_0-\epsilon_0, \tau_0+\epsilon_0) & \to & C^{0,0}_{2\pi} \\ & (h, \boldsymbol{a}, \tau) & \mapsto & (\alpha_0+h) - K^{2\pi}_{\boldsymbol{a},\tau}(\alpha_0+h). \end{array}$$

It holds that $F(0,\alpha_0,\tau_0) = 0$. Moreover

$$D_h F(0,\alpha_0,\tau_0) = I - K^{2\pi}_{\alpha_0,\tau_0} \in \mathcal{L}(C^{0,0}_{2\pi}, C^{0,0}_{2\pi}),$$

which is invertible with continuous inverse by Lemma 3.12. So the implicit function theorem applies: there exists ($V^{0,0}_{2\pi}$, $V^0_{2\pi}$, V_{τ_0}) open neighborhoods of $(0,\alpha_0,\tau_0)$ in $C^{0,0}_{2\pi} \times$

$C^0_{2\pi} \times \mathbb{R}^*_+$ and a $\mathcal{C}^2$-Fréchet differentiable function $U : V^0_{2\pi} \times V_{\tau_0} \to V^{0,0}_{2\pi}$ such that

$$\forall h, \boldsymbol{a}, \tau \in V^{0,0}_{2\pi} \times V^0_{2\pi} \times V_{\tau_0}, \quad F(h, \boldsymbol{a}, \tau) = 0 \iff h = U(\boldsymbol{a}, \tau).$$

By uniqueness of the invariant measure of the Markov chain with transition kernel $K^{2\pi}_{\boldsymbol{a},\tau}$, we deduce that

$$\pi_{\boldsymbol{a},\tau} = \alpha_0 + U(\boldsymbol{a}, \tau),$$

which is a $\mathcal{C}^2$-Fréchet differentiable function of $(\boldsymbol{a}, \tau)$. □

Proof of Proposition 3.11. Recall that $\rho_{\boldsymbol{a},\tau} = \frac{\pi_{\boldsymbol{a},\tau}}{c_{\boldsymbol{a},\tau}}$, where the constant $c_{\boldsymbol{a},\tau}$ is given by

$$c_{\boldsymbol{a},\tau} = \tau H^{2\pi}_{\boldsymbol{a},\tau}(\pi_{\boldsymbol{a},\tau}).$$

Furthermore, it holds that $\pi_{\alpha_0,\tau_0} = \frac{1}{2\pi}$ and $\rho_{\alpha_0,\tau_0} = \gamma(\alpha_0)$ (see (3.26)). So $c_{\alpha_0,\tau_0} = \frac{1}{2\pi\gamma(\alpha_0)} > 0$. So for ϵ_0, η_0 small enough, it holds that

$$\forall \boldsymbol{a} \in B^{2\pi}_{\eta_0}(\alpha_0), \forall \tau \in (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0), \quad c_{\boldsymbol{a},\tau} > 0.$$

Using Lemmas 3.13 and 3.14, it holds that c and ρ are $\mathcal{C}^2$, which ends the proof. □

As a first application of this result, we prove that the mean number of spikes of a neuron driven by a periodic input only depends on the mean of the input current.

Proposition 3.16. *Grant Assumptions 1.1, 1.2 and let $\alpha_0 > 0$ such that Assumption 1.5 holds. Let $\tau_0 > 0$ and consider η_0 be given by Proposition 3.11. Let $h \in C^{0,0}_{2\pi}$ such that $\alpha_0 + h \in B^{2\pi}_{\eta_0}(\alpha_0)$. It holds that*

$$c_{\alpha_0+h,\tau_0} = c_{\alpha_0,\tau_0} = \frac{1}{2\pi\gamma(\alpha_0)}.$$

We denote by c_{α_0} this last quantity. In particular, the mean number of spikes per period:

$$\frac{1}{2\pi}\int_0^{2\pi} \rho_{\alpha_0+h,\tau_0}(u)du = \gamma(\alpha_0)$$

only depends on α_0 (which is the mean of the external current $(\alpha_0 + h(t))_{t\in[0,2\pi]}$).

Proof. Let $\boldsymbol{a} \in B^{2\pi}_{\eta_0}(\alpha_0)$. We prove that

$$\forall h \in C^{0,0}_{2\pi}, \quad D_{\boldsymbol{a}} c_{\boldsymbol{a},\tau_0} \cdot h = 0.$$

We have $c_{\boldsymbol{a},\tau_0} = \tau_0 H^{2\pi}_{\boldsymbol{a},\tau_0}(\pi_{\boldsymbol{a},\tau_0})$. Differentiating with respect to $\boldsymbol{a}$, one gets

$$D_{\boldsymbol{a}} c_{\boldsymbol{a},\tau_0} \cdot h = \tau_0 \left[D_{\boldsymbol{a}} H^{2\pi}_{\boldsymbol{a},\tau_0} \cdot h\right](\pi_{\boldsymbol{a},\tau_0}) + \tau_0 H^{2\pi}_{\boldsymbol{a},\tau_0} D_{\boldsymbol{a}} \pi_{\boldsymbol{a},\tau_0} \cdot h.$$

Recall that $\pi_{\boldsymbol{a},\tau_0} = K^{2\pi}_{\boldsymbol{a},\tau_0} \pi_{\boldsymbol{a},\tau_0}$ so

$$D_{\boldsymbol{a}} \pi_{\boldsymbol{a},\tau_0} \cdot h = \left[D_{\boldsymbol{a}} K^{2\pi}_{\boldsymbol{a},\tau_0} \cdot h\right] \pi_{\boldsymbol{a},\tau_0} + K^{2\pi}_{\boldsymbol{a},\tau_0} \left[D_{\boldsymbol{a}} \pi_{\boldsymbol{a},\tau_0} \cdot h\right].$$

So, using Lemma 3.12, one has

$$D_{\boldsymbol{a}} \pi_{\boldsymbol{a},\tau_0} \cdot h = \left[I - K^{2\pi}_{\boldsymbol{a},\tau_0}\right]^{-1} \left[D_{\boldsymbol{a}} K^{2\pi}_{\boldsymbol{a},\tau_0} \cdot h\right] \pi_{\boldsymbol{a},\tau_0}. \tag{3.31}$$

Define on $C^{0,0}_{2\pi}$ the linear operator

$$\forall h \in C^{0,0}_{2\pi}, \quad \mathbb{1}^{2\pi}(h)(t) := \int_0^{2\pi} \mathbb{1}_{\{t \geq s\}} h(s) ds = \int_0^t h(s) ds.$$

We have

$$1 * K_{\boldsymbol{a},\tau_0} = 1 - H_{\boldsymbol{a},\tau_0}, \tag{3.32}$$

so on $C^{0,0}_{2\pi}$,

$$H^{2\pi}_{\boldsymbol{a},\tau_0} = \mathbb{1}^{2\pi}\left[I - K^{2\pi}_{\boldsymbol{a},\tau_0}\right]. \tag{3.33}$$

So

$$H^{2\pi}_{\boldsymbol{a},\tau_0}\left[I - K^{2\pi}_{\boldsymbol{a},\tau_0}\right]^{-1} = \mathbb{1}^{2\pi}.$$

Consequently, we have

$$D_{\boldsymbol{a}} c_{\boldsymbol{a},\tau_0} \cdot h = \tau_0 \left[D_{\boldsymbol{a}} H^{2\pi}_{\boldsymbol{a},\tau_0} \cdot h\right](\pi_{\boldsymbol{a},\tau_0}) + \tau_0 \mathbb{1}^{2\pi}\left[D_{\boldsymbol{a}} K^{2\pi}_{\boldsymbol{a},\tau_0} \cdot h\right]\pi_{\boldsymbol{a},\tau_0}$$

Differentiating (3.33), one has

$$D_{\boldsymbol{a}} H^{2\pi}_{\boldsymbol{a},\tau_0} \cdot h = -\mathbb{1}^{2\pi}\left[D_{\boldsymbol{a}} K^{2\pi}_{\boldsymbol{a},\tau_0} \cdot h\right],$$

and so for all $h \in C^{0,0}_{2\pi}$, $D_{\boldsymbol{a}} c_{\boldsymbol{a},\tau_0} \cdot h = 0$. Then for all $h \in C^{0,0}_{2\pi}$ such that $\alpha_0 + h \in B^{2\pi}_{\eta_0}(\alpha_0)$, one has

$$c_{\alpha_0+h,\tau_0} - c_{\alpha_0,\tau_0} = \int_0^1 \left[D_{\boldsymbol{a}} c_{\alpha_0+th,\tau_0} \cdot h\right] dt = 0.$$

Finally we have $\pi_{\alpha_0,\tau_0} = \frac{1}{2\pi}$ and, by (3.26), $\rho_{\alpha_0,\tau_0} = \gamma(\alpha_0)$. By definition (3.30), we have $c_{\alpha_0,\tau_0} = \frac{\pi_{\alpha_0,\tau_0}}{\rho_{\alpha_0,\tau_0}}$. It ends the proof. □

3.6 Strategy to handle the non-linear equation (1.1)

Grant Assumptions 1.1, 1.2 and let $\alpha_0 > 0$ such that Assumption 1.5 holds. Let $\tau_0 > 0$ be given by Assumption 2.2. For $\eta_0, \epsilon_0 > 0$, define $G : B^{2\pi}_{\eta_0}(\alpha_0) \cap C^{0,0}_{2\pi} \times (\alpha_0 - \eta_0, \alpha_0 + \eta_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0) \to C^{0,0}_{2\pi}$ such that

$$G(h, \alpha, \tau) := (\alpha + h) - J(\alpha)\rho_{\alpha+h,\tau}. \tag{3.34}$$

Using Propositions 3.11 and 3.16, we choose η_0, ϵ_0 small enough such that G is $\mathcal{C}^2$-Fréchet differentiable and indeed takes values in $C^{0,0}_{2\pi}$. For any constant $\alpha, \tau > 0$, we have, by (3.26), $\rho_{\alpha,\tau} = \gamma(\alpha)$. Recalling that $J(\alpha)\gamma(\alpha) = \alpha$, we have

$$\forall(\alpha, \tau) \in (\alpha_0 - \eta_0, \alpha_0 + \eta_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0), \quad G(0, \alpha, \tau) = 0. \tag{3.35}$$

Those are the trivial roots of G. To construct the periodic solutions to (1.1), we find the non-trivial roots of G. In fact, Theorem 2.8 is deduced from the following proposition.

Proposition 3.17. *Consider b, f and $\alpha_0, \tau_0 > 0$ such that Assumptions 1.1, 1.2, 1.5, 2.2, 2.3 and 2.6 hold. Let G be defined by* (3.34). *There exists $X \times V_{\alpha_0} \times V_{\tau_0}$ an open neighborhood of $(0, \alpha_0, \tau_0)$ in $(C^{0,0}_{2\pi}, ||\cdot||_\infty) \times \mathbb{R}^*_+ \times \mathbb{R}^*_+$ such that:*

1. *There exists a continuous curve $\{(h_v, \alpha_v, \tau_v),\ v \in (-v_0, v_0)\}$ passing through $(0, \alpha_0, \tau_0)$ at $v = 0$ and such that for all $v \in (-v_0, v_0)$*

   $$(h_v, \alpha_v, \tau_v) \in X \times V_{\alpha_0} \times V_{\tau_0} \quad \text{and} \quad G(h_v, \alpha_v, \tau_v) = 0.$$

   Moreover, it holds that

   $$\forall v \in (-v_0, v_0), \quad \frac{1}{2\pi}\int_0^{2\pi} h_v(t)\cos(t)dt = v \quad \text{and} \quad \frac{1}{2\pi}\int_0^{2\pi} h_v(t)\sin(t)dt = 0.$$

   In particular, $h_v \not\equiv 0$ for $v \neq 0$.

2. *For all $(h, \alpha, \tau) \in X \times V_{\alpha_0} \times V_{\tau_0}$, with $h \not\equiv 0$, it holds that*

   $$G(h, \alpha, \tau) = 0 \iff \left[\exists v \in (-v_0, v_0), \exists \theta \in [0, 2\pi), \quad (h, \alpha, \tau) \equiv (S_\theta(h_v), \alpha_v, \tau_v)\right].$$

We here prove that our main result is a consequence of this proposition.

Proof that Proposition 3.17 implies Theorem 2.8. Let (h_v, α_v, τ_v) be the continuous curve given by Proposition 3.17. Define $\boldsymbol{a}_v$

$$\forall t \in \mathbb{R}, \quad a_v(t) := \alpha_v + h_v(t/\tau_v).$$

The function $\boldsymbol{a}_v$ is $2\pi\tau_v$-periodic and continuous. From $G(h_v, \alpha_v, \tau_v) = 0$, we deduce that $\boldsymbol{a}_v$ solves (2.3):

$$\boldsymbol{a}_v = J(\alpha_v)\rho_{\boldsymbol{a}_v}.$$

Consider $\tilde{\nu}_{\boldsymbol{a}_v}$ defined by (3.24). By Proposition 3.9, $(\tilde{\nu}_{\boldsymbol{a}_v}(t))$ is a $2\pi\tau_v$-periodic solution of (1.1) and $(\tilde{\nu}_{\boldsymbol{a}_v}, \alpha_v, \tau_v)$ satisfies all the properties stated in Theorem 2.8: this gives the existence part of the proof. We now prove uniqueness.

Let $\epsilon_0 > 0$ small enough such that $(\tau_0 - \epsilon_0, \tau_0 + \epsilon_0) \subset V_{\tau_0}$, V_{τ_0} being given by Proposition 3.17. Let $J, \tau > 0$ be fixed, consider $\nu(t)$ a $2\pi\tau$-periodic solution of (1.1) such that

$$|\tau - \tau_0| < \epsilon_0 \quad \text{and} \quad \sup_{t \in [0, 2\pi\tau]} \left| J \int_{\mathbb{R}_+} f(x)\nu(t, dx) - \alpha_0 \right| < \epsilon_1,$$

for some constant $\epsilon_1 > 0$ to be specified later. Define $\boldsymbol{a}$

$$\forall t \in \mathbb{R}, \quad a(t) := J \int_{\mathbb{R}_+} f(x)\nu(t, dx).$$

The function $\boldsymbol{a}$ is $2\pi\tau$-periodic. Let $(X_t)_{t\geq 0}$ be the solution of the non-linear equation (1.1), starting with the initial condition $\nu(0) \in \mathcal{M}(f^2)$. The arguments of [5, Lem. 24] show that, under Assumptions 1.1 and 1.2, the function $t \mapsto \mathbb{E}\, f(X_t)$ is continuous, and so $\boldsymbol{a} \in C^0_{2\pi\tau}$. We write

$$a(t) =: \alpha + h(t/\tau),$$

for some constant α and some $h \in C^{0,0}_{2\pi}$. Because $\nu(t)$ is a periodic solution of (1.1), it holds that

$$\boldsymbol{a} = J\rho_{\boldsymbol{a}},$$

or equivalently,

$$\alpha + h = J\rho_{\alpha+h,\tau}. \tag{3.36}$$

We have by assumption

$$|\alpha - \alpha_0| = \left| \frac{1}{2\pi}\int_0^{2\pi} J \int_{\mathbb{R}_+} f(x)\nu(\tau u, dx)du - \frac{1}{2\pi}\int_0^{2\pi} J(\alpha_0) \int_{\mathbb{R}_+} f(x)\nu^\infty_{\alpha_0}(dx)du \right| < \epsilon_1.$$

Recall that α_0 satisfies Assumption 1.5. By Lemma 3.1 and using the continuity of b', we can assume that ϵ_1 is small enough such that Assumption 1.5 is also satisfied by α. Let η_0 be given by Proposition 3.11 (η_0 only depends on b, f, α_0 and τ_0). Provided that $\epsilon_1 \leq \eta_0$, we can apply Proposition 3.16 at (α, τ). It holds that

$$\frac{1}{2\pi}\int_0^{2\pi} \rho_{\alpha+h,\tau}(u)du = \gamma(\alpha),$$

so (3.36) gives

$$\alpha = J\gamma(\alpha).$$

This proves that $J = J(\alpha)$ and so (3.36) yields $G(h, \alpha, \tau) = 0$. By the uniqueness part of Proposition 3.17, there exists $\theta \in [0, 2\pi)$ and $v \in (-v_0, v_0)$ such that

$$\forall t, \quad h(t) = h_v(t + \theta), \quad \alpha = \alpha_v, \quad \tau = \tau_v.$$

So, we deduce that $a(t) = \alpha_v + h_v\left(\frac{t+\theta}{\tau_v}\right)$ and $J = J(\alpha_v)$. This ends the proof. $\square$

It remains to prove Proposition 3.17.

3.7 Linearization of G.

Define:

$$\forall t \in \mathbb{R}, \quad \Theta_{\alpha,\tau}(t) := \tau \Theta_\alpha(\tau t) \mathbb{1}_{\mathbb{R}_+}(t), \tag{3.37}$$

where Θ_α is given by (1.17). The main result of this section is the following.

Proposition 3.18. *Let $h \in C^{0,0}_{2\pi}$. It holds that*

$$[D_h G(0, \alpha, \tau) \cdot h] (t) = h(t) - J(\alpha) \int_{\mathbb{R}} \Theta_{\alpha,\tau}(t-s) h(s) ds.$$

The proof of this proposition relies on Lemmas 3.19 and 3.20 below. Let $h \in C^{0,0}_{2\pi}$. The definition of G yields

$$D_h G(0, \alpha, \tau) \cdot h = h - J(\alpha) D_{\boldsymbol{a}} \rho_{\alpha,\tau} \cdot h.$$

By equation (3.30) and Proposition 3.16, one has

$$D_{\boldsymbol{a}} \rho_{\alpha,\tau} \cdot h = \frac{1}{c_\alpha} D_{\boldsymbol{a}} \pi_{\alpha,\tau} \cdot h.$$

Recall that π_α is the uniform law on $[0, 2\pi]$. To compute $D_{\boldsymbol{a}} \pi_{\alpha,\tau} \cdot h$, we use (3.31) with $\boldsymbol{a} \equiv \alpha$:

$$D_{\boldsymbol{a}} \pi_{\alpha,\tau} \cdot h = (I - K^{2\pi}_{\alpha,\tau})^{-1} \left[D_{\boldsymbol{a}} K^{2\pi}_{\alpha,\tau} \cdot h \right] (\tfrac{1}{2\pi}). \tag{3.38}$$

The next lemma is devoted to the computation of $(I - K^{2\pi}_{\alpha,\tau})^{-1}$. Consider $t \mapsto r_\alpha(t)$ the solution of the convolution Volterra integral equation (1.6) (with $\nu = \delta_0$ and $\boldsymbol{a} = \alpha$). That is, r_α solves $r_\alpha = K_\alpha + K_\alpha * r_\alpha$. By [5, Prop. 37], there exists a function $\xi_\alpha \in L^1(\mathbb{R}_+)$ such that for all $t \geq 0$,

$$r_\alpha(t) = \gamma(\alpha) + \xi_\alpha(t).$$

Define for all $t \geq 0$, $r_{\alpha,\tau}(t) := \tau r_\alpha(\tau t)$. It solves

$$r_{\alpha,\tau} = K_{\alpha,\tau} + K_{\alpha,\tau} * r_{\alpha,\tau}, \tag{3.39}$$

where $K_{\alpha,\tau}$ is given by (3.27). Similarly, let $\xi_{\alpha,\tau}(t) := \tau \xi_\alpha(\tau t)$. We have

$$r_{\alpha,\tau}(t) = \tau \gamma(\alpha) + \xi_{\alpha,\tau}(t).$$

Recall that by definition, we have

$$K^{2\pi}_{\alpha,\tau}(h)(t) = \int_0^{2\pi} K^{2\pi}_{\alpha,\tau}(t,s) h(s) ds = \int_{-\infty}^t K_{\alpha,\tau}(t-s) h(s) ds.$$

and

$$H^{2\pi}_{\alpha,\tau}(h)(t) = \int_0^{2\pi} H^{2\pi}_{\alpha,\tau}(t,s) h(s) ds = \int_{-\infty}^t H_{\alpha,\tau}(t-s) h(s) ds.$$

Lemma 3.19. *The inverse of the linear operator $I - K^{2\pi}_{\alpha,\tau} : C^{0,0}_{2\pi} \to C^{0,0}_{2\pi}$ is given by $I + r^{2\pi}_{\alpha,\tau}$ where for all $h \in C^{0,0}_{2\pi}$ and $t \in [0, 2\pi]$*

$$\begin{aligned}
r^{2\pi}_{\alpha,\tau}(h) &:= \tau \gamma(\alpha) \Gamma(h) + \xi^{2\pi}_{\alpha,\tau}(h), \\
\Gamma(h)(t) &:= \int_0^t h(s) ds - \frac{1}{2\pi} \int_0^{2\pi} \int_0^s h(u) du ds, \\
\xi^{2\pi}_{\alpha,\tau}(h)(t) &:= \int_{-\infty}^t \xi_{\alpha,\tau}(t-s) h(s) ds.
\end{aligned}$$

Proof. Note that $\Gamma(h)$ is the only primitive of h which belongs to $C^{0,0}_{2\pi}$. Moreover, because $t \mapsto \xi_{\alpha,\tau}(t) \in L^1(\mathbb{R}_+)$, we have for $h \in C^{0,0}_{2\pi}$:

$$\begin{aligned}\int_0^{2\pi}\int_{-\infty}^t \xi_{\alpha,\tau}(t-s)h(s)dsdt &= \int_0^{2\pi}\int_0^{\infty} \xi_{\alpha,\tau}(u)h(t-u)dudt \\ &= \int_0^{\infty}\int_0^{2\pi} \xi_{\alpha,\tau}(u)h(t-u)dtdu = 0.\end{aligned}$$

So $\xi^{2\pi}_{\alpha,\tau}(h) \in C^{0,0}_{2\pi}$. Altogether, $r^{2\pi}_{\alpha,\tau} \in C^{0,0}_{2\pi}$. To conclude, we have to show that on $C^{0,0}_{2\pi}$

$$K^{2\pi}_{\alpha,\tau} \circ r^{2\pi}_{\alpha,\tau} = r^{2\pi}_{\alpha,\tau} \circ K^{2\pi}_{\alpha,\tau} = r^{2\pi}_{\alpha,\tau} - K^{2\pi}_{\alpha,\tau}.$$

Note that for all $t \in [0, 2\pi]$,

$$\frac{d}{dt}\left[\Gamma(h)(t) - H^{2\pi}_{\alpha,\tau}(h)(t)\right] = K^{2\pi}_{\alpha,\tau}(h)(t).$$

Because $\Gamma(h), H^{2\pi}_{\alpha,\tau}(h) \in C^{0,0}_{2\pi}$, we deduce that

$$\Gamma(K^{2\pi}_{\alpha,\tau}(h)) = \Gamma(h) - H^{2\pi}_{\alpha,\tau}.$$

Moreover, we have (using that $\xi_{\alpha,\tau}, K_{\alpha,\tau} \in L^1(\mathbb{R}_+)$)

$$\begin{aligned}\xi^{2\pi}_{\alpha,\tau}(K^{2\pi}_{\alpha,\tau}(h))(t) &= \int_{-\infty}^t \xi_{\alpha,\tau}(t-s)\int_{-\infty}^s K_{\alpha,\tau}(s-u)h(u)duds \\ &= \int_{-\infty}^t h(u)\int_u^t \xi_{\alpha,\tau}(t-s)K_{\alpha,\tau}(s-u)dsdu \\ &= \int_{-\infty}^t h(u)(\xi_{\alpha,\tau} * K_{\alpha,\tau})(t-u)du.\end{aligned}$$

Using (3.32) and (3.39), we deduce the identity

$$K_{\alpha,\tau} * \xi_{\alpha,\tau} = \xi_{\alpha,\tau} * K_{\alpha,\tau} = \xi_{\alpha,\tau} - K_{\alpha,\tau} + \tau\gamma(\alpha)H_{\alpha,\tau}. \tag{3.40}$$

So

$$\xi^{2\pi}_{\alpha,\tau}(K^{2\pi}_{\alpha,\tau}(h)) = \xi^{2\pi}_{\alpha,\tau}(h) - K^{2\pi}_{\alpha,\tau}(h) + \tau\gamma(\alpha)H^{2\pi}_{\alpha,\tau}(h).$$

Altogether,

$$r^{2\pi}_{\alpha,\tau}(K^{2\pi}_{\alpha,\tau}(h)) = r^{2\pi}_{\alpha,\tau}(h) - K^{2\pi}_{\alpha,\tau}(h).$$

We now prove that $K^{2\pi}_{\alpha,\tau}(r^{2\pi}_{\alpha,\tau}(h)) = r^{2\pi}_{\alpha,\tau}(h) - K^{2\pi}_{\alpha,\tau}(h)$. Using (3.40), we have $K^{2\pi}_{\alpha,\tau}(\xi^{2\pi}_{\alpha,\tau}(h)) = \xi^{2\pi}_{\alpha,\tau}(K^{2\pi}_{\alpha,\tau}(h))$. Moreover, because $K^{2\pi}_{\alpha,\tau}(1) = 1$, we have

$$\begin{aligned}K^{2\pi}_{\alpha,\tau}(\Gamma(h))(t) &= \int_{-\infty}^t K_{\alpha,\tau}(t-s)\int_0^s h(u)duds - \frac{1}{2\pi}\int_0^{2\pi}\int_0^s h(u)duds \\ &= \left[H_{\alpha,\tau}(t-s)\int_0^s h(u)du\right]_{-\infty}^t - \int_{-\infty}^t H_{\alpha,\tau}(t-s)h(s)ds \\ &\quad - \frac{1}{2\pi}\int_0^{2\pi}\int_0^s h(u)duds \\ &= \Gamma(h)(t) - H^{2\pi}_{\alpha,\tau}(h)(t) = \Gamma(K^{2\pi}_{\alpha,\tau}(h))(t).\end{aligned}$$

It ends the proof. □

Let $(\alpha, \tau) \in (\alpha_0 - \eta_0, \alpha_0 + \eta_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0)$ and $h \in C^{0,0}_{2\pi}$. Using (3.30) and Proposition 3.16, it holds that

$$D_{\boldsymbol{a}}\rho_{\alpha,\tau} \cdot h = \frac{1}{c_{\alpha,\tau}} D_{\boldsymbol{a}}\pi_{\alpha,\tau} \cdot h \overset{(3.38)}{=} (I + r^{2\pi}_{\alpha,\tau}) \left[D_{\boldsymbol{a}} K^{2\pi}_{\alpha,\tau} \cdot h \right] (\gamma(\alpha)). \tag{3.41}$$

Consider $\Xi_\alpha(t)$ be defined by (1.21) and define for all $t \geq 0$, $\Xi_{\alpha,\tau}(t) := \tau \Xi_\alpha(\tau t)$. We also denote by $\Xi^{2\pi}_{\alpha,\tau}$ the linear operator

$$\forall h \in C^0_{2\pi}, \forall t \in [0, 2\pi], \quad \Xi^{2\pi}_{\alpha,\tau}(h)(t) := \int_{-\infty}^{t} \Xi_{\alpha,\tau}(t-u) h(u) du.$$

Lemma 3.20. *For all* $h \in C^0_{2\pi}$ *we have* $\left[D_{\boldsymbol{a}} K^{2\pi}_{\alpha,\tau} \cdot h \right] (\gamma(\alpha)) = \Xi^{2\pi}_{\alpha,\tau}(h)$.

Proof. Given $h \in C^0_{2\pi}$, we have

$$\left[D_{\boldsymbol{a}} K^{2\pi}_{\alpha,\tau} \cdot h \right] (\gamma(\alpha))(t) = \gamma(\alpha) \int_{-\infty}^{t} \left[D_{\boldsymbol{a}} K_{\alpha,\tau} \cdot h \right] (t,s) ds$$

So we have to prove that

$$\forall h \in C^0_{2\pi}, \quad \gamma(\alpha) \int_{-\infty}^{t} \left[D_{\boldsymbol{a}} K_{\alpha,\tau} \cdot h \right] (t,s) ds = \int_{-\infty}^{t} \Xi_{\alpha,\tau}(t-s) h(s) ds. \tag{3.42}$$

When $\tau = 1$, we know by Lemma 3.10 that $K_{\alpha,1} = K_\alpha$, $H_{\alpha,1} = H_\alpha$, etc.. In [6], eq. (72) gives

$$\gamma(\alpha) \int_{-\infty}^{t} \left[D_{\boldsymbol{a}} H_\alpha \cdot h \right] (t,s) ds = - \int_{\mathbb{R}} \Psi_\alpha(t-s) h(s) ds,$$

where $\Psi_\alpha(t)$ is given by (1.20). Using that $\Psi_\alpha(0) = 0$, $\Xi_\alpha(t) \overset{(1.21)}{=} \frac{d}{dt}\Psi_\alpha(t)$ and

$$\int_{-\infty}^{t} \left[D_{\boldsymbol{a}} K_\alpha \cdot h \right] (t,s) ds = - \frac{d}{dt} \int_{-\infty}^{t} \left[D_{\boldsymbol{a}} H_\alpha \cdot h \right] (t,s) ds,$$

we deduce (3.42) with $\tau = 1$. The result for $\tau \neq 1$ can be deduced from the case $\tau = 1$. Indeed, given $\alpha > 0$ and $h \in C^0_{2\pi}$, define $\tilde{f} := \tau f$, $\tilde{b} := \tau b$, $\tilde{\alpha} := \tau \alpha$, and $\tilde{h} := \tau h$. By applying the result for $\tilde{\tau} := 1$, $\tilde{b}$, $\tilde{f}$, $\tilde{\alpha}$ and $\tilde{h}$, we obtain exactly the stated equality. $\square$

Proof of Proposition 3.18. We use Lemma 3.20 together with (3.41). For all $h \in C^{0,0}_{2\pi}$, one obtains

$$D_{\boldsymbol{a}}\rho_{\alpha,\tau} \cdot h = \Xi^{2\pi}_{\alpha,\tau}(h) + r^{2\pi}_{\alpha,\tau}(\Xi^{2\pi}_{\alpha,\tau}(h)).$$

The definition of $r^{2\pi}_{\alpha,\tau}$ yields

$$r^{2\pi}_{\alpha,\tau}(\Xi^{2\pi}_{\alpha,\tau}(h)) = \tau \gamma(\alpha) \Gamma(\Xi^{2\pi}_{\alpha,\tau}(h)) + \xi^{2\pi}_{\alpha,\tau}(\Xi^{2\pi}_{\alpha,\tau}(h)).$$

Let $\Psi_{\alpha,\tau}(t) := \Psi_\alpha(\tau t)$, such that $\frac{d}{dt}\Psi_{\alpha,\tau}(t) = \Xi_{\alpha,\tau}(t)$. From the identity

$$\frac{d}{dt} \int_{-\infty}^{t} \Psi_{\alpha,\tau}(t-u) h(u) du = \int_{-\infty}^{t} \Xi_{\alpha,\tau}(t-u) h(u) du,$$

we find that

$$\Gamma(\Xi^{2\pi}_{\alpha,\tau}(h))(t) = \int_{-\infty}^{t} \Psi_{\alpha,\tau}(t-u) h(u) du = \int_{-\infty}^{t} (1 * \Xi_{\alpha,\tau})(t-u) h(u) du.$$

So

$$[D_{\boldsymbol{a}}\rho_{\alpha,\tau}\cdot h](t) = \int_{-\infty}^{t}\Xi_{\alpha,\tau}(t-u)h(u)du + \tau\gamma(\alpha)\int_{-\infty}^{t}(1*\Xi_{\alpha,\tau})(t-u)h(u)du$$
$$+\int_{-\infty}^{t}\xi_{\alpha,\tau}(t-u)\int_{-\infty}^{u}\Xi_{\alpha,\tau}(u-\theta)h(\theta)d\theta du.$$

Fubini's Theorem yields

$$\int_{-\infty}^{t}\xi_{\alpha,\tau}(t-u)\int_{-\infty}^{u}\Xi_{\alpha,\tau}(u-\theta)h(\theta)d\theta du = \int_{-\infty}^{t}(\xi_{\alpha,\tau}*\Xi_{\alpha,\tau})(t-\theta)h(\theta)d\theta.$$

Finally, we have

$$\Xi_{\alpha,\tau}+\tau\gamma(\alpha)(1*\Xi_{\alpha,\tau})+\xi_{\alpha,\tau}*\Xi_{\alpha,\tau} = \Xi_{\alpha,\tau}+r_{\alpha,\tau}*\Xi_{\alpha,\tau} \quad \text{(because } r_{\alpha,\tau}=\tau\gamma(\alpha)+\xi_{\alpha,\tau}\text{)}$$
$$\overset{(1.23)}{=}\Theta_{\alpha,\tau},$$

so

$$[D_{\boldsymbol{a}}\rho_{\alpha,\tau}\cdot h](t) = \int_{-\infty}^{t}\Theta_{\alpha,\tau}(t-u)h(u)du.$$

It ends the proof. □

3.8 The linearization of G at $(0,\alpha_0,\tau_0)$ is a Fredholm operator

For notational convenience we now write

$$B_0 := D_h G(0,\alpha_0,\tau_0).$$

Proposition 3.21. *We have $N(B_0)=R(Q)$, $R(B_0)=N(Q)$, where Q is the following projector on $C^{0,0}_{2\pi}$:*

$$\forall z\in C^{0,0}_{2\pi},\quad Q(z)(t) := \left[\frac{1}{2\pi}\int_0^{2\pi}z(s)e^{-is}ds\right]e^{it}+\left[\frac{1}{2\pi}\int_0^{2\pi}z(s)e^{is}ds\right]e^{-it}. \tag{3.43}$$

Remark 3.22. In particular, $B_0\in\mathcal{L}(C^{0,0}_{2\pi},C^{0,0}_{2\pi})$ is a Fredholm operator of index 0, with dim $N(B_0)=2$.

Proof. First, let $h\in N(B_0)$. One has for all $t\in\mathbb{R}$

$$h(t) = J(\alpha_0)\int_{\mathbb{R}}\Theta_{\alpha_0,\tau_0}(t-s)h(s)ds.$$

Consider for all $n\in\mathbb{Z}$

$$\tilde{h}_n := \frac{1}{2\pi}\int_0^{2\pi}h(s)e^{-ins}ds$$

the n-th Fourier coefficient of h. We have

$$\forall n\in\mathbb{Z},\quad \tilde{h}_n = J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(in)\tilde{h}_n.$$

Assumption 2.3 ensures that

$$\forall n\in\mathbb{Z}\backslash\{-1,1\},\quad J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(in)\neq 1,$$

and so

$$\forall n\in\mathbb{Z}\backslash\{-1,1\},\quad \tilde{h}_n = 0.$$

We deduce that $h \in R(Q)$. Conversely, if $h \in R(Q)$, there exists $c \in \mathbb{C}$ such that

$$h(t) = ce^{it} + \bar{c}e^{-it}$$

and so

$$\begin{aligned} J(\alpha_0) \int_{\mathbb{R}} \Theta_{\alpha_0,\tau_0}(t-s)h(s)ds &= ce^{it} J(\alpha_0) \int_{\mathbb{R}} \Theta_{\alpha_0,\tau_0}(s)e^{-is}ds + \bar{c}e^{-it} J(\alpha_0) \int_{\mathbb{R}} \Theta_{\alpha_0,\tau_0}(s)e^{is}ds \\ &= ce^{it} J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(i) + \bar{c}e^{-it} J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(-i) \\ &= h(t). \end{aligned}$$

We used here that $J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(i) = J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(-i) = 1$ (Assumption 2.2). This proves that $N(B_0) = R(Q)$. Consider now $k \in R(B_0)$, there exists $h \in C^{0,0}_{2\pi}$ such that $B_0(h) = k$. We have for all $t \in \mathbb{R}$

$$h(t) - J(\alpha_0) \int_{\mathbb{R}} \Theta_{\alpha_0,\tau_0}(t-s)h(s)ds = k(t).$$

Using that $J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(i) = 1$, we deduce that

$$\frac{1}{2\pi} \int_0^{2\pi} k(s)e^{-is}ds = \left[\frac{1}{2\pi} \int_0^{2\pi} h(s)e^{-is}ds \right] (1 - J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(i)) = 0.$$

Similarly, $\frac{1}{2\pi} \int_0^{2\pi} k(s)e^{is}ds = 0$ and so $k \in N(Q)$. It remains to show that $N(Q) \subset R(B_0)$. Consider $h \in N(Q)$ and let

$$\tilde{h}_n := \frac{1}{2\pi} \int_0^{2\pi} h(s)e^{-ins}ds$$

be its n-th Fourier coefficient. We have $\tilde{h}_1 = \tilde{h}_{-1} = 0$. The function h is continuous, and so h belongs to $L^2([0, 2\pi])$. We deduce that

$$\sum_{n \in \mathbb{Z} \setminus \{-1,1\}} |\tilde{h}_n|^2 < \infty.$$

Define

$$\forall n \in \mathbb{Z}\setminus\{-1, 1\}, \quad \epsilon_n := \frac{J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(in)}{1 - J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(in)}.$$

Using [6, Lem. 33, 34], the function $t \mapsto \Psi_{\alpha_0}(t)$, explicitly given by (1.20), is $\mathcal{C}^1$ and its derivative $\Xi_{\alpha_0}(t) = \frac{d}{dt}\Psi_{\alpha_0}(t)$ belongs to $L^1(\mathbb{R}_+)$. The same holds true for $t \mapsto \Xi_{\alpha_0}(t)$. So, using (1.24) and (3.37), we deduce that $t \mapsto \Theta_{\alpha_0,\tau_0}(t)$ is $\mathcal{C}^1$ and its derivative belongs to $L^1(\mathbb{R}_+)$. This gives the existence of a constant C such that for $n \in \mathbb{Z}$,

$$|n| > 1 \implies |\epsilon_n| \le \frac{C}{|n|}.$$

We deduce that

$$\sum_{n \in \mathbb{Z} \setminus \{-1,1\}} |n\epsilon_n \tilde{h}_n|^2 < \infty.$$

Consequently, defining

$$\forall t \in \mathbb{R}, \quad w(t) := \sum_{n \in \mathbb{Z} \setminus \{-1,1\}} \epsilon_n \tilde{h}_n e^{int},$$

it holds that $w \in H^1([0, 2\pi])$, and so w is continuous (see for instance [1, Th. 8.2]). Finally, let $k := h + w$. It holds that $k \in C^{0,0}_{2\pi}$ and the n-th Fourier coefficient of k is equals to $\frac{\tilde{h}_n}{1 - J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(in)}$. We deduce that $B_0(k) = h$. This ends the proof. □

3.9 The Lyapunov-Schmidt reduction method

The problem of finding the roots of G defined by (3.34) is an infinite dimensional problem. We use the method of Lyapunov-Schmidt to obtain an equivalent problem of finite-dimension - here of dimension 2. The equation $G = 0$ is equivalent to

$$\begin{aligned} QG(Qh + (I-Q)h, \alpha, \tau) = 0 \\ (I-Q)G(Qh + (I-Q)h, \alpha, \tau) = 0 \end{aligned}$$

where the projector Q is defined by (3.43). Define the following function W:

$$\begin{array}{rccc} W: & U_2 \times W_2 \times V_{\alpha_0} \times V_{\tau_0} & \to & R(B_0) \\ & (v, w, \alpha, \tau) & \mapsto & (I-Q)G(v + w, \alpha, \tau), \end{array}$$

where $U_2 \times W_2$ are open neighborhood of $(0,0)$ in $N(B_0) \times R(B_0)$.

We have $W(0, 0, \alpha_0, \tau_0) = 0$ and $D_w W(0, 0, \alpha_0, \tau_0) = (I-Q)D_h G(0, \alpha_0, \tau_0) = (I-Q)B_0 \in \mathcal{L}(R(B_0), R(B_0))$ which is bijective with continuous inverse. The implicit function theorem applies: there exists a $\mathcal{C}^1$ function $\psi : N(B_0) \times V_{\alpha_0} \times V_{\tau_0} \mapsto R(B_0)$ such that

> $W(v, w, \alpha, \tau) = 0$ for $(v, w, \alpha, \tau) \in U_2 \times W_2 \times V_{\alpha_0} \times V_{\tau_0}$ is equivalent to
> $w = \psi(v, \alpha, \tau)$.

Again, the neighborhoods $U_2, W_2, V_{\tau_0}, V_{\alpha_0}$ may be shrunk in this construction. We deduce that

$$G(h, \alpha, \tau) = 0 \text{ for } (h, \alpha, \tau) \in X \times V_{\alpha_0} \times V_{\tau_0} \text{ is equivalent to} \tag{3.44}$$

$$QG(Qh + \psi(Qh, \alpha, \tau), \alpha, \tau) = 0. \tag{3.45}$$

Indeed, if $G(h, \alpha, \tau) = 0$, we have in particular $W(Qh, (I-Q)h, \alpha, \tau) = 0$ and so $(I-Q)h = \psi(Qh, \alpha, \tau)$: this gives (3.45). Reciprocally, if (3.45) holds, we set $h = Qh + \psi(Qh, \alpha, \tau)$ and obtain (3.44). Note that for all $\theta \in \mathbb{R}$, we have for all $\tau > 0$ and $\boldsymbol{a} \in C^0_{2\pi},\ \rho_{S_\theta(\boldsymbol{a}), \tau} = S_\theta(\rho_{\boldsymbol{a}, \tau})$. It follows by definition of G that

$$G(S_\theta(h), \alpha, \tau) = S_\theta(G(h, \alpha, \tau)).$$

Moreover, it is clear that the projection Q commutes with S_θ (for all $\theta \in \mathbb{R},\ S_\theta Q = Q S_\theta$) and by the local uniqueness of the implicit function theorem, we deduce that

$$\psi(S_\theta(v), \alpha, \tau) = S_\theta(\psi(v, \alpha, \tau)).$$

Using that any element $Qh \in N(B_0)$ can be written

$$Qh = t \mapsto c e^{it} + \bar{c} e^{-it} := c e_0 + \bar{c} \bar{e}_0$$

for some $c \in \mathbb{C}$ and using the definition of Q, we deduce that (3.44) is equivalent to the complex equation:

$$\hat{\Phi}(c, \alpha, \tau) = 0 \text{ for } (c, \alpha, \tau) \in V_0 \times V_{\alpha_0} \times V_{\tau_0}, \text{ where}$$

$$\hat{\Phi}(c, \alpha, \tau) := \frac{1}{2\pi} \int_0^{2\pi} G(c e_0 + \bar{c} \bar{e}_0 + \psi(c e_0 + \bar{c} \bar{e}_0, \alpha, \tau), \alpha, \tau)_t e^{-it} dt$$

and V_0 is an open neighborhood of 0 in $\mathbb{C}$. We have moreover

$$\forall \theta \in \mathbb{R}, \quad \hat{\Phi}(c e^{i\theta}, \alpha, \tau) = e^{i\theta} \hat{\Phi}(c, \alpha, \tau),$$

and so (3.44) is equivalent to

$$\hat{\Phi}(v, \alpha, \tau) = 0 \text{ for } v \in (-v_0, v_0).$$

Note that $\hat{\Phi}(-v, \alpha, \tau) = -\hat{\Phi}(v, \alpha, \tau)$ and in particular

$$\forall \alpha, \tau \in V_{\alpha_0} \times V_{\tau_0}, \quad \hat{\Phi}(0, \alpha, \tau) = 0.$$

This is coherent with (3.35). In order to eliminate these trivial solutions, following [19], we set for $v \in (-v_0, v_0) \setminus \{0\}$:

$$\begin{aligned}\tilde{\Phi}(v, \alpha, \tau) :=& \frac{\hat{\Phi}(v, \alpha, \tau)}{v} \\ =& \int_0^1 D_v \hat{\Phi}(\theta v, \alpha, \tau) d\theta.\end{aligned}$$

To summarize, we have proved that

Lemma 3.23. *There exists $v_0 > 0$ and open neighborhoods $X \times V_{\alpha_0} \times V_{\tau_0}$ of $(0, \alpha_0, \tau_0)$ in $C^{0,0}_{2\pi} \times \mathbb{R}^*_+ \times \mathbb{R}^*_+$ such that the problem*

$$G(h, \alpha, \tau) = 0 \textit{ for } (h, \alpha, \tau) \in X \times V_{\alpha_0} \times V_{\tau_0} \textit{ with } h \neq 0$$

is equivalent to

$$\tilde{\Phi}(v, \alpha, \tau) = 0 \textit{ for } (v, \alpha, \tau) \in (-v_0, v_0) \times V_{\alpha_0} \times V_{\tau_0}.$$

The next section is devoted to the study of this reduced problem.

3.10 Study of the reduced 2D-problem

We denote by $\cos$ the cosinus function, such that $ve_0 + v\bar{e_0} = 2v \cos$.

Lemma 3.24. *We have:*

1. $\tilde{\Phi}(0, \alpha_0, \tau_0) = 0$.
2. $D_\tau \tilde{\Phi}(0, \alpha_0, \tau_0) = \frac{1}{2\pi} \int_0^{2\pi} \left[D^2_{h\tau} G(0, \alpha_0, \tau_0) \cdot 2\cos\right]_t e^{-it} dt$.
3. $D_\alpha \tilde{\Phi}(0, \alpha_0, \tau_0) = \frac{1}{2\pi} \int_0^{2\pi} \left[D^2_{h\alpha} G(0, \alpha_0, \tau_0) \cdot 2\cos\right]_t e^{-it} dt$.

Proof. We have $\tilde{\Phi}(0, \alpha_0, \tau_0) = D_v \hat{\Phi}(0, \alpha_0, \tau_0)$ and

$$D_v \hat{\Phi}(0, \alpha_0, \tau_0) = \frac{1}{2\pi} \int_0^{2\pi} D_h G(0, \alpha_0, \tau_0) \cdot \left[2\cos + D_v \psi(0, \alpha_0, \tau_0) \cdot 2\cos\right]_t e^{-it} dt.$$

Moreover, it holds that (see [19, Coroll. 1.2.4])

$$D_v \psi(0, \alpha_0, \tau_0) \cdot \cos = 0$$

and $\cos \in N(D_h G(0, \alpha_0, \tau_0))$, so $\tilde{\Phi}(0, \alpha_0, \tau_0) = 0$. To prove the second point (the third point is proved similarly), we have $D_\tau \tilde{\Phi}(0, \alpha_0, \tau_0) = D^2_{v\tau} \hat{\Phi}(0, \alpha_0, \tau_0)$. Moreover,

$$\begin{aligned}D_\tau \hat{\Phi}(v, \alpha, \tau) =& \frac{1}{2\pi} \int_0^{2\pi} D_\tau G(2v\cos + \psi(2r\cos, \alpha, \tau), \alpha, \tau)_t e^{-it} dt \\ &+ \frac{1}{2\pi} \int_0^{2\pi} \left[D_h G(2r\cos + \psi(2v\cos, \alpha, \tau), \alpha, \tau) \cdot D_\tau \psi(2v\cos, \alpha, \tau)\right]_t e^{-it} dt.\end{aligned}$$

So

$$\begin{aligned}
D^2_{v\tau}\hat{\Phi}(0,\alpha_0,\tau_0) &= \frac{1}{2\pi}\int_0^{2\pi}\left[D^2_{h\tau}G(0,\alpha_0,\tau_0)\cdot(2\cos+D_v\psi(0,\alpha_0,\tau_0)\cdot 2\cos)\right]_t e^{-it}dt \\
&\quad+\frac{1}{2\pi}\int_0^{2\pi}\left[D_hG(0,\alpha_0,\tau_0)\cdot D^2_{v\tau}\psi(0,\alpha_0,\tau_0)\cdot 2\cos\right]_t e^{-it}dt \\
&\quad+\frac{1}{2\pi}\int_0^{2\pi} D^2_{hh}G(0,\alpha_0,\tau_0) \\
&\qquad\cdot[2\cos+D_v\psi(0,\alpha_0,\tau_0)\cdot 2\cos, D_\tau\psi(0,\alpha_0,\tau_0)]_t e^{-it}dt.
\end{aligned}$$

Note that for all α,τ in the neighborhood of α_0,τ_0, one has

$$\psi(0,\alpha,\tau)=0,$$

so $D_\tau\psi(0,\alpha_0,\tau_0)=0$. Consequently the third term is null. Recall now that $B_0 := D_hG(0,\alpha_0,\tau_0)$ and by Proposition 3.21, it holds that $QB_0=0$. So the second term is also null. Finally, using again that $D_v\psi(0,\alpha_0,\tau_0)\cdot\cos=0$ we obtain the stated formula. □

By Proposition 3.18, we have for all $h\in C^{0,0}_{2\pi}$

$$D_hG(0,\alpha,\tau)\cdot h = h - J(\alpha)\Theta_{\alpha,\tau}*h,$$

where the function $\Theta_{\alpha,\tau}$ is given by equation (3.37). It follows that

$$D^2_{h\tau}G(0,\alpha_0,\tau_0)\cdot 2\cos = -2J(\alpha_0)\frac{\partial}{\partial\tau}\left.(\Theta_{\alpha_0,\tau}*\cos)\right|_{\tau=\tau_0},$$

and so we have

$$D_\tau\tilde{\Phi}(0,\alpha_0,\tau_0) = -J(\alpha_0)\frac{\partial}{\partial\tau}\left.\widehat{\Theta}_{\alpha_0,\tau}(i)\right|_{\tau=\tau_0}.$$

Similarly,

$$D_\alpha\tilde{\Phi}(0,\alpha_0,\tau_0) = -\frac{\partial}{\partial\alpha}\left.\left(J(\alpha)\widehat{\Theta}_{\alpha,\tau_0}(i)\right)\right|_{\alpha=\alpha_0}.$$

Lemma 3.25. *Write $J(\alpha_0)\frac{\partial}{\partial z}\widehat{\Theta}_{\alpha_0}(\frac{i}{\tau_0}) =: x_0+iy_0$. It holds that*

1. $D_\tau\tilde{\Phi}(0,\alpha_0,\tau_0)=(ix_0-y_0)/\tau_0^2$.
2. $D_\alpha\tilde{\Phi}(0,\alpha_0,\tau_0)=\mathfrak{Z}_0'(\alpha_0)(x_0+iy_0)$, *where $\mathfrak{Z}_0'(\alpha_0)$ is defined in Lemma 2.5.*

Proof. From $\Theta_{\alpha,\tau}(t)=\tau\Theta_\alpha(\tau t)$, we have

$$\frac{\partial}{\partial\tau}\Theta_{\alpha,\tau}(t)=\tfrac{1}{\tau}\left[\tau\Theta_\alpha(\tau t)+\tau\Pi_\alpha(\tau t)\right],\quad \text{with}\quad \Pi_\alpha(t):=t\frac{\partial}{\partial t}\Theta_\alpha(t).$$

So

$$\left[\widehat{\frac{\partial}{\partial\tau}\Theta_{\alpha,\tau}}\right](z)=\frac{1}{\tau}\left[\widehat{\Theta}_\alpha\left(\frac{z}{\tau}\right)+\widehat{\Pi}_\alpha\left(\frac{z}{\tau}\right)\right].$$

Moreover, an integration by parts shows that

$$\begin{aligned}
\widehat{\Pi}_\alpha(z) &= \int_0^\infty e^{-zt}t\frac{\partial}{\partial t}\Theta_\alpha(t)dt \\
&= -\widehat{\Theta}_\alpha(z)+z\int_0^\infty e^{-zt}t\Theta_\alpha(t)dt. \\
&= -\widehat{\Theta}_\alpha(z)-z\frac{\partial}{\partial z}\widehat{\Theta}_\alpha(z).
\end{aligned}$$

Choosing $z = i$ ends the proof of the first point. Define now

$$\Delta(z, \alpha) := J(\alpha)\widehat{\Theta}_\alpha(z) - 1.$$

By the definition of $\mathfrak{Z}_0(\alpha)$ (see Lemma 2.5), we have

$$\forall \alpha \in V_{\alpha_0}, \quad \Delta(\mathfrak{Z}_0(\alpha), \alpha) = 0.$$

We differentiate with respect to α and obtain

$$\frac{\partial}{\partial z}\Delta(\mathfrak{Z}_0(\alpha), \alpha)\mathfrak{Z}_0'(\alpha) + \frac{\partial}{\partial \alpha}\Delta(\mathfrak{Z}_0(\alpha), \alpha) = 0.$$

Evaluating this expression at $\alpha = \alpha_0$ gives

$$\frac{\partial}{\partial \alpha}\left(J(\alpha)\widehat{\Theta}_\alpha\right)\Big|_{\alpha=\alpha_0}(\tfrac{i}{\tau_0}) = -\mathfrak{Z}_0'(\alpha_0)(x_0 + iy_0),$$

which concludes the proof. □

Lemma 3.26. *There exists $v_0 > 0$, an open neighborhood $V_{\alpha_0} \times V_{\tau_0}$ of (α_0, τ_0) in $(\mathbb{R}_+^*)^2$ and two functions $v \mapsto \tau_v, \alpha_v \in \mathcal{C}^1((-v_0, v_0))$ such that for all $(v, \alpha, \tau) \in (-v_0, v_0) \times V_{\alpha_0} \times V_{\tau_0}$ we have*

$$\tilde{\Phi}(v, \alpha, \tau) = 0 \iff \tau = \tau_v \text{ and } \alpha = \alpha_v.$$

Proof. We decompose $\tilde{\Phi}$ into real part and imaginary part (without changing the notations), such that now

$$\tilde{\Phi} : (-v_0, v_0) \times V_{\alpha_0} \times V_{\tau_0} \to \mathbb{R}^2.$$

We have $\tilde{\Phi}(0, \alpha_0, \tau_0) = 0$ and

$$\begin{aligned} D_{(\alpha,\tau)}\tilde{\Phi}(0, \alpha_0, \tau_0) &= \begin{pmatrix} \Re D_\alpha \tilde{\Phi}(0, \alpha_0, \tau_0) & \Re D_\tau \tilde{\Phi}(0, \alpha_0, \tau_0) \\ \Im D_\alpha \tilde{\Phi}(0, \alpha_0, \tau_0) & \Im D_\tau \tilde{\Phi}(0, \alpha_0, \tau_0) \end{pmatrix} \\ &= \begin{pmatrix} x_0 \Re \mathfrak{Z}_0'(\alpha_0) - y_0 \Im \mathfrak{Z}_0'(\alpha_0) & -\frac{y_0}{\tau_0^2} \\ x_0 \Im \mathfrak{Z}_0'(\alpha_0) + y_0 \Re \mathfrak{Z}_0'(\alpha_0) & \frac{x_0}{\tau_0^2} \end{pmatrix}. \end{aligned}$$

The determinant of this matrix is $\frac{\Re \mathfrak{Z}_0'(\alpha_0)}{\tau_0^2}(x_0^2 + y_0^2)$ and this quantity is non-null by Assumptions 2.2 and 2.6. Consequently, the implicit function theorem applies and gives the result. □

The proof of Proposition 3.17 then follows immediately from this result and Lemma 3.23. This ends the proof of Theorem 2.8.

4 An explicit example

We now give a simple example of functions f and b such that Hopf bifurcations occurs and such that the spectral assumptions of Theorem 2.8 can be analytically verified. First, by [6, Th. 21], if

$$\forall x \geq 0, \quad f(x) + b'(x) \geq 0, \tag{4.1}$$

then any invariant probability measure of (1.1) is locally stable. So, to have Hopf bifurcations, the drift b has to be sufficiently attractive to break (4.1). Our minimal example satisfies all the assumptions of Theorem 2.8, except Assumption 1.2, because the function f we consider is not continuous. Indeed, to simplify the computation, we consider the step function

$$\forall x \in \mathbb{R}_+, \quad f(x) := \begin{cases} 0 & \text{for } 0 \leq x < 1, \\ 1/\beta & \text{for } x \geq 1, \end{cases}$$

where $\beta > 0$ is a (small) parameter of the model.

4.1 Some generalities when f is a step function

We shall specify later the exact shape of b, for now we only assume that

$$\inf_{x\in[0,1]} b(x) > 0.$$

This ensures in particular that the Dirac mass at 0 is not an invariant measure. We now consider some fixed constant $\alpha \geq 0$. Let, for all $x \in [0,1]$

$$t^*_\alpha(x) := \inf\{t \geq 0,\ \varphi^\alpha_t(x) = 1\},$$

the time required for the deterministic flow to hit 1, starting from x. A simple computation shows that

$$t^*_\alpha(x) = \int_x^1 \frac{dy}{b(y)+\alpha}.$$

Let $H^x_\alpha(t)$ be defined by (1.5) (with $\nu = \delta_x,\ \boldsymbol{a} \equiv \alpha$ and $s = 0$). Using the explicit shape of f, we find for all $x \in [0,1]$,

$$H^x_\alpha(t) := \begin{cases} 1 & \text{for } 0 \leq t < t^*_\alpha(x), \\ e^{-\frac{t-t^*_\alpha(x)}{\beta}} & \text{for } t \geq t^*_\alpha(x). \end{cases}$$

Moreover,

$$\forall x > 1, \quad H^x_\alpha(t) = e^{-t/\beta}. \tag{4.2}$$

Altogether,

$$\forall z \in \mathbb{C} \text{ with } \Re(z) > -1/\beta \quad \widehat{H}_\alpha(z) = \frac{1-e^{-zt^*_\alpha(0)}}{z} + \frac{e^{-zt^*_\alpha(0)}}{z+1/\beta}.$$

Note that in particular (using that $1/\gamma(\alpha) = \widehat{H}_\alpha(0)$)

$$1/\gamma(\alpha) = t^*_\alpha(0) + \beta.$$

So

$$J(\alpha) := \frac{\alpha}{\gamma(\alpha)} = \int_0^1 \frac{dy}{1+b(y)/\alpha} + \alpha\beta \tag{4.3}$$

is a strictly increasing function of α: for a fixed value of $J > 0$, there is a unique $\alpha > 0$ solution of $\alpha = J\gamma(\alpha)$ and the corresponding ν^∞_α is the unique invariant measure of (1.1). Let $\sigma_\alpha = \lim_{t\to\infty} \varphi^\alpha_t(0)$. This invariant measure is given by

$$\nu^\infty_\alpha(x) = \begin{cases} \frac{\gamma(\alpha)}{b(x)+\alpha} & \text{for } x \in [0,1), \\ \frac{\gamma(\alpha)}{b(x)+\alpha} \exp\left(-\frac{1}{\beta}\int_1^x \frac{dy}{b(y)+\alpha}\right) & \text{for } x \in [1,\sigma_\alpha) \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for $x \in [0,1]$ and $t > t^*_\alpha(x)$,

$$\frac{d}{dx} H^x_\alpha(t) = -\frac{1}{\beta}\frac{e^{-\frac{t-t^*_\alpha(x)}{\beta}}}{b(x)+\alpha}.$$

So the Laplace transform of $\frac{d}{dx}H^x_\alpha(t)$ is, for all $z \in \mathbb{C}$ with $\Re(z) > -1/\beta$

$$\forall x \in [0,1], \quad \int_0^\infty e^{-zt}\frac{d}{dx}H^x_\alpha(t)dt = -\frac{e^{-t^*_\alpha(x)z}}{b(x)+\alpha}\frac{1}{1+\beta z}.$$

Let Ψ_α be defined by (1.19). For all $z \in \mathbb{C}$ with $\Re(z) > -\beta$, one has

$$J(\alpha)\widehat{\Psi}_\alpha(z) = -\frac{\alpha}{\gamma(\alpha)} \int_0^{\sigma_\alpha} \int_0^\infty e^{-zt} \frac{d}{dx} H_\alpha^x(t) dt \, \nu_\alpha^\infty(x) dx$$
$$= \frac{\alpha}{1+\beta z} \int_0^1 \frac{e^{-t_\alpha^*(x) z}}{(b(x)+\alpha)^2} dx.$$

Indeed, using (4.2), it holds that $\frac{d}{dx} H_\alpha^x(t) = 0$ for $x > 1$. Finally, the change of variable

$$x = \varphi_u^\alpha(0), \quad u \in [0, t_\alpha^*(0)),$$

such that $t_\alpha^*(x) = t_\alpha^*(0) - u$, shows that

$$J(\alpha)\widehat{\Psi}_\alpha(z) = \frac{\alpha e^{-z t_\alpha^*(0)}}{1+\beta z} \int_0^{t_\alpha^*(0)} \frac{e^{uz}}{b(\varphi_u^\alpha(0)) + \alpha} du.$$

Using [6, Remark 35], the (local) stability of the invariant measure ν_α^∞ is given by the location of the roots of the following holomorphic function, defined for all $\Re(z) > -1/\beta$:

$$\boxed{J(\alpha)\widehat{\Psi}_\alpha(z) - \widehat{H}_\alpha(z) = \frac{\alpha e^{-z t_\alpha^*(0)}}{1+\beta z} \int_0^{t_\alpha^*(0)} \frac{e^{uz}}{b(\varphi_u^\alpha(0)) + \alpha} du - \frac{1 - e^{-z t_\alpha^*(0)}}{z} - \frac{\beta e^{-z t_\alpha^*(0)}}{1+\beta z}}.$$

4.2 A linear drift b.

We now specify the shape of b. For some parameter $m > 1$, we choose:

$$\forall x \geq 0, \quad b(x) = m - x,$$

It holds that $b(x) + \alpha = \sigma_\alpha - x$ with $\sigma_\alpha = m + \alpha$. We have $\varphi_u^\alpha(0) = \sigma_\alpha(1 - e^{-u})$ and so

$$t_\alpha^*(0) = \log\left(\frac{\sigma_\alpha}{\sigma_\alpha - 1}\right).$$

Finally

$$\int_0^{t_\alpha^*(0)} \frac{e^{uz}}{b(\varphi_u^\alpha(0)) + \alpha} du = \frac{1}{\sigma_\alpha} \int_0^{t_\alpha^*(0)} e^{(z+1)u} du = \frac{1}{\sigma_\alpha} \frac{e^{(z+1)t_\alpha^*(0)} - 1}{z+1},$$

so

$$J(\alpha)\widehat{\Psi}_\alpha(z) - \widehat{H}_\alpha(z) = \frac{\alpha}{\sigma_\alpha} \frac{e^{t_\alpha^*(0)} - e^{-z t_\alpha^*(0)}}{(1+\beta z)(z+1)} - \frac{1 - e^{-z t_\alpha^*(0)}}{z} - \frac{\beta e^{-z t_\alpha^*(0)}}{1+\beta z}.$$

Consequently, we have to study the complex solutions of

$$\Re(z) > -1/\beta, \quad \frac{\alpha}{m+\alpha-1} \frac{1 - \left(\frac{m+\alpha}{m+\alpha-1}\right)^{-(z+1)}}{(1+\beta z)(z+1)} - \frac{1 - \left(\frac{m+\alpha}{m+\alpha-1}\right)^{-z}}{z} - \frac{\beta \left(\frac{m+\alpha}{m+\alpha-1}\right)^{-z}}{1+\beta z} = 0. \tag{4.4}$$

Remark 4.1. In fact this analysis can be easily extended to any linear drift

$$b(x) = \kappa(m - x),$$

with $\kappa, m \in \mathbb{R}$. Indeed, adapting slightly the proof of [6, Th. 21] when $\kappa \leq 0$, it holds that $f + b' \geq 0$ and so the unique non trivial invariant measure is locally stable: there is no Hopf bifurcation. If on the other hand $\kappa > 0$, by setting

$$\tilde{\kappa} = 1, \quad \tilde{\alpha} = \frac{\alpha}{\kappa}, \quad \tilde{m} = m \quad \tilde{\beta} = \kappa\beta,$$

we can easily reduce the problem to $\kappa = 1$.

We now make the following change of variable

$$\omega := \log\left(\frac{m+\alpha}{m+\alpha-1}\right) \quad \text{and} \quad \delta := \frac{\alpha}{m+\alpha-1},$$

with $\omega > 0$ et $\delta \in (0,1)$. That is, we have

$$\alpha = \frac{\delta}{e^{\omega}-1} \quad \text{and} \quad m = 1 + \frac{1-\delta}{e^{\omega}-1}. \tag{4.5}$$

With this change of variable, (4.4) becomes

$$\Re(z) > -1/\beta, \quad \delta\frac{1}{1+\beta z}\frac{1-e^{-\omega(z+1)}}{1+z} - \frac{1-e^{-\omega z}}{z} - \frac{\beta e^{-\omega z}}{1+\beta z} = 0. \tag{4.6}$$

Recall that the strictly increasing function $\alpha \mapsto J(\alpha)$ is given by (4.3). With (4.5), we have

$$J'(\alpha) = \beta + \omega - \delta(1-e^{-\omega}) \neq 0.$$

We deduce that $z=0$ is not a solution of (4.6). Multiplying by $(1+\beta z)z$ on both side of (4.6), we finally find that we have to study the zeros of

$$\Re(z) > -1/\beta, \quad U(\beta,\delta,\omega,z) = 0,$$

with

$$\boxed{U(\beta,\delta,\omega,z) := \delta\frac{z}{z+1}(1-e^{-\omega(z+1)}) + e^{-\omega z} - (1+\beta z).} \tag{4.7}$$

4.3 On the roots of U

An explicit parametrization of the purely imaginary roots

We now describe all the imaginary roots of U. If $z = iy,\ y \geq 0$, the equation $U(\beta,\delta,\omega,z) = 0$ yields

$$\begin{cases} \cos(\omega y) + \sin(\omega y)y(1-\delta e^{-\omega}) &= 1-\beta y^2 \\ -\sin(\omega y) + \cos(\omega y)y(1-\delta e^{-\omega}) &= y(1+\beta-\delta). \end{cases} \tag{4.8}$$

For $\omega > 0$ et $y \geq 0$ fixed, (4.8) admits a unique solution in (β,δ), given by

$$\begin{aligned} \beta^0_\omega(y) &:= \frac{(1+e^{\omega})(1-\cos(\omega y)) - (e^{\omega}-1)y\sin(\omega y)}{y^2e^{\omega} - y^2\cos(\omega y) - y\sin(\omega y)}, \\ \delta^0_\omega(y) &:= \frac{e^{\omega}(1+y^2)(1-\cos(\omega y))}{y^2e^{\omega} - y^2\cos(\omega y) - y\sin(\omega y)}. \end{aligned} \tag{4.9}$$

Proposition 4.2. *The parametric curve $(\beta^0_\omega(y), \delta^0_\omega(y))_{y>0}$ admits exactly two multiple points given by*

$$(0,0) \quad \textit{and} \quad (0, \frac{2}{1+e^{-\omega}}).$$

Apart from those two points, the curve does not intersect itself.

Proof. Squaring the two equations of (4.8) and summing the result, one gets

$$1 + y^2(1-\delta e^{-\omega})^2 = (1-\beta y^2)^2 + y^2(1+\beta-\delta)^2,$$

that is

$$(1-\delta e^{-\omega})^2 = -2\beta + \beta^2 y^2 + (1+\beta-\delta)^2. \tag{4.10}$$

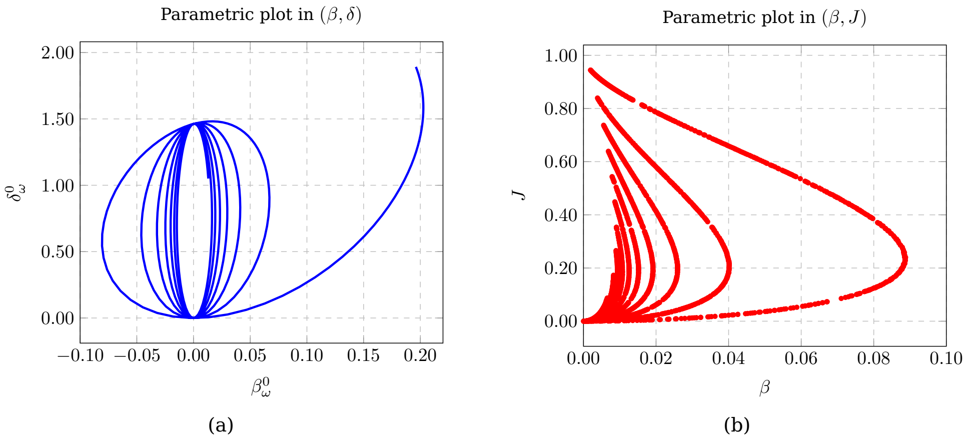

Figure 2: Description of the purely imaginary roots of U. (a) The parametric curve $(\beta^0_\omega(y), \delta^0_\omega(y))$, plotted with $\omega = 1$ and $y \in [0, 15.5\pi]$. Each point of the curve corresponds to a purely imaginary roots of U. (b) Purely imaginary solutions of U plotted in the plane (β, J), the value of m being fixed ($m = 3/2$).

Note that if $\beta \neq 0$, for fixed values of δ, β, there is a unique y satisfying this equation. This proves that all the multiple points are located on the axis $\beta = 0$. When $\beta = 0$, the equation becomes

$$(1 - \delta e^{-\omega})^2 = (1 - \delta)^2,$$

whose solutions are

$$\delta = 0 \quad \text{and} \quad \delta = \frac{2}{1 + e^{-\omega}}.$$

Those are indeed multiple points. For $(0, 0)$ for instance, it suffices to consider $y = \frac{2\pi k}{\omega}, k \in \mathbb{N}^*$. This ends the proof. $\square$

4.4 Construction of the bifurcation point satisfying all the spectral assumptions.

Let $\omega_0 > 0$ being fixed, chosen arbitrarily. Let $y_0 := \frac{2\pi}{\omega_0}(1 - \frac{\epsilon_0}{\omega_0})$ with $\epsilon_0 > 0$ (small) to be chosen later. Let $\beta_0 := \beta^0_{\omega_0}(y_0)$ and $d_0 := \delta^0_{\omega_0}(y_0)$. We have

$$\beta_0 = \epsilon_0 + \mathcal{O}(\epsilon_0^2) \quad \text{as } \epsilon_0 \to 0.$$

and

$$d_0 = \frac{e^{\omega_0}}{2(e^{\omega_0} - 1)} \left(1 + \frac{(2\pi)^2}{\omega_0^2}\right) \epsilon_0^2 + \mathcal{O}(\epsilon_0^3) \quad \text{as } \epsilon_0 \to 0.$$

We then have from (4.7)

$$\frac{\partial U}{\partial z}(\beta_0, d_0, \omega_0, iy_0) = -\omega_0 - (1 + 2i\pi)\epsilon_0 + \mathcal{O}(\epsilon_0^2) \quad \text{as } \epsilon_0 \to 0.$$

This quantity is non-null provided that ϵ_0 is sufficiently small. The implicit function theorem applies and gives the existence of a $\mathcal{C}^1$ function $(\beta, \delta, \omega) \mapsto z_0(\beta, \delta, \omega)$ defined in the neighborhood of (β_0, d_0, ω_0) such that

$$U(\beta, \delta, \omega, z_0(\beta, \delta, \omega)) = 0, \quad \text{with} \quad z_0(\beta_0, d_0, \omega_0) = iy_0.$$

Furthermore, one has

$$\frac{\partial}{\partial \delta} z_0(\beta_0, d_0, \omega_0) = -\frac{\frac{\partial U}{\partial \delta}(\beta_0, d_0, \omega_0, iy_0)}{\frac{\partial U}{\partial z}(\beta_0, d_0, \omega_0, iy_0)} \overset{(4.7)}{=} 2\pi \frac{1 - e^{-\omega_0}}{\omega_0} \frac{2\pi + i\omega_0}{(2\pi)^2 + \omega_0^2} + \mathcal{O}(\epsilon_0) \quad \text{as } \epsilon_0 \to 0$$

and

$$\frac{\partial}{\partial \omega} z_0(\beta_0, d_0, \omega_0) = -\frac{\frac{\partial U}{\partial \omega}(\beta_0, d_0, \omega_0, iy_0)}{\frac{\partial U}{\partial z}(\beta_0, d_0, \omega_0, iy_0)} \overset{(4.7)}{=} -\frac{2i\pi}{\omega_0} + \mathcal{O}(\epsilon_0) \quad \text{as } \epsilon_0 \to 0.$$

We finally set

$$\alpha_0 := \frac{d_0}{e^{\omega_0} - 1}, \quad m_0 := 1 + \frac{1 - d_0}{e^{\omega_0} - 1},$$

and

$$\mathfrak{z}_0(\alpha) := z_0(\beta_0, \frac{\alpha}{m_0 + \alpha - 1}, \log\left(\frac{m_0 + \alpha}{m_0 + \alpha - 1}\right)),$$

such that

$$\begin{aligned} \frac{d}{d\alpha} \mathfrak{z}_0(\alpha_0) =& 2\pi \frac{1 - e^{-\omega_0}}{\omega_0} \frac{2\pi + i\omega_0}{(2\pi)^2 + \omega_0^2} \frac{m_0 - 1}{(m_0 - 1 - \alpha_0)^2} \\ &+ \frac{2i\pi}{\omega_0} \frac{1}{(m_0 - 1 + \alpha_0)(m_0 + \alpha_0)} + \mathcal{O}(\epsilon_0) \quad \text{as } \epsilon_0 \to 0. \end{aligned}$$

The second term on the right hand side is purely imaginary. So

$$\Re \frac{d}{d\alpha} \mathfrak{z}_0(\alpha_0) = \frac{1 - e^{-\omega_0}}{\omega_0} \frac{(2\pi)^2}{(2\pi)^2 + \omega_0^2} \frac{m_0 - 1}{(m_0 - 1 - \alpha_0)^2} + \mathcal{O}(\epsilon_0) \quad \text{as } \epsilon_0 \to 0.$$

This quantity is strictly positive provided that ϵ_0 is small enough. By choosing the parameters of the model to be $\beta = \beta_0$ and $m = m_0$, the Assumptions 2.2, 2.3 and 2.6 are satisfied at the point $\alpha = \alpha_0$. In particular, Assumption 2.3 follows from Proposition 4.2.

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