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The boundary of the range of a random walk and the Følner property*

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Abstract

The range process R_n of a random walk is the collection of sites visited by the random walk up to time n. In this paper we deal with the question of whether the range process of a random walk or the range process of a cocycle over an ergodic transformation is almost surely a Følner sequence and show the following results: (a) The size of the inner boundary $|\partial R_n|$ of the range of recurrent aperiodic random walks on $\mathbb{Z}^2$ with finite variance and aperiodic random walks in $\mathbb{Z}$ in the standard domain of attraction of the Cauchy distribution, divided by $n/\log^2(n)$, converges to a constant almost surely. (b) We establish a formula for the Følner asymptotic of transient cocycles over an ergodic probability preserving transformation and use it to show that for admissible transient random walks on finitely generated groups, the range is never a Følner sequence unless the walk is a skip-free random walk on $\mathbb{Z}$. (c) For strongly aperiodic random walks in the domain of attraction of symmetric α-stable distributions with $1 < \alpha \leq 2$, we prove a sharp polynomial upper bound for the decay at infinity of $|\partial R_n|/|R_n|$. This last result shows that the range process of these random walks is almost surely a Følner sequence.

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1 Introduction and main results

Let G be a countable group with identity element id_{G}, $\xi_1, \xi_2, \ldots,$ be i.i.d. G-valued random variables and define the random walk $(S_n)_n$, where $S_0 := \mathrm{id}_{\mathsf{G}}$ and $S_n = \xi_1 \cdot \xi_2 \cdot \ldots \cdot \xi_n$ for $n \geq 1$. The *range* of the random walk, denoted

$$R_n := \{S_1, \ldots, S_n\},$$

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is the random subset of G which consists of the sites visited by the random walk up to time n. The case where $\mathsf{G} = \mathbb{Z}^d$ will serve as a motivating and recurring example in this paper; as this group is Abelian we will denote the random walk in this case by $S_n = \sum_{i=1}^n \xi_i$.

The range is a natural object to study and understanding its size and shape is of great interest for a variety of models in probability theory such as *random walk in random scenery*; see for example [1, 16] where the size of the range is shown to determine the leading term of the asymptotic growth rate of the information arising from the scenery. The size of the range and its fluctuations have been extensively treated in the literature starting with the seminal paper [17] where the authors obtained strong laws in the case of the simple random walk on $\mathbb{Z}^2$, see also [18] and [19]. A central limit theorem for the range was obtained in [24], whereas the case of random walks with stable increments was treated in [28].

More recently, the focus has shifted towards more involved objects, still related to the range. For example [7] studies the entropy of the range, [5, 4] its *capacity*, [37] the largest gap problem in the range [37]; finally [3] and [30] study the boundary of the range, henceforth denoted by ∂R_n, that is the sites in the range with at least one neighbour not visited by the random walk. Apart from its intrinsic interest, the motivation for studying the range and its relatives often stems from their relevance in more intricate models; the capacity of the range is of interest in *random interlacements* (see [35]), whereas the range itself is relevant in the study of random walks in random scenery.

The main focus of this paper is on asymptotic size of the boundary of the range and its relation to the Følner property of the range. We first present our results in the case of recurrent walks.

1.1 Recurrent walks in $\mathbb{Z}$ and $\mathbb{Z}^2$

The first and last authors first studied the Følner property of the range in [16], where in the case of the simple random walk on $\mathbb{Z}^2$, it was shown that the range is almost surely a Følner sequence, that is almost surely

$$\lim_{n\to\infty} \frac{|R_n \triangle (R_n + x)|}{|R_n|} = 0, \quad \text{for all } x \in \mathbb{Z}^d, \tag{1.1}$$

where for two subsets $A, B \subset \mathbb{Z}^d$, we write $A \triangle B$ for the symmetric difference $A \triangle B := (A \cup B) \setminus (A \cap B)$.

One can easily see that the Følner property captures the asymptotic relation between the size of the range $|R_n|$ and that of its boundary $|\partial R_n|$ defined by

$$\partial R_n := \left\{ x \in R_n : \exists y \in \mathbb{Z}^d \setminus R_n,\ |x - y| = 1 \right\}. \tag{1.2}$$

Indeed let $\mathbb{S}_d$ denote the usual generators of $\mathbb{Z}^d$, that is $\mathbb{S}_d := \{\pm e_i : 1 \le i \le d\}$, where e_i are the standard unit basis vectors. For $v \in \mathbb{Z}^d$, we also define the *v-boundary* of the range as

$$\partial_v R_n := R_n \setminus \big(R_n + v\big).$$

Using the above notation, the boundary ∂R_n can also be written in the form

$$\partial R_n = \bigcup_{v \in \mathbb{S}_d} R_n \setminus (R_n + v) = \bigcup_{v \in \mathbb{S}_d} \partial_v R_n.$$

From the above it is clear that for any $v \in \mathbb{S}_d$

$$|\partial_v R_n| \le |\partial R_n| \le \sum_{v \in \mathbb{S}_d} |\partial_v R_n|. \tag{1.3}$$

Notice that a similar relation also holds for any countable, finitely generated group G, with $\mathbb{S}_d$ replaced by the generators of G.

In addition, for any $a, b \in \mathbb{Z}^d$ it can be easily shown that

$$|\partial_a R_n \backslash \partial_b R_n| \leq |\partial_{a-b} R_n|, \qquad |\partial_{a+b} R_n| \leq |\partial_b R_n| + |\partial_a R_n|.$$

In particular (for countable finitely generated groups) the Følner property is equivalent to $|\partial R_n|/|R_n| \to 0$ as $n \to \infty$. The Følner property is important in ergodic theory and information theory as it implies for example that the partial sums of any G-indexed random field along (F_n) satisfies the mean ergodic theorem [13] and the Shannon-McMillan-Breiman Theorem [27, 29].

A rate, at least in expectation, is given by [7], who proved for the simple random walk on $\mathbb{Z}^2$, that $\mathbb{E}\,|\partial R_n| \sim Cn/\log^2(n)$, whereas [30] proved that

$$\frac{\pi^2}{2} \leq \lim_{n\to\infty} \frac{(\log n)^2}{n}\,\mathbb{E}\,|\partial R_n| \leq 2\pi^2.$$

We strengthen the results mentioned above by obtaining a strong law of large numbers for the boundary $|\partial R_n|$ of a recurrent random walk in $\mathbb{Z}^2$, with finite second moments, or of a random walk in $\mathbb{Z}$ in the domain of attraction of the symmetric Cauchy law. Before introducing our first main result we give a precise statement of our assumptions.

Let $\xi, \xi_1, \xi_2, \cdots$ be i.i.d. $\mathbb{Z}^d$-valued random variables satisfying one of the following:

(A1) $d = 2$ and there exists a nonsingular covariance matrix $\Sigma \in M_{2\times 2}(\mathbb{R})$ such that for all $t \in [-\pi, \pi]^2$,

$$\phi(t) = \mathbb{E}\left(\exp(i\,\langle t, \xi\rangle)\right) = 1 - \langle \Sigma t, t\rangle + o\left(|t|^2\right);$$

(A2) $d = 1$ and there exists $\gamma > 0$ such that for $t \in [-\pi, \pi]$,

$$\phi(t) = \mathbb{E}\left(\exp(it\xi)\right) = 1 - \gamma|t| + o(|t|).$$

It is worth mentioning under (A1) the random variables $\xi_i \in \mathbb{Z}^2$ have zero mean and finite second moments and therefore $S_n/\sqrt{n}$ converges to a Gaussian random vector in $\mathbb{R}^2$. For example Simple Random Walk in $\mathbb{Z}^2$ satisfies (A1). On the other hand, random walk satisfying assumption (A2) are in the domain of attraction of the symmetric Cauchy distribution, that is if $S_n := X_1 + \cdots + X_n$, then $S_n/n \to Z$, where Z is a real random variable with characteristic function $\mathbb{E}\exp(\mathrm{i}tZ) = \exp(-|t|)$

We will always assume that the random walk is *aperiodic*, namely that there is no proper subgroup of $\mathbb{Z}^d$ containing the support of ξ_1. By [34, Theorem 7.1] this is equivalent to requiring that $\phi(t) = 1$ for $t \in [-\pi, \pi]^d$ if and only if $t = 0$. Notice that when $\xi_i \in \mathbb{Z}^2$ has finite second moments, aperiodicity implies that the covariance matrix Σ in (A2) is non-singular.

Our first main result is the following.

Theorem 1.1. *Let $\xi_1, \xi_2, ..., \xi_n, ..$ be i.i.d. $\mathbb{Z}^d$-valued aperiodic random variables satisfying Assumption (A1) or Assumption (A2). Then for every nonzero $v \in \mathbb{Z}^d$, there exist constants $C_v, C > 0$ such that almost surely,*

$$\lim_{n\to\infty} \frac{\log^2(n)}{n}\,|\partial_v R_n| = C_v, \qquad \lim_{n\to\infty} \frac{\log^2(n)}{n}\,|\partial R_n| = C.$$

In addition $\mathbb{E}(|\partial_v R_n|) \sim \frac{C_v n}{\log^2(n)}$ and $\mathbb{E}(|\partial R_n|) \sim \frac{Cn}{\log^2(n)}$ as $n \to \infty$.

In the cases covered by Theorem 1.1 the random walks are recurrent, since $\mathrm{Re}\int[1-\phi(t)]^{-1}\mathrm{d}t=+\infty$, see [34, II.8.P1]. By [17, 24] and [28] we almost surely have $|R_n|\sim dn/\log(n)$ for some $d>0$; thus for all nonzero $v\in\mathbb{Z}^2$ (or $\mathbb{Z}$ in the Cauchy case), there exists a $C>0$ such that almost surely for n large enough,

$$C^{-1}\left(\frac{1}{\log(n)}\right)\leq\frac{|R_n\triangle(R_n+v)|}{|R_n|}\leq C\left(\frac{1}{\log(n)}\right),$$

providing a sharp quantitative version of the Følner property for these random walks.

We complete our study of recurrent random walks, by considering $\mathbb{Z}$-valued random walks in the domain of attraction of the symmetric, α-stable distribution with $1<\alpha\leq 2$. In these cases Le Gall and Rosen showed the range when scaled appropriately converges in distribution to a nontrivial random variable, namely the Lebesgue measure of the set $W_\alpha([0,1]):=\{W_\alpha(t):\ 0\leq t\leq 1\}$, where W_α is the symmetric, α-stable Lévy process [28].

Theorem 1.2. *Let S_n be a recurrent, aperiodic random walk on $\mathbb{Z}$ in the domain of attraction of a nondegenerate, symmetric, α-stable distribution with $1<\alpha\leq 2$. Then for all $\epsilon>0$, almost surely*

$$\frac{|\partial R_n|}{|R_n|}=o\left(n^{\frac{1}{\alpha}-1+\epsilon}\right).$$

For $\alpha=2$ the quantitative upper bound of Theorem 1.2 is optimal in the polynomial term as can be deduced from the case of the simple random walk on $\mathbb{Z}$, see Remark 4.3. Theorem 1.2 shows that for this class of random walks, the range process is almost surely a Følner sequence. This in turn can be used, for example, to greatly simplify the calculation of the upper bound of the relative complexity of the scenery in [1], which is quite technical and is based on dyadic partitions, as Kieffer's Shannon-McMillan-Breiman formula directly applies to the sequence of sets $(R_n)_n$.

1.2 Transient walks

Kaimanovich in a private communication asked which groups admit random walks whose range satisfies the almost sure Følner property. Most groups only carry transient random walks. Indeed, the only finitely generated groups on which there are recurrent random walks are the groups which are virtually cyclic or virtually $\mathbb{Z}^2$, see [38, Theorem 3.24] meaning that there exists H a finite index subgroup of G which is isomorphic to $\{0\}$, $\mathbb{Z}$ or $\mathbb{Z}^2$. As for transient random walks the range grows linearly, Kaimanovich's question is naturally linked to the asymptotic size of the boundary of the range of transient random walks; see the discussion in Section 1.1.

Kesten, Spitzer and Whitman's showed that for all random walks,

$$\frac{|R_n|}{n}\to\mathbb{P}\left(S_n\neq 0,\text{ for all }n\geq 1\right),$$

almost surely, where clearly the limit is positive for transient random walks, see [34, p. 38]. In the case of transient random walks on $\mathbb{Z}^d$ with $d\geq 1$, Okada showed in [30] that if $X_1,X_2,\cdots$, is a sequence of i.i.d. random variables in $\mathbb{Z}^d$, then almost surely

$$\lim_{n\to\infty}\frac{|\partial R_n|}{n}=\mathbb{P}\Big(\Big\{\forall k\in\mathbb{N},S_k\neq 0\Big\}\cap\Big\{\exists y\in\{\pm e_i;\ 1\leq i\leq d\},\forall k\in\mathbb{Z},\ S_k\neq y\Big\}\Big),$$

where $\{S_{-n};n\geq 0\}$ is an independent copy of $\{-S_n;n\geq 0\}$.

We take a more general viewpoint and consider random walks on groups. Let G be a discrete group, $\{S_n:n\geq 0\}$ be a G-valued random walk and $R_n:=\{S_1,\dots,S_n\}$. We first show that

$$\lim_{n\to\infty}\frac{|R_n\triangle(R_ng)|}{|R_n|}=c(g),\quad\text{a.s. for all }g\in\mathsf{G},\tag{1.4}$$

where $c(g)$ is given explicitly in terms of return probabilities and is thus closely related with the Green function $G(g) = \sum_{n=0}^{\infty} \mathbb{P}(S_n = g)$. In particular, Proposition 5.4 shows that if the Green function of a random walk vanishes at infinity then its range is almost surely not a Følner sequence.

Let us say that a random walk is *admissible* if the walk starting from the origin can reach any point with positive probability. A function $f : \mathsf{G} \to \mathbb{R}$ satisfies $\lim_{g\to\infty} f(g) = 0$ if for all $\epsilon > 0$, we have $|f(g)| < \epsilon$ for all but finitely many elements of G. We then establish the following.

Theorem 1.3. *Let p be an admissible probability measure on a finitely generated group G. Assume that p defines a transient random walk, and that G is not virtually cyclic. Then the Green function $G(g)$ tends to 0 when g tends to infinity.*

The condition that G is not virtually cyclic is necessary for the theorem: in $\mathbb{Z}$, the Green function of a non-centered random walk with finite first moment does not tend to 0 at infinity, by the renewal theorem (and this statement can be extended to virtually cyclic groups).

It is worth noting that stronger, quantitative results, are available if one restricts the class of groups under consideration, see [38]. For instance, when G is non-amenable the probabilities $p_n(\mathrm{id}_{\mathsf{G}}, x)$ vanish exponentially fast, uniformly in x, from which the result follows readily. On amenable groups, for symmetric walks with finite support or more generally a second moment, one can sometimes use isoperimetric techniques to obtain much stronger results, specifying the rate of decay of $p_n(\mathrm{id}_{\mathsf{G}}, x)$ and of the Green function at infinity. However, in general, there is no hope to get a quantitative version of Theorem 1.3 as we have made no moment assumption; it suffices to consider p such that ξ_1 is at distance 2^{2^n} of id_{G} with probability $1/n^2$ – then the Green function decays at most like $1/\log\log d(\mathrm{id}_{\mathsf{G}}, x)$. Moreover, surprisingly few tools apply in all classes of groups, regardless of their geometry.

After completing our proof, we learned that part of the statement of Theorem 1.3, namely if G is not virtually $\mathbb{Z}^2$, can also be deduced from results of Coulhon on isoperimetric inequalities (see [9, Proposition 3.1]). We nevertheless decided to keep our argument for this case also since it is shorter and more elementary.

There is a simple class of transient random walks for which R_n is a Følner sequence. Let $X_1, X_2, \ldots$ be a $\mathbb{Z}$ valued i.i.d sequence with finite first moment, positive expectation, and

$$\mathbb{P}(X_1 > 1) = 0.$$

Let $S_n = \sum_{k=1}^{n} X_k$. Then for almost all $\omega \in \Omega$, $\{R_n\}_{n=1}^{\infty}$ is an eventually monotone sequence of growing intervals with $|R_n| \to \infty$ as $n \to \infty$. Consequently the range process is almost surely a Følner sequence, and $|\partial R_n|$ remains almost surely bounded. A similar statement holds for walks with $S_n \to -\infty$ as $n \to \infty$ and $\mathbb{P}(X_1 < -1) = 0$. We say that these walks are the *skip-free* walks on $\mathbb{Z}$.

With Theorem 1.3 at hand together with a specific study of virtually cyclic groups, we show the following.

Theorem 1.4. *Let S_n be a transient admissible random walk on a finitely generated group G. If it is not a skip-free random walk on $\mathbb{Z}$, then almost surely $\{R_n\}_{n=1}^{\infty}$ is not a Følner sequence.*

In particular, for all $d \geq 3$, the range of all random walks on $\mathbb{Z}^d$ is almost surely not a Følner sequence. This answers the question of Kaimanovich in full generality for all transient random walks on finitely generated groups. We obtain in particular that for transient random walks, either $|\partial R_n|$ is eventually bounded, or it grows linearly, with no intermediate behavior.

The remainder of the paper is structured as follows. In Section 2 we introduce the necessary notation, collect some preliminary results and set the setting for proving the main results. Also in Section 2 we extend the results of [30] to transient co-cycles in discrete groups. The added generality of considering co-cycles is not strictly necessary, but it comes at no additional technical complexity and we think is of independent interest. The proofs of the main results, along with the relevant definitions and more precise statements, are given in Sections 3, 4 and 5 respectively. The Appendix contains some auxiliary results.

1.2.1 Notation

Given two sequences $(a_n)_n$ and $(b_n)_n$ of positive numbers we will write $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ and $a_n \lesssim b_n$ if there exists $C > 0$ such that for all $n \in \mathbb{N}$, $a_n \le C b_n$. For $x \in \mathbb{R}$, $\lfloor x \rfloor \in \mathbb{Z}$ denotes the lower floor function of x.

2 Preliminary results

As mentioned in the introduction we will present the results of this Section in the context of group-valued co-cycles. The abstraction to co-cycles is not strictly necessary for the rest of the paper but it does not add to the complexity of the paper and we think is of independent interest.

Let T be an invertible bi-measurable, ergodic, measure preserving transformation of a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, meaning that $\mathbb{P} \circ T^{-1} = \mathbb{P}$. In this case we will refer to $(\Omega, \mathcal{B}, \mathbb{P}, T)$ as a probability preserving transformation. The transformation is ergodic if for all $A \in \mathcal{B}$, $T^{-1}A = A$ implies that $\mathbb{P}(A) = 0$ or $\mathbb{P}(\Omega \setminus A) = 0$. In this section we will make use of the pointwise ergodic theorem: if a measurable $f : \Omega \to \mathbb{R}$ is $\mathbb{P}$ integrable and T is ergodic, then for $\mathbb{P}$ almost every $\omega \in \Omega$,

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} f \circ T^k(\omega)}{n} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} f \circ T^{-k}(\omega)}{n} = \mathbb{E}_{\mathbb{P}}(f).$$

Let $(\mathsf{G}, \times)$ be a countable discrete group and denote by id_{G} and m_{G} the identity element and the Haar measure of G respectively. A G-valued cocycle is a function $F : \mathbb{Z} \times \Omega \to \mathsf{G}$ which satisfies the cocycle identity: for all $m, n \in \mathbb{Z}$ and $\omega \in \Omega$,

$$F(n + m, \omega) = F(m, \omega) \times F(n, T^m \omega).$$

Any measurable function $f : \Omega \to \mathsf{G}$ determines a G-valued cocycle F via

$$F_f(n, \omega) = \begin{cases} f(\omega) \times f \circ T(\omega) \times \cdots \times f \circ T^{n-1}(\omega) & n \in \mathbb{N} \\ \mathrm{id}_{\mathsf{G}} & n = 0 \\ \left(f \circ T^{-1}(\omega)\right)^{-1} \times \cdots \times \left(f \circ T^{-n}(\omega)\right)^{-1} & n \in \mathbb{Z}_-. \end{cases}$$

In fact the above representation is general; any G-valued cocycle F is of the form F_f for f defined by $f(\omega) = F(1, \omega)$. These cocycles appear in the projection to the second coordinate of the skew product map $T_f : \Omega \times \mathsf{G} \to \Omega \times \mathsf{G}$ defined by $T_f(\omega, g) = (T\omega, g f(\omega))$. Note that T_f preserves the σ-finite measure $\mathbb{P} \times m_{\mathsf{G}}$. A cocycle F is recurrent if almost surely $N(\omega) := \#\{n \in \mathbb{N} : F(n, \omega) = \mathrm{id}_{\mathsf{G}}\} = \infty$ and transient if $N(\omega) < \infty$ almost surely. We would like to point out that there is no general necessary and sufficient criteria for recurrence of cocycles and that it follows from Herman's (unpublished notes) and [20, 2] that all amenable groups admit a recurrent cocycle. As the class of amenable groups includes all Abelian groups as well as some groups of exponential growth the latter class is much more diverse than the class of groups of polynomial growth at most

2. We conclude that there are many groups which admit a recurrent cocycle and all their random walks are transient. The papers [6, 31, 32, 14] provide some sufficient or necessary criteria for recurrence of cocycles.

Let
$$R_n(\omega) := \{F(k,\omega);\ 1 \le k \le n\}$$
be the range, or trace, of the co-cycle up to time n. When no confusion is possible we will write R_n for $R_n(\omega)$. For a finite subset $A \subset \mathsf{G}$, we write $|A|$ for its cardinality.

The next proposition was proved in Spitzer [34, pp. 38-40] [1] [2] for random walks in $\mathbb{Z}^d$ and the proof of Proposition 2.1 below follows its lines. The case of Random walks on $\mathbb{Z}^d$ can be recovered by taking $\Omega = (\mathbb{Z}^d)^{\mathbb{Z}}$, $\mathbb{P}$ is a product measure (distribution of an i.i.d. sequence), T is the Bernoulli shift (ergodic) and $f(\omega) = \omega_0$, where $\omega = (\cdots, \omega_{-1}, \omega_0, \omega_1, \cdots)$.

Proposition 2.1 (pp. 38-40 [34]). *Let $(\Omega, \mathcal{B}, \mathbb{P}, T)$ be an ergodic, probability-preserving transformation, G a countable group and $F : \mathbb{Z} \times \Omega \to \mathsf{G}$ a cocycle. Then for $\mathbb{P}$-almost every ω,*
$$\lim_{n\to\infty} \frac{|R_n(\omega)|}{n} = \mathbb{P}\left(\omega \in \Omega;\ \forall n \in \mathbb{N},\ F(n,\omega) \ne \mathrm{id}_{\mathsf{G}}\right).$$

Proof. Define
$$A_n = \{\omega' \in \Omega : \forall k \in [1,n] \cap \mathbb{N},\ F(k,\omega') \ne \mathrm{id}_{\mathsf{G}}\}$$
and $A = A_\infty$. Counting each $z \in R_n$ according to the last time it has been visited in $[1,n] \cap \mathbb{N}$, it follows that
$$|R_n(\omega)| = \sum_{k=1}^{n} \mathbb{1}_{[\forall k<j\le n, F(j,\omega)\ne F(k,\omega)]} = \sum_{k=1}^{n} \mathbb{1}_{[\forall 0<j\le n-k, F(j,T^k\omega)\ne \mathrm{id}_{\mathsf{G}}]}$$
and for all $n \in \mathbb{N}$ and $N < n$
$$\sum_{k=1}^{n} \mathbb{1}_A \circ T^k \le |R_n| = \sum_{k=1}^{n} \mathbb{1}_{A_{n-k}} \circ T^k \le N + \sum_{k=1}^{n-N} \mathbb{1}_{A_N} \circ T^k.$$
By the pointwise ergodic theorem, dividing all sides of the inequality by n, one has that for all $N \in \mathbb{N}$, for almost every $\omega \in \Omega$,
$$\mathbb{P}(A) \le \underline{\lim}_{n\to\infty} \frac{|R_n(\omega)|}{n} \le \overline{\lim}_{n\to\infty} \frac{|R_n(\omega)|}{n} \le \mathbb{P}\left(A_N\right).$$
Noting that $A_N \downarrow A$ as $N \to \infty$ and thus $\mathbb{P}(A_N) \xrightarrow[N\to\infty]{} \mathbb{P}(A)$ the conclusion follows. □

The following corollary is almost immediate.

Corollary 2.2. *Let $(\Omega, \mathcal{B}, \mathbb{P}, T)$ be an ergodic probability preserving transformation, G a countable group and $F : \mathbb{Z} \times \Omega \to \mathsf{G}$ a cocycle. If F is recurrent then $|R_n| = o(n)$ almost surely. In the transient case there exists $c > 0$ such that $|R_n| \sim cn$ almost surely.*

Proof. It remains to show that if F is transient then
$$c := \mathbb{P}\left(\omega \in \Omega :\ \forall n \in \mathbb{N},\ F(n,\omega) \ne \mathrm{id}_{\mathsf{G}}\right) > 0.$$
To see this let $l : \Omega \mapsto \mathbb{N} \cup 0$ be defined as
$$l(\omega) := \sup\left\{n \in \mathbb{N} \cup \{0\} :\ F(n,\omega) = \mathrm{id}_{\mathsf{G}}\right\}.$$

[1] Although it is stated for $\mathbb{Z}^d$, it holds for general countable groups as pointed out by Flatto [19].
[2] Spitzer attributes the proof to an unpublished manuscript with Kesten and Whitman.

We need to show that $\mathbb{P}(l=0)>0$. Letting $B_N=\{\omega\in\Omega;\ l=N\}$, the transience of F implies that there exists $N\in\mathbb{N}\cup\{0\}$ such that $\mathbb{P}(B_N)>0$. For all $\omega\in B_N$ and $j\in\mathbb{N}$,

$$\mathrm{id}_{\mathsf{G}}\neq F(N+j,\omega)=F(N,\omega)\times F\left(j,T^N\omega\right)=F\left(j,T^N\omega\right).$$

The latter implies that $T^NB_N\subset\{\omega\in\Omega: l=0\}$ and since T preserves $\mathbb{P}$ we have $\mathbb{P}(l=0)>0$. □

2.1 The asymptotic size of the boundary of transient cocycles

[7] and [30] studied the asymptotic size of the boundary of a random walk in $\mathbb{Z}^d$. In the case of transient random walks on $\mathbb{Z}^d$, Okada obtained a strong law of the boundary of the range [30]. The following is a similar result for transient cocycles on arbitrary countable groups. The proof follows closely that of [30, Theorem 2.1]. Note that the proof [30, Theorem 2.1] contains a small issue since the shift operator as defined in the beginning of [30, Section 3.1] is measure-preserving with respect to an infinite measure. This however can be easily fixed by using instead the setup mentioned just before Proposition 2.1.

Proposition 2.3. *Let $(\Omega,\mathcal{B},\mathbb{P},T)$ be an ergodic probability preserving transformation, G a countable group and $F:\mathbb{Z}\times\Omega\to\mathsf{G}$ a cocyle. Then for all $g\in\mathsf{G}$, for $\mathbb{P}$-almost every $\omega\in\Omega$,*

$$\lim_{n\to\infty}\frac{|R_n(\omega)\setminus(R_n(\omega)g)|}{n}=\mathbb{P}\left(\omega\in\Omega;\ \forall n\in\mathbb{N},\ F(n,\omega)\neq\mathrm{id}_{\mathsf{G}},\forall n\in\mathbb{Z},\ F(n,\omega)\neq g^{-1}\right).$$

Proof. Let $g\in\mathsf{G}$ and $\omega\in\Omega$. By definition, $z\in R_n(\omega)\setminus(R_n(\omega)g)$ if and only if there exists $1\le k\le n$ such that for all $1\le j\le n$,

$$F(j,\omega)g\neq z=F(k,\omega).$$

By considering the maximal $k\le n$ for which $F(k,\omega)=z$, we get $|R_n(\omega)\setminus(R_n(\omega)g)|=\sum_{k=1}^n\mathbb{1}_{B_n(k)}$ where

$$B_n(k)=\left\{\omega\in\Omega;\forall j\in[1,n]\cap\mathbb{N},F(j,\omega)\neq F(k,\omega)g^{-1},\ \forall k<j\le n,\ F(j,\omega)\neq F(k,\omega)\right\}.$$

As for all $k,j\in\mathbb{Z}$, $F(k,\omega)^{-1}F(j,\omega)=F(j-k,T^k\omega)$, for all $k\in\{1,\ldots,n\}$ the set $B_n(k)$ is equal to

$$T^{-k}\Big\{\omega\in\Omega:\forall j\in[-k,n-k]\cap\mathbb{Z},\ F(j,\omega)\neq g^{-1},\forall j\in[1,n-k]\cap\mathbb{N},\ F(j,\omega)\neq\mathrm{id}_{\mathsf{G}}\Big\}.$$

This implies that for all $N<n/2$,

$$\sum_{k=1}^n\mathbb{1}_C\circ T^k\le|R_n\setminus(R_ng)|\le 2N+\sum_{k=N+1}^{n-N}\mathbb{1}_{C_N}\circ T^k,$$

where

$$C_N:=\left\{\omega\in\Omega;\ \forall j\in[-N,N]\cap\mathbb{Z},\ F(j,\omega)\neq g^{-1},\ \forall j\in[1,N]\cap\mathbb{N},\ F(j,\omega)\neq\mathrm{id}_{\mathsf{G}}\right\}$$

and $C=C_\infty=\cap_NC_N$. The conclusion follows from the ergodic theorem as in the proof of Proposition 2.1. □

The next Corollary follows easily from a combination of Propositions 2.1, 2.3 and Corollary 2.2.

Corollary 2.4. *Let $(\Omega,\mathcal{B},\mathbb{P},T)$ be an ergodic probability-preserving transformation, G a countable group and $F:\mathbb{Z}\times\Omega\to\mathsf{G}$ a cocycle. If F is transient then for all $g\in\mathsf{G}$,*

$$\frac{|R_n\setminus(R_ng)|}{|R_n|}\xrightarrow[n\to\infty]{a.s.}\mathbb{P}\left(\forall k\in\mathbb{Z},\ F(k,\cdot)\neq g^{-1}\middle|\forall k\in\mathbb{N},\ F(k,\cdot)\neq\mathrm{id}_{\mathsf{G}}\right).$$

2.2 The Følner property

A sequence $\{K_n\}_{n\in\mathbb{N}}$ of subsets of G is a *right Følner sequence* in G if for all $n \in \mathbb{N}$, K_n is a nonempty finite set and for all $g \in \mathsf{G}$,

$$\frac{|K_n \triangle (K_n g)|}{|K_n|} \xrightarrow[n\to\infty]{} 0.$$

The existence of Følner sequences is equivalent to amenability of the group (existence of a right invariant mean on G). In [16], it was shown that the range of the symmetric random walk in $\mathbb{Z}^2$ is almost surely a Følner sequence. Kaimanovich, in a private communication, asked which random walks on general countable groups have almost surely Følner ranges and Dolgopyat suggested to generalise this question to which cocycles have almost surely Følner ranges. The following is a partial advance on Dolgopyat's question for transient co-cycles as it establishes a relatively simple criterion for checking the Følner property. This criterion will be used afterwards to give a complete answer to Kaimanovich's question.

Proposition 2.5. *Let $(\Omega, \mathcal{B}, \mathbb{P}, T)$ be an ergodic probability preserving transformation, G a countable group and $F : \mathbb{Z} \times \Omega \to \mathsf{G}$ a cocycle. If F is transient then for all $g \in \mathsf{G}$, almost surely,*

$$\lim_{n\to\infty} \frac{|R_n \triangle R_n g|}{|R_n|} = \Phi(g) + \Phi\left(g^{-1}\right),$$

where $\Phi(g) := \mathbb{P}\left(\forall n \in \mathbb{Z},\ F(n,\cdot) \neq g | \forall k \in \mathbb{N},\ F(k,\cdot) \neq \mathrm{id}_{\mathsf{G}}\right)$.

Proof. It is easy to see, by multiplying by g^{-1}, that for all $n \in \mathbb{N}$,

$$|R_n g \setminus R_n| = \left|R_n \setminus \left(R_n g^{-1}\right)\right|.$$

Since $(R_n \triangle (R_n g)) = (R_n \setminus (R_n g)) \uplus ((R_n g) \setminus R_n)$, the result follows from Corollary 2.4. $\square$

3 Proof of Theorem 1.1

In [7] it was shown that in the case of the symmetric random walk on $\mathbb{Z}^2$, $\mathbb{E}|\partial R_n|$ is proportional to the entropy of the range (at time n) and it is of order constant times $n/\log^2(n)$. [30] has shown that $2^{-1}\pi^2 \le C \le 2\pi^2$. The previous Theorem is a law of large numbers type result for a more general class of random walks which includes random walks in the normal domain of attraction of a symmetric Cauchy distribution. It also gives a more precise estimate on the almost sure Følner property of the range of such random walks; see [16] for some consequences of the Følner property of the range in the model of random walks in random sceneries. The proof goes by first establishing an upper bound for the variance of $|\partial R_n|$ which gives convergence in probability. After that we use a method from [19] which improves the asymptotics of the rate of convergence in probability, thus enabling us to use the Borel-Cantelli lemma for showing convergence almost surely.

Recall that for $v \in \mathbb{Z}^d$, the v-*boundary* of the range is defined as $\partial_v R_n = R_n \setminus \left(R_n + v\right)$, whereas for $V \subset \mathbb{Z}^d$, we write $\partial_V R_n := \cap_{v\in V} \partial_v R_n$. For $d = 1, 2$, write $\mathbb{S}_d := \{\pm e_i : 1 \le i \le d\}$, where e_i are the usual generators of $\mathbb{Z}^d$. The boundary ∂R_n can also be written in the form

$$\partial R_n = \bigcup_{v\in\mathbb{S}_d} R_n \setminus (R_n + v) = \bigcup_{v\in\mathbb{S}_d} \partial_v R_n. \tag{3.1}$$

To prove the statement in Theorem 1.1 for ∂R_n, we need first the statement for $\partial_v R_n$ and in fact a more general statement for $\partial_V R_n$, where $V \subset \mathbb{Z}^d$. The proof of the statement for ∂R_n will then follow from the inclusion-exclusion principle.

3.1 Auxiliary results

We now state and prove a number of auxiliary results that will be used in the proof of Theorem 1.1, that we could not find in the literature.

Proposition 3.1. *Under the assumptions of Theorem 1.1, for all $j \in \mathbb{Z}^d \setminus \{0\}$ there exist $c_j, d_j > 0$ such that*

$$\mathbb{P}\left(\forall 1 \le k \le n,\ S_k \notin \{0, j\}\right) \sim \frac{c_j}{\log(n)} \text{ as } n \to \infty.$$

and

$$\mathbb{P}\left(\forall 1 \le k \le n,\ S_k \neq j\right) \sim \frac{d_j}{\log(n)} \text{ as } n \to \infty.$$

Remark 3.2. Proposition 3.1 in the case of symmetric random walks on $\mathbb{Z}^2$ with finite variance was (essentially) treated in [30].

Proof. The proof is a simple application of [26, Theorem 4a]. This theorem states that for any aperiodic random walk in $\mathbb{Z}^d$, $W \subset \mathbb{Z}^d$ a finite subset, and any $x \in \mathbb{Z}^d$ we have

$$\lim_{n\to\infty} \frac{\mathbb{P}^x\left(S_k \notin W, k = 1, \dots, n\right)}{\mathbb{P}^0\left(S_k \neq 0, k = 1, \dots, n\right)} = \tilde{g}_W(x), \tag{3.2}$$

where in the cases of interest to us, by [26, Eq.(1.16)], for $x, y \in \mathbb{Z}^d$

$$\begin{aligned} \tilde{g}_W(x,y) &:= \mathbb{1}_{[x=y]} + \sum_{n=1}^{\infty} \mathbb{P}^x\left(S_n = y, S_k \notin W, 1 \le k \le n-1\right), \\ \tilde{g}_W(x) &= \lim_{|y|\to\infty} \tilde{g}_W(x,y). \end{aligned} \tag{3.3}$$

Under assumptions (A1) or (A2), it follows from the local limit theorem that

$$\sum_{j=0}^{n} \mathbb{P}(S_j = 0) \sim c \log n. \tag{3.4}$$

Using this asymptotics, a standard argument, see for example [17] and [23, Lemma 2.3], shows that there exists a constant $\gamma_d > 0$, depending on the random walk, such that

$$\mathbb{P}^0\left\{S_k \neq 0,\ k = 1, \dots, n\right\} \sim \frac{\gamma_d}{\log n}. \tag{3.5}$$

When $W = \{0, j\}$, from [26, Eq.(5.17) and (5.18)] we have that

$$\tilde{g}_{\{0,j\}}(j) = \frac{a(j)}{a(j) + a(-j)}, \quad \tilde{g}_{\{0,j\}}(0) = \frac{a(-j)}{a(j) + a(-j)},$$

where

$$a(j) = \sum_{n=0}^{\infty} \left[\mathbb{P}^0(S_n = 0) - \mathbb{P}^j(S_n = 0)\right] = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{1 - \exp(\mathrm{i}jt)}{1 - \phi(t)} \mathrm{d}t.$$

Under our assumptions, we have $a(x) > 0$ for all $x \neq 0$, by [34, Proposition 11.7] for $d = 2$ and [34, Proposition 30.2] for $d = 1$, whence $c_j, d_j > 0$. The result follows from this and (3.2). ☐

Lemma 3.3. *Under the assumptions of Theorem 1.1, for any nonempty finite subset $O \subset \mathbb{Z}^d$ and any $x, y \in \mathbb{Z}^d$, there exists $C = C(O, x, y)$ such that for all $n \in \mathbb{N}$,*

$$\mathbb{P}^x\left[S_n = y, S_j \notin O,\ 1 \le j \le n-1\right] \le \frac{C}{n \log^2(n)}.$$

Proof. Let

$$Q^n_O(x,y) := \mathbb{P}^x\left[S_n = y, S_j \notin O,\, 1 \le j \le n-1\right].$$

Assume first that (A1) holds, i.e., $d=1$, and that the walk is *strongly aperiodic*, which is equivalent to $|\phi(t)| < 1$ for all $t \in (-\pi,\pi)^d \setminus \{0\}$. For aperiodic random walks, this amounts to the condition that the greatest common divisor of return times to the origin is 1. For instance, the simple random walk is aperiodic, but not strongly aperiodic.

By strong aperiodicity of the walk condition (6.6) in [25] is satisfied. By [25, Theorems 8 and 9] we have

$$\lim_{n\to\infty} \frac{Q^n_O(x,y)}{\pi\gamma} n\log(n)^2 = 1,$$

which implies the result.

Under (A2), still with strong aperiodicity, the estimate

$$\sum_{u,v\in O} Q^n_O(u,v) \sim \frac{2\pi \det(\Sigma)^{1/2}}{n\log(n)^2},$$

follows from [25, Theorem 9] and [24, Theorem 4.1]. Then

$$Q^n_O(x,y) \sim \frac{2\pi \det(\Sigma)^{1/2}}{n\log(n)^2}\tilde{g}_O(x)\tilde{g}_{-O}(-y),$$

follows from the above and [25, Theorem 6a]. Indeed, although [25, Theorem 9] is stated only for one-dimensional walks in the domain of attraction of a symmetric stable law, as explained in the proof the result remains true for any recurrent random walk satisfying [25, Equation (11.1)]. This has been established under (A2) for strongly aperiodic random walks in [24, Theorem 4.1].

Under (A1) or (A2), but with strong aperiodicity, we have proved the result of the lemma. We claim that it still holds if one weakens strong aperiodicity to aperiodicity, but a little extra work is needed. Let $\{\tilde{S}_j\}_{j\ge 0}$ be a *lazy* version $\{S_j\}_{j\ge 0}$. In particular, if $S_j = \xi_1 + \cdots + \xi_j$, we let $\tilde{S}_j = \tilde{\xi}_1 + \cdots + \tilde{\xi}_j$, where $\tilde{\xi}_j = B_j\xi_j$ where $(B_j)_{j=1}^\infty$ are i.i.d. random variables, independent of the random walk, with $\mathbb{P}\{B_j = 1\} = \rho \in (0,1)$ and $\mathbb{P}\{B_j = 0\} = 1-\rho$. It can be easily checked that $\{\tilde{S}_j\}_{j\ge 0}$ is then strongly aperiodic and therefore that there exists $C>0$ such that for all $n \in \mathbb{N}$,

$$\tilde{Q}^n_O(x,y) := \mathbb{P}^x\left[\tilde{S}_n = y, \tilde{S}_j \notin O,\, 1 \le j \le n-1\right] \le \frac{C}{n\log(n)^2}. \tag{3.6}$$

In addition let

$$T_0 := 0, \qquad T_n := \inf\{n > T_{n-1} : \tilde{\xi}_n \neq 0\},\, n \ge 1.$$

It is then clear that for all $j \ge 1$, $T_j - T_{j-1}$ are i.i.d. geometrically distributed on the positive integers. With this notation, we can embed a path of $\{S_j\}_{j\ge 0}$ into a path of $\{\tilde{S}_j\}_{j\ge 0}$ by letting $S_j = \tilde{S}_{T_j}$ for $j \ge 0$. Up to a time-change, the paths of $\{S_j\}$ and $\{\tilde{S}_j\}$ coincide and thus

$$\begin{aligned}
Q^n_O(x,y) &= \mathbb{P}^x\left[S_n = y, S_j \notin O,\, 1 \le j \le n-1\right]\\
&= \mathbb{P}^x\left[\tilde{S}_{T_n} = y, S_j \notin O,\, 1 \le j \le T_n - 1\right]\\
&= \sum_{k=n}^\infty \mathbb{P}^x\left[\tilde{S}_k = y, S_j \notin O,\, 1 \le j \le k-1\right]\mathbb{P}(T_n = k)\\
&\le_{(3.6)} C\sum_{k=n}^\infty \frac{\mathbb{P}(T_n = k)}{k\log(k)^2} \le \frac{C}{n\log(n)^2}\sum_{k=n}^\infty \mathbb{P}(T_n = k) \le \frac{C}{n\log(n)^2}.
\end{aligned}$$

□

As $\mathbb{Z}^d$ is an Abelian group, in this case we consider instead a bi-infinite i.i.d. sequence $\{X_i\}_{i\in\mathbb{Z}}$ and write for $n \in \mathbb{N}$,

$$S_n^{(-)} = -\sum_{k=-n}^{-1} X_k.$$

In what follows we will make use of the fact that for $k < n$ and $v \in \mathbb{Z}^d$, for all $\omega \in \Omega$,

$$S_k(\omega) = S_n(\omega) + v \quad \text{iff} \quad S_{n-k}^{(-)}(T^n(\omega)) = v \quad \text{iff} \quad S_{n-k}\left(T^k(\omega)\right) = -v.$$

Proposition 3.4. *Under the assumptions of Theorem 1.1, there exists a constant $C > 0$ such that*

$$\mathbb{E}\left(|\partial R_n|\right) \sim \frac{Cn}{\log^2(n)}, \text{ as } n \to \infty. \tag{3.7}$$

In addition, there exists $M > 0$ such that

$$\mathrm{Var}\left(|\partial R_n|\right) \lesssim \frac{Mn^2 \log\log(n)}{\log^5(n)} \quad \text{as } n \to \infty. \tag{3.8}$$

Proof. It follows easily from the equality $\partial R_n = \bigcup_{v\in\mathbb{S}_d} \partial_v R_n$ in (3.1) that for any $v \in \mathbb{S}_d$ we have

$$|\partial_v R_n| \leq |\partial R_n| \leq \sum_{v\in\mathbb{S}_d} |\partial_v R_n|. \tag{3.9}$$

Using the inclusion-exclusion principle and enumerating the elements of $\mathbb{S}_d$ as $v_1, \ldots, v_{2d}$ we have

$$|\partial R_n| = \sum_{V\subset\mathbb{S}_d} (-1)^{|V|+1} \left|\partial_V R_n\right|, \tag{3.10}$$

recalling that for a collection, $V := \{v_1, \ldots, v_l\}$ say, of distinct vectors in $\mathbb{Z}^d$, we have that

$$\partial_V R_n := \bigcap_{v\in V} \partial_v R_n = R_n \cap \bigcap_{v\in V} (R_n + v)^c$$

whence we can write as in the proof of Proposition 2.3

$$|\partial_V R_n| = \sum_{k=1}^{n} \mathbb{1}_{A_{k,V}(k)}(\omega)\mathbb{1}_{B_{k,V}(n-k)}(\omega) \tag{3.11}$$

where

$$A_{0,V}(L) = \left\{\omega \in \Omega : \ \forall 0 < l < L, S_l^{(-)} \notin V\right\}, \qquad A_{k,V}(L) = T^{-k}A_{0,V}(L),$$

and

$$B_{0,V}(L) = \left\{\omega \in \Omega : \ \forall 0 < l < L, \ S_l \notin V \cup \{0\}\right\}, \quad B_{k,V}(L) = T^{-k}B_{0,V}(L).$$

To see why notice that

$$A_{k,V}(k) \cap B_{k,V}(n-k) = \left\{\omega \in \Omega : \forall l \in [1,n], \ S_k \notin S_l + V, \ \forall m \in [k+1,n], \ S_m \neq S_k\right\}.$$

Let us first consider the case $V = \{v\}$. Then one has

$$\left|\partial_v R_n\right|(\omega) = \sum_{k=1}^{n} \mathbb{1}_{A_k(k)}(\omega)\mathbb{1}_{B_k(n-k)}(\omega),$$

where $A_k(k) := A_{k,\{v\}}(k)$, $B_k(n-k) := B_{k,\{v\}}(n-k)$. By the Markov property for the random walk, for all $1 \le k \le n$, $A_k(k)$ and $B_k(n-k)$ are independent, thus

$$\begin{aligned}\mathbb{E}\left(|\partial_v R_n|\right) &= \sum_{k=1}^{n} \mathbb{P}\left(A_k(k)\right)\mathbb{P}\left(B_k(n-k)\right)\\ &= \sum_{k=1}^{n} \mathbb{P}\Big(\forall 1 \le l \le k, S_l^{(-)} \neq v\Big)\mathbb{P}\Big(\forall 1 \le l \le n-k,\ S_l \notin \{0,v\}\Big)\\ &= \sum_{k=2}^{n-2} \frac{c_v d_v}{\log(k)\log(n-k)}(1+o(1)),\end{aligned}$$

as $n \to \infty$, where $c_v, d_v > 0$ by Proposition 3.1. An easy calculation, using the fact that $\log$ is slowly varying, shows that

$$\begin{aligned}\sum_{k=2}^{n-2} \frac{1}{\log(k)\log(n-k)} &= \sum_{k=\lfloor n/\log^2 n\rfloor}^{n-\lfloor n/\log^2 n\rfloor} \frac{1}{\log(k)\log(n-k)} + O\left(\frac{n}{(\log n)^3}\right)\\ &= \frac{n}{\log(n)^2}(1+o(1)) + O\left(\frac{n}{(\log n)^3}\right),\end{aligned}$$

where we write $\lfloor x \rfloor$ for the integer part of x.

Assume now $|V| > 1$. For any finite set W, the proof of Proposition 3.1 and in particular (3.2) imply that, as $n \to \infty$

$$\mathbb{P}\left(\forall 1 \le k \le n,\ S_k \notin W\right) = \frac{\tilde{g}_W(0)\gamma_d}{\log(n)} + o\left(\log(n)^{-1}\right) \text{ as } n \to \infty, \tag{3.12}$$

with γ_d and $\tilde{g}_W(0)$ as defined in (3.5) and (3.3) respectively. Therefore similar arguments show that for any $V \subset \mathbb{S}_d$ we have that

$$\mathbb{E}\,|\partial_V R_n| \sim \gamma_d^2\,\tilde{g}_V(0)\tilde{g}_{V\cup\{0\}}(0)\,\frac{n}{\log^2(n)}, \tag{3.13}$$

and when $\tilde{g}_V(0)\tilde{g}_{V\cup\{0\}}(0) = 0$,

$$\mathbb{E}\,|\partial_V R_n| = o\left(\frac{n}{\log^2(n)}\right).$$

Going back to ∂R_n, we thus have that

$$\lim_{n\to\infty} \frac{\log(n)^2}{n}\,\mathbb{E}\,|\partial R_n| = \sum_{V\subseteq \mathbb{S}_d} (-1)^{|V|+1}\gamma_d^2\,\tilde{g}_V(0)\tilde{g}_{V\cup\{0\}}(0) \ge \max_{v\in V} c_{\{v\}}d_{\{v\}} > 0 \tag{3.14}$$

where the last two inequalities follow from (3.9) and Proposition 3.1. This proves (3.7). Note that this equation gives a semi-explicit formula for the constant in the asymptotics of $\mathbb{E}\,|\partial R_n|$ in (3.7). We do not know any simpler formula for this constant ([7] only provides upper and lower bounds, but no exact formula).

Variance. For the second part, from (3.10) we have that

$$\mathrm{Var}\left(|\partial R_n|\right) = \mathrm{Var}\left[\sum_{V\subseteq \mathbb{S}_d} |\partial_V R_n|\right] \le C \sum_{V\subseteq\mathbb{S}_d} \mathrm{Var}\left[|\partial_V R_n|\right].$$

Similarly to the proof of (3.7), we first work out in detail the case $V = \{v\}$, for $v \in \mathbb{S}_d$ arbitrary. First notice that

$$\mathbb{E}\left(|\partial_v R_n|^2\right) = \mathbb{E}\sum_{k=1}^{n}\left(\mathbb{1}_{A_k(k)}\mathbb{1}_{B_k(n-k)}\right) + 2\sum_{1\le k<m\le n}\mathbb{E}\left(\mathbb{1}_{A_k(k)}\mathbb{1}_{B_k(n-k)}\mathbb{1}_{A_m(m)}\mathbb{1}_{B_m(n-m)}\right).$$

The first term is equal to $\mathbb{E}\left(|\partial_v R_n|\right)$. For the second term, notice that for $1 \le k < m \le n$,

$$\mathbb{1}_{B_k(n-k)}\mathbb{1}_{A_m(m)} \le \mathbb{1}_{B_k(\lfloor (m-k)/2\rfloor)}\mathbb{1}_{A_m(\lfloor (m-k)/2\rfloor)}, \tag{3.15}$$

since for any k, m, $B_k(\cdot)$, $A_m(\cdot)$ are decreasing sequences of sets. To keep notation concise, for any integers l, k we will denote $\mathbb{P}(A_k(l))$ by $\psi(l)$ and $\mathbb{P}(B_k(l))$ by $\theta(l)$, where we can drop the dependence on k since $\mathbb{P}\circ T^{-1} = \mathbb{P}$. Since for $1 \le k \le m \le n$ the events

$$A_k(k),\, B_k\left(\lfloor (m-k)/2\rfloor\right),\, A_m\left(\lfloor (m-k)/2\rfloor\right),\, B_m(n-m),$$

are independent, we have

$$\mathbb{E}\left(\mathbb{1}_{A_k(k)}\mathbb{1}_{B_k(n-k)}\mathbb{1}_{A_m(m)}\mathbb{1}_{B_m(n-m)}\right) \le \psi(k)\theta\left(\left\lfloor \frac{m-k}{2}\right\rfloor\right)\psi\left(\left\lfloor \frac{m-k}{2}\right\rfloor\right)\theta(n-m).$$

This shows that for $k < m$,

$$\iota(k,m) := \mathbb{E}\left(\mathbb{1}_{A_k(k)}\mathbb{1}_{B_k(n-k)}\mathbb{1}_{A_m(m)}\mathbb{1}_{B_m(n-m)}\right) - \mathbb{E}\left(\mathbb{1}_{A_k(k)}\mathbb{1}_{B_k(n-k)}\right)\mathbb{E}\left(\mathbb{1}_{A_m(m)}\mathbb{1}_{B_m(n-m)}\right)$$

is bounded from above by

$$\psi(k)\theta(n-m)\Bigg\{\psi\left(\left\lfloor \frac{m-k}{2}\right\rfloor\right)\left[\theta\left(\left\lfloor \frac{m-k}{2}\right\rfloor\right) - \theta(n-k)\right] + \theta(n-k)\left[\psi\left(\left\lfloor \frac{m-k}{2}\right\rfloor\right) - \psi(m)\right]\Bigg\}$$

Denote by

$$D(n) := \left\{(k,m) \in [1,n]^2 :\ k, n-m > \sqrt{n} \ \text{ and } \ m-k \ge \frac{n}{\log^5(n)}\right\}.$$

Since for n large enough

$$\log(n) - 5\log\log(n) - \log(2) \ge \frac{\log(n)}{2},$$

it follows that for all $(k,m) \in D(n)$,

$$\psi\left(\left\lfloor \frac{m-k}{2}\right\rfloor\right) \sim \frac{c_v}{\log((m-k)/2)} \le \frac{c_v}{\log(n) - 5\log\log(n) - \log(2)} \le \frac{2c_v}{\log(n)}, \tag{3.16}$$

and

$$\theta\left(\left\lfloor \frac{m-k}{2}\right\rfloor\right) \sim \frac{d_v}{\log((m-k)/2)} \le \frac{2d_v}{\log(n)}. \tag{3.17}$$

with c_v, d_v as in Proposition 3.1. Similarly, for all $(k,m) \in D(n)$ and n large enough we have that

$$\begin{aligned}\theta\left(\left\lfloor \frac{m-k}{2}\right\rfloor\right) - \theta(n-k) &= \sum_{j=\lfloor \frac{m-k}{2}\rfloor}^{n-k-1} \left(\theta(j) - \theta(j+1)\right)\\ &= \sum_{j=\lfloor \frac{m-k}{2}\rfloor}^{n-k-1} \mathbb{P}^0\left[S_{j+1} \in \{0,v\}, S_l \notin \{0,v\}, \forall 1 \le l \le j\right]\\ &= \sum_{j=\lfloor \frac{m-k}{2}\rfloor}^{n-k-1}\left(\sum_{w\in\{0,v\}} \mathbb{P}^0\left[S_{j+1} = w, S_l \notin \{0,v\}, 1 \le l \le j\right]\right).\end{aligned}$$

By Lemma 3.3, the term in the sum is bounded by $C/(j\log^2(j))$. Thus, since $(m-k)/2 \to \infty$ when $n\to\infty$,

$$
\begin{aligned}
\theta\left(\left\lfloor\frac{m-k}{2}\right\rfloor\right)-\theta(n-k) &\le C\sum_{j=\lfloor\frac{m-k}{2}\rfloor}^{n-k-1}\frac{1}{j\log(j)^2}\\
&\lesssim \int_{s=\lfloor\frac{m-k}{2}\rfloor}^{n-k}\frac{\mathrm{d}s}{s\log(s)^2}\\
&\lesssim \left[\frac{1}{\log((m-k)/2)}-\frac{1}{\log(n-k)}\right]\\
&\le \left[\frac{1}{\log n-5\log\log n-\log 2}-\frac{1}{\log n}\right]\\
&\le \frac{C\log\log(n)}{\log^2(n)}, \qquad (3.18)
\end{aligned}
$$

and similarly

$$
\psi\left(\left\lfloor\frac{m-k}{2}\right\rfloor\right)-\psi(m)\lesssim\frac{C\log\log(n)}{\log^2(n)}. \qquad (3.19)
$$

since for $(k,m)\in D(n)$, it holds that $n/\log^5(n)\le m\le n$.

Combining (3.16), (3.17), (3.18) and (3.19), we obtain a global constant $M>0$ such that for large n, for all $(k,m)\in D(n)$

$$
\iota(k,m)\lesssim\frac{M\log\log(n)}{\log^3(n)}\frac{1}{\log(n-m)}\frac{1}{\log(k)}\le\frac{4M\log\log(n)}{\log^5(n)}. \qquad (3.20)
$$

Since $\iota(k,m)\le 1$ for all k,m, we have

$$
\begin{aligned}
\sum_{1\le k<m\le n}\iota(k,m) &\le \#\left\{(k,l)\in[1,n]^2\setminus D(n):\ k<l\right\}+\sum_{(k,m)\in D(n)}\iota(k,m)\\
&\le\left(2n^{3/2}+\frac{n^2}{\log^5(n)}\right)+\frac{Mn^2\log\log(n)}{\log^5(n)}.
\end{aligned}
$$

This together with (3.7) implies for any $v\in\mathbb{S}^d$

$$
\operatorname{Var}\left(|\partial_v R_n|\right)\lesssim\frac{Mn^2\log\log(n)}{\log^5(n)}\ \text{ as } n\to\infty.
$$

Notice that for general V, we can essentially repeat the same proof using (3.11) and $\psi_V(l):=\mathbb{P}(A_{k,V}(l))$, $\theta_V(l):=\mathbb{P}(B_{k,V}(l))$ in place of $\psi(l)$, $\theta(l)$. In particular notice that $\psi_V(n),\theta_V(n)\le C/\log(n)$ follows from (3.16), (3.17) along with (3.12), whereas the bounds (3.18) and (3.19) also follow from Lemma 3.3 which holds for general V. This completes the proof. □

Corollary 3.5. *Under the assumptions of Theorem 1.1, there exists a constant $C>0$ such that*

$$
\lim_{n\to\infty}\frac{|\partial R_n|}{\left(n/\log^2(n)\right)}=C,\quad\textit{in probability.}
$$

Proof. It follows from Proposition 3.4 that there exists $M>0$ such that,

$$
\operatorname{Var}(|\partial R_n|)\le M\frac{\mathbb{E}\left(\partial R_n\right)^2\log\log(n)}{\log(n)}.
$$

By Markov's inequality,

$$\mathbb{P}\left(\Big||\partial R_n| - \mathbb{E}\left(|\partial R_n|\right)\Big| > \epsilon\mathbb{E}\left(|\partial R_n|\right)\right) \leq \frac{M\log\log(n)}{\epsilon^2\log(n)} \xrightarrow[n\to\infty]{} 0, \tag{3.21}$$

whence $|\partial R_n|/\,\mathbb{E}\,|\partial R_n| \to 1$ in probability. In particular, from (3.7), there exists $C > 0$ such that $\log^2(n)|\mathbb{E}(|\partial R_n|)/n \to C$, and thus by elementary arguments

$$\frac{\log^2(n)}{n}|\partial R_n| \xrightarrow[n\to\infty]{} C \text{ in probability.} \tag{3.22}$$

□

In order to prove Theorem 1.1 we would like to improve the convergence in probability result to almost sure convergence. The gap of order $\log\log(n)/\log(n)$ in the decay of these probabilities is enough to guarantee almost sure convergence of $|\partial R_{N_k}|$ for $N_k = \exp\left(n^a\right)$ with $a > 1$. By a different method looking at

$$Z_n = \frac{|\partial R_n|}{\mathbb{E}\left(|\partial R_n|\right)} - 1,$$

one can show almost sure convergence of Z_{N_k} to 0 where N_k is of the form $[\exp(n^a)]$ with $a > 1/2$. This is since for $a > 1/2$, $\sum_{k=1}^{\infty}\mathbb{E}\left(|Z_{N_k}|^2\right) < \infty$.

Unfortunately as $|\partial R_n|$ is not necessarily monotone, this subsequence is too thin in order to interpolate the almost sure convergence from the subsequence to almost sure convergence along the whole sequence. To that end we use a method from [19].

Definition 3.6. *(i) Given $\delta > 0$, the sequence of random variables $\{X_n; n \geq 0\}$ satisfies property $\mathbf{A}(\delta)$ if for all $\epsilon_0 > 0$ there exists $C = C(\epsilon_0, \delta) > 0$ such that for all $n \geq 2$ and $\epsilon \geq \epsilon_0$,*

$$\mathbb{P}\left(X_n > (1+\epsilon)\mathbb{E}\left(X_n\right)\right) \leq \frac{C}{\epsilon^2\log^{\delta}(n)}.$$

(ii) The sequence of random variables $\{X_n; n \geq 0\}$ satisfies property $\mathbf{D}(\delta)$ if for all $\epsilon > 0$,

$$\mathbb{P}\left(X_n < (1-\epsilon)\mathbb{E}\left(X_n\right)\right) = O\left(\frac{1}{\log^{\delta}(n)}\right)$$

Theorem A. [Theorem 4.2 in [19]] Let $V \subset \mathbb{S}_d$. For all $\delta > 0$, if $|\partial_V R_n|$ satisfies property $\mathbf{A}(\delta)$ then it satisfies property $\mathbf{A}(4\delta/3)$.

Theorem B. [Bound (1.9) in [19]] Under the assumptions of Theorem 1.1, for all $V \subset \mathbb{Z}^d$, $|\partial_V R_n|$ satisfies property $\mathbf{D}(\delta)$ for all $\delta > 0$.

Remark 3.7. It is worth saying a few words about the fascinating method by Flatto [19] that produces Theorems A, B. Our intuition of why it works is the following. First notice that if we decompose the random walk path into blocks, the range/boundary of the full path can then be compared with the union of the corresponding objects of the blocks after correcting for pair-wise interactions. The first key ingredient is the fact that these interactions are weak, therefore the enhanced bounds guaranteed by Flatto's method can be seen as a form of concentration. The second key ingredient is slow variation of the original bound which allows one to control the effect of the interactions between the blocks and makes this approach possible.

It follows from (3.21) that for all $\delta \in (0,1)$, the sequence $|\partial R_n|$ satisfies properties $\mathbf{A}(\delta)$ and $\mathbf{D}$.

Corollary 3.8. *Let $V \subset \mathbb{S}_d$. For all $\delta > 0$, the sequence $|\partial_V R_n|$ satisfies properties $\mathbf{A}(\delta)$ and $\mathbf{D}(\delta)$. Consequently, taking $\delta = 5$, for all $\varepsilon > 0$ there exists $C = C(\varepsilon)$ such that for all*

$n \geq 2$,

$$\mathbb{P}\left(\Big||\partial_V R_n| - \mathbb{E}\left(|\partial_V R_n|\right)\Big| > \varepsilon \mathbb{E}\left(|\partial_V R_n|\right)\right) \leq \frac{C}{\varepsilon^2 \log^5(n)}. \tag{3.23}$$

The proof of Theorems A and B is similar to the proof of [19, Thm. 4.2], hence it is postponed to the appendix. We note that the proof of Theorem B is easier for each individual term $|\partial_V R_n|$ separately then for $|\partial R_n|$ which is a weighted sum of $\{|\partial_V R_n| : V \subset \mathbb{S}_d\}$.

Proof of Thm. 1.1. Notice that if we manage prove that $\log^2(n)|\partial R_n|/n \to C$ almost surely, then the fact that $C > 0$ follows from Corollary 3.5. In addition, similarly to the proof of Corollary 3.5 it suffices to prove that for any $V \subset \mathbb{S}_d$ we have

$$\frac{\log^2(n)}{n}|\partial_V R_n| \xrightarrow[n\to\infty]{a.s.} C_V.$$

By (3.13) it further suffices to show that

$$Z_n := \left|\frac{|\partial_V R_n|}{\mathbb{E}\left(|\partial_V R_n|\right)} - 1\right| \xrightarrow[n\to\infty]{a.s.} 0.$$

Let $\epsilon > 0$ and write $N_k := \left\lfloor \exp\left(\sqrt[4]{k}\right)\right\rfloor$. By Corollary 3.8 there exists $C > 0$ such that for all $n \geq 2$,

$$\mathbb{P}\left(Z_{N_k} > \epsilon\right) \leq \frac{C}{\epsilon^2 \log^5(N_k)} \leq \frac{C}{k^{5/4}}.$$

It follows from the Borel-Cantelli lemma that for almost every $w \in \Omega$,

$$\limsup_{k\to\infty} Z_{N_k} \leq \epsilon.$$

As $Z_n \geq 0$ and ϵ is arbitrary it follows that for almost every $w \in \Omega$,

$$\lim_{n\to\infty} Z_{N_k} = 0.$$

Now for a general $n \in \mathbb{N}$ there exists a unique $m = m(n) \in \mathbb{N}$ such that $N_m \leq n < N_{m+1}$. Since $e^x - 1 \leq 2x$ for all $0 \leq x \leq 1$ and for all m large $\sqrt[4]{m+1} - \sqrt[4]{m} \leq 1/3m^{-3/4}$, it follows that for all large m,

$$\begin{aligned} N_{m+1} - N_m &\leq \exp(\sqrt[4]{m+1}) - \exp(\sqrt[4]{m+1}) + 2 \\ &= \exp\left(\sqrt[4]{m}\right)\left[\exp\left(\sqrt[4]{m+1} - \sqrt[4]{m}\right) - 1\right] + 2 \leq m^{-3/4} N_m. \end{aligned} \tag{3.24}$$

In particular, $N_{m+1}/N_m \to 1$. By (3.13), $\mathbb{E}(|\partial_V R_n|) \sim Cn/(\log^2(n))$ is regularly varying. Therefore, since $N_m \leq n \leq N_{m+1}$,

$$\mathbb{E}(|\partial_V R_n|)/\mathbb{E}(|\partial_V R_{N_m}|) \to 1. \tag{3.25}$$

From the trivial bound

$$|\partial_V R_n| - 2dm \leq |\partial_V R(n+m)| \leq |\partial_V R_n| + 2dm$$

for all m large enough and $N_m \leq n \leq N_{m+1}$, we have using (3.24)

$$\frac{\big||\partial_V R_n| - |\partial_V R_{N_m}|\big|}{\mathbb{E}(|\partial_V R_{N_m}|)} \leq \frac{2d(N_{m+1} - N_m)}{\mathbb{E}(|\partial_V R_{N_m}|)} \leq \frac{2dm^{-3/4} N_m}{CN_m/\log^2(N_m)} \lesssim \frac{m^{1/2}}{m^{3/4}} \to 0.$$

Since $|\partial_V R_{N_m}|/\mathbb{E}(|\partial_V R_{N_m}|)$ tends almost surely to 1, we obtain that $|\partial_V R_n|/\mathbb{E}(|\partial_V R_{N_m}|)$ also tends to 1. Together with (3.25), this gives $|\partial_V R_n|/\mathbb{E}(|\partial_V R_n|) \to 1$ almost surely as required. It then follows from (3.13) that for all $V \subset \mathbb{S}_d$, there exists $C_V \geq 0$ such that almost surely,

$$\lim_{n\to\infty} \left(\frac{\log^2(n)}{n} |\partial_V R_n| \right) = C_V,$$

and consequently,

$$\frac{\log^2(n)}{n} |\partial R_n| = \sum_{V \subset \mathbb{S}_d} \frac{\log^2(n)}{n} |\partial_V R_n| \xrightarrow[n\to\infty]{} C = \sum_{V \subset \mathbb{S}_d} C_V > 0,$$

almost surely as required. □

4 Proof of Theorem 1.2

Let $\{X_n\}_{n=-\infty}^{\infty}$ be as sequence of i.i.d., centered, $\mathbb{Z}$-valued random variables, and define the two-sided random walk $\{S_n\}_{n\in\mathbb{Z}}$ as follows, $S_0 = 0$, and for $n \geq 1$ let $S_n = X_1 + \cdots + X_n$ and $S_n^{(-)} = -X_{-1} - \cdots - X_{-n}$. We assume that $\{S_n\}_n$ is aperiodic in the sense of Section 3 and that the random variables $\{X_i\}_i$ belong to the domain of attraction of a non-degenerate, symmetric, α-stable distribution with $1 < \alpha \leq 2$. This implies that the X_i are centered, and there exists a positive, slowly varying (at ∞) function $L : \mathbb{R}_+ \to \mathbb{R}_+$ such that,

$$Y_n := \frac{S_n}{n^{\frac{1}{\alpha}} L(n)} \xrightarrow[n\to\infty]{d} Z_\alpha,$$

where Z_α is a real random variable with characteristic function $\mathbb{E}(e^{itZ_\alpha}) = e^{-|t|^\alpha}$. By Lévy's continuity theorem, writing $\phi(t) := \mathbb{E}(e^{itX_1})$, we see that for all $t > 0$,

$$\mathbb{E}\left(e^{itY_n}\right) = \phi\left(\frac{t}{n^{\frac{1}{\alpha}} L(n)}\right)^n \xrightarrow[n\to\infty]{} e^{-|t|^\alpha}.$$

From this and a Tauberian theorem it follows that, see e.g. [22, Theorem 2.6.5],

$$\phi(t) = 1 - |t|^\alpha L(1/|t|) \left[1 + o(1)\right], \quad t \to 0. \tag{4.1}$$

If $A \subset \mathbb{Z}$ is a finite subset we define

$$r_n(x, A) := \mathbb{P}^x \left(S_k \notin A,\ 1 \leq k \leq n\right),$$

and write $r_n := r_n(0, \{0\})$.

We will need the following result.

Lemma 4.1. *Let $\{X_i\}_{i\in\mathbb{Z}}$ be i.i.d. $\mathbb{Z}$-valued random variables with characteristic function ϕ satisfying* (4.1) *for a positive slowly varying function L and $\alpha \in (1,2)$. Assume in addition that the corresponding random walk is aperiodic. Then r_n is regularly varying of index $1/\alpha - 1$ as $n \to \infty$; that is there exists another positive slowly varying (at ∞) function M such that, for any $x \in \mathbb{Z}$ and any finite nonempty $A \subseteq \mathbb{Z}$,*

$$\lim_{n\to\infty} n^{1-1/\alpha} M(n) r_n(x, A) = C(x, A), \tag{4.2}$$

for some constant $C(x, A) \geq 0$.

Proof. When $\lim_{s\to\infty} L(s)$, where L is the first slowly varying function in the statement, exists this result is [25, Theorem 8]. So we will consider the case where $L(\cdot)$ does not have a limit at infinity. Similarly to the proof of [25, Theorem 8] we define for $\lambda \in [0,1)$

$$U(\lambda) := \sum_{n=0}^{\infty} \lambda^n \, \mathbb{P}(S_n = 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}t}{1 - \lambda\phi(t)},$$
$$R(\lambda) := \sum_{n=0}^{\infty} \lambda^n r_n = (1-\lambda)^{-1} U(\lambda)^{-1}.$$

First we will study the asymptotic behaviour of $U(\lambda)$ as $\lambda \to 1$.

Notice that since $|t|^\alpha L(1/t)$ is regularly varying at the origin, by [8, Theorem 1.5.3] there exists a monotone, α-regularly varying function $g(t)$, such that $g(t) \sim C|t|^\alpha L(1/t)$ as $t \to 0$, for some $C > 0$. Therefore $\phi(t) = 1 - g(t)\,[1 + o\,(1)]$, as $t \to 0$. Next, we use aperiodicity to concentrate on the behaviour around the origin. In particular for any $\epsilon > 0$, there exists a constant $C(\epsilon) > 0$ such that $|\phi(t)| < 1 - C(\epsilon)$ for all $\epsilon \le |t| \le \pi$. The main contribution to the asymptotic behavior of $U(\lambda)$ as $\lambda \to 1$ will come from the integral over $|t| < \epsilon$.

We claim that the function

$$\tilde{U}_\epsilon(\delta) := \int_0^\epsilon \frac{\mathrm{d}t}{\delta + g(t)}$$

is regularly varying of index $1/\alpha - 1$ when $\delta \to 0+$ and that its asymptotic behaviour at $0+$ does not depend on ϵ. Let us prove this claim.

Since g is α-regularly varying and monotone, its generalised inverse $f(u) := g^{-1}(u)$ will be monotone and regularly varying of index $1/\alpha$, [8, see Theorem 1.5.12]. Letting $t = g^{-1}(z)$ we have

$$\tilde{U}_\epsilon(\delta) = \int_{z=0}^{g(\epsilon)} \frac{\mathrm{d}g^{-1}(z)}{\delta + z} = \int_{z=0}^{\infty} \frac{\mathrm{d}g_\epsilon^{-1}(z)}{\delta + z},$$

interpreting the integral in the Stieltjes sense, and letting $g_\epsilon^{-1}(s) := g^{-1}(s)$ for $s < g(\epsilon)$ and $g_\epsilon^{-1}(s) := g^{-1}(g(\epsilon))$ for $s \ge g(\epsilon)$. With this definition it is clear that g_ϵ^{-1} is also monotone and that $g_\epsilon^{-1}(s) \sim g^{-1}(s)$ as $s \to 0+$. In particular its behaviour near the origin is independent of ϵ which shows that $\lim_{\delta\to 0+} \tilde{U}_\epsilon(\delta)/\tilde{U}_{\epsilon'}(\delta) = 1$ for all $\epsilon, \epsilon' > 0$.

The rest is fairly similar to the proof of [8, Theorem 1.7.4]. First notice that

$$\begin{aligned}
\tilde{U}_\epsilon(\delta) &= \int_{z=0}^{\infty} \frac{\mathrm{d}g_\epsilon^{-1}(z)}{\delta + z} \\
&= \int_{z=0}^{\infty} \int_{u=0}^{\infty} \mathrm{e}^{-(\delta+z)u} \mathrm{d}g_\epsilon^{-1}(z)\mathrm{d}u \\
&= \int_{u=0}^{\infty} \mathrm{e}^{-\delta u} \int_{z=0}^{\infty} \mathrm{e}^{-zu} \mathrm{d}g_\epsilon^{-1}(z)\mathrm{d}u \\
&= \int_{u=0}^{\infty} \mathrm{e}^{-\delta u} V_\epsilon(u)\mathrm{d}u,
\end{aligned}$$

where

$$V_\epsilon(u) := \int_{z=0}^{\infty} \mathrm{e}^{-zu} \mathrm{d}g_\epsilon^{-1}(z).$$

By [8, Theorem 1.7.1'], since $g_\epsilon^{-1}(z)$ is regularly varying of index $1/\alpha$ at the origin, we have that $V_\epsilon(u)$ is regularly varying of index $-1/\alpha$ as $u \to \infty$. In turn this implies that

$$W_\epsilon(u) := \int_0^u V_\epsilon(s)\mathrm{d}s,$$

is regularly varying of index $1 - 1/\alpha$ as $u \to \infty$ and since

$$\tilde{U}_\epsilon(\delta) = \int_{u=0}^{\infty} \mathrm{e}^{-\delta u} V_\epsilon(u)\mathrm{d}u = \int_{u=0}^{\infty} \mathrm{e}^{-\delta u} W_\epsilon(\mathrm{d}u),$$

by Karamata's Tauberian theorem [8, 1.7.1] we have that $\tilde{U}_\epsilon(\delta)$ is regularly varying of index $1/\alpha - 1$ as $\delta \to 0$ with asymptotic behaviour at $0+$ independent of ϵ. That is we have shown that there exists a $(1/\alpha - 1)$-regularly varying function $\tilde{U}$ such that for all $\epsilon > 0$ we have $\lim_{\delta \to 0} \tilde{U}_\epsilon(\delta)/\tilde{U}(\delta) = 1$.

Having proved the claim, let us now prove the main result. We write

$$\begin{aligned} U(\lambda) &= \frac{\tilde{U}_\epsilon(1-\lambda)}{\pi} + \frac{1}{2\pi}\left(\int_0^\epsilon \frac{\mathrm{d}t}{1-\lambda\phi(t)} - \tilde{U}_\epsilon(1-\lambda)\right) \\ &\quad + \frac{1}{2\pi}\left(\int_{-\epsilon}^0 \frac{\mathrm{d}t}{1-\lambda\phi(t)} - \tilde{U}_\epsilon(1-\lambda)\right) + \frac{1}{2\pi}\int_{\pi > |t| > \epsilon} \frac{\mathrm{d}t}{1-\lambda\phi(t)} \\ &=: J_1(\epsilon,\lambda) + J_2(\epsilon,\lambda) + J_3(\epsilon,\lambda) + J_4(\epsilon,\lambda). \end{aligned}$$

We claim that the term $J_1(\epsilon, \lambda)$ on the right dominates the other ones when λ tends to 1. First, it dominates the last one as $\tilde{U}_\epsilon(1-\lambda)$ tends to infinity while the last term remains bounded. The other two terms are similar, so let us handle the first one. We write the difference as

$$\begin{aligned} J_2(\epsilon,\lambda) = \int_0^\epsilon \frac{\mathrm{d}t}{1-\lambda\phi(t)} - \tilde{U}_\epsilon(1-\lambda) &= \int_0^\epsilon \left(\frac{1}{1-\lambda\phi(t)} - \frac{1}{1-\lambda+g(t)}\right)\mathrm{d}t \\ &= \int_0^\epsilon \frac{g(t) - \lambda(1-\phi(t))}{(1-\lambda\phi(t))(1-\lambda+g(t))}\mathrm{d}t. \end{aligned}$$

By choice of g we can write $g(t) = [1-\phi(t)][1+o(1)]$ as $t \to 0+$ and thus

$$g(t) - \lambda(1-\phi(t)) = [1-\phi(t)](1-\lambda+o(1)).$$

In particular for $0 < t < \epsilon$ we can write

$$|g(t) - \lambda(1-\phi(t))| \le |1-\phi(t)|(1-\lambda+\eta(\epsilon)),$$

for some positive function $\eta(\cdot)$ such that $\eta(\epsilon) \to 0$ as $\epsilon \to 0+$. Therefore

$$\begin{aligned} |J_2(\epsilon,\lambda)| &\le \int_0^\epsilon \frac{[1-\lambda+\eta(\epsilon)]|1-\phi(t)|}{|1-\lambda\phi(t)|\,|1-\lambda+g(t)|}\mathrm{d}t \\ &\le [1-\lambda+\eta(\epsilon)]\int_0^\epsilon \frac{\mathrm{d}t}{|1-\lambda+g(t)|} = [1-\lambda+\eta(\epsilon)]\tilde{U}_\epsilon(1-\lambda). \end{aligned}$$

We can similarly bound the other term

$$J_3(\epsilon,\lambda) \le [1-\lambda+\eta(\epsilon)]\tilde{U}_\epsilon(1-\lambda).$$

Therefore we have for $i = 2, 3$ that

$$\limsup_{\lambda \to 1} \frac{2|J_i(\epsilon,\lambda)|}{|J_1(\epsilon,\lambda)|} \le \eta(\epsilon),$$

and since we have shown above that $\lim_{\delta \to 0} \tilde{U}_\epsilon(\delta)/\tilde{U}(\delta) = 1$ we also have that

$$\limsup_{\lambda \to 1} \frac{|J_i(\epsilon,\lambda)|}{\tilde{U}(1-\lambda)} \le \eta(\epsilon),$$

Overall what we have just shown is that for some $(1/\alpha - 1)$ regularly varying function $\tilde{U}$ we have

$$\limsup_{\lambda\to 1} \frac{\left|U(\lambda) - \frac{1}{\pi}\tilde{U}(1-\lambda)\right|}{\tilde{U}(1-\lambda)} = \limsup_{\lambda\to 1}\left|\frac{U(\lambda)}{\tilde{U}(1-\lambda)} - \frac{1}{\pi}\right| \le \eta(\epsilon).$$

Since $\epsilon > 0$ is arbitrary and $\eta(\epsilon) \to 0$ with $\epsilon \to 0$ this proves that

$$\lim_{\lambda\to 1} \frac{U(\lambda)}{\tilde{U}(1-\lambda)} = \frac{1}{\pi},$$

whence it follows that $U(\lambda)$ is regularly varying of index $1/\alpha - 1$. Thus $R(\lambda) \sim C(1-\lambda)^{-1/\alpha}M(1/(1-\lambda))$, and since r_n is monotone it follows that $r_n \sim cn^{1/\alpha-1}M(n)$ for some constants $c, C > 0$ and a positive, regularly varying function M, by [8, Corollary 1.7.3]. Having established this, (4.2) follows from [26, Theorem 4a]. ☐

Remark 4.2. The following alternative approach using a local limit theorem is possible. Firstly, by [21],

$$|\mathbb{P}(S_n = m) - \mathbb{P}(S_n = 0)| = O\left(\frac{1}{a_n^2}\right), \quad as\ n \to \infty \tag{4.3}$$

where $m \in \mathbb{Z}$ and a_n is a $1/\alpha$ regularly varying sequence. Then a similar argument as in [17] shows that

$$r_n(m) := \mathbb{P}(S_k \neq m \text{ for all } 1 \le k < n) \sim r_n(0) \ \text{ as } n \to \infty. \tag{4.4}$$

Remark 4.3. This theorem, in the case of i.i.d. random variables with $\mathbb{E}(X_1) = 0$ and $\mathbb{E}\left(X_1^2\right) = D < \infty$ gives the rate $o(n^p)$ for every $p < \frac{1}{2}$. We claim that this is the optimal rate in the polynomial exponent. Indeed, for the simple random walk on $\mathbb{Z}$, R_n is an interval and thus $|\partial R_n| = 2$. By Theorem 3 in [11],

$$\limsup_{n\to\infty} \frac{\sqrt{n}\,|\partial R_n|}{|R_n|} = \limsup_{n\to\infty} \frac{2\sqrt{n}}{|R_n|} = \infty.$$

Before proceeding with the proof of Theorem 1.2, we need a number of auxiliary results.

Proposition 4.4. *Let S_n be an aperiodic random walk on $\mathbb{Z}$ in the domain of attraction of a nondegenerate, symmetric, α-stable distribution with $1 < \alpha \le 2$, then for any $\epsilon > 0$, $k \in \mathbb{N}$ and $x \in \mathbb{Z} \setminus \{0\}$ there exists $M > 0$ such that*

$$\mathbb{E}\left(|R_n \setminus (R_n - x))|^k\right) \le Mn^{k\left(\frac{2}{\alpha}-1+\epsilon\right)}.$$

Proof. Let $\epsilon > 0$ and $j \in \mathbb{Z}\setminus\{0\}$. In the course of the proof C will denote a global positive constant whose value can change (increase) from line to line. We have the bound

$$|R_n \setminus (R_n - x)| \le \sum_{j=1}^{n} \mathbb{1}_{A_n(j)} \circ T^j$$

where

$$A_n(m) = \{w \in \Omega : S_j \neq x, \ \ \forall j \in [-m, n-m]\} = D_x^-(m) \cap D_x^+(n-m),$$

where for $y \in \mathbb{Z}$ and $n \ge 0$ we define

$$D_y^-(n) := \{\omega \in \Omega : S_j^{(-)} \neq y,\ j \in [1,n]\}, \quad D_y^+(n) := \{\omega \in \Omega : S_j \neq y,\ j \in [1,n]\}.$$

Note that D_x^- and D_x^+ are independent, thus

$$\mathbb{P}\left(A_n(m)\right) = \mathbb{P}\left(D_x^-(m)\right)\mathbb{P}\left(D_x^+(n-m)\right).$$

As Z_α is symmetric, X_1 is in the domain of attraction of Z_α if and only if $-X_1$ is in the domain of attraction of Z_α. This together with Lemma 4.1 implies that there exist positive, slowly varying functions L_+, L_- and constants possibly depending on x such that

$$\mathbb{P}\left(D_x^\pm(n)\right) \le \frac{C(x)L_\pm(n)}{n^{1-1/\alpha}},$$

for $n \ge 1$.

We will now show that for all $k \in \mathbb{N}$ there exists $C > 0$ such that,

$$\mathbb{E}\left(|R_n \setminus (R_n - x))|^k\right) \le C\left[n^{k\left(\frac{2}{\alpha}-1+\epsilon\right)} + 5k^2\mathbb{E}\left(|R_n \setminus (R_n - x)|^{k-1}\right)\right].$$

The latter inequality proves the proposition by a simple induction argument. Writing

$$\Delta_k(n) := \left\{(m_1, m_2, .., m_k) \in \{1, \dots, n\}^k : \ \forall 1 < l \le k, \ m_l - m_{l-1} \ge 3\right\},$$

and

$$\Lambda_k(n) := \left\{(m_1, m_2, .., m_k) \in \{1, \dots, n\}^k : \ \exists 1 < l < l' \le k, \ |m_l - m_{l'}| \le 2\right\},$$

Then

$$|R_n \setminus (R_n - x))|^k \le k! \sum_{(m_1,\dots,m_k)\in\Delta_k(n)} \prod_{l=1}^k \mathbb{1}_{A_n(m_l)} \circ T^{m_l} + k! \sum_{\Lambda_k(n)} \prod_{l=1}^k \mathbb{1}_{A_n(m_l)} \circ T^{m_l}$$

For every $(m_1, ..., m_k) \in \Lambda_k(n)$ there exists a minimal $1 \le \mathfrak{l} < k$ such that there exists a minimal $l' > \mathfrak{l}$ with $|m_\mathfrak{l} - m_{l'}| \le 2$. Since for all $1 \le \mathfrak{l} < k$

$$\prod_{l=1}^k \mathbb{1}_{A_n(m_l)} \circ T^{m_l} \le \prod_{l\in\{1,..k\}\setminus\{\mathfrak{l}\}} \mathbb{1}_{A_n(m_l)} \circ T^{m_l},$$

and,

$$\left\{(m_j)_{j\in\{1,\dots,k\}\setminus\{\mathfrak{l}\}} : \ (m_j)_{j=1}^k \in \Lambda_k(N)\right\} \subset \{1, \dots, n\}^{k-1},$$

and there are 5 possible values for $m_\mathfrak{l}$ given $m_{l'} \in \{1, \dots, n\}$, we can see that

$$\begin{aligned}\mathbb{E}\left(\sum_{\Lambda_k(n)} \prod_{l=1}^k \mathbb{1}_{A_n(m_l)} \circ T^{m_l}\right) &\le 5\binom{k}{2}\mathbb{E}\left(\sum_{m_1,\dots,m_{k-1}=1}^n \prod_{l=1}^{k-1} \mathbb{1}_{A_n(m_l)} \circ T^{m_l}\right) \\ &= 5\binom{k}{2}\mathbb{E}\left(|R_n \setminus (R_n - x))|^{k-1}\right).\end{aligned}$$

It remains to bound the other term. For $(m_1, ..., m_k) \in \Delta_k(n)$, letting $q_1 = m_1$, $q_{k+1} = n - m_k$ and for $2 \le j \le k$, $q_j = \lfloor (m_j - m_{j-1})/2 \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x, we deduce the inequality

$$\prod_{l=1}^k \mathbb{1}_{A_n(m_l)} \circ T^{m_l} \le \mathbb{1}_{D_x^-(m_1)} \circ T^{m_1} \left(\prod_{l=2}^k \left[\mathbb{1}_{D_x^+(q_l)} \circ T^{m_{l-1}} \mathbb{1}_{D_x^-(q_l)} \circ T^{m_l}\right]\right) \mathbb{1}_{D_x^+(q_{k+1})} \circ T^{m_k}$$

by replacing the restriction that $S_j + x \ne S_{m_l}$ for $j \in [1, n]$ by the weaker one that $S_j + x \ne S_{m_l}$ for $j \in [m_{l-1} + q_l, m_l + q_{l+1}]$; see Figure 1 where the random walk started

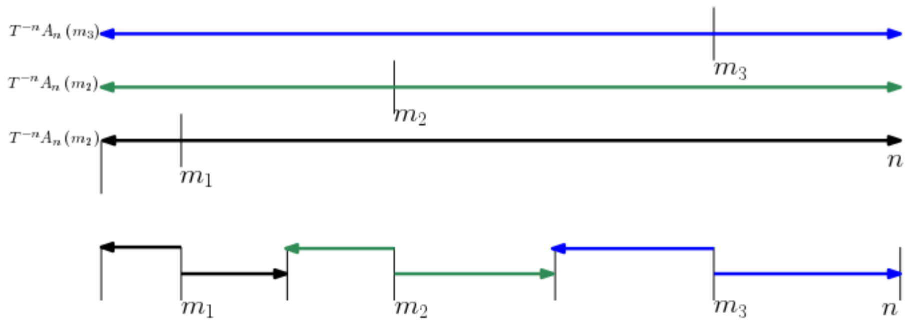

Figure 1:

at the marked point is constrained to not visit x for the time indicated by the arrows. The bound above is essentially a product of events that depend on non-overlapping sequences of the random variables $\{X_j\}_{j\in\mathbb{Z}}$ and therefore by independence we have that

$$\mathbb{E}\left(\prod_{l=1}^{k} \mathbb{1}_{A_n(m_l)} \circ T^{m_l}\right) \le \mathbb{P}\left\{D_x^-(q_1)\right\} \prod_{l=2}^{k} \left[\mathbb{P}\left\{D_x^+(q_l)\right\} \mathbb{P}\left\{D_x^-(q_l)\right\}\right] \mathbb{P}\left\{D_x^+(q_{k+1})\right\}$$
$$\le C(x) \prod_{l=2}^{k+1} \frac{L_-(q_{l-1})}{(q_{l-1})^{1-1/\alpha}} \frac{L_+(q_l)}{(q_l)^{1-1/\alpha}},$$

where we write $C(x)$ for a generic positive constant depending on x. In particular since $L_\pm(\cdot)$ are slowly varying, for any $\epsilon > 0$ we can find a positive constant $C > 0$ such that $L_\pm(n) \le C n^\epsilon$ and thus

$$\mathbb{E}\left(\prod_{l=1}^{k} \mathbb{1}_{A_n(m_l)} \circ T^{m_l}\right) \le C(x) \prod_{l=2}^{k+1} \frac{C}{(q_{l-1}q_l)^{1-1/\alpha-\epsilon}}$$
$$= C(x) \left[q_1 q_{k+1}\right]^{1/\alpha-1+\epsilon} \prod_{l=2}^{k} (q_l)^{2/\alpha-2+2\epsilon}.$$

Therefore,

$$\mathbb{E}\left(\sum_{\Delta_k(n)} \prod_{l=1}^{k} \mathbb{1}_{A_n(m_l)} \circ T^{m_l}\right) \le C(x) \sum_{q_1,\ldots,q_{k+1}} \left[q_1 q_{k+1}\right]^{\frac{1}{\alpha}-1+\epsilon} \prod_{l=2}^{k} q_l^{\frac{2}{\alpha}-2+2\epsilon}.$$

The sum is restricted to the values of q_i that can be produced by the above process. They satisfy $q_1 + 2q_2 + \cdots + 2q_k + q_{k+1} \le n$ and $q_1 + 2q_2 + \cdots + 2q_k + q_{k+1} \ge n - k$, because of the integer parts in the definition of q_j. We have that

$$\sum_{n-k \le q_1+\ldots+q_{k+1} \le n} \left[q_1 q_{k+1}\right]^{\frac{1}{\alpha}-1+\epsilon} \prod_{l=2}^{k} q_l^{\frac{2}{\alpha}-2+2\epsilon}$$
$$= \sum_{l=1}^{k} \sum_{q_1+\ldots+q_{k+1}=n-l} \left[q_1 q_{k+1}\right]^{\frac{1}{\alpha}-1+\epsilon} \prod_{l=2}^{k} q_l^{\frac{2}{\alpha}-2+2\epsilon}$$
$$\le k \sum_{q_1+\ldots+q_{k+1}=n} \left[q_1 q_{k+1}\right]^{\frac{1}{\alpha}-1+\epsilon} \prod_{l=2}^{k} q_l^{\frac{2}{\alpha}-2+2\epsilon} \le C k n^\rho$$

by Lemma 4.5 below, since $1/\alpha - 1 + \epsilon > -1$, $2/\alpha - 2 + 2\epsilon > -1$ for small enough ϵ, where

$$\rho = k + 2 \cdot \left(\frac{1}{\alpha} - 1 + \epsilon\right) + (k-1) \cdot \left(\frac{2}{\alpha} - 2 + 2\epsilon\right) = k \cdot \left(\frac{2}{\alpha} - 1 + 2\epsilon\right).$$

The result follows with $\epsilon' = 2\epsilon$. $\square$

Lemma 4.5. *Let $k \in \mathbb{Z}$, $k > 1$, and suppose that $\alpha_1, \dots, \alpha_k > -1$. Then*

$$\sum_{\substack{q_1+\cdots+q_k=n \\ q_1,\dots,q_k \geq 0}} \prod_{l=1}^{k} q_l^{\alpha_l} \leq C n^{(k-1)+\sum_{l=1}^{k} \alpha_l}.$$

Proof of Lemma 4.5. We will proceed by induction on k. Let $k = 2$ and notice that

$$\begin{aligned}
\sum_{q_1+q_2=n} q_1^{\alpha_1} q_2^{\alpha_2} &= \sum_{k=1}^{n} k^{\alpha_1} (n-k)^{\alpha_2} \\
&= n^{1+\alpha_1+\alpha_2} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^{\alpha_1} \left(1 - \frac{k}{n}\right)^{\alpha_2} \frac{1}{n} \\
&\leq C n^{1+\alpha_1+\alpha_2} \int_{x=0}^{1} x^{\alpha_1} (1-x)^{\alpha_2} \mathrm{d}x \leq C\beta(\alpha_1, \alpha_2) n^{1+\alpha_1+\alpha_2},
\end{aligned}$$

where $\beta(\alpha_1, \alpha_2)$ is the value of the β-function at (α_1, α_2).

Suppose now that the result holds for all integers $\leq k$ and notice that

$$\begin{aligned}
\sum_{q_1+\cdots+q_{k+1}=n} \prod_{l=1}^{k+1} q_l^{\alpha_l} &= \sum_{m=1}^{n} m^{\alpha_{k+1}} \sum_{q_1+\cdots+q_k=n-m} \prod_{l=1}^{k+1} q_l^{\alpha_l} \\
&\leq C \sum_{m=1}^{n} m^{\alpha_{k+1}} (n-m)^{k-1+\alpha_1+\cdots+\alpha_k} \leq C n^{k+\alpha_1+\cdots+\alpha_k+\alpha_{k+1}},
\end{aligned}$$

where the last two inequalities follow from the inductive hypothesis for $j = k$ and $j = 2$ respectively. $\square$

Proposition 4.6. *Let S_n be an aperiodic random walk on $\mathbb{Z}$ in the domain of attraction of a nondegenerate, symmetric, α-stable distribution with $1 < \alpha \leq 2$. Then for all $\epsilon > 0$, almost surely*

$$\lim_{n\to\infty} \frac{|R_n|}{n^{\frac{1}{\alpha}-\epsilon}} = \infty.$$

Proof. Let $\epsilon > 0$. Le Gall and Rosen have shown in [28] that there exists a $1/\alpha$-regularly varying sequence a_n such that

$$\frac{|R_n|}{a_n} \xrightarrow[n\to\infty]{dist.} \mathrm{Leb}_{\mathbb{R}} \left(W_\alpha[0,1]\right), \tag{4.5}$$

where $W_\alpha[0,1]$ is the range of the symmetric α-stable Lévy motion up to time 1. It is well known, see for example [10], that the occupation measure of a one-dimensional α-stable process defined by

$$\mu(A) := \int_0^1 \mathbb{1}_A \circ W_\alpha(s) \mathrm{d}s,$$

is almost surely absolutely continuous with respect to Lebesgue measure[3] for $\alpha > 1$. As $\mu(W_\alpha[0,1]) = 1$ this implies that

$$\mathbb{P}\left\{\mathrm{Leb}_{\mathbb{R}} \left(W_\alpha[0,1]\right) > 0\right\} = 1. \tag{4.6}$$

Since a_n is $1/\alpha$-regularly varying, setting $t_n := \lfloor n^\kappa \rfloor$, where $\kappa > 0$, we have that a_{t_n} is κ/α-regularly varying. Let $\kappa = 1 - \alpha\epsilon/2$, so that $\kappa/\alpha > 1/\alpha - \epsilon$. Then for n large enough

[3] Equivalently, almost surely possesses a continuous local time $x \mapsto L_\alpha(1, x)$

we have $a_{t_n} > n^{1/\alpha-\epsilon}$. Decompose the interval $[0,n]\cap\mathbb{Z}$ into sub-intervals $[jt_n,(j+1)t_n]$, $j=0,1,..,\lfloor n/t_n\rfloor$ of length t_n, plus perhaps a remainder interval which will be ignored. Writing $R(n,m)=\{S_{n+1},..,S_m\}$, for $\delta>0$ and all n large enough we have that

$$|R_n| \geq \max_{j\in\left\{1,..,\left\lfloor \frac{n}{t_n}\right\rfloor\right\}} |R\left(jt_n,(j+1)t_n\right)|,$$

and therefore

$$\begin{aligned}\mathbb{P}\left(|R_n| < \delta n^{\frac{1}{\alpha}-\epsilon}\right) &\leq \mathbb{P}\Big(\max_{j\in\{1,..,\left\lfloor \frac{n}{t_n}\right\rfloor\}} |R\left(jt_n,(j+1)t_n\right)| < \delta a_{t_n}\Big)\\ &\leq \mathbb{P}\left(|R_{t_n}| < \delta a_{t_n}\right)^{\left\lfloor \frac{n}{t_n}\right\rfloor}.\end{aligned}$$

By (4.6) we can choose $\delta>0$ small enough so that $\mathbb{P}\left(\mathrm{Leb}_{\mathbb{R}}\left(W_\alpha[0,1]\right)<\delta\right)<1$ and by (4.5) we can choose N_0 large enough so that for all $n\geq N_0$ we have

$$\mathbb{P}\left(|R_{t_n}| < \delta a_{t_n}\right) \leq \rho < 1.$$

Therefore for all $n>N_0$, from the definition of t_n and the above it follows that

$$\mathbb{P}\left(|R_n| < \delta n^{\frac{1}{\alpha}-\epsilon}\right) \leq \rho^{\left\lfloor \frac{n}{t_n}\right\rfloor}.$$

Since $n/t_n \sim n^{\kappa'}$ where $\kappa' = \epsilon\alpha/2$, we have that

$$\sum_{n\geq 1}\mathbb{P}\left(|R_n| < \delta n^{\frac{1}{\alpha}-\epsilon}\right) \leq \sum_{n\geq 1}\rho^{\left\lfloor \frac{n}{t_n}\right\rfloor} < \infty,$$

and thus from the Borel-Cantelli Lemma we have that almost surely

$$\varliminf_{n\to\infty}\frac{|R_n|}{n^{\frac{1}{\alpha}-\epsilon}} \geq \delta.$$

As $\epsilon>0$ is arbitrary the conclusion follows. □

Finally, we are now able to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Fix $\delta>0$. By Proposition 4.6, it remains to show that

$$\lim_{n\to\infty}\frac{|\partial R_n|}{n^{\frac{2}{\alpha}-1+\delta}} = 0. \tag{4.7}$$

First note that

$$|\partial R_n| \leq |R_n\setminus(R_n-1)| + |R_n\setminus(R_n+1)| = V_n(1)+V_n(2).$$

Let $i\in\{1,2\}$. By Proposition 4.4 for any $\epsilon>0$ and $k\in\mathbb{N}$ there exists $M>0$ such that for all $n\in\mathbb{N}$ and $i\in\{1,2\}$,

$$\mathbb{E}\left(\left(\frac{V_n(i)}{n^{\frac{2}{\alpha}-1+\delta}}\right)^k\right) \leq Mn^{k(\epsilon-\delta)}.$$

Therefore, choosing $\epsilon<\delta$, there exist $k\in\mathbb{N}$ and $M>0$ such that for all $n\in\mathbb{N}$ and $i\in\{1,2\}$,

$$\mathbb{E}\left(\left(\frac{V_n(i)}{n^{\frac{2}{\alpha}-1+\delta}}\right)^k\right) \leq Mn^{-2}.$$

A standard use of Markov's inequality and the Borel Cantelli Lemma shows that for any $\delta > 0$ almost surely

$$\lim_{n\to\infty} \frac{V_n(1)}{n^{\frac{2}{\alpha}-1+\delta}} = \lim_{n\to\infty} \frac{V_n(2)}{n^{\frac{2}{\alpha}-1+\delta}} = 0. \tag{4.8}$$

Therefore we have that almost surely for all $\delta > 0$ small enough

$$\lim_{n\to\infty} \frac{|\partial R_n|}{|R_n|\, n^{\frac{1}{\alpha}-1+\delta}} = \lim_{n\to\infty} \frac{|\partial R_n|\,/n^{\frac{2}{\alpha}-1+\delta/2}}{|R_n|\,/n^{\frac{1}{\alpha}-\delta/2}} = 0,$$

which proves the theorem. □

5 Proof of Theorem 1.4

5.1 Følner property and transient random walks

We now turn to an application of Proposition 2.5 in the context of random walks in groups.

Let G be a countable group. Given p a probability measure on G and $\xi_1, \xi_2, \ldots$ an i.i.d. sequence with marginals distributed as p, let $S_n = \xi_1 \cdot \xi_2 \cdot \ldots \cdot \xi_n$ be the corresponding random walk and $R_n := \{S_1, \ldots, S_n\}$ be its range process. Let us say that a random walk driven by a measure p on the group G is *admissible* (or *irreducible*) if the walk starting from the identity can reach any point in the group, i.e. the semigroup generated by the support of p is the whole group. This is a natural irreducibility assumption.

We define the Green function, $G: \ \mathsf{G} \mapsto [0,\infty]$, of a transient random walk on G, by

$$G(g) := \mathbb{E}\left(L(g)\right) = \sum_{n=0}^{\infty} \mathbb{P}\left(S_n = g\right), \tag{5.1}$$

where $L(g) := |\{n \in \mathbb{N} \cup \{0\},\ S_n = g\}| : \Omega \to \mathbb{N} \cup \{0\}$. For $g \in G$ define

$$q(g) := \mathbb{P}\left(\forall n \in \mathbb{N},\ S_n \neq g\right),$$

and write $q := q\left(\mathrm{id}_{\mathsf{G}}\right)$.

The next Lemma is well known and we only include its proof for completeness.

Lemma 5.1. *If S_n is a G-valued transient random walk then for all $g \in \mathsf{G}$,*

$$G(g) = \frac{1-q(g)}{q}.$$

Proof. Let $g \in \mathsf{G}$. By the Markov property for the random walk,

$$\begin{aligned} &\mathbb{P}(L(g) = 0) = q(g) \\ &\mathbb{P}(L(g) = j) = (1-q(g))(1-q)^{j-1}q, \quad j \geq 1. \end{aligned}$$

and thus

$$\begin{aligned} \mathbb{E}(L(g)) &= q(g)\cdot 0 + (1-q(g))\sum_{j=1}^{\infty} j(1-q)^{j-1}q \\ &= (1-q(g))\,\mathbb{E}\,L(0) = \frac{1-q(g)}{q}. \end{aligned}$$

□

The ergodic theoretic model of the random walk is the following skew product transformation. Let $\Omega := \mathsf{G}^{\mathbb{Z}}$, $\mathbb{P} = p^{\otimes \mathsf{G}}$ the product measure on Ω with marginals distributed as p and $T : \Omega \to \Omega$ the full shift defined by $(T\omega)_n = \omega_{n+1}$. The dynamical system, $\left(\mathsf{G}^{\mathbb{Z}}, \mathcal{B}, p^{\otimes \mathsf{G}}, T\right)$ is a stationary Bernoulli shift, hence ergodic, see for example

[15, p. 180]. Writing m_{G} for the Haar measure of G, and $f : \Omega \to \mathsf{G}$, $f(\omega) := \omega(0)$, the skew product transformation $T_f : \Omega \times \mathsf{G} \to \Omega \times \mathsf{G}$ satisfies

$$\pi_{\mathsf{G}}(T_f^n) \stackrel{d}{=} S_n,$$

here $\pi_{\mathsf{G}}(\omega, h) = h$ is the projection to the G coordinate. The advantage of working with the skew product is that the cocycle identity indicates what is the relevant random walk in inverse time. In this case, write for $n \in \mathbb{Z}$,

$$S_n^{(-)} = S_n^{(-)}(\omega) := F(-n, \omega) = \omega(-1)^{-1}\omega(-2)^{-1} \cdots \omega(-n)^{-1}. \tag{5.2}$$

Corollary 2.4 gives the following extension of Okada's result.

Corollary 5.2. *Let* G *be a discrete countable group and* p *a probability measure on* G *and* $\xi_1, \xi_2, \ldots$ *an i.i.d. sequence with marginal* p*. Then for any* $g \neq \mathrm{id}_{\mathsf{G}}$*, almost everywhere*

$$\lim_{n\to\infty} \frac{|R_n \triangle (R_n \cdot g)|}{|R_n|} = \mathbb{P}\left(\forall n \in \mathbb{N},\ S_n^{(-)} \neq g\right) \mathbb{P}\left(\forall n \in \mathbb{N}, S_n \neq g \middle| \forall n \in \mathbb{N},\ S_n \neq \mathrm{id}_{\mathsf{G}}\right)$$

$$+ \mathbb{P}\left(\forall n \in \mathbb{N},\ S_n^{(-)} \neq g^{-1}\right) \mathbb{P}\left(\forall n \in \mathbb{N}, S_n \neq g^{-1} \middle| \forall n \in \mathbb{N},\ S_n \neq \mathrm{id}_{\mathsf{G}}\right).$$

Proof. This is a direct consequence of Corollary 2.4 and the fact that $\{S_n\}_{n=1}^{\infty}$ and $\{S_n^{(-)}\}_{n=1}^{\infty}$ are independent. ☐

Remark 5.3. In the case where G is Abelian, $S_n^{(-)} \stackrel{d}{=} (S_n)^{-1}$ and thus for all $g \in \mathsf{G}$,

$$\mathbb{P}\left(\forall n \in \mathbb{N},\ S_n \neq g^{-1}\right) = \mathbb{P}\left(\forall n \in \mathbb{N},\ S_n^{(-)} \neq g\right).$$

The statement of Corollary 5.2 can be simplified in this case. Note that for a general group $(S_n^{(-)})^{-1}$ is a (multiplication from the) left random walk and S_n is a (multiplication from the) right random walk and their distribution as processes may no longer coincide.

Our next result links the Følner property of a transient G-valued random walk, with the decay of its Green function at infinity.

Proposition 5.4. *Let* S_n *be a* G *valued transient random walk and suppose there exists a sequence* $\{g_n\}_{n=1}^{\infty} \subset \mathsf{G}$ *such that*

$$\lim_{n\to\infty} \max(G(g_n), G^{(-)}(g_n)) = 0,$$

where $G^{(-)}$ *denotes the Green function of* $S_n^{(-)}$*, defined in* (5.2)*. Then* $\{R_n\}_{n=1}^{\infty}$ *is almost surely not a Følner sequence. Furthermore if* $g \in \mathsf{G}$ *is of infinite order in* G *and*

$$\lim_{n\to\infty} \max(G(g^n), G^{(-)}(g^n)) = 0,$$

then

$$\lim_{n\to\infty} \frac{|R_n \triangle (R_n g)|}{|R_n|} > 0 \quad \text{a.e.}$$

Proof. It follows from Lemma 5.1 that $G(g_n) \xrightarrow[n\to\infty]{} 0$ implies that $q(g_n) \xrightarrow[n\to\infty]{} 1$. Letting

$$A_n := \{\forall k \in \mathbb{N},\ S_k \neq g_n\}, \qquad B := \{\forall k \in \mathbb{N},\ S_k \neq \mathrm{id}_{\mathsf{G}}\},$$

we have that $\mathbb{P}(A_n) = q(g_n) \to 1$ and $\mathbb{P}(B) = q$, whence it easily follows that

$$\lim_{n\to\infty} \mathbb{P}\left(\forall k \in \mathbb{N},\ S_k \neq g_n \middle| \forall k \in \mathbb{N},\ S_k \neq \mathrm{id}_{\mathsf{G}}\right) = \lim_{n\to\infty} \frac{\mathbb{P}(A_n \cap B)}{\mathbb{P}(B)} = 1.$$

A similar reasoning shows that $G^{(-)}(g_n) \xrightarrow[n\to\infty]{} 0$ implies that

$$\lim_{n\to\infty} \mathbb{P}\left(\forall k \in \mathbb{N},\ S_k^{(-)} \neq g_n\right) = 1.$$

By this

$$\lim_{n\to\infty} \left[\mathbb{P}\left(\forall k \in \mathbb{N},\ S_k^{(-)} \neq g_n\right) \mathbb{P}\left(\forall k \in \mathbb{N},\ S_k \neq g_n | \forall k \in \mathbb{N},\ S_k \neq \mathrm{id}_{\mathsf{G}}\right)\right] = 1,$$

and an application of Corollary 5.2 shows that for all large $n \in \mathbb{N}$, almost surely

$$\lim_{k\to\infty} \frac{|R_k \triangle (R_k \cdot g_n)|}{|R_k|} > \frac{1}{2}.$$

We conclude that almost surely the range is not a Følner sequence.

In order to show the second part let g be of infinite order. Consequently, $g^n \xrightarrow[n\to\infty]{} \infty$ and there exists $n \in \mathbb{N}$ such that almost everywhere,

$$\lim_{k\to\infty} \frac{|R_k \triangle (R_k \cdot g^n)|}{|R_k|} > \frac{1}{2}. \tag{5.3}$$

By the triangle inequality for cardinality of symmetric differences of sets, for all $k \in \mathbb{N}$,

$$\begin{aligned} |R_k \triangle (R_k \cdot g^n)| &\leq |R_k \triangle (R_k \cdot g)| + \sum_{j=2}^{n} \left|\left(R_k g^{j-1}\right) \triangle \left(R_k \cdot g^j\right)\right| \\ &= |R_k \triangle (R_k \cdot g)| + \sum_{j=2}^{n} \left|\left(R_k \triangle (R_k \cdot g)\right) g^{j-1}\right| \\ &= n\,|R_k \triangle (R_k \cdot g)|\,, \end{aligned}$$

since for all $h \in G$ and $A \subset G$, $|Ah| = |A|$. We conclude that for all $k \in \mathbb{N}$,

$$\frac{|R_k \triangle (R_k \cdot g^n)|}{|R_k|} \leq n \frac{|R_k \triangle (R_k \cdot g)|}{|R_k|}$$

Taking limits as $k \to \infty$ we see that almost everywhere,

$$n \cdot \lim_{k\to\infty} \frac{|R_k \triangle (R_k g)|}{|R_k|} \geq \lim_{k\to\infty} \frac{|R_k \triangle (R_k g^n)|}{|R_k|} > \frac{1}{2},$$

where the last inequality is (5.3). This concludes the proof of the second part. □

Corollary 5.5. *Let* G *be a torsion free countable group. Let* S_n *be a* G *valued transient random walk for which the Green function tends to* 0 *at infinity. For all* $g \in \mathsf{G} \setminus \{id\}$*, there exists* $c(g) > 0$ *such that*

$$c(g) = \lim_{n\to\infty} \frac{|R_n \triangle (R_n g)|}{|R_n|} \quad \textit{a.e.}$$

5.2 Transient random walks on virtually cyclic groups

Let us show that the skip-free random walks (defined just before Theorem 1.4) are the only transient $\mathbb{Z}$-valued random walks with almost surely Følner range.

Proposition 5.6. *Let* $S_n = \sum_{k=1}^{n} X_k$ *be a transient* $\mathbb{Z}$*-valued random walk. If* $\{R_n\}_{n=1}^{\infty}$ *is almost surely a Følner sequence then* S_n *is a skip-free random walk.*

Proof. If $\mathbb{E}(|X_1|) = \infty$ then by [34, P8, page 287] the Green function of S_n and $S_n^{(-)}$ decays at infinity. By Corollary 5.5, $\lim_{n\to\infty} (|R_n \triangle (R_n+1)| / |R_n|) > 0$.

It remains to treat the case when $\mathbb{E}(|X_1|) < \infty$. If $\mathbb{E}(X_1) = 0$, it follows from the Chung Fuchs Theorem [12] that the walk is recurrent, which we exclude. Therefore, by the strong law of large numbers, either $\lim_{n\to\infty} S_n = \infty$ almost surely or $\lim_{n\to\infty} S_n = -\infty$ almost surely according to whether $\mathbb{E}(X_1) > 0$ or $\mathbb{E}(X_1) < 0$. It remains to show that if $S_n \to \infty$ almost surely and $\mathbb{P}(X_1 > 1) > 0$ then the range process is almost surely not a Følner sequence, the opposite case being similar.

Now if the random walk is transient and and $S_n \to \infty$ almost surely then[4]

$$\mathbb{P}(\forall n \in \mathbb{N},\ S_n > 0) > 0.$$

Since $\mathbb{P}(X_1 > 1) > 0$, there exists $\mathbb{Z} \ni j > 1$ such that $\mathbb{P}(X_1 = j) > 0$. Therefore,

$$\begin{aligned}\mathbb{P}(\forall n \in \mathbb{N},\ S_n > 1) &\geq \mathbb{P}(X_1 = j \text{ and } \forall 2 \leq n \in \mathbb{N}, S_n - X_1 > 0)\\ &= \mathbb{P}(X_1 = j)\,\mathbb{P}(\forall n \in \mathbb{N},\ S_n > 0), \text{ by the Markov property of } S_n.\end{aligned}$$

It follows that $\mathbb{P}(\forall n \in \mathbb{N},\ S_n > 1) > 0$ and

$$\mathbb{P}(\forall n \in \mathbb{N}, S_n \neq 1 | \forall n \in \mathbb{N},\ S_n \neq 0) \geq \frac{\mathbb{P}(X_1 = j)\,\mathbb{P}(\forall n \in \mathbb{N},\ S_n > 0)}{\mathbb{P}(\forall n \in \mathbb{N},\ S_n \neq 0)} > 0.$$

As the distributions of $\left\{S_n^{(-)}\right\}_{n=1}^{\infty}$ and $\{-S_n\}_{n=1}^{\infty}$ are the same

$$\mathbb{P}\left(\forall n \in \mathbb{N},\ S_n^{(-)} \neq 1\right) = \mathbb{P}(\forall n \in \mathbb{N},\ S_n \neq -1) \geq \mathbb{P}(\forall n \in \mathbb{N},\ S_n > 0) > 0.$$

We have shown that

$$\mathbb{P}\left(\forall n \in \mathbb{N},\ S_n^{(-)} \neq 1\right)\mathbb{P}(\forall n \in \mathbb{N}, S_n \neq 1 | \forall n \in \mathbb{N},\ S_n \neq 0) > 0.$$

By Corollary 5.2, $\lim_{n\to\infty} (|R_n \triangle (R_n+1)| / |R_n|) > 0$ almost surely and $\{R_n\}_{n=1}^{\infty}$ is almost surely not a Følner sequence. ☐

Let us show that, on groups which are virtually $\mathbb{Z}$ but not $\mathbb{Z}$, there is no transient walk for which the range is Følner.

Proposition 5.7. *Let S_n be a transient random walk on a group which is virtually $\mathbb{Z}$, but not isomorphic to $\mathbb{Z}$. Then the range $\{R_n\}_{n=1}^{\infty}$ is almost surely not a Følner sequence.*

Proof. By [33, Theorem 5.12], there is a surjective morphism π from G to either $\mathbb{Z}$ or the infinite dihedral group $\mathbb{Z} \rtimes \mathbb{Z}/2$, with finite kernel F. Since the kernel is finite, the image under π of the transient random walk on G is still a transient random walk in the image group. We will treat separately the two cases.

Assume first that $\pi(\mathsf{G}) = \mathbb{Z}$. Let p denote the measure on G driving the random walk S_n, and p^π its image in $\mathbb{Z}$ driving the image random walk S_n^π. Write G, resp. G^π for the Green function of S_n, resp. S_n^π. If X_1^π has no first moment, then the Green function of S_n^π decays at infinity, by [34, P8, page 287]. Since $G(g) \leq G^\pi(\pi(g))$, it follows that the Green function of S_n also decays at infinity, since π has finite kernel. Then Proposition 5.4 shows that R_n is not a Følner sequence. If X_1^π has a first moment, then $\mathbb{E}(X_1^\pi)$ has to be nonzero by [12] since the walk S_n^π is transient. In this case, by the strong law of large numbers, the probability that $S_n^\pi \neq 0$ for all $n \in \mathbb{Z} \setminus \{0\}$ is positive, as in the proof of Proposition 5.6. Since we are assuming $\mathsf{G} \neq \mathbb{Z}$, the kernel F is nontrivial. Therefore, we

[4] See the proof of Corollary 2.2 where it shown that $\mathbb{P}(l = 0) > 0$.

may choose $g \in F \setminus \{\mathrm{id}_\mathsf{G}\}$. With positive probability, we have $S_n \neq g$ for all $n \in \mathbb{Z} \setminus \{0\}$, as this follows from the fact that $S_n^\pi \neq 0 = \pi(g)$ for the projected random walk. Then Corollary 5.2 yields that $|R_n \triangle (R_n \cdot g)| / |R_n|$ converges to a positive limit, and therefore R_n is not Følner.

Consider now the case where $\pi(\mathsf{G}) = \mathbb{Z} \rtimes \mathbb{Z}/2$. The dihedral group $\mathbb{Z} \rtimes \mathbb{Z}/2$ is made of two copies of $\mathbb{Z}$ that we denote by $\mathbb{Z}$ and $\mathbb{Z} \times \{\rho\}$ where ρ is the nontrivial element of $\mathbb{Z}/2$, acting on $\mathbb{Z}$ by sign reversal. Decomposing p^π as a sum of two measures on these two copies of $\mathbb{Z}$, we write $p^\pi = \alpha p_1 + (1-\alpha)\rho_* p_2$ for some probability measures p_1, p_2 on $\mathbb{Z}$ and some $\alpha \in [0,1)$ (the value $\alpha = 1$ being excluded as the walk is admissible, so it can not remain stuck in $\mathbb{Z}$). Let $\check{p}_i$ be the reversed measure, given by $\check{p}_i(n) = p_i(-n)$. The first return of the random walk S_n^π to $\mathbb{Z}$ is distributed according to the measure

$$p' = \alpha p_1 + (1-\alpha)^2 \sum_{k=0}^{\infty} \alpha^k (p_2 * \check{p}_1^{*k} * \check{p}_2),$$

where the k-th term corresponds to trajectories of the random walk jumping to the second copy of $\mathbb{Z}$, making k steps there, and then coming back. Here, $*$ denotes the convolution of measures, i.e., $\mu_1 * \mu_2$ is the distribution of the sum of two independent random variables distributed respectively as μ_1 and μ_2. The presence of the reversed measures is coming from the action of ρ that reverses signs.

If p_1 or p_2 has an infinite moment, then so does the measure p'. Therefore, the Green function associated to p' tends to zero at infinity on $\mathbb{Z}$, by [34, P8, page 287]. Since, for points in $\mathbb{Z}$, it coincides with the Green function of the random walk S_n^π, it follows that $G^\pi(g)$ also tends to zero at infinity along $\mathbb{Z}$. As $G(g) \leq G^\pi(\pi(g))$, we get a sequence tending to infinity in G along which $G(g_n)$ and $G^{(-)}(g_n)$ tend to 0. By Proposition 5.4, R_n is almost surely not a Følner sequence.

Assume now that both p_1 and p_2 have a finite moment. Then so does p'. Moreover, the expectation of a variable distributed according to p' is given by

$$\alpha \mathbb{E}(p_1) + (1-\alpha)^2 \sum_{k=0}^{\infty} \alpha^k (\mathbb{E}(p_2) + k\mathbb{E}(\check{p}_1) + \mathbb{E}(\check{p}_2)).$$

As $\mathbb{E}(\check{p}_i) = -\mathbb{E}(p_i)$, this reduces to

$$\alpha \mathbb{E}(p_1) + (1-\alpha)^2 \sum_{k=0}^{\infty} \alpha^k (\mathbb{E}(p_2) - k\mathbb{E}(p_1) - \mathbb{E}(p_2)) = \mathbb{E}(p_1)\Big(\alpha - (1-\alpha)^2 \sum_{k=0}^{\infty} k\alpha^k\Big) = 0.$$

Therefore, by [12], the random walk associated to p' is recurrent. Then so is S_n^π. This is a contradiction, concluding the proof. □

5.3 Green functions vanish at infinity

The proof of Theorem 1.4 will essentially follow from Proposition 5.4 and the results of Subsection 5.2, once we establish the decay of the Green function of any admissible transient random walks on non-virtually cyclic groups. This is the main goal of this section.

As in Section 5.1 we consider a probability measure p on a group G, and let $S_n = \xi_1 \cdots \xi_n$ be the corresponding random walk, where ξ_i are i.i.d. random variables distributed as p. When this random walk is transient, the Green function $G(g)$ of the walk is defined as in (5.1), by $G(g) = \sum_{n=0}^\infty p_n(\mathrm{id}_\mathsf{G}, g)$, where $p_n(\mathrm{id}_\mathsf{G}, g) = \mathbb{P}(S_n = g)$. More generally, let $G(g,h) = G(g^{-1}h)$. This is the average time that the walk starting from g spends at h.

The main result of this section is a proof of Theorem 1.3, asserting that if G is not virtually cyclic (i.e., there is no finite index subgroup which is isomorphic to $\mathbb{Z}$) then the Green function $G(g)$ tends to 0 when g tends to infinity. By this, we mean that for any $\epsilon > 0$ there are only finitely many points g with $G(g) \geq \epsilon$, so this notion does not depend of the choice of a distance on the group.

In the proof, we will have to separate the case where G is virtually $\mathbb{Z}^2$. Let us start with this case.

Lemma 5.8. *Assume that G is virtually $\mathbb{Z}^2$, and that the admissible probability measure p on G defines a transient random walk. Then its Green function tends to 0 at infinity.*

Proof. Assume first that $\mathsf{G} = \mathbb{Z}^2$. Then the convergence to 0 at infinity of the Green function is [34, 24.P.5], as p is admissible and therefore aperiodic.

Assume now that G has a finite index subgroup H which is isomorphic to $\mathbb{Z}^2$. Replacing H with the intersection of its (finitely many) conjugates, one can even assume that H is normal in G. The measure p induces a measure on the group G/H, which defines a recurrent walk as G/H is finite. In particular, almost every trajectory of the random walk returns to H. The distribution of this first return is an admissible probability measure p_H on H, to which one can apply the previous result: its Green function tends to 0 at infinity. Moreover, the Green functions of p and p_H coincide on H as the trajectories of the random walk associated to p_H can be obtained from the trajectories of the random walk for p by restricting to the times where the walk is in H. It follows that $G(g)$ tends to 0 when g tends to infinity along H.

By Harnack's inequality [38, 25.1], there exists a constant C such that, for all $g_1, g_2 \in \mathsf{G}$, one has $G(g_1) \leq C^{d(g_1,g_2)} G(g_2)$. This inequality is also easy to prove directly, by concatenating a path from id_{G} to g_1 with a path γ from g_1 to g_2 to obtain $G(g_2) \geq G(g_1) p_\gamma$, where p_γ is the probability to follow γ. Thanks to admissibility, one can choose such a path γ with $p_\gamma \geq c^{d(g_1,g_2)}$ for some $c > 0$, proving Harnack's inequality.

As H has finite index in G, every point of G is within uniformly bounded distance of H. Therefore, the convergence to 0 of the Green function along H extends to the whole group. □

To prove Theorem 1.3, we can therefore assume that G is virtually neither $\mathbb{Z}$ or $\mathbb{Z}^2$. Then all admissible random walks on G, including the simple random walk, are transient, by [38, Theorem 3.24]. For symmetric walks (i.e., such that $p(g^{-1}) = p(g)$), the decay of the Green function at infinity is easy, as shown in the next lemma. The main point of the argument will be to deduce this also for non-symmetric walks by a comparison argument explained below.

Lemma 5.9. *Assume that a symmetric random walk on a group G is transient. Then its Green function tends to 0 at infinity.*

Proof. Let $p_n(g,h)$ denote the probability that the walk starting at g is at position h at time n. Then it is a standard fact that $p_{2n}(\mathrm{id}_{\mathsf{G}}, g)$ is maximal for $g = \mathrm{id}_{\mathsf{G}}$. This is proved using Cauchy-Schwarz inequality and the symmetry of the walk as follows:

$$\begin{aligned}
p_{2n}(\mathrm{id}_{\mathsf{G}}, g) &= \sum_h p_n(\mathrm{id}_{\mathsf{G}}, h) p_n(h,g) \leq \left(\sum_h p_n(\mathrm{id}_{\mathsf{G}}, h)^2 \right)^{1/2} \left(\sum_h p_n(h,g)^2 \right)^{1/2} \\
&= \left(\sum_h p_n(\mathrm{id}_{\mathsf{G}}, h) p_n(h, \mathrm{id}_{\mathsf{G}}) \right)^{1/2} \left(\sum_h p_n(g,h) p_n(h,g) \right)^{1/2} \\
&= p_{2n}(\mathrm{id}_{\mathsf{G}}, \mathrm{id}_{\mathsf{G}})^{1/2} p_{2n}(g,g)^{1/2} = p_{2n}(\mathrm{id}_{\mathsf{G}}, \mathrm{id}_{\mathsf{G}}).
\end{aligned}$$

Conditioning on the position of the walk at time 1, one gets $p_{2n+1}(\mathrm{id}_{\mathsf{G}}, g) \leq p_{2n}(\mathrm{id}_{\mathsf{G}}, \mathrm{id}_{\mathsf{G}})$. Therefore, for any g and any N,

$$\sum_{n=2N}^{\infty} p_n(\mathrm{id}_{\mathsf{G}}, g) \leq 2 \sum_{n=2N}^{\infty} p_n(\mathrm{id}_{\mathsf{G}}, \mathrm{id}_{\mathsf{G}}).$$

The right hand side is the tail of the converging series $\sum_n p_n(\mathrm{id}_{\mathsf{G}}, \mathrm{id}_{\mathsf{G}}) = G(\mathrm{id}_{\mathsf{G}})$. If N is large enough, it is bounded by an arbitrarily small constant ϵ. For each $n < 2N$, the measure $p_n(\mathrm{id}_{\mathsf{G}}, \cdot)$ is a probability measure on G. Hence, there are only finitely many points g for which $p_n(\mathrm{id}_{\mathsf{G}}, g) > \epsilon/(2N)$. Summing over $n \leq 2N$, it follows that for all but finitely many points one has $\sum_{n<2N} p_n(\mathrm{id}_{\mathsf{G}}, g) \leq \epsilon$, and therefore $G(g) = \sum_n p_n(\mathrm{id}_{\mathsf{G}}, g) \leq 2\epsilon$ for all but finitely many points. □

The next lemma is the main step of the proof of Theorem 1.3.

Lemma 5.10. *Consider a finitely generated group G, with a finite symmetric generating set S. Assume that the simple random walk on G (driven by the uniform measure on S) is transient. Let $c > 0$. Consider a probability measure p on G with $p(s) \geq c$ for all $s \in S$. Then its Green function tends to 0 at infinity.*

When the simple random walk is transient, all the admissible random walks on G are transient, by [38, Theorem 3.24]. Hence, under the assumptions of the lemma, the Green function associating to p is finite, and it makes sense to ask if it tends to zero at infinity.

Proof. The proof will rely on a classical comparison lemma, making it possible to relate general random walks to symmetric ones. Denote by G the Green function associated to p, and by G_S the Green function associated to the simple random walk. Under the assumptions of the lemma, [36, Proposition, Page 251] ensures that for any nonnegative square-integrable function f on G,

$$\sum_{g,h\in\mathsf{G}} f(g)G(g,h)f(h) \leq c^{-1} \sum_{g,h\in\mathsf{G}} f(g)G_S(g,h)f(h). \tag{5.4}$$

Let $F(g, h)$ be the probability to reach h starting from g. By Lemma 5.1, we have $F(g, h) = qG(g, h)$ for a fixed $q > 0$, independent of g and h.

Let $\Omega = \mathsf{G}^{\mathbb{N}}$ be the space of all possible trajectories, endowed with the probability measure $\mathbb{P}$ corresponding to the distribution of the random walk given by p starting from id_{G}. A cylinder set is a set of the form

$$[g_0, \dots, g_n] = \{\omega \in \Omega \mid \omega_0 = g_0, \cdots, \omega_n = g_n\} \subseteq \Omega.$$

Assume by contradiction that $G(g)$ does not tend to 0 at infinity. Then one can find $\epsilon > 0$ and an infinite set $I \subseteq \mathsf{G}$ on which $F(\mathrm{id}_{\mathsf{G}}, g) = qG(g) > \epsilon$. Let $M > 0$ be large enough (how large will be specified at the end of the argument). We define a sequence h_n of elements of I as follows.

- First, take $h_0 = \mathrm{id}_{\mathsf{G}}$. Let also $T_0 = [\mathrm{id}_{\mathsf{G}}] \subseteq \Omega$ be the set of all trajectories starting from id_{G}, and $R_0 = \{\mathrm{id}_{\mathsf{G}}\}$.
- Then, take an $h_1 \in I$ at distance at least M of h_0. As the probability to reach h_1 starting from id_{G} is $> \epsilon$ by definition of I, one can find (by throwing away very long trajectories or very unlikely trajectories in finite time) a finite number of cylinder sets starting at id_{G} and ending at h_1 with total probability $> \epsilon$. Denote this set of trajectories by T_1, with $\mathbb{P}(T_1) > \epsilon$. Let R_1 be the set of points that trajectories in T_1 reach before h_1. As T_1 is a finite union of cylinders, R_1 is finite.

- Then, take an $h_2 \in I$ at distance at least M of h_0 and h_1. We can also require that it does not belong to $R_0 \cup R_1$, as this set is finite while I is infinite. As above, we can then define a set T_2 which is a finite union of cylinders ending at h_2 with $\mathbb{P}(T_2) > \epsilon$, and R_2 the finite set of points reached by these trajectories before h_2.
- The construction goes on inductively to define h_n.

The point of the previous construction is that for $i \leq j$ we have the inequality

$$\mathbb{P}(T_i \cap T_j) \leq F(h_i, h_j). \tag{5.5}$$

Indeed, this is clear for $i = j$. For $i < j$, note that trajectories in $T_i \cap T_j$ reach h_i and then h_j, in this order as $h_j \notin R_i$. Therefore,

$$\mathbb{P}(T_i \cap T_j) \leq \mathbb{P}(\exists n < m, S_n = h_i \text{ and } S_m = h_j) = F(\mathrm{id}_{\mathsf{G}}, h_i) \cdot F(h_i, h_j) \leq F(h_i, h_j),$$

where the central equality follows from the Markov property. This proves (5.5).

Let us now take N large, and apply the inequality (5.4) to the characteristic function of $\{h_0, \cdots, h_{N-1}\}$. We obtain

$$\sum_{i,j<N} G(h_i, h_j) \leq c^{-1} \sum_{i,j<N} G_S(h_i, h_j). \tag{5.6}$$

We will bound the left hand side from below and the right hand side from above to get a contradiction. Thanks to (5.5), we have

$$\begin{aligned}
\sum_{i,j<N} G(h_i, h_j) &\geq \sum_{i\leq j<N} G(h_i, h_j) = q^{-1} \sum_{i\leq j<N} F(h_i, h_j) \\
&\geq q^{-1} \sum_{i\leq j<N} \mathbb{P}(T_i \cap T_j) \\
&\geq \frac{q^{-1}}{2} \sum_{i,j<N} \mathbb{P}(T_i \cap T_j) = \frac{q^{-1}}{2} \int \left(\sum_{i<N} 1_{T_i} \right)^2 \mathrm{d}\mathbb{P} \\
&\geq \frac{q^{-1}}{2} \left(\int \sum_{i<N} 1_{T_i} \mathrm{d}\mathbb{P} \right)^2 \\
&= \frac{q^{-1}}{2} \left(\sum_{i<N} \mathbb{P}(T_i) \right)^2 \geq \frac{q^{-1}}{2} \epsilon^2 N^2.
\end{aligned}$$

The Green function G_S tends to 0 at infinity, by Lemma 5.9. As the distance between h_i and h_j is at least M for $i \neq j$ by construction, it follows that $G_S(h_i, h_j) \leq \eta(M)$ where η tends to 0 with M. We obtain

$$\sum_{i,j<N} G_S(h_i, h_j) \leq \sum_{i=j} G_S(h_i, h_j) + \sum_{i\neq j} G_S(h_i, h_j) \leq N G_S(\mathrm{id}_{\mathsf{G}}) + N^2 \eta(M).$$

Combining these two estimates with (5.6) yields

$$\frac{q^{-1}}{2} \epsilon^2 N^2 \leq c^{-1} N G_S(\mathrm{id}_{\mathsf{G}}) + c^{-1} N^2 \eta(M).$$

We obtain a contradiction by taking M large enough so that $c^{-1}\eta(M) < q^{-1}\epsilon^2/2$, and then letting N tend to infinity. □

Proof of Theorem 1.3. The result follows from Lemma 5.8 if G is virtually $\mathbb{Z}^2$. Hence, we can assume that this is not the case, and therefore that the simple random walk on G is transient by [38, Theorem 3.24].

The Green functions for the probability measures p and $(p+\delta_{\mathrm{id}_\mathsf{G}})/2$ are related by the identity $G_{(p+\delta_{\mathrm{id}_\mathsf{G}})/2}(g) = 2G_p(g)$, by [38, Lemma 9.2]. Without loss of generality, we can therefore replace p with $(p+\delta_{\mathrm{id}_\mathsf{G}})/2$ and assume $p(\mathrm{id}_\mathsf{G}) > 0$. As p is admissible, it follows that there exists N with $p_N(\mathrm{id}_\mathsf{G}, s) > 0$ for all s in the generating set S. By Lemma 5.10, the Green function associated to p_N, denoted by G_N, tends to 0 at infinity.

To compute $G(g)$, split the arrival times to g according to their values modulo N. For times of the form $i + kN$, such arrivals can be realized by following p for i steps, and then p_N for k steps. It follows that

$$G(g) = \sum_{h\in\mathsf{G}} \sum_{i<N} p_i(\mathrm{id}_\mathsf{G}, h) G_N(h^{-1}g).$$

Let $\epsilon > 0$. Take a finite set $F \subset \mathsf{G}$ such that $\sum_{h\notin F}\sum_{i<N} p_i(\mathrm{id}_\mathsf{G}, h) < \epsilon$. Then

$$G(g) \le \sum_{h\in F} \sum_{i<N} p_i(\mathrm{id}_\mathsf{G}, h) G_N(h^{-1}g) + \epsilon \|G_N\|_{L^\infty}.$$

When g tends to infinity, the first term tends to 0 as this is a finite sum and G_N tends to 0 at infinity. For large enough g, we get $G(g) \le 2\epsilon\|G_N\|_{L^\infty}$. □

We are now ready to complete the proof of Theorem 1.4.

Proof of Theorem 1.4. For groups which are not virtually $\mathbb{Z}$, the result follows directly from Proposition 5.4 and Theorem 1.3 showing that the Green functions G and $G^{(-)}$ tend to 0 at infinity. Groups which are virtually $\mathbb{Z}$ but not $\mathbb{Z}$ are handled in Proposition 5.7. Finally, the case of $\mathbb{Z}$ is done in Proposition 5.6. □

A Flatto's inequality enhancement procedure

Proof of Theorem A. Assume that $\mathbf{A}(\delta)$ holds for $|\partial R_n|$. Fix $\epsilon_0 > 0$ and denote by $\kappa > 0$ the unique constant, see Proposition 3.4, so that

$$\mathbb{E}\left(|\partial R_n|\right) \sim \frac{\kappa n}{\log^2(n)} \text{ as } n \to \infty.$$

For $n \in \mathbb{N}$ we will write $N = N(n) = \left\lfloor \log^{\delta/3}(n) \right\rfloor$. For $1 \le i \le N$, write $n_i = \lfloor ni/N \rfloor$ and divide the range R_n into N-blocks,

$$X_{n,i} := \left\{S_{n_{i-1}+1}, S_{n_{i-1}+2}, \dots, S_{n_i}\right\}, \quad 1 \le i \le N.$$

As before

$$\partial_v X_{n,i} := X_{n,i} \setminus \{X_{n,i} + v\}, \qquad \partial X_{n,i} = \bigcup_{v\in\mathbb{E}_d} \partial_v X_{n,i}.$$

Clearly

$$|\partial R_n| \le \sum_{i=1}^N |\partial X_{n,i}|.$$

Let $\epsilon \ge \epsilon_0$ and set

$$A_i := \left\{\omega \in \Omega : \ |\partial X_{n,i}| \ge \left(1 + \frac{\epsilon}{2}\right) \frac{\kappa n}{N \log^2(n)}\right\}$$

and

$$B_i := \left\{\omega \in \Omega : \ |\partial X_{n,i}| \geq \left(1 + \frac{\epsilon N}{2}\right) \frac{\kappa n}{N \log^2(n)}\right\},$$

for $1 \leq i \leq N$. Then a simple combinatorial argument (see equation (4.9) in [19]) gives,

$$\left[|\partial R_n| > (1+\epsilon)\frac{\kappa n}{\log^2(n)}\right] \subset \left(\bigcup_{1 \leq i < j \leq N} A_i \cap A_j\right) \cup \left(\bigcup_{i=1}^{N} B_i\right) \tag{A.1}$$

First we estimate $\mathbb{P}(A_i)$, $\mathbb{P}(B_i)$ from above. Writing $m_i = m_i(n) = n_i - n_{i-1}$ for $1 \leq i \leq N$, $|\partial X_{n,i}|$ is equal in distribution to $|\partial R(m_i)|$. In addition,

$$\lim_{n\to\infty} \frac{m_i/\log^2(m_i)}{n/(N\log^2(n))} = 1$$

uniformly in $1 \leq i \leq N$. In addition, by (3.7), since $m_i \to \infty$ as $n \to \infty$,

$$\lim_{n\to\infty} \frac{\mathbb{E}(|\partial R(m_i)|)}{\kappa m_i/\log^2(m_i)} = 1.$$

Consequently there exists $\mathbf{n} = \mathbf{n}(\epsilon_0)$, such that for all $\epsilon > \epsilon_0$ and $n > \mathbf{n}$,

$$A_i \subset \left[|\partial X_{n,i}| \geq \left(1 + \frac{\epsilon}{3}\right) \frac{\kappa m_i}{\log^2(m_i)}\right] \subset \left[|\partial X_{n,i}| \geq \left(1 + \frac{\epsilon}{4}\right) \mathbb{E}(|\partial R(m_i)|)\right]. \tag{A.2}$$

We deduce that for all $\epsilon > \epsilon_0$, $n > \mathbf{n}$ and $1 \leq i \leq N$,

$$\begin{aligned}
\mathbb{P}(A_i) &\leq \mathbb{P}\left(|\partial X_{n,i}| > \left(1 + \frac{\epsilon}{4}\right) \mathbb{E}(|\partial R(m_i)|)\right) \\
&= \mathbb{P}\left(|\partial R(m_i)| > \left(1 + \frac{\epsilon}{4}\right) \mathbb{E}(|\partial R(m_i)|)\right), \quad \text{by property } \mathbf{A}(\delta) \\
&\leq \frac{16C(\epsilon_0/4, \delta)}{\epsilon^2 \log^\delta(m_i)}.
\end{aligned}$$

Finally, as $\log^\delta(m_i) \sim \log^\delta(n)$ as $n \to \infty$, we can enlarge $\mathbf{n}$ such that for all $\epsilon > \epsilon_0$, and $n > \mathbf{n}$,

$$\mathbb{P}(A_i) \leq \frac{32C(\epsilon_0/4, \delta)}{\epsilon^2 \log^\delta(n)} \leq \frac{32C(\epsilon_0/4, \delta)}{\epsilon_0 \epsilon \log^\delta(n)} \tag{A.3}$$

To bound $\mathbb{P}(B_i)$ from above notice that by similar considerations as in (A.2), for all $\epsilon > \epsilon_0$ and $n > \mathbf{n}$,

$$B_i \subset \left[|\partial X_{n,i}| \geq \left(1 + \frac{\epsilon N}{4}\right) \mathbb{E}(|\partial R(m_i)|)\right],$$

consequently for all $1 \leq i \leq N$,

$$\mathbb{P}(B_i) \leq \mathbb{P}\left(|\partial R_{m_i}| \geq \left(1 + \frac{\epsilon N}{4}\right) \mathbb{E}(|\partial R_{m_i}|)\right) \leq \frac{16C(\epsilon_0/4, \delta)}{\epsilon^2 N^2 \log^\delta(m_i)}.$$

Now as $N \sim \log^{\delta/3}(n)$ as $n \to \infty$, by enlarging $\mathbf{n}$ if needed, we can assume that for all $n > \mathbf{n}$ uniformly on $1 \leq i \leq N$ and $\epsilon > \epsilon_0$,

$$\mathbb{P}(B_i) \leq \frac{32C(\epsilon_0/4, \delta)}{\epsilon^2 N \log^{4\delta/3}(n)}. \tag{A.4}$$

Since for $1 \le i < j \le N$, the events A_i and A_j are independent it follows from (A.1), (A.3) and (A.4) that for all $\epsilon > \epsilon_0$ and $n > \mathbf{n}$,

$$\begin{aligned}\mathbb{P}\left(|\partial R_n| > (1+\epsilon)\mathbb{E}\left(|\partial R_n|\right)\right) &\le \sum_{1\le i<j\le N} \mathbb{P}(A_i)\mathbb{P}(A_j) + \sum_{k=1}^{N} \mathbb{P}(B_k) \\ &\le \left(\frac{32C(\epsilon_0/4,\delta)}{\epsilon_0}\right)^2 \frac{N^2}{\epsilon^2 \log^{2\delta}(n)} + \frac{32C(\epsilon_0/4,\delta)}{\epsilon^2 \log^{4\delta/3}(n)} \\ &\sim \left(\left(\frac{32C(\epsilon_0/4,\delta)}{\epsilon_0}\right)^2 + 32C(\epsilon_0/4,\delta)\right) \frac{1}{\epsilon^2 \log^{4\delta/3}(n)}\end{aligned}$$

as $n \to \infty$. It follows that there exists $C(\epsilon_0, 4\delta/3)$ such that for all $\epsilon \ge \epsilon_0$ and $n \ge 2$

$$\mathbb{P}\left(|\partial R_n| > (1+\epsilon)\mathbb{E}\left(|\partial R_n|\right)\right) \le \frac{C(\epsilon_0, 4\delta/3)}{\epsilon^2 \log^{4\delta/3}(n)}.$$

As ϵ_0 is arbitrary this concludes the proof of Theorem A. □

Proof of Theorem B. The proof essentially follows Sections 5,6 in [19], which is quite lengthy and technical. Rather than reproducing the full argument we opt instead to point out the necessary modifications for the statements of the relevant results in [19] to apply to the objects of interest in our case. Divide the random walk path $S = \{S_j; 1 \le j \le n\}$ into $N = [n/L] + 1$ blocks of size $L = [n/\log\log n]$, where $[x]$ denotes the integer part of x. We will write $S^{(i)}$ for the i-th block and $R_n^{(i)}$ for its range. That is $R_n^{(i)} = \{S_k^{(i)} : k = 0, L-1\}$, with $S_k^{(i)} = S_{iL+k}$ for $k = 0, \dots, L-1$.

One may follow [19, Section 6] replacing the objects T_n^p by $|\partial_V R_n|$, T_i by $\left|\partial_V R_n^{(i)}\right|$. Finally the object T_{ij} must be replaced by

$$T_{i,j} := \left|\left\{x \in \partial_V R_n^{(i)} : R_n^{(j)} \cap (x + (V \cup \{0\})) \ne \emptyset\right\}\right|.$$

With this definition we have

$$|\partial_V R_n| \ge \sum_{i=1}^{N} \left|\partial_V R_n^{(i)}\right| - \sum_{1\le i<j\le N} T_{i,j},$$

and given the analogous version of [19, Theorems 5.1, 5.2] one can proceed verbatim as in [19, Section 6].

We will now briefly explain how one can adapt the proof of [19, Theorems 5.1] for the $T_{i,j}$ written above. This is possible by replacing the events $A(i,j;\mu,\mu;x)$ defined therein by the events

$$\begin{aligned}A'(i,j;\mu,\mu';x) := \{S_\mu^{(i)} = x\} &\cap \left\{S_\mu^{(i)} - S_l^{(i)} \notin V,\, 0 \le l \le \mu\right\} \\ &\cap \left\{S_l^{(i)} - S_\mu^{(i)} \notin \{0\} \cup V,\, \mu < l \le L-1\right\} \cap \left\{S_l^{(j)} - S_\mu^{(i)} \notin \{0\} \cup V,\, 0 < l \le \mu'\right\} \\ &\cap \left\{S_{\mu'}^{(j)} - S_\mu^{(i)} \in \{0\} \cup V\right\}.\end{aligned}$$

For a collection of distinct positive integers $i_1, \dots, i_{2m}$, $1 \le \mu_1, \dots, \mu_{2m} \le L$ and $x_1, \dots, x_m \in \mathbb{Z}^2$ let σ be the unique permutation of $\{1, \dots, 2m\}$ such that letting $j_k := i_{\sigma(k)}$ we have $j_1 < \dots < j_{2m}$, i.e. the unique increasing re-arragement of $i_1, \dots, i_{2m}$. Let $\nu_k = \mu_{\sigma(k)}$ and $z_k := x_{[(\sigma(k)+1)/2]}$ and

$$\Delta := \left\{(y_i)_{i=1}^{2m} : \ \forall i \in \{1, \dots, 2m\}, y_i - z_i \in V \cup \{0\}\right\}.$$

Then, again following [19, Theorem 5.1] we have

$$\bigcap_{k=1}^{m} A(i_{2k-1}, i_{2k}; \mu_{2k-1}, \mu_{2k}; x_k)$$
$$\subseteq \bigcup_{(y_i)_{i=1}^{2m} \in \Delta} \left(B(\nu_1; y_1) \cap \bigcap_{j=2}^{2m} B(j-1; \nu_{j-1}, \nu_j; y_{j-1}, y_j) \cap B'(\nu_{2m}; y_{2m}) \right)$$

where the events

$$B(\nu; y) := \{S_\nu^{(j_1)} = y\} \cap \left\{ S_l^{(j_1)} \notin y + V,\ 0 \le l < \nu \right\}$$
$$B(k; \nu_k, \nu_{k+1}; y_k, y_{k+1}) := \left\{ S_{\nu_k}^{(j_k)} = y_k,\ \text{and } S_l^{(j_k)} \notin y_k + (V \cup \{0\}),\ \nu_k < l \le L - \frac{L - \nu_k}{2} \right\}$$
$$\cap \{S_{\nu_{k+1}}^{(j_{k+1})} = y_{k+1}\} \cap \left\{ S_l^{(j_{k+1})} \notin y_{k+1} - V,\ \frac{\nu_{k+1}}{2} \le l < \nu_{k+1} \right\},$$
$$B'(\nu; y) := \left\{ S_l^{(j_{2m})} - S_\nu^{(j_{2m})} \notin V \cup \{0\} \right\} \cap \left\{ S_\nu^{(j_{2m})} = y \right\},$$

are independent since they depend on disjoint blocks of random variables. The calculations then are similar to [19], subject to routine modifications to [19, Lemma 5.1]. Indeed, writing $s_k = (j_{k+1} - j_k)\, L + \nu' - \nu$, and using [19, Lemma 5.1], it is straightforward to prove that there exists $C > 0$ such that for all $1 \le k \le 2m - 1$, writing $M_k = s_k - [(L-\nu)/2] - [(\nu'-1)/2]$,

$$\mathbb{P}\left(B(k; \nu, \nu'; y_k, y_{k+1})\right) \le \frac{C}{M_k} \frac{1}{\log(L-\nu)\log(\nu')}$$

e.g. proving (5.1) of Flatto. Since $j_{k+1} > j_k$ then,

$$M \ge L + \nu' - \nu - \frac{L - \nu}{2} - \frac{\nu'}{2} \ge \frac{L + (\nu' - \nu)}{2},$$

then

$$\mathbb{P}\left(B(k; \nu, \nu'; y_k, y_{k+1})\right) \le \frac{2}{(L + \nu' - \nu)\log(L-\nu)\log(\nu')}. \tag{A.5}$$

We get an extra $\Delta_m| = (|V| + 1)^2 m$ constant from the summation on Δ_m.

One can similarly modify the proof of [19, Theorem 5.2]. □

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