Electronic Journal of Probability

Electron. J. Probab. **26** (2021), article no. 90, 1–41.
ISSN: 1083-6489 https://doi.org/10.1214/21-EJP658

American options in nonlinear markets*

Edward Kim[†] Tianyang Nie[‡] Marek Rutkowski[§]

Abstract

We study unilateral valuation problems for American options within the framework of a general nonlinear market by extending results from Bielecki et al. [9, 12] who examined contracts of European style. A BSDE approach is used to establish more explicit pricing, hedging and exercising results when solutions to reflected BSDEs have additional desirable properties. We employ for this purpose results on solutions to BSDEs and reflected BSDEs driven by RCLL martingales obtained by Nie and Rutkowski [62, 63].

Keywords: nonlinear market; American option; nonlinear evaluation; reflected BSDE.
MSC2020 subject classifications: 91G40; 60J28; 91G30; 60H30; 60H10.
Submitted to EJP on March 9, 2020, final version accepted on June 7, 2021.

1 Introduction

Contracts of American style are asymmetric between the two counterparties, commonly referred to as the *issuer* and the *holder*, not only due to the opposite directions of contractual cash flows, but also due to the fact that only one party, the holder, has the right to *exercise* an American contract before its expiration date. The arbitrage-free pricing and rational exercising of American options within the framework of a linear market model have been studied in numerous papers, to mention just a few: Bensoussan [5], Bouchard and Nam [13], El Karoui et al. [30], Jaillet et al. [39], Karatzas [42], Karatzas and Kou [43], Kallsen and Kühn [41], Mulinacci and Pratelli [56], Myneni [57] and Szimayer [70]. The authors of these papers usually adopted the classical Black and Scholes setup, though it was sometimes complemented by specific trading constraints.

*The research of T. Nie was supported by the National Natural Science Foundation of China (Nos. 12022108, 11971267, 11831010, 61961160732), Natural Science Foundation of Shandong Provincial (No. ZR2019ZD42), China Postdoctoral Science Foundation (Nos. 2018M640620 and 2019T120580) and Qilu Young Scholars Program and Distinguished Young Scholars Program of Shandong University. The research of M. Rutkowski was supported by the Australian Research Council Discovery Project DP200101550.

[†]School of Mathematics and Statistics, University of Sydney, Australia. E-mail: edward.kim@sydney.edu.au

[‡]Corresponding author. School of Mathematics, Shandong University, Jinan, China. E-mail: nietianyang@sdu.edu.cn

[§]School of Mathematics and Statistics, University of Sydney, Australia and Faculty of Mathematics and Information Science, Warsaw University of Technology, Poland. E-mail: marek.rutkowski@sydney.edu.au

The goal of this work is to re-examine and extend the findings from the recent paper by Dumitrescu et al. [26] who applied the nonlinear pricing approach developed in El Karoui and Quenez [32]. In contrast to [26] where a particular model with a single jump of the underlying asset was studied, we place ourselves within the setup of a general nonlinear arbitrage-free market with possibly discontinuous asset prices, as introduced in Bielecki et al. [9, 12] and we examine *unilateral acceptable prices* for American contracts. We obtain results regarding the pricing, hedging, break-even times and rational exercise times using results on backward stochastic differential equations (BSDEs) from Nie and Rutkowski [62, 63], but without explicitly specifying the dynamics of underlying risky assets and funding accounts. We focus instead on unilateral nonlinear evaluations generated by BSDEs associated with the issuer's and holder's wealth processes and thus our results are model-free. Consequently, they can be applied to American options in a wide spectrum of nonlinear (and, obviously, also linear) market models.

In the aftermath of the global financial crisis of 2007-2009, there was a rapidly growing interest in financial models accounting for the counterparty credit risk, collateralization, differential funding costs and other trading adjustments; see, e.g., Bichuch et al. [7, 8], Brigo and Pallavicini [14], Burgard and Kjaer [15], Capponi [16], Crépey [18, 19], Crépey et al. [20], Pallavicini et al. [64], and Piterbarg [67]. Due to abovementioned intricacies of trading, the problem of risk mitigation through hedging of a financial contract is no longer as straightforward as it was in the past. Therefore, complex market models including risky assets, multiple funding accounts and dedicated margin accounts for collateral either pledged or accepted need to be studied in the nonlinear framework, where the nonlinearity arises due to the differential borrowing and lending interest rates, idiosyncratic funding costs for risky assets via secured accounts driven by the repo rates, netting of portfolio positions and asymmetric remuneration of margin accounts, and possibly also an endogenous specification of collateral amount. We give some explicit examples of nonlinear markets in Section 2, to wit, the model with a partial netting presented in Subsection 2.3.1 and the model with idiosyncratic funding costs and collateral described in Subsection 2.3.2. These examples are merely special cases of a generic nonlinear market, which is formally introduced in Subsection 2.3.3 and studied throughout the present paper.

In fact, the case of a financial market with a nonlinear trading has attracted attention of researchers since mid-1990s. First, Bergman [6] and Korn [50] investigated the range of arbitrage-free prices for European options under different lending and borrowing rates and, more recently, Mercurio [55] extended Bergman's results by examining the pricing of collateralized European options. In the context of counterparty credit risk, Crépey [18, 19] and Crépey et al. [20] analyzed the pricing and hedging of the CVA (Credit Valuation Adjustment) term of the price for European options under funding constraints through nonlinear BSDEs and quasi-linear PDEs. A more systematic study was undertaken by Bielecki and Rutkowski [12] (see also the follow-up work by Bielecki et al. [9]) who introduced a generic nonlinear trading model for collateralized contracts and attempted to develop a unified framework for the nonlinear approach to hedging and pricing of over-the-counter (OTC) financial contracts in the spirit of the seminal work by El Karoui and Quenez [32].

It is expected that, due to the nonlinearity of the market model, the issuer's and holder's unilateral prices are likely to diverge. The inequalities satisfied by unilateral prices and the range fair bilateral prices were studied in papers by Nie and Rutkowski [60, 61] for models with either an exogenous or endogenous collateralization, respectively. More recently, Bichuch et al. [7, 8] (see also Lee et al. [52] and Lee and Zhou [53] for related studies) explicitly addressed the issue of hedging the counterparty credit risk and analyze the CVA for European claims in the Black-Scholes model complemented by defaultable bonds issued by the counterparties and they also examined bounds for fair bilateral prices. We stress that the above-mentioned papers are mainly concerned with prices of contingent claims of a European style and thus it is natural to ask analogous questions regarding American contingent claims in a nonlinear market model with idiosyncratic funding costs, counterparty credit risk and other market frictions affecting the trading mechanism.

In this paper, we will provide a thorough study of unilateral valuation problems for American options within the framework of a general nonlinear market by extending results from Bielecki et al. [9, 12] who examined European style contracts. We will also use a BSDE approach to establish more explicit pricing, hedging and exercising results. Let us point out that there are some interesting and challenging features of arbitrage-free pricing of financial derivatives within the framework of a generic nonlinear market and thus, in the first step, one needs to properly define the concepts of a fair price and replication of an American contract in a nonlinear setup.

In particular, we carefully distinguish between different kinds of prices and hedging costs, including the concepts of a fair price, superhedging cost, strict superhedging cost, replication cost and, finally, a unilaterally acceptable price. It is interesting to examine whether they coincide or certain additional assumptions about trading need be added to ensure the consistency of pricing. In Subsections 2.2 and 3.1, we show that maximum fair price, minimum superhedging costs, minimum strict superhedging cost, minimum replication cost and acceptable price coincide in an abstract nonlinear market under fairly mild assumptions about the dynamics of wealth processes of trading strategies.

To cover also the counterparty credit risk, we study in Section 4 the case of an American contract with extraneous risks. In Example 4.5, we show how to apply our results directly when a vulnerable contract pays a predetermined recovery at the counterparty's default. Example 4.6 demonstrates that our generic nonlinear model can also be applied to general American contracts with extraneous risks when the recovery payoff at time of default does not depend on the jump of the reference defaultable asset. In particular, the model studied by Szimayer [70] can be seen as a very special case of our generic market model with extraneous risks.

Notice also in this regard that Dumitrescu et al. [25, 26] also consider American and game options with default risk but their study focuses on contracts subject to the third-party credit risk, as opposed to the counterparty credit risk. In Remark 4.7, we mention that another justification for the study of wealth processes driven by a general RCLL martingale comes from the fact that, typically, several dependent defaults are present in the market (see [11]). Finally, we note that in the case of a diffusion-type model (see, e.g., Example 2.15), one may apply a PDE approach to identify the issuer's and holder's acceptable prices through solutions to parabolic nonlinear PDEs with obstacle and thus to extend the classical PDE approach to pricing of American options (see Remark 2.24).

Due to diverging funding costs and asymmetry of collateralization, a nonempty interval of *bilaterally profitable prices* may arise when the ask price set by the issuer is below the bid price computed by a potential buyer of an American option. Therefore, the obvious feature that the two parties need to address their respective valuation and hedging problems unilaterally and, typically, using two different proprietary models with idiosyncratic funding costs, does not necessarily mean that it will be impossible to enter into a bilaterally profitable trade. On the contrary, the ubiquity of the over-the-counter market may support our view that the theory of unilateral pricing could provide a sound theoretical foundation for the existence bilaterally beneficial trades in imperfect markets. For more comments on this issue in the context of European contracts, the reader is referred to Section 3.2 in [61] (see also Propositions 4.6 and 4.13 in [60] for a detailed study of collateralized contracts).

In practice, it is conceivable to enter into a contract in which each party is exposed to a potential future loss in adverse market circumstances, provided that its size and real-world probability of occurrence are deemed to be bearable. Therefore, the concept of a unilaterally acceptable price should not be seen as a stringent constraint, which cannot be violated, but rather as a preliminary step to real-world pricing where mathematical models are blended with expert opinions, trading experience and, last but not least, trader's speculative anticipations based on his constantly updated information about the currently prevailing market sentiment and future economic perspectives.

The structure of the paper is as follows. In Sections 2 and 3, we re-examine and extend a BSDE approach to the valuation of American options in nonlinear market initiated by El Karoui and Quenez [32] and continued in a recent paper by Dumitrescu et al. [26]. We first work in an abstract nonlinear setup, meaning that we only make fairly general assumptions about the nonlinear dynamics of the wealth process of self-financing strategies. The main postulate of that kind is the strict monotonicity property of the wealth process (see Assumption 2.3). We examine general properties of unilateral superhedging costs and acceptable prices for the two counterparties, the issuer and the holder. In particular, we built upon papers by Bielecki et al. [9, 12] where the arbitrage-free valuation of European contingent claims in nonlinear markets was examined. Since the proofs of all results in Sections 2.2 and 3.1 are elementary, they are not provided here and thus the interested reader is referred to Kim et al. [44] for detailed demonstrations and further comments. Our main goal is to show that *unilateral acceptable prices* for an American contract $\mathcal{C}^a$, which are introduced in Definitions 2.11 and 3.9, can be characterized in terms of solutions to reflected BSDEs driven by a multi-dimensional (and possibly discontinuous) semimartingale S. We also study the *break-even times* for the issuer and the *rational exercise times* for the holder, as given by Definitions 2.9 and 3.20, respectively. For the sake of generality, we do not focus on any particular market model, but instead we study a generic nonlinear market under mild assumptions. For related results on BSDEs and reflected BSDEs driven by multi-dimensional RCLL martingales, the reader is referred to Nie and Rutkowski [62, 63]. In Section 4, we study American contracts with extraneous risks and thus we also cover vulnerable American options with the counterparty credit risk. It should be acknowledged that we do not cover an important issue of a model risk and thus it is worth to mention that, to the best of our knowledge, the existing literature in the very active field of robust finance is still largely limited to the case of classical linear models, although some authors cover special kinds of trading constraints such as, e.g., restrictions on short-selling of shares and/or borrowing of cash (see, e.g., Aksamit et al. [1] or Bayraktar and Zhou [4]). Some useful results on the nonlinear optimal stopping are collected in the appendix.

2 Issuer's pricing and hedging problems

The goal of this section is to re-examine and extend a BSDE approach to the valuation of American options in nonlinear market, which was initiated by El Karoui and Quenez [32] for models driven by a Brownian motion and recently extended by Dumitrescu et al. [26] to a nonlinear model with a single jump. Our main goal is to show that the *issuer's acceptable price* for an American contract $\mathcal{C}^a$ is unique and can be characterized in terms of solutions to reflected BSDEs driven by an n-dimensional, RCLL martingale.

2.1 Setup and notation

We first describe the setup and notation used throughout the paper. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a probability space where the filtration $\mathbb{F}$ satisfies the usual conditions of right-continuity and $\mathbb{P}$-completeness, and the initial σ-field $\mathcal{F}_0$ is trivial. All processes introduced in what follows are assumed to be $\mathbb{F}$-adapted and we denote by $\mathcal{T} = \mathcal{T}_{[0,T]}$ for the class of all $\mathbb{F}$-stopping times taking values in $[0,T]$. By convention, the contractual cash flows are given from the issuer's perspective. Hence when a cash flow is positive, then the cash amount is paid by the holder and received by the issuer and, if a cash flow is negative, then the cash amount is transferred from the issuer to the holder. For instance, when dealing with an American put option written on the stock S, we assume that the cash flow to the issuer (resp. the holder) equals $X^h_\tau = -(K-S_\tau)^+$ (resp. $X^h_\tau = (K-S_\tau)^+$) if the option is exercised at time $\tau \in \mathcal{T}$ by the holder. This is formalized through the following abstract definition where the superscript h in X^h is used to stress that only the holder has the right to exercise the contract (this should be contrasted with the case of game option studied in [45]) but no assumption about the sign of the payoff X^h is made.

Definition 2.1. An *American contingent claim* with expiration date T and the $\mathbb{F}$-adapted, RCLL payoff process X^h is a contract between the issuer and holder in which the latter has the right to exercise the contract by selecting an $\mathbb{F}$-stopping time $\tau \in \mathcal{T}$. Then, at time τ, the issuer gets the amount X^h_τ or, equivalently, transfers to the holder the amount of $-X^h_\tau$ where the $\mathbb{F}$-adapted payoff process X^h_t, $t \in [0,T]$ is specified by the contract.

More generally, an *American contract* is defined as a triplet $\mathcal{C}^a = (A, X^h, \mathcal{T})$ where an $\mathbb{F}$-adapted, RCLL process A, which is predetermined in the contract, represents the *cumulative cash flows* from time 0 till the contract's maturity date T. In the financial interpretation, the process A represents all external cash flows of a given American contract, which are either paid out from or added to the issuer's wealth via the value process of his portfolio of traded assets. By symmetry, an analogous interpretation applies to the holder of an American contract and, obviously, any amount received (resp. paid) by one of the parties is paid (resp. received) by the counterparty. We stress that the price of the contract $\mathcal{C}^a$, which is exchanged at its initiation (by convention, at time 0), is not included in the process A so that we set $A_0 = 0$. This choice is motivated by the fact that the contract's price before the trade is yet unspecified and it needs to be determined through negotiations between the counterparties. We will argue that unilateral pricing is feasible but it is unlikely to yield a single value for the initial price of an American contract in a nonlinear framework.

When examining the valuation of an American contract at any time $t \in [0,T]$, we implicitly assume that it has not yet been exercised and thus the set of exercise times available at time t to its current holder is the class $\mathcal{T}_{[t,T]}$ of all $\mathbb{F}$-stopping times taking values in $[t,T]$. In principle, one could consider two alternative conventions regarding the payoff upon exercise: either (a) the cash flow upon exercise at time t equals $A_t - A_{t-} + X^h_t$ or (b) if a contract is exercised at time t, then the cash flow $A_t - A_{t-}$ is waived, so the only cash flow occurring at time t is X^h_t. Unless explicitly stated otherwise, we work under covenant (a) and we acknowledge that the choice of a particular settlement rule may result in a different value for the price of an American contract $\mathcal{C}^a$, in general. Of course, this choice is immaterial when the process A is continuous or, simply, when it vanishes, so that the contract reduces to a pair $(X^h, \mathcal{T})$.

An important original feature of the nonlinear arbitrage-free pricing is the concept of the *benchmark wealth* $\bar{V}(x)$ (also known as the *legacy portfolio*) with respect to which arbitrage opportunities of a trader are quantified and assessed. As in [9, 12], as an easily manageable candidate for the benchmark wealth we may propose the process

$$\bar{V}_t(x) := xB^{0,l}_t \mathbb{1}_{\{x\geq 0\}} + xB^{0,b}_t \mathbb{1}_{\{x<0\}}$$

where the risk-free *lending* (resp. *borrowing*) *cash account* $B^{0,l}$ (resp. $B^{0,b}$) is used for unsecured lending (resp. borrowing) of cash.

Under that stylized convention for assessing bank's profits and losses, the process $\bar{V}(x)$ represents the wealth of a trader who commits himself at time 0 to keep his initial cash endowment x in either the lending (when $x \geq 0$) or borrowing (when $x < 0$) cash account and abstains from all other trading activities between time 0 and T. Notice that the concept of the benchmark wealth is irrelevant for classical arbitrage-free pricing in a linear market model even though it corresponds to the natural economic concept of opportunity cost. In what follows, the real numbers x^i and x^h represent the initial endowment of the issuer and holder, respectively, and the processes $\bar{V}^i(x^i)$ and $\bar{V}^h(x^h)$ denote their respective benchmark wealths (or future values of legacy portfolios).

Although one could object that the idea of unilateral pricing based on the bank's legacy portfolio is hard to implement in practice, we stress that even if we postulate that $x^i = x^h = 0$ and $\bar{V}^i_t(0) = \bar{V}^h_t(0) = 0$ for all $t \in [0,T]$, the asymmetry in unilateral pricing would still hold since it is a direct result of the nonlinear dynamics of wealth processes.

2.2 Issuer's unilateral pricing

Let $\mathcal{M}^i = (\mathcal{S}, \mathcal{B}^i, \Psi^i)$ be an issuer's nonlinear market model, which is assumed to be arbitrage-free with respect to European contracts, in the sense of Bielecki et al. [9, 12]. Here $\mathcal{S}$ (resp. $\mathcal{B}^i$) denotes the collection of *primary traded assets* (resp. the collection of issuer's *funding accounts*) and Ψ^i stands for the class of all issuer's *admissible* trading strategies. We denote by $\Psi^i(y, A)$ the set of all trading strategies from Ψ^i with an initial wealth $y \in \mathbb{R}$ and an external cash flow stream A. For any trading strategy $\varphi \in \Psi^i(y, A)$, we write $V^i(y, \varphi, A)$ to denote the *wealth process* of φ. Obviously, the equality $V^i_0(y, \varphi, D) = y$ holds for all pairs $(y, \varphi) \in (\mathbb{R}, \Psi^i(y, A))$.

From now on, it is assumed throughout that the processes A, X^h, $\bar{V}^i(x^i)$ and the wealth process $V^i(y, \varphi, A)$ are $\mathbb{F}$-adapted and RCLL. However, it is not hard to check that our results still hold under the assumption that the processes $\bar{V}^i(x^i) - X^h - A$ and $V^i(y, \varphi, A) - A$ are $\mathbb{F}$-adapted and RCLL. In addition, we will gradually impose more conditions on the nonlinear dynamics of the wealth process. In Section 2.2, we work under Assumption 2.3 of the strict forward monotonicity of the wealth and thus all results in this section are model independent. For simple proofs of all results in this section, the reader is referred to [44].

Let us consider the extended market model $\mathcal{M}^{i,p}(\mathcal{C}^a)$ in which an American contract $\mathcal{C}^a$ is traded at time 0 at some initial price p where p is an arbitrary real number. We first give a preliminary analysis of unilateral pricing of an American contract by its issuer who is endowed with the pre-trading initial wealth $x^i \in \mathbb{R}$ and thus employs $\bar{V}^i(x^i)$ as the benchmark wealth. Since the process A is fixed throughout, to alleviate notation, we will frequently write $V^i(x^i + p, \varphi)$ instead of $V^i(x^i + p, \varphi, A)$ when dealing with the issuer when no confusion may arise. We first introduce conditions associated with the study of the issuer's pricing and hedging problems.

For brevity, we say that a triplet $(p, \varphi, \tau) \in \mathbb{R} \times \Psi^i(x^i + p, A) \times \mathcal{T}$ satisfies:

$$\text{(AO)} \iff V^i_\tau(x^i + p, \varphi) + X^h_\tau \geq \bar{V}^i_\tau(x^i) \text{ and } \mathbb{P}\big(V^i_\tau(x^i + p, \varphi) + X^h_\tau > \bar{V}^i_\tau(x^i)\big) > 0,$$

$$\text{(SH)} \iff V^i_\tau(x^i + p, \varphi) + X^h_\tau \geq \bar{V}^i_\tau(x^i),$$

$$\text{(BE)} \iff V^i_\tau(x^i + p, \varphi) + X^h_\tau = \bar{V}^i_\tau(x^i),$$

$$\text{(NA)} \iff V^i_\tau(x^i + p, \varphi) + X^h_\tau = \bar{V}^i_\tau(x^i) \text{ or } \mathbb{P}\big(V^i_\tau(x^i + p, \varphi) + X^h_\tau < \bar{V}^i_\tau(x^i)\big) > 0,$$

where (AO) stands for *arbitrage opportunity*, (SH) for *superhedging*, (BE) for *break-even* and (NA) for *no-arbitrage*. We write $(p, \varphi, \tau) \in$ (AO) if a triplet (p, φ, τ) satisfies condition (AO); an analogous convention applies to other conditions. For instance, we say that a pair $(p, \varphi) \in \mathbb{R} \times \Psi^i(x^i + p, A)$ is an *issuer's arbitrage opportunity* in $\mathcal{M}^{i,p}(\mathcal{C}^a)$ and we write $(p, \varphi) \in$ (AO) if $(p, \varphi, \tau) \in$ (AO) for every $\tau \in \mathcal{T}$.

2.2.1 Issuer's fair prices

We first introduce the notion of an issuer's *fair price*, by which we mean any level of the initial price p at which an arbitrage opportunity for the issuer is excluded. By convention, we henceforth set $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. A real number $p^{f,i}(x^i, \mathcal{C}^a)$ is an *issuer's fair price* for $\mathcal{C}^a$ if no issuer's arbitrage opportunity (p, φ) exists in $\mathcal{M}^{i,p}(\mathcal{C}^a)$ when $p = p^{f,i}(x^i, \mathcal{C}^a)$. The set of all issuer's fair prices is given by

$$\mathcal{H}^{f,i}(x^i) := \big\{ p \in \mathbb{R} \,|\, \forall\, \varphi \in \Psi^i(x^i + p, A)\, \exists\, \tau \in \mathcal{T} \colon (p, \varphi, \tau) \in \text{(NA)} \big\}$$

and the upper bound for issuer's fair prices equals $\overline{p}^{f,i}(x^i, \mathcal{C}^a) := \sup\, \mathcal{H}^{f,i}(x^i)$. Notice that the superscript f stands here for *fair* and i for *issuer*. Similar notational conventions will be applied to other instances of issuer's prices and costs without special mentioning.

Definition 2.2. If the equality $\overline{p}^{f,i}(x^i, \mathcal{C}^a) = \max\, \mathcal{H}^{f,i}(x^i)$ holds, then $\overline{p}^{f,i}(x^i, \mathcal{C}^a)$ is denoted as $\widehat{p}^{f,i}(x^i, \mathcal{C}^a)$ and called the *issuer's maximum fair price* for $\mathcal{C}^a$.

The following basic property of trading strategies is satisfied in a vast majority of existing market models (see Lemma 2.23 for a fairly general result in this vein).

Assumption 2.3. The following *strict forward monotonicity* property holds: for all $x, p \in \mathbb{R}$, $\varphi \in \Psi^i(x + p, A)$ and $p' > p$ (resp. $p' < p$), there exists a trading strategy $\varphi' \in \Psi^i(x + p', A)$ such that $V^i_t(x + p', \varphi', A) > V^i_t(x + p, \varphi, A)$ (resp. $V^i_t(x + p', \varphi', A) < V^i_t(x + p, \varphi, A)$) for every $t \in [0, T]$.

Lemma 2.4. *If Assumption 2.3 holds and $p \in \mathcal{H}^{f,i}(x^i)$, then $p' \in \mathcal{H}^{f,i}(x^i)$ for every $p' < p$ and thus, if $\mathcal{H}^{f,i}(x^i) \neq \emptyset$, then either $\mathcal{H}^{f,i}(x^i) = (-\infty, \widehat{p}^{f,i}(x^i, \mathcal{C}^a)]$ for some $\widehat{p}^{f,i}(x^i, \mathcal{C}^a) \in \mathbb{R}$ or $\mathcal{H}^{f,i}(x^i) = (-\infty, \overline{p}^{f,i}(x^i, \mathcal{C}^a))$ where $\overline{p}^{f,i}(x^i, \mathcal{C}^a) \in \mathbb{R} \cup \{\infty\}$.*

2.2.2 Issuer's superhedging costs

The concepts of (strict) superhedging strategies and their respective costs for the issuer and holder are fairly standard. As usual, for the issuer we impose conditions for every $\tau \in \mathcal{T}$ whereas for the holder it suffices to postulate that analogous conditions are satisfied for at least one $\tau \in \mathcal{T}$ (see Definitions 3.5 and 3.6).

If property (SH) is satisfied by a triplet (p, φ, τ), then we say that an *issuer's superhedging at time τ* arises. From the optional section theorem, condition (SH) holds for a pair $(p, \varphi) \in \mathbb{R} \times \Psi^i(x^i + p, A)$ and *all* $\tau \in \mathcal{T}$ if and only if (p, φ) is such that the inequality $V^i_t(x^i + p, \varphi) + X^h_t \geq \bar{V}^i_t(x^i)$ is valid for all $t \in [0, T]$. This justifies the following definition.

Definition 2.5. We say that pair $(p, \varphi) \in \mathbb{R} \times \Psi^i(x^i + p, A)$ is an *issuer's superhedging strategy* in $\mathcal{M}^{i,p}(\mathcal{C}^a)$ and we write $(p, \varphi) \in$ (SH) if the inequality $V^i_t(x^i + p, \varphi) + X^h_t \geq \bar{V}^i_t(x^i)$ holds for all $t \in [0, T]$, that is,

$$\mathbb{P}(V^i_t(x^i + p, \varphi) + X^h_t \geq \bar{V}^i_t(x^i),\, \forall\, t \in [0, T]) = 1.$$

The lower bound for *issuer's superhedging costs* for $\mathcal{C}^a$ is given by $\underline{p}^{s,i}(x^i, \mathcal{C}^a) := \inf\, \mathcal{H}^{s,i}(x^i)$ where

$$\mathcal{H}^{s,i}(x^i) := \big\{ p \in \mathbb{R} :\, \exists\, \varphi \in \Psi^i(x^i + p, A) \colon (p, \varphi) \in \text{(SH)} \big\}.$$

Definition 2.6. If the equality $\underline{p}^{s,i}(x^i, \mathcal{C}^a) = \min\, \mathcal{H}^{s,i}(x^i)$ holds, then $\underline{p}^{s,i}(x^i, \mathcal{C}^a)$ is denoted as $\check{p}^{s,i}(x^i, \mathcal{C}^a)$ and called the *issuer's minimum superhedging cost* for $\mathcal{C}^a$.

We also examine issuer's strict superhedging strategies and related costs. The lower bound for *issuer's strict superhedging costs* for $\mathcal{C}^a$ is given by $\underline{p}^{a,i}(x^i, \mathcal{C}^a) := \inf\, \mathcal{H}^{a,i}(x^i)$ where

$$\mathcal{H}^{a,i}(x^i) := \big\{ p \in \mathbb{R} :\, \exists\, \varphi \in \Psi^i(x^i + p, A) \colon (p, \varphi) \in \text{(AO)} \big\}.$$

Definition 2.7. If $\underline{p}^{a,i}(x^i, \mathcal{C}^a) = \min\, \mathcal{H}^{a,i}(x^i)$, then it is denoted as $\breve{p}^{a,i}(x^i, \mathcal{C}^a)$ and called the *issuer's minimum strict superhedging cost* for $\mathcal{C}^a$.

It is readily seen that $\mathcal{H}^{a,i}(x^i)$ is the complement of $\mathcal{H}^{f,i}(x^i)$ and thus, in view of Lemma 2.4, the equality $\underline{p}^{a,i}(x^i, \mathcal{C}^a) = \overline{p}^{f,i}(x^i, \mathcal{C}^a)$ holds under Assumption 2.3.

Assumption 2.3 entails the following lemma (for its proof, see [44]).

Lemma 2.8. *If Assumption 2.3 is satisfied, then the equality* $\underline{p}^{s,i}(x^i, \mathcal{C}^a) = \underline{p}^{a,i}(x^i, \mathcal{C}^a)$ *holds and thus* $\overline{p}^{f,i}(x^i, \mathcal{C}^a) = \underline{p}^{s,i}(x^i, \mathcal{C}^a) = \underline{p}^{a,i}(x^i, \mathcal{C}^a)$.

2.2.3 Issuer's acceptable price

Our next goal is to examine the following question: under which assumptions a suitably defined *replication cost* of an American contract is at the same time the maximum fair price and the minimum superhedging cost for the issuer? The answer to this question and an analogous one for the holder (see Section 3.1.3) leads to the important concept of a *unilateral acceptable price*. In the issuer's case, we will also examine important stopping times related to the *break-even* condition (BE).

Definition 2.9. If condition (BE) is satisfied by $(p, \varphi, \tau) \in \mathbb{R} \times \Psi^i(x^i + p, A) \times \mathcal{T}$, then the stopping time $\tau \in \mathcal{T}$ is called an *issuer's break-even time* for the pair $(p, \varphi) \in \mathbb{R} \times \Psi^i(x^i + p, A)$.

Even when a pair (p, φ) is fixed, the uniqueness of an issuer's break-even time τ is not ensured, in general. Any issuer's break-even time can be seen as a holder's exercise time but we will argue that an issuer's break-even time is unlikely to be a *rational exercise time* for the holder. In fact, it may not be advantageous for the holder to exercise at a stopping time that causes the issuer to break even or prohibits the issuer's arbitrage opportunities. Firstly, usually the holder is not informed about the issuer's trading strategy. Secondly, the holder should be behaving in a rational way for his own payoff and hedging abilities. A holder's rational exercise time can be identified with a particular instance of a *holder's break-even time*, which is introduced in Definition 3.1, see Remark 3.23 for more details.

We work hereafter under Assumption 2.3 and thus, in view of Lemma 2.8, the following equalities are valid

$$\overline{p}^{f,i}(x^i, \mathcal{C}^a) = \underline{p}^{s,i}(x^i, \mathcal{C}^a) = \underline{p}^{a,i}(x^i, \mathcal{C}^a).$$

The lower bound for *issuer's replication costs* for $\mathcal{C}^a$ is given by $\underline{p}^{r,i}(x^i, \mathcal{C}^a) := \inf\, \mathcal{H}^{r,i}(x^i)$ where

$$\mathcal{H}^{r,i}(x^i) := \big\{ p \in \mathbb{R} \,|\, \exists\, (\varphi, \tau) \in \Psi^i(x^i + p, A) \times \mathcal{T} \colon (p, \varphi) \in \text{(SH)} \;\&\; (p, \varphi, \tau) \in \text{(BE)} \big\}.$$

Definition 2.10. If the equality $\underline{p}^{r,i}(x^i, \mathcal{C}^a) = \min\, \mathcal{H}^{r,i}(x^i)$ holds, then $\underline{p}^{r,i}(x^i, \mathcal{C}^a)$ is denoted as $\breve{p}^{r,i}(x^i, \mathcal{C}^a)$ and called the *issuer's minimum replication cost* for $\mathcal{C}^a$.

Definition 2.10 focuses on a particular issuer's superhedging strategy for which a break-even time exists and we do not impose any restrictions on other issuer's trading strategies. Hence, in principle, it may happen that for $p \in \mathcal{H}^{r,i}(x^i)$ there exists another pair, say (p, ψ), which is an issuer's strict superhedging strategy and this would mean that p would fail to be an issuer's fair price for $\mathcal{C}^a$. To eliminate this potential shortcoming of Definition 2.10 we impose, in addition, the no-arbitrage condition on all issuer's trading strategies associated with p. The lower bound for *issuer's fair replication costs* for $\mathcal{C}^a$ is given by $\underline{p}^{f,r,i}(x^i, \mathcal{C}^a) := \inf\, \mathcal{H}^{f,r,i}(x^i)$ where

$$\begin{aligned} \mathcal{H}^{f,r,i}(x^i) := \big\{ & p \in \mathbb{R} \,|\, \exists\, (\varphi, \tau) \in \Psi^i(x^i + p, A) \times \mathcal{T} \colon (p, \varphi) \in \text{(SH)} \;\&\; (p, \varphi, \tau) \in \text{(BE)}; \\ & \forall\, \varphi' \in \Psi^i(x^i + p, A)\, \exists\, \tau' \in \mathcal{T} \colon\, (p, \varphi', \tau') \in \text{(NA)} \big\}. \end{aligned}$$

Definition 2.11. If the equality $\underline{p}^{f,r,i}(x^i, \mathcal{C}^a) = \min \mathcal{H}^{f,r,i}(x^i)$ holds, then $\underline{p}^{f,r,i}(x^i, \mathcal{C}^a)$ is denoted as $\breve{p}^{f,r,i}(x^i, \mathcal{C}^a)$ and it is called the *issuer's minimum fair replication cost* for $\mathcal{C}^a$. If the set $\mathcal{H}^{f,r,i}(x^i)$ has a unique element, then it is denoted as $p^i(x^i, \mathcal{C}^a)$ and called the *issuer's acceptable price* for $\mathcal{C}^a$.

It is easy to check that the following inclusions and equality are valid

$$\mathcal{H}^{s,i}(x^i) \supseteq \mathcal{H}^{r,i}(x^i) \supseteq \mathcal{H}^{f,r,i}(x^i) = \mathcal{H}^{f,i}(x^i) \cap \mathcal{H}^{r,i}(x^i)$$

and thus we obtain

$$\overline{p}^{f,i}(x^i, \mathcal{C}^a) = \underline{p}^{s,i}(x^i, \mathcal{C}^a) \leq \underline{p}^{r,i}(x^i, \mathcal{C}^a) \leq \underline{p}^{f,r,i}(x^i, \mathcal{C}^a). \tag{2.1}$$

The following main result of Section 2.2 summarizes the fundamental properties of various issuer's costs associated with the pricing and hedging of an American contract $\mathcal{C}^a$. It shows that the issuer's acceptable price $p^i(x^i, \mathcal{C}^a)$, provided that it exists, satisfies all desirable properties of a contract's price from the issuer's perspective. For the detailed proof of Proposition 2.12, the interested reader is referred to Kim et al. [44].

Proposition 2.12. *If Assumption 2.3 is satisfied and $\mathcal{H}^{f,r,i}(x^i) \neq \emptyset$, then it has a unique element. Moreover, the issuer's acceptable price $p^i(x^i, \mathcal{C}^a)$ is finite and satisfies $p^i(x^i, \mathcal{C}^a) = \widehat{p}^{f,i}(x^i, \mathcal{C}^a) = \breve{p}^{r,i}(x^i, \mathcal{C}^a) = \breve{p}^{s,i}(x^i, \mathcal{C}^a)$.*

It is clear that in order to make use of Proposition 2.12, we need to show that the set $\mathcal{H}^{f,r,i}(x^i)$ is nonempty. This goal will be achieved in Section 2.4 using a BSDE approach in a fairly general setup. Subsequently, in Section 2.5, we will study the properties of the issuer's break-even times.

2.3 Dynamics of the wealth process

Before examining a BSDE approach to the issuer's valuation, we need first to describe the main features of the mechanism of nonlinear trading. For concreteness, we introduce explicit notation for cash accounts, risky assets, and funding accounts associated with risky assets. However, our further developments are independent of the choice of any particular model for primary traded assets and thus the approach to American contracts developed in the present paper is capable of covering a broad spectrum of nonlinear market models. To emphasize this universality, we will later postulate that the wealth dynamics for the issuer's and holder's trading strategies are given by generic SDEs (2.7) and (3.5), respectively, rather than any more specific equations.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions of right-continuity and completeness. The processes $S^1, S^2, \dots, S^n$ model prices of arbitrary traded securities, such as, stocks, stock options, interest rate swaps, currency options, cross-currency swaps, CDSs, etc., and thus they are not assumed to be strictly positive. Let B^l and B^b, denote the *lending* and *borrowing* unsecured cash accounts, respectively. We postulate that: (a) for $i = 1, 2, \dots, n$, the process S^i is an $\mathbb{F}$-semimartingale and the cumulative dividend stream A^i is a process of finite variation with $A^i_0 = 0$, (b) B^l, B^b are strictly positive, continuous processes of finite variation with $B^l_0 = B^b_0 = 1$.

Due to peculiarities in the wealth dynamics under nowadays ubiquitous market frictions, one needs to study the wealth dynamics for self-financing strategies under alternative assumptions about trading and netting. To illustrate the concept of nonlinear trading through explicit examples, we will consider two instances; first, a model with partial netting and, second, a model with idiosyncratic funding costs and collateralization. In the first model, we will consider the contract without collateralization and, in the second one, we will introduce a collateralized contract to show how the presence of the margin account affects the dynamics of the wealth process.

2.3.1 Market model with partial netting

In this subsection, we focus on a market model with partial netting, which was introduced in Bielecki and Rutkowski [12]. Specifically, we assume that:
(a) short cash positions in risky assets $S^1, S^2, \dots, S^n$ are aggregated and the proceeds from short-selling are available for trading.
(b) long cash positions in risky assets S^i are funded from their respective funding accounts $B^{i,b}$, which can be interpreted as secured loans in the repo market. Here $B^{i,b}$ is a strictly positive, continuous process of finite variation with $B^{i,b}_0 = 1$.
(c) all positive and negative cash flows from the external cash flow stream A of the contract and a trading strategy φ, inclusive of the proceeds from short-selling of risky assets, are reinvested in traded assets.

An issuer's *trading strategy* is formally composed of his initial endowment x, a process $\varphi = \big(\xi^1, \dots, \xi^n, \psi^l, \psi^b, \psi^{1,b}, \dots, \psi^{n,b},\big)$ and the cash flow A. The components of φ represent positions in the risky assets S^i, $i = 1, 2, \dots, n$, the unsecured lending cash account B^l, the unsecured borrowing cash account B^b, the funding accounts $B^{i,b}$, $i = 1, 2, \dots, n$ for risky assets. Consistently with (a)–(c), we postulate that:
(i) $\psi^l_t \geq 0$, $\psi^b_t \leq 0$ and $\psi^l_t \psi^b_t = 0$ for all $t \in [0, T]$,
(ii) $\psi^{i,b}_t = -(B^{i,b}_t)^{-1} (\xi^i_t S^i_t)^+$ for every $i = 1, 2, \dots, n$ and all $t \in [0, T]$.

The issuer's *initial endowment* x is interpreted as either a positive or negative amount of cash he owns before entering into a contract. Hence after he engages in a transaction at time 0, his *initial wealth* becomes $V_0 := x + p_0$ where p_0 is the initial price of the contract, as seen by the issuer.

The next definition introduces a suitable version of the self-financing property for a trading strategy in the model with partial netting.

Definition 2.13. An issuer's trading strategy (x, φ, A) is *self-financing* whenever the *issuer's wealth* $V(x, \varphi, A)$, which is given by

$$V_t(x, \varphi, A) = \sum_{i=1}^n \xi^i_t S^i_t + \sum_{i=1}^n \psi^{i,b}_t B^{i,b}_t + \psi^{0,l}_t B^l_t + \psi^{0,b}_t B^b_t, \tag{2.2}$$

satisfies, for every $t \in [0, T]$,

$$V_t(x, \varphi, A) = x + \sum_{i=1}^n \int_{(0,t]} \xi^i_u \, d(S^i_u + A^i_u) + \sum_{i=1}^n \int_0^t \psi^{i,b}_u \, dB^{i,b}_u + \int_0^t \psi^l_u \, dB^l_u + \int_0^t \psi^b_u \, dB^b_u + A_t.$$

From $\psi^{i,b}_t = -(B^{i,b}_t)^{-1} (\xi^i_t S^i_t)^+$, $i = 1, 2, \dots, n$ and (2.2), we obtain

$$V_t(x, \varphi, A) = \psi^l_t B^l_t + \psi^b_t B^b_t - \sum_{i=1}^n (\xi^i_t S^i_t)^-.$$

Since we postulated that $\psi^l_t \geq 0$, $\psi^b_t \leq 0$ and $\psi^l_t \psi^b_t = 0$ for all $t \in [0, T]$, we also have that

$$\psi^l_t = (B^l_t)^{-1} \Big(V_t(x, \varphi, A) + \sum_{i=1}^n (\xi^i_t S^i_t)^- \Big)^+$$

and

$$\psi^b_t = -(B^b_t)^{-1} \Big(V_t(x, \varphi, A) + \sum_{i=1}^n (\xi^i_t S^i_t)^- \Big)^-.$$

Consequently, we obtain the following result showing that, for a given (x, A), the choice of the process ξ uniquely determines the trading strategy (x, φ, A) and thus also the unique value process $V(x, \varphi, A)$.

Lemma 2.14. *The dynamics of a self-financing trading strategy* (x,φ,A) *are uniquely determined by the initial endowment* x *and processes* ξ *and* A *through the following equation*

$$\begin{aligned} dV_t(x,\varphi,A) = &\sum_{i=1}^n \xi_t^i\,(dS_t^i + dA_t^i) - \sum_{i=1}^n (\xi_t^i S_t^i)^+ (B_t^{i,b})^{-1}\,dB_t^{i,b} + dA_t \\ &+ \Big(V_t(x,\varphi,A) + \sum_{i=1}^n (\xi_t^i S_t^i)^-\Big)^+ (B_t^l)^{-1}\,dB_t^l \\ &- \Big(V_t(x,\varphi,A) + \sum_{i=1}^n (\xi_t^i S_t^i)^-\Big)^- (B_t^b)^{-1}\,dB_t^b. \end{aligned} \tag{2.3}$$

The following standard assumption will allow us to derive more explicit expressions for the wealth dynamics and thus also to compute the generator for the associated BSDE. We postulate that the processes B^l, B^b and $B^{i,b}$ are continuous with $B_0^l = B_0^b = B_0^{i,b} = 1$ and such that

$$dB_t^l = r_t^l B_t^l\,dt, \quad dB_t^b = r_t^b B_t^b\,dt, \quad dB_t^{i,b} = r_t^{i,b} B_t^{i,b}\,dt,$$

for some $\mathbb{F}$-adapted processes r^l, r^b and $r^{i,b}$ such that $0 \le r^l \le r^b$ and $r^l \le r^{i,b}$ for every $i = 1,2,\dots,n$.

Let the discounted cumulative prices of risky assets be given by the following expression

$$\widetilde{S}_t^{i,l,\mathrm{cld}} := (B_t^l)^{-1} S_t^i + \int_{(0,t]} (B_u^l)^{-1}\,dA_u^i.$$

The process $\widetilde{S}^{i,l,\mathrm{cld}}$ is aptly specified for the study the non-arbitrage property of market for the issuer and holder with a non-negative initial endowment. Indeed, if there exists a probability measure $\widetilde{\mathbb{P}}^l$, which is equivalent to $\mathbb{P}$ on $(\Omega,\mathcal{F}_T)$, and such that the processes $\widetilde{S}^{i,l,\mathrm{cld}}$, $i = 1,2,\dots,n$ are $(\widetilde{\mathbb{P}}^l,\mathbb{F})$-local martingales, then no extended arbitrage opportunity exists (see [12, 60] for details). To examine the existence of a probability measure $\widetilde{\mathbb{P}}^l$, we recall the following example of a diffusion-type model (see Remark 4.3 in [60]).

Example 2.15. We consider the classical case where the prices of risky assets are given by the diffusion-type model. We may assume that each risky asset S^i for $i = 1,2,\dots,n$ has the ex-dividend price under the real-world probability $\mathbb{P}$ governed by

$$dS_t^i = S_t^i\Big(\mu_t^i\,dt + \sum_{j=1}^n \sigma_t^{ij}\,dW_t^j\Big)$$

with $S_0^i > 0$. Equivalently, the n-dimensional process $S = (S^1, S^2, \dots, S^n)^*$ satisfies

$$dS_t = \mathbb{S}_t(\mu_t\,dt + \sigma_t\,dW_t)$$

where $\mathbb{S} = \mathrm{diag}\,(S^1,S^2,\dots,S^n)$ (the diagonal matrix with the entries $S^1,S^2,\dots,S^n$) and $W = (W^1,W^2,\dots,W^n)^*$ is an n-dimensional Brownian motion. Furthermore, $\mu = (\mu^1,\dots,\mu^n)^*$ is an $\mathbb{R}^n$-valued, $\mathbb{F}^W$-adapted process, $\sigma = [\sigma^{ij}]$ is an n-dimensional square matrix of $\mathbb{F}^W$-adapted processes satisfying the ellipticity condition: there exists a constant $\Lambda > 0$ such that $\sum_{i,j=1}^n (\sigma_t\sigma_t^*)_{ij}\,a_i a_j \ge \Lambda\|a\|^2 = \Lambda a^* a$ for all $t \in [0,T]$ and every $a \in \mathbb{R}^n$. For simplicity of presentation, we also assume that the processes μ, σ and κ are bounded. We now set $\mathbb{F} = \mathbb{F}^W$ where $\mathbb{F}^W$ is the natural filtration of W. Recall that the Brownian motion W enjoys the predictable representation property with respect to $\mathbb{F}$ and this property is shared by the process $\widetilde{W}$ given by equality (2.4).

Assuming that the dividend processes satisfy $A^i_t = \int_0^t \kappa^i_u S^i_u \, du$ for all $i = 1, 2, \dots, n$, we obtain

$$d\widetilde{S}^{i,l,\mathrm{cld}}_t = (B^l_t)^{-1}\big(dS^i_t + dA^i_t - r^l_t S^i_t \, dt\big)$$

or, more explicitly,

$$d\widetilde{S}^{i,l,\mathrm{cld}}_t = (B^l_t)^{-1} S^i_t \Big(\big(\mu^i_t + \kappa^i_t - r^l_t\big)\, dt + \sum_{j=1}^n \sigma^{ij}_t \, dW^j_t\Big).$$

Hence, if we denote $\widetilde{S}^{l,\mathrm{cld}} := (\widetilde{S}^{1,l,\mathrm{cld}}, \widetilde{S}^{2,l,\mathrm{cld}}, \dots, \widetilde{S}^{n,l,\mathrm{cld}})^*$ and

$$\mu + \kappa - r^l := \big(\mu^1 + \kappa^1 - r^l, \mu^2 + \kappa^2 - r^l, \dots, \mu^n + \kappa^n - r^l\big)^*,$$

then we may write

$$d\widetilde{S}^{l,\mathrm{cld}}_t = (B^l_t)^{-1} \mathbb{S}_t \Big(\big(\mu_t + \kappa_t - r^l_t\big)\, dt + \sigma_t \, dW_t\Big).$$

Let us denote $l_t := \sigma_t^{-1}(\mu_t + \kappa_t - r^l_t)$ for all $t \in [0,T]$. Since μ, σ, κ are assumed to be bounded and σ to satisfy the ellipticity condition, we see that the process l is bounded as well and thus we can define the probability measure $\widetilde{\mathbb{P}}^l$ on $(\Omega, \mathcal{F}^W_T)$ by setting

$$\frac{d\widetilde{\mathbb{P}}^l}{d\mathbb{P}} = \exp\bigg(- \int_0^T l_t \, dW_t - \frac{1}{2}\int_0^T |l_t|^2 \, dt\bigg).$$

Then the probability measure $\widetilde{\mathbb{P}}^l$ is equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}^W_T)$ and, from Girsanov's theorem, the process $\widetilde{W}^l := (\widetilde{W}^{l,1}, \widetilde{W}^{l,2}, \dots, \widetilde{W}^{l,n})^*$ where

$$d\widetilde{W}^l_t := dW_t + l_t \, dt = dW_t + \sigma_t^{-1}(\mu_t + \kappa_t - r^l_t)\, dt \tag{2.4}$$

is an n-dimensional Brownian motion under $\widetilde{\mathbb{P}}^l$. It is clear that under $\widetilde{\mathbb{P}}^l$

$$d\widetilde{S}^{l,\mathrm{cld}}_t = (B^l_t)^{-1} \mathbb{S}_t \sigma_t \, d\widetilde{W}^l_t.$$

Hence the price processes $\widetilde{S}^{i,l,\mathrm{cld}}$, $i = 1, 2, \dots, n$ are continuous, square-integrable, $(\widetilde{\mathbb{P}}^l, \mathbb{F})$-martingales and the quadratic variation of $\widetilde{S}^{l,\mathrm{cld}}$ equals $\langle \widetilde{S}^{l,\mathrm{cld}} \rangle_t = \int_0^t m^l_u (m^l_u)^* \, du$ where $m^l(m^l)^* = \mathbb{S}\gamma\gamma^*\mathbb{S}$ and where we write $\gamma := (B^l)^{-1}\sigma$.

Remark 2.16. More generally, the process $\widetilde{S}^{l,\mathrm{cld}}$ can be assumed to be an RCLL martingale, which corresponds to the study of an imperfect market model with credit risk. For more details, see Example 4.5 where the price of a defaultable risky assets is assumed to be driven by a Wiener process and a jump martingale associated with an extraneous event, which may represent the default event of either the third party or the counterparty in an American contract under study.

In view (2.3), we have the following result yielding the dynamics of the wealth process in the market model with partial netting.

Lemma 2.17. *The wealth* $Y := V(x, \varphi, A)$ *satisfies*

$$dY_t = \sum_{i=1}^n B^l_t \xi^i_t \, d\widetilde{S}^{i,l,cld}_t + f_l(t, Y^l_t, \xi_t)\, dt + dA_t$$

where $f_l : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *equals*

$$f_l(t,y,z) := \sum_{i=1}^n r^l_t z^i S^i_t - \sum_{i=1}^n r^{i,b}_t (z^i S^i_t)^+ + r^l_t \Big(y + \sum_{i=1}^n (z^i S^i_t)^-\Big)^+ - r^b_t \Big(y + \sum_{i=1}^n (z^i S^i_t)^-\Big)^-.$$

We can see that the nonlinearity of the wealth dynamics in the present model arises due to the different borrowing and lending interest rates (that is, $r^b \neq r^l$) and the postulated netting of positions in risky assets.

2.3.2 Market model with idiosyncratic funding costs and collateralization

Let us now examine the case of a collateralized contract with the *collateral process* C represents the margin account. It is convenient to decompose the process C as follows $C_t = C_t \mathbb{1}_{\{C_t \geq 0\}} + C_t \mathbb{1}_{\{C_t < 0\}} = C_t^+ - C_t^-$ where by convention, $C_t^+ := C_t \mathbb{1}_{\{C_t \geq 0\}}$ is the cash collateral accepted at time t by the issuer whereas $C_t^- := -C_t \mathbb{1}_{\{C_t < 0\}}$ represents the cash collateral pledged at time t by the issuer. The equality $C_T = 0$ is imposed to formally ensure that the collateral amount is returned in full to the pledging party at the contract's maturity date T. Since the contractual cash flows A are now supplemented by the collateral process C, the contract is formally represented as $(A, C, X^h, \mathcal{T})$.

We work throughout under the standing assumption of *full rehypothecation*, which means that the cash collateral can be used for trading by the receiving party without any restrictions; this convention should be contrasted with the case of *segregated* collateral (see, e.g. [12]). We denote by B^c the process specifying the interest paid/received on the margin account. For simplicity, the issuer and holder are implicitly assumed to be default-free before the maturity date T of a contract at hand. In the presence of a default event, we would need to specify also the close-out payoff, as in Section 4.

In this subsection, we consider a market with idiosyncratic funding of risky assets. Specifically, we henceforth postulate that: (a) all positive and negative cash flows from (A, C) and a trading strategy φ are immediately reinvested in traded assets; (b) long cash positions in risky assets S^i are assumed to be funded from their respective *funding accounts* $B^{i,b}$, which can be interpreted as secured loans in the repo market; (c) cash amounts from short positions in risky assets S^i are kept in segregated accounts and remunerated at interest rates implied by respective *remuneration accounts* $B^{i,l}$.

We also assume that the processes $B^l, B^b, B^{i,l}, B^{i,b}$ and B^c are strictly positive, continuous processes of finite variation with $B_0^l = B_0^b = B_0^{i,l} = B_0^{i,b} = B_0^c = 1$ such that

$$dB_t^l = r_t^l B_t^l \, dt, \; dB_t^b = r_t^b B_t^b \, dt, \; dB_t^{i,l} = r_t^{i,l} B_t^{i,l} \, dt, \; dB_t^{i,b} = r_t^{i,b} B_t^{i,b} \, dt, \; dB_t^c = r_t^c B_t^c \, dt$$

for some $\mathbb{F}$-adapted and bounded processes r^l, r^b, $r^{i,l}$, $r^{i,b}$ and r^c satisfying $0 \leq r_t^l \leq r_t^b$ and $0 \leq r_t^{i,l} \leq r^l \leq r_t^{i,b}$ for all $t \in [0, T]$.

In general, an issuer's *trading strategy* (x, φ, A, C) is composed of his initial endowment x, a process $\varphi = \big(\xi^1, \dots, \xi^n, \psi^{1,l}, \dots, \psi^{n,l}, \psi^{1,b}, \dots, \psi^{n,b}, \psi^l, \psi^b, \eta\big)$ and cash flow (A, C). First, the components $\xi^1, \xi^2, \dots, \xi^n$ specify the number of shares of risky assets $S^1, S^2, \dots, S^n$ and the processes $\psi^{i,l}$ and $\psi^{i,b}$ represent respective positions in the remuneration and funding accounts $B^{i,l}$ and $B^{i,b}$ for the ith risky assets. Second, ψ^l and ψ^b are positions in the unsecured lending cash account B^l and the unsecured borrowing cash account B^b, respectively. Finally, the process η is given in terms of the collateral account B^c and the collateral process C through the equality $\eta = -(B^c)^{-1} C$ where the minus sign means that the interest payments are made by the receiver of the collateral. The *portfolio's value* at time t is denoted as $V_t^p(x, \varphi, A, C)$ and it equals

$$V_t^p(x, \varphi, A, C) = \sum_{i=1}^n \xi_t^i S_t^i + \sum_{i=1}^n \psi_t^{i,l} B_t^{i,l} + \sum_{i=1}^n \psi_t^{i,b} B_t^{i,b} + \psi_t^l B_t^l + \psi_t^b B_t^b. \tag{2.5}$$

Consistently with the financial interpretation, we postulate that $\psi_t^{i,l} \geq 0, \psi_t^{i,b} \leq 0$, $\psi_t^l \geq 0$ and $\psi_t^b \leq 0$ for all $t \in [0, T]$. Let $V_t(x, \varphi, A, C)$ stand for the *issuer's wealth* at time t. Notice that it is not equal to the portfolio's value since it accounts for the fact that collateral amount is merely pledged, but not granted, to the receiver and thus it does not constitute a legitimate part of his wealth. Under the standing assumption of full rehypothecation, the equality $V_t(x, \varphi, A, C) = V_t^p(x, \varphi, A, C) - C_t$ is known to hold for every $t \in [0, T]$ (see [12]) and thus we have the following definition of a self-financing strategy under idiosyncratic funding of risky assets.

Definition 2.18. *A trading strategy* (x, φ, A, C) *where*

$$\varphi = (\xi^1, \xi^2, \ldots, \xi^n, \varphi^{1,l}, \varphi^{2,l}, \ldots, \varphi^{n,l}, \varphi^{1,b}, \varphi^{2,b}, \ldots, \varphi^{n,b}, \varphi^l, \varphi^b, \eta)$$

is self-financing if the portfolio's value $V^p(x, \varphi, A, C)$*, which is given by* (2.5)*, satisfies the following conditions, for every* $t \in [0, T]$*,* $\psi^l_t \geq 0$*,* $\psi^b_t \leq 0$*,* $\psi^l_t \psi^b_t = 0$*,*

$$\psi^{i,l}_t = (B^{i,l}_t)^{-1}(\xi^i_t S^i_t)^-, \quad \psi^{i,b}_t = -(B^{i,b}_t)^{-1}(\xi^i_t S^i_t)^+,$$

and

$$V^p_t(x, \varphi, A, C) = x + \sum_{i=1}^n \int_{(0,t]} \xi^i_u \, d(S^i_u + A^i_u) + \sum_{i=1}^n \int_0^t \psi^{i,l}_u \, dB^{i,l}_u + \sum_{i=1}^n \int_0^t \psi^{i,b}_u \, dB^{i,b}_u + \int_0^t \psi^l_u \, dB^l_u + \int_0^t \psi^b_u \, dB^b_u + A^C_t$$

where we denote $A^C := A + C + F^C$ *and where the remuneration process* F^C *for the margin account equals* $F^C_t := -\int_0^t r^c_u C_u \, du$*.*

The following result is a straightforward consequence of Definition 2.18.

Lemma 2.19. *For any self-financing trading strategy* (x, φ, A, C)*, the portfolio's value process* $Y^{p,l} := V^p(x, \varphi, A, C)$ *satisfies*

$$dY^{p,l}_t = \sum_{i=1}^n B^l_t \xi^i_t \, d\widetilde{S}^{i,l,\mathrm{cld}}_t + G_{p,l}(t, Y^{p,l}_t, \xi_t) \, dt + dA^C_t$$

where the mapping $G_l : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *is given by*

$$G_{p,l}(t, y, z) := \sum_{i=1}^n r^l_t z^i S^i_t + \sum_{i=1}^n r^{i,l}_t (z^i S^i_t)^- - \sum_{i=1}^n r^{i,b}_t (z^i S^i_t)^+ + \big(r^l_t B^l_t y^+ - r^b_t B^l_t y^-\big).$$

Moreover, the wealth process $Y^l := V(x, \varphi, A, C)$ *satisfies*

$$dY^{p,l}_t = \sum_{i=1}^n B^l_t \xi^i_t \, d\widetilde{S}^{i,l,\mathrm{cld}}_t + G_l(t, Y^{p,l}_t + C_t, \xi_t) \, dt + dA_t \tag{2.6}$$

where $G_l(t, y, z) := G_{p,l}(t, y, z) - r^c_t C_t$*.*

Let us now consider the case of an endogenously determined collateral amount. Specifically, we examine the case of *issuer's collateral*, that is, a particular situation where the collateral amount is computed in reference to the issuer's wealth, but is independent of the holder's wealth. If $V(x)$ is the issuer's wealth process with an initial endowment x, then we say that C is the *issuer's collateral* (see [60]) if $C_t = q(\bar{V}_t(x) - V_t)$ for $t \in [0, T)$ where $q : \mathbb{R} \to \mathbb{R}$ is some uniformly Lipschitz continuous function such that $q(0) = 0$. For example, by setting $q(y) = (1 + \alpha_1)y^+ - (1 + \alpha_2)y^-$ for some constant *haircuts* $\alpha_1 > -1$ and $\alpha_2 > -1$ we obtain the collateral process C specified as in [12]. In particular, the case of a fully collateralized contract from the perspective of the issuer is obtained by taking $q(y) = y$, that is, by setting $\alpha_1 = \alpha_2 = 0$.

Under the assumed convention of issuer's collateral, we deduce from (2.6) that the wealth process $Y := V(x, \varphi, A, C)$ is governed by

$$dY_t = \sum_{i=1}^n B^l_t \xi^i_t \, d\widetilde{S}^{i,l,\mathrm{cld}}_t + G(t, Y_t, \xi_t) \, dt + dA_t$$

where $G(t, y, z) := G_{p,l}(t, y + q(-y + \bar{V}_t(x)), z) - r^c_t q(-y + \bar{V}_t(x))$. We thus see that the nonlinearity of the market may come from the different borrowing and lending interest rates when $r^b \neq r^l$, different funding costs for risky assets when $r^{i,b} \neq r^{i,l}$, but also from the endogenous collateral when q is a nonlinear function.

2.3.3 Generic market model

In the preceding subsections, we have shown that, due to additional trading constraints, the level of an initial wealth y and the choice of an n-dimensional process ξ are sufficient to uniquely determine the wealth process of an issuer's self-financing strategy $\varphi \in \Psi^i(y,A)$ and thus, for our further purposes, the pair (y,φ) is formally identified with the pair (y,ξ). In addition, we need also to introduce some kind of *admissibility* of a trading strategy and to postulate that the issuer's market model $\mathcal{M}^i$ where the class $\Psi^i(A) = \cup_{y\in\mathbb{R}} \Psi^i(y,A)$ of all issuer's admissible trading strategies is arbitrage-free in a suitable sense. We refer the reader to Bielecki et al. [9, 12] for general versions of the self-financing property of a trading strategy (see, e.g., Definition 1 in [9] or Definition 4.5 in [12]) and to Nie and Rutkowski [58, 60, 61] for explicit examples of arbitrage-free nonlinear markets with funding costs and collateralization.

Remark 2.20. Notice that we do not assume that the trading arrangements are identical for the two parties and thus they use distinct market models $\mathcal{M}^i$ and $\mathcal{M}^h$. Although one may postulate that they can access the same set of risky assets $S^1, S^2, \dots, S^n$, it is also reasonable to assume that the cash and funding accounts are different in their respective markets, denoted as $\mathcal{M}^i$ and $\mathcal{M}^h$. Similarly, it is natural to assume that both parties can observe the processes $S^1, S^2, \dots, S^n, X^h$ and A, but it would be far-stretched to imagine that they are fully aware about trading conditions and hedging strategy of the other party.

Several variants of the arbitrage-free property for nonlinear markets were first examined by El Karoui and Quenez [32] and subsequently extended by Bielecki et al. [9, 12]. Hence we do not elaborate on that important issue here and we simply postulate that a market model $\mathcal{M}^i$ is *regular* (hence arbitrage-free), in the sense of Definition 19 in Bielecki et al. [9]. Notice that the regularity of the issuer's market model holds if the associated BSDE for the wealth process enjoys the strict comparison property and thus we will focus on the latter property in what follows. Analogous assumptions are implicitly postulated to be satisfied by the holder's market model without further explicit mentioning.

In most financial models (in particular, in the two models outlined in preceding subsections) it can be deduced that the wealth process satisfies the SDE

$$V^i_t = y - \int_0^t h^i(u, V^i_u, \xi^i_u)\, dQ_u + \int_0^t \xi^{i*}_u \, d\widetilde{S}_u + A_t$$

for some mapping $h^i : \Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ and some $\mathbb{F}$-adapted, increasing, continuous, and bounded process Q. Although in most cases one may take $Q_t = t$ for all $t \in [0,T]$, in fact other choices may be more convenient in some circumstances. Let $\widetilde{S}$ denote the process of discounted cumulative prices of risky assets. Regarding the dynamics of prices of risky assets, we assume that $S^j = S^j_0 + N^j + D^j$ where D^j, $j = 1,2,\dots,n$ are $\mathbb{F}$-adapted, continuous processes of finite variation and N^j, $j = 1,2,\dots,n$ are $(\mathbb{P},\mathbb{F})$-local martingales, which are not necessarily continuous.

To ensure the arbitrage-free property of a nonlinear market, it is common to postulate the existence of a probability measure $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ on $(\Omega,\mathcal{F}_T)$ (as, for instance, in Example 2.15) such that the process $\widetilde{S}$ is a $(\widetilde{\mathbb{P}},\mathbb{F})$-local martingale (see [12, 60]) and then the pricing and hedging can be studied on the probability space $(\Omega,\mathcal{F}_T,\widetilde{\mathbb{P}})$. To alleviate the notation, we henceforth assume, without loss of generality, that $\widetilde{S} = M$ where M is a $(\mathbb{P},\mathbb{F})$-local martingale defined on $(\Omega,\mathcal{F}_T,\mathbb{P})$. If we assume that $\widetilde{S} = M$, then we can use results from [62, 63] regarding the existence, uniqueness and comparison property of solutions to BSDEs and reflected BSDEs in suitable spaces of stochastic processes.

To be more specific, we assume that there exists an $\mathbb{R}^{n\times n}$-valued, $\mathbb{F}$-predictable process m and an $\mathbb{F}$-adapted, continuous, nondecreasing process Q with $Q_0 = 0$ such that, for all $t \in [0,T]$, $\langle M \rangle_t = \int_0^t m_u m_u^* \, dQ_u$. Let us denote by $\mathcal{S}^2$ the space of all real-valued, RCLL, $\mathbb{F}$-adapted processes X with the norm $\| \cdot \|_{\mathcal{S}^2}$ given by

$$\|X\|_{\mathcal{S}^2}^2 := \mathbb{E}_{\mathbb{P}} \left[\sup_{t \in [0,T]} X_t^2 \right] < \infty$$

and $\mathcal{H}^2(Q)$ is the space of equivalence classes of all real-valued, $\mathbb{F}$-progressively measurable processes X with respect to the pseudo-norm $\| \cdot \|_{\mathcal{H}^2(Q)}$ given by

$$\|X\|_{\mathcal{H}^2(Q)}^2 := \mathbb{E}_{\mathbb{P}} \left[\int_0^T X_t^2 \, dQ_t \right] < \infty.$$

We denote by $\mathcal{L}^2(M)$ the space of all $\mathbb{R}^n$-valued, $\mathbb{F}$-predictable processes X with the pseudo-norm $\| \cdot \|_{\mathcal{L}^2(M)}$ given by

$$\|X\|_{\mathcal{L}^2(M)}^2 := \mathbb{E}_{\mathbb{P}} \left[\int_0^T \|m_t X_t\|^2 \, dQ_t \right] < \infty.$$

As usual, $L^2(\mathcal{F}_T)$ stands for the class of all real-valued, $\mathcal{F}_T$-measurable random variables η such that $\|\eta\|_{L^2(\mathcal{F}_T)}^2 = \mathbb{E}_{\mathbb{P}}(\eta^2) < \infty$. Let $\mathcal{A}^2$ be the class of nondecreasing, RCLL, $\mathbb{F}$-predictable processes such that $A_0 = 0$ and $\mathbb{E}_{\mathbb{P}}(A_T^2) < +\infty$.

Assumption 2.21. For every $\mathbb{F}$-predictable process ξ^i such that the integral $\int_0^\cdot \xi_u^{i*} \, dM_u$ is well defined, the wealth $V^i = V^i(y, \varphi, A)$ of the issuer's admissible trading strategy $(y, \varphi, A) \in \Psi^i(y, A)$ is $\mathbb{F}$-adapted and it is a unique strong solution to the SDE

$$V_t^i = y - \int_0^t g_i(u, V_u^i, \xi_u^i) \, dQ_u + \int_0^t \xi_u^{i*} \, dM_u + A_t. \tag{2.7}$$

Remark 2.22. For each fixed ω, we can solve (2.7) as a deterministic differential equation whose well-posedness holds under some conditions, for instance, when g is uniformly m-Lipschitz continuous and $g(\cdot, 0, \xi_u^i) \in \mathcal{H}^2(Q)$. Moreover, if the process M is RCLL, then the process $V^i - A$ is RCLL as well.

The following lemma addresses the important issue of the (strict) monotonicity of the issuer's and holder's wealth processes, which are driven by SDEs (2.7) and (3.5), respectively. Note that since the process z is assumed to be given in the statement of Lemma 2.23, we may interpret the SDE (2.8) as a deterministic integral equation, which is assumed to be satisfied for almost all $\omega \in \Omega$. For an elementary proof of Lemma 2.23, we refer to [44].

Lemma 2.23. *Assume that z is an $\mathbb{R}^n$-valued, $\mathbb{F}$-predictable stochastic process, M is an $\mathbb{R}^n$-valued $\mathbb{F}$-martingale, the $\mathbb{F}$-adapted process $k = k^1 - k^2$ is nondecreasing with $k_0 = 0$, and A is an RCLL, $\mathbb{F}$-adapted process. Let $g_l : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $l = 1, 2$ be $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R})$-measurable. Assume that the SDE*

$$v_t^l = y_l - \int_0^t g_l(u, v_u^l, z_u) \, dQ_u + k_t^l + \int_0^t z_u^* \, dM_u + A_t \tag{2.8}$$

has a unique solution v^l, for $l = 1, 2$. If $g_1(t, v_t^2, z_t) \leq g_2(t, v_t^2, z_t)$, $d\ell \otimes d\mathbb{P}$-a.e. and $y_1 \geq y_2$ (resp. $y_1 > y_2$), then $v_t^1 \geq v_t^2$ (resp. $v_t^1 > v_t^2$) for all $t \in [0,T]$.

By applying Lemma 2.23 with $y_1 = x + p < x + p' = y_2$, $g_1 = g_2 = g_i$ and $z = \xi^i$, it is easy to check that Assumption 2.3 is met when the wealth process $V^i(y, \varphi, A)$ is a unique solution to (2.7). Therefore, all results on the issuer's valuation established in Section 2.2 are valid in the present framework.

Remark 2.24. For the diffusion-type model from Example 2.15, one can develop a PDE approach to issuer's and holder's acceptable prices for the model with partial netting from Subsection 2.3.1 and the model with idiosyncratic funding costs and collateral from Subsection 2.3.2. Indeed, with the help of nonlinear Feynman-Kac theorem as in [29], one can link a solution to reflected BSDE to a parabolic PDE with obstacle. Then, similarly as in [61], the acceptable price and the replicating strategy can be obtained through a solution of a nonlinear PDE and thus the classical PDE approach to American options can be extended to nonlinear markets. Another inspiration for the study of a generic market model comes from the need to analyze American contracts with extraneous risk (see Examples 4.5, 4.6 and Remark 4.7 in Section 4).

2.4 Issuer's acceptable price via a reflected BSDE

In view of its financial interpretation, the nonlinear evaluation $\mathcal{E}^{g_i,A}$ associated with the BSDE

$$Y_t = \zeta_s + \int_t^s g_i(u, Y_u, Z_u)\, dQ_u - \int_t^s Z_u^*\, dM_u - (A_s - A_t) \tag{2.9}$$

is denoted by $\mathcal{E}^i$ and called the *issuer's evaluation*. To keep the presentation concise, we directly postulate here that the BSDE (2.9) has the desirable properties, such as: the existence, uniqueness, and strict comparison property of solutions and we refer the reader to [62, 63] for the proofs of respective results for BSDEs and reflected BSDEs (RBSDEs) driven by a multidimensional RCLL martingale.

In particular, we introduce the following assumption, which is justified by Theorem 4.1 in [62] (see also [17, 24, 28, 31, 32, 68] for analogous results in various frameworks). Notice that $\mathcal{E}^i_{t,s}(\zeta_s) = Y_s$ where (Y, Z) is the unique solution to BSDE (2.9).

Assumption 2.25. For every $(s, \zeta_s) \in [0, T] \times L^2(\mathcal{F}_s)$, the BSDE (2.9) has a unique solution (Y, Z) on $[0, s]$ such that $(Y, Z) \in \mathcal{H}^2(Q) \times \mathcal{L}^2(M)$ and $Y - A$ is RCLL, so that the issuer's evaluation $\mathcal{E}^i$ is well defined.

If the inequality $\zeta_s \geq \hat{\zeta}_s$ implies that $\mathcal{E}^i_{t,s}(\zeta_s) \geq \mathcal{E}^i_{t,s}(\hat{\zeta}_s)$ for all $t \in [0, s]$, then we say that the *comparison property* of $\mathcal{E}^i$ is valid. If, in addition, the equality $\mathcal{E}^i_{0,s}(\zeta_s) = \mathcal{E}^i_{0,s}(\hat{\zeta}_s)$ implies that $\mathcal{E}^i_{t,s}(\zeta_s) = \mathcal{E}^i_{t,s}(\hat{\zeta}_s)$ for all $t \in [0, s]$, then we say that the *strict comparison property* of $\mathcal{E}^i$ holds. Observe that the strict comparison property of $\mathcal{E}^i$ holds, provided that suitable assumptions are satisfied by a financial model of our interest (for a fairly general result, see Theorem 6.1 in [62]).

Let us recall the following definition related to nonlinear evaluations (see, e.g., Peng [65]).

Definition 2.26. We say that an $\mathbb{F}$-optional process η is an $\mathcal{E}^i$-*supermartingale* (resp. an $\mathcal{E}^i$-*submartingale*, an $\mathcal{E}^i$-*martingale*) on $[0, T]$ if $\eta_s \geq \mathcal{E}^i_{s,t}(\eta_t)$ (resp. $\eta_s \leq \mathcal{E}^i_{s,t}(\eta_t)$, $\eta_s = \mathcal{E}^i_{s,t}(\eta_t)$) for $0 \leq s \leq t \leq T$.

We henceforth denote by $X(x^i) := \bar{V}^i(x^i) - X^h$ the *issuer's relative reward* and we assume that $X(x^i)$ is a square-integrable process belonging to $\mathcal{H}^2(Q)$. Then, by Assumption 2.25, the BSDE on $[0, T]$

$$Y_t = X_T(x^i) + \int_t^T g_i(u, Y_u, Z_u)\, dQ_u - \int_t^T Z_u^*\, dM_u - (A_T - A_t)$$

has a unique solution $(Y, Z) = (Y^{x^i}, Z^{x^i}) \in \mathcal{H}^2(Q) \times \mathcal{L}^2(M)$. Furthermore, we postulate that the process $X - A$ belongs to the space $\mathcal{S}^2$ and we work under the following postulate, which is justified by Theorem 3.1 in [63] (see also [2, 3, 21, 33, 34, 36, 37, 46, 47, 48, 54, 59, 66, 69] for various results on reflected BSDEs).

Assumption 2.27. For a fixed $x^i \in \mathbb{R}$, the RBSDE with the lower obstacle $X_t := X_t(x^i) = \bar{V}^i_t(x^i) - X^h_t$ for all $t \in [0,T]$

$$\begin{cases} dY_t = -g_i(t, Y_t, Z_t)\, dQ_t + Z^*_t\, dM_t + dA_t - dK_t, \\ Y_T = X_T, \quad Y_t \geq X_t, \quad \int_0^T (Y_t - X_t)\, dK^c_t = 0, \quad \Delta K^d_t = K^d_t \mathbb{1}_{\{Y_{t-} = X_{t-}\}}, \end{cases} \tag{2.10}$$

has a unique solution $(Y,Z,K) = (Y^{x^i}, Z^{x^i}, K^{x^i})$ such that $(Y,Z,K) \in \mathcal{H}^2(Q) \times \mathcal{L}^2(M) \times \mathcal{A}^2$ where Z is an $\mathbb{F}$-predictable process and $Y - A$ is an RCLL process.

The following definition hinges on the concept of the nonlinear optimal stopping problem studied in Section 5. Although the issuer cannot exercise the contract, Definition 2.28 is essential in the analysis of his pricing and hedging problem.

Definition 2.28. We say that $\overline{v}^i(x^i, \mathcal{C}^a) \in \mathbb{R}$ is the *value* of the issuer's optimal stopping problem for $\mathcal{C}^a$ if

$$\overline{v}^i(x^i, \mathcal{C}^a) = \sup_{\tau \in \mathcal{T}} \mathcal{E}^i_{0,\tau}(X_\tau(x^i))$$

where $X_t(x^i) = \bar{V}^i_t(x^i) - X^h_t$ for all $t \in [0,T]$.

The following assumption is justified by Theorem 5.3 from the appendix, which is valid under suitable assumptions on the RBSDE (2.10).

Assumption 2.29. The value $\overline{v}^i(x^i, \mathcal{C}^a)$ to the issuer's optimal stopping problem exists and satisfies $\overline{v}^i(x^i, \mathcal{C}^a) = Y_0$.

We are ready to analyze the issuer's minimum superhedging cost. Although the issuer's initial endowment x^i and his benchmark wealth $\bar{V}^i(x^i)$ are not considered in Dumitrescu et al. [26], the proofs of Proposition 2.30 and Theorem 3.4 in [26] are based on similar arguments and thus the proof of Proposition 2.30 is given here for the sake of completeness.

Proposition 2.30. *If Assumptions 2.3–2.29 are satisfied and $\mathcal{E}^i$ has the comparison property, then the issuer's minimum superhedging cost is well defined and satisfies*

$$\breve{p}^{s,i}(x^i, \mathcal{C}^a) = \overline{v}^i(x^i, \mathcal{C}^a) - x^i = Y_0 - x^i$$

where $(Y,Z,K) = (Y^{x^i}, Z^{x^i}, K^{x^i})$ is the unique solution to the RBSDE (2.10).

Proof. We first prove that $\underline{p}^{s,i}(x^i, \mathcal{C}^a) \leq Y_0 - x^i$. It suffices to show that for the initial value $p' := Y_0 - x^i$, we can find an issuer's superhedging strategy, that is, there exists a trading strategy $\varphi' \in \Psi^i(x^i + p', A)$ such that $V^i_t(x^i + p', \varphi') \geq X_t(x^i)$ for all $t \in [0,T]$. To this end, we set $(p', \varphi') = (Y_0 - x^i, Z)$ where $(Y,Z,K) = (Y^{x^i}, Z^{x^i}, K^{x^i})$ is the unique solution to the RBSDE (2.10). Then, on the one hand, the value process $V^i = V^i(x^i + p', \varphi')$ is a unique solution to the following SDE where the initial value $V^i_0 = Y_0$ and the process Z are fixed

$$dV^i_t = -g_i(t, V^i_t, Z_t)\, dQ_t + Z^*_t\, dM_t + dA_t. \tag{2.11}$$

On the other hand, if (Y,Z,K) solves the RBSDE (2.10), then the process $\widetilde{Y} = Y$ can also be seen as a unique strong solution to the following SDE

$$d\widetilde{Y}_t = -g_i(t, \widetilde{Y}_t, Z_t)\, dQ_t + Z^*_t\, dM_t + dA_t - dK_t$$

where, once again, the initial value $\widetilde{Y}_0 = Y_0$ and the processes Z and K are given. Therefore, from Lemma 2.23 with $g_1 = g_2 = g$ we infer that $V^i_t \geq \widetilde{Y}_t = Y_t$ for all $t \in [0,T]$. Since $Y_t \geq X_t(x^i)$ for all $t \in [0,T]$, we conclude that $V^i_t \geq X_t(x^i)$ for all $t \in [0,T]$. Consequently, $(x^i + p', \varphi') = (Y_0, Z)$ is an issuer's superhedging strategy and thus $\underline{p}^{s,i}(x^i, \mathcal{C}^a) \leq Y_0 - x^i$.

We will now show that $\underline{p}^{s,i}(x^i,\mathcal{C}^a) \geq Y_0 - x^i$. Let us consider an arbitrary $p \in \mathbb{R}$ for which there exists $\varphi \in \Psi^i(x^i+p,A)$ such that (p,φ) satisfy (SH). If we can show that $x^i + p \geq Y_0$, then the inequality $\underline{p}^{s,i}(x^i,\mathcal{C}^a) \geq Y_0 - x^i$ will hold by the definition of the lower bound $\underline{p}^{s,i}(x^i,\mathcal{C}^a)$. To this end, we observe that $V^i_\tau(x^i+p,\varphi) \geq X_\tau(x^i)$ for every $\tau \in \mathcal{T}$ since, by Definition 2.5, we have that $V^i_t(x^i+p,\varphi) \geq X_t(x^i)$ for all $t \in [0,T]$.

Consequently, by applying the mapping $\mathcal{E}^i$ to both sides and using the comparison property of $\mathcal{E}^i$, we obtain

$$x^i + p = \mathcal{E}^i_{0,\tau}(V^i_\tau(x^i+p,\varphi)) \geq \mathcal{E}^i_{0,\tau}(X_\tau(x^i)).$$

Since $\tau \in \mathcal{T}$ is arbitrary, we conclude that $x^i + p \geq \sup_{\tau\in\mathcal{T}} \mathcal{E}^i_{0,\tau}(X_\tau(x^i)) = \overline{v}^i(x^i,\mathcal{C}^a) = Y_0$ where the second equality follows from Assumption 2.29. Hence $\underline{p}^{s,i}(x^i,\mathcal{C}^a) \geq Y_0 - x^i$ and thus we conclude that the equality $\underline{p}^{s,i}(x^i,\mathcal{C}^a) = Y_0 - x^i$ is valid.

Finally, from the first part of the proof, we know that for $p' = Y_0 - x^i$ there exists a trading strategy $\varphi' = Z \in \Psi^i(x^i+p',A)$ such that $V^i_t(x^i+p',\varphi') \geq X_t(x^i)$ for all $t \in [0,T]$ so that $Y_0 - x^i \in \mathcal{H}^{s,i}(x^i)$. Consequently, we have that $\underline{p}^{s,i}(x^i,\mathcal{C}^a) = \breve{p}^{s,i}(x^i,\mathcal{C}^a) = Y_0 - x^i$. □

Definition 2.31. A stopping time $\tau^* \in \mathcal{T}$ is called a *solution* to the issuer's optimal stopping problem if $\overline{v}^i(x^i,\mathcal{C}^a) = \widehat{v}^i(x^i,\mathcal{C}^a)$ where

$$\widehat{v}^i(x^i,\mathcal{C}^a) := \mathcal{E}^i_{0,\tau^*}(X_{\tau^*}(x^i)) = \max_{\tau\in\mathcal{T}} \mathcal{E}^i_{0,\tau}(X_\tau(x^i))$$

where $X_t(x^i) = \bar{V}^i_t(x^i) - X^h_t$ for all $t \in [0,T]$.

For the first main result in this section, Theorem 2.34, we also need the following assumption.

Assumption 2.32. The stopping time $\tau^i := \inf\{t \in [0,T] \,|\, Y_t = X_t(x^i)\}$ is a (not necessarily unique) solution to the issuer's optimal stopping problem so that $\widehat{v}^i(x^i,\mathcal{C}^a) = \mathcal{E}^i_{0,\tau^i}(X_{\tau^i}(x^i))$.

Remark 2.33. It is possible to check that Assumption 2.32 is valid when the process $X - A$ is left-upper-semicontinuous along stopping times. Indeed, on the one hand, from Remark 5.4, we know that under such assumption, the process K is continuous. On the other hand, using the definition of τ^i and recalling the right-continuity of Y_{τ^i} and $X_{\tau^i}(x^i)$, we deduce that $Y_{\tau^i} = X_{\tau^i}(x^i)$ and, by the minimality conditions in (2.10), we have that $K = 0$ on $[0,\tau^i)$. The continuity of K implies in turn that $K = 0$ on $[0,\tau^i]$, so that Y is an $\mathcal{E}^i$-martingale on $[0,\tau^i]$, we get $\mathcal{E}^i_{0,\tau^i}(Y_{\tau^i}) = Y_0$. In view of Assumption 2.29, we have that $Y_0 = \overline{v}^i(x^i,\mathcal{C}^a)$ and thus the equalities $\mathcal{E}^i_{0,\tau^i}(X_{\tau^i}(x^i)) = \overline{v}^i(x^i,\mathcal{C}^a) = \widehat{v}^i(x^i,\mathcal{C}^a)$ hold, which means that τ^i is a solution to the issuer's optimal stopping problem.

Theorem 2.34. *Let Assumptions 2.3–2.32 be satisfied and let* $(Y,Z,K) = (Y^{x^i},Z^{x^i},K^{x^i})$ *be the unique solution to the RBSDE* (2.10)*. If* $\mathcal{E}^i$ *has the strict comparison property, then the following assertions are valid:*
(i) the pair $(Y_0 - x^i, Z)$ *is an issuer's replicating strategy for* $\mathcal{C}^a$ *and* τ^i *is an issuer's break-even time for the pair* $(Y_0 - x^i, Z)$*,*
(ii) the issuer's minimum superhedging and replication costs satisfy

$$\breve{p}^{r,i}(x^i,\mathcal{C}^a) = \breve{p}^{s,i}(x^i,\mathcal{C}^a) = Y_0 - x^i = \mathcal{E}^i_{0,\tau^i}(X_{\tau^i}(x^i)) - x^i = \widehat{v}^i(x^i,\mathcal{C}^a) - x^i,$$

(iii) the issuer's acceptable price $p^i(x^i,\mathcal{C}^a)$ *is well defined and*

$$p^i(x^i,\mathcal{C}^a) = \widehat{p}^{f,i}(x^i,\mathcal{C}^a) = \breve{p}^{r,i}(x^i,\mathcal{C}^a) = \breve{p}^{s,i}(x^i,\mathcal{C}^a). \tag{2.12}$$

Proof. Consider the solution $(Y,Z,K) = (Y^{x^i},Z^{x^i},K^{x^i})$ to the RBSDE (2.10). From (2.1) and Proposition 2.30, we already know that

$$Y_0 - x^i = \underline{p}^{s,i}(x^i,\mathcal{C}^a) = \breve{p}^{s,i}(x^i,\mathcal{C}^a) \leq \underline{p}^{r,i}(x^i,\mathcal{C}^a).$$

Therefore, to establish the equality $\breve{p}^{r,i}(x^i,\mathcal{C}^a) = \breve{p}^{s,i}(x^i,\mathcal{C}^a)$, it is enough to show that the trading strategy $(p',\varphi') = (Y_0 - x^i, Z)$, which is already known to be an issuer's superhedging strategy (see the proof of Proposition 2.30), is also an issuer's replicating strategy for $\mathcal{C}^a$. We first note that the definition of τ^i and the right-continuity of the issuer's relative reward $X(x^i)$ and the solution Y to the BSDE yield the equality $X_{\tau^i}(x^i) = Y_{\tau^i}$. Consequently, we have that

$$Y_0 = \widehat{v}^i(x^i,\mathcal{C}^a) = \mathcal{E}^i_{0,\tau^i}(X_{\tau^i}(x^i)) = \mathcal{E}^i_{0,\tau^i}(Y_{\tau^i})$$

where the first two equalities follow from Assumptions 2.29 and 2.32, respectively. We will now show that $K_{\tau^i} = 0$. Since (Y,Z,K) solves the RBSDE (2.10), we know that

$$Y_0 = Y_{\tau^i} + \int_0^{\tau^i} g_i(u,Y_u,Z_u)\,dQ_u - \int_0^{\tau^i} Z^*_u\,dM_u - A_{\tau^i} + K_{\tau^i}.$$

Hence $Y_0 = \mathcal{E}^i_{0,\tau^i}(Y_{\tau^i} + K_{\tau^i})$ so that $\mathcal{E}^i_{0,\tau^i}(Y_{\tau^i}) = \mathcal{E}^i_{0,\tau^i}(Y_{\tau^i} + K_{\tau^i})$. From the strict comparison property of $\mathcal{E}^i$, we conclude that $K_{\tau^i} = 0$ and thus, for all $t \in [0,\tau^i]$,

$$Y_t = Y_0 - \int_0^t g_i(u,Y_u,Z_u)\,dQ_u + \int_0^t Z^*_u\,dM_u + A_t.$$

Finally, using the equality $V^i_0(Y_0,Z) = Y_0$ and the postulated uniqueness of a solution to the SDE (2.11), we obtain the equality $V^i_t(Y_0,Z) = Y_t$ on $[0,\tau^i]$ and thus, in particular, $V^i_{\tau^i}(Y_0,Z) = Y_{\tau^i} = X_{\tau^i}(x^i)$. We have thus shown that τ^i is an issuer's break-even time for the pair $(Y_0 - x^i, Z)$ so that the pair $(p',\varphi') = (Y_0,Z)$ is an issuer's replicating strategy for $\mathcal{C}^a$. Assertion (ii) now follows easily from Proposition 2.30.

For part (iii), it suffices to show that $\breve{p}^{r,i}$, where the variables $(x^i,\mathcal{C}^a)$ are suppressed, belongs to $\mathcal{H}^{f,i}(x^i)$ or, equivalently, that $\breve{p}^{r,i} < p$ for every $p \in \mathcal{H}^{a,i}(x^i)$ (recall that $\mathcal{H}^{a,i}(x^i)$ is the complement of $\mathcal{H}^{f,i}(x^i)$). To this end, we will argue by contradiction. Assume that $\breve{p}^{r,i} \in \mathcal{H}^{a,i}(x^i)$ so that there exists a strategy $\breve{\varphi} \in \Psi^i(x^i + \breve{p}^{r,i}, A)$ such that $(\breve{p}^{r,i},\breve{\varphi})$ satisfy (AO). Then we have, for every $\tau \in \mathcal{T}$,

$$\mathbb{P}\big(V^i_\tau(x^i + \breve{p}^{r,i},\breve{\varphi}) \geq X_\tau(x^i)\big) = 1 \;\text{ and }\; \mathbb{P}\big(V^i_\tau(x^i + \breve{p}^{r,i},\breve{\varphi}) > X_\tau(x^i)\big) > 0.$$

Let us now take $\tau = \tau^i$. By applying the mapping $\mathcal{E}^i$ to both sides, we obtain

$$x^i + \breve{p}^{r,i} = \mathcal{E}^i_{0,\tau^i}\big(V^i_{\tau^i}(x^i + \breve{p}^{r,i},\breve{\varphi})\big) > \mathcal{E}^i_{0,\tau^i}(X_{\tau^i}(x^i)) = x^i + \breve{p}^{r,i}$$

where the last equality follows from part (ii). This is an obvious contradiction and thus we have shown that $\breve{p}^{r,i}$ is not in $\mathcal{H}^{a,i}(x^i)$. Recall that either $\mathcal{H}^{a,i}(x^i) = [\underline{p}^{a,i},\infty)$ or $\mathcal{H}^{a,i}(x^i) = (\underline{p}^{a,i},\infty)$ and we claim that in fact the latter case is true. Indeed, from Assumption 2.25, Lemma 2.8 and part (ii), we have $\breve{p}^{r,i} = \breve{p}^{s,i} = \underline{p}^{a,i}$ and, since $\breve{p}^{r,i}$ is not in $\mathcal{H}^{a,i}(x^i)$, we have that $\mathcal{H}^{a,i}(x^i) = (\underline{p}^{a,i},\infty)$. Obviously, $\breve{p}^{r,i} < p$ for every $p \in \mathcal{H}^{a,i}(x^i)$ and thus $\breve{p}^{r,i}$ belongs to $\mathcal{H}^{f,i}(x^i)$ meaning that the set $\mathcal{H}^{f,r,i}(x^i)$ is nonempty. All equalities in (2.12) now follow from Proposition 2.12. □

2.5 Issuer's break-even times

Throughout this section, we postulate that the assumptions of Theorem 2.34 are satisfied and the contract $\mathcal{C}^a$ is traded at time 0 at the issuer's acceptable price $p^i = p^i(x^i,\mathcal{C}^a)$. From Definition 2.10 and the proof of Theorem 2.34, we know that there exists a pair $(\varphi',\tau^i) \in \Psi^i(x^i + p^i, A) \times \mathcal{T}$ such that (p^i,φ') satisfy (SH) and (p^i,φ',τ^i) satisfy (BE), specifically, $\varphi' = Z$ and $p^i = Y_0 - x^i$ where (Y,Z,K) is the unique solution to the reflected BSDE (2.10).

Our next goal is to provide a detailed characterization of all *issuer's break-even times* (see Definition 2.9) associated with his replicating strategy (p^i,φ'). Notice that an issuer's break-even time is a purely theoretical concept and it should not be confused with the holder's *rational exercise time* (see Definition 3.20).

To establish Theorem 2.37, which is the second main result, we will use the following property of solutions to the issuer's BSDE, whose validity is supported by Theorem 6.1 in Nie and Rutkowski [62].

Assumption 2.35. The following variant of the comparison property holds: if for $l=1,2$

$$\begin{cases} dY^l_s = -g_i(s,Y^l_s,Z^l_s)\,dQ_s + Z^{l*}_s\,dM_s + dA^l_s, \\ Y^l_\rho = \xi_l, \end{cases}$$

where $\rho\in\mathcal{T}$, $\xi_1\ge\xi_2$ and the process A^1-A^2 is nonincreasing, then $Y^1_s\ge Y^2_s$ for all $s\in[0,\rho]$.

Lemma 2.36. *If (Y,Z,K) is a unique solution to the RBSDE* (2.10) *and Assumption 2.35 is valid, then the process Y is a strong $\mathcal{E}^i$-supermartingale on $[0,T]$, in the sense that $Y_\sigma\ge\mathcal{E}^i_{\sigma,\rho}(Y_\rho)$ for every $\sigma,\rho\in\mathcal{T}$ such that $\sigma\le\rho$.*

Proof. On the one hand, for any fixed $t\in(0,T]$ the process $\bar Y_s := \mathcal{E}^i_{s,t}(Y_t),\ s\in[0,t]$ solves the BSDE

$$\begin{cases} d\bar Y_s = -g_i(s,\bar Y_s,\bar Z_s)\,dQ_s + \bar Z^*_s\,dM_s + dA_s, \\ \bar Y_t = Y_t. \end{cases}$$

On the other hand, if (Y,Z,K) solves the RBSDE (2.10), then, for any fixed $[0,t]$, the pair $(\widetilde Y,\widetilde Z)=(Y,Z)$ is a unique solution to the BSDE

$$\begin{cases} d\widetilde Y_s = -g_i(s,\widetilde Y_s,\widetilde Z_s)\,dQ_s + \widetilde Z^*_s\,dM_s + dA_s - dK_s, \\ \widetilde Y_t = Y_t, \end{cases}$$

where K is a predetermined nondecreasing process. Therefore, in view of Assumption 2.35 with $A^1=-K$ and $A^2=0$, the inequality $Y_s\ge\mathcal{E}^i_{s,t}(Y_t)$ holds for all $s\in[0,t]$ and thus Y is an $\mathcal{E}^i$-supermartingale. Using similar arguments, one can show that for any $\sigma,\rho\in\mathcal{T}$ such that $\sigma\le\rho$, the inequality $Y_\sigma\ge\mathcal{E}^i_{\sigma,\rho}(Y_\rho)$ holds. □

The following result shows that, under mild assumptions, an issuer's break-even time can be identified with a solution to the issuer's optimal stopping problem. We stress once again that an issuer's break-even time cannot be identified with the *holder's rational exercise time* introduced in Definition 3.8 in [26], and thus Theorem 2.37 does not support Proposition 3.9 in [26]. We write $p^i=p^i(x^i,\mathcal{C}^a)$.

Theorem 2.37. *Let Assumptions 2.3–2.35 be satisfied and the strict comparison property of $\mathcal{E}^i$ hold. If $(Y,Z,K)=(Y^{x^i},Z^{x^i},K^{x^i})$ is the unique solution to the RBSDE* (2.10)*, for the process $\varphi'=Z\in\Psi^i(x^i+p^i,A)$ and an arbitrary $\tau'\in\mathcal{T}$ the following assertions are equivalent:*
(i) τ' is an issuer's break-even time for the pair $(p^i,\varphi')\in\mathbb{R}\times\Psi^i(x^i+p^i,A)$,
(ii) the triplet (p^i,φ',τ') satisfies (NA)*,*
(iii) the equality $V^i_{\tau'}(x^i+p^i,\varphi')=X_{\tau'}(x^i)$ holds,
(iv) $X_{\tau'}(x^i)=Y_{\tau'}$ and $K_{\tau'}=0$ and thus Y is an $\mathcal{E}^i$-martingale on $[0,\tau']$,
(v) τ' is a solution to the issuer's optimal stopping problem so that $\mathcal{E}^i_{0,\tau'}(X_{\tau'}(x^i)) = \widehat v^i(x^i,\mathcal{C}^a)$.
The stopping time $\tau^i=\inf\{t\in[0,T]\,|\,Y_t=X_t(x^i)\}$ is the earliest issuer's break-even time for (p^i,φ').

Proof. Recall that if $\varphi' = Z$, then the pair (p^i, φ') is an issuer's superhedging strategy for $\mathcal{C}^a$ (see the proof of Proposition 2.30). It is thus clear that assertions (i), (ii) and (iii) are equivalent.
(iii) $\Rightarrow$ (iv). From the proof of Proposition 2.30, we know that $V^i_t(x^i + p^i, \varphi') \geq Y_t \geq X_t(x^i)$ for all $t \in [0,T]$ and thus, in particular, the inequality $V^i_\tau(x^i + p^i, \varphi') \geq Y_\tau \geq X_\tau(x^i)$ holds for every $\tau \in \mathcal{T}$. Since we assumed that (iii) holds, we have $V^i_{\tau'}(x^i + p^i, \varphi') = X_{\tau'}(x^i)$ and thus $V^i_{\tau'}(Y_0, \varphi') = Y_{\tau'} = X_{\tau'}(x^i)$ (recall from Theorem 2.34 that $p^i = Y_0 - x^i$).

It thus remains to show that $K_{\tau'} = 0$. Since the process $V^i = V^i(Y_0, \varphi')$ satisfies the SDE (2.11), it is an $\mathcal{E}^i$-martingale and thus we obtain the following equalities

$$\mathcal{E}^i_{0,\tau'}(Y_{\tau'}) = \mathcal{E}^i_{0,\tau'}\big(V^i_{\tau'}(Y_0, \varphi')\big) = Y_0.$$

Using similar arguments as in the proof of Theorem 2.34, one can now show that $K_{\tau'} = 0$.
(iv) $\Rightarrow$ (iii). By assumption, $Y_{\tau'} = X_{\tau'}(x^i)$ and $K_{\tau'} = 0$ and thus the RBSDE (2.10) reduces to the following BSDE on $[0, \tau']$

$$\begin{cases} dY_t = -g_i(t, Y_t, Z_t)\, dQ_t + Z^*_t\, dM_t + dA_t, \\ Y_{\tau'} = X_{\tau'}(x^i). \end{cases}$$

The above BSDE can also be represented in the forward manner on the stochastic interval $[0, \tau']$

$$dY_t = -g_i(t, Y_t, Z_t)\, dQ_t + Z^*_t\, dM_t + dA_t$$

where the initial value Y_0 and the process Z are given. Similarly, the wealth process $V^i := V^i(x^i + p^i, \varphi') = V^i(Y_0, Z)$ solves the following SDE, for $t \in [0,T]$,

$$dV^i_t = -g_i(t, V^i_t, Z_t)\, dQ_t + Z^*_t\, dM_t + dA_t$$

with initial condition $V^i_0 = Y_0$. From the uniqueness of a solution to the above SDE, we deduce that $V^i_t = Y_t$ for $t \in [0, \tau']$. In particular, $V^i_{\tau'}(x^i + p^i, \varphi') = Y_{\tau'} = X_{\tau'}(x^i)$, as was required to show.
(iv) $\Rightarrow$ (v). The $\mathcal{E}^i$-martingale property of Y on $[0, \tau']$ gives $\mathcal{E}^i_{0,\tau'}(Y_{\tau'}) = Y_0$. In view of Assumption 2.29, we have that $Y_0 = \overline{v}^i(x^i, \mathcal{C}^a)$ and thus the equalities $\mathcal{E}^i_{0,\tau'}(X_{\tau'}) = \overline{v}^i(x^i, \mathcal{C}^a) = \widehat{v}^i(x^i, \mathcal{C}^a)$ hold, which means that τ' is a solution to the issuer's optimal stopping problem.
(v) $\Rightarrow$ (iv). From condition (v) and Assumption 2.29, we obtain the equality $Y_0 = \mathcal{E}^i_{0,\tau'}(X_{\tau'}(x^i))$. If $Y_{\tau'} \geq X_{\tau'}$ and $Y_{\tau'} \neq X_{\tau'}(x^i)$, then the strict comparison property of $\mathcal{E}^i$ yields

$$Y_0 = \mathcal{E}^i_{0,\tau'}(Y_{\tau'}) > \mathcal{E}^i_{0,\tau'}(X_{\tau'}) = Y_0,$$

which is a contradiction. This shows that $Y_{\tau'} = X_{\tau'}(x^i)$. As in the proof of the Theorem 2.34, $Y_0 = \mathcal{E}^i_{0,\tau'}(X_{\tau'}(x^i))$ yields $K_{\tau'} = 0$ and thus Y is an $\mathcal{E}^i$-martingale on $[0, \tau']$.

It remains to show that the last assertion is valid. In view of Assumption 2.32, τ^i is a solution to the issuer's optimal stopping problem and thus, from part (v), τ^i is an issuer's break-even time for (p^i, φ'). In view of part (iv), for any break-even time for (p^i, φ'), we have that $X_{\tau'}(x^i) = Y_{\tau'}$. The definition of τ^i now shows that it is the earliest issuer's break-even time for $\mathcal{C}^a$. □

3 Holder's pricing, hedging and exercising

After examining the issuer's problem, we now address the issues of pricing, hedging and exercising of an American contract from the perspective of the holder. Although some arguments used in this section are similar to those used when analyzing the issuer's problems, it is clear that essential modifications of definitions formulated for the issuer are required since the holder has also the right to exercise an American style option before its expiration date T.

3.1 Holder's unilateral pricing

We consider the holder's market model $\mathcal{M}^h = (\mathcal{S}, \mathcal{B}^h, \Psi^h)$, which is assumed to be arbitrage-free and may coincide with the issuer's model $\mathcal{M}^i$. We assume that he is endowed with the pre-trading initial wealth $x^h \in \mathbb{R}$ and his computations refer to the benchmark wealth process $\bar{V}^h(x^h)$. We use the shorthand notation $V^h(x^h - p, \psi) := V^h(x^h - p, \psi, -A)$ when there is no danger of confusion. Let us consider the extended market model $\mathcal{M}^{h,p}(\mathcal{C}^a)$ in which an American contract $\mathcal{C}^a$ is traded by the holder at time 0 at some initial price p where p is an arbitrary real number.

We first introduce the terminology used in the analysis of the holder's pricing, hedging and exercising problems. We say that $(p, \psi, \tau) \in \mathbb{R} \times \Psi^h(x^h - p, -A) \times \mathcal{T}$ satisfy:

$$\text{(AO}'\text{)} \iff V^h_\tau(x^h - p, \psi) - X^h_\tau \geq \bar{V}^h_\tau(x^h) \text{ and } \mathbb{P}\big(V^h_\tau(x^h - p, \psi) - X^h_\tau > \bar{V}^h_\tau(x^h)\big) > 0,$$

$$\text{(SH}'\text{)} \iff V^h_\tau(x^h - p, \psi) - X^h_\tau \geq \bar{V}^h_\tau(x^h),$$

$$\text{(BE}'\text{)} \iff V^h_\tau(x^h - p, \psi) - X^h_\tau = \bar{V}^h_\tau(x^h),$$

$$\text{(NA}'\text{)} \iff V^h_\tau(x^h - p, \psi) - X^h_\tau = \bar{V}^h_\tau(x^h) \text{ or } \mathbb{P}\big(V^h_\tau(x^h - p, \psi) - X^h_\tau < \bar{V}^h_\tau(x^h)\big) > 0.$$

Similarly to the issuer's case, property (AO′) (resp. (SH′)) is called the *arbitrage opportunity* (resp. *superhedging*) condition for the holder. Condition (BE′) leads to the following definition.

Definition 3.1. If the equality $V^h_{\tau'}(x^h - p, \psi) - X^h_{\tau'} = \bar{V}^h_{\tau'}(x^h)$ holds, then the stopping time $\tau' \in \mathcal{T}$ is called a *holder's break-even time* for the pair $(p, \psi) \in \mathbb{R} \times \Psi^h(x^h - p, -A)$.

The concept of a holder's arbitrage opportunity reflects the fact that the holder has the right to exercise an American contract, that is, to conveniently choose a stopping time τ at which the contract is settled and terminated. Specifically, a *holder's arbitrage opportunity in* $\mathcal{M}^{h,p}(\mathcal{C}^a)$ is a triplet $(p, \psi, \tau) \in \mathbb{R} \times \Psi^h(x^h - p, -A) \times \mathcal{T}$ satisfying condition (AO′). The following assumption is a holder's counterpart of Assumption 2.3.

Assumption 3.2. The following *strict forward monotonicity* property holds: for all $x, p \in \mathbb{R}$, $\varphi \in \Psi^h(x - p, A)$ and $p' < p$ (resp. $p' > p$), there exists a trading strategy $\varphi' \in \Psi^h(x - p', A)$ such that $V^h_t(x - p', \varphi', A) > V^h_t(x - p, \varphi, A)$ (resp. $V^h_t(x - p', \varphi', A) < V^h_t(x - p, \varphi, A)$) for every $t \in [0, T]$.

From now on, it is assumed throughout that the processes $A, X^h, \bar{V}^h$ and the wealth process $V^h(y, \varphi, A)$ are $\mathbb{F}$-adapted and RCLL although, in fact, it would be enough to postulate that the processes $\bar{V}^h + X^h + A$ and $V^h(y, \varphi, A) + A$ are $\mathbb{F}$-adapted and RCLL.

3.1.1 Holder's fair prices

The concept of a holder's fair price is different from the corresponding notion for the issuer, not only quantitatively but also qualitatively, since only the holder has the right to exercise the contract and thus our results obtained for the issuer cannot be applied to the holder's pricing problem. A real number $p^{f,h}(x^h, \mathcal{C}^a)$ is a *holder's fair price* for $\mathcal{C}^a$ if no holder's arbitrage opportunity (p, ψ, τ) arises in the extended market $\mathcal{M}^{h,p}(\mathcal{C}^a)$ when $p = p^{f,h}(x^h, \mathcal{C}^a)$. Hence the set of all holder's fair prices equals

$$\mathcal{H}^{f,h}(x^h) := \big\{ p \in \mathbb{R} \,|\, \forall\, (\psi, \tau) \in \Psi^h(x^h - p, -A) \times \mathcal{T} \colon (p, \psi, \tau) \in \text{(NA}'\text{)} \big\}$$

and the lower bound for the holder's fair prices is given by

$$\underline{p}^{f,h}(x^h, \mathcal{C}^a) := \inf \big\{ p \in \mathbb{R} \,|\, p \text{ is a holder's fair price for } \mathcal{C}^a \big\} = \inf\, \mathcal{H}^{f,h}(x^h). \tag{3.1}$$

Definition 3.3. If the equality $\underline{p}^{f,h}(x^h, \mathcal{C}^a) = \min\, \mathcal{H}^{f,h}(x^h)$ holds, then $\underline{p}^{f,h}(x^h, \mathcal{C}^a)$ is denoted as $\check{p}^{f,h}(x^h, \mathcal{C}^a)$ and called the *holder's minimum fair price* for $\mathcal{C}^a$.

Lemma 3.4. *Let Assumption 3.2 be satisfied. If $p \in \mathcal{H}^{f,h}(x^h)$, then for any $p' > p$ we have that $p' \in \mathcal{H}^{f,h}(x^h)$ and thus, if $\mathcal{H}^{f,h}(x^h) \neq \emptyset$, then either $\mathcal{H}^{f,h}(x^h) = [\,\underline{p}^{f,h}(x^h, \mathcal{C}^a), \infty) = [\,\breve{p}^{f,h}(x^h, \mathcal{C}^a), \infty)$ or $\mathcal{H}^{f,h}(x^h) = (\underline{p}^{f,h}(x^h, \mathcal{C}^a), \infty)$.*

3.1.2 Holder's superhedging costs

As for the issuer, we also introduce the notion of a *superhedging cost* for the holder. The upper bound for holder's superhedging costs for $\mathcal{C}^a$ equals $\overline{p}^{s,h}(x^h, \mathcal{C}^a) := \sup\, \mathcal{H}^{s,h}(x^h)$ where

$$\mathcal{H}^{s,h}(x^h) := \big\{p \in \mathbb{R} \,|\, \exists\, (\psi, \tau) \in \Psi^h(x^h - p, -A) \times \mathcal{T} \colon (p, \psi, \tau) \in (\text{SH}')\big\}.$$

Definition 3.5. If the equality $\overline{p}^{s,h}(x^h, \mathcal{C}^a) = \max\, \mathcal{H}^{s,h}(x^h)$ holds, then $\overline{p}^{s,h}(x^h, \mathcal{C}^a)$ is denoted as $\widehat{p}^{s,h}(x^h, \mathcal{C}^a)$ and called the *holder's maximum superhedging cost* for $\mathcal{C}^a$.

The upper bound for holder's *strict superhedging costs* for $\mathcal{C}^a$ is given by $\overline{p}^{a,h}(x^h, \mathcal{C}^a) := \sup\, \mathcal{H}^{a,h}(x^h)$ where

$$\mathcal{H}^{a,h}(x^h) := \big\{p \in \mathbb{R} \,|\, \exists\, (\psi, \tau) \in \Psi^h(x^h - p, -A) \times \mathcal{T} \colon (p, \psi, \tau) \in (\text{AO}')\big\}.$$

Definition 3.6. If the equality $\overline{p}^{a,h}(x^h, \mathcal{C}^a) = \max\, \mathcal{H}^{a,h}(x^h)$ holds, then $\overline{p}^{a,h}(x^h, \mathcal{C}^a)$ is denoted as $\widehat{p}^{a,h}(x^h, \mathcal{C}^a)$ and called the *holder's maximum strict superhedging cost* for $\mathcal{C}^a$.

Since $\mathcal{H}^{a,h}(x^h)$ is the complement of $\mathcal{H}^{f,h}(x^h)$, we deduce from Lemma 3.4 that the equality $\overline{p}^{a,h}(x^h, \mathcal{C}^a) = \underline{p}^{f,h}(x^h, \mathcal{C}^a)$ is satisfied if Assumption 3.2 holds for Ψ^h and $-A$. Moreover, we have that either

$$\mathcal{H}^{a,h}(x^h) = (-\infty, \overline{p}^{a,c}(x^h, \mathcal{C}^a)) \text{ and } \mathcal{H}^{f,h}(x^h) = [\,\breve{p}^{f,h}(x^h, \mathcal{C}^a), \infty) \tag{3.2}$$

or

$$\mathcal{H}^{a,h}(x^h) = (-\infty, \widehat{p}^{a,c}(x^h, \mathcal{C}^a)\,] \text{ and } \mathcal{H}^{f,h}(x^h) = (\underline{p}^{f,h}(x^h, \mathcal{C}^a), \infty). \tag{3.3}$$

Lemma 3.7. *If Assumption 3.2 is valid, then the equality $\overline{p}^{s,h}(x^h, \mathcal{C}^a) = \overline{p}^{a,h}(x^h, \mathcal{C}^a)$ holds and thus $\underline{p}^{f,h}(x^h, \mathcal{C}^a) = \overline{p}^{s,h}(x^h, \mathcal{C}^a) = \overline{p}^{a,h}(x^h, \mathcal{C}^a)$.*

3.1.3 Holder's acceptable price

The next step is to examine costs of the holder's replication. The upper bound for *holder's replication costs* for $\mathcal{C}^a$ is given by $\overline{p}^{r,h}(x^h, \mathcal{C}^a) := \sup\, \mathcal{H}^{r,h}(x^h)$ where

$$\mathcal{H}^{r,h}(x^h) := \big\{p \in \mathbb{R} \,|\, \exists\, (\psi, \tau) \in \Psi^h(x^h - p, -A) \times \mathcal{T} \colon (p, \psi, \tau) \in (\text{BE}')\big\}.$$

Definition 3.8. If the equality $\overline{p}^{r,h}(x^h, \mathcal{C}^a) = \max\, \mathcal{H}^{r,h}(x^h)$ holds, then $\overline{p}^{r,h}(x^h, \mathcal{C}^a)$ is denoted as $\widehat{p}^{r,h}(x^h, \mathcal{C}^a)$ and called the *holder's maximum replication cost* for $\mathcal{C}^a$.

To establish the existence of the holder's acceptable price, as given by Definition 3.9 below, we employ the idea of the holder's fair replication costs. The upper bound for *holder's fair replication costs* for $\mathcal{C}^a$ is given by $\overline{p}^{f,r,h}(x^h, \mathcal{C}^a) := \sup\, \mathcal{H}^{f,r,h}(x^h)$ where

$$\begin{aligned}\mathcal{H}^{f,r,h}(x^h) := \big\{p \in \mathbb{R} \,|\, &\exists\, (\psi, \tau) \in \Psi^h(x^h - p, -A) \times \mathcal{T} \colon (p, \psi, \tau) \in (\text{BE}')\ \& \\ &\forall\, (\psi', \tau') \in \Psi^h(x^h - p, -A) \times \mathcal{T} \colon (p, \psi', \tau') \in (\text{NA}')\big\}.\end{aligned}$$

Definition 3.9. If the equality $\overline{p}^{f,r,h}(x^h, \mathcal{C}^a) = \max\, \mathcal{H}^{f,r,h}(x^h)$ holds, then $\overline{p}^{f,r,h}(x^h, \mathcal{C}^a)$ is denoted as $\widehat{p}^{f,r,h}(x^h, \mathcal{C}^a)$ and called the *holder's maximum fair replication cost* for $\mathcal{C}^a$. If the set $\mathcal{H}^{f,r,h}(x^h)$ has a unique element, then it is denoted as $p^h(x^h, \mathcal{C}^a)$ and called the *holder's acceptable price*.

It is elementary to check that $\mathcal{H}^{s,h}(x^h) \supseteq \mathcal{H}^{r,h}(x^h) \supseteq \mathcal{H}^{f,r,h}(x^h) = \mathcal{H}^{r,h}(x^h) \cap \mathcal{H}^{f,h}(x^h)$ and thus, in view of Lemma 3.7, we have

$$\underline{p}^{f,h}(x^h, \mathcal{C}^a) = \overline{p}^{s,h}(x^h, \mathcal{C}^a) \geq \overline{p}^{r,h}(x^h, \mathcal{C}^a) \geq \overline{p}^{f,r,h}(x^h, \mathcal{C}^a). \tag{3.4}$$

The following result corresponds to Proposition 2.12 for the issuer.

Proposition 3.10. *If Assumption 3.2 is satisfied and $\mathcal{H}^{f,r,h}(x^h) \neq \emptyset$, then it has a unique element. Moreover, the holder's acceptable price $p^h(x^h, \mathcal{C}^a)$ is finite and satisfies $p^h(x^h, \mathcal{C}^a) = \breve{p}^{f,h}(x^h, \mathcal{C}^a) = \widehat{p}^{r,h}(x^h, \mathcal{C}^a) = \widehat{p}^{s,h}(x^h, \mathcal{C}^a)$.*

3.2 Holder's acceptable price via a reflected BSDE

We maintain the setup described in Section 2.3 and we focus on holder's wealth process and related BSDE. The following Assumptions 3.11–3.13 are natural holder's counterparts of the issuer's Assumptions 2.21–2.27. Let $\Psi^h(y, -A)$ be the class of all holder's admissible trading strategies with the initial wealth y.

Assumption 3.11. For every $\mathbb{F}$-predictable process ξ^h such that the integral $\int_0^\cdot \xi_u^{h*}\, dM_u$ is well defined, the wealth $V^h = V^h(y, \psi, -A)$ of the holder's admissible trading strategy $(y, \psi) \in \Psi^h(y, -A)$ is a unique $\mathbb{F}$-adapted solution to the SDE

$$V_t^h = y - \int_0^t g_h(u, V_u^h, \xi_u^h)\, dQ_u + \int_0^t \xi_u^{h*}\, dM_u - A_t. \tag{3.5}$$

In view of Lemma 2.23, it is clear that Assumption 3.2 is met when the holder's wealth $V^h(y, \psi, -A)$ is given by a unique solution to the SDE (3.5). The nonlinear evaluation $\mathcal{E}^{g_h, -A}$, which is defined through solutions to the BSDE

$$y_t = \zeta_s + \int_t^s g_h(u, y_u, z_u)\, dQ_u - \int_t^s z_u^*\, dM_u + A_s - A_t, \tag{3.6}$$

is henceforth denoted by $\mathcal{E}^h$ and called the *holder's evaluation*.

Assumption 3.12. For every $(s, \zeta_s) \in [0, T] \times L^2(\mathcal{F}_s)$, the BSDE (3.6) has a unique solution (y, z) on $[0, s]$ such that $(y, z) \in \mathcal{H}^2(Q) \times \mathcal{L}^2(M)$ and $y + A$ is an RCLL process so that the holder's evaluation $\mathcal{E}^h$ is well defined.

We assume that the *holder's relative reward*, which is given by $x_t(x^h) := \bar{V}_t^h(x^h) + X_t^h$ for all $t \in [0, T]$, belongs to $\mathcal{H}^2(Q)$.

Assumption 3.13. The RBSDE with the upper obstacle $x_t(x^h) = \bar{V}_t^h(x^h) + X_t^h$ for all $t \in [0, T]$

$$\begin{cases} dy_t = -g_h(t, y_t, z_t)\, dQ_t + z_t^*\, dM_t - dA_t + dk_t, \\ y_T = x_T, \quad y_t \leq x_t, \quad \int_0^T (x_t - y_t)\, dk_t^c = 0, \quad \Delta k_t^d = \Delta k_t^d \mathbb{1}_{\{y_{t-} = x_{t-}\}}, \end{cases} \tag{3.7}$$

has the unique solution $(y, z, k) = (y^{x^h}, z^{x^h}, k^{x^h})$ in the space $\mathcal{H}^2(Q) \times \mathcal{L}^2(M) \times \mathcal{A}^2$ where z is an $\mathbb{F}$-predictable process and $y + A$ is an RCLL process.

The following definition corresponds to Definition 2.28 for the issuer.

Definition 3.14. We say that $\underline{v}^h(x^h, \mathcal{C}^a) \in \mathbb{R}$ is the *value* of the holder's optimal stopping problem for $\mathcal{C}^a$ if

$$\underline{v}^h(x^h, \mathcal{C}^a) = \inf_{\tau \in \mathcal{T}} \mathcal{E}_{0,\tau}^h(x_\tau(x^h))$$

where $x_t(x^h) = \bar{V}_t^h(x^h) + X_t^h$ for all $t \in [0, T]$.

Assumption 3.15. The value $\underline{v}^h(x^h, \mathcal{C}^a)$ to the holder's optimal stopping problem exists and satisfies $\underline{v}^h(x^h, \mathcal{C}^a) = y_0$.

Proposition 3.16. *If Assumptions 3.2–3.15 are satisfied and the comparison property of $\mathcal{E}^h$ holds, then*

$$\overline{p}^{s,h}(x^h, \mathcal{C}^a) \leq x^h - \underline{v}^h(x^h, \mathcal{C}^a) = x^h - y_0 \tag{3.8}$$

where $(y, z, k) = (y^{x^h}, z^{x^h}, k^{x^h})$ *is the unique solution to the RBSDE* (3.7).

Proof. We will show that $\overline{p}^{s,h}(x^h, \mathcal{C}^a) \leq x^h - y_0$. By the definition of the supremum, it is enough to show that $x^h - y_0 \geq p$ for all $p \in \mathcal{H}^{s,h}(x^h)$. From the definition of $\mathcal{H}^{s,h}(x^h)$, we know that for any $p \in \mathcal{H}^{s,h}(x^h)$, there exists a pair $(\psi, \tau) \in \Psi^h(x^h - p, -A) \times \mathcal{T}$ such that $V^h_\tau(x^h - p, \psi) \geq x_\tau$. The comparison property of $\mathcal{E}^h$ gives

$$x^h - p = \mathcal{E}^h_{0,\tau}\big(V^h_\tau(x^h - p, \psi)\big) \geq \mathcal{E}^h_{0,\tau}(x_\tau(x^h))$$

and thus

$$x^h - p \geq \inf_{\tau \in \mathcal{T}} \mathcal{E}^h_{0,\tau}\big(V^h_\tau(x^h - p, \psi)\big) \geq \inf_{\tau \in \mathcal{T}} \mathcal{E}^h_{0,\tau}(x_\tau(x^h)) = \underline{v}^h(x^h, \mathcal{C}^a) = y_0$$

where the last equality follows from Assumption 3.15. We have thus shown that $\overline{p}^{s,h}(x^h, \mathcal{C}^a) \leq x^h - y_0 = x^h - \underline{v}^h(x^h, \mathcal{C}^a)$. □

Definition 3.17. A stopping time $\tau^* \in \mathcal{T}$ is called a *solution* to the holder's optimal stopping problem if $\underline{v}^h(x^h, \mathcal{C}^a) = \breve{v}^h(x^h, \mathcal{C}^a)$ where

$$\breve{v}^h(x^h, \mathcal{C}^a) := \mathcal{E}^h_{0,\tau^*}(x_{\tau^*}(x^h)) = \min_{\tau \in \mathcal{T}} \mathcal{E}^h_{0,\tau}(x_\tau(x^h)). \tag{3.9}$$

Recall that $x_t(x^h) = \bar{V}^h_t(x^h) + X^h_t$ for all $t \in [0, T]$.

In the next result, we work under the following postulate, which is known to hold under mild assumptions (see the appendix).

Assumption 3.18. The stopping time $\tau^h := \inf\{t \in [0, T] \,|\, y_t = x_t(x^h)\}$ is a solution to the holder's optimal stopping problem.

The following theorem deals with a solution to the holder's pricing problem for an American contract $\mathcal{C}^a$. One of our goals is to show that the set $\mathcal{H}^{f,r,h}(x^h)$ is nonempty and thus, in view of Proposition 3.10, the holder's acceptable price is well defined.

Theorem 3.19. *Let Assumption 3.2–3.18 be satisfied and let* $(y, z, k) = (y^{x^h}, z^{x^h}, k^{x^h})$ *be a unique solution to the RBSDE* (3.7). *If* $\mathcal{E}^h$ *has the strict comparison property, then:*
(i) $(x^h - y_0, z, \tau^h)$ *is a holder's replicating strategy for* $\mathcal{C}^a$*,*
(ii) the holder's maximum replication cost is well defined and satisfies

$$\widehat{p}^{r,h}(x^h, \mathcal{C}^a) = \widehat{p}^{s,h}(x^h, \mathcal{C}^a) = x^h - y_0 = x^h - \breve{v}^h(x^h, \mathcal{C}^a) = x^h - \mathcal{E}^h_{0,\tau^h}(x_{\tau^h}(x^h)),$$

(iii) the holder's acceptable price $p^h(x^h, \mathcal{C}^a)$ *is well defined and*

$$p^h(x^h, \mathcal{C}^a) = \breve{p}^{f,h}(x^h, \mathcal{C}^a) = \widehat{p}^{r,h}(x^h, \mathcal{C}^a) = \widehat{p}^{s,h}(x^h, \mathcal{C}^a).$$

Proof. We already know that $x^h - y_0 \geq \overline{p}^{s,h}(x^h, \mathcal{C}^a) \geq \overline{p}^{r,h}(x^h, \mathcal{C}^a)$ (see (3.4) and (3.8)). Hence to establish (i) and (ii), it suffices to show that if (y, z, k) is the unique solution to the RBSDE (3.7), then $(p', \psi', \tau') = (x^h - y_0, z, \tau^h)$ is a holder's replicating strategy. The wealth process $V^h = V^h(x^h - p', \psi')$ satisfies the SDE

$$dV^h_t = -g_h(t, V^h_t, z_t)\, dQ_t + z^*_t\, dM_t - dA_t \tag{3.10}$$

where the initial value $V^h_0 = y_0$ and the process z are given. The definition of τ^h and the right-continuity of the processes y and $x(x^h)$ ensure that $x_{\tau^h}(x^h) = y_{\tau^h}$ so that

$$y_0 = \breve{v}^h(x^h, \mathcal{C}^a) = \mathcal{E}^h_{0,\tau^h}(x_{\tau^h}(x^h)) = \mathcal{E}^h_{0,\tau^h}(y_{\tau^h})$$

where the second equality is a consequence of Assumption 3.18, and thus we see that $y_0 = \mathcal{E}^h_{0,\tau^h}(y_{\tau^h})$. Therefore, using the strict comparison property of $\mathcal{E}^h$ and simple arguments analogous to those used in the derivation of the equality $K_{\tau^i} = 0$ in the proof of Theorem 2.34, we obtain the equality $k_{\tau^h} = 0$. Since $k_t = 0$ on $[0, \tau^h]$, the RBSDE (3.7) can be interpreted on $[0, \tau^h]$ as the SDE

$$dy_t = -g_h(t, y_t, z_t)\, dQ_t + z^*_t\, dM_t - dA_t$$

where the initial value $y_0 = V^h_0$ and the process z are given. From the uniqueness of a solution to the SDE (3.10), it follows that $V^h = y$ on $[0, \tau^h]$. Hence $V^h_{\tau^h} = y_{\tau^h} = x_{\tau^h}(x^h)$ and thus the triplet $(p', \psi', \tau') = (x^h - y_0, z, \tau^h)$ is indeed a holder's replicating strategy.

For part (iii), we will first show that $\widehat{p}^{r,h}(x^h, \mathcal{C}^a) \in \mathcal{H}^{f,h}(x^h)$. In view of (3.2) and (3.3), it is enough to prove that $\widehat{p}^{r,h}(x^h, \mathcal{C}^a) > p$ for every $p \in \mathcal{H}^{a,h}(x^h)$. To this end, we argue by contradiction. Let us write $\widehat{p} = \widehat{p}^{r,h}(x^h, \mathcal{C}^a)$. Assume that $\widehat{p} \in \mathcal{H}^{a,h}(x^h)$ so that there exists $(\widehat{\varphi}, \widehat{\tau}) \in \Psi^h(x^h - \widehat{p}, -A) \times \mathcal{T}$ such that $(\widehat{p}, \widehat{\varphi}, \widehat{\tau})$ satisfies (AO′), that is,

$$\mathbb{P}\big(V^h_{\widehat{\tau}}(x^h - \widehat{p}, \widehat{\varphi}) \geq x_{\widehat{\tau}}(x^h)\big) = 1 \;\text{ and }\; \mathbb{P}\big(V^h_{\widehat{\tau}}(x^h - \widehat{p}, \widehat{\varphi}) > x_{\widehat{\tau}}(x^h)\big) > 0.$$

By applying the mapping $\mathcal{E}^h$, we obtain

$$x^h - \widehat{p} = \mathcal{E}^h_{0,\widehat{\tau}}\big(V^h_{\widehat{\tau}}(x^h - \widehat{p}^{r,h}, \widehat{\varphi})\big) > \mathcal{E}^h_{0,\widehat{\tau}}(x_{\widehat{\tau}}(x^h)) \geq \inf_{\tau \in \mathcal{T}} \mathcal{E}^h_{0,\tau}(x_\tau(x^h)) = \mathcal{E}^h_{0,\tau^h}(x_{\tau^h}(x^h)) = x^h - \widehat{p}$$

where the last equality comes from part (ii). This is a clear contradiction and thus we see that $\widehat{p}^{r,h}(x^h, \mathcal{C}^a) \notin \mathcal{H}^{a,h}(x^h)$. In general, either $\mathcal{H}^{a,h}(x^h) = (-\infty, \overline{p}^{a,h}(x^h, \mathcal{C}^a)]$ or $\mathcal{H}^{a,h}(x^h) = (-\infty, \overline{p}^{a,h}(x^h, \mathcal{C}^a))$ and we argue that the latter situation occurs. Indeed, from Lemma 3.7, Proposition 3.16 and part (ii), we obtain $\widehat{p}^{r,h}(x^h, \mathcal{C}^a) = \widehat{p}^{s,h}(x^h, \mathcal{C}^a) = \overline{p}^{a,h}(x^h, \mathcal{C}^a)$ and thus, since $\widehat{p}^{r,h}(x^h, \mathcal{C}^a)$ is not in $\mathcal{H}^{a,h}(x^h)$, we conclude that $\mathcal{H}^{a,h}(x^h) = (-\infty, \overline{p}^{a,h}(x^h, \mathcal{C}^a))$. It is also clear that $\widehat{p}^{r,h}(x^h, \mathcal{C}^a) > p$ for every $p \in \mathcal{H}^{a,h}(x^h)$ and thus $\widehat{p}^{r,h}(x^h, \mathcal{C}^a)$ belongs to $\mathcal{H}^{f,h}(x^h)$ so that $\mathcal{H}^{f,r,h}(x^h) \neq \emptyset$. We complete the proof by making use of Proposition 3.10. □

3.3 Holder's rational exercise times

A salient feature of an American contract is a holder's *rational exercise time*, which in our framework is defined as follows.

Definition 3.20. We say that $\tau \in \mathcal{T}$ is a *rational exercise time* for the holder of $\mathcal{C}^a$ if the contract is traded at the holder's maximum superhedging cost $\widehat{p}^{s,h} = \widehat{p}^{s,h}(x^h, \mathcal{C}^a)$ at time 0 and there exists a strategy $\psi \in \Psi^h(x^h - \widehat{p}^{s,h}, -A)$ such that $V^h_\tau(x^h - \widehat{p}^{s,h}, \psi) \geq x_\tau(x^h)$.

In fact, we will use Definition 3.20 within the setup where the equality $\widehat{p}^{r,h}(x^h, \mathcal{C}^a) = \widehat{p}^{s,h}(x^h, \mathcal{C}^a)$ holds. If, in addition, the strict comparison property for the BSDE with the driver g is satisfied, then the inequality $V^h_\tau(x^h - \widehat{p}^{r,h}, \psi) \geq x_\tau(x^h)$ can be replaced by the equality $V^h_\tau(x^h - \widehat{p}^{r,h}, \psi) = x_\tau(x^h)$ so that a rational exercise time is also a holder's break-even time (see Remark 3.23).

Note that in Theorem 3.22 we work under the assertions of Theorem 3.19. We thus already know that the equality $\widehat{p}^{r,h}(x^h, \mathcal{C}^a) = \widehat{p}^{s,h}(x^h, \mathcal{C}^a)$ holds and thus a stopping time $\tau \in \mathcal{T}$ is a holder's rational exercise time if the contract is traded at the holder's maximum replication cost $\widehat{p}^{r,h} = \widehat{p}^{r,h}(x^h, \mathcal{C}^a)$ at time 0 and there exists a strategy $\psi \in \Psi^h(x^h - \widehat{p}^{r,h}, -A)$ such that $V^h_\tau(x^h - \widehat{p}^{r,h}, \psi) = x_\tau(x^h)$. We thus deal with a natural extension of the classical concept of a rational exercise time for the holder of an American option when the underlying market model is linear. Notice that in any complete linear market, but not in a general nonlinear market, any holder's rational exercise time is also an issuer's break-even time (in particular, the equality $\tau^h = \tau^i$ is satisfied).

Our next goal is to characterise all holder's rational exercise times and describe the earliest and the latest rational exercise times. Results of this kind are well known from the existing literature on the classical optimal stopping problem based on the expected value (see, for instance, Kobylanski and Quenez [49]). Similarly to Assumption 2.35, we now introduce the following assumption for the holder.

Assumption 3.21. The following comparison property holds: if for $l=1,2$

$$\begin{cases} dy^l_s = -g_h(s,y^l_s,z^l_s)\,dQ_s + z^{l*}_s\,dM_s + dA^l_s, \\ y^l_\tau = \xi_l, \end{cases}$$

where $\tau\in\mathcal{T}$, $\xi_1\geq\xi_2$ and the process A^1-A^2 is nonincreasing, then $y^1_s\geq y^2_s$ for all $s\in[0,\tau]$.

Recall that an $\mathbb{F}$-optional process η is said to be an $\mathcal{E}^h$-submartingale (resp. an $\mathcal{E}^h$-martingale) on $[0,T]$ if $\eta_s\leq\mathcal{E}^h_{s,t}(\eta_t)$ (resp. $\eta_s=\mathcal{E}^h_{s,t}(\eta_t)$) for all $0\leq s\leq t\leq T$.

Theorem 3.22. *Let Assumptions 3.2–3.21 be satisfied. We suppose that $\mathcal{E}^h$ has the strict comparison property. In particular, let $(y,z,k)=(y^{x^h},z^{x^h},k^{x^h})$ be the unique solution to the RBSDE* (3.7)*. Then a stopping time $\tau'\in\mathcal{T}$ is a holder's rational exercise time if and only if the following conditions are met:*
(i) y is an $\mathcal{E}^h$-martingale on $[0,\tau']$, that is, $k_{\tau'}=0$,
(ii) the equality $y_{\tau'}=x_{\tau'}(x^h)$ holds.
The earliest holder's rational exercise time equals $\tau^h:=\inf\{t\in[0,T]\,|\,y_t=x_t(x^h)\}$. If, in addition, k is continuous, then $\bar{\tau}^h:=\inf\{t\in[0,T]\,|\,k_t>0\}$ is the latest holder's rational exercise time.

Proof. Let $\tau'\in\mathcal{T}$ be any stopping time such that conditions (i) and (ii) are met. Since $y_{\tau'}=x_{\tau'}(x^h)$ and $k_{\tau'}=0$, we see that the triplet (y,z,k) solves the following BSDE on $[0,\tau']$

$$\begin{cases} dy_t = -g_h(t,y_t,z_t)\,dQ_t + z^*_t\,dM_t - dA_t, \\ y_{\tau'} = x_{\tau'}(x^h), \end{cases}$$

which can also be written in the forward manner, for all $t\in[0,\tau']$,

$$dy_t = -g_h(t,y_t,z_t)\,dQ_t + z^*_t\,dM_t - dA_t$$

where initial condition y_0 and the process z are given. Now we take $\psi=z$ and we recall from Theorem 3.19 that $\widehat{p}^{\,r,h}(x^h,\mathcal{C}^a)=x^h-y_0$. Hence the wealth process $V^h=V^h(x^h-\widehat{p}^{\,r,h}(x^h,\mathcal{C}^a),\psi)$ satisfies the following SDE for all $t\in[0,T]$

$$dV^h_t = -g_h(t,V^h_t,z_t)\,dQ_t + z^*_t\,dM_t - dA_t$$

with initial condition $V^h_0=y_0$. From the uniqueness of a solution to the above SDE, we infer that $V^h_t=y_t\leq x_t(x^h)$ for every $t\in[0,\tau']$. In particular, $V^h_{\tau'}=y_{\tau'}=x_{\tau'}(x^h)$ and thus τ' is a rational exercise time for the holder of $\mathcal{C}^a$.

Let us now assume that τ' is a rational exercise time for the holder of $\mathcal{C}^a$. From Definition 3.20, it follows that for $p=\widehat{p}^{\,r,h}(x^h,\mathcal{C}^a)=x^h-y_0$ there exists a strategy $\psi\in\Psi^h(x^h-p,-A)$ such that $V^h_{\tau'}(x^h-p,\psi)\geq x_{\tau'}(x^h)$. The comparison property of $\mathcal{E}^h$ yields

$$y_0 = x^h - p = \mathcal{E}^h_{0,\tau'}\big(V^h_{\tau'}(x^h-p,\psi)\big) \geq \mathcal{E}^h_{0,\tau'}(x_{\tau'}(x^h)) \geq \mathcal{E}^h_{0,\tau'}(y_{\tau'}) \tag{3.11}$$

where the last inequality is valid since $x_{\tau'}(x^h)\geq y_{\tau'}$. For any fixed $t\in(0,T]$, the process $\bar{y}_s:=\mathcal{E}^h_{s,t}(y_t)$ solves the following BSDE on $[0,t]$

$$\begin{cases} d\bar{y}_s = -g_h(s,\bar{y}_s,\bar{z}_s)\,dQ_s + \bar{z}^*_s\,dM_s - dA_s, \\ \bar{y}_t = y_t. \end{cases}$$

If (y, z, k) is a solution to the RBSDE (3.7), then y satisfies the following BSDE on $[0, t]$

$$\begin{cases} dy_s = -g_h(s, y_s, z_s)\, dQ_s + z_s^*\, dM_s - dA_s + dk_s, \\ y_t = y_t. \end{cases}$$

Using Assumption 3.21, we get $y_s \leq \bar{y}_s = \mathcal{E}^h_{s,t}(y_t)$ for all $s \in [0, t]$ and thus y is an $\mathcal{E}^h$-submartingale. In fact, by slightly modifying the above proof, one can show that for any $\rho \in \mathcal{T}$ we have $\mathcal{E}^h_{\sigma,\rho}(y_\rho) \geq y_\sigma$ for all $\sigma \leq \rho$ so that y is a strong $\mathcal{E}^h$-submartingale on $[0, \rho]$. We claim that from (3.11) and the assumed strict comparison property of $\mathcal{E}^h$, we may deduce that, for every $0 \leq s \leq \tau'$,

$$\mathcal{E}^h_{s,\tau'}(y_{\tau'}) = y_s. \tag{3.12}$$

Suppose, on the contrary, that this is not true. Then the strict comparison property of $\mathcal{E}^h$ would yield

$$\mathcal{E}^h_{0,\tau'}(y_{\tau'}) = \mathcal{E}^h_{0,s}(\mathcal{E}^h_{s,\tau'}(y_{\tau'})) > \mathcal{E}^h_{0,s}(y_s) \geq y_0,$$

which would clearly contradict (3.11). We now claim that for $0 \leq s \leq t \leq \tau'$, we have that $\mathcal{E}^h_{s,t}(y_t) = y_s$. To show this, we observe that (3.12) yields $\mathcal{E}^h_{t,\tau'}(y_{\tau'}) = y_t$ and thus

$$\mathcal{E}^h_{s,t}(y_t) = \mathcal{E}^h_{s,t}(\mathcal{E}^h_{t,\tau'}(y_{\tau'})) = \mathcal{E}^h_{s,\tau'}(y_{\tau'}) = y_s$$

where the last equality also comes from (3.12). We thus see that y is an $\mathcal{E}^h$-martingale on $[0, \tau']$ and thus $k_{\tau'} = 0$. In particular, we have $\mathcal{E}^h_{0,\tau'}(y_{\tau'}) = y_0$ and thus, using (3.11), we obtain

$$y_0 = \mathcal{E}^h_{0,\tau'}(x_{\tau'}(x^h)) = \mathcal{E}^h_{0,\tau'}(y_{\tau'}) = \mathcal{E}^h_{0,\tau'}\big(V^h_{\tau'}(x^h - p, \psi)\big). \tag{3.13}$$

By combining this equality with the inequality $y_{\tau'} \leq x_{\tau'}(x^h)$ and the strict comparison property of $\mathcal{E}^h$, we conclude that $y_{\tau'} = x_{\tau'}(x^h)$. We have thus shown that if τ' is a rational exercise time, then conditions (i)–(ii) are valid.

Let us show that τ^h is a rational exercise time. From the definition of τ^h and the right-continuity of y and $x(x^h)$, we infer that $y_{\tau^h} = x_{\tau^h}(x^h)$. Equality $k_{\tau^h} = 0$ has been already established in the proof of Theorem 3.19. Hence τ^h satisfies conditions (i)–(ii) and thus it is one of the holder's rational exercise times and it is the earliest one, since $y_t < x_t$ for all $t \in [0, \tau^h)$.

It remains to prove that $\bar{\tau}^h$ is the latest rational exercise time under an additional assumption that the process k is continuous so that $k = k^c$. We need to show that $y_{\bar{\tau}^h} = x_{\bar{\tau}^h}(x^h)$. For an arbitrary $\varepsilon > 0$, there exists $\delta \in [0, \varepsilon]$ such that $k_{\bar{\tau}^h+\delta} > 0$. Since

$$\int_0^T (x_t(x^h) - y_t)\, dk_t = 0,$$

from the right-continuity of processes $x(x^h)$ and y and the inequality $x(x^h) \geq y$, we obtain the equality $y_{\bar{\tau}^h} = x_{\bar{\tau}^h}(x^h)$. Since, obviously, $k_t = 0$ for $t \in [0, \bar{\tau}^h)$, we also have $k_{\bar{\tau}^h} = 0$. This shows that $\bar{\tau}^h$ is one of the holder's rational exercise times. Moreover, it is the latest one since, if $\tau' \in \mathcal{T}$ is such that $\mathbb{P}(\tau' > \bar{\tau}^h) > 0$, then $\mathbb{P}(k_{\tau'} > 0) > 0$ and thus the equality $k_{\tau'} = 0$ cannot hold. Observe that if the continuity of k is not postulated, then it may happen that $k_{\bar{\tau}^h} \neq 0$ in which case $\bar{\tau}^h$ fails to be a rational exercise time (for instance, such properties are always true if $k = k^d$). □

Remark 3.23. From the proof of Theorem 3.22 (see, in particular, equation (3.13)), one can see that the inequality $V^h_{\tau'}(x^h - p, \psi) \geq x_{\tau'}(x^h)$ and the strict comparison property of $\mathcal{E}^h$ imply that when the equality $\widehat{p}^{r,h}(x^h, \mathcal{C}^a) = \widehat{p}^{s,h}(x^h, \mathcal{C}^a)$ holds, then for any rational exercise time given by Definition 3.20 we have that $V^h_\tau(x^h - \widehat{p}^{r,h}(x^h, \mathcal{C}^a), \psi) = x_\tau(x^h)$, meaning that a rational exercise time is also a holder's break-even time. It is also obvious that a holder's break-even time is a rational exercise time.

Thus when the equality $\widehat{p}^{r,h}(x^h, \mathcal{C}^a) = \widehat{p}^{s,h}(x^h, \mathcal{C}^a)$ is satisfied, then the inequality $V^h_\tau(x^h - \widehat{p}^{r,h}(x^h, \mathcal{C}^a), \psi) \geq x_\tau(x^h)$ in Definition 3.20 can be replaced by the equality $V^h_\tau(x^h - \widehat{p}^{r,h}(x^h, \mathcal{C}^a), \psi) = x_\tau(x^h)$. Note that this is fully consistent with the definition of a holder's rational exercise time in a complete, linear market model.

Before concluding this section, let us make some comments on differences between the present work and the related paper by Dumitrescu et al. [26]. Notice that the concept of the issuer's break-even time introduced in Definition 2.9 differs from the notion of the *buyer's rational exercise time* introduced in Definition 3.8 in [26] since the latter is directly defined as an $\mathbb{F}$-stopping time $\widehat{\tau}$ such that $\mathcal{E}^i_{0,\widehat{\tau}}(\xi_{\widehat{\tau}}) = \sup_{\tau \in \mathcal{T}} \mathcal{E}^i_{0,\tau}(\xi_\tau)$ where ξ is the exercise payoff to the holder. Then the buyer's rational exercise time can be characterized using directly the optimality criterion for the nonlinear optimal stopping problem (see Proposition 3.9 in [26]). Definition 3.8 in [26] hinges on the argument that if the buyer purchases the option at the seller's superhedging price, then his rational exercise time can be identified in the classical way by comparing the exercise payoff with the continuation value of the option.

4 American options with extraneous risks

Our final goal is to examine the case of contracts of American style that are subject to extraneous risks. As an example of an extraneous risk, one may consider a vulnerable American option with a possibility of issuer's default, under the assumption that the time when default event occurs cannot be chosen by the issuer. Similarly as in the paper by Szimayer [70], we adopt the reduced-form approach to credit risk modeling where the default event is triggered by certain unforeseen circumstances. Recall that a random time ϑ when the issuer's default occurs is modelled in [70] using the concept of stochastic intensity with respect to a reference filtration and it is postulated that the *hypothesis* (H) is valid. In the reduced-form approach to modelling of extraneous risks, it is convenient to introduce two filtrations, hereafter denoted by $\mathbb{F}$ and $\mathbb{G}$, respectively, where $\mathbb{F}$ is a subfiltration of $\mathbb{G}$. In a typical default risk model, the filtration $\mathbb{G}$ is defined as the *progressive enlargement* of $\mathbb{F}$ with observations of a random time ϑ. Hence the default time ϑ is given as a $\mathbb{G}$-stopping time, which is usually assumed to be totally inaccessible with respect to $\mathbb{G}$, but which is not an $\mathbb{F}$-stopping time. In addition, we will postulate that the filtration $\mathbb{F}$ and $\mathbb{G}$ satisfy the hypothesis (H), which is also known as the *immersion property* between $\mathbb{F}$ and $\mathbb{G}$ (see, for instance, [11, 12]).

We now denote by $\mathcal{T}^{\mathbb{F}}$ and $\mathcal{T}^{\mathbb{G}}$ the classes of all $\mathbb{F}$-stopping times and all $\mathbb{G}$-stopping times taking values in $[0, T]$, respectively. Let $\mathcal{T}^e \subseteq \mathcal{T}^{\mathbb{G}}$ stand for the class of all possible random times of occurrence of some extraneous event, which is supposed to forcibly terminate a contract of American style and affect its closeout payoff. Similarly, we denote by $\mathcal{T}^h \subseteq \mathcal{T}^{\mathbb{F}}$ the set of all possible exercise times that can be chosen by the holder to stop and settle the contract. Unless we deal with a contract of a Bermudan style, it is natural to assume that $\mathcal{T}^h = \mathcal{T}^{\mathbb{F}}$, that is, to postulate that the holder's decision when to exercise the contract is unrestricted although, as usual, it should rely on the information conveyed by the filtration $\mathbb{F}$. Obviously, the specification of the class $\mathcal{T}^e$ will depend on the financial interpretation of an extraneous event.

Let X^i, X^h and X^b stand for the $\mathbb{F}$-predictable, RCLL processes representing the payoffs to the issuer up to time $\vartheta \wedge \tau \wedge T$ where T is the contract's notional maturity. A role played by each of these processes can be recognized from Definition 4.1. Notice that although the payoff given by (4.1) is similar to the payoff of a game option, due to the fact that ϑ is not chosen by the issuer, the contract given by Definition 4.1 cannot be directly addressed by applying results from papers on game contracts, although we will employ a formal connection to a game contract in Section 4.2.

Definition 4.1. An *American contract with extraneous risks*, which is denoted as $\mathcal{C}^v$, is given by:
(i) the class $\mathcal{T}^h \subseteq \mathcal{T}^{\mathbb{F}}$ of possible exercise times τ by the holder,
(ii) the class $\mathcal{T}^e \subseteq \mathcal{T}^{\mathbb{G}}$ of random times ϑ when an extraneous event may occur,
(iii) the $\mathbb{F}$-optional process A with $A_0 = 0$ of external cash flows stopped strictly before $\vartheta \wedge \tau$,
(iv) the terminal payoff to the issuer occurs at time $\vartheta \wedge \tau \wedge T$ and equals, for every $\vartheta \in \mathcal{T}^e$ and $\tau \in \mathcal{T}^h$,

$$I(X^i, X^h, X^b, \vartheta, \tau) := X^i_\vartheta \mathbb{1}_{\{\vartheta < \tau\}} + X^h_\tau \mathbb{1}_{\{\tau < \vartheta\}} + X^b_\vartheta \mathbb{1}_{\{\tau = \vartheta\}}. \tag{4.1}$$

Remark 4.2. American and game options with credit risk were also studied in recent papers by Dumitrescu et al. [25, 26]. It should be stressed, however, that their default risk model concentrates on the third-party credit risk, which is formally represented by a sudden decline to zero of the price process of a reference defaultable risky asset whereas Szimayer [70] and the present work focus on the counterparty credit risk, that is, a possible failure of a party in an option contract to fulfil his obligations.

We will now describe more explicitly the class $\mathcal{T}^e$ by focusing first on the case when $\mathcal{T}^e$ is a singleton. Suppose that we are given a process Γ, which is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and satisfies, for every $t \in \mathbb{R}_+$

$$\Gamma_t = \int_0^t \gamma_u \, du$$

for some strictly positive, $\mathbb{F}$-progressively measurable process γ, which is called the $\mathbb{F}$-*intensity*. To provide an explicit construction of a random time ϑ with the $\mathbb{F}$-intensity γ, we postulate that the underlying probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is sufficiently rich to support a random variable ξ, which is uniformly distributed on the interval $[0,1]$ and independent of $\mathcal{F}_\infty$ under $\mathbb{P}$. Then we define the random time $\vartheta : \Omega \to \mathbb{R}_+$ by setting

$$\vartheta := \inf\{t \in \mathbb{R}_+ : e^{-\Gamma_t} \leq \xi\} = \inf\{t \in \mathbb{R}_+ : \Gamma_t \geq \zeta\} \tag{4.2}$$

where the random variable $\zeta = -\ln \xi$ has a unit exponential law under $\mathbb{P}$. Then we have, for any two dates $0 \leq t \leq u$,

$$\mathbb{P}(\vartheta \leq t \,|\, \mathcal{F}_\infty) = \mathbb{P}(\vartheta \leq t \,|\, \mathcal{F}_u) = \mathbb{P}(\vartheta \leq t \,|\, \mathcal{F}_t) = 1 - e^{-\Gamma_t}.$$

It is worth noting that ϑ is a $\mathbb{G}$-totally inaccessible stopping time with the $\mathbb{G}$-compensator $\Gamma_{t\wedge\tau}$ meaning that the process $\widetilde{M}_t := \mathbb{1}_{\{\vartheta \leq t\}} - \Gamma_{t\wedge\tau}$ is a $\mathbb{G}$-martingale.

Remark 4.3. More generally, one may assume that the $\mathbb{F}$-intensity process is not unique but belongs to some set of intensity processes so that the above construction generates a whole class $\mathcal{T}^e$ of random times. It is not necessary to assume that a random variable ξ is identical for all random times from $\mathcal{T}^e$. It is easy to check that the immersion property between $\mathbb{F}$ and $\mathbb{G}$ still holds when $\mathbb{G}$ is defined as the progressive enlargement of $\mathbb{F}$ through observations of all random times from the class $\mathcal{T}^e$. Then one may assume, for instance, that the contract is stopped at a random time equal to $\min\{\vartheta : \vartheta \in \mathcal{T}^e\}$. This could cover the case of two defaultable counterparties but also the case of reference credit risk.

To cover models with several sources of extraneous risks, we make the following generic assumption regarding the class $\mathcal{T}^e$ and its impact on an American contract.

Assumption 4.4. The class $\mathcal{T}^e$ is the set of random times ϑ, which is given by (4.2) with a strictly positive $\mathbb{F}$-intensity process γ, and such that the contract is stopped at $\vartheta \wedge \tau$ when it is exercised at $\tau \in \mathcal{T}^{\mathbb{F}}$ by its holder.

For brevity, we will sometimes write $I(\vartheta,\tau)$ instead of $I(X^i,X^h,X^b,\vartheta,\tau)$. We henceforth postulate that $X^b=X^i$ since when an extraneous event occurs and a contract is terminated, the concurrent holder's decision to exercise a contract does not affect the recovery payoff. In fact, according to our assumptions regarding ϑ, we have that $\mathbb{P}(\vartheta=\tau)=0$ for every $\mathbb{F}$-stopping time τ and thus the process X^b is immaterial anyway. We thus see that the payoff to the issuer at time $\tau\wedge\vartheta\wedge T$ can be represented as follows

$$I(\vartheta,\tau)=X^i_\vartheta\mathbb{1}_{\{\vartheta\leq\tau\}}+X^h_\tau\mathbb{1}_{\{\tau<\vartheta\}}=X^i_\vartheta\mathbb{1}_{\{\vartheta<\tau\}}+X^h_\tau\mathbb{1}_{\{\tau<\vartheta\}}$$

and, obviously, the payoff to the holder equals $-I(\vartheta,\tau)$. For the sake of generality, we do not postulate that $-X^h\geq -X^i$, although this property is expected to hold for a vulnerable American call (or put) option as a result of the natural assumption that the issuer's recovery rate takes values in the interval $[0,1]$. For instance, in the case of a vulnerable American call option we have $X^h_t=-(S_t-K)^+$ and $X^i_t=\delta_tX^h_t$ for some $\mathbb{F}$-adapted, RCLL recovery rate δ. Hence the issuer's payoff equals

$$I(\vartheta,\tau)=-\delta_\vartheta(S_\vartheta-K)^+\mathbb{1}_{\{\vartheta<\tau\}}-(S_\tau-K)^+\mathbb{1}_{\{\tau<\vartheta\}}.$$

Example 4.5. In the above example, the recovery process is given as a function of the price process of a certain default-free asset. Let us observe that the case where a defaultable asset S^{n+1} pays a predetermined recovery as default is covered by the following setup (see [10]). To this end, we consider a financial market with a defaultable asset S^{n+1} satisfying

$$dS^{n+1}_t=S^{n+1}_{t-}\Big(\mu^{n+1}_t\,dt+\sum_{j=1}^n\sigma^{n+1,j}_t\,dW^j_t+\widetilde{\kappa}_t\,d\widetilde{M}_t\Big)$$

with $S^{n+1}_0>0$. The case of a constant recovery payoff $\delta\geq 0$ corresponds to $\widetilde{\kappa}_t=\delta(S^{n+1}_{t-})^{-1}-1$ so that

$$dS^{n+1}_t=S^{n+1}_{t-}\Big(\mu^{n+1}_t\,dt+\sum_{j=1}^n\sigma^{n+1,j}_t\,dW^j_t+(\delta(S^{n+1}_{t-})^{-1}-1)\,d\widetilde{M}_t\Big).$$

Alternatively, if the recovery payoff is proportional to the pre-default value $S^{n+1}_{\vartheta-}$, we have $\widetilde{\kappa}_t=\delta-1$ and thus S^{n+1} satisfies

$$dS^{n+1}_t=S^{n+1}_{t-}\Big(\mu^{n+1}_t\,dt+\sum_{j=1}^n\sigma^{n+1,j}_t\,dW^j_t+(\delta-1)\,d\widetilde{M}_t\Big).$$

Finally, in the special case where $\widetilde{\kappa}=-1$ (i.e., $\delta=0$), we deal with the zero recovery scheme.

As in Example 2.15, by using the martingale approach of the defaultable asset S^{n+1} (see [11]) as well as the general Girsanov's theorem (see Theorem 3.4.1 of [11]), we can also find an probability measure $\widetilde{\mathbb{P}}^l$ which is equivalent to $\mathbb{P}$ on $(\Omega,\mathcal{G}_T)$ such that the processes $\widetilde{S}^{i,l,\mathrm{cld}}$, $i=1,2,\dots,n$ are continuous, square-integrable, $(\widetilde{\mathbb{P}}^l,\mathbb{G})$-martingales and $\widetilde{S}^{n+1,l,\mathrm{cld}}$ is an RCLL, $(\widetilde{\mathbb{P}}^l,\mathbb{G})$-martingale. Then, according to [12, 60], there will be no extended arbitrage opportunity in our general nonlinear market model (in particular, models from Subsections 2.3.1 and 2.3.2). Hence results obtained in Sections 2 and 3 can be applied directly to the pricing and hedging of American options in this kind of a nonlinear market model with default.

Example 4.6. Formally, the recovery payoff X^i_ϑ at time of default is specified by a predetermined recovery process X^i, which is independent of the jump of the defaultable asset (see, e.g., [70]). Note that the model and some results from Szimayer [70] are special cases of our model and results. In particular, we show in Section 4.1 that the issuer's pricing and hedging problem formally reduces to the case of a standard American contract with the issuer's reward process equal to the minimum of X^i and X^h.

Therefore, for that kind of an American option with extraneous risks, we can use the theory of reflected BSDEs driven by an RCLL martingale (see equation (2.10)) to obtain issuer's acceptable price, which thus also corresponds to the value of the issuer's nonlinear optimal stopping problem. For example, if the model with partial netting from Section 2.3.1 is assumed, the reflected BSDE (2.10) has the form

$$\begin{cases} dY_t = \sum_{i=1}^n B_t^l \xi_t^i \, d\widetilde{S}_t^{i,l,\mathrm{cld}} + f_l(t, Y_t^l, \xi_t)\, dt + dA_t - dK_t, \\ Y_T = X_T, \quad Y_t \geq X_t, \quad \int_0^T (Y_t - X_t)\, dK_t^c = 0, \quad \Delta K_t^d = K_t^d \mathbb{1}_{\{Y_{t-} = X_{t-}\}}, \end{cases}$$

where

$$f_l(t,y,z) := \sum_{i=1}^n r_t^l z^i S_t^i - \sum_{i=1}^n r_t^{i,b} (z^i S_t^i)^+ + r_t^l \Big(y + \sum_{i=1}^n (z^i S_t^i)^- \Big)^+ - r_t^b \Big(y + \sum_{i=1}^n (z^i S_t^i)^- \Big)^-.$$

Clearly, similar results can be obtained for the model with idiosyncratic funding costs and collateral from Section 2.3.2 and, in the special case where the model is linear, one can obtain Theorem 2 of [70]. It should be noted that in [70] the author considered the pricing and hedging problem of the American options with extraneous risks from the perspective of issuer only and, as expected, the holder's pricing and hedging problem is quite different. We will see in Section 4.2 that it is associated with a game contract with reward processes X^i and X^h and the holder's acceptable price is given by a solution to a doubly reflected BSDE driven by an RCLL martingale (see, e.g., [63]), which also means that it coincides with the upper value of the related nonlinear Dynkin game, which was studied in Kim et al. [45].

Remark 4.7. We mention that in our generic model (see Assumption 2.21 in Subsection 2.3.3), the issuer's wealth V^i satisfies the following SDE

$$V_t^i = y - \int_0^t g_i(u, V_u^i, \xi_u^i)\, dQ_u + \int_0^t \xi_u^{i*}\, dM_u + A_t$$

where M is a general RCLL martingale which can have several jumps. This allows us to apply the results of the present paper to study the pricing and hedging American contracts in the market model with dependent defaults (see Chapter 5 in [11]). By combining this feature of the market with Examples 2.15, 4.5 and 4.6, we argue that it is necessary to study market models where the wealth dynamic is driven by RCLL martingales. This gives us another motive to study a generic market model.

4.1 Issuer's pricing and hedging

We first analyze the issuer's pricing and hedging problem for an American contract with extraneous risks. Our goal is to show that it can be reduced to the case of a standard American contract with the issuer's reward process equal to the minimum of X^i and X^h. Indeed, since the issuer has neither the ability to exercise the option nor to control the timing of an extraneous event, he needs to hedge not only against the event of early exercise of the option by its holder, but also protect himself from an extraneous event, which is triggered by a dummy player (or nature).

This observation leads to the following definition of an issuer's superhedging strategy for $\mathcal{C}^v$. Notice that the class $\mathcal{T}^e$ is henceforth specified as in Remark 4.3 but in the proofs of Propositions 4.9 and 4.11, it suffices to assume that $\mathcal{T}^e$ is nonempty and focus on a random time from Assumption 4.4.

Definition 4.8. A pair $(p,\varphi) \in \mathbb{R} \times \Psi^i(x^i+p,A)$ is an *issuer's superhedging strategy* for $\mathcal{C}^v$ if, for every $\vartheta \in \mathcal{T}^e$ and $\tau \in \mathcal{T}^h$,

$$V^i_{\vartheta\wedge\tau}(x^i+p,\varphi) + I(\vartheta,\tau) \geq \bar{V}^i_{\vartheta\wedge\tau}(x^i). \tag{4.3}$$

We denote $\widehat{I}_t := \min(X^i_t, X^h_t)$ and

$$\widehat{X}_t(x^i) := \bar{V}^i_t(x^i) - \widehat{I}_t = \bar{V}^i_t(x^i) - \min(X^i_t, X^h_t) = \bar{V}^i_t(x^i) + \max(-X^i_t, -X^h_t).$$

We henceforth assume that $\mathcal{T}^h = \mathcal{T}^{\mathbb{F}}$. Moreover, we assume that the perfect hedging of extraneous risks is impossible. To this end, we postulate that the price processes of traded assets, and thus also the wealth process of an issuer's trading strategy, are $\mathbb{F}$-adapted.

Proposition 4.9. *A pair $(p,\varphi) \in \mathbb{R} \times \Psi^i(x^i+p,A)$ is an issuer's superhedging strategy for $\mathcal{C}^v$ if and only if $V^i_t(x^i+p,\varphi) \geq \widehat{X}_t(x^i)$ for all $t \in [0,T]$, that is, $\mathbb{P}(V^i_t(x^i+p,\varphi) \geq \widehat{X}_t(x^i),\, \forall\, t \in [0,T]) = 1$.*

Proof. Since $I(\vartheta,\tau) \geq \widehat{I}_{\vartheta\wedge\tau}$, it is clear that if the inequality $V^i_t(x^i+p,\varphi) \geq \widehat{X}_t(x^i)$ holds for all $t \in [0,T]$, then (p,φ) satisfies (4.3). Conversely, let us assume that a pair (p,φ) satisfies Definition 4.8. Since $V^i(x^i+p,\varphi)$ and $\widehat{X}(x^i)$ are $\mathbb{F}$-optional processes, it suffices to show that $V^i_\tau(x^i+p,\varphi) \geq \widehat{X}_\tau(x^i)$ for every $\tau \in \mathcal{T}^{\mathbb{F}}$. Assume, on the contrary, that there exists $\tau \in \mathcal{T}^{\mathbb{F}}$ such that $\mathbb{P}(A) > 0$ where $A := \{V^i_\tau(x^i+p,\varphi) < \widehat{X}_\tau(x^i)\}$. Let us denote $C_1 := \{X^h_\tau \leq X^i_\tau\}$ and $C_2 := \{X^h_\tau > X^i_\tau\}$. Then either $\mathbb{P}(A_1) > 0$ or $\mathbb{P}(A_2) > 0$ where we denote

$$A_1 := A \cap C_1 = \{V^i_\tau(x^i+p,\varphi) + X^h_\tau < \bar{V}^i_\tau(x^i)\}$$

and

$$A_2 := A \cap C_2 = \{V^i_\tau(x^i+p,\varphi) + X^i_\tau < \bar{V}^i_\tau(x^i)\}.$$

Our goal is to show that if $\mathbb{P}(A) > 0$, then there exists an $\mathbb{F}$-stopping time $\widehat{\tau}$ such that

$$\mathbb{P}\big(V^i_{\vartheta\wedge\widehat{\tau}}(x^i+p,\varphi) + I(\vartheta,\widehat{\tau}) < \bar{V}^i_{\vartheta\wedge\widehat{\tau}}(x^i)\big) > 0, \tag{4.4}$$

which contradicts (4.3). We first assume that $\mathbb{P}(A_1) > 0$. In that case, we observe that $\mathbb{P}(A_1 \cap \{\tau < \vartheta\}) > 0$ since $A_1 \in \mathcal{F}_\tau \subset \mathcal{F}_\infty$ and ζ is independent of $\mathcal{F}_\infty$. It is thus sufficient to observe that $I(\vartheta,\tau) = X^h_\tau$ on the event $\{\tau < \vartheta\}$ so that (4.4) is valid with $\widehat{\tau} = \tau$.

Let us now assume that $\mathbb{P}(A_2) > 0$. Although $I(\vartheta,\tau) = X^i_\vartheta$ on the event $\{\tau \geq \vartheta\}$, it is not sufficient to show that $\mathbb{P}(A_2 \cap \{\tau \geq \vartheta\}) > 0$ since manifestly $\mathbb{P}(A_2 \cap \{\tau = \vartheta\}) \leq \mathbb{P}(\tau = \vartheta) = 0$. However, since the processes $V^i(x^i+p,\varphi)$, X^i and $\bar{V}^i(x^i)$ are RCLL, there exists an $\mathbb{F}$-stopping time $\widehat{\tau}$ such that $\widehat{\tau} > \tau$ on A_2 and $\mathbb{P}(\widehat{A}_2) > 0$ where

$$\widehat{A}_2 = \big\{V^i_t(x^i+p,\varphi) + X^i_t < \bar{V}^i_t(x^i),\, \forall\, t \in [\tau,\widehat{\tau}]\big\}.$$

Similar arguments as in the first step show that $\mathbb{P}(\widehat{A}_2 \cap \{\tau < \vartheta < \widehat{\tau}\}) > 0$, which in turn implies that (4.4) holds with $\widehat{\tau}$. We conclude that $\mathbb{P}(A) = 0$ for every $\mathbb{F}$-stopping time τ and thus the proof is completed. □

Proposition 4.9 shows that the issuer's pricing and hedging problem for an American contract with extraneous risks reduces to the case of a standard American contract $\mathcal{C}^a = (A, \widehat{I}, \mathcal{T})$ where $\widehat{I} = \min(X^h, X^i)$. Hence all results from Section 2 can be applied, provided that suitable modifications of assumptions introduced in Section 2 are valid when the process $\widehat{I}$ is substituted for X^h so that the issuer's relative reward $X(x^i) = \bar{V}^i(x^i) - X^h$ is replaced by the process $\widehat{X}(x^i) := \bar{V}^i(x^i) - \widehat{I}$.

In particular, we say that $(\overline{v}^i(x^i,\mathcal{C}^v),\tau^{*,i}) \in \mathbb{R}\times\mathcal{T}^{\mathbb{F}}$ is a *solution* to the issuer's optimal stopping problem for $\mathcal{C}^v$ if $\overline{v}^i(x^i,\mathcal{C}^v) = \mathcal{E}^i_{0,\tau}(\widehat{X}_{\tau^{*,i}}(x^i))$ where

$$\overline{v}^i(x^i,\mathcal{C}^v) = \max_{\tau\in\mathcal{T}} \mathcal{E}^i_{0,\tau}(\widehat{X}_\tau(x^i)) = \max_{\tau\in\mathcal{T}} \mathcal{E}^i_{0,\tau}\big(\bar{V}^i_\tau(x^i) + \max\left(-X^i_\tau, -X^h_\tau\right)\big)$$

and the issuer's acceptable price for $\mathcal{C}^v$ satisfies $p^i(x^i,\mathcal{C}^v) = \overline{v}^i(x^i,\mathcal{C}^v) - x^i$.

4.2 Holder's pricing, hedging and exercising

The holder makes decisions about his hedging and exercising policies but, of course, the timing of an extraneous event is beyond his control. Hence we adopt the following definition of a holder's superhedging for an American contract with extraneous risks in which, as in the issuer's case, we assume that $\mathcal{T}^h = \mathcal{T}^{\mathbb{F}}$.

Definition 4.10. A triplet $(p,\psi,\tau) \in \mathbb{R}\times\Psi^h(x^h-p,-A)\times\mathcal{T}^{\mathbb{F}}$ is called a *holder's superhedging strategy* for $\mathcal{C}^v$ if the inequality $V^h_{\vartheta\wedge\tau}(x^h-p,\psi) - I(\vartheta,\tau) \geq \bar{V}^h_{\vartheta\wedge\tau}(x^h)$ is satisfied for every $\vartheta \in \mathcal{T}^e$.

It is clear that Definition 4.10 differs from the standard holder's superhedging problem for an American contract, which was examined in Section 3.1.2. Although a random time ϑ is not chosen by the issuer, the contract described in Definition 4.1 may be intuitively interpreted as an abstract game option between the holder and a dummy player who has no objective and thus his only role is to choose the timing of an extraneous event. More formally, since an extraneous event comes as a surprise to the holder, his superhedging problem can be solved by studying an associated game contract in which a random time ϑ is replaced by the family of all $\mathbb{F}$-stopping times associated with the dummy player. This conjecture is formally justified by the following result.

Proposition 4.11. *A triplet $(p,\psi,\tau) \in \mathbb{R}\times\Psi^h(x^h-p,-A)\times\mathcal{T}^{\mathbb{F}}$ is a holder's superhedging strategy for $\mathcal{C}^v$ if and only if the inequality $V^h_{\sigma\wedge\tau}(x^h-p,\psi) - I(\sigma,\tau) \geq \bar{V}^h_{\sigma\wedge\tau}(x^h)$ holds for every $\sigma \in \mathcal{T}^{\mathbb{F}}$ such that $\mathbb{P}(\sigma=\tau)=0$.*

Proof. We first assume that (p,ψ,τ) satisfies Definition 4.10 and we prove by contradiction the 'only if' part using similar arguments as in the proof of Proposition 4.9. Suppose that there exists $\sigma \in \mathcal{T}^{\mathbb{F}}$ such that $\mathbb{P}(\sigma=\tau)=0$ and $\mathbb{P}(A) > 0$ where $A := \{V^h_{\sigma\wedge\tau}(x^h-p,\psi) - I(\sigma,\tau) < \bar{V}^h_{\sigma\wedge\tau}(x^h)\}$. Let us denote $D_1 := \{\tau < \sigma\}$ and $D_2 := \{\sigma < \tau\}$. Then either $\mathbb{P}(A_1) > 0$ or $\mathbb{P}(A_2) > 0$ where we denote

$$A_1 := A \cap D_1 = \{V^h_\tau(x^h-p,\psi) - X^h_\tau < \bar{V}^h_\tau(x^h)\}$$

and

$$A_2 := A \cap D_2 = \{V^h_\sigma(x^h-p,\psi) - X^i_\sigma < \bar{V}^h_\sigma(x^h)\}.$$

Our goal is to show that if $\mathbb{P}(A) > 0$, then

$$\mathbb{P}\big(V^h_{\vartheta\wedge\tau}(x^h-p,\psi) - I(\vartheta,\tau) < \bar{V}^h_{\vartheta\wedge\tau}(x^h)\big) > 0, \tag{4.5}$$

which would contradict Definition 4.10. We first assume that $\mathbb{P}(A_1) > 0$ and we observe that $\mathbb{P}(A_1 \cap \{\tau < \vartheta\}) > 0$ since $A_1 \in \mathcal{F}_\tau \subset \mathcal{F}_\infty$ and ζ is independent of $\mathcal{F}_\infty$. Furthermore, $I(\vartheta,\tau) = X^h_\tau$ on the event $\{\tau < \vartheta\}$ and thus (4.5) is valid.

Let us now assume that $\mathbb{P}(A_2) > 0$. Since $V^h(x^h-p,\psi), X^h$ and $\bar{V}^h(x^h)$ are RCLL processes, there exists an $\mathbb{F}$-stopping time $\widehat{\sigma}$ such that $\sigma < \widehat{\sigma} \leq \tau$ on A_2 and $\mathbb{P}(\widehat{A}_2) > 0$ where

$$\widehat{A}_2 = \big\{V^h_t(x^h-p,\psi) + X^h_t < \bar{V}^h_t(x^h),\, \forall\, t \in [\sigma,\widehat{\sigma}]\big\}.$$

Consequently, $\mathbb{P}(\widehat{A}_2 \cap \{\sigma < \vartheta < \widehat{\sigma}\}) > 0$ and thus (4.5) holds since $I(\vartheta,\tau) = X^i_\vartheta$ on the event $\{\vartheta < \tau\}$. We conclude that if $\mathbb{P}(A) > 0$, then (4.5) is valid, which completes the proof of the first implication.

To establish the converse implication, let us assume that (p, ψ, τ) is such that the inequality $V^h_{\sigma\wedge\tau}(x^h - p, \psi) - I(\sigma, \tau) \geq \bar{V}^h_{\sigma\wedge\tau}(x^h)$ holds for every $\sigma \in \mathcal{T}^{\mathbb{F}}$ such that $\mathbb{P}(\sigma = \tau) = 0$. Then

$$\mathbb{P}\big(V^h_{t\wedge\tau}(x^h - p, \psi) - I(t, \tau) \geq \bar{V}^h_{t\wedge\tau}(x^h),\, \forall\, t \in \mathbb{R}_+,\, t \neq \tau(\omega)\big) = 1.$$

Since $\mathbb{P}(\vartheta = \tau) = 0$, it is now easy to see that the triplet (p, ψ, τ) fulfils Definition 4.10. □

Proposition 4.11 gives a formal link between the holder's superhedging problem and an associated game option in which the respective reward processes are X^i and X^h and where, by design, the two parties cannot stop the contract simultaneously. Although the latter condition is not imposed in existing papers on game options, we observe that the so-called rational stopping times are known to satisfy that condition under mild assumptions on relative reward processes, even when exercising decisions by either of the two parties are a priori unrestricted.

Remark 4.12. For more details on the pricing, hedging and exercising of game contracts in a nonlinear market, we refer to Dumitrescu et al. [25] and Kim et al. [45] who study, in particular, the unilateral pricing of game contracts through a doubly reflected BSDE. Let us only observe that Definition 3.14 of the holder's optimal stopping problem needs to be adjusted. We denote $J^h(\sigma, \tau, x^h) = \bar{V}^h_{\sigma\wedge\tau}(x^h) + I(\sigma, \tau)$ and we say that a triplet $(\underline{v}^h(x^h, \mathcal{C}^v), \sigma^{*,h}, \tau^{*,h}) \in \mathbb{R} \times \mathcal{T}^{\mathbb{F}} \times \mathcal{T}^{\mathbb{F}}$ is a solution to the *holder's optimal replication problem* for the contract $\mathcal{C}^v$ if $\underline{v}^h(x^h, \mathcal{C}^v) = \mathcal{E}^h_0(J^h(\sigma^{*,h}, \tau^{*,h}, x^h))$ where the $\mathbb{F}$-stopping times $\sigma^{*,h}$ and $\tau^{*,h}$ are such that

$$\mathcal{E}^h_0(J^h(\sigma^{*,h}, \tau^{*,h}, x^h)) = \min_{\tau\in\mathcal{T}} \max_{\sigma\in\mathcal{T}} \mathcal{E}^h_0(J^h(\sigma, \tau, x^h)) = \min_{\tau\in\mathcal{T}} \max_{\sigma\in\mathcal{T}} \mathcal{E}^h_0\big(\bar{V}^h_{\sigma\wedge\tau}(x^h) + I(\sigma, \tau)\big).$$

Finally, the holder's acceptable price for $\mathcal{C}^v$ is given by $p^h(x^h, \mathcal{C}^v) = x^h - \underline{v}^h(x^h, \mathcal{C}^v)$.

References

[1] Aksamit, A., Deng, S., Obłój, J., Tan, X.: The robust pricing-hedging duality for American options in discrete time financial markets. *Math. Finance* **28(3)**, (2018), 1–37. MR-3974286

[2] Baadi, B. and Ouknine, Y.: Reflected BSDEs when the obstacle is not right-continuous in a general filtration. *ALEA Lat. Am. J. Probab. Math. Stat.* **14(1)**, (2017), 201–218. MR-3655127

[3] Bayraktar, E. and Yao, S.: Quadratic reflected BSDEs with unbounded obstacles. *Stoch. Process. Appl.* **122(4)**, (2012), 1155–1203. MR-2914748

[4] Bayraktar, E. and Zhou, Z.: On arbitrage and duality under model uncertainty and portfolio constraints. *Math. Finance* **27(4)**, (2017), 988–1012. MR-3705160

[5] Bensoussan, A.: On the theory of option pricing. *Acta Appl. Math.* **2(2)**, (1984), 139–158. MR-0748007

[6] Bergman, Y. Z.: Option pricing with differential interest rates. *Rev. Finan. Stud.* **8**, (1995), 475–500.

[7] Bichuch, M., Capponi, A., Sturm, S.: Arbitrage-free XVA. *Math. Finance* **28(2)**, (2018), 582–620. MR-3780968

[8] Bichuch, M., Capponi, A., Sturm, S.: Robust XVA. *Math. Finance* **30(3)**, (2020), 738–781. MR-4116450

[9] Bielecki, T. R., Cialenco, I., Rutkowski, M.: Arbitrage-free pricing of derivatives in nonlinear market models. *Probab. Uncertain. Quant. Risk* **3**, (2018), Paper No. 2, 56 pp. MR-3796722

[10] Bielecki, T. R., Jeanblanc, M., Rutkowski, M.: PDE approach to valuation and hedging of credit derivatives. *Quant. Finance* **5(3)**, (2005), 257–270. MR-2238536

[11] Bielecki, T. R., Jeanblanc, M., Rutkowski, M.: *Credit Risk Modeling.* Osaka University Press, Osaka, 2009. MR-1848558

[12] Bielecki, T. R. and Rutkowski, M.: Valuation and hedging of contracts with funding costs and collateralization. *SIAM J. Financial Math.* **6(1)**, (2015), 594–655. MR-3372104

[13] Bouchard, B. and Nam, V. T.: The American version of the geometric dynamic programming principle: Application to the pricing of American options under constraints. *Applied Math. Optim.* **61(2)**, (2010), 235–265. MR-2585143

[14] Brigo, D. and Pallavicini, A.: Non-linear consistent valuation of CCP cleared or CSA bilateral trades with initial margins under credit, funding and wrong-way risks. *J. Finan. Engrg.* **1(1)**, (2014), 1–60. MR-3942984

[15] Burgard, C. and Kjaer, M.: Partial differential equations representations of derivatives with counterparty risk and funding costs. *J. Credit Risk* **7**, (2011), 1–19.

[16] Capponi, A.: Pricing and mitigation of counterparty credit exposure. In *Handbook on Systemic Risk*. J. P. Fouque and J. A. Langsam (Eds.). *Cambridge University Press*, Cambridge, 2013, pp. 495–511. MR-3087758

[17] Carbone, R., Ferrario, B., Santacroce, M.: Backward stochastic differential equations driven by càdlàg martingales. *Theory Probab. Appl.* **52(2)**, (2008), 304–314. MR-2742510

[18] Crépey, S.: Bilateral counterparty risk under funding constraints – Part I: Pricing. *Math. Finance* **25(1)**, (2015), 1–22. MR-3312194

[19] Crépey, S.: Bilateral counterparty risk under funding constraints – Part II: CVA. *Math. Finance* **25(1)**, (2015), 23–50. MR-3312195

[20] Crépey, S., Bielecki, T. R., Brigo, D.: *Counterparty Risk and Funding: A Tale of Two Puzzles. Chapman & Hall/CRC Financial Mathematics Series*, Boca Raton, FL, 2014. xxii+365 pp. ISBN: 978-1-4665-1645-8. MR-3222744

[21] Crépey, S. and Matoussi, A.: Reflected and doubly reflected BSDEs with jumps: a priori estimates and comparison. *Ann. Probab.* **18(5)**, (2008), 2041–2069. MR-2462558

[22] Cvitanić, J. and Karatzas, I.: Backward stochastic differential equations with reflection and Dynkin games. *Ann. Probab.* **24(4)**, (1996), 2024–2056. MR-1415239

[23] Dellacherie, C., and Meyer, P. A.: *Probabilities and Potential. B. Theory of Martingales. North-Holland Publishing Co.*, Amsterdam, 1982. xvii+463 pp. ISBN: 0-444-86526-8. MR-0745449

[24] Dumitrescu, R., Grigorova, M., Quenez, M. C., Sulem, A.: BSDEs with default jump. In *Computation and Combinatorics in Dynamics, Stochastics and Control, Abel Symposia, Vol. 13*, E. Celledoni, G. Di Nunno, K. Ebrahimi-Fard, and H. Munthe-Kaas (Eds). *Springer*, Cham, 2018, pp. 233–263. MR-3967386

[25] Dumitrescu, R., Quenez, M. C., Sulem, A.: Game options in an imperfect market with default. *SIAM J. Financial Math.* **8(1)**, (2017), 532–559. MR-3679314

[26] Dumitrescu, R., Quenez, M. C., Sulem, A.: American options in an imperfect complete market with default. *ESAIM Proc. Surveys* **64**, (2018), 93–110. MR-3883982

[27] El Karoui, N.: Les aspects probabilistes du contrôle stochastique. In *Lecture Notes in Math. 876, Ecole d'Eté de Probabilités de Saint-Flour IX, 1979*, P.-L. Hennequin (Ed.). *Springer*, Berlin- New York, 1981, pp. 73–238. MR-0637471

[28] El Karoui, N. and Huang, S. J.: A general result of existence and uniqueness of backward stochastic differential equations. In *Backward Stochastic Differential Equations, Pitman Research Notes in Mathematics Series 364*, N. El Karoui and L. Mazliak (Eds.). *Addison Wesley Longman Ltd*, Harlow, Essex, 1997, pp. 27–36. MR-1752673

[29] El Karoui, N., Kapoudjian, C., Pardoux, E., Peng, S., Quenez, M. C.: Reflected solutions of backward SDE's, and related obstacle problems for PDE's. *Ann. Probab.* **25(2)**, (1997), 702–737. MR-1434123

[30] El Karoui, N., Pardoux, E., Quenez, M. C.: Reflected backward SDEs and American options. In *Numerical Methods in Finance*, L. C. G. Rogers and D. Talay (Eds.). *Cambridge University Press*, Cambridge, 1997, pp. 215–231. MR-1470516

[31] El Karoui, N., Peng, S., Quenez, M. C.: Backward stochastic differential equations in finance. *Math. Finance* **7(1)**, (1997), 1–71. MR-1434407

[32] El Karoui, N. and Quenez, M. C.: Non-linear pricing theory and backward stochastic differential equations. In *Lecture Notes in Math. 1656*, W. J. Runggaldier (Ed.). *Springer*, Berlin, 1997, pp. 191–246. MR-1478202

[33] Essaky, E. H.: Reflected backward stochastic differential equation with jumps and RCLL obstacle. *Bull. Sci. Math.* **132(8)**, (2008), 690–710. MR-2474488

[34] Grigorova, M., Imkeller, P., Offen, E., Ouknine, Y., Quenez, M. C.: Reflected BSDEs when the obstacle is not right-continuous and optimal stopping. *Ann. Appl. Probab.* **27(5)**, (2017), 3153–3188. MR-3719955

[35] Grigorova, M., Imkeller, P., Ouknine, Y., Quenez, M. C.: Optimal stopping with f-expectations: the irregular case. *Stoch. Process. Appl.* **130(3)**, (2020), 1258–1288. MR-4058273

[36] Hamadène, S.: Reflected BSDE's with discontinuous barrier and application. *Stoch. Stoch. Rep.* **74(3-4)**, (2002), 571–596. MR-1943580

[37] Hamadène, S. and Ouknine, Y.: Reflected backward SDEs with general jumps. *Theory Probab. Appl.* **60(2)**, (2016), 263–280. MR-3568776

[38] He S., Wang, J., Yan, J.: *Semimartingale Theory and Stochastic Calculus*. *Science Press and CRS Press*, Beijing and Boca Raton, FL, 1992. xiv+546 pp. ISBN: 7-03-003066-4. MR-1219534

[39] Jaillet, P., Lamberton, D., Lapeyre, B.: Variational inequalities and the pricing of American options. *Acta Appl. Math.* **21(3)**, (1990), 263–289. MR-1096582

[40] Jeanblanc, M. and Le Cam, Y.: Immersion property and credit risk modelling. In *Optimality and Risk – Modern Trends in Math. Finance: The Kabanov Festschrift,* F. Delbaen, M. Rasonyi, and C. Stricker (Eds.). *Springer*, Berlin, 2009, pp. 99–131. MR-2648600

[41] Kallsen, J. and Kühn, C.: Pricing derivatives of American and game type in incomplete markets. *Finance Stoch.* **8(2)**, (2004), 261–284. MR-2048831

[42] Karatzas, I.: On the pricing of American options. *Appl. Math. Optim.* **17(1)**, (1988), 37–60. MR-0908938

[43] Karatzas, I. and Kou, S.: Hedging American contingent claims with constrained portfolios. *Finance Stoch.* **2(3)**, (1998), 215–258. MR-1809521

[44] Kim, E., Nie, T., Rutkowski, M.: Arbitrage-free pricing of American options in nonlinear markets. Working paper, 2018 (arXiv:1804.10753v2).

[45] Kim, E., Nie, T., Rutkowski, M.: Valuation and hedging of game options in nonlinear models. Working paper, 2018 (arXiv:1807.05448v1).

[46] Klimsiak, T.: Reflected BSDEs with monotone generator. *Electron. J. Probab.* **17**, (2012), no. 107, 25 pp. MR-3015691

[47] Klimsiak, T.: Reflected BSDEs on filtered probability spaces. *Stoch. Process. Appl.* **125(11)**, (2015), 4204–4241. MR-3385601

[48] Klimsiak, T., Rzymowski, M., and Słomiński, L.: Reflected BSDEs with regulated trajectories. *Stoch. Process. Appl.* **129(4)**, (2019), 1153–1184. MR-3926552

[49] Kobylanski, M. and Quenez, M. C.: Optimal stopping time problem in a general framework. *Electron. J. Probab.* **17**, (2012), no. 72, 28 pp. MR-2968679

[50] Korn, R.: Contingent claim valuation in a market with different interest rates. *Math. Methods Oper. Res.* **42(3)**, (1995), 255–274. MR-1358829

[51] Kusuoka, S.: A remark on default risk models. *Adv. Math. Econ.* **1**, (1999), 69–82. MR-1722700

[52] Lee, J., Sturm, S., Zhou, C.: A risk-sharing framework of bilateral contracts. *SIAM J. Financial Math.* **11(2)**, (2020), 385–410. MR-4083341

[53] Lee, J. and Zhou, C.: Binary funding impacts in derivative valuation. *Math. Finance* **31(1)**, (2021), 242–278. MR-4205883

[54] Lepeltier, J. P. and Xu, M.: Penalization method for reflected backward stochastic differential equations with one r.c.l.l. barrier. *Statist. Probab. Lett.* **75(1)**, (2005), 58–66. MR-2185610

[55] Mercurio, F.: Bergman, Piterbarg and beyond: Pricing derivatives under collateralization and differential rates. In *Actuarial Sciences and Quantitative Finance, Springer Proceedings in Mathematics and Statistics, Vol. 135. Springer*, Berlin, 2015, pp. 65–95.

[56] Mulinacci, S. and Pratelli, M.: Functional convergence of Snell envelopes: Applications to American options approximations. *Finance Stoch.* **2(3)**, (1998), 311–327. MR-1809524

[57] Myneni, R.: The pricing of the American option. *Ann. Appl. Probab.* **2(1)**, (1992), 1–23. MR-1143390

[58] Nie, T. and Rutkowski, M.: Fair bilateral prices in Bergman's model with exogenous collateralization. *Internat. J. Theor. Appl. Finance* **18**, (2015), no. 7, 1550048, 26 pp. MR-3423185

[59] Nie, T. and Rutkowski, M.: BSDEs driven by multidimensional martingales and their applications to markets with funding costs. *Theory Probab. Appl.* **60(4)**, (2016), 604–630. MR-3583450

[60] Nie, T. and Rutkowski, M.: A BSDE approach to fair bilateral pricing under endogenous collateralization. *Finance Stoch.* **20(4)**, (2016), 855–900. MR-3551855

[61] Nie, T. and Rutkowski, M.: Fair bilateral prices under funding costs and exogenous collateralization. *Math. Finance* **28(2)**, (2018), 621–655. MR-3780969

[62] Nie, T. and Rutkowski, M.: Existence, uniqueness and strict comparison theorems for backward stochastic differential equations driven by RCLL martingales. Working paper, 2021 (arXiv:2103.08898v1).

[63] Nie, T. and Rutkowski, M.: Reflected BSDEs and doubly reflected BSDEs driven by RCLL martingales. Working paper, 2021 (arXiv:2103.08917v1).

[64] Pallavicini, A., Perini, D. and Brigo, D.: Funding, collateral and hedging: uncovering the mechanics and the subtleties of funding valuation adjustments. Working paper, 2012 (arXiv:1210.3811).

[65] Peng, S.: Nonlinear expectations, nonlinear evaluations and risk measures. In *Lecture Notes in Math. 1856*, M. Frittelli and W. Runggaldier (Eds.), *Springer*, Berlin, 2004, pp. 165–253. MR-2113723

[66] Peng, S. and Xu, M.: The smallest g-supermartingale and reflected BSDE with single and double L^2 obstacles. *Ann. Inst. Henri Poincaré Probab. Stat.* **41(3)**, (2005), 605–630. MR-2139035

[67] Piterbarg, V.: Funding beyond discounting: collateral agreements and derivatives pricing. *Risk*, February (2010), 97–102.

[68] Quenez, M. C. and Sulem, A.: BSDEs with jumps, optimization and applications to dynamic risk measures. *Stoch. Process. Appl.* **123(8)**, (2013), 3328–3357. MR-3062447

[69] Quenez, M. C. and Sulem, A.: Reflected BSDEs and robust optimal stopping time for dynamic risk measures with jumps. *Stoch. Process. Appl.* **124(9)**, (2014), 3031–3054. MR-3217432

[70] Szimayer, A.: Valuation of American options in the presence of event risk. *Finance Stoch.* **9(1)**, (2005), 89–107. MR-2210929

5 Appendix: Nonlinear optimal stopping problem

For relationships between reflected BSDEs and nonlinear optimal stopping problems, the reader may consult, e.g., Cvitanić and Karatzas [22], El Karoui et al. [29], Grigorova et al. [34, 35], and Quenez and Sulem [69] who studied nonlinear optimal stopping problems under various regularity conditions imposed on the reward process ξ.

Let us recall the terminology related to nonlinear evaluations generated by solutions to BSDEs (see Peng [65]). We consider the following BSDE on $[0,s]$

$$Y_t = \xi_s + \int_t^s g(u, Y_u, Z_u)\, dQ_u - \int_t^s Z_u^*\, dM_u - (A_s - A_t), \tag{5.1}$$

and, for every $0 \le t \le s \le T$ and $\xi_s \in L^2(\mathcal{F}_s)$, we define $\mathcal{E}^{g,A}_{t,s}(\xi_s) := Y_t$ where (Y,Z) is a unique solution to (5.1). Then the system of operators $\mathcal{E}^{g,A}_{t,s} : L^2(\mathcal{F}_s) \to L^2(\mathcal{F}_t)$ is called the $\mathcal{E}^{g,A}$*-evaluation*. It is clear that deterministic dates $t \le s$ appearing in (5.1) can be replaced by $\mathbb{F}$-stopping times $\tau \le \sigma$ from $\mathcal{T}$ and thus the $\mathcal{E}^{g,A}$-evaluation can be extended to all stopping times with values in $[0,T]$ yielding the system of operators $\mathcal{E}^{g,A}_{\tau,\sigma} : L^2(\mathcal{F}_\sigma) \to L^2(\mathcal{F}_\tau)$ for all $\tau, \sigma \in \mathcal{T}$. The following definitions are standard.

Definition 5.1. The *comparison property* of $\mathcal{E}^{g,A}$ holds if, for every stopping time $\tau \in \mathcal{T}$ and any random variables $\xi^1_\tau, \xi^2_\tau \in L^2(\mathcal{F}_\tau)$, the following property is valid: if $\xi^1_\tau \geq \xi^2_\tau$, then $\mathcal{E}^{g,A}_{0,\tau}(\xi^1_\tau) \geq \mathcal{E}^{g,A}_{0,\tau}(\xi^2_\tau)$. The *strict comparison property* of $\mathcal{E}^{g,A}$ holds if, for every $\tau \in \mathcal{T}$ and arbitrary $\xi^1_\tau, \xi^2_\tau \in L^2(\mathcal{F}_\tau)$, if $\xi^1_\tau \geq \xi^2_\tau$ and $\xi^1_\tau \neq \xi^2_\tau$, then $\mathcal{E}^{g,A}_{0,\tau}(\xi^1_\tau) > \mathcal{E}^{g,A}_{0,\tau}(\xi^2_\tau)$.

Definition 5.2. The *value* of the $\mathcal{E}^{g,A}$-max stopping problem with *reward* ξ is given by

$$\overline{v}_0(\xi) = \sup_{\tau \in \mathcal{T}} \mathcal{E}^{g,A}_{0,\tau}(\xi_\tau) \tag{5.2}$$

and an $\mathbb{F}$-stopping time $\tau^* \in \mathcal{T}$ is a *maximizer* if

$$\overline{v}_0(\xi) = \mathcal{E}^{g,A}_{0,\tau^*}(\xi_{\tau^*}) = \max_{\tau \in \mathcal{T}} \mathcal{E}^{g,A}_{0,\tau}(\xi_\tau).$$

For a fixed horizon date $T > 0$, we introduce the following BSDE on $[0,T]$ with data (g, η, A)

$$\begin{cases} dY_t = -g(t, Y_t, Z_t)\, dQ_t + Z^*_t\, dM_t + dA_t, \\ Y_T = \eta, \end{cases} \tag{5.3}$$

where A is a given real-valued, $\mathbb{F}$-adapted process or, more explicitly, for every $t \in [0,T]$,

$$Y_t = \eta + \int_t^T g(t, Y_u, Z_u)\, dQ_u - \int_t^T Z^*_u\, dM_u - (A_T - A_t)$$

where, as usual, the equality is assumed to hold $\mathbb{P}$-a.s.. We also consider the reflected BSDE on $[0,T]$ with data (g, η, A, ξ)

$$\begin{cases} dY_t = -g(t, Y_t, Z_t)\, dQ_t + Z^*_t\, dM_t + dA_t - dK_t, \\ Y_T = \eta, \quad Y_t \geq \xi_t, \\ \int_0^T (Y_t - \xi_t)\, dK^c_t = 0, \quad \Delta K^d_\tau = \Delta K^d_\tau \mathbb{1}_{\{Y_{\tau-} = \xi_{\tau-}\}}, \ \forall\, \tau \in \mathcal{T}_p, \end{cases} \tag{5.4}$$

where $\mathcal{T}_p$ is the class of all $\mathbb{F}$-predictable stopping times taking values in $[0,T]$ and K is a nondecreasing, RCLL, $\mathbb{F}$-predictable process such that $K_0 = 0$. The continuous and discontinuous components of sample paths of the process K are denoted by K^c and K^d, respectively.

The following result is a counterpart of Theorem 3.3 in Quenez and Sulem [69] where the case of an RCLL reward ξ was examined and the nonlinear evaluation $\mathcal{E}^g$ (i.e., $\mathcal{E}^{g,A}$ with $A = 0$) was assumed to be generated by a BSDE driven by a Brownian motion and Poisson random measure. Suppose that, for every $\mathbb{F}$-stopping time τ, the triplets $(g, Y_\tau, A - K)$ and (g, ξ_τ, A) satisfy the assumptions of Theorem 6.1 in [62].

Theorem 5.3. *Let $\xi - A \in \mathcal{S}^2$ and let (Y, Z, K) be the unique solution to the reflected BSDE* (5.4) *such that $Y - A$ is an RCLL process. Then the following assertions are valid:*
(i) Y_0 is the value of the $\mathcal{E}^{g,A}$-max stopping problem with the reward ξ, that is, $Y_0 = \overline{v}_0(\xi)$,
(ii) an $\mathbb{F}$-stopping time $\widehat{\tau}$ is a maximizer in (5.2) *if and only if $Y_t = \widehat{X}_t$ on $[0, \widehat{\tau}]$ where $(\widehat{X}, \widehat{Z})$ is a solution to the BSDE* (5.3) *with $X_{\widehat{\tau}} = \xi_{\widehat{\tau}}$,*
(iii) the $\mathbb{F}$-stopping time $\tau^* := \inf\{t \in [0,T] \,|\, Y_t = \xi_t\}$ is a maximizer in (5.2) *provided that the equality $K_{\tau^*} = 0$ holds.*

Proof. Although the present setup differs from that examined in [69], most of the arguments needed for the proof of Theorem 5.3 are identical to those used to establish Theorem 3.3, Lemma 3.4 and Proposition 3.5 in [69]. Therefore, we omit the details and we present the main steps only.

(i) The inequality $Y_0 \geq \overline{v}_0(\xi)$ is an immediate consequence of (5.2), the definition of the nonlinear evaluation $\mathcal{E}^{g,A}$ and the comparison theorem. Specifically, for every $\tau \in \mathcal{T}$, it suffices to apply part (a) of Theorem 6.1 in [62] to the BSDE (5.3) on $[0,\tau]$ with the terminal condition ξ_τ and the processes $A^1 := A - K$ and $A^2 := A$.

The converse inequality requires a bit more work. We first note that Lemma 3.4 in [69] is still valid in our setup and thus if we set, for any $\tau^\varepsilon := \inf\{t \in [0,T] \,|\, Y_t \le \xi_t + \varepsilon\}$ for any fixed $\varepsilon > 0$ then, by noticing the right-continuity of $Y - A$ and $\xi - A$, we have that $Y_{\tau^\varepsilon} \le \xi_{\tau^\varepsilon} + \varepsilon$ and $Y_t = \mathcal{E}^{g,A}_{t,\tau^\varepsilon}(Y_{\tau^\varepsilon})$ for $0 \le t \le \tau^\varepsilon$. The comparison property now yields

$$Y_0 = \mathcal{E}^{g,A}_{0,\tau^\varepsilon}(Y_{\tau^\varepsilon}) \le \mathcal{E}^{g,A}_{0,\tau^\varepsilon}(\xi_{\tau^\varepsilon} + \varepsilon)$$

and, in view of the stability result for the BSDE (5.3) (see Remark 3.1 in [62]), there exists a positive constant C such that

$$\left|\mathcal{E}^{g,A}_{0,\tau^\varepsilon}(\xi_{\tau^\varepsilon} + \varepsilon) - \mathcal{E}^{g,A}_{0,\tau^\varepsilon}(\xi_{\tau^\varepsilon})\right| \le C\varepsilon.$$

Consequently, $Y_0 \le \mathcal{E}^{g,A}_{0,\tau^\varepsilon}(\xi_{\tau^\varepsilon}) + C\varepsilon$, meaning that τ^ε is a $(C\varepsilon)$-optimal time for the optimal stopping problem (5.2). Since ε was arbitrary, it is now easy to conclude that $Y_0 \le \overline{v}_0(\xi)$, which ends the proof of the equality $Y_0 = \overline{v}_0(\xi)$.
(ii) The second assertion can be restated as follows: an $\mathbb{F}$-stopping time $\widehat{\tau} \in \mathcal{T}$ is such that $Y_0 = \mathcal{E}^{g,A}_{0,\widehat{\tau}}(\xi_{\widehat{\tau}})$ if and only if $Y_t = \mathcal{E}^{g,A}_{t,\widehat{\tau}}(\xi_{\widehat{\tau}})$ on $[0,\widehat{\tau}]$. Of course, it suffices to show that the equality $Y_0 = \mathcal{E}^{g,A}_{0,\widehat{\tau}}(\xi_{\widehat{\tau}})$ implies that $Y_t = \mathcal{E}^{g,A}_{t,\widehat{\tau}}(\xi_{\widehat{\tau}})$ on $[0,\widehat{\tau}]$. To this end, it suffices to apply the strict comparison property established in part (b) of Theorem 6.1 in [62] to the BSDE (5.3) on $[0,\widehat{\tau}]$ with $Y_{\widehat{\tau}} \ge \xi_{\widehat{\tau}}$ and $A^1 = A^2 = A$.

(iii) It suffices to notice that, under the assumptions of part (iii), we immediately obtain the equality $Y_0 = \mathcal{E}^{g,A}_{0,\tau^*}(\xi_{\tau^*})$ and thus, in view of part (i) in the theorem, we also have that $\overline{v}_0(\xi) = \mathcal{E}^{g,A}_{0,\tau^*}(\xi_{\tau^*})$, which means that the stopping time τ^*, which is given by the equality $\tau^* := \inf\{t \in [0,T] \,|\, Y_t = \xi_t\}$, is indeed a maximizer for the stopping problem (5.2). □

Remark 5.4. Using Theorem 10 in Chapter VII of [23] (see also Theorem 2.6 in [69] or Proposition B.10 in [49] for a general case), it can be shown that the increasing process K in the solution (Y,Z,K) to the reflected BSDE (5.4) is continuous (and thus $K_{\tau^*} = 0$) if the process $\xi - A$ is left-upper-semicontinuous along stopping times.

A result analogous to Theorem 5.3 can be established for the $\mathcal{E}^{g,A}$-min stopping problem, that is, the stopping problem corresponding to the holder's valuation of an American contract. We now say that $\underline{v}_0(\xi)$ is the *value* of the $\mathcal{E}^{g,A}$-min stopping problem with *cost* ξ if

$$\underline{v}_0(\xi) = \inf_{\tau \in \mathcal{T}} \mathcal{E}^{g,A}_{0,\tau}(\xi_\tau)$$

and an $\mathbb{F}$-stopping time $\tau_* \in \mathcal{T}$ is a *minimizer* if

$$\underline{v}_0(\xi) = \mathcal{E}^{g,A}_{0,\tau_*}(\xi_{\tau_*}) = \min_{\tau \in \mathcal{T}} \mathcal{E}^{g,A}_{0,\tau}(\xi_\tau).$$

As in the proof of Theorem 5.3, one may show that the value and minimizer for the $\mathcal{E}^{g,A}$-min stopping problem are associated with a solution (y,z,k) to the reflected BSDE on $[0,T]$ with the upper obstacle ξ

$$\begin{cases} dy_t = -g(t,y_t,z_t)\,dQ_t + z_t^*\,dM_t + dA_t + dk_t, \\ y_T = \xi_T, \quad y_t \le \xi_t, \\ \int_0^T (\xi_t - y_t)\,dk_t^c = 0, \quad \Delta k_\tau^d = \Delta k_\tau^d \mathbb{1}_{\{y_{\tau-} = \xi_{\tau-}\}}, \ \forall\, \tau \in \mathcal{T}_p, \end{cases}$$

where k is a nondecreasing, RCLL, $\mathbb{F}$-predictable process with $k_0 = 0$. Then the process k is continuous if the process $A - \xi$ is left-upper-semicontinuous along stopping times.