

Poisson approximation and connectivity in a scale-free random connection model*

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Abstract

We study an inhomogeneous random connection model in the connectivity regime. The vertex set of the graph is a homogeneous Poisson point process \mathcal{P}_s of intensity $s > 0$ on the unit cube $S = (-\frac{1}{2}, \frac{1}{2}]^d$, $d \geq 2$. Each vertex is endowed with an independent random weight distributed as W , where $P(W > w) = w^{-\beta} 1_{[1, \infty)}(w)$, $\beta > 0$. Given the vertex set and the weights an edge exists between $x, y \in \mathcal{P}_s$ with probability $\left(1 - \exp\left(-\frac{\eta W_x W_y}{(d(x,y)/r)^\alpha}\right)\right)$, independent of everything else, where $\eta, \alpha > 0$, $d(\cdot, \cdot)$ is the toroidal metric on S and $r > 0$ is a scaling parameter. We derive conditions on α, β such that under the scaling $r_s(\xi)^d = \frac{1}{c_0 s} (\log s + (k-1) \log \log s + \xi + \log(\frac{\alpha \beta}{k! d}))$, $\xi \in \mathbb{R}$, the number of vertices of degree k converges in total variation distance to a Poisson random variable with mean $e^{-\xi}$ as $s \rightarrow \infty$, where c_0 is an explicitly specified constant that depends on α, β, d and η but not on k . In particular, for $k = 0$ we obtain the regime in which the number of isolated nodes stabilizes, a precursor to establishing a threshold for connectivity. We also derive a sufficient condition for the graph to be connected with high probability for large s . The Poisson approximation result is derived using the Stein's method.

Keywords: scale-free networks; Poisson point process; inhomogeneous random connection model; Poisson convergence; Stein's method; connectivity.

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1 Introduction

Social, financial and other networks such as the internet and wireless networks have been objects of much interest among researchers and practitioners in various fields in recent years. This is largely due to following stylized features observed in empirical data

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(See [15], and Section 1.3, in [8]). *Small world effect*: Typically the number of ‘links’ required to connect two distant vertices is very small (see [20]). *Clustering property*: Linked vertices have mutual connections. *Heavy-tailed degree distribution*: It has been observed that the degree distribution is heavy-tailed with tail parameter between 1 and 2 (see [8]).

These real-life networks naturally possess long range connections among the vertices. The homogeneous long range percolation model on \mathbb{Z} was first introduced by Zhang in [21]. The model can be extended to a random graph with vertex set \mathbb{Z}^d . Vertices x, y are connected by an edge with probability proportional to $\eta|x - y|^{-\alpha}$ as $|x - y| \rightarrow \infty$ for some $\eta, \alpha > 0$. Since nearby points are connected with higher probability, this form of connection probability leads to the graph having the clustering property. For certain range of values of the parameter α a small world effect has also been observed [4]. The continuum version of the above graph called the random connection model was introduced in [16]. Such models have found applications in the study of wireless communication networks, models for spread of epidemic and interactions among molecules [9]. The vertex set of the graph is a homogeneous Poisson point process \mathcal{P}_λ of intensity $\lambda > 0$ in \mathbb{R}^d . $x, y \in \mathcal{P}_\lambda$ are connected by an edge with probability $g(x - y)$ where $g : \mathbb{R}^d \rightarrow [0, 1]$. It was shown that a non-trivial phase transition occurs if and only if g is integrable [16]. This set-up where the expected degree is finite is referred to as the thermodynamic regime. Of interest are a non-trivial threshold for percolation, the degree distribution and the graph distance.

The random graph models described above do not exhibit a heavy-tailed degree distribution. To overcome this, Deijfen et. al. in [7] proposed an inhomogeneous version of this long-range percolation model on \mathbb{Z}^d and called it the scale-free percolation model. The inhomogeneity was introduced by assigning independent and identically distributed weights W_x at each vertex $x \in \mathbb{Z}^d$ representing the importance of a vertex. Given any two points $x, y \in \mathbb{Z}^d$ and corresponding weights W_x, W_y , the probability that there is an edge between these two points equals $1 - \exp\left(-\frac{\eta W_x W_y}{|x - y|^\alpha}\right)$ where $\eta, \alpha > 0$ and $P(W_x > w) = w^{-\beta} 1_{[1, \infty)}(w)$ for some $\beta > 0$. For the graph to be non-trivial (finite degrees) one must have $\min\{\alpha, \alpha\beta\} > d$. This model is studied in great detail in [5]. When $\alpha\beta < 2d$ the degree distribution is heavy tailed. Surprisingly, under this condition the graph also shows an ultra small world effect, that is, the graph distance between two far away points grows doubly logarithmically in the Euclidean distance. A non-trivial phase transition occurs in the model if $\alpha\beta > 2d$. A non-trivial phase transition for the above model refers to the existence of a critical value $\eta_c \in (0, \infty)$ such that for any $\eta < \eta_c$ all components in the graph are finite and for all $\eta > \eta_c$, there is, with probability one, an infinite component in the graph. In addition, if $\alpha \in (d, 2d)$ the graph displays a small-world effect. Thus this model is rich enough to exhibit all the stylized features of real-world networks. The continuity of the percolation function which was conjectured in [7] was proved in [5]. The above results on \mathbb{Z}^d were extended to the continuum in [6].

The graph in the thermodynamic regime is however far from being connected. To obtain connectivity, one needs to scale the connection function suitably. As is the case with the Erdős-Rényi random graph, though much harder to prove, the main obstacle to connectivity in certain random geometric graphs is the presence of isolated nodes (see Chapter 13, [17], [18]). A scaling in the connectivity regime under which the number of isolated vertices converges to a Poisson random variable for the random connection model is obtained in [14]. Suppose that the graph is connected with probability approaching one in the absence of isolated nodes. The Poisson convergence result then yields the asymptotic probability that the graph is connected. Such a result is available only in some restricted cases for the random connection model. The soft random geometric

graph was considered by Penrose in [18]. In this model each pair of n points distributed uniformly in the unit square are connected with probability p provided the inter-point distance between them is at most r . Conditions for a Poisson approximation for the number of isolated nodes and the asymptotic probability of connectivity are derived for arbitrary sequences of parameters (p_n, r_n) . The former result is extended to a larger class of connection functions in higher dimensions. A sufficient condition for the random connection model to be connected asymptotically with high probability is derived in [11].

Another model of interest with a different type of inhomogeneity is a general random connection model studied by Penrose in [19] in the connectivity regime. Non-uniformity in this graph comes from two sources: the vertices of the graph form a Poisson point process with intensity measure $s\mu$ where μ is a probability measure and $s > 0$ is a parameter. Given a realization of the point process, vertices located at x, y are connected with probability $\phi_s(x, y)$, where $\phi_s : \mathbb{X} \times \mathbb{X} \rightarrow [0, 1]$ is a symmetric function. Of interest are the number of nodes of a certain degree and the number of components of a given size. It is shown that the number of vertices of a fixed degree converge (as $s \rightarrow \infty$) to a Poisson random variable when expected number of such vertices converge as $s \rightarrow \infty$. A key assumption under which the result is proved is that the connection function satisfies $\max_{x,y} \phi_s(x, y) < 1 - \epsilon$ for some $\epsilon > 0$, that is, it stays bounded away from one. Thus it does not include the simple Gilbert's disk model where an edge exists between any pair of nodes that are within a specified distance from each other. That the above condition does not hold for our model complicates the computations.

We shall study, for the inhomogeneous random graph considered in [6], the asymptotic behavior of the number of isolated nodes and vertices of arbitrary degree in the connectivity regime. As is often the case, it is convenient to study the connectivity problem on the unit cube. The vertex set is assumed to be distributed according a homogeneous Poisson point process \mathcal{P}_s of intensity s on the unit cube. To avoid boundary effects, we equip the space with the toroidal metric $d(\cdot, \cdot)$ (to be specified below). The connection function is given by

$$g(x, W_x; y, W_y) = 1 - \exp\left(-\frac{\eta W_x W_y}{(d(x, y))^\alpha}\right), \quad (1.1)$$

where $\eta, \alpha > 0$ and $P(W_x > w) = w^{-\beta} 1_{[1, \infty)}(w)$ for some $\beta > 0$. To stabilize the expected number of isolated nodes or vertices of fixed degree we must scale the connection function by a parameter depending on the intensity that we denote by r_s . The modified connection function will be denoted by

$$g_s(x, W_x; y, W_y) = 1 - \exp\left(-\frac{\eta W_x W_y}{\left(\frac{d(x, y)}{r_s}\right)^\alpha}\right). \quad (1.2)$$

We derive an explicit expression for the scaling parameter r_s and show that the number of vertices of arbitrary degree converges to a Poisson random variable under certain condition on the parameters. The Poisson convergence results are proved using the Stein's method and thus give a bound on the rate of convergence in the total variation distance. We adapt Theorem 3.1 from [19] to account for the inhomogeneity arising from the random weights associated with the vertices of the graph. Much of the effort in proving this result lies in overcoming the challenge posed by the fact that the connection function given by (1.2) can take values arbitrarily close to one and the presence of weights on the vertices. The connectivity problem for geometric random graphs is, in general hard. Having derived the Poisson convergence for the number of isolated nodes, one would like to show that the graph is connected with high probability whenever the isolated nodes vanish. We derive a sufficient condition for the graph to be connected with high probability under mild conditions on the parameters.

2 The inhomogeneous random connection model

We now provide a precise definition of the model that is the object of our study. We denote by $S = (-\frac{1}{2}, \frac{1}{2}]^d$ the unit cube centered at the origin. To avoid “boundary effects” we equip S with the toroidal metric $d(\cdot, \cdot) : S \times S \rightarrow \mathbb{R}^+ \cup \{0\}$ defined by $d(x, y) = \inf\{\|x - y - z\| : z \in \mathbb{Z}^d\}$, where $\|\cdot\|$ is the Euclidean norm. Let \mathcal{P}_s be a Poisson point process with intensity s on S . Consider the random graph with vertex set \mathcal{P}_s . To each $x \in \mathcal{P}_s$ we associate independent random weights with probability distribution satisfying $P(W > w) = w^{-\beta} 1_{[1, \infty)}(w)$, $\beta > 0$. Given any two points in \mathcal{P}_s along with their associated weights, the probability that there is an edge is a function of the distance between them as well as the weights. For fixed $\alpha, \eta > 0$, the probability that there is an edge between vertices (located at) $x, y \in \mathcal{P}_s$ is given by (1.2) independent of everything else. We denote the resulting random graph by $G(\mathcal{P}_s, r_s)$. The parameter η is not important for our results and can be set equal to one. For the model described earlier on the lattice \mathbb{Z}^d , results in the thermodynamic regime such as phase transitions are stated in terms of η . For the continuum model such results can be stated either in terms of η or the intensity of the underlying Poisson process.

3 Statements of main results

Let $D_k = D_{k,s}$ be the number of vertices of degree k in $G(\mathcal{P}_s, r_s)$. We derive a scaling regime in which $E[D_k]$ converges under certain conditions on the parameters. In this regime we show that D_k converges in distribution to a Poisson random variable under some additional conditions on the parameters. For fixed $\xi \in \mathbb{R}$ consider the scaling $r_s \equiv r_s(\xi)$, defined by

$$r_s(\xi)^d = \frac{1}{c_0 s} \left(\log s + (k-1) \log \log s + \xi + \log \left(\frac{\alpha\beta}{k!d} \right) \right), \quad (3.1)$$

where $c_0 = \frac{\theta_d \alpha \beta}{\alpha \beta - d} \eta^{\frac{d}{\alpha}} \Gamma(1 - \frac{d}{\alpha})$ and θ_d is the volume of d -dimensional unit ball. For $\lambda > 0$, let $Po(\lambda)$ denote a Poisson random variable with mean λ and \xrightarrow{d} denotes convergence in distribution.

Theorem 3.1. *Consider the random graph $G(\mathcal{P}_s, r_s)$ with the connection function g_s of the form (1.2). Suppose $\alpha > d$, $\beta > 1$ and the scaling parameter r_s is as defined by (3.1). Then for any $k \geq 0$ we have*

$$E[D_k] \rightarrow e^{-\xi} \text{ as } s \rightarrow \infty. \quad (3.2)$$

Note that we need $\min\{\alpha, \alpha\beta\} > d$ for the vertices to have finite degrees. This holds since we also require that the weights have a finite mean ($\beta > 1$). Our next result shows that under some additional conditions, D_k converges in distribution to a Poisson random variable. The condition $\alpha\beta > 2d$ in Theorem 3.2 is required for the degree distribution to have a finite variance [6].

Theorem 3.2. *Consider the random graph $G(\mathcal{P}_s, r_s)$ with the connection function g_s of the form (1.2). Suppose $\alpha > d$, $\beta > 1$ and the scaling parameter r_s is as defined by (3.1). If for any $k \geq 0$, $\alpha\beta > \max\{2d, (k+1)d, (2k+3)(\alpha-d)\}$, then,*

$$D_k \xrightarrow{d} Po(e^{-\xi}) \text{ as } s \rightarrow \infty. \quad (3.3)$$

Consider now the problem of connectivity. For random geometric graphs, that is, when the connection function is the indicator function on the unit ball, it is known (see

13.37, [17]) that the graph is connected with high probability in the absence of isolated nodes. Such a result would imply that under the conditions of Theorem 3.2 with $k = 0$, $G(\mathcal{P}_s, r_s)$ is connected with probability close to $\exp(-e^{-\xi})$ for large s . Though such a result is out of our reach at the moment, the next result provides a sufficient condition for the graph $G(\mathcal{P}_s, \cdot)$ to be connected with high probability. To state the result we need some notation. The idea is to choose the scaling parameter in such a way that there is a one hop path (a two path) from each vertex to every one of its neighbours within a distance that guarantees connectivity in the usual random geometric graph with high probability. Let E^W denote the expectation with respect to the weight W . Let $B(x, r)$ be the unit ball of radius r centered at x with respect to the Euclidean metric and the origin by O . For $\gamma > 0$, let

$$\hat{r}_s(\gamma)^d := \frac{\gamma \log s}{\kappa s}, \quad \text{where} \quad \kappa := \int_{B(O,1)} E^{W_z} \left[1 - \exp\left(-\frac{\eta W_z}{|z|^\alpha}\right) \right] dz. \quad (3.4)$$

Note that $\kappa < \infty$ if $\beta > 1$ and $\alpha > d$. Let $\hat{g}_s(x, W_x; y, W_y)$ be the connection probability between two points x, y with r_s replaced by $\hat{r}_s(\gamma)$ ($\equiv \hat{r}_s$) in (1.2). Denote the random graph with the connection function \hat{g}_s by $G(\mathcal{P}_s, \hat{r}_s)$. Let θ_d be the volume of the d -dimensional unit ball. Define the functions

$$T(\gamma) := 1 - \exp \left\{ -\eta \left(1 + \left(\frac{\kappa}{\gamma \theta_d} \right)^{\frac{1}{d}} \right)^{-\alpha} \right\} \quad \text{and} \quad Q(\gamma) := \gamma T(\gamma). \quad (3.5)$$

Observe that $Q(0) = 0, Q(\infty) = \infty$ and $Q'(\gamma) > 0$ so that ρ satisfying $Q(\rho) = 1$ is uniquely defined.

Theorem 3.3. *Let Q be as defined in (3.5) and ρ satisfy $Q(\rho) = 1$. Suppose $\alpha > d, \beta > 1$. Then for the sequence of graphs $G(\mathcal{P}_s, \hat{r}_s(\gamma))$*

$$P(G(\mathcal{P}_s, \hat{r}_s(\gamma)) \text{ is connected}) \rightarrow 1, \quad (3.6)$$

as $s \rightarrow \infty$, for all $\gamma > \rho$.

4 Proofs

In all the proofs c, c_0, c_1, c_2, \dots and C, C_0, C_1, C_2, \dots will denote constants whose values will change from place to place. Let d_1 be the toroidal metric on $A_s := r_s^{-1}S$ defined as

$$d_1(x, y) := \inf_{z \in r_s^{-1}\mathbb{Z}^d} \|x - y - z\|, \quad x, y \in A_s. \quad (4.1)$$

In the proofs we often make the change of variable from $r_s^{-1}x$ to x and $r_s^{-1}y$ to y and hence shall refer to this as the standard change of variables. Such a change of variables transforms the connection function given by (1.2) to $1 - \exp\left(-\frac{\eta W_x W_y}{(d_1(x, y))^\alpha}\right)$ which is independent of s and hence we shall denote it by $\tilde{g}(x, W_x; y, W_y)$.

4.1 Proof of Theorem 3.1

Fix $\xi \in \mathbb{R}$ and let r_s be as defined in (3.1). By the Campbell-Mecke formula we have

$$\begin{aligned} E[D_k] &= E \left[\sum_{X \in \mathcal{P}_s} 1_{\{\deg(X)=k \text{ in } G(\mathcal{P}_s, r_s)\}} \right] = s P^o(\deg(O) = k \text{ in } G(\mathcal{P}_s, r_s)) \\ &= s E^{W_0} \left[\frac{1}{k!} \left(s \int_S E^{W_x} [g_s(O, W_0; x, W_x)] dx \right)^k \right. \\ &\quad \left. \times \exp \left(-s \int_S E^{W_x} [g_s(O, W_0; x, W_x)] dx \right) \right]. \end{aligned} \quad (4.2)$$

By the standard change of variables we obtain

$$E[D_k] = sE^{W_0} \left[\frac{1}{k!} \left(sr_s^d \int_{A_s} E^{W_x} [\tilde{g}(W_0; x, W_x)] dx \right)^k \times \exp \left(-sr_s^d \int_{A_s} E^{W_x} [\tilde{g}(W_0; x, W_x)] dx \right) \right], \quad (4.3)$$

where we have written $E^{W_x} [\tilde{g}(W_0; x, W_x)]$ for $E^{W_x} [\tilde{g}(O, W_0; x, W_x)]$. We shall compute upper and lower bounds for the expression on the right in (4.3). Let $D_1 := B(O, \frac{1}{2})$. We start with the following trivial inequalities.

$$\int_{r_s^{-1}D_1} E^{W_x} [\tilde{g}(W_0; x, W_x)] dx \leq \int_{A_s} E^{W_x} [\tilde{g}(W_0; x, W_x)] dx \leq \int_{\mathbb{R}^d} E^{W_x} [\tilde{g}(W_0; x, W_x)] dx. \quad (4.4)$$

Consider the upper bound in (4.4). Using the probability density function of the weights and the fact that $d_1(O, x) = |x|$ we obtain

$$\int_{\mathbb{R}^d} E^{W_x} [\tilde{g}(W_0; x, W_x)] dx = \int_{\mathbb{R}^d} \int_1^\infty \left(1 - \exp \left(-\frac{\eta W_0 w}{|x|^\alpha} \right) \right) \beta w^{-\beta-1} dw dx.$$

Switching to hyper-spherical coordinates then applying the Fubini's theorem and subsequently making the change of variables $t = r^{-\alpha}$ in the above equation yields

$$\begin{aligned} \int_{\mathbb{R}^d} E^{W_x} [\tilde{g}(W_0; x, W_x)] dx &= C_d \int_0^\infty \int_1^\infty r^{d-1} \left(1 - \exp \left(-\frac{\eta W_0 w}{r^\alpha} \right) \right) \beta w^{-\beta-1} dw d\theta dr \\ &= \frac{C_d \beta}{\alpha} \int_1^\infty \int_0^\infty t^{-\frac{d}{\alpha}-1} (1 - e^{-\eta W_0 w t}) w^{-\beta-1} dt dw, \end{aligned} \quad (4.5)$$

where $C_d = 2\pi \prod_{i=1}^{d-2} \int_0^\pi \sin^{d-1-i} \phi_i d\phi_i = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$. Using integration by parts the inner integral in (4.5) can be evaluated to yield

$$\begin{aligned} \int_0^\infty t^{-\frac{d}{\alpha}-1} (1 - e^{-\eta W_0 w t}) dt &= -\frac{\alpha}{d} (1 - e^{-\eta W_0 w t}) t^{-\frac{d}{\alpha}} \Big|_0^\infty + \frac{\alpha}{d} (\eta W_0 w) \int_0^\infty t^{-\frac{d}{\alpha}} e^{-\eta W_0 w t} dt \\ &= \frac{\alpha}{d} (\eta W_0 w)^{\frac{d}{\alpha}} \int_0^\infty u^{-\frac{d}{\alpha}} e^{-u} du \\ &= \frac{\alpha}{d} (\eta W_0 w)^{\frac{d}{\alpha}} \Gamma \left(1 - \frac{d}{\alpha} \right), \end{aligned} \quad (4.6)$$

where we have used the assumption that $\alpha > d$. Substituting from (4.6) in (4.5) we obtain

$$\int_{\mathbb{R}^d} E^{W_x} [\tilde{g}(W_0; x, W_x)] dx = \frac{C_d \beta}{d} (\eta W_0)^{\frac{d}{\alpha}} \Gamma \left(1 - \frac{d}{\alpha} \right) \int_1^\infty w^{\frac{d}{\alpha}-\beta-1} = c_0 W_0^{\frac{d}{\alpha}}, \quad (4.7)$$

where we have used the fact that $\alpha\beta > d$ and have set $c_0 = \frac{\theta_d \alpha \beta}{\alpha \beta - d} \eta^{\frac{d}{\alpha}} \Gamma \left(1 - \frac{d}{\alpha} \right)$ since $\theta_d = \frac{C_d}{d}$. We now bound the integral on the left in (4.4) from below. Proceeding as above and using (4.7) we obtain

$$\int_{r_s^{-1}D_1} E^{W_x} [\tilde{g}(W_0; x, W_x)] dx = c_0 W_0^{\frac{d}{\alpha}} - C_d \beta I_2, \quad (4.8)$$

where by a use of the Fubini's theorem we can write

$$I_2 = \int_1^\infty \int_{\frac{1}{2r_s}}^\infty \left[1 - \exp \left(-\frac{\eta W_0 w}{r^\alpha} \right) \right] r^{d-1} w^{-\beta-1} dr dw. \quad (4.9)$$

To compute the inner integral on the right hand side in (4.9) we make the change of variable $t = r^{-\alpha}$ and use the assumption that $\alpha > d$. This yields

$$\begin{aligned} & \frac{1}{\alpha} \int_0^{(2r_s)^\alpha} (1 - e^{-\eta W_0 w t}) t^{-\frac{d}{\alpha}-1} dt \\ &= -\frac{1}{d} (1 - e^{-\eta W_0 w t}) t^{-\frac{d}{\alpha}} \Big|_0^{(2r_s)^\alpha} + \frac{1}{d} (\eta W_0 w) \int_0^{(2r_s)^\alpha} t^{-\frac{d}{\alpha}} e^{-\eta W_0 w t} dt \\ &\leq \frac{1}{d} (\eta W_0 w) \int_0^{(2r_s)^\alpha} t^{-\frac{d}{\alpha}} dt = c_1 W_0 w r_s^{\alpha-d}, \end{aligned} \tag{4.10}$$

where $c_1 = \frac{\alpha \eta 2^{\alpha-d}}{d(\alpha-d)}$. From (4.9), (4.10) and the fact that $\beta > 1$ we obtain

$$I_2 \leq \int_1^\infty c_1 \beta W_0 r_s^{\alpha-d} w^{-\beta} = c_2 W_0 r_s^{\alpha-d}, \tag{4.11}$$

where $c_2 = \frac{\beta c_1}{\beta-1}$. From (4.8), (4.11) and the fact that $\tilde{g} > 0$ we obtain

$$\int_{r_s^{-1} D_1} E^{W_x} [\tilde{g}(W_0; x, W_x)] dx \geq 0 \vee (c_0 W_0^{\frac{d}{\alpha}} - c_3 W_0 r_s^{\alpha-d}), \tag{4.12}$$

where $c_3 = C_d \beta c_2$. For $w \geq 1$, let $\Lambda_s(w) := 0 \vee (c_0 w^{\frac{d}{\alpha}} - c_3 w r_s^{\alpha-d})$. Substituting from (4.7) and (4.12) in (4.4) we obtain

$$\Lambda_s(W_0) \leq \int_{A_s} E^{W_x} [\tilde{g}(W_0; x, W_x)] dx \leq c_0 W_0^{\frac{d}{\alpha}}. \tag{4.13}$$

It follows from (4.3) and (4.13) that

$$\frac{s}{k!} E^{W_0} \left[(s r_s^d \Lambda_s(W_0))^k e^{-c_0 s r_s^d W_0^{\frac{d}{\alpha}}} \right] \leq E[D_k] \leq \frac{s}{k!} E^{W_0} \left[(c_0 s r_s^d W_0^{\frac{d}{\alpha}})^k e^{-s r_s^d \Lambda_s(W_0)} \right]. \tag{4.14}$$

The next step is to prove that both the upper and lower bounds in (4.14) converge to $e^{-\xi}$ as $s \rightarrow \infty$. For later use we shall state the convergence of the lower and upper bounds in somewhat greater generality as separate lemmas.

Theorem 3.1 follows from (4.14) and Lemmas 4.1–4.3 stated below. For the case $k = 0$ we use the first assertion in Lemma 4.1 and Lemma 4.2 with $j = 1$. For $k \geq 1$ we use the second assertion in Lemma 4.1 and Lemma 4.3 with $j = 1$ and $m = k$. \square

Lemma 4.1. (i) Let r_s be as defined in (3.1) with $k = 0$.

$$\lim_{s \rightarrow \infty} s E^{W_0} \left[\exp \left(-c_0 s r_s^d W_0^{\frac{d}{\alpha}} \right) \right] = e^{-\xi}. \tag{4.15}$$

(ii) Let r_s be as defined in (3.1). Suppose $\alpha > d$ and $k \geq 1$, then

$$\lim_{s \rightarrow \infty} \frac{s}{k!} E^{W_0} \left[(s r_s^d \Lambda_s(W_0))^k \exp \left(-c_0 s r_s^d W_0^{\frac{d}{\alpha}} \right) \right] = e^{-\xi}. \tag{4.16}$$

Lemma 4.2. Let r_s be as defined in (3.1) with $k = 0$. Suppose $j \geq 1$, $\alpha > d$ and $\alpha\beta > jd$, then as $s \rightarrow \infty$

$$s^j E^{W_0} \left[e^{-j s r_s^d \Lambda_s(W_0)} \right] = \frac{1}{j} \left(\frac{\alpha\beta}{d} \right)^{-j+1} e^{-j\xi} (\log s)^{j-1} + o(1). \tag{4.17}$$

Lemma 4.3. Let r_s be as defined in (3.1) with $k \geq 1$. Suppose $\alpha > d, j \geq 1$ and $m \geq 1$, then as $s \rightarrow \infty$

$$\left(\frac{s}{k!}\right)^j E^{W_0} \left[\left(c_0 s r_s^d W_0^{\frac{d}{\alpha}}\right)^{jm} e^{-j s r_s^d \Lambda_s(W_0)} \right] = \frac{1}{j} \left(\frac{\alpha\beta}{d}\right)^{-j+1} e^{-j\xi} (\log s)^{j-1} + o(1). \quad (4.18)$$

The following observations will be invoked several times in the proofs.

$$\begin{aligned} c_0 s r_s^d &= \log s + (k-1) \log \log s + \xi + \log \left(\frac{\alpha\beta}{k!d}\right), \\ (c_0 s r_s^d)' &:= \frac{d}{ds} (c_0 s r_s^d) = s^{-1}(1 + o(1)). \end{aligned} \quad (4.19)$$

Proof of Lemma 4.1(i). Using the density of W_0 we obtain

$$\begin{aligned} s E^{W_0} \left[\exp \left(-c_0 s r_s^d W_0^{\frac{d}{\alpha}}\right) \right] &= s \int_1^\infty \exp \left(-c_0 s r_s^d w^{\frac{d}{\alpha}}\right) \beta w^{-\beta-1} dw \\ &= \frac{\alpha\beta}{d} s (c_0 s r_s^d)^{\frac{\alpha\beta}{d}} \int_{c_0 s r_s^d}^\infty e^{-t} t^{-\frac{\alpha\beta}{d}-1} dt \\ &= \frac{\alpha\beta}{d} \frac{\int_{c_0 s r_s^d}^\infty e^{-t} t^{-\frac{\alpha\beta}{d}-1} dt}{s^{-1} (c_0 s r_s^d)^{-\frac{\alpha\beta}{d}}}. \end{aligned} \quad (4.20)$$

Since $s r_s^d \rightarrow \infty$ as $s \rightarrow \infty$ we can apply to L'Hospital's rule to conclude that the limit of the last expression in (4.20) equals the limit of the ratio

$$\frac{\alpha\beta}{d} \frac{-e^{-c_0 s r_s^d} (c_0 s r_s^d)^{-\frac{\alpha\beta}{d}-1} (c_0 s r_s^d)'}{-s^{-2} (c_0 s r_s^d)^{-\frac{\alpha\beta}{d}} - \frac{\alpha\beta}{d} s^{-1} (c_0 s r_s^d)^{-\frac{\alpha\beta}{d}-1} (c_0 s r_s^d)'} = \frac{\alpha\beta}{d} \frac{s^2 e^{-c_0 s r_s^d} (c_0 s r_s^d)'}{(c_0 s r_s^d) + s \frac{\alpha\beta}{d} (c_0 s r_s^d)'}$$

Lemma 4.1(i) now follows from (4.19) with $k = 0$.

Proof of Lemma 4.1(ii). Substituting for the density of W_0 , the expression in the limits on the left of (4.16) equals

$$\begin{aligned} \frac{s}{k!} \int_1^\infty \left(s r_s^d \left(0 \vee \left(c_0 w^{\frac{d}{\alpha}} - c_3 w r_s^{\alpha-d} \right) \right) \right)^k \exp \left(-c_0 s r_s^d w^{\frac{d}{\alpha}}\right) \beta w^{-\beta-1} dw \\ = \frac{s}{k!} \int_1^{C r_s^{-\alpha}} \left(c_0 s r_s^d w^{\frac{d}{\alpha}} - c_3 s w r_s^\alpha \right)^k \exp \left(-c_0 s r_s^d w^{\frac{d}{\alpha}}\right) \beta w^{-\beta-1} dw, \end{aligned} \quad (4.21)$$

where $C = \left(\frac{c_0}{c_3}\right)^{\frac{\alpha}{\alpha-d}}$. By the change of variable $t = c_0 s r_s^d w^{\frac{d}{\alpha}}$ the right hand side in (4.21) equals

$$\begin{aligned} \frac{\alpha\beta}{k!d} s (c_0 s r_s^d)^{\frac{\alpha\beta}{d}} \int_{c_0 s r_s^d}^{c_4 s} \left(t - c_3 c_0^{-\frac{\alpha}{d}} \frac{t^{\frac{\alpha}{d}}}{s^{\frac{\alpha}{d}-1}} \right)^k e^{-t} t^{-\frac{\alpha\beta}{d}-1} dt \\ = \frac{\alpha\beta}{k!d} \frac{\int_{c_0 s r_s^d}^{c_4 s} \left(t - c_5 \frac{t^{\frac{\alpha}{d}}}{s^{\frac{\alpha}{d}-1}} \right)^k e^{-t} t^{-\frac{\alpha\beta}{d}-1} dt}{s^{-1} (c_0 s r_s^d)^{-\frac{\alpha\beta}{d}}}, \end{aligned} \quad (4.22)$$

where $c_4 = c_0^{\frac{\alpha}{\alpha-d}} c_3^{-\frac{d}{\alpha-d}}$ and $c_5 = c_3 c_0^{-\frac{\alpha}{d}}$. Again we can apply L'Hospital's rule to conclude that the limit of the ratio on the right in (4.22) equals the limit of

$$\begin{aligned} \frac{\alpha\beta}{k!d} \frac{\left(c_4 s - c_5 c_4^{\frac{\alpha}{d}} s \right)^k e^{-c_4 s} (c_4 s)^{-\frac{\alpha\beta}{d}-1} c_4}{-s^{-2} (c_0 s r_s^d)^{-\frac{\alpha\beta}{d}} - \frac{\alpha\beta}{d} s^{-1} (c_0 s r_s^d)^{-\frac{\alpha\beta}{d}-1} (c_0 s r_s^d)'} \\ - \frac{\alpha\beta}{k!d} \frac{(c_0 s r_s^d - c_3 s r_s^\alpha)^k e^{-c_0 s r_s^d} (c_0 s r_s^d)^{-\frac{\alpha\beta}{d}-1} (c_0 s r_s^d)'}{-s^{-2} (c_0 s r_s^d)^{-\frac{\alpha\beta}{d}} - \frac{\alpha\beta}{d} s^{-1} (c_0 s r_s^d)^{-\frac{\alpha\beta}{d}-1} (c_0 s r_s^d)'} \end{aligned} \quad (4.23)$$

The first term in (4.23) is zero since $c_4 - c_5 c_4^{\frac{\alpha}{d}} = 0$. The second term in (4.23) simplifies to

$$\frac{\alpha\beta}{k!d} \frac{s^2 (c_0 sr_s^d - c_3 sr_s^\alpha)^k e^{-c_0 sr_s^d} (c_0 sr_s^d)'}{(c_0 sr_s^d) + s \frac{\alpha\beta}{d} (c_0 sr_s^d)'} = \frac{\alpha\beta}{k!d} \frac{s^2 (c_0 sr_s^d)^k e^{-c_0 sr_s^d} (c_0 sr_s^d)'}{(c_0 sr_s^d) + s \frac{\alpha\beta}{d} (c_0 sr_s^d)'} \left(1 - \frac{c_3}{c_0} r_s^{\alpha-d}\right)^k.$$

Lemma 4.1(ii) now follows from (4.19) and observing that $\alpha > d$ and $r_s \rightarrow 0$. □

Proof of Lemma 4.2. Note that $\{c_0 W_0^{\frac{d}{\alpha}} - c_3 W_0 r_s^{\alpha-d} \geq 0\} = \{W_0 \leq C r_s^{-\alpha}\}$ where $C = \left(\frac{c_0}{c_3}\right)^{\frac{\alpha}{\alpha-d}}$. The left hand side of (4.17) equals

$$\begin{aligned} & s^j E^{W_0} \left[\exp\left(-j c_0 sr_s^d W_0^{\frac{d}{\alpha}} + j c_3 W_0 r_s^\alpha\right); W_0 \leq C r_s^{-\alpha} \right] + s^j P(W_0 > C r_s^{-\alpha}) \\ &= s^j \int_1^{C r_s^{-\alpha}} \exp\left(-j c_0 sr_s^d w^{\frac{d}{\alpha}} + j c_3 sr_s^\alpha w\right) \beta w^{-\beta-1} dw + s^j C^{-\beta} r_s^{\alpha\beta}. \end{aligned} \tag{4.24}$$

The second term in (4.24) converges to zero by (3.1) since $\alpha\beta > jd$. By the change of variable $t = j c_0 sr_s^d w^{\frac{d}{\alpha}}$ the first term on the right in (4.24) equals

$$\begin{aligned} & \frac{\alpha\beta}{d} s^j (j c_0 sr_s^d)^{\frac{\alpha\beta}{d}} \int_{j c_0 sr_s^d}^{c_4 s} e^{-t+c_5 t^{\frac{\alpha}{d}} s^{-\frac{\alpha}{d}+1}} t^{-\frac{\alpha\beta}{d}-1} dt \\ &= \frac{\alpha\beta}{d} \frac{\int_{j c_0 sr_s^d}^{c_4 s} e^{-t+c_5 t^{\frac{\alpha}{d}} s^{-\frac{\alpha}{d}+1}} t^{-\frac{\alpha\beta}{d}-1} dt}{s^{-j} (j c_0 sr_s^d)^{-\frac{\alpha\beta}{d}}}, \end{aligned} \tag{4.25}$$

where $c_4 = j c_0 C^{\frac{d}{\alpha}}$ and $c_5 = j c_3 (j c_0)^{-\frac{\alpha}{d}}$. By the L'Hospital's rule the limit of the expression on the right in (4.25) equals the limit of

$$\begin{aligned} & \frac{\alpha\beta}{d} \frac{e^{-c_4 s+c_5 (c_4 s)^{\frac{\alpha}{d}} s^{-\frac{\alpha}{d}+1}} (c_4 s)^{-\frac{\alpha\beta}{d}-1} c_4}{-j s^{-j-1} (j c_0 sr_s^d)^{-\frac{\alpha\beta}{d}} - \frac{\alpha\beta}{d} s^{-j} (j c_0 sr_s^d)^{-\frac{\alpha\beta}{d}-1} (j c_0 sr_s^d)'} \\ & \quad - \frac{\alpha\beta}{d} \frac{e^{-j c_0 sr_s^d} (j c_0 sr_s^d)^{-\frac{\alpha\beta}{d}-1} (j c_0 sr_s^d)' e^{c_5 (j c_0)^{\frac{\alpha}{d}} sr_s^\alpha}}{-j s^{-j-1} (j c_0 sr_s^d)^{-\frac{\alpha\beta}{d}} - \frac{\alpha\beta}{d} s^{-j} (j c_0 sr_s^d)^{-\frac{\alpha\beta}{d}-1} (j c_0 sr_s^d)'} \\ &= C_1 \frac{e^{-c_4 s+c_5 c_4^{\frac{\alpha}{d}} s^{-\frac{\alpha}{d}+1}} (j c_0 sr_s^d)^{\frac{\alpha\beta}{d}+1}}{j (j c_0 sr_s^d) + s \frac{\alpha\beta}{d} (j c_0 sr_s^d)'} + \frac{\alpha\beta s^{j+1} e^{-j c_0 sr_s^d} (j c_0 sr_s^d)' e^{c_5 (j c_0)^{\frac{\alpha}{d}} sr_s^\alpha}}{j (j c_0 sr_s^d) + s \frac{\alpha\beta}{d} (j c_0 sr_s^d)'}, \end{aligned} \tag{4.26}$$

where C_1 is a constant. Since $c_4 - c_5 c_4^{\frac{\alpha}{d}} = 0$, the first term on the right in (4.26) simplifies to

$$C_1 \frac{s^{-\frac{\alpha\beta}{d}+j} (j c_0 sr_s^d)^{\frac{\alpha\beta}{d}+1}}{j (j c_0 sr_s^d) + s \frac{\alpha\beta}{d} (j c_0 sr_s^d)'} \rightarrow 0 \text{ as } s \rightarrow \infty,$$

by (4.19) and the assumption that $\alpha\beta > jd$. Lemma 4.2 now follows by using (4.19) in the second term on the right in (4.26) and the assumption that $\alpha > d$. □

Proof of Lemma 4.3. By the observation at the beginning of the proof of Lemma 4.2, the left hand side of (4.18) equals

$$\begin{aligned} & \left(\frac{s}{k!}\right)^j \int_1^{C r_s^{-\alpha}} \left(c_0 sr_s^d w^{\frac{d}{\alpha}}\right)^{jm} e^{-j c_0 sr_s^d w^{\frac{d}{\alpha}} + j c_3 sr_s^\alpha w} \beta w^{-\beta-1} dw \\ & \quad + \left(\frac{s}{k!}\right)^j \int_{C r_s^{-\alpha}}^\infty \left(c_0 sr_s^d w^{\frac{d}{\alpha}}\right)^{jm} \beta w^{-\beta-1} dw. \end{aligned} \tag{4.27}$$

By changing the variable $t = c_0 sr_s^d w^{\frac{d}{\alpha}}$ in the second term in (4.27) we obtain

$$\frac{\alpha\beta}{d(k!)^j} s^j (c_0 sr_s^d)^{\frac{\alpha\beta}{d}} \int_{c_0 C^{\frac{d}{\alpha}} s}^\infty t^{jm-\frac{\alpha\beta}{d}-1} dt = \frac{\alpha\beta}{(k!)^j d} \frac{\int_{c_4 s}^\infty t^{jm-\frac{\alpha\beta}{d}-1} dt}{s^{-j} (c_0 sr_s^d)^{-\frac{\alpha\beta}{d}}}, \tag{4.28}$$

where $c_4 = c_0 C_{\frac{d}{\alpha}}$. To apply the L'Hospital's rule differentiate the numerator and denominator of the expression on the right in (4.28) to obtain

$$\begin{aligned} & \frac{\alpha\beta}{d(k!)^j} \frac{-(c_4 s)^{jm - \frac{\alpha\beta}{d} - 1} c_4}{-j s^{-j-1} (c_0 s r_s^d)^{-\frac{\alpha\beta}{d}} - \frac{\alpha\beta}{d} s^{-j} (c_0 s r_s^d)^{-\frac{\alpha\beta}{d} - 1} (c_0 s r_s^d)'} \\ &= c_4^{jm - \frac{\alpha\beta}{d}} \frac{\alpha\beta}{d(k!)^j} \frac{s^{jm - \frac{\alpha\beta}{d} + j} (c_0 s r_s^d)^{\frac{\alpha\beta}{d} + 1}}{j(c_0 s r_s^d) + s \frac{\alpha\beta}{d} (c_0 s r_s^d)'}. \end{aligned} \tag{4.29}$$

By making the same change of variable $t = c_0 s r_s^d w^{\frac{d}{\alpha}}$ in the first term in (4.27) we obtain

$$\begin{aligned} & \frac{\alpha\beta}{d(k!)^j} s^j (c_0 s r_s^d)^{\frac{\alpha\beta}{d}} \int_{c_0 s r_s^d}^{c_4 s} e^{-jt + c_5 j t^{\frac{\alpha}{d}} s^{-\frac{\alpha}{d} + 1}} t^{jm - \frac{\alpha\beta}{d} - 1} dt \\ &= \frac{\alpha\beta}{d(k!)^j} \frac{\int_{c_0 s r_s^d}^{c_4 s} e^{-jt + c_5 j t^{\frac{\alpha}{d}} s^{-\frac{\alpha}{d} + 1}} t^{jm - \frac{\alpha\beta}{d} - 1} dt}{s^{-j} (c_0 s r_s^d)^{-\frac{\alpha\beta}{d}}}, \end{aligned} \tag{4.30}$$

where $c_5 = c_3 c_0^{-\frac{\alpha}{d}}$. Differentiating the numerator and denominator of the expression on the right in (4.30) we obtain

$$\frac{\alpha\beta}{d(k!)^j} \frac{e^{-c_4 j s + c_5 j (c_4 s)^{\frac{\alpha}{d}} s^{-\frac{\alpha}{d} + 1}} (c_4 s)^{jm - \frac{\alpha\beta}{d} - 1} c_4 - e^{-j c_0 s r_s^d + j c_5 c_0^{\frac{\alpha}{d}} s r_s^{\alpha}} (c_0 s r_s^d)^{jm - \frac{\alpha\beta}{d} - 1} (c_0 s r_s^d)'}{-j s^{-j-1} (c_0 s r_s^d)^{-\frac{\alpha\beta}{d}} - \frac{\alpha\beta}{d} s^{-j} (c_0 s r_s^d)^{-\frac{\alpha\beta}{d} - 1} (c_0 s r_s^d)'}$$

The above term equals to the sum of

$$\begin{aligned} & - \frac{\alpha\beta}{d(k!)^j} c_4^{jm - \frac{\alpha\beta}{d}} \frac{e^{-c_4 j s + c_5 c_4^{\frac{\alpha}{d}} j s} s^{jm - \frac{\alpha\beta}{d} + j} (c_0 s r_s^d)^{\frac{\alpha\beta}{d} + 1}}{j(c_0 s r_s^d) + s \frac{\alpha\beta}{d} (c_0 s r_s^d)'} \\ &= - \frac{\alpha\beta}{d(k!)^j} c_4^{jm - \frac{\alpha\beta}{d}} \frac{s^{jm - \frac{\alpha\beta}{d} + j} (c_0 s r_s^d)^{\frac{\alpha\beta}{d} + 1}}{j(c_0 s r_s^d) + s \frac{\alpha\beta}{d} (c_0 s r_s^d)'}, \end{aligned} \tag{4.31}$$

since $c_4 - c_5 c_4^{\frac{\alpha}{d}} = 0$ and

$$\begin{aligned} & \frac{\alpha\beta}{d(k!)^j} \frac{s^{j+1} e^{-j c_0 s r_s^d} (c_0 s r_s^d)^{jm} (c_0 s r_s^d)'}{j(c_0 s r_s^d) + s \frac{\alpha\beta}{d} (c_0 s r_s^d)'} e^{j c_5 c_0^{\frac{\alpha}{d}} s r_s^{\alpha}} \\ &= \frac{\alpha\beta}{d(k!)^j} \frac{s^{j+1} e^{-j c_0 s r_s^d} (c_0 s r_s^d)^{jm} (c_0 s r_s^d)'}{j(c_0 s r_s^d) + s \frac{\alpha\beta}{d} (c_0 s r_s^d)'} e^{j c_5 c_0^{\frac{\alpha}{d}} s r_s^{\alpha}}. \end{aligned} \tag{4.32}$$

Adding (4.29) to the last expressions in (4.31) and (4.32) and observing that the expression on the right in (4.29) cancels with the term on the right in (4.31) we obtain by the L'Hospital's rule that the limit of the expression on the left in (4.18) equals the limit of

$$\frac{\alpha\beta}{d(k!)^j} \frac{s^{j+1} \exp(-j c_0 s r_s^d) (c_0 s r_s^d)^{jm} (c_0 s r_s^d)'}{j(c_0 s r_s^d) + s \frac{\alpha\beta}{d} (c_0 s r_s^d)'} e^{j c_5 c_0^{\frac{\alpha}{d}} s r_s^{\alpha}}. \tag{4.33}$$

Lemma 4.3 now follows by using (4.19) and the assumption that $\alpha > d$ in (4.33). □

4.2 Stein's method for poisson convergence

To prove the Poisson convergence results stated in Theorem 3.2 we adapt Theorem 3.1 from [19]. The proof of the later result which uses the Stein's method needs to be modified to account for the inhomogeneity arising from the random weights associated with the vertices of the graph.

Let $\tilde{\mathcal{P}}_s$ be a W -marking of \mathcal{P}_s . That is, suppose $\mathcal{P}_s = \sum_{i=1}^{N_s} \delta_{X_n}$ where N_s is a Poisson random variable with mean s and X_1, X_2, \dots is a sequence of independent uniformly distributed random variables taking values in S . Then $\tilde{\mathcal{P}}_s = \sum_{i=1}^{N_s} \delta_{(X_n, W_n)}$ where W_1, W_2, \dots is a sequence of independent random variables with probability distribution satisfying $P(W > w) = w^{-\beta} 1_{[1, \infty)}(w)$ for some $\beta > 0$ and independent of the random variable N_s and the sequence $\{X_n\}_{n \geq 1}$. We write

$$D_k = D_k(\tilde{\mathcal{P}}_s) = \sum_{y \in \mathcal{P}_s} f(y, W_y, \tilde{\mathcal{P}}_s \setminus \{(y, W_y)\}), \tag{4.34}$$

where $f(y, W_y, \tilde{\mathcal{P}}_s \setminus \{(y, W_y)\})$ equals one if the degree of y equals k in the graph $G(\mathcal{P}_s \cup \{y\}, r_s)$ and zero otherwise. We shall abbreviate $f(y, W_y, \tilde{\mathcal{P}}_s \setminus \{(y, W_y)\})$ to $f(y, \mathcal{P}_s \setminus \{y\})$. Note that $f(y, \mathcal{P}_s \setminus \{y\}) = f(y, \mathcal{P}_s)$. We use the former notation to avoid confusion while applying the Campbell-Mecke formula. Set $\tilde{p}_s(x, W_x) = E[f(x, \mathcal{P}_s) | W_x]$, $x \in S$. Since the underlying point process is homogeneous and the metric is toroidal, \tilde{p} depends on s, W_x and not on x . Hence we shall write $\tilde{p}_s(W_x)$ instead of $\tilde{p}_s(x, W_x)$. By the Campbell-Mecke formula $\nu := E[D_k]$ satisfies

$$\nu = E \left[\sum_{y \in \mathcal{P}_s} f(y, W_y, \tilde{\mathcal{P}}_s \setminus \{(y, W_y)\}) \right] = s \int_S E^{W_x} [\tilde{p}_s(x, W_x)] dx = s E^{W_o} [\tilde{p}_s(W_o)]. \tag{4.35}$$

Let d_{TV} denote the total variation distance, Z_ν be a Poisson random variable with mean ν and F_{D_k}, F_{Z_ν} denote the distribution functions of D_k, Z_ν respectively. For any function $\phi : \mathbb{N} \cup \{0\} \rightarrow \mathbb{R}$, let $\Delta\phi(i) = \phi(i + 1) - \phi(i)$ and $\|\cdot\|_\infty$ denote the sup norm.

Theorem 4.4. *Suppose that for almost every $x \in S$ we can find a random variable $V_x = V_x(W_x)$ coupled with D_k such that conditional on W_x and the event $\{f(x, \mathcal{P}_s \setminus \{x\}) = 1\}$, $1 + V_x$ has the same distribution as $D_k(\{(x, W_x)\} \cup \tilde{\mathcal{P}}_s)$. Then*

$$d_{TV}(F_{D_k}, F_{Z_\nu}) \leq (1 \wedge \nu^{-1}) s \int_S E^{W_x} [E[|D_k - V_x| | W_x] \tilde{p}_s(W_x)] dx. \tag{4.36}$$

Proof of Theorem 4.4. Let $\phi : \mathbb{N} \cup \{0\} \rightarrow \mathbb{R}$ be a bounded function. By using the definition of D_k and the Campbell-Mecke formula we have

$$\begin{aligned} E[D_k \phi(D_k)] &= E \left[\sum_{x \in \mathcal{P}_s} f(x, \mathcal{P}_s \setminus \{x\}) \phi(D_k(\tilde{\mathcal{P}}_s)) \right] \\ &= s \int_S E^{W_x} \left[E \left[f(x, \mathcal{P}_s) \phi \left(D_k(\{(x, W_x)\} \cup \tilde{\mathcal{P}}_s) \right) \mid W_x \right] \right] dx \\ &= s \int_S E^{W_x} \left[E \left[\phi \left(D_k(\{(x, W_x)\} \cup \tilde{\mathcal{P}}_s) \right) \mid \{f(x, \mathcal{P}_s) = 1\}, W_x \right] \tilde{p}_s(W_x) \right] dx \\ &= s \int_S E^{W_x} [E[\phi(V_x + 1) | W_x] \tilde{p}_s(W_x)] dx. \end{aligned} \tag{4.37}$$

From (4.35) we have

$$E[\nu \phi(D_k + 1)] = s \int_S E^{W_x} [\phi(D_k + 1) \tilde{p}_s(W_x)] dx. \tag{4.38}$$

Using (4.37) and (4.38) we obtain

$$\begin{aligned} & |E[\nu\phi(D_k + 1) - D_k\phi(D_k)]| \\ & \leq s \int_S E^{W_x} [|E[\phi(D_k + 1)|W_x] - E[\phi(V_x + 1)|W_x]| \tilde{p}_s(W_x)] dx \\ & \leq s\|\Delta\phi\|_\infty \int_S E^{W_x} [E[|D_k - V_x||W_x] \tilde{p}_s(W_x)] dx, \end{aligned} \tag{4.39}$$

where in the last step we have used the fact that $|\phi(i) - \phi(j)| \leq \|\Delta\phi\|_\infty |i - j|$ for all $i, j \in \mathbb{N} \cup \{0\}$. Given $A \subset \mathbb{N} \cup \{0\}$ choose $\phi : \mathbb{N} \cup \{0\} \rightarrow \mathbb{R}$ such that for each $i \in \mathbb{N} \cup \{0\}$

$$1_A(i) - 1_A(Z_\nu) = \nu\phi(i + 1) - i\phi(i) \quad \text{and} \quad \phi(0) = 0. \tag{4.40}$$

By Lemma 1.1.1 of [3] ϕ is bounded and satisfies $\|\Delta\phi\|_\infty \leq 1 \wedge \nu^{-1}$. The assertion in (4.36) now follows by replacing i by D_k and taking expectations in (4.40) and then using (4.39). \square

To use Theorem 4.4 we need to construct a random variables V_x coupled with D_k for any $x \in S$. The heart of the problem lies in estimating an uniform upper bound for $E[|D_k - V_x||W_x]$. The difficulty in the computations are due to the presence of random weights at each vertex and the fact that the connection function can be arbitrarily close to one. In [19] the connection function is assumed to be uniformly bounded away from one and the inhomogeneity in the graph arises from the non-uniform intensity and the fact that the (non-random) connection function is location dependent.

4.3 Proof of Theorem 3.2

Let x be a point in S and W_x be an independent random variable with the same distribution as the distribution of the weights. For $k \geq 1$ consider an extra sequence of independent and identically distributed points $Y_1, Y_2, \dots, Y_k \in S$ with associated independent weights W_1, \dots, W_k such that

$$P(Y_i \in dy|W_i) = \frac{g_s(x, W_x; y, W_i) dy}{\int_S E^{W_i} [g_s(x, W_x; w, W_i)] dw}.$$

For a graph G let $E(G)$ denotes its edge set. Denote by G_s the graph with vertex set $\mathcal{P}_s \cup \{x\} \cup \{Y_1, Y_2, \dots, Y_k\}$ and edge set

$$E(G_s) = E(G(\mathcal{P}_s \cup \{x\} \cup \{Y_1, Y_2, \dots, Y_k\}, r_s)) \cup \{\{x, Y_1\}, \{x, Y_2\}, \dots, \{x, Y_k\}\}.$$

Let $\mathcal{P}_{s,x} := \{X \in \mathcal{P}_s : X \text{ is neighbour of } x \text{ in } G_s\}$ and $\mathcal{P}_s^x := \mathcal{P}_s \setminus \mathcal{P}_{s,x}$. Given the weight W_x at x the two Poisson point processes $\mathcal{P}_{s,x}$ and \mathcal{P}_s^x are independent. Note that

$$D_k = D_k(\tilde{\mathcal{P}}_s) := \#\{X \in \mathcal{P}_s : \text{deg}(X) = k \text{ in the graph } G_s \text{ induced by } \mathcal{P}_s\}.$$

We construct a coupled point process $\mathcal{P}^* \subset \mathcal{P}_s \cup \{Y_1, Y_2, \dots, Y_k\}$. If $|\mathcal{P}_{s,x}| > k$, then choose $|\mathcal{P}_{s,x}| - k$ many points from $\mathcal{P}_{s,x}$ uniformly at random and discard it from the collection $\mathcal{P}_s \cup \{Y_1, Y_2, \dots, Y_k\}$. Call the resulting collection of points as \mathcal{P}^* . If $k > 0$ and $|\mathcal{P}_{s,x}| < k$, then take $\mathcal{P}^* = \mathcal{P}_s \cup \{Y_1, Y_2, \dots, Y_{k-|\mathcal{P}_{s,x}|}\}$. Let $\mathcal{Y}_x := \{Y_1, Y_2, \dots, Y_{k-|\mathcal{P}_{s,x}|}\}$. Denote the subgraph of G_s induced by $\mathcal{P}^* \cup \{x\}$ by G_s^* . Observe that $\text{deg}(x) = k$ in G_s^* . Let

$$V_x := \#\{X \in \mathcal{P}^* : \text{deg}(X) = k \text{ in the graph } G_s^*\}.$$

Given W_x , the random variable V_x has the same distribution as $D_k(\tilde{\mathcal{P}}_s \cup \{(x, W_x)\}) - 1$ conditioned on the event $\{f(x, \mathcal{P}_s \setminus \{x\}) = 1\}$.

If $|\mathcal{P}_{s,x}| > k$ we can write $|D_k - V_x| \leq U'_x + V'_x$, where U'_x is the number of vertices $y \in \mathcal{P}_{s,x}$ such that y is connected to k points in \mathcal{P}_s and V'_x is the number of pairs of

vertices (y, z) , $y \in \mathcal{P}_{s,x}$, $z \in \mathcal{P}_s$ with $y \neq z$, z is connected to y and z having at most k neighbours in \mathcal{P}_s^x .

If $|\mathcal{P}_{s,x}| < k$ then $|D_k - V_x| \leq U_x'' + V_x''$, where U_x'' is the number of vertices $y \in \mathcal{Y}_x$ such that y is connected to at most k points in \mathcal{P}_s and V_x'' is the number of pairs of vertices (y, z) , $y \in \mathcal{Y}_x$, $z \in \mathcal{P}_s$ with $y \neq z$, z is connected to y and z having at most k neighbours in \mathcal{P}_s .

By the Theorem 4.4

$$\begin{aligned} (1 \wedge \nu_s^{-1})^{-1} d_{TV}(F_{D_k}, F_{Z_\nu}) &\leq s \int_S E^{W_x} [E[|D_k - V_x| | W_x] \tilde{p}_s(W_x)] dx \\ &= s \int_S E^{W_x} [E[|D_k - V_x|; \{|\mathcal{P}_{s,x}| > k\} | W_x] \tilde{p}_s(W_x)] dx \\ &\quad + s \int_S E^{W_x} [E[|D_k - V_x|; \{|\mathcal{P}_{s,x}| < k\} | W_x] \tilde{p}_s(W_x)] dx \\ &\leq s \int_S E^{W_x} [E[U_x' + V_x' | W_x] \tilde{p}_s(W_x)] dx \\ &\quad + s \int_S E^{W_x} [E[U_x'' + V_x'' | W_x] \tilde{p}_s(W_x)] dx, \end{aligned} \tag{4.41}$$

where

$$\begin{aligned} \tilde{p}_s(w) &= E \left[1_{\{deg(O) = k \text{ in } G(\tilde{\mathcal{P}}_s \cup \{(O, W_o)\}, r_s)\}} \middle| W_o = w \right] \\ &= \frac{1}{k!} \left(s \int_S E^{W_y} [g_s(O, w; y, W_y)] dy \right)^k e^{-s \int_S E^{W_y} [g_s(O, w; y, W_y)] dy} \end{aligned} \tag{4.42}$$

and $\nu_s = E[D_k]$. By the standard change of variables

$$\tilde{p}_s(w) = \frac{1}{k!} \left(sr_s^d \int_{A_s} E^{W_y} [\tilde{g}(O, w; y, W_y)] dy \right)^k e^{-sr_s^d \int_{A_s} E^{W_y} [\tilde{g}(O, w; y, W_y)] dy}. \tag{4.43}$$

Using the fact that the metric d_1 (defined in (4.1)) on $A_s := r_s^{-1}S$ is toroidal and writing $\tilde{g}(w; y, W_y)$ for $\tilde{g}(O, w; y, W_y)$ we obtain

$$\begin{aligned} \tilde{p}_s(w) &= \frac{1}{k!} \left(sr_s^d \int_{A_s} E^{W_y} [\tilde{g}(w; y, W_y)] dy \right)^k \exp \left(-sr_s^d \int_{A_s} E^{W_y} [\tilde{g}(w; y, W_y)] dy \right) \\ &\leq \frac{1}{k!} \left(c_0 sr_s^d w^{\frac{d}{\alpha}} \right)^k \exp(-sr_s^d \Lambda_s(w)), \end{aligned} \tag{4.44}$$

where the last inequality in (4.44) follows from the bounds in (4.13). By Theorem 3.1 we have $E[D_k] \rightarrow e^{-\xi}$ as $s \rightarrow \infty$. Thus it suffices to show that the right hand side of (4.41) converges to zero. The computations involved in showing this are somewhat tedious. We shall state the requisite bounds obtained upon computing the right hand side of (4.41) in Lemma 4.5 below. For the case $k = 0$, $U_x'' \equiv 0$, $V_x'' \equiv 0$ and thus the conditions required for the theorem to hold will be determined only by the first term on the right in (4.41). In order to state Lemma 4.5 we need some notation.

Recall that for $w \geq 1$,

$$\Lambda_s(w) := 0 \vee \left(c_0 w^{\frac{d}{\alpha}} - c_3 wr_s^{\alpha-d} \right).$$

For $s > 0, j \geq 1, m \in \mathbb{N}, w \geq 1$ and $T > 0$ define the functions,

$$f_j(s, m) := E^W \left[\left(c_0 sr_s^d W^{\frac{d}{\alpha}} \right)^{jm} e^{-j sr_s^d \Lambda_s(W)} \right], \tag{4.45}$$

and

$$\phi_s(w, T) := \exp\left(-C_0 s r_s^d \left(1 - \exp\left(-\frac{\eta}{2T}\right)\right) \exp\left(-\frac{\eta w E[W]}{T}\right) T^{\frac{d}{\alpha}}\right). \quad (4.46)$$

Lemma 4.5. *Let $\alpha > d, \beta > 1$. Then for all s sufficiently large and some constant $C > 0$ we have*

(i)

$$s r_s^d \int_{A_s} E^{W_x} [E [U'_x | W_x] \tilde{p}_s(W_x)] dx \leq C r_s^{-d} f_1(s, k + 1)^2, \quad (4.47)$$

(ii)

$$s r_s^d \int_{A_s} E^{W_x} [E [U''_x | W_x] \tilde{p}_s(W_x)] dx \leq C s^{-1} r_s^{-2d} f_1(s, k + 1)^2. \quad (4.48)$$

(iii) *If in addition $\alpha\beta > \max\{2d, (k + 1)d\}$ then*

$$s r_s^d \int_{A_s} E^{W_x} [E [V'_x | W_x] \tilde{p}_s(W_x)] dx \leq C s (s r_s^d)^{k+1} (E^{W_x} [\phi_s(W_x, T)^2] f_2(s, k + 1))^{\frac{1}{2}}. \quad (4.49)$$

(iv) *If in addition $\alpha\beta > 2d$ then*

$$s r_s^d \int_{A_s} E^{W_x} [E [V''_x | W_x] \tilde{p}_s(W_x)] dx \leq C s f_1(s, k + 1)^2. \quad (4.50)$$

We first use Lemma 4.5 to complete the proof of Theorem 3.2. The proof of Lemma 4.5 will follow subsequently. Since $\alpha > d$, by Lemma 4.3 with $j = 1$ and $m = k + 1$, $f_1(s, k + 1) \leq \frac{c_1}{s}$, for all s sufficiently large. Hence by Lemma 4.5 the expression on the right in (4.47), (4.48) and (4.50) converge to zero as $s \rightarrow \infty$. Note that for the last assertion to hold in (4.47), (4.48) we only require that $\alpha > d$ and $\beta > 1$. For it to hold in (4.50) we need the additional condition that $\alpha\beta$ be strictly larger than $2d$.

It remains to prove that the term involving $E [V'_x | W_x]$ in (4.41) converges to zero as $s \rightarrow \infty$. By Lemma 4.5 (iii)

$$s r_s^d \int_{A_s} E^{W_x} [E [V'_x | W_x] \tilde{p}_s(W_x)] dx \leq C_3 s (s r_s^d)^{k+1} (E^{W_x} [\phi_s(W_x, T)^2] f_2(s, k + 1))^{\frac{1}{2}}. \quad (4.51)$$

We now derive an upper bound for $E^{W_x} [\phi_s(W_x, T)^2]$. From (4.46) we have for some constants c_1, c_4 that

$$\begin{aligned} E^{W_x} [\phi_s(W_x, T)^2] &= E^{W_x} \left[\exp\left(-c_4 s r_s^d \left(\left(1 - \exp\left(-\frac{\eta}{2T}\right)\right) \exp\left(-\frac{\eta c_1 W_x}{T}\right) \right) T^{\frac{d}{\alpha}} \right) \right] \\ &= E^{W_x} \left[\exp\left(-c_4 s r_s^d \left(\left(1 - \exp\left(-\frac{\eta}{2T}\right)\right) \exp\left(-\frac{\eta c_1 W_x}{T}\right) \right) T^{\frac{d}{\alpha}} \right); W_x \leq T \right] \\ &\quad + E^{W_x} \left[\exp\left(-c_4 s r_s^d \left(\left(1 - \exp\left(-\frac{\eta}{2T}\right)\right) \exp\left(-\frac{\eta c_1 W_x}{T}\right) \right) T^{\frac{d}{\alpha}} \right); W_x > T \right] \\ &\leq E^{W_x} \left[\exp\left(-c_4 s r_s^d \left(\left(1 - \exp\left(-\frac{\eta}{2T}\right)\right) \exp(-\eta c_1) \right) T^{\frac{d}{\alpha}} \right) \right] + P(W_x > T) \\ &\leq \exp\left(-c_5 s r_s^d \left(\frac{1}{T} - \frac{c}{T^2} \right) T^{\frac{d}{\alpha}} \right) + P(W_x > T) = R_T(s) \text{ (say)}. \end{aligned} \quad (4.52)$$

Substituting from (4.52) in (4.51) we obtain

$$s r_s^d \int_{A_s} E^{W_x} [E [V'_x | W_x] \tilde{p}_s(W_x)] dx \leq C_4 (s r_s^d)^{k+1} (R_T(s))^{\frac{1}{2}} (s^2 f_2(s, k + 1))^{\frac{1}{2}}. \quad (4.53)$$

By the definition of $R_T(s)$ and the fact that $sr_s^d \leq C \log s$ for $s \geq 2$, we have

$$(sr_s^d)^{k+1} [R_T(s)]^{\frac{1}{2}} \leq C_5 \left[(\log s)^{2k+2} \exp \left(-c_5 sr_s^d \left(\frac{1}{T} - \frac{c}{T^2} \right) T^{\frac{d}{\alpha}} \right) + (\log s)^{2k+2} P(W_x > T) \right]^{\frac{1}{2}}. \tag{4.54}$$

Since $\beta > (2k + 3) \left(1 - \frac{d}{\alpha} \right)$, choose $\epsilon > 0$ such that

$$\frac{2k + 3 + \epsilon}{\beta} \left(1 - \frac{d}{\alpha} \right) < 1 \tag{4.55}$$

and set $T = (\log s)^{\frac{2k+3+\epsilon}{\beta}}$. Since $P(W_x > T) = T^{-\beta}$ we have

$$(\log s)^{2k+2} P(W_x > T) = \frac{1}{(\log s)^{1+\epsilon}}. \tag{4.56}$$

Since $T \rightarrow \infty$ and $sr_s^d \geq \frac{\log s}{2}$ we have for all s sufficiently large

$$\begin{aligned} & (\log s)^{2k+2} \exp \left(-c_5 sr_s^d \left(\frac{1}{T} - \frac{c}{T^2} \right) T^{\frac{d}{\alpha}} \right) \\ & \leq (\log s)^{2k+2} \exp \left(-c_6 sr_s^d T^{\frac{d}{\alpha}-1} \right) \\ & \leq (\log s)^{2k+2} \exp \left(-c_7 \log s \left((\log s)^{-\frac{2k+3+\epsilon}{\beta} \left(1 - \frac{d}{\alpha} \right)} \right) \right). \end{aligned} \tag{4.57}$$

Since $\alpha > d$, we have from Lemma 4.3 with $j = 2$ and $m = k + 1$ that

$$s^2 f_2(s, k + 1) \leq C \log s, \tag{4.58}$$

for all s sufficiently large and some constant C . Thus it follows from (4.55)–(4.58) that the expression on the right in (4.53) converges to zero as $s \rightarrow \infty$ and hence

$$sr_s^d \int_{A_s} E^{W_x} [E[V'_x | W_x] \tilde{p}_s(W_x)] dx \rightarrow 0 \text{ as } s \rightarrow \infty. \tag{4.59}$$

In proving (4.59) we have used all the three conditions mentioned in the statement of the theorem. This completes the proof of Theorem 3.2. \square

Proof of Lemma 4.5 (i). Applying the Campbell-Mecke formula we obtain

$$E[U'_x | W_x] = s \int_S E^{W_y} [g_s(x, W_x; y, W_y) \Phi_s(y; W_y, k)] dy, \tag{4.60}$$

where

$$\Phi_s(y; W_y, k) := \frac{1}{k!} \left(s \int_S E^{W_z} [g_s(y, W_y; z, W_z)] dz \right)^k e^{-s \int_S E^{W_z} [g_s(y, W_y; z, W_z)] dz}. \tag{4.61}$$

Making change of variables $r_s^{-1}y \rightarrow y$ and $r_s^{-1}z \rightarrow z$ and using the fact that the metric is toroidal we obtain

$$E[U'_x | W_x] = sr_s^d \int_{A_s} E^{W_y} [\tilde{g}(W_x; y, W_y) \bar{\Phi}_s(W_y, k)] dy, \tag{4.62}$$

where

$$\bar{\Phi}_s(W_y, k) := \frac{1}{k!} \left(sr_s^d \int_{A_s} E^{W_z} [\tilde{g}(W_y; z, W_z)] dz \right)^k e^{-sr_s^d \int_{A_s} E^{W_z} [\tilde{g}(W_y; z, W_z)] dz}. \tag{4.63}$$

Bounding the expression in (4.63) using (4.13) and substituting in (4.62) yields

$$\begin{aligned} E[U'_x|W_x] &\leq \frac{sr_s^d}{k!} \int_{\mathbb{R}^d} E^{W_y} \left[\left(1 - \exp\left(-\frac{\eta W_x W_y}{|y|^\alpha}\right)\right) \left(c_0 sr_s^d W_y^{\frac{d}{\alpha}}\right)^k e^{-sr_s^d \Lambda_s(W_y)} \right] dy \\ &= \frac{sr_s^d}{k!} \int_{\mathbb{R}^d} \int_1^\infty \left(1 - \exp\left(-\frac{\eta W_x w}{|y|^\alpha}\right)\right) \left(c_0 sr_s^d w^{\frac{d}{\alpha}}\right)^k e^{-sr_s^d \Lambda_s(w)} \beta w^{-\beta-1} dw dy. \end{aligned} \tag{4.64}$$

Interchanging the integrals by Fubini's theorem we have

$$E[U'_x|W_x] \leq \frac{sr_s^d}{k!} \int_1^\infty \int_{\mathbb{R}^d} \left(1 - \exp\left(-\frac{\eta W_x w}{|y|^\alpha}\right)\right) \left(c_0 sr_s^d w^{\frac{d}{\alpha}}\right)^k e^{-sr_s^d \Lambda_s(w)} \beta w^{-\beta-1} dy dw. \tag{4.65}$$

Changing to hyper-spherical coordinates in the inner integral in (4.65) and then making a change of variable $t = r^{-\alpha}$ we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \left(1 - \exp\left(-\frac{\eta W_x w}{|y|^\alpha}\right)\right) dy &= C_d \int_0^\infty r^{d-1} \left(1 - \exp\left(-\frac{\eta W_x w}{r^\alpha}\right)\right) dr \\ &= \frac{C_d}{\alpha} \int_0^\infty t^{-\frac{d}{\alpha}-1} (1 - e^{-\eta W_x w t}) dt, \end{aligned} \tag{4.66}$$

where $C_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$. Using integration by parts the integral in the last expression in (4.66) equals

$$-\frac{\alpha}{d} (1 - e^{-\eta W_x w t}) t^{-\frac{d}{\alpha}} \Big|_0^\infty + \frac{\alpha}{d} (\eta W_x w) \int_0^\infty t^{-\frac{d}{\alpha}} e^{-\eta W_x w t} dt. \tag{4.67}$$

Since $\alpha > d$, the first term in (4.67) equals zero while the second term evaluates to

$$\frac{\alpha}{d} (\eta W_x w)^{\frac{d}{\alpha}} \int_0^\infty t^{-\frac{d}{\alpha}} e^{-t} dt = \frac{\alpha}{d} (\eta W_x w)^{\frac{d}{\alpha}} \Gamma\left(1 - \frac{d}{\alpha}\right). \tag{4.68}$$

Substituting from (4.68) in (4.66) and then the resulting expression in (4.65) yields

$$\begin{aligned} E[U'_x|W_x] &\leq c_0 sr_s^d (\alpha\beta - d) \frac{1}{k!} W_x^{\frac{d}{\alpha}} \int_1^\infty w^{\frac{d}{\alpha}} \left(c_0 sr_s^d w^{\frac{d}{\alpha}}\right)^k e^{-sr_s^d \Lambda_s(w)} w^{-\beta-1} dw \\ &\leq \frac{(\alpha\beta - d)}{\beta k!} W_x^{\frac{d}{\alpha}} E^{W_y} \left[\left(c_0 sr_s^d W_y^{\frac{d}{\alpha}}\right)^{k+1} e^{-sr_s^d \Lambda_s(W_y)} \right] \\ &= C_1 f_1(s, k + 1) W_x^{\frac{d}{\alpha}}, \end{aligned} \tag{4.69}$$

where $f_1(s, k + 1)$ is as defined in (4.45). Note that $f_1(s, k + 1)$ does not depend on W_x . Using (4.69) and (4.44) we obtain

$$\begin{aligned} sr_s^d \int_{A_s} E^{W_x} [E[U'_x|W_x] \tilde{p}_s(W_x)] dx &\leq C_3 r_s^{-d} f_1(s, k + 1) E^{W_x} \left[\left(c_0 sr_s^d W_x^{\frac{d}{\alpha}}\right) \left(c_0 sr_s^d W_x^{\frac{d}{\alpha}}\right)^k \exp(-sr_s^d \Lambda_s(W_x)) \right] \\ &= C_3 r_s^{-d} f_1(s, k + 1)^2. \end{aligned} \tag{4.70}$$

This proves Lemma 4.5 (i).

Proof of Lemma 4.5 (ii). By applying the Campbell-Mecke formula

$$E[U''_x|W_x] = k \sum_{i=0}^k \int_S E^{W_y} \left[\frac{g_s(x, W_x; y, W_y)}{\int_S E^{W_w} [g_s(x, W_x; w, W_w)] dw} \Phi_s(y; W_y, i) \right] dy, \tag{4.71}$$

where $\Phi_s(y; W_y, i)$ is defined as in (4.61). Making standard change of variables $r_s^{-1}y \rightarrow y$, $r_s^{-1}z \rightarrow z$ and $r_s^{-1}w \rightarrow w$ and using the toroidal metric we can write (4.71) as

$$E[U_x''|W_x] = k \sum_{i=0}^k r_s^d \int_{A_s} E^{W_y} \left[\frac{\tilde{g}(W_x; y, W_y)}{r_s^d \int_{A_s} E^{W_w} [\tilde{g}(W_x; w, W_w)] dw} \bar{\Phi}_s(W_y, i) \right] dy, \quad (4.72)$$

where $\bar{\Phi}_s(W_y, i)$ is defined as in (4.63). Using the fact that the weights are larger than one in (4.72) we have

$$E[U_x''|W_x] \leq k \sum_{i=0}^k \int_{A_s} E^{W_y} \left[\frac{\tilde{g}(W_x; y, W_y)}{\int_{A_s} \left[1 - \exp\left(-\frac{\eta}{|w|^\alpha}\right) \right] dw} \bar{\Phi}_s(W_y, i) \right] dy. \quad (4.73)$$

Since $\alpha > d$

$$\int_{A_s} \left[1 - \exp\left(-\frac{\eta}{|w|^\alpha}\right) \right] dw \rightarrow \int_{\mathbb{R}^d} \left[1 - \exp\left(-\frac{\eta}{|w|^\alpha}\right) \right] dw < \infty, \quad (4.74)$$

and consequently we obtain the bound

$$E[U_x''|W_x] \leq C_1 \sum_{i=0}^k \int_{A_s} E^{W_y} [\tilde{g}(W_x; y, W_y) \bar{\Phi}_s(W_y, i)] dy. \quad (4.75)$$

Using the bounds from (4.13) and the fact that the weights are larger than one we get

$$\begin{aligned} E[U_x''|W_x] &\leq C_2 \sum_{i=0}^k \int_{A_s} E^{W_y} \left[\tilde{g}(W_x; y, W_y) \left(c_0 s r_s^d W_y^{\frac{d}{\alpha}} \right)^i e^{-s r_s^d \Lambda_s(W_y)} \right] dy \\ &\leq C_3 \int_{\mathbb{R}^d} E^{W_y} \left[\tilde{g}(W_x; y, W_y) \left(c_0 s r_s^d W_y^{\frac{d}{\alpha}} \right)^k e^{-s r_s^d \Lambda_s(W_y)} \right] dy, \end{aligned} \quad (4.76)$$

for all s sufficiently large. Comparing (4.76) with the first inequality in (4.64) and proceeding as we did to derive (4.70) we obtain

$$s r_s^d \int_{A_s} E^{W_x} [E[U_x''|W_x] \tilde{p}_s(W_x)] dx \leq C_3 s^{-1} r_s^{-2d} f_1(s, k+1)^2. \quad (4.77)$$

This proves Lemma 4.5 (ii).

Proof of Lemma 4.5 (iii). By the Campbell-Mecke formula we obtain

$$\begin{aligned} E[V_x'|W_x] &= \sum_{i=0}^k s^2 \int_S \int_S E^{W_y W_z} \left[g_s(x, W_x; y, W_y) g_s(y, W_y; z, W_z) \right. \\ &\quad \left. \times \frac{1}{i!} f_s(x, W_x; z, W_z; i) \right] dz dy, \end{aligned} \quad (4.78)$$

where

$$\begin{aligned} f_s(x, W_x; z, W_z; i) &= \left(s \int_S E^{W_w} [g_s(z, W_z; w, W_w) (1 - g_s(x, W_x; w, W_w))] dw \right)^i \\ &\quad \times \exp\left(-s \int_S E^{W_w} [g_s(z, W_z; w, W_w) (1 - g_s(x, W_x; w, W_w))] dw\right). \end{aligned}$$

If the connection function g were to be uniformly bounded away from one, then replacing $1 - g$ by some $\epsilon > 0$ in f_s above simplifies the computations considerably as in [19]. Making the standard change of variables in (4.78) yields

$$\begin{aligned} E[V_x'|W_x] &= \sum_{i=0}^k \frac{s^2 r_s^{2d}}{i!} \int_{A_s} \int_{A_s} E^{W_y W_z} [\tilde{g}(x, W_x; y, W_y) \tilde{g}(y, W_y; z, W_z) \\ &\quad \times h_s(x, W_x; z, W_z; i)] dz dy, \end{aligned} \quad (4.79)$$

where

$$h_s(x, W_x; z, W_z; i) = \left(sr_s^d \int_{A_s} E^{W_w} [\tilde{g}(z, W_z; w, W_w) (1 - \tilde{g}(x, W_x; w, W_w))] dw \right)^i \times \exp \left(-sr_s^d \int_{A_s} E^{W_w} [\tilde{g}(z, W_z; w, W_w) (1 - \tilde{g}(x, W_x; w, W_w))] dw \right).$$

Using the fact that the weights are larger than one and the metric is toroidal we obtain the bound

$$h_s(x, W_x; z, W_z; i) \leq \left(sr_s^d \int_{A_s} \tilde{g}(W_z; w, W_w) dw \right)^i \times \exp \left(-sr_s^d \int_{A_s} E^{W_w} \left[\left(1 - e^{-\frac{\eta}{d_1(z,w)^\alpha}} \right) e^{-\frac{\eta W_x W_w}{d_1(x,w)^\alpha}} \right] dw \right). \quad (4.80)$$

Substituting the upper bound from (4.13) in the first factor on the right in (4.80) yields

$$h_s(x, W_x; z, W_z; i) \leq \left(c_0 sr_s^d W_z^{\frac{d}{\alpha}} \right)^i \exp \left(-sr_s^d \int_{A_s} \left(1 - e^{-\frac{\eta}{d_1(z,w)^\alpha}} \right) E^{W_w} \left[e^{-\frac{\eta W_x W_w}{d_1(x,w)^\alpha}} \right] dw \right). \quad (4.81)$$

Since $\beta > 1$ we have $c_1 = E[W_w] < \infty$. By the Jensen's inequality

$$E^{W_w} \left[\exp \left(-\frac{\eta W_x W_w}{d_1(x,w)^\alpha} \right) \right] \geq \exp \left(-\frac{\eta W_x c_1}{d_1(x,w)^\alpha} \right). \quad (4.82)$$

Substituting from (4.82) in (4.81) we obtain

$$h_s(x, W_x; z, W_z; i) \leq \left(c_0 sr_s^d W_z^{\frac{d}{\alpha}} \right)^i \exp \left(-sr_s^d \int_{A_s} \left(1 - e^{-\frac{\eta}{d_1(z,w)^\alpha}} \right) e^{-\frac{\eta c_1 W_x}{d_1(x,w)^\alpha}} dw \right). \quad (4.83)$$

For $z, x \in A_s$ and $T \equiv T(s)$ to be specified later, define

$$D(z, x, s) := A_s \cap B(x, T^{\frac{1}{\alpha}})^c \cap \left\{ B(z, (2T)^{\frac{1}{\alpha}}) \setminus B(z, T^{\frac{1}{\alpha}}) \right\},$$

where $B(x, r)$ is the ball centered at x and radius r with respect to the toroidal metric on A_s . Observe that for $w \in D(z, x, s)$, $d_1(z, w)^\alpha \leq 2T$ and $d_1(x, w)^\alpha \geq T$. Further $|D(z, x, s)| \geq c_2 T^{\frac{d}{\alpha}}$ for some constant $c_2 > 0$. Using the above observations we obtain

$$\begin{aligned} \int_{A_s} \left(1 - e^{-\frac{\eta}{d_1(z,w)^\alpha}} \right) e^{-\frac{\eta c_1 W_x}{d_1(x,w)^\alpha}} dw &\geq \int_{D(z,x,s)} \left(1 - e^{-\frac{\eta}{d_1(z,w)^\alpha}} \right) e^{-\frac{\eta c_1 W_x}{d_1(x,w)^\alpha}} dw \\ &\geq \int_{D(z,x,s)} \left(1 - e^{-\frac{\eta}{2T}} \right) e^{-\frac{\eta c_1 W_x}{T}} dw \\ &\geq \left(1 - e^{-\frac{\eta}{2T}} \right) e^{-\frac{\eta c_1 W_x}{T}} c_2 T^{\frac{d}{\alpha}}. \end{aligned} \quad (4.84)$$

Substituting from (4.84) in (4.83) we obtain

$$\begin{aligned} h_s(x, W_x; z, W_z; i) &\leq \left(c_0 sr_s^d W_z^{\frac{d}{\alpha}} \right)^i \exp \left(-c_3 sr_s^d \left(1 - \exp \left(-\frac{\eta}{2T} \right) \right) \exp \left(-\frac{\eta c_1 W_x}{T} \right) T^{\frac{d}{\alpha}} \right) \\ &= \left(c_0 sr_s^d W_z^{\frac{d}{\alpha}} \right)^i \phi_s(W_x, T), \end{aligned} \quad (4.85)$$

where $\phi_s(W_x, T)$ is defined in (4.46). Since the bound in (4.85) does not depend on x, z

we can use the toroidal metric and substitute in (4.79) to obtain

$$\begin{aligned}
 E[V'_x|W_x] &\leq \sum_{i=0}^k \frac{s^2 r_s^{2d}}{i!} \int_{A_s} \int_{A_s} E^{W_y W_z} \left[\tilde{g}(W_x; y, W_y) \tilde{g}(W_y; z, W_z) \right. \\
 &\quad \left. \times \left(c_0 s r_s^d W_z^{\frac{d}{\alpha}} \right)^i \phi_s(W_x, T) \right] dz dy \\
 &\leq C_0 (s r_s^d)^{2+k} \phi_s(W_x, T) \int_{A_s} E^{W_y} \left[\tilde{g}(W_x; y, W_y) \right. \\
 &\quad \left. \times \int_{A_s} E^{W_z} \left[\tilde{g}(W_y; z, W_z) W_z^{\frac{k d}{\alpha}} \right] dz \right] dy, \quad (4.86)
 \end{aligned}$$

for sufficiently large s . Since $\alpha\beta > (k + 1)d$, proceeding as we did in (4.65)–(4.69) yields

$$\begin{aligned}
 \int_{A_s} E^{W_z} \left[\tilde{g}(W_y; z, W_z) W_z^{\frac{k d}{\alpha}} \right] dz &\leq (\alpha\beta - d) c_0 W_y^{\frac{d}{\alpha}} E^{W_z} \left[W_z^{\frac{(k+1)d}{\alpha}} \right] \\
 &= \frac{\alpha\beta - d}{\alpha\beta - (k + 1)d} c_0 W_y^{\frac{d}{\alpha}}. \quad (4.87)
 \end{aligned}$$

Substituting from (4.87) in (4.86) we have

$$\begin{aligned}
 E[V'_x|W_x] &\leq C_1 (s r_s^d)^{2+k} \phi_s(W_x, T) \int_{A_s} E^{W_y} \left[\tilde{g}(W_x; y, W_y) W_y^{\frac{d}{\alpha}} \right] dy \\
 &\leq C_2 (s r_s^d)^{2+k} \phi_s(W_x, T) c_0 W_x^{\frac{d}{\alpha}} E^{W_y} \left[W_y^{\frac{2d}{\alpha}} \right] \\
 &= C_3 (s r_s^d)^{2+k} \phi_s(W_x, T) W_x^{\frac{d}{\alpha}}, \quad (4.88)
 \end{aligned}$$

where we use the fact that $E^{W_y} \left[W_y^{\frac{2d}{\alpha}} \right] < \infty$ since $\alpha\beta > 2d$. Using (4.88) and (4.44) we obtain

$$\begin{aligned}
 s r_s^d \int_{A_s} E^{W_x} [E[V'_x|W_x] \tilde{p}_s(W_x)] dx &\leq C_4 s (s r_s^d)^{k+2} E^{W_x} \left[\phi_s(W_x, T) W_x^{\frac{d}{\alpha}} \left(c_0 s r_s^d W_x^{\frac{d}{\alpha}} \right)^k e^{-s r_s^d \Lambda_s(W_x)} \right] \\
 &= C_5 s (s r_s^d)^{k+1} E^{W_x} \left[\phi_s(W_x, T) \left(c_0 s r_s^d W_x^{\frac{d}{\alpha}} \right)^{k+1} e^{-s r_s^d \Lambda_s(W_x)} \right]. \quad (4.89)
 \end{aligned}$$

An application of Cauchy-Schwarz inequality yields

$$s r_s^d \int_{A_s} E^{W_x} [E[V'_x|W_x] \tilde{p}_s(W_x)] dx \leq C_5 s (s r_s^d)^{k+1} \left(E^{W_x} [\phi_s(W_x, T)^2] f_2(s, k + 1) \right)^{\frac{1}{2}}, \quad (4.90)$$

where $f_2(s, k + 1)$ is defined in (4.45). This proves Lemma 4.5 (iii).

Proof of Lemma 4.5 (iv). By the Campbell-Mecke formula and the union bound

$$\begin{aligned}
 E[V''_x|W_x] &\leq k \sum_{i=0}^k s \int_S \int_S E^{W_y W_z} \left[\frac{g_s(x, W_x; y, W_y)}{\int_S E^{W_w} [g_s(x, W_x; w, W_w)] dw} \right. \\
 &\quad \left. \times g_s(y, W_y; z, W_z) \Phi_s(z; W_z, i) \right] dz dy, \quad (4.91)
 \end{aligned}$$

where $\Phi_s(z; W_z, i)$ is defined as in (4.61). By making the standard change of variable the expression inside the sum on the right in (4.91) can be bounded by some constant times

$$s r_s^{2d} \int_{A_s} \int_{A_s} E^{W_y W_z} \left[\frac{\tilde{g}(W_x; y, W_y) \tilde{g}(W_y; z, W_z)}{r_s^d \int_{A_s} E^{W_w} [\tilde{g}(W_x; w, W_w)] dw} \bar{\Phi}_s(W_z, i) \right] dz dy, \quad (4.92)$$

where $\bar{\Phi}_s(W_z, i)$ is defined as in (4.63). Using the fact that the weights are larger than one the expression in (4.92) is bounded by

$$\begin{aligned} & sr_s^d \int_{A_s} \int_{A_s} E^{W_y W_z} \left[\frac{\tilde{g}(W_x; y, W_y)}{\int_{A_s} \left[1 - \exp\left(-\frac{\eta}{|w|^\alpha}\right) \right] dw} \tilde{g}(W_y; z, W_z) \bar{\Phi}_s(W_z, i) \right] dz dy \\ & \leq C_0 sr_s^d \int_{A_s} E^{W_y} \left[\tilde{g}(W_x; y, W_y) \int_{A_s} E^{W_z} [\tilde{g}(W_y; z, W_z) \bar{\Phi}_s(W_z, i)] dz \right] dy, \end{aligned} \quad (4.93)$$

where the last inequality follows from (4.74). Using (4.13) and the fact that weights are larger than one, we bound the inner integral in (4.93) by,

$$\begin{aligned} & \int_{A_s} E^{W_z} [\tilde{g}(W_y; z, W_z) \bar{\Phi}_s(W_z, i)] dz \\ & \leq C_1 \int_{A_s} E^{W_z} \left[\tilde{g}(W_y; z, W_z) \left(c_0 sr_s^d W_z^{\frac{d}{\alpha}} \right)^i e^{-sr_s^d \Lambda_s(W_z)} \right] dz \\ & \leq C_1 \int_{A_s} E^{W_z} \left[\tilde{g}(W_y; z, W_z) \left(c_0 sr_s^d W_z^{\frac{d}{\alpha}} \right)^k e^{-sr_s^d \Lambda_s(W_z)} \right] dz, \end{aligned} \quad (4.94)$$

for sufficiently large s . Comparing (4.94) with the first inequality in (4.64) and proceeding as we did to derive (4.69) we obtain

$$\int_{A_s} E^{W_z} \left[\tilde{g}(W_y; z, W_z) \left(c_0 sr_s^d W_z^{\frac{d}{\alpha}} \right)^k e^{-sr_s^d \Lambda_s(W_z)} \right] dz \leq C_2 f_1(s, k + 1) c_0 W_y^{\frac{d}{\alpha}}, \quad (4.95)$$

where $f_1(s, k + 1)$ is defined in (4.45). Substituting from (4.94) and (4.95) in (4.93) we obtain

$$\begin{aligned} E[V_x'' | W_x] & \leq C_3 sr_s^d \int_{A_s} E^{W_y} \left[\tilde{g}(W_x; y, W_y) f_1(s, k + 1) c_0 W_y^{\frac{d}{\alpha}} \right] dy \\ & = C_3 f_1(s, k + 1) c_0 sr_s^d \int_{A_s} E^{W_y} \left[\tilde{g}(W_x; y, W_y) W_y^{\frac{d}{\alpha}} \right] dy \\ & \leq C_4 sr_s^d f_1(s, k + 1) W_x^{\frac{d}{\alpha}} E^{W_y} \left[W_y^{\frac{2d}{\alpha}} \right] = C_5 sr_s^d f_1(s, k + 1) W_x^{\frac{d}{\alpha}}, \end{aligned} \quad (4.96)$$

since $\alpha\beta > 2d$. From (4.96) and (4.44) we get

$$\begin{aligned} & sr_s^d \int_{A_s} E^{W_x} [E[V_x'' | W_x] \tilde{p}_s(W_x)] dx \\ & \leq C_5 s^2 r_s^d f_1(s, k + 1) E^{W_x} \left[W_x^{\frac{d}{\alpha}} \left(c_0 sr_s^d W_x^{\frac{d}{\alpha}} \right)^k e^{-sr_s^d \Lambda_s(W_x)} \right] \\ & = C_6 s f_1(s, k + 1) E^{W_x} \left[c_0 sr_s^d W_x^{\frac{d}{\alpha}} \left(c_0 sr_s^d W_x^{\frac{d}{\alpha}} \right)^k e^{-sr_s^d \Lambda_s(W_x)} \right] \\ & = C_6 s f_1(s, k + 1)^2. \end{aligned} \quad (4.97)$$

This completes the proof of Lemma 4.5. □

4.4 Proof of Theorem 3.3

Let $\hat{r}_s(\gamma), \kappa, T, Q$ be as defined in (3.4), (3.5) and ρ satisfy $Q(\rho) = 1$. Fix $\gamma > \rho$. Since Q is continuous and increasing (see remark below (3.5)), we can and do choose $b > 1$ such that $\gamma T(\frac{\gamma}{b}) > 1$. Let $\tilde{r}_s(b)^d := \frac{b \log s}{\theta_d s}$. In any graph G with vertex set V and edge set E , a one-hop path between distinct vertices $x, y \in V$ is a two-path comprising of edges $\{x, z\}, \{z, y\} \in E$ for some $z \in V$. Let $E_s \equiv E_s(\gamma, b)$ be the event that there is a vertex

$X \in \mathcal{P}_s$ such that X does not have a one-hop path to some vertex in $\mathcal{P}_s \cap B(X, \tilde{r}_s(b))$ in the graph $G(\mathcal{P}_s, \hat{r}_s(\gamma))$. The result is not altered if we restrict attention only to those vertices in $\mathcal{P}_s \cap B(X, \tilde{r}_s(b))$ that do not have a direct connection to X .

If $P(E_s) \rightarrow 0$ as $s \rightarrow \infty$, then every vertex X in the graph $G(\mathcal{P}_s, \hat{r}_s(\gamma))$ is connected to every point in $\mathcal{P}_s \cap B(X, \tilde{r}_s(b))$ through a one-hop path whp. The result then follows since the existence of a path to all the $\tilde{r}_s(b)$ -neighbours for some $b > 1$ will imply that the graph $G(\mathcal{P}_s, \hat{r}_s(\gamma))$ is connected whp (Theorem 13.7, [17]). Thus it suffices to show that $P(E_s) \rightarrow 0$ as $s \rightarrow \infty$.

Let $h(x, \mathcal{P}_s)$ equal one if x is not connected to at least one vertex in $\mathcal{P}_s \cap B(x, \tilde{r}_s(b))$ via a one hop path in the graph $G(\mathcal{P}_s \cup \{x\}, \hat{r}_s(\gamma))$ and zero otherwise. By the Campbell-Mecke formula we have

$$P(E_s) \leq E \left[\sum_{x \in \mathcal{P}_s} h(x, \mathcal{P}_s) \right] = sE^o[h(O, \mathcal{P}_s)]. \tag{4.98}$$

Let $H : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be defined by $H(x) = 1 - x + x \log x$. Choose a large enough such that $bH(\frac{a}{b}) > 1$. Define the event $F_s := \{0 < \mathcal{P}_s(B(O, \tilde{r}_s(b))) < a \log s\}$. From (4.98) we have

$$P(E_s) \leq sE^o[h(O, \mathcal{P}_s); F_s] + sP(F_s^c). \tag{4.99}$$

By the Chernoff bound (see Lemma 1.2 of [17]),

$$\begin{aligned} sP(\mathcal{P}_s(B(O, \tilde{r}_s(b))) \geq a \log s) &\leq s \exp \left(-s\theta \tilde{r}_s(b)^d H \left(\frac{a \log s}{s\theta \tilde{r}_s(b)^d} \right) \right) \\ &= s^{-bH(\frac{a}{b})+1} \rightarrow 0, \end{aligned} \tag{4.100}$$

as $s \rightarrow \infty$, since $bH(\frac{a}{b}) > 1$. Hence $sP(F_s^c) \rightarrow 0$ as $s \rightarrow \infty$. We now evaluate $E^o[h(O, \mathcal{P}_s); F_s]$. Let Y_1, Y_2, \dots be a sequence of independent random variables distributed uniformly in $B(O, \tilde{r}_s(b))$. Let N_s be an independent Poisson random variable with mean $\theta_d s \tilde{r}_s(b)^d$. Set \mathcal{P}_s to be $\{Y_1, Y_2, \dots, Y_{N_s}\}$ on $B(O, \tilde{r}_s(b))$. Let $A_s^{(i)}$ be the event that O is not connected to Y_i via a one-hop path in the graph $\tilde{G}_s := G(\mathcal{P}_s \cup \{O, Y_{N_s+1}, Y_{N_s+2}, \dots\}, \hat{r}_s(\gamma))$ using only the vertices $\mathcal{P}_s \cup \{O, Y_i\}$. Regard $G(\mathcal{P}_s, \hat{r}_s(\gamma))$ as a subgraph of \tilde{G}_s . We then have

$$\begin{aligned} E^o[h(O, \mathcal{P}_s); F_s] &\leq E^o \left[\sum_{j=1}^{N_s} 1_{A_s^{(j)}}; F_s \right] \\ &= E^o \left[\sum_{j=1}^{\lfloor a \log s \rfloor} 1_{A_s^{(j)}}; F_s \right] \leq a \log s E [P^{o,Y}(A_s)], \end{aligned} \tag{4.101}$$

where A_s is the event that O is not connected to point Y chosen independently and uniformly at random in $B(O, \tilde{r}_s(b))$ via a one-hop path in the graph $G(\mathcal{P}_s, \hat{r}_s(\gamma))$ under the Palm measure $P^{o,Y}$. By the thinning theorem (see Proposition 5.5, Theorem 5.8, [13]) we have

$$E [P^{o,Y}(A_s)] = \frac{1}{\theta_d \tilde{r}_s(b)^d} \int_{B(O, \tilde{r}_s(b))} E^{W_0 W_y} \left[e^{-s \int_S E^{W_z} [\hat{g}_s(O, W_o; z, W_z) \hat{g}_s(y, W_y; z, W_z)] dz} \right] dy. \tag{4.102}$$

Since the weights are all greater than one, the integral inside the exponential in (4.102) can be bounded from below as follows. Since $\hat{r}_s(\gamma) \rightarrow 0$ as $s \rightarrow \infty$, we have for all s

sufficiently large,

$$\begin{aligned} & \int_S E^{W_z} \left[\left(1 - \exp \left(- \frac{\eta W_0 W_z}{\left(\frac{d(O,z)}{\hat{r}_s(\gamma)} \right)^\alpha} \right) \right) \left(1 - \exp \left(- \frac{\eta W_y W_z}{\left(\frac{d(y,z)}{\hat{r}_s(\gamma)} \right)^\alpha} \right) \right) \right] dz \\ & \geq \int_{B(y, \hat{r}_s(\gamma))} \left(1 - \exp \left(- \frac{\eta}{\left(\frac{d(O,z)}{\hat{r}_s(\gamma)} \right)^\alpha} \right) \right) E^{W_z} \left[1 - \exp \left(- \frac{\eta W_z}{\left(\frac{d(y,z)}{\hat{r}_s(\gamma)} \right)^\alpha} \right) \right] dz. \end{aligned} \quad (4.103)$$

Using the triangle inequality in the first factor inside the integral and a change of variable the expression on the right in (4.103) can be bounded from below by

$$\begin{aligned} & \left(1 - \exp \left(- \eta \left| \frac{\hat{r}_s(\gamma) + \tilde{r}_s(b)}{\hat{r}_s(\gamma)} \right|^{-\alpha} \right) \right) \hat{r}_s(\gamma)^d \int_{B(O,1)} E^{W_z} \left[1 - \exp \left(- \frac{\eta W_z}{|z|^\alpha} \right) \right] dz \\ & = \kappa T \left(\frac{\gamma}{b} \right) \hat{r}_s(\gamma)^d, \end{aligned} \quad (4.104)$$

since $\frac{\tilde{r}_s(b)}{\hat{r}_s(\gamma)} = \left(\frac{b\kappa}{\gamma\theta_d} \right)^{\frac{1}{d}}$. From (4.101)–(4.104) we obtain

$$\begin{aligned} s E^o[h(O, \mathcal{P}_s); F_s] & \leq as \log s \exp \left(- \kappa T \left(\frac{\gamma}{b} \right) s \hat{r}_s(\gamma)^d \right) \\ & = as^{-\gamma T \left(\frac{\gamma}{b} \right) + 1} \log s \rightarrow 0, \end{aligned} \quad (4.105)$$

as $s \rightarrow \infty$, since $\gamma T \left(\frac{\gamma}{b} \right) > 1$. It follows from (4.99), (4.100) and (4.105) that $P(E_s) \rightarrow 0$ as $s \rightarrow \infty$. This completes the proof of Theorem 3.3. \square

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