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Scaling limits of crossing probabilities in metric graph GFF*

Mingchang Liu† Hao Wu‡

Abstract

We consider metric graph Gaussian free field (GFF) defined on polygons of $\delta\mathbb{Z}^2$ with alternating boundary data. The crossing probabilities for level-set percolation of metric graph GFF have scaling limits. When the boundary data is well-chosen, the scaling limits of crossing probabilities can be explicitly constructed as "fusion" of multiple SLE_4 pure partition functions.

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1 Introduction

This article concerns crossing probability of level-set percolation of Gaussian free field (GFF) on the square lattice $\mathbb{Z}^2$. For $L > 0$, consider the rectangle

$$R_L = \{z : 0 < \Re z < L, 0 < \Im z < 1\}.$$

Let y_1, y_2, y_3, y_4 be its four corners, listed in counterclockwise order with $y_2 = 0$. For $\delta > 0$, let $V_\delta = R_L \cap \delta\mathbb{Z}^2$ and let $y_1^\delta, y_2^\delta, y_3^\delta, y_4^\delta$ be its four corners, listed in counterclockwise order such that y_2^δ is closest to y_2. For two vertices $u, v \in \partial V_\delta$, we denote by (uv) the arc of ∂V_δ from u to v in counterclockwise order. Let Γ^δ be a discrete GFF (see Section 5.1) on V_δ with alternating boundary data:

$$\mu \text{ on } (y_1^\delta y_2^\delta) \cup (y_3^\delta y_4^\delta), \quad -\mu \text{ on } (y_2^\delta y_3^\delta) \cup (y_4^\delta y_1^\delta),$$

where $\mu > 0$ is a positive constant. Let $\tilde{\Gamma}^\delta$ be the GFF on the metric graph $\tilde{V}_\delta$ (see Section 5.1) with the same boundary data. We are interested in the event that there

*Supported by Beijing Natural Science Foundation (JQ20001, Z180003) and Thousand Talents Plan for Young Professionals.
†Tsinghua University, China. E-mail: liumc18@mails.tsinghua.edu.cn
‡Tsinghua University, China. E-mail: hao.wu.proba@gmail.com

exists a path in V_δ (resp. $\tilde{V}_\delta$) from $(y_1^\delta y_2^\delta)$ to $(y_3^\delta y_4^\delta)$ such that Γ^δ (resp. $\tilde{\Gamma}^\delta$) is non-negative on this path. We denote this event by

$$(y_1^\delta y_2^\delta) \overset{\Gamma\geq 0}{\longleftrightarrow} (y_3^\delta y_4^\delta) \quad \text{and} \quad (y_1^\delta y_2^\delta) \overset{\tilde{\Gamma}\geq 0}{\longleftrightarrow} (y_3^\delta y_4^\delta)$$

for Γ^δ and $\tilde{\Gamma}^\delta$ respectively. Although, both discrete GFF Γ^δ and metric graph GFF $\tilde{\Gamma}^\delta$ converge as distributions to the continuum GFF as $\delta \to 0$, the probabilities for such crossing events have distinct scaling limits, as proved in [DWW20, Theorem 1.2]. It is then natural to ask whether we are able to give explicit formula for scaling limits of such crossing probabilities.

The answer to this question relies on Schramm-Sheffield's famous work on level lines of GFF. We call $(\Omega; x, y)$ a Dobrushin domain if $\Omega \subsetneq \mathbb{C}$ is non-empty simply connected and x, y are distinct boundary points. In [SS09], the authors prove that there exists $\lambda = \lambda(\mathbb{Z}^2) > 0$ such that the zero level line of discrete GFF on Dobrushin domains of $\delta\mathbb{Z}^2$ with boundary data $\pm\lambda$ converges in distribution to Schramm-Loewner Evolution (SLE_4, see Section 2.3). Based on this result, one is able to show that [DWW20, Theorem 1.3], when $\mu = \lambda$,

$$\lim_{\delta\to 0} \mathbb{P}\left[(y_1^\delta y_2^\delta) \overset{\Gamma\geq 0}{\longleftrightarrow} (y_3^\delta y_4^\delta)\right] = q, \tag{1.1}$$

where q is the cross-ratio of the rectangle: let φ be any conformal map from R_L onto the upper-half plane $\mathbb{H}$ with $\varphi(y_1) < \varphi(y_2) < \varphi(y_3) < \varphi(y_4)$, then

$$q = \frac{(\varphi(y_2) - \varphi(y_1))(\varphi(y_4) - \varphi(y_3))}{(\varphi(y_3) - \varphi(y_1))(\varphi(y_4) - \varphi(y_2))}. \tag{1.2}$$

This gives answer to the case of discrete GFF. The authors in [DWW20] derive (1.1) by showing that the scaling limit of the crossing probability in discrete GFF is the same as the one for continuum GFF whose crossing probability is calculated in [PW19, Theorem 1.4]. Such probability is also calculated in [KW11]. It remains to answer the question for the case of metric graph GFF.

The goal of this article is to derive explicit formula for scaling limits of crossing probability in metric graph GFF. We will show that, when $\mu = 2\lambda$,

$$\lim_{\delta\to 0} \mathbb{P}\left[(y_1^\delta y_2^\delta) \overset{\tilde{\Gamma}\geq 0}{\longleftrightarrow} (y_3^\delta y_4^\delta)\right] = q^4, \tag{1.3}$$

where q is the cross-ratio of the rectangle as in (1.2). In fact, we are able to give answer in a more general setting: we can calculate the scaling limits of crossing probabilities for the metric graph GFF with alternating boundary data on a polygon with $2N$ marked points on the boundary. To state our main result, we first introduce some notations about planar link patterns.

For $p \in \mathbb{Z}_{>0}$, we call $(\Omega; x_1, \ldots, x_p)$ a polygon if $\Omega \subsetneq \mathbb{C}$ is non-empty simply connected and $x_1, \ldots, x_p$ are p boundary points in counterclockwise order lying on locally connected boundary segments. We first introduce planar pair partitions. Suppose $p = 2N$ is even and suppose there are N non-intersecting simple curves in Ω connecting the $2N$ boundary points pairwise. These N curves form a planar pair partition that we denote by $\alpha = \{\{a_1, b_1\}, \ldots \{a_N, b_N\}\}$ with $\{a_1, b_1, \ldots, a_N, b_N\} = \{1, 2, \ldots, 2N\}$. We call the pairs $\{a, b\}$ in α links. We denote by $\mathrm{PP_N}$ the set of planar pair partitions with $2N$ points and set $\mathrm{PP} = \bigsqcup_{N\geq 0} \mathrm{PP_N}$.

Next, we introduce general planar link patterns. The planar pair partitions then arise as a special case. Suppose $(\Omega; x_1, \ldots, x_p)$ is a polygon. Fix a multiindex $\varsigma = (s_1, \ldots, s_p) \in$

$\mathbb{Z}^{p}_{>0}$ such that $\sum_{i=1}^{p} s_i$ is even and denote by

$$\ell = \frac{1}{2}\sum_{i=1}^{p} s_i \in \mathbb{Z}_{>0}.$$

Suppose there are ℓ simple curves in Ω connecting the p boundary points pairwise such that they do not intersect except at their common end points. These ℓ curves form a planar link pattern that we call planar ℓ-link patterns of p points. Precisely, we call planar ℓ-link patterns of p points with index valences $\varsigma = (s_1, \ldots, s_p)$ as collections $\omega = \{\{a_1, b_1\}, \ldots, \{a_\ell, b_\ell\}\}$ of ℓ-links $\{a, b\}$ which connect a pair of distinct indices $a, b \in \{1, 2, \ldots, p\}$ such that, for any $i \in \{1, 2, \ldots, p\}$, the index i is an endpoint of exactly s_i links and that none of the links intersect except at their common endpoints. We denote the collection of ℓ-link patterns of p points with index valences ς by LP_ς. With such definition, when $p = 2N$ is even, the planar N-link pattern of $2N$ points with index valences $\varsigma = (1, \ldots, 1)$ is a planar pair partition and $\mathrm{LP}_{(1,\ldots,1)} = \mathrm{PP_N}$.

In this article, we are interested in planar $2N$-link patterns of $2N$ points with index valences $\varsigma = (2, \ldots, 2)$, see Figure 1 for $N = 2$. With the above definition, the collection of such planar link patterns is denoted by $\mathrm{LP}_{(2,\ldots,2)}$ where the index has length $2N$.

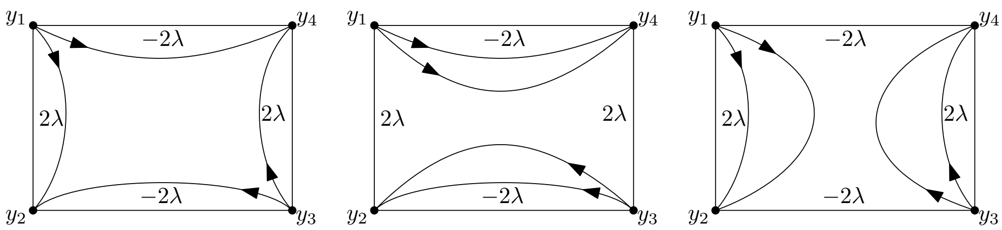

Figure 1: Consider metric graph GFF in rectangle with alternating boundary data when $\mu = 2\lambda$. Consider the positive first passage sets attached to $(y^\delta_1 y^\delta_2)$ and to $(y^\delta_3 y^\delta_4)$ and consider the negative first passage sets attached to $(y^\delta_2 y^\delta_3)$ and to $(y^\delta_4 y^\delta_1)$. Their frontier form a planar 4-link pattern of 4 points with index valences $\varsigma = (2, 2, 2, 2)$. There are three possibilities as indicated in the figure. In the right panel, there is negative vertical crossing of the metric graph GFF. In the middle panel, there is positive horizontal crossing. In the left panel, there is neither positive horizontal crossing nor negative vertical crossing. As $\delta \to 0$, the frontier converges to level lines of continuum GFF with the same boundary data and the four level lines are as follows: there are two level lines starting from y_1 (resp. from y_3), one has height $-\lambda$ and the other one has height λ.

Fix a polygon $(\Omega; y_1, \ldots, y_{2N})$ such that $\Omega \subset [-C, C]^2$ for some $C > 0$. Suppose $\{(\Omega^\delta; y^\delta_1, \ldots, y^\delta_{2N})\}_{\delta>0}$ are polygons such that $\Omega^\delta \subset [-C, C]^2$ for all $\delta > 0$. Suppose $(\Omega^\delta; y^\delta_1, \ldots, y^\delta_{2N})$ converges to $(\Omega; y_1, \ldots, y_{2N})$ as $\delta \to 0$ in the following sense:

$$\begin{aligned}&[-C, C]^2 \setminus \Omega^\delta \text{ converges to } [-C, C]^2 \setminus \Omega \text{ in Hausdorff metric}\\ &\text{and } y^\delta_i \to y_i \text{ for each } 1 \le i \le 2N.\end{aligned} \tag{1.4}$$

Consider metric graph GFF $\tilde{\Gamma}^\delta$ in $(\Omega^\delta; y^\delta_1, \ldots, y^\delta_{2N})$ with alternating boundary data:

$$2\lambda \text{ on } (y^\delta_{2j-1} y^\delta_{2j}), \quad \text{and} \quad -2\lambda \text{ on } (y^\delta_{2j} y^\delta_{2j+1}), \quad \text{for } j \in \{1, \ldots, N\}, \tag{1.5}$$

where $y_{2N+1} = y_1$ by convention. Consider positive first passage set (see Section 5.2) of $\tilde{\Gamma}^\delta$ attached to the boundary segments $(y^\delta_{2j-1} y^\delta_{2j})$, and negative first passage set attached to the boundary segments $(y^\delta_{2j} y^\delta_{2j+1})$, for $j \in \{1, \ldots, N\}$. The frontier of these first passage sets is a collection of $2N$ curves connecting the $2N$ boundary points so that their end points form a planar $2N$-link pattern of $2N$ points with index valences $\varsigma = (2, 2, \ldots, 2)$. See Figure 1. We denote the link pattern by $\mathcal{A}^\delta$. Our main result is the following.

Theorem 1.1. *Fix $N \geq 1$ and the index valences $\varsigma = (2, \ldots, 2)$ of length $2N$. Consider the frontier of first passage sets of metric graph GFF in Ω^δ with alternating boundary data (1.5). The frontier is a collection of $2N$ curves connecting the $2N$ boundary points whose end points form a planar link pattern $\mathcal{A}^\delta \in \mathrm{LP}_\varsigma$. We have*

$$\lim_{\delta \to 0} \mathbb{P}[\mathcal{A}^\delta = \hat{\alpha}] = \mathcal{M}_{\omega,\tau(\hat{\alpha})} \frac{\mathcal{Z}_{\hat{\alpha}}(\Omega; y_1, \ldots, y_{2N})}{\mathcal{Z}^{(N)}_{\mathrm{mGFF}}(\Omega; y_1, \ldots, y_{2N})}, \quad \textit{for all } \hat{\alpha} \in \mathrm{LP}_\varsigma,$$

where the coefficient $\mathcal{M}_{\omega,\tau(\hat{\alpha})}$ is given by Lemma 5.9, the function $\mathcal{Z}_{\hat{\alpha}}$ is given by Proposition 5.6 and Corollary 5.8, and

$$\mathcal{Z}^{(N)}_{\mathrm{mGFF}}(\Omega; y_1, \ldots, y_{2N}) = \sum_{\hat{\alpha} \in \mathrm{LP}_\varsigma} \mathcal{M}_{\omega,\tau(\hat{\alpha})} \mathcal{Z}_{\hat{\alpha}}(\Omega; y_1, \ldots, y_{2N}).$$

The definition for $\mathcal{M}_{\omega,\tau(\hat{\alpha})}$ and $\mathcal{Z}_{\hat{\alpha}}$ is quite involved, and we omit it from the introduction. Nevertheless, let us mention in the introduction nice properties that $\mathcal{Z}_{\hat{\alpha}}$ enjoys. First of all, they are conformally covariant: for any polygon $(\Omega; y_1, \ldots, y_{2N})$ such that $y_1, \ldots, y_{2N}$ lie on sufficiently regular segments of $\partial\Omega$ (e.g. $C^{1+\epsilon}$ for some $\epsilon > 0$) and any conformal map φ on Ω, we have

$$\mathcal{Z}_{\hat{\alpha}}(\Omega; y_1, \ldots, y_{2N}) = \prod_{i=1}^{2N} \varphi'(y_i) \times \mathcal{Z}_{\hat{\alpha}}(\varphi(\Omega); \varphi(y_1), \ldots, \varphi(y_{2N})). \tag{1.6}$$

When $\Omega = \mathbb{H}$ and $y_1 < \cdots < y_{2N}$, we write

$$\mathcal{Z}_{\hat{\alpha}}(y_1, \ldots, y_{2N}) = \mathcal{Z}_{\hat{\alpha}}(\mathbb{H}; y_1, \ldots, y_{2N}), \quad \mathcal{Z}^{(N)}_{\mathrm{mGFF}}(y_1, \ldots, y_{2N}) = \mathcal{Z}^{(N)}_{\mathrm{mGFF}}(\mathbb{H}; y_1, \ldots, y_{2N}).$$

Then, we have

$$\mathcal{Z}^{(N)}_{\mathrm{mGFF}}(y_1, \ldots, y_{2N}) = \prod_{1 \leq i < j \leq 2N} (y_j - y_i)^{2(-1)^{j-i}}. \tag{1.7}$$

Proposition 1.2. *Fix $N \geq 1$ and the index valences $\varsigma = (2, \ldots, 2)$ of length $2N$. For any $\hat{\alpha} \in \mathrm{LP}_\varsigma$, the function $\mathcal{Z}_{\hat{\alpha}} : \mathfrak{X}_{2N} \to \mathbb{R}_{>0}$ given by Proposition 5.6 and Corollary 5.8 satisfies the following PDE system: for all $j \in \{1, \ldots, 2N\}$,*

$$\left[\frac{\partial^3}{\partial y_j^3} - 4\mathcal{L}^{(j)}_2 \frac{\partial}{\partial y_j} + 2\mathcal{L}^{(j)}_3 \right] \mathcal{Z}_{\hat{\alpha}}(y_1, \ldots, y_{2N}) = 0, \tag{1.8}$$

where $$\mathcal{L}^{(j)}_2 := \sum_{i \neq j} \left(\frac{1}{(y_i - y_j)^2} - \frac{1}{y_i - y_j} \frac{\partial}{\partial y_i} \right),$$
$$\mathcal{L}^{(j)}_3 := \sum_{i \neq j} \left(\frac{2}{(y_i - y_j)^3} - \frac{1}{(y_i - y_j)^2} \frac{\partial}{\partial y_i} \right).$$

In Section 2, we will give preliminaries on planar link patterns and SLEs. In Section 3, we will introduce multiple SLE partition functions and prove a preliminary result about "fusion" of partition functions—Proposition 3.1. In Section 4, we will introduce continuum GFF and prove a result on connection probabilities—Theorem 4.1. In Section 5, we will introduce metric graph GFF and complete the proof of Theorem 1.1 and (1.6), (1.7) and (1.8) by combining the results from preceding sections.

The proof for Theorem 1.1 relies on the following three ingredients: a). Aru-Lupu-Sepúlveda's work on the convergence of first passage sets of metric graph GFF [ALS20], see Sections 5.2 and 5.3. b). A generalization of Peltola and the second author's work [PW19] on crossing probabilities in continuum GFF, see Theorem 4.1. c). Analysis on the asymptotics of multiple SLE_4 partition functions, see Proposition 5.6. With these three at hand, let us briefly describe how we derive Theorem 1.1 with $N = 2$ which gives (1.3). The proof for general N uses a similar idea. Our strategy is as follows: First, we use a) to show that the frontiers of first passage set of metric graph GFF converge to level lines of continuum GFF with boundary data $(-2\lambda, 2\lambda, -2\lambda, 2\lambda)$ and proper heights, see Figure 1. Second, we use b) to calculate the crossing probabilities in continuum GFF with boundary data $(-2\lambda, 0, 2\lambda, 0, -2\lambda, 0, 2\lambda, 0)$, see Figure 5. Finally, in Figure 5, we let $x_1, x_2 \to y_1$, and $x_3, x_4 \to y_2$ and $x_5, x_6 \to y_3$ and $x_7, x_8 \to y_4$, then the four level lines in Figure 5 become the level lines in Figure 1 and, due to c), the crossing probabilities calculated in the second step admit limits which give the desired probability in (1.3). See Corollary 5.10.

The proof for Proposition 1.2 relies on Proposition 3.1. Note that the third order PDEs are not surprising. SLE partition functions are solutions to 2nd order PDEs (3.2) and they can be understood as certain correlation functions in terms of conformal field theory (CFT). Then the third order PDEs can be obtained by specific fusion channel, see [BB03, BB04, BBK05, FK04, Dub15, KP16, Pel20, Pel19]. See also discussion after Proposition 3.1. The 2nd order PDEs (3.2) arise from stochastic differentials of certain local martingales and SLE partition functions are related to crossing probabilities for the critical statistical physics models, see [FSKZ17, KKP20, PW19, PW18]. However, there seems no known direct probabilistic interpretation of higher order PDEs of CFT before. In this sense, our work provides an example that gives a probabilistic interpretation to higher order PDEs of CFT.

2 Preliminaries

2.1 Planar pair partitions and Dyck paths

In this section, we will give a one-to-one correspondence between planar pair partitions and Dyck paths. A Dyck path is a walk on $\mathbb{Z}_{\ge 0}$ with steps of length one, starting and ending at zero: $\alpha : \{0, 1, \ldots, 2N\} \to \mathbb{Z}_{\ge 0}$ such that $\alpha(0) = \alpha(2N) = 0$, and $|\alpha(k) - \alpha(k-1)| = 1$ for all $k \in \{1, 2, \ldots, 2N\}$. For $N \ge 1$, we denote the set of all Dyck paths of $2N$ steps by $\mathrm{DP_N}$. There is a natural partial order on Dyck paths:

$$\alpha \preceq \beta \quad \text{if and only if} \quad \alpha(k) \le \beta(k), \text{ for all } k \in \{0, 1, \ldots, 2N\}. \tag{2.1}$$

We set $\mathrm{DP} = \bigsqcup_{N \ge 0} \mathrm{DP_N}$.

To each planar pair partition $\alpha \in \mathrm{PP_N}$, we write it as

$$\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\}, \tag{2.2}$$
$$\text{where } a_1 < a_2 < \cdots < a_N \text{ and } a_j < b_j, \text{ for all } j \in \{1, \ldots, N\}.$$

We associate it with a Dyck path, also denoted by $\alpha \in \mathrm{DP_N}$, as follows. We set $\alpha(0) = 0$ and, for all $k \in \{1, \ldots, 2N\}$, we set

$$\alpha(k) = \begin{cases} \alpha(k-1) + 1, & \text{if } k \in \{a_1, a_2, \ldots, a_N\}, \\ \alpha(k-1) - 1, & \text{if } k \in \{b_1, b_2, \ldots, b_N\}. \end{cases} \tag{2.3}$$

One may check, this defines a Dyck path $\alpha \in \mathrm{DP_N}$. Conversely, for any Dyck path $\alpha : \{0, 1, \ldots, 2N\} \to \mathbb{Z}_{\ge 0}$, we associate a planar pair partition α by giving to each up-step (i.e., step away from zero) an index a_r, for $r = 1, 2, \ldots, N$, and to each down-step (i.e., step towards zero) an index b_s, for $s = 1, 2, \ldots, N$, and setting $\alpha := \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\}$. These two mappings define a bijection between $\mathrm{PP_N}$ and $\mathrm{DP_N}$. We thus identify the elements α of these two sets and use the indistinguishable notation $\alpha \in \mathrm{PP_N}$ and $\alpha \in \mathrm{DP_N}$ for both. See Figure 2.

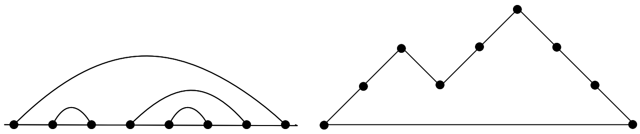

Figure 2: Illustration of the bijection $\mathrm{PP_N} \leftrightarrow \mathrm{DP_N}$, identifying planar pair partition and Dyck path for $\alpha = \{\{1, 8\}, \{2, 3\}, \{4, 7\}, \{5, 6\}\}$.

For a Dyck path $\alpha \in \mathrm{DP_N}$, we say that α has a local maximum at j if $\alpha(j) - \alpha(j-1) = 1$ and $\alpha(j+1) - \alpha(j) = -1$, and we denote $\wedge^j \in \alpha$; we say that α has a local minimum at j if $\alpha(j) - \alpha(j-1) = -1$ and $\alpha(j+1) - \alpha(j) = 1$, and we denote $\vee_j \in \alpha$; we say that α has a slope at j if otherwise, and we denote $\times_j \in \alpha$. We say that α has a local extremum at j if α has a local minimum or maximum at j, and we denote $\quad_j \in \alpha$.

If a planar pair partition $\alpha \in \mathrm{PP_N}$ has a link $\{j, j+1\} \in \alpha$, then $\wedge^j \in \alpha$. Let $\alpha/\{j, j+1\}$ denote the planar pair partition by removing from α the link $\{j, j+1\}$ and relabelling the remaining indices by $1, 2, \ldots, 2N-2$. In terms of Dyck path, we denote this operation by $\alpha/\wedge^j \in \mathrm{DP_{N-1}}$. We define operation $\alpha/\vee_j \in \mathrm{DP_{N-1}}$ analogously when α has a local minimum at j. When α has a local extremum at j, we denote such operation by $\alpha/\quad_j$. If α has a local minimum at j, we associate α with another Dyck path by converting the local minimum at j to local maximum, and denote this operation by $\alpha \uparrow \quad_j$.

2.2 From planar link pattern to planar pair partition

Fix an index valences $\varsigma = (s_1, \ldots, s_p) \in \mathbb{Z}^p_{>0}$ such that $\sum_{i=1}^p s_i$ is even and we denote this even number by 2ℓ. Recall that LP_ς is the collection of all planar ℓ-link patterns of p points with index valences ς. We define a natural map which associates to each planar link pattern a planar pair partition. This map, denoted by

$$\tau : \mathrm{LP}_\varsigma \to \mathrm{PP}_\ell, \quad \hat{\alpha} \mapsto \tau(\hat{\alpha}),$$

is defined as following: in $\hat{\alpha}$, for each $j \in \{1, 2, \ldots, p\}$, we split the jth point to s_j distinct points and attach the s_j links of $\hat{\alpha}$ ending there to these new s_j points so that each of them has valence one. See Figure 3.

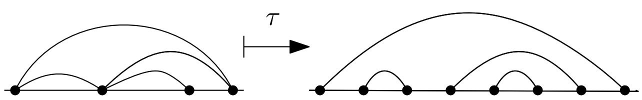

Figure 3: In this figure, we have a planar link pattern with index valences $\varsigma = (2, 3, 1, 2)$. It is associated to a planar pair partition by splitting the four points into eight points according to the valences and attaching the corresponding links.

In this article, we are interested in planar link patterns with index valences $\varsigma = (2, \ldots, 2)$. Fix $N \geq 1$ and the index valences $\varsigma = (2, \ldots, 2)$ of length $2N$. Then τ introduces a bijection between LP_ς and the collection $\{\beta \in \mathrm{PP_{2N}} : \wedge_{2j-1} \notin \beta, \text{ for all } 1 \leq j \leq N\}$.

2.3 Loewner chain and SLE

We call a compact subset K of $\overline{\mathbb{H}}$ an $\mathbb{H}$-hull if $\mathbb{H} \setminus K$ is simply connected. Riemann's mapping theorem implies that there exists a unique conformal map g_K from $\mathbb{H} \setminus K$ onto $\mathbb{H}$ with the property that $\lim_{z \to \infty} |g_K(z) - z| = 0$. We say that g_K is normalized at ∞.

Consider families of conformal maps $(g_t, t \geq 0)$ obtained by solving the Loewner equation: for each $z \in \mathbb{H}$,

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \qquad g_0(z) = z,$$

where $(W_t, t \geq 0)$ is a real-valued continuous function, which we call the driving function. Let T_z be the swallowing time of z defined as $\sup\{t \geq 0 \colon \inf_{s \in [0,t]} |g_s(z) - W_s| > 0\}$. Denote $K_t := \overline{\{z \in \mathbb{H} : T_z \leq t\}}$. Then, g_t is the unique conformal map from $H_t := \mathbb{H} \setminus K_t$ onto $\mathbb{H}$ normalized at ∞. The collection of $\mathbb{H}$-hulls $(K_t, t \geq 0)$ associated with such maps is called a Loewner chain.

Fix $\kappa > 0$. The Schramm-Loewner Evolution SLE_κ in $\mathbb{H}$ from 0 to ∞ is the random Loewner chain $(K_t, t \geq 0)$ driven by $W_t = \sqrt{\kappa} B_t$, where $(B_t, t \geq 0)$ is the standard one-dimensional Brownian motion. Rohde-Schramm prove in [RS05] that $(K_t, t \geq 0)$ is almost surely generated by a continuous transient curve, i.e., there almost surely exists a continuous curve η such that for each $t \geq 0$, H_t is the unbounded connected component of $\mathbb{H} \setminus \eta[0,t]$ and $\lim_{t \to \infty} |\eta(t)| = \infty$. This random curve is called the SLE_κ trace in $\mathbb{H}$ from 0 to ∞. When $\kappa \in (0,4]$, the SLE_κ curves are simple; when $\kappa \in (4,8)$, they have self-touchings; when $\kappa \geq 8$, they are space-filling. In this article, we focus on $\kappa = 4$ as SLE_4 is the level line of Gaussian free field, see Section 4.

3 Partition functions for multiple SLEs

At the beginning of this section, we will give a summary on "pure partition functions". As it is more convenient to see the connection to previous works, we write the summary for general κ. Fix

$$\kappa \in (0,6], \quad h = \frac{6-\kappa}{2\kappa}, \quad H = \frac{8-\kappa}{\kappa}. \tag{3.1}$$

Pure partition functions for multiple SLE_κ is a collection of smooth functions

$$\mathcal{Z}_\alpha \colon \mathfrak{X}_{2N} \to \mathbb{R}$$

defined on the configuration space $\mathfrak{X}_{2N} := \{(x_1, \ldots, x_{2N}) \in \mathbb{R}^{2N} \colon x_1 < \cdots < x_{2N}\}$ and indexed by planar pair partitions $\alpha \in \mathrm{PP_N}$ and satisfying the normalization $\mathcal{Z}_\emptyset = 1$ and the following properties:

- Partial differential equations of second order (PDE): for all $j \in \{1, \ldots, 2N\}$,

$$\left[\frac{\kappa}{2}\frac{\partial^2}{\partial x_j^2} + \sum_{i \neq j}\left(\frac{2}{x_i - x_j}\frac{\partial}{\partial x_i} - \frac{2h}{(x_i - x_j)^2}\right)\right] \mathcal{Z}(x_1, \ldots, x_{2N}) = 0. \tag{3.2}$$

- Möbius covariance (COV): For all Möbius maps φ of $\mathbb{H}$ such that $\varphi(x_1) < \cdots < \varphi(x_{2N})$,

$$\mathcal{Z}(x_1, \ldots, x_{2N}) = \prod_{i=1}^{2N} \varphi'(x_i)^h \times \mathcal{Z}(\varphi(x_1), \ldots, \varphi(x_{2N})). \tag{3.3}$$

- Asymptotics (ASY): For all $\alpha \in \mathrm{PP_N}$ and for all $j \in \{1, \ldots, 2N-1\}$ and $\xi \in (x_{j-1}, x_{j+2})$, we have

$$\lim_{x_j, x_{j+1} \to \xi} \frac{\mathcal{Z}_\alpha(x_1, \ldots, x_{2N})}{(x_{j+1}-x_j)^{-2h}} \tag{3.4}$$
$$= \begin{cases} 0, & \text{if } \{j, j+1\} \notin \alpha, \\ \mathcal{Z}_{\alpha/\{j,j+1\}}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}), & \text{if } \{j, j+1\} \in \alpha, \end{cases}$$

  where $\alpha/\{j, j+1\} \in \mathrm{PP_{N-1}}$ denotes the link pattern obtained from α by removing the link $\{j, j+1\}$ and relabelling the remaining indices by $1, 2, \ldots, 2N-2$, as defined in Section 2.1.

- Power law bound: For all $\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\} \in \mathrm{PP_N}$,

$$0 < \mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) \leq \prod_{i=1}^{N} |x_{b_i} - x_{a_i}|^{-2h}. \tag{3.5}$$

The uniqueness of such collection of smooth functions was proved in [FK15][1] and the existence of such collection was proved in [Wu20] for $\kappa \leq 6$. See [Dub06], [Dub07], [BBK05], [KP16], and [PW19] for earlier works on partition functions.

In (3.4), we see that, if $\{j, j+1\} \in \alpha$, we normalize the function $\mathcal{Z}_\alpha$ by $(x_{j+1}-x_j)^{-2h}$ and we obtain the limiting function $\mathcal{Z}_{\alpha/\{j,j+1\}}$. The goal of this section is to investigate the correct normalization of $\mathcal{Z}_\alpha$ when $\{j, j+1\} \notin \alpha$ and to analyze the limiting function.

Proposition 3.1. *Fix $\kappa = 4$. For $\alpha \in \mathrm{PP_N}$ and for $j \in \{1, 2, \ldots, 2N-1\}$, we assume $\{j, j+1\} \notin \alpha$. For all $\xi \in (x_{j-1}, x_{j+2})$, the following limit exists:*

$$\mathcal{Z}_{\alpha/\amalg_j}(x_1, \ldots, x_{j-1}, \xi, x_{j+2}, \ldots, x_{2N}) := \lim_{x_j, x_{j+1} \to \xi} \frac{\mathcal{Z}_\alpha(x_1, \ldots, x_{2N})}{(x_{j+1}-x_j)^{2/\kappa}}. \tag{3.6}$$

Furthermore, the limiting function $\mathcal{Z}_{\alpha/\amalg_j}$ satisfies the following system of $(2N-1)$ PDEs [2] and the conformal covariance with $\kappa = 4$.

- *Partial differential equations of second order (PDE): for $n \in \{1, \ldots, 2N\} \setminus \{j, j+1\}$, we have*

$$\left(\frac{\partial^2}{\partial x_n^2} - \frac{4}{\kappa}\mathcal{L}^{(n)}_{-2}\right) \mathcal{Z}_{\alpha/\amalg_j}(x_1, \ldots, x_j, x_{j+2}, \ldots, x_{2N}) = 0, \tag{3.7}$$

  where $\mathcal{L}^{(n)}_{-2} = \sum_{\substack{1 \leq i \leq 2N, \\ i \neq j, j+1, n}} \left(\frac{h}{(x_i-x_n)^2} - \frac{1}{x_i-x_n}\frac{\partial}{\partial x_i}\right) + \left(\frac{H}{(x_j-x_n)^2} - \frac{1}{x_j-x_n}\frac{\partial}{\partial x_j}\right).$

[1] In fact, [FK15, Lemma 1] proves a much stronger uniqueness, and such stronger uniqueness plays essential role in deriving (3.12). As we will not need this stronger uniqueness directly in the current article, we do not include the precise statement and refer interested readers to [FK15, Lemma 1] and [PW19].

[2] Note that, the operators $\mathcal{L}^{(j)}_{-2}$ and $\mathcal{L}^{(j)}_{-3}$ in Proposition 3.1 are distinct from the ones in Proposition 1.2. In fact, this kind of operators also depends on the index valences of planar link patterns. To simplify notations, we omit the dependence.

- *Partial differential equation of third order (PDE):*

$$\left[\frac{\partial^3}{\partial x_j^3} - \frac{16}{\kappa}\mathcal{L}^{(j)}_{2}\frac{\partial}{\partial x_j} + \frac{8(8-\kappa)}{\kappa^2}\mathcal{L}^{(j)}_{3}\right]\mathcal{Z}_{\alpha/\wp_j}(x_1,\ldots,x_j,x_{j+2},\ldots,x_{2N})=0,$$

where $$\mathcal{L}^{(j)}_{2} = \sum_{\substack{1\le i\le 2N,\\ i\neq j,j+1}}\left(\frac{h}{(x_i-x_j)^2} - \frac{1}{(x_i-x_j)}\frac{\partial}{\partial x_i}\right), \tag{3.8}$$

$$\mathcal{L}^{(j)}_{3} = \sum_{\substack{1\le i\le 2N,\\ i\neq j,j+1}}\left(\frac{2h}{(x_i-x_j)^3} - \frac{1}{(x_i-x_j)^2}\frac{\partial}{\partial x_i}\right).$$

- *Möbius covariance (COV): For all Möbius maps* φ *of* $\mathbb{H}$ *such that* $\varphi(x_1) < \cdots < \varphi(x_{2N})$,

$$\begin{aligned}&\mathcal{Z}_{\alpha/\wp_j}(x_1,\ldots,x_j,x_{j+2},\ldots,x_{2N}) \\ &= \prod_i \varphi'(x_i)^{\Delta_i} \times \mathcal{Z}_{\alpha/\wp_j}(\varphi(x_1),\ldots,\varphi(x_j),\varphi(x_{j+2}),\ldots,\varphi(x_{2N})),\end{aligned} \tag{3.9}$$

where $\Delta_i = h$ *for* $i\in\{1,\ldots,j-1,j+2,\ldots,2N\}$ *and* $\Delta_j = H$.

The connection of SLE_κ with conformal field theory (CFT) is now well-known [BB03, BB04, BBK05, FK04]. In that sense, solutions to PDE (3.2) correspond to correlation functions in CFT with central charge $c = (3\kappa-8)(6-\kappa)/2\kappa$. Then PDE (3.7) and (3.8) come as specific fusion channel of correlation functions in terms of CFT [BBK05, Dub15, KP16]. Note that the parameters h, H in (3.1) coincide with the Kac conformal weights $h_{1,2}$ and $h_{1,3}$. In fact, Peltola proves in [Pel20] a more general conclusion for $\kappa\in(0,8)\setminus\mathbb{Q}$. From there, all conclusions in Proposition 3.1 hold for $\kappa\in(0,8)\setminus\mathbb{Q}$. Our results indicate that a similar conclusion as in [Pel20] also holds for $\kappa=4$. Our method is straight forward but is specific for $\kappa=4$, as our proof uses the explicit formulae for SLE_4 partition functions constructed in [PW19]. The explicit formulae involve “conformal block functions” which we will introduce in Section 3.1.

Finally, let us describe the connection between the 2nd order PDE (3.2) and the third order PDE (1.8). Consider $y_1 < \ldots < y_{2N}$ and $x_1 < x_2 < \cdots < x_{4N-1} < x_{4N}$. Fix $\alpha\in\mathrm{PP}_{2\mathrm{N}}$ and suppose $\mathcal{Z}_\alpha(x_1,\ldots,x_{4N})$ satisfies the 2nd order PDE (3.2). We take limits $x_{2n-1}, x_{2n}\to y_n$ for all $n\in\{1,2,\ldots,2N\}$ and normalize $\mathcal{Z}_\alpha(x_1,\ldots,x_{4N})$ properly. From (3.6), we see that the proper normalization should be $\prod_{1\le n\le 2N}(x_{2n}-x_{2n-1})^{2/\kappa}$. We will show in Proposition 5.6 that the function $\mathcal{Z}_\alpha(x_1,x_2,\ldots,x_{4N-1},x_{4N})$ normalized by $\prod_{1\le n\le 2N}(x_{2n}-x_{2n-1})^{2/\kappa}$ admits a limit and then show that the limit satisfies the third order PDE (1.8). To check the third order PDE (1.8), we will use PDE (3.8). See Proof of Proposition 1.2 in Section 5.4. Again, our proof is specific for $\kappa=4$. The same conclusion holds for $\kappa\in(0,8)\setminus\mathbb{Q}$ due to [Pel20].

In the rest of the article, we fix $\kappa=4$.

3.1 Conformal block functions

For $\alpha=\{\{a_1,b_1\},\ldots,\{a_N,b_N\}\}\in\mathrm{DP}_\mathrm{N}$ ordered as in (2.2), we define conformal block function $\mathcal{U}_\alpha:\mathsf{X}_{2N}\to\mathbb{R}_{>0}$ as follows:

$$\mathcal{U}_\alpha(x_1,\ldots,x_{2N}) := \prod_{1\le i<j\le 2N}(x_j-x_i)^{\frac{1}{2}\vartheta_\alpha(i,j)}, \tag{3.10}$$

where $$\vartheta_\alpha(i,j) := \begin{cases} +1, & \text{if } i,j\in\{a_1,a_2,\ldots,a_N\}, \text{ or } i,j\in\{b_1,b_2,\ldots,b_N\},\\ -1, & \text{otherwise.}\end{cases}$$

The function $\mathcal{U}_\alpha$ satisfies the second order PDEs (3.2), see [PW19, Lemma 6.4]. These functions appear in CFT as "conformal blocks". In particular, there are analog of such functions for $\kappa \in (0,8) \setminus \mathbb{Q}$ discussed in [KKP19] in terms of CFT.

Next, we give the relation between the two collections $\{\mathcal{Z}_\alpha : \alpha \in \mathrm{PP_N}\}$ and $\{\mathcal{U}_\alpha : \alpha \in \mathrm{DP_N}\}$: they are related by a linear transformation. To give the transformation, we introduce a binary relation $\overset{()}{\longleftarrow}$. Let $\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\} \in \mathrm{PP_N}$ be ordered as in (2.2). Let $\beta \in \mathrm{PP_N}$. Then, $\alpha \overset{()}{\longleftarrow} \beta$ if and only if there exists a permutation $\sigma \in \mathsf{S}_N$ such that

$$\beta = \{\{a_1, b_{\sigma(1)}\}, \ldots, \{a_N, b_{\sigma(N)}\}\}.$$

Note that the right-hand side in the above expression may not be ordered as in (2.2). We denote by $\mathcal{M} = (\mathcal{M}_{\alpha,\beta})$ the $C_N \times C_N$ incidence matrix of this relation:

$$\mathcal{M}_{\alpha,\beta} = \begin{cases} 1, & \text{if } \alpha \overset{()}{\longleftarrow} \beta; \\ 0, & \text{if else.} \end{cases} \tag{3.11}$$

We collect some properties of $\mathcal{M}$ in the following lemma. Recall from Section 2.1 that each planar pair partition $\alpha \in \mathrm{PP_N}$ is associated with a Dyck path which we also denote by $\alpha \in \mathrm{DP_N}$.

Lemma 3.2. *The matrix $\mathcal{M}$ is invertible and we denote its inverse by $\mathcal{M}^{-1} = (\mathcal{M}^{-1}_{\alpha,\beta})$. The entry $\mathcal{M}^{-1}_{\alpha,\beta}$ is non-zero if and only if $\alpha \preceq \beta$ as in (2.1). Furthermore, we have the following properties of $\mathcal{M}^{-1}$. Suppose $\alpha, \beta \in \mathrm{DP_N}$.*

- *Suppose $\wedge^j \notin \alpha$ and $\vee_j \in \beta$. Then $\alpha \preceq \beta$ if and only if $\alpha \preceq \beta \uparrow \vee_j$.*
- *Suppose $\wedge^j \notin \alpha$, $\vee_j \in \beta$ and $\alpha \preceq \beta$. Then $\mathcal{M}^{-1}_{\alpha,\beta} = -\mathcal{M}^{-1}_{\alpha,\beta\uparrow\vee_j}$.*

Proof. See [PW19, Proposition 2.9 and Lemma 2.10]. ☐

Now, we are ready to state the linear transformation between the two collections $\{\mathcal{Z}_\alpha : \alpha \in \mathrm{PP_N}\}$ and $\{\mathcal{U}_\alpha : \alpha \in \mathrm{DP_N}\}$: (see [PW19, Theorem 1.5])

$$\begin{cases} \mathcal{U}_\alpha(x_1, \ldots, x_{2N}) = \sum_{\beta \in \mathrm{PP_N}} \mathcal{M}_{\alpha,\beta} \mathcal{Z}_\beta(x_1, \ldots, x_{2N}), \\ \mathcal{Z}_\alpha(x_1, \ldots, x_{2N}) = \sum_{\beta \in \mathrm{DP_N}} \mathcal{M}^{-1}_{\alpha,\beta} \mathcal{U}_\beta(x_1, \ldots, x_{2N}). \end{cases} \tag{3.12}$$

3.2 Asymptotics of partition functions

In this section, we will analyze the asymptotics of pure partition functions and conformal block functions as $x_j, x_{j+1} \to \xi$. Note that, we will use the following basic facts about ϑ_α through calculation without notice: for distinct $i, s, t \in \{1, 2, \ldots, 2N\}$, we have

$$\vartheta_\alpha(t,s)^2 = 1, \quad \vartheta_\alpha(t,i)\vartheta_\alpha(s,i) = \vartheta_\alpha(t,s).$$

Lemma 3.3. *The collection $\{\mathcal{U}_\alpha : \alpha \in \mathrm{DP}\}$ of conformal block functions satisfy the following asymptotic property: for any $j \in \{1, \ldots, 2N-1\}$ and $x_1 < x_2 < \cdots < x_{j-1} < \xi < x_{j+2} < \cdots < x_{2N}$,*

$$\lim_{\substack{\tilde{x}_j, \tilde{x}_{j+1} \to \xi, \\ \tilde{x}_i \to x_i \text{ for } i \neq j, j+1}} \frac{\mathcal{U}_\alpha(\tilde{x}_1, \ldots, \tilde{x}_{2N})}{(\tilde{x}_{j+1} - \tilde{x}_j)^{-1/2}} = \begin{cases} \mathcal{U}_{\alpha/\wedge^j}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}), & \textit{if } \wedge^j \in \alpha, \\ \mathcal{U}_{\alpha/\vee_j}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}), & \textit{if } \vee_j \in \alpha, \end{cases} \tag{3.13}$$

$$\lim_{\substack{\tilde{x}_j, \tilde{x}_{j+1} \to \xi, \\ \tilde{x}_i \to x_i \text{ for } i \neq j, j+1}} \frac{\mathcal{U}_\alpha(\tilde{x}_1, \ldots, \tilde{x}_{2N})}{(\tilde{x}_{j+1} - \tilde{x}_j)^{1/2}} = \mathcal{U}_{\alpha/\times_j}(x_1, \ldots, x_{j-1}, \xi, x_{j+2}, \ldots, x_{2N}), \quad \textit{if } \times_j \in \alpha, \tag{3.14}$$

where

$$\mathcal{U}_{\alpha/\times_j}(x_1,\ldots,x_{j-1},\xi,x_{j+2},\ldots,x_{2N}) := \prod_{\substack{1\le t<s\le 2N\\ t,s\neq j,j+1}} (x_s-x_t)^{\frac{1}{2}\vartheta_\alpha(t,s)} \prod_{\substack{1\le i\le 2N\\ i\neq j,j+1}} |x_i-\xi|^{\vartheta_\alpha(i,j)}. \tag{3.15}$$

Proof. The asymptotics in (3.13) is proved in [PW19, Lemma 6.6]. It remains to show (3.14). By definition,

$$\frac{\mathcal{U}_\alpha(\tilde{x}_1,\ldots,\tilde{x}_{2N})}{(\tilde{x}_{j+1}-\tilde{x}_j)^{1/2}} = \prod_{\substack{1\le t<s\le 2N\\ t,s\neq j,j+1}} (\tilde{x}_s-\tilde{x}_t)^{\frac{1}{2}\vartheta_\alpha(t,s)} \prod_{\substack{1\le i\le 2N\\ i\neq j,j+1}} |\tilde{x}_i-\tilde{x}_j|^{\frac{1}{2}\vartheta_\alpha(i,j)} \prod_{\substack{1\le i\le 2N\\ i\neq j,j+1}} |\tilde{x}_i-\tilde{x}_{j+1}|^{\frac{1}{2}\vartheta_\alpha(i,j+1)}.$$

Since $\times_j \in \alpha$, we have $\vartheta_\alpha(i,j)=\vartheta_\alpha(i,j+1)$. By taking limit, we obtain (3.14). ☐

Lemma 3.4. *The collection* $\{\mathcal{Z}_\alpha : \alpha\in\mathrm{PP}\}$ *of pure partition functions satisfy the following asymptotic property: for any* $j\in\{1,\ldots,2N-1\}$ *and* $x_1<x_2<\cdots<x_{j-1}<\xi<x_{j+2}<\cdots<x_{2N}$,

$$\lim_{\substack{\tilde{x}_j,\tilde{x}_{j+1}\to\xi,\\ \tilde{x}_i\to x_i \text{ for } i\neq j,j+1}} \frac{\mathcal{Z}_\alpha(\tilde{x}_1,\ldots,\tilde{x}_{2N})}{(\tilde{x}_{j+1}-\tilde{x}_j)^{-1/2}} = \mathcal{Z}_{\alpha/\wedge_j}(x_1,\ldots,x_{j-1},x_{j+2},\ldots,x_{2N}), \qquad \text{if } \{j,j+1\}\in\alpha, \tag{3.16}$$

$$\lim_{\substack{\tilde{x}_j,\tilde{x}_{j+1}\to\xi,\\ \tilde{x}_i\to x_i \text{ for } i\neq j,j+1}} \frac{\mathcal{Z}_\alpha(\tilde{x}_1,\ldots,\tilde{x}_{2N})}{(\tilde{x}_{j+1}-\tilde{x}_j)^{1/2}} = \mathcal{Z}_{\alpha/\curlyvee_j}(x_1,\ldots,x_{j-1},\xi,x_{j+2},\ldots,x_{2N}), \quad \text{if } \{j,j+1\}\notin\alpha, \tag{3.17}$$

where

$$\mathcal{Z}_{\alpha/\curlyvee_j} := \sum_{-_j\in\beta} \mathcal{M}^{-1}_{\alpha,\beta}\mathcal{V}_{\beta/-_j} + \sum_{\times_j\in\beta} \mathcal{M}^{-1}_{\alpha,\beta}\mathcal{U}_{\beta/\times_j}, \tag{3.18}$$

$$\mathcal{V}_{\beta/-_j}(x_1,\ldots,x_{j-1},\xi,x_{j+2},\ldots,x_{2N}) := \prod_{\substack{1\le t<s\le 2N\\ t,s\neq j,j+1}} (x_s-x_t)^{\frac{1}{2}\vartheta_\beta(t,s)} \sum_{\substack{1\le i\le 2N\\ i\neq j,j+1}} \frac{\vartheta_\beta(i,j)}{x_i-\xi}.$$

Proof. The asymptotics in (3.16) is proved in [PW19, Lemma 6.7]. It remains to show (3.17). In the following, we assume $\{j,j+1\}\notin\alpha$. From Lemma 3.2 and (3.12), we have

$$\begin{aligned} &\mathcal{Z}_\alpha(\tilde{x}_1,\ldots,\tilde{x}_{2N}) \\ &= \sum_{\substack{\alpha\preceq\beta\\ -_j\in\beta}} \mathcal{M}^{-1}_{\alpha,\beta}\mathcal{U}_\beta(\tilde{x}_1,\ldots,\tilde{x}_{2N}) + \sum_{\substack{\alpha\preceq\beta\\ \wedge_j\in\beta}} \mathcal{M}^{-1}_{\alpha,\beta}\mathcal{U}_\beta(\tilde{x}_1,\ldots,\tilde{x}_{2N}) + \sum_{\substack{\alpha\preceq\beta\\ \times_j\in\beta}} \mathcal{M}^{-1}_{\alpha,\beta}\mathcal{U}_\beta(\tilde{x}_1,\ldots,\tilde{x}_{2N}). \end{aligned} \tag{3.19}$$

From Lemma 3.2, for every $\beta\in\mathrm{DP_N}$ with $\vee_j\in\beta$, we have $\alpha\preceq\beta$ if and only if $\alpha\preceq\beta\uparrow\diamond_j$. In such case, we have further that $\mathcal{M}^{-1}_{\alpha,\beta}=-\mathcal{M}^{-1}_{\alpha,\beta\uparrow\diamond_j}$. For the first two sums in the right hand side of (3.19), we have

$$\begin{aligned} &\sum_{\substack{\alpha\preceq\beta\\ -_j\in\beta}} \mathcal{M}^{-1}_{\alpha,\beta}\mathcal{U}_\beta(\tilde{x}_1,\ldots,\tilde{x}_{2N}) + \sum_{\substack{\alpha\preceq\beta\\ \wedge_j\in\beta}} \mathcal{M}^{-1}_{\alpha,\beta}\mathcal{U}_\beta(\tilde{x}_1,\ldots,\tilde{x}_{2N}) \\ &= \sum_{\substack{\alpha\preceq\beta\\ -_j\in\beta}} \mathcal{M}^{-1}_{\alpha,\beta}(\mathcal{U}_\beta-\mathcal{U}_{\beta\uparrow\diamond_j})(\tilde{x}_1,\ldots,\tilde{x}_{2N}). \end{aligned}$$

Fix β such that $\alpha \preceq \beta$ and $\vee_j \in \beta$, we have

$$
\begin{aligned}
&(\mathcal{U}_\beta - \mathcal{U}_{\beta\wedge_j})(\tilde{x}_1,\ldots,\tilde{x}_{2N})\\
=&(\tilde{x}_{j+1}-\tilde{x}_j)^{-\frac{1}{2}} \times \left(\prod_{i\neq j,j+1} \left(\frac{\tilde{x}_i-\tilde{x}_j}{\tilde{x}_i-\tilde{x}_{j+1}}\right)^{\frac{1}{2}\vartheta_\beta(i,j)} - \prod_{i\neq j,j+1} \left(\frac{\tilde{x}_i-\tilde{x}_{j+1}}{\tilde{x}_i-\tilde{x}_j}\right)^{\frac{1}{2}\vartheta_\beta(i,j)} \right)\\
&\times \prod_{\substack{1\le t<s\le 2N\\ t,s\neq j,j+1}} (\tilde{x}_s-\tilde{x}_t)^{\frac{1}{2}\vartheta_\beta(t,s)}.
\end{aligned}
$$

Dividing by $(\tilde{x}_{j+1}-\tilde{x}_j)^{1/2}$, we have

$$
\lim_{\substack{\tilde{x}_j,\tilde{x}_{j+1}\to\xi,\\ \tilde{x}_i\to x_i \text{ for } i\neq j,j+1}} \frac{(\mathcal{U}_\beta - \mathcal{U}_{\beta\wedge_j})(\tilde{x}_1,\ldots,\tilde{x}_{2N})}{(\tilde{x}_{j+1}-\tilde{x}_j)^{\frac{1}{2}}} = \prod_{\substack{1\le t<s\le 2N\\ t,s\neq j,j+1}} (x_s-x_t)^{\frac{1}{2}\vartheta_\beta(t,s)} \sum_{\substack{1\le i\le 2N\\ i\neq j,j+1}} \frac{\vartheta_\beta(i,j)}{x_i-\xi}. \tag{3.20}
$$

For the third sum in the right hand side of (3.19), by (3.14), we have

$$
\lim_{\substack{\tilde{x}_j,\tilde{x}_{j+1}\to\xi,\\ \tilde{x}_i\to x_i \text{ for } i\neq j,j+1}} \frac{\mathcal{U}_\beta(\tilde{x}_1,\ldots,\tilde{x}_{2N})}{(\tilde{x}_{j+1}-\tilde{x}_j)^{1/2}} = \mathcal{U}_{\beta/\amalg_j}(x_1,\ldots,x_{j-1},\xi,x_{j+2},\ldots,x_{2N}). \tag{3.21}
$$

Plugging (3.20) and (3.21) into (3.19), we obtain (3.17). □

Note that, we use the notation α/II_j in (3.18). It can be understood as a general link pattern. For a planar pair partition $\alpha\in\mathrm{PP_N}$, suppose $\times_j\in\alpha$ or $\vee_j\in\alpha$, we define α/II_j to be the N-link pattern of $(2N-1)$ points with index valences $\varsigma=(1,\ldots,1,2,1,\ldots,1)$ obtained from α by merging the points j and $j+1$ and relabelling the remaining $(2N-1)$ indices so that they are the first $(2N-1)$ integers.

3.3 Fusion of partition functions

In this section, we will show that the functions defined in (3.15) and (3.18) satisfy the system of $(2N-1)$ PDEs in (3.7) and (3.8), and complete the proof of Proposition 3.1.

Lemma 3.5. *The function $\mathcal{U}_{\alpha/\amalg_j}$ defined in* (3.15) *satisfies the second order PDE* (3.7) *with $\kappa=4$ for $n\in\{1,\ldots,2N\}\setminus\{j,j+1\}$.*

Proof. Without loss of generality, we assume $j=1$. Note that $h=1/4$ and $H=1$ when $\kappa=4$. The second order PDE (3.7) becomes the following: for $n\in\{3,4,\ldots,2N\}$,

$$
\left(\frac{\partial^2}{\partial x_n^2} - \mathcal{L}_2^{(n)}\right) F(x_1,x_3,\ldots,x_{2N}) = 0, \tag{3.22}
$$

where $\mathcal{L}_2^{(n)} = \sum_{\substack{3\le i\le 2N,\\ i\neq n}} \left(\frac{\frac{1}{4}}{(x_i-x_n)^2} - \frac{1}{x_i-x_n}\frac{\partial}{\partial x_i}\right) + \left(\frac{1}{(x_1-x_n)^2} - \frac{1}{x_1-x_n}\frac{\partial}{\partial x_1}\right).$

The function in (3.15) with $j=1$ becomes

$$
\mathcal{U}_{\alpha/\amalg_1}(x_1,x_3,\ldots,x_{2N}) := \prod_{3\le t<s\le 2N} (x_s-x_t)^{\frac{1}{2}\vartheta_{\alpha/\amalg_1}(t,s)} \prod_{3\le i\le 2N} (x_i-x_1)^{\vartheta_{\alpha/\amalg_1}(i,1)}. \tag{3.23}
$$

It suffices to show that the function in (3.23) solves the second order PDE (3.22).

We write $\boldsymbol{x}=(x_1,x_3,\ldots,x_{2N})$. We have, for $i\in\{3,4,\ldots,2N\}$,

$$
\frac{\frac{\partial}{\partial x_i}\mathcal{U}_{\alpha/\amalg_1}(\boldsymbol{x})}{\mathcal{U}_{\alpha/\amalg_1}(\boldsymbol{x})} = \sum_{\substack{3\le s\le 2N,\\ s\neq i}} \frac{\frac{1}{2}\vartheta_\alpha(s,i)}{x_i-x_s} + \frac{\vartheta_\alpha(i,1)}{x_i-x_1}; \quad \frac{\frac{\partial}{\partial x_1}\mathcal{U}_{\alpha/\amalg_1}(\boldsymbol{x})}{\mathcal{U}_{\alpha/\amalg_1}(\boldsymbol{x})} = \sum_{3\le s\le 2N} \frac{-\vartheta_\alpha(s,1)}{x_s-x_1}.
$$

Then, we have

$$\frac{\mathcal{L}^{(n)}_2\mathcal{U}_{\alpha/\,1}(\boldsymbol{x})}{\mathcal{U}_{\alpha/\,1}(\boldsymbol{x})} = \sum_{\substack{3\le i\le 2N,\\ i\neq n}} \frac{\frac{1}{4}-\frac{1}{2}\vartheta_\alpha(n,i)}{(x_n-x_i)^2} + \frac{1-\vartheta_\alpha(n,1)}{(x_n-x_1)^2} + \sum_{\substack{3\le t<s\le 2N,\\ t,s\neq n}} \frac{\frac{1}{2}\vartheta_\alpha(t,s)}{(x_n-x_t)(x_n-x_s)} + \sum_{\substack{3\le i\le 2N,\\ i\neq n}} \frac{\vartheta_\alpha(i,1)}{(x_n-x_i)(x_n-x_1)};$$

$$\frac{\frac{\partial^2}{\partial x_n^2}\mathcal{U}_{\alpha/\,1}(\boldsymbol{x})}{\mathcal{U}_{\alpha/\,1}(\boldsymbol{x})} = \sum_{\substack{3\le i\le 2N,\\ i\neq n}} \frac{\frac{1}{4}-\frac{1}{2}\vartheta_\alpha(n,i)}{(x_n-x_i)^2} + \frac{1-\vartheta_\alpha(n,1)}{(x_n-x_1)^2} + \sum_{\substack{3\le t<s\le 2N,\\ t,s\neq n}} \frac{\frac{1}{2}\vartheta_\alpha(s,t)}{(x_n-x_t)(x_n-x_s)} + \sum_{\substack{3\le i\le 2N,\\ i\neq n}} \frac{\vartheta_\alpha(i,1)}{(x_n-x_i)(x_n-x_1)}.$$

These give $\left[\frac{\partial^2}{\partial x_n^2}-\mathcal{L}^{(n)}_2\right]\mathcal{U}_{\alpha/\,1}=0$ as desired. □

Lemma 3.6. *The function* $\mathcal{U}_{\alpha/\,j}$ *defined in* (3.15) *satisfies the third order PDE* (3.8) *with* $\kappa=4$*.*

Proof. Without loss of generality, we assume $j=1$. The third order PDE (3.8) becomes the following:

$$\left(\frac{\partial^3}{\partial x_1^3} - 4\mathcal{L}^{(1)}_2\frac{\partial}{\partial x_1} + 2\mathcal{L}^{(1)}_3\right) F(x_1,x_3,\ldots,x_{2N}) = 0, \tag{3.24}$$

where $\mathcal{L}^{(1)}_2 = \sum_{3\le i\le 2N}\left(\frac{\frac{1}{4}}{(x_i-x_1)^2} - \frac{1}{(x_i-x_1)}\frac{\partial}{\partial x_i}\right),$

$$\mathcal{L}^{(1)}_3 = \sum_{3\le i\le 2N}\left(\frac{\frac{1}{2}}{(x_i-x_1)^3} - \frac{1}{(x_i-x_1)^2}\frac{\partial}{\partial x_i}\right).$$

It suffices to show that the function in (3.23) solves the third order PDE (3.24). We write $\boldsymbol{x}=(x_1,x_3,\ldots,x_{2N})$. We have

$$\frac{2\mathcal{L}^{(1)}_3\mathcal{U}_{\alpha/\,1}(\boldsymbol{x})}{\mathcal{U}_{\alpha/\,1}(\boldsymbol{x})} = \sum_{3\le i\le 2N}\frac{1-2\vartheta_\alpha(i,1)}{(x_i-x_1)^3} + \sum_{3\le t<s\le 2N}\frac{\vartheta_\alpha(s,t)(x_t-x_1+x_s-x_1)}{(x_t-x_1)^2(x_s-x_1)^2};$$

$$\frac{4\mathcal{L}^{(1)}_2\frac{\partial}{\partial x_1}\mathcal{U}_{\alpha/\,1}(\boldsymbol{x})}{\mathcal{U}_{\alpha/\,1}(\boldsymbol{x})} = \sum_{3\le i\le 2N}\frac{4-5\vartheta_\alpha(i,1)}{(x_i-x_1)^3} + \sum_{3\le t<s\le 2N}\frac{4\vartheta_\alpha(s,t)(x_t-x_1+x_s-x_1)-3\vartheta_\alpha(t,1)(x_t-x_1)-3\vartheta_\alpha(s,1)(x_s-x_1)}{(x_t-x_1)^2(x_s-x_1)^2} + \sum_{3\le t<s<n\le 2N}\frac{(-6)\vartheta_\alpha(t,1)\vartheta_\alpha(s,1)\vartheta_\alpha(n,1)}{(x_t-x_1)(x_s-x_1)(x_n-x_1)};$$

$$
\begin{aligned}
&\frac{\frac{\partial^3}{\partial x_1^3}\mathcal{U}_{\alpha/\ _1}(\boldsymbol{x})}{\mathcal{U}_{\alpha/\ _1}(\boldsymbol{x})}\\
&=\sum_{3\le i\le 2N}\frac{3-3\vartheta_\alpha(i,1)}{(x_i-x_1)^3}\\
&+\sum_{3\le t<s\le 2N}\frac{3(\vartheta_\alpha(s,t)-\vartheta_\alpha(t,1))(x_t-x_1)+3(\vartheta_\alpha(s,t)-\vartheta_\alpha(s,1))(x_s-x_1)}{(x_t-x_1)^2(x_s-x_1)^2}\\
&+\sum_{3\le t<s<n\le 2N}\frac{(-6)\vartheta_\alpha(t,1)\vartheta_\alpha(s,1)\vartheta_\alpha(n,1)}{(x_t-x_1)(x_s-x_1)(x_n-x_1)}.
\end{aligned}
$$

These give $\left[\frac{\partial^3}{\partial x_1^3}-4\mathcal{L}^{(1)}_2\frac{\partial}{\partial x_1}+2\mathcal{L}^{(1)}_3\right]\mathcal{U}_{\alpha/\ _1}=0$ as desired. □

Lemma 3.7. *The function* $\mathcal{Z}_{\alpha/q_j}$ *defined in* (3.18) *satisfies the second order PDE* (3.7) *with* $\kappa=4$ *for* $n\in\{1,\ldots,2N\}\setminus\{j,j+1\}$.

Proof. Without loss of generality, we assume $j=1$. The function in (3.18) with $j=1$ becomes $\mathcal{Z}_{\alpha/q_1}=\sum_{\beta:\_1\in\beta}\mathcal{M}^{-1}_{\alpha,\beta}\mathcal{V}_{\beta/\_1}+\sum_{\beta:\ _1\in\beta}\mathcal{M}^{-1}_{\alpha,\beta}\mathcal{U}_{\beta/\ _1}$ where

$$
\mathcal{V}_{\beta/\_1}(x_1,x_3,\ldots,x_{2N}):=\prod_{3\le t<s\le 2N}(x_s-x_t)^{\frac{1}{2}\vartheta\ (t,s)}\sum_{3\le i\le 2N}\frac{\vartheta_\beta(i,1)}{x_i-x_1}. \tag{3.25}
$$

We will show that $\mathcal{Z}_{\alpha/q_1}$ satisfies the second order PDE (3.22). From Lemma 3.5, the function $\mathcal{U}_{\beta/\ _1}$ satisfies PDE (3.22). It suffcies to show that the function $\mathcal{V}_{\beta/\_1}$ in (3.25) satisfies PDE (3.22).

We write $\boldsymbol{x}=(x_1,x_3,\ldots,x_{2N})$ and set

$$
\sigma(\boldsymbol{x})=\sum_{3\le i\le 2N}\frac{\vartheta_\beta(i,1)}{x_i-x_1}. \tag{3.26}
$$

We have, for $i\in\{3,4,\ldots,2N\}$,

$$
\begin{aligned}
\frac{\frac{\partial}{\partial x_i}\mathcal{V}_{\beta/\_1}(\boldsymbol{x})}{\mathcal{V}_{\beta/\_1}(\boldsymbol{x})}&=\sum_{\substack{3\le s\le 2N,\\ s\neq i}}\frac{\frac{1}{2}\vartheta_\beta(i,s)}{x_i-x_s}+\frac{-\vartheta_\beta(i,1)}{\sigma(\boldsymbol{x})(x_i-x_1)^2},\\
\frac{\frac{\partial}{\partial x_1}\mathcal{V}_{\beta/\_1}(\boldsymbol{x})}{\mathcal{V}_{\beta/\_1}(\boldsymbol{x})}&=\sum_{3\le s\le 2N}\frac{\vartheta_\beta(s,1)}{\sigma(\boldsymbol{x})(x_s-x_1)^2}.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
&\frac{\mathcal{L}^{(n)}_2\mathcal{V}_{\beta/\_1}(\boldsymbol{x})}{\mathcal{V}_{\beta/\_1}(\boldsymbol{x})}\\
&=\sum_{\substack{3\le i\le 2N,\\ i\neq n}}\frac{\frac{1}{4}-\frac{1}{2}\vartheta_\beta(i,n)}{(x_n-x_i)^2}+\frac{1}{(x_n-x_1)^2}+\frac{\vartheta_\beta(n,1)}{\sigma(\boldsymbol{x})(x_n-x_1)^3}\\
&+\sum_{\substack{3\le t<s\le 2N,\\ t,s\neq n}}\frac{\frac{1}{2}\vartheta_\beta(t,s)}{(x_n-x_s)(x_n-x_t)}+\sum_{\substack{3\le i\le 2N,\\ i\neq n}}\frac{-\vartheta_\beta(i,1)}{\sigma(\boldsymbol{x})(x_n-x_i)(x_i-x_1)(x_n-x_1)};
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\frac{\partial^2}{\partial x_n^2}\mathcal{V}_{\beta/\wedge_1}(\boldsymbol{x})}{\mathcal{V}_{\beta/\wedge_1}(\boldsymbol{x})}\\
=&\sum_{\substack{3\le i\le 2N,\\ i\neq n}}\frac{\frac{1}{4}-\frac{1}{2}\vartheta_\beta(i,n)}{(x_n-x_i)^2}+\frac{2\vartheta_\beta(n,1)}{\sigma(\boldsymbol{x})(x_n-x_1)^3}\\
&+\sum_{\substack{3\le t<s\le 2N,\\ t,s\neq n}}\frac{\frac{1}{2}\vartheta_\beta(t,s)}{(x_n-x_s)(x_n-x_t)}+\sum_{\substack{3\le i\le 2N,\\ i\neq n}}\frac{-\vartheta_\beta(i,1)}{\sigma(\boldsymbol{x})(x_n-x_i)(x_n-x_1)^2}.
\end{aligned}
$$

Taking the difference, we have

$$
\begin{aligned}
&\frac{\frac{\partial^2}{\partial x_n^2}\mathcal{V}_{\beta/\wedge_1}(\boldsymbol{x})}{\mathcal{V}_{\beta/\wedge_1}(\boldsymbol{x})}-\frac{\mathcal{L}^{(n)}_2\mathcal{V}_{\beta/\wedge_1}(\boldsymbol{x})}{\mathcal{V}_{\beta/\wedge_1}(\boldsymbol{x})}\\
=&\frac{-1}{(x_n-x_1)^2}+\frac{\vartheta_\beta(n,1)}{\sigma(\boldsymbol{x})(x_n-x_1)^3}+\sum_{\substack{3\le i\le 2N,\\ i\neq n}}\frac{-\vartheta_\beta(i,1)}{\sigma(\boldsymbol{x})(x_n-x_i)(x_n-x_1)^2}\\
&+\sum_{\substack{3\le i\le 2N,\\ i\neq n}}\frac{\vartheta_\beta(i,1)}{\sigma(\boldsymbol{x})(x_n-x_i)(x_i-x_1)(x_n-x_1)}\\
=&\frac{-1}{(x_n-x_1)^2}+\frac{\vartheta_\beta(n,1)}{\sigma(\boldsymbol{x})(x_n-x_1)^3}+\sum_{\substack{3\le i\le 2N,\\ i\neq n}}\frac{\vartheta_\beta(i,1)}{\sigma(\boldsymbol{x})(x_i-x_1)(x_n-x_1)^2}\\
=&\frac{-1}{(x_n-x_1)^2}+\frac{\vartheta_\beta(n,1)}{\sigma(\boldsymbol{x})(x_n-x_1)^3}+\frac{\sigma(\boldsymbol{x})-\frac{\vartheta_\beta(n,1)}{x_n-x_1}}{\sigma(\boldsymbol{x})(x_n-x_1)^2}=0.
\end{aligned}
$$

This completes the proof. ☐

Lemma 3.8. *The function* $\mathcal{Z}_{\alpha/q_j}$ *defined in* (3.18) *satisfies the third order PDE* (3.8) *with* $\kappa=4$.

Proof. Without loss of generality, we assume $j=1$. From Lemma 3.6, the function $\mathcal{U}_{\alpha/\wedge_1}$ satisfies PDE (3.24). It suffices to show that the function $\mathcal{V}_{\beta/\wedge_1}$ in (3.25) satisfies the third order PDE (3.24). We write $\boldsymbol{x}=(x_1,x_3,\ldots,x_{2N})$ and set $\sigma(\boldsymbol{x})$ as in (3.26). Then we have

$$
\begin{aligned}
\frac{2\mathcal{L}^{(1)}_3\mathcal{V}_{\beta/\wedge_1}(\boldsymbol{x})}{\mathcal{V}_{\beta/\wedge_1}(\boldsymbol{x})}=&\sum_{3\le i\le 2N}\frac{1}{(x_i-x_1)^3}+\sum_{3\le i\le 2N}\frac{2\vartheta_\beta(i,1)}{\sigma(\boldsymbol{x})(x_i-x_1)^4}\\
&+\sum_{3\le t<s\le 2N}\frac{\vartheta_\beta(s,t)(x_t-x_1+x_s-x_1)}{(x_t-x_1)^2(x_s-x_1)^2};\\
\frac{4\mathcal{L}^{(1)}_2\frac{\partial}{\partial x_1}\mathcal{V}_{\beta/\wedge_1}(\boldsymbol{x})}{\mathcal{V}_{\beta/\wedge_1}(\boldsymbol{x})}=&\sum_{3\le i\le 2N}\frac{8\vartheta_\beta(i,1)}{\sigma(\boldsymbol{x})(x_i-x_1)^4}+\sigma(\boldsymbol{x})\frac{\partial\sigma(\boldsymbol{x})}{\partial x_1};\\
\frac{\frac{\partial^3}{\partial x_1^3}\mathcal{V}_{\beta/\wedge_1}(\boldsymbol{x})}{\mathcal{V}_{\beta/\wedge_1}(\boldsymbol{x})}=&\sum_{3\le i\le 2N}\frac{6\vartheta_\beta(i,1)}{\sigma(\boldsymbol{x})(x_i-x_1)^4}.
\end{aligned}
$$

Therefore,

$$
\frac{\frac{\partial^3}{\partial x_1^3}\mathcal{V}_{\beta/\wedge_1}(\boldsymbol{x})}{\mathcal{V}_{\beta/\wedge_1}(\boldsymbol{x})}-\frac{4\mathcal{L}^{(1)}_2\frac{\partial}{\partial x_1}\mathcal{V}_{\beta/\wedge_1}(\boldsymbol{x})}{\mathcal{V}_{\beta/\wedge_1}(\boldsymbol{x})}+\frac{2\mathcal{L}^{(1)}_3\mathcal{V}_{\beta/\wedge_1}(\boldsymbol{x})}{\mathcal{V}_{\beta/\wedge_1}(\boldsymbol{x})}
$$

$$= \sum_{3 \le i \le 2N} \frac{1}{(x_i - x_1)^3} + \sum_{3 \le t < s \le 2N} \frac{\vartheta_\beta(s,t)(x_t - x_1 + x_s - x_1)}{(x_t - x_1)^2 (x_s - x_1)^2} - \sigma(\boldsymbol{x}) \frac{\partial \sigma(\boldsymbol{x})}{\partial x_1}$$
$$= \left(\sum_{3 \le t \le 2N} \frac{\vartheta_\beta(t,1)}{x_t - x_1} \right) \left(\sum_{3 \le s \le 2N} \frac{\vartheta_\beta(s,1)}{(x_s - x_1)^2} \right) - \sigma(\boldsymbol{x}) \frac{\partial \sigma(\boldsymbol{x})}{\partial x_1} = 0.$$

This completes the proof. □

Proof of Proposition 3.1. The existence of the limit (3.6) is a consequence of (3.17). The limiting function satisfies the PDE system due to Lemmas 3.7 and 3.8. COV (3.9) is a consequence of COV (3.3) and the existence of the limit (3.6). □

4 Connection probabilities for level lines in GFF

In this section, we first introduce continuum GFF and level lines in Section 4.1. Then we state the main conclusion of the section—Theorem 4.1—in Section 4.2. This theorem gives the connection probabilities for level lines of GFF in polygons with boundary data given by Dyck paths. The proof of Theorem 4.1 involves several technical lemmas which we find not instructive to include in the main text. We put the proof of these technical lemmas to Appendix A.

4.1 Continuum GFF and level lines

In this section, we introduce the Gaussian free field and its level lines. We refer to the literature [She07, SS13, MS16, WW17] for details. Let $\Omega \subsetneq \mathbb{C}$ be a non-empty domain. We denote by $H_s(\Omega)$ the space of real-valued smooth functions which are compactly supported in Ω. We equip the space with Dirichlet inner product

$$(f,g)_\nabla := \frac{1}{2\pi} \int_\Omega \nabla f(z) \cdot \nabla g(z) d^2 z.$$

We denote by $H(\Omega)$ the Hilbert space completion of $H_s(\Omega)$ with respect to the Dirichlet inner product. A (zero-boundary) Gaussian free field (GFF) Γ is an $H(\Omega)$-indexed linear space of random variables, denoted by $(\Gamma, f)_\nabla$ for each $f \in H(\Omega)$, such that the map $f \mapsto (\Gamma, f)_\nabla$ is linear and each $(\Gamma, f)_\nabla$ is a centered Gaussian with variance $(f,f)_\nabla$. In general, for any harmonic function u on Ω, we define the GFF with boundary data u by $\Gamma + u$ where Γ is the zero-boundary GFF on Ω.

Next, we introduce SLE with force points. We set

$$\underline{y}^L = (y^{L,l} < \cdots < y^{L,1} \le 0) \quad \text{and} \quad \underline{y}^R = (0 \le y^{R,1} < \cdots < y^{R,r}),$$

and

$$\underline{\rho}^L = (\rho^{L,l}, \cdots, \rho^{L,1}) \quad \text{and} \quad \underline{\rho}^R = (\rho^{R,1}, \cdots, \rho^{R,r}),$$

where $\rho^{q,i} \in \mathbb{R}$, for $q \in \{L, R\}$ and $i \in \mathbb{Z}_{>0}$. An $\mathrm{SLE}_\kappa(\underline{\rho}^{\mathrm{L}}; \underline{\rho}^{\mathrm{R}})$ process with force points $(\underline{y}^L; \underline{y}^R)$ is the Loewner evolution driven by W_t that solves the following system of integrated SDEs:

$$\begin{aligned} W_t =& \sqrt{\kappa} B_t + \int_0^t \frac{\rho^{L,i} ds}{W_s - V_s^{L,i}} + \sum_{i=1} \int_0^t \frac{\rho^{R,i} ds}{W_s - V_s^{R,i}}, \\ V_t^{q,i} =& y^{q,i} + \int_0^t \frac{2 ds}{V_s^{q,i} - W_s}, \quad \text{for } q \in \{L, R\} \text{ and } i \in \mathbb{Z}_{>0}, \end{aligned} \tag{4.1}$$

where B_t is the one-dimensional Brownian motion. Note that the process $V_t^{q,i}$ is the evolution of the point $y^{q,i}$, and we may write $g_t(y^{q,i})$ for $V_t^{q,i}$. We define the continuation threshold of the $\mathrm{SLE}_\kappa(\underline{\rho}^{\mathrm{L}};\underline{\rho}^{\mathrm{R}})$ to be the infimum of the time t for which

$$\text{either} \quad \sum_{i:V_t^{L,i}=W_t} \rho^{L,i} \le -2, \quad \text{or} \quad \sum_{i:V_t^{R,i}=W_t} \rho^{R,i} \le -2.$$

By [MS16], the $\mathrm{SLE}_\kappa(\underline{\rho}^{\mathrm{L}};\underline{\rho}^{\mathrm{R}})$ process is well-defined up to the continuation threshold, and it is almost surely generated by a continuous curve up to and including the continuation threshold.

Now, we are ready to introduce level lines of GFF. Let $K = (K_t, t \ge 0)$ be an $\mathrm{SLE}_4(\underline{\rho}^{\mathrm{L}};\underline{\rho}^{\mathrm{R}})$ process with force points $(\underline{y}^L;\underline{y}^R)$, with $W, V^{q,i}$ solving the SDE system (4.1) with $\kappa = 4$. Let $(g_t, t \ge 0)$ be the corresponding family of conformal maps and set $f_t := g_t - W_t$. Let u_t^0 be the harmonic function on $\mathbb{H}$ with boundary data

$$\begin{cases} -\lambda(1+\sum_{i=0}^{j}\rho^{L,i}), & \text{if } x \in (f_t(y^{L,j+1}), f_t(y^{L,j})), \\ +\lambda(1+\sum_{i=0}^{j}\rho^{R,i}), & \text{if } x \in (f_t(y^{R,j}), f_t(y^{R,j+1})), \end{cases}$$

where $\lambda = \pi/2$ and $\rho^{L,0} = \rho^{R,0} = 0$, $y^{L,0} = 0^-$, $y^{L,l+1} = -\infty$,$y^{R,0} = 0^+$, and $y^{R,r+1} = \infty$ by convention. Define $u_t(z) := u_t^0(f_t(z))$. By [Dub09, SS13], there exists a coupling (Γ, K), where Γ is a zero-boundary GFF on $\mathbb{H}$, such that the following is true. Let τ be any η-stopping time before the continuation threshold. Then, the conditional law of $\Gamma + u_0$ restricted to $\mathbb{H} \setminus K_\tau$ given K_τ is the same as the law of $\Gamma^0 \circ f_\tau + u_\tau$ where Γ^0 is a zero-boundary GFF. Furthermore, in this coupling, the process K is almost surely determined by Γ. We refer to the $\mathrm{SLE}_4(\underline{\rho}^{\mathrm{L}};\underline{\rho}^{\mathrm{R}})$ in this coupling as the level line of the field $\Gamma + u_0$. In general, for $a \in \mathbb{R}$, the level line of $\Gamma + u_0$ with height a is the level line of $\Gamma + u_0 - a$.

4.2 Connection probabilities

For $\alpha \in \mathrm{PP}_\mathrm{N}$, recall from Section 2.1 that α also denotes the corresponding Dyck path in DP_N. Let u_α be the harmonic function on $\mathbb{H}$ with the following boundary data: ($x_0 = -\infty$ and $x_{2N+1} = \infty$ by convention)

$$2\lambda(\alpha(k) - 1) \text{ on } (x_k, x_{k+1}), \quad \text{for all } k \in \{0, 1, 2, \ldots, N\}. \tag{4.2}$$

With such choice, we see that u_α has boundary data -2λ on $(-\infty, x_1) \cup (x_{2N}, \infty)$, and has boundary data 0 on $(x_1, x_2) \cup (x_{2N-1}, x_{2N})$. Define

$$\mathcal{H}_\alpha(k) = \lambda(\alpha(k-1) + \alpha(k) - 2), \quad \text{for all } k \in \{1, , 2 \ldots, 2N\}. \tag{4.3}$$

We write $\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\}$ as ordered in (2.2). Suppose Γ is zero-boundary GFF on $\mathbb{H}$, and consider level lines of $\Gamma + u_\alpha$. Let η_{a_i} be the level line of $\Gamma + u_\alpha$ starting from x_{a_i} with height $\mathcal{H}_\alpha(a_i)$. With such choice, the boundary data to the left side of η_{a_i} is $2\lambda(\alpha(k-1) - 1)$ and the boundary data to the right side of η_{a_i} is $2\lambda(\alpha(k) - 1)$. Then the N curves $\{\eta_{a_1}, \eta_{a_2}, \ldots, \eta_{a_N}\}$ are non-intersecting simple curves and their end points form a planar pair partition of the $2N$ boundary points. We denote this planar pair partition by $\mathcal{A} = \mathcal{A}(\eta_{a_1}, \ldots, \eta_{a_N}) \in \mathrm{PP}_\mathrm{N}$. See Figures 4–5. The goal of this section is to derive the probabilities for $\mathbb{P}[\mathcal{A} = \beta]$.

Theorem 4.1. *Fix $\alpha \in \mathrm{PP}_\mathrm{N}$. Let $\Gamma + u_\alpha$ be the GFF on $\mathbb{H}$ with boundary data given by (4.2). Consider the planar pair partition $\mathcal{A}$ formed by its level lines described as above. Then we have*

$$\mathbb{P}[\mathcal{A} = \beta] = \mathcal{M}_{\alpha,\beta} \frac{\mathcal{Z}_\beta(x_1, \ldots, x_{2N})}{\mathcal{U}_\alpha(x_1, \ldots, x_{2N})}, \quad \textit{for all } \beta \in \mathrm{PP}_\mathrm{N}, \tag{4.4}$$

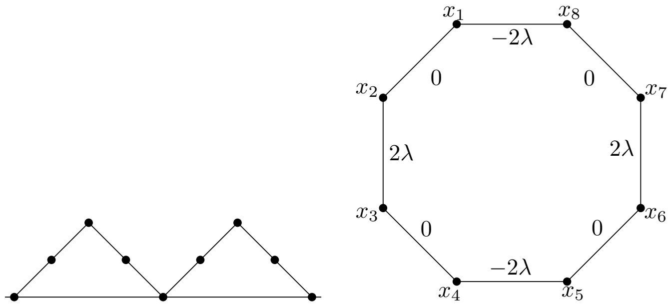

Figure 4: Illustration of the boundary data with $N = 4$: the planar pair partition for boundary data is $\alpha = \{\{1,4\},\{2,3\},\{5,8\},\{6,7\}\}$ as ordered in (2.2).

where $\{\mathcal{Z}_\beta : \beta \in \mathrm{PP_N}\}$ *are pure partition functions for multiple* SLE_4, $\{\mathcal{U}_\alpha : \alpha \in \mathrm{DP_N}\}$ *are conformal block functions defined in* (3.10)*, and* $\{\mathcal{M}_{\gamma,\beta} : \gamma, \beta \in \mathrm{PP_N}\}$ *is the incidence matrix defined through* (3.11)*.*

Theorem 4.1 is a generalization of [PW19, Theorem 1.4] where the authors derive the connection probabilities for $\alpha = \{\{1,2\},\{3,4\},\ldots,\{2N-1,2N\}\}$.

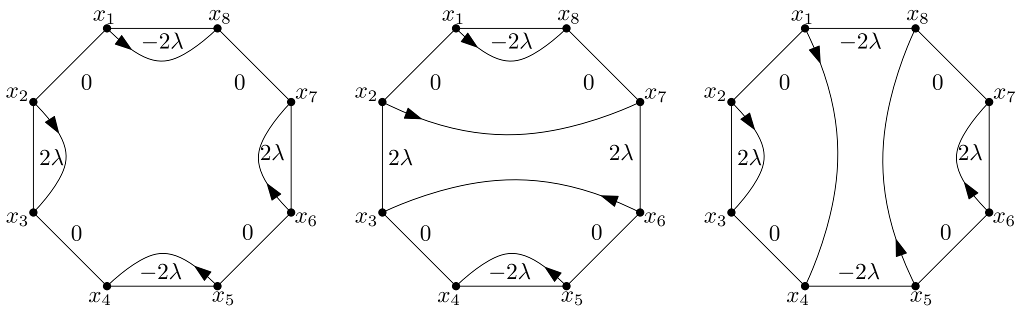

Figure 5: Figure 4 continued: the curve η_i is the level line starting from x_i with height $-\lambda$ for $i = 1,5$; the curve η_i is the level line starting from x_i with height λ for $i = 2,6$. The four curves $\eta_1,\eta_2,\eta_5,\eta_6$ connect the eight boundary points. Their end points give a planar pair partition, and there are three possibilities as indicated in the figure. From left to right, the three planar pair partitions are $\beta_1 = \{\{1,8\},\{2,3\},\{5,4\},\{6,7\}\}$, $\beta_2 = \{\{1,8\},\{2,7\},\{5,4\},\{6,3\}\}$, $\beta_3 = \{\{1,4\},\{2,3\},\{5,8\},\{6,7\}\}$. Note that $\alpha \overset{()}{\longleftarrow} \beta_i$ for $i = 1,2,3$.

Lemma 4.2. *Let* $\eta = \eta_1$ *be the level line of* $\Gamma + u_\alpha$ *starting from* x_1 *with height* $-\lambda$*, let* $(W_t, t \geq 0)$ *be the driving function, and* $(g_t, t \geq 0)$ *be the corresponding conformal maps, and* T *be the continuation threshold. For a smooth function* $F : \mathsf{X}_{2N} \to \mathbb{R}$*, the process*

$$M_t := \frac{F(W_t, g_t(x_2), \ldots, g_t(x_{2N}))}{\mathcal{U}_\alpha(W_t, g_t(x_2), \ldots, g_t(x_{2N}))}$$

is a local martingale if and only if F *satisfies PDE* (3.2) *with* $\kappa = 4$ *and* $i = 1$*.*

Proof. The level line of $\Gamma + u_\alpha$ starting from x_1 with height $-\lambda$ is the $\mathrm{SLE}_4(\rho_2,\ldots,\rho_{2\mathrm{N}})$ process with force points $(x_2,\ldots,x_{2N})$ and $\rho_i = 2(\alpha(i) - \alpha(i-1))$. Recalling from (4.1), its driving function W_t satisfies the following intergrated SDEs up to the continuation threshold T:

$$W_t = 2B_t + x_1 + \sum_{i=2}^{2N} \int_0^t \frac{\rho_i ds}{W_s - g_s(x_i)}, \quad g_t(x_i) = x_i + \int_0^t \frac{2ds}{g_s(x_i) - W_s}, \quad \text{for } 2 \le i \le 2N.$$

We denote $\mathsf{Y} = (W_t, g_t(x_2), \ldots, g_t(x_{2N}))$ and $X_{i1} = g_t(x_i) - W_t$ for $2 \le i \le 2N$. In this proof, we write ∂_i for $\frac{\partial}{\partial x_i}$ as there is no ambiguity. We denote the differential operator in (3.2) with $\kappa = 4$ and $i = 1$ by

$$\mathcal{D}^{(1)} := 2\partial_1^2 + \sum_{i=2}^{2N} \left(\frac{2\partial_i}{x_i - x_1} - \frac{1}{2(x_i - x_1)^2} \right).$$

By Itô's formula, we have

$$\begin{aligned} dF(\mathsf{Y}) =& 2\partial_1 F(\mathsf{Y}) dB_t + \left(2\partial_1^2 + \sum_{i=2}^{2N} \frac{2\partial_i}{X_{i1}} - \frac{\rho_i \partial_1}{X_{i1}} \right) F(\mathsf{Y}) dt, \\ =& 2\partial_1 F(\mathsf{Y}) dB_t + \left(\mathcal{D}^{(1)} + \sum_{i=2}^{2N} \frac{1}{2X_{i1}^2} - \frac{\rho_i \partial_1}{X_{i1}} \right) F(\mathsf{Y}) dt. \end{aligned}$$

We also have

$$\frac{d\mathcal{U}_\alpha(\mathsf{Y})}{\mathcal{U}_\alpha(\mathsf{Y})} = -\sum_{i=2}^{2N} \frac{\vartheta_\alpha(1,i)}{X_{i1}} dB_t + \left(\sum_{i=2}^{2N} \frac{1+\vartheta_\alpha(1,i)\rho_i}{2X_{i1}^2} + \sum_{2 \le i \ne j \le 2N} \frac{\frac{1}{2}(\vartheta_\alpha(1,j)\rho_i + \vartheta_\alpha(1,i)\rho_j)}{2X_{i1}X_{j1}} \right) dt.$$

By definition, we have $\vartheta_\alpha(1,i)\rho_i = 2$ for $2 \le i \le 2N$ and $\vartheta_\alpha(1,j)\rho_i + \vartheta_\alpha(1,i)\rho_j = 4\vartheta_\alpha(i,j)$ for $i \ne j$. Thus

$$\frac{d\mathcal{U}_\alpha(\mathsf{Y})}{\mathcal{U}_\alpha(\mathsf{Y})} = -\sum_{i=2}^{2N} \frac{\vartheta_\alpha(1,i)}{X_{i1}} dB_t + \left(\sum_{i=2}^{2N} \frac{3}{2X_{i1}^2} + \sum_{2 \le i \ne j \le 2N} \frac{\vartheta_\alpha(i,j)}{X_{i1}X_{j1}} \right) dt.$$

Therefore, we have

$$\begin{aligned} \frac{dM_t}{M_t} =& \frac{dF(\mathsf{Y})}{F(\mathsf{Y})} - \frac{d\mathcal{U}_\alpha(\mathsf{Y})}{\mathcal{U}_\alpha(\mathsf{Y})} + 4\left(\frac{\partial_1 \mathcal{U}_\alpha(\mathsf{Y})}{\mathcal{U}_\alpha(\mathsf{Y})} \right)^2 dt - 4\left(\frac{\partial_1 \mathcal{U}_\alpha(\mathsf{Y})}{\mathcal{U}_\alpha(\mathsf{Y})} \right)\left(\frac{\partial_1 F(\mathsf{Y})}{F(\mathsf{Y})} \right) dt \\ =& \left(\frac{2\partial_1 F(\mathsf{Y})}{F(\mathsf{Y})} - \frac{2\partial_1 \mathcal{U}_\alpha(\mathsf{Y})}{\mathcal{U}_\alpha(\mathsf{Y})} \right) dB_t + \frac{\mathcal{D}^{(1)} F(\mathsf{Y})}{F(\mathsf{Y})} dt. \end{aligned}$$

Thus M_t is a local martingale if and only if F satisfies PDE (3.2) with $\kappa = 4$ and $i = 1$. □

Proof of Theorem 4.1. We prove by induction on N. We write $\alpha = \{\{a_1, b_1\}, \ldots, \{a_N, b_N\}\}$ as ordered in (2.2). We first show the conclusion for $\beta \in \mathrm{PP_N}$ such that $\mathcal{M}_{\alpha,\beta} = 1$. There exists $j \in \{1,,2\ldots,2N-1\}$ such that $\{j, j+1\} \in \beta$. In this case, we have $\wedge^j \in \beta$ and $\vee_j \in \alpha$. If $\wedge^j \in \alpha$, we let $\eta = \eta_j$ be the level line of $\Gamma + u_\alpha$ starting from x_j with height $\mathcal{H}_\alpha(j)$. If $\vee_j \in \alpha$, we let $\eta = \eta_{j+1}$ be the level line of $\Gamma + u_\alpha$ starting from x_{j+1} with height $\mathcal{H}_\alpha(j+1)$. The second case can be proved in a similar way as the first case. So we only give proof for the first case: we may assume $\wedge^j \in \alpha$. Let $\eta = \eta_j$ be the level line of $\Gamma + u_\alpha$ starting from x_j with height $\mathcal{H}_\alpha(j)$. Let $(W_t, t \ge 0)$ be the driving function, and $(g_t, t \ge 0)$ be the corresponding conformal maps, and T be the continuation threshold.

Define

$$M_t := \frac{\mathcal{Z}_\beta(g_t(x_1), \ldots, g_t(x_{j-1}), W_t, g_t(x_{j+1}), g_t(x_{2N}))}{\mathcal{U}_\alpha(g_t(x_1), \ldots, g_t(x_{j-1}), W_t, g_t(x_{j+1}), g_t(x_{2N}))}.$$

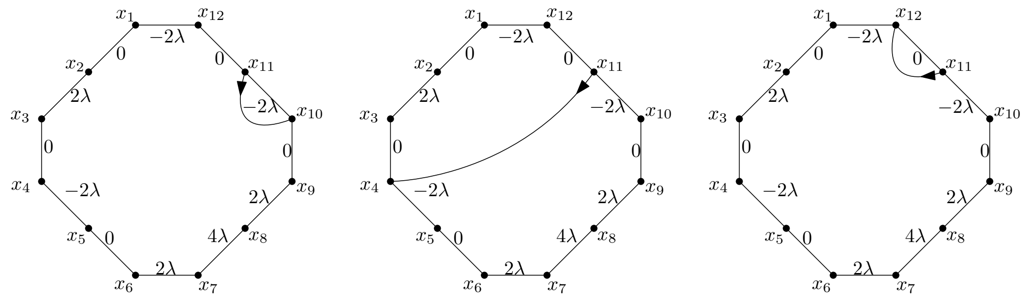

Figure 6: Suppose $\alpha = \{\{1,4\},\{2,3\},\{5,10\},\{6,9\},\{7,8\},\{11,12\}\}$ as ordered in (2.2). We see that $\wedge^{11} \in \alpha$. Consider the level line η_{11} starting from x_{11} with height $-\lambda$. It may terminate at x_n with positive chance for $n \in \{4,10,12\}$. In all three cases, the boundary data on (x_n, x_{n+1}) is the same as the boundary data on (x_{10}, x_{11}), and the boundary data on (x_{n-1}, x_n) is the same as the boundary data on (x_{11}, x_{12}). Therefore, we have $\alpha(n) = \alpha(10)$ and $\alpha(n-1) = \alpha(11)$.

From a similar calculation as in Lemma 4.2, this is a local martingale. From (3.12), this is a bounded martingale. Optional stopping theorem gives $M_0 = \mathbb{E}[M_T]$. We will analyze the behavior of the process as $t \to T$. Consider the level line η, it will terminate at a point x_n such that $\alpha(n-1) = \alpha(j)$ and $\alpha(n) = \alpha(j-1)$. See Figure 6.

If $\eta(T) = x_{j+1}$, from (3.13) and (3.16), we have, as $t \to T$, almost surely,

$$\begin{aligned} M_t =& \frac{(g_t(x_{j+1}) - W_t)^{1/2} \mathcal{Z}_\beta(g_t(x_1), \ldots, g_t(x_{j-1}), W_t, g_t(x_{j+1}), g_t(x_{2N}))}{(g_t(x_{j+1}) - W_t)^{1/2} \mathcal{U}_\alpha(g_t(x_1), \ldots, g_t(x_{j-1}), W_t, g_t(x_{j+1}), g_t(x_{2N}))} \\ \to& \frac{\mathcal{Z}_{\beta/\wedge_j}(g_T(x_1), \ldots, g_T(x_{j-1}), g_T(x_{j+2}), \ldots, g_T(x_{2N}))}{\mathcal{U}_{\alpha/\wedge_j}(g_T(x_1), \ldots, g_T(x_{j-1}), g_T(x_{j+2}), \ldots, g_T(x_{2N}))}. \end{aligned}$$

If $\eta(T) = x_n$ with $n \neq j+1$, from Lemma A.3 and (3.5), we have $\lim_{t \to T} M_t = 0$ almost surely. In summary, we have

$$M_0 = \mathbb{E}[M_T] = \mathbb{E}\left[\mathbb{1}_{\{\eta(T) = x_{j+1}\}} \frac{\mathcal{Z}_{\beta/\wedge_j}(g_T(x_1), \ldots, g_T(x_{j-1}), g_T(x_{j+2}), \ldots, g_T(x_{2N}))}{\mathcal{U}_{\alpha/\wedge_j}(g_T(x_1), \ldots, g_T(x_{j-1}), g_T(x_{j+2}), \ldots, g_T(x_{2N}))} \right].$$

By induction hypothesis, we have

$$\mathbb{P}[\mathcal{A} = \beta \mid \eta[0,T]] = \frac{\mathcal{Z}_{\beta/\wedge_j}(g_T(x_1), \ldots, g_T(x_{j-1}), g_T(x_{j+2}), \ldots, g_T(x_{2N}))}{\mathcal{U}_{\alpha/\wedge_j}(g_T(x_1), \ldots, g_T(x_{j-1}), g_T(x_{j+2}), \ldots, g_T(x_{2N}))}.$$

Therefore, $M_0 = \mathbb{P}[\mathcal{A} = \beta]$ as desired.

The above argument gives (4.4) for $\beta \in \mathrm{PP_N}$ such that $\mathcal{M}_{\alpha,\beta} = 1$. Since $\mathcal{U}_\alpha = \sum_\beta \mathcal{M}_{\alpha,\beta} \mathcal{Z}_\beta$, we have

$$\sum_{\beta : \mathcal{M}_{\alpha,\beta} = 1} \mathbb{P}[\mathcal{A} = \beta] = \sum_{\beta : \mathcal{M}_{\alpha,\beta} = 1} \frac{\mathcal{Z}_\beta}{\mathcal{U}_\alpha} = 1.$$

This implies $\mathbb{P}[\mathcal{A} = \gamma] = 0$ for all $\gamma \in \mathrm{PP_N}$ with $\mathcal{M}_{\alpha,\gamma} = 0$. This completes the proof for (4.4). □

5 Metric graph GFF and first passage sets

In this section, we first introduce discrete GFF and metric graph GFF in Section 5.1, and then we introduce first passage set in Section 5.2. In Section 5.3, we show that the crossing probabilities in metric graph GFF converges to the probability of certain connection probabilities in continuum GFF, see Proposition 5.2. This gives the first half of the proof of Theorem 1.1. In order to calculate the desired connection probabilities in continuum GFF, we use Theorem 4.1 and a result about asymptotics of pure partition functions—Proposition 5.6. Section 5.4 proves Proposition 5.6 and Proposition 1.2, and it is quite independent of the rest of the section. Finally, we complete the proof of Theorem 1.1 in Section 5.5.

5.1 Discrete GFF and metric graph GFF

In this section, we review basic definition and properties of discrete GFF and metric graph GFF. We refer to [SS09, ALS20] for details. Suppose $\mathcal{G} = (V, E)$ is a connected planar graph, and $\partial\mathcal{G}$ is a given subset of V which we call the boundary of $\mathcal{G}$. Let Δ be the discrete Laplacian on $\mathcal{G}$:

$$(\Delta f)(x) = \sum_{y \sim x} (f(y) - f(x)), \; \forall x \in V \setminus \partial\mathcal{G}.$$

The discrete Green's function $G_{\mathcal{G}}$ is the inverse of $-\Delta$ with zero-boundary condition on $\partial\mathcal{G}$. The discrete GFF is the centered Gaussian process $\{\Gamma^{\mathcal{G}}(v) : v \in V\}$ with covariance given by Green's function:

$$\mathbb{E}\left[\Gamma^{\mathcal{G}}(x)\Gamma^{\mathcal{G}}(y)\right] = G_{\mathcal{G}}(x, y), \quad \forall x, y \in V.$$

Suppose $\mathcal{G} = (V, E)$ is a connected planar graph with boundary $\partial\mathcal{G}$. For each $e \in E$, we view it as a line segment in the plane, and for every $x', y' \in e$, we define[3]

$$m([x', y']) = \frac{|x' - y'|}{|x - y|}.$$

This defines a length measure dm on $\mathcal{G}$. We call $(\mathcal{G}, dm)$ metric graph of $\mathcal{G}$ and we denote it by $\tilde{\mathcal{G}}$.

The metric graph GFF $\Gamma^{\tilde{\mathcal{G}}}$ can be constructed as follows, see [Lup16]. First, we sample the discrete GFF $\{\Gamma^{\mathcal{G}}(v) : v \in V\}$. Then, conditional on $\{\Gamma^{\mathcal{G}}(v) : v \in V\}$, for each $e = \{x, y\} \in E$, we sample an independent Brownian bridge with length $m([x, y])$ and two terminal values $\Gamma^{\mathcal{G}}(x)$ and $\Gamma^{\mathcal{G}}(y)$. This defines the metric graph GFF with zero-boundary condition and we denote it by $\{\Gamma^{\tilde{\mathcal{G}}}(z) : z \in \tilde{\mathcal{G}}\}$. Given a function $u : \partial\mathcal{G} \to \mathbb{R}$, we choose the discrete harmonic extension of u to $V \setminus \partial\mathcal{G}$ and then extend it inside each edge by linear interpolation. We still denote this function by u and view it as the harmonic function on the metric graph. We call $\Gamma^{\tilde{\mathcal{G}}} + u$ the metric graph GFF with boundary data u.

5.2 First passage sets

In this section, we introduce first passage sets for metric graph GFF. Suppose $\Gamma^{\tilde{\mathcal{G}}} + u$ is the metric graph GFF with boundary data u. For every $a \in \mathbb{R}$, the first passage set above $-a$ is defined by

$$\tilde{\mathbb{A}}^u_{a} := \{x \in \tilde{\mathcal{G}} \,|\, \exists \text{ a continous path } \gamma \text{ from } x \text{ to } \partial\mathcal{G} \text{ in } \tilde{\mathcal{G}} \text{ such that } \Gamma^{\tilde{\mathcal{G}}} + u \geq -a \text{ along } \gamma\}.$$

Note that, conditional on $\tilde{\mathbb{A}}^u_{a}$, the closure of $\tilde{\mathcal{G}} \setminus \tilde{\mathbb{A}}^u_{a}$ is also a metric graph with length measure inherited from $\tilde{\mathcal{G}}$. According to [ALS20, Proposition 2.1], metric graph GFF satisfies the following space Markov property:

$$\Gamma^{\tilde{\mathcal{G}}} = \Gamma^{\tilde{\mathcal{G}}}_{\tilde{\mathbb{A}}^u_{a}} + \Gamma^{\tilde{\mathcal{G}}, \tilde{\mathbb{A}}^u_{a}},$$

[3] Here we use the normalization in [ALS20] which is distinct from the one in [Lup16].

where $\Gamma^{\tilde{\mathcal{G}},\tilde{\mathbb{A}}^u{}_a}$ is the metric graph GFF with zero-boundary condition on the closure of $\tilde{\mathcal{G}} \setminus \tilde{\mathbb{A}}^u{}_a$ conditional on $\tilde{\mathbb{A}}^u{}_a$, and $\Gamma^{\tilde{\mathcal{G}}}_{\tilde{\mathbb{A}}^u{}_a}$ is defined as follows: it is $\Gamma^{\tilde{\mathcal{G}}}$ on $\tilde{\mathbb{A}}^u{}_a$ and it is the harmonic function with boundary value given by $\Gamma^{\tilde{\mathcal{G}}}$ on $\tilde{\mathcal{G}} \setminus \tilde{\mathbb{A}}^u{}_a$.

We also need the following description of first passage set by clusters of loops and excursions. The Brownian loop measure and Brownian excursion measure are conformally invariant measures on Brownian paths in the plane. In this article, we do not need the precise definition of these measures, so we content ourselves with referring their definition to [ALS20, Section 2.2]. We denote by $\mu^{\tilde{\mathcal{G}}}_{\text{loop}}$ the Brownian loop measure on $\tilde{\mathcal{G}}$. Suppose u is non-negative, and we denote by $\mu^{\tilde{\mathcal{G}},u}_{\text{exc}}$ the Brownian excursion measure on $\tilde{\mathcal{G}}$ with boundary data u. We sample Poisson point process with intensity measure $\frac{1}{2}\mu^{\tilde{\mathcal{G}}}_{\text{loop}}$, and denote it by $\mathcal{L}^{\tilde{\mathcal{G}}}_{1/2}$. We sample an independent Poisson point process with intensity measure $\mu^{\tilde{\mathcal{G}},u}_{\text{exc}}$ and denote it by $\Xi^{\tilde{\mathcal{G}}}_u$. We denote by $\tilde{\mathcal{A}}(\mathcal{L}^{\tilde{\mathcal{G}}}_{1/2}, \Xi^{\tilde{\mathcal{G}}}_u)$ the closure of union of clusters formed by loops and excursions that contain at least one excursion connected to $\partial\mathcal{G}$. As shown in [ALS20, Proposition 2.5], the set $\tilde{\mathcal{A}}(\mathcal{L}^{\tilde{\mathcal{G}}}_{1/2}, \Xi^{\tilde{\mathcal{G}}}_u)$ has the same law as the first passage set $\tilde{\mathbb{A}}^u_0$.

Next, we introduce the first passage set for continuum GFF. To this end, we first introduce local set. Suppose $\Omega \subsetneq \mathbb{C}$ is a simply connected domain and let Γ be a continuum GFF on Ω with zero-boundary condition. We call a random closed set $A \subset \overline{\Omega}$ is a local set of Γ, if $\Gamma = \Gamma_A + \Gamma^A$, where Γ_A and Γ^A are two random distributions such that Γ_A is harmonic in $\Omega \setminus A$ and, conditional on (A, Γ_A), the function Γ^A is the GFF with zero-boundary condition in $\Omega \setminus A$. Suppose h_A is defined as follows: it is Γ_A on $\Omega \setminus A$ and it is 0 on A. Then we have the following description of the first passage set.

Theorem 5.1. *Suppose $\Omega \subsetneq \mathbb{C}$ is a simply connected domain and let Γ be a continuum GFF on Ω with zero-boundary condition. Suppose u is a bounded harmonic function with piecewise constant boundary data.*[4] *The first passage set $\mathbb{A}^u{}_a$ is the local set of Γ containing $\partial\Omega$ with the following two properties:*

- *The function $h_{\mathbb{A}^u{}_a} + u$ is harmonic in $\Omega \setminus \mathbb{A}^u{}_a$ such that it equals $-a$ on $\partial\mathbb{A}^u{}_a \setminus \partial\Omega$ and it equals u on $\partial(\Omega \setminus \mathbb{A}^u{}_a) \cap \partial\Omega$. Moreover, $h_{\mathbb{A}^u{}_a} + u \leq -a$.*
- *We have $\Gamma_{\mathbb{A}^u{}_a} - h_{\mathbb{A}^u{}_a} \geq 0$. I.e. for any positive smooth function f with compact support, we have $(\Gamma_{\mathbb{A}^u{}_a} - h_{\mathbb{A}^u{}_a}, f) \geq 0$.*

For all $a \geq 0$, the first passage set $\mathbb{A}^u{}_a$ exists. Moreover, the set $\mathbb{A}^u{}_a$ is the unique local set which satisfies the above two properties and is measurable with respect to Γ.

Proof. See [ALS20, Theorem 3.5]. □

Now, we are ready to state the convergence of the first passage set of the metric graph GFF to the first passage set of the continuum GFF. Fix a bounded simply connected domain Ω such that $\Omega \subset [-C, C]^2$ for some $C > 0$. Suppose $\{\Omega^\delta\}_{\delta>0}$ is a sequence of simply connected domains such that $\Omega^\delta \subset [-C, C]^2$ for all $\delta > 0$. Suppose Ω^δ converges to Ω as $\delta \to 0$ in the following sense:

$$[-C, C]^2 \setminus \Omega^\delta \text{ converges to } [-C, C]^2 \setminus \Omega \text{ in Hausdorff metric.}$$

We denote by $\delta\tilde{\mathbb{Z}}^2$ the corresponding metric graph. We define $\tilde{\Omega}^\delta$ to be the closure of $\Omega^\delta \cap \delta\tilde{\mathbb{Z}}^2$. It is also a metric graph with metric inherited from $\delta\tilde{\mathbb{Z}}^2$. We define its boundary by $\partial\tilde{\Omega}^\delta := \tilde{\Omega}^\delta \cap \partial\Omega^\delta$. We have the following setup for the convergence.

[4] Throughout the article, by piecewise constant boundary data, we mean that the boundary data is piecewise constant and it changes only finitely many times.

- Suppose Γ is the continuum GFF on Ω and $\tilde{\Gamma}^\delta$ is the metric grpah GFF on $\tilde{\Omega}^\delta$ with zero-boundary condition. We extend Γ to $(-C,C)^2$ such that it is zero outside Ω, and we still denote the extension by Γ. We define $\hat{\Gamma}^\delta$ on $(-C,C)^2$ as follows: it equals $\tilde{\Gamma}^\delta$ on $\tilde{\Omega}^\delta$ and it is harmonic in $(-C,C)^2 \setminus \tilde{\Omega}^\delta$ which equals zero along $\partial(-C,C)^2$.
- Suppose u is a harmonic function on Ω with piecewise constant boundary data and u^δ is a harmonic function on $\tilde{\Omega}^\delta$ for every $\delta > 0$ such that u^δ converges to u uniformly as $\delta \to 0$.
- For $a \in \mathbb{R}$, suppose $\mathbb{A}^u{}_a$ is the first passage set of Γ on Ω and $\tilde{\mathbb{A}}^u{}_a$ is the first passage set of $\tilde{\Gamma}^\delta$ on $\tilde{\Omega}^\delta$. We extend $\Gamma_{\mathbb{A}^u{}_a}$ to $(-C,C)^2$ such that it is zero outside Ω, and we still denote the extension by $\Gamma_{\mathbb{A}^u{}_a}$. We define $\hat{\Gamma}^\delta_{\tilde{\mathbb{A}}^u{}_a}$ on $(-C,C)^2$ as follows: it equals $\tilde{\Gamma}^\delta_{\tilde{\mathbb{A}}^u{}_a}$ on $\tilde{\Omega}^\delta$ and it is harmonic in $(-C,C)^2 \setminus \tilde{\Omega}^\delta$ which equals zero along $\partial(-C,C)^2$.

Proposition 5.2. *We have the following convergence in law:*

$$\hat{\Gamma}^\delta, \hat{\Gamma}^\delta_{\tilde{\mathbb{A}}^u{}_a}, \tilde{\mathbb{A}}^u{}_a \quad \to \quad \Gamma, \Gamma_{\mathbb{A}^u{}_a}, \overline{\mathbb{A}^u{}_a \cap \Omega} \quad , \quad \text{as } \delta \to 0.$$

Furthermore, if we couple $\{\hat{\Gamma}^\delta\}_{\delta>0}$ and Γ together such that $\hat{\Gamma}^\delta \to \Gamma$ in probability as distributions on $[-C,C]^2$, then $(\hat{\Gamma}^\delta, \tilde{\mathbb{A}}^u{}_a) \to (\Gamma, \overline{\mathbb{A}^u{}_a \cap \Omega})$ in probability.

Proof. See [ALS20, Proposition 4.7 and Lemma 4.9]. □

5.3 Convergence of the connection probability

Fix a bounded polygon $(\Omega; y_1, \ldots, y_{2N})$ and suppose $\Omega^\delta; y_1^\delta, \ldots, y_{2N}^\delta$ converges to $(\Omega; y_1, \ldots, y_{2N})$ as $\delta \to 0$ in the sense of (1.4). We have the following setup.

- Suppose $\tilde{\Gamma}^\delta$ is the zero-boundary metric graph GFF on $\tilde{\Omega}^\delta$ and let u^δ be the harmonic function with boundary data (1.5). Suppose Γ is zero-boundary GFF on Ω and let u be the harmonic function with the same boundary data.
- We call the first passage set above 0 of $\tilde{\Gamma}^\delta + u^\delta$ the positive first passage set and we denote it by $\tilde{\mathbb{A}}^\delta$. We call the first passage set above 0 of $-(\tilde{\Gamma}^\delta + u^\delta)$ the negative first passage set and we denote it by $\tilde{\mathbb{V}}^\delta$. Similarly, we can also define the positive first passage set and the negative first passage set for the continuum GFF in Ω, and we denote them by $\mathbb{A}$ and $\mathbb{V}$ respectively.

Note that the frontier of these first passage sets is a collection of $2N$ curves connecting the $2N$ boundary points so that their end points form a planar $2N$-link pattern of $2N$ points with index valences $\varsigma = (2, \ldots, 2)$, see Figure 1. We denote the link pattern by $\mathcal{A}^\delta$ for metric graph GFF and by $\mathcal{A}$ for cotinuum GFF. The goal of this section is to prove the following convergence.

Proposition 5.3. *Fix $N \geq 1$ and $\varsigma = (2, \ldots, 2)$ of length $2N$. For all $\hat{\alpha} \in \mathrm{LP}_\varsigma$, we have*

$$\lim_{\delta \to 0} \mathbb{P}[\mathcal{A}^\delta = \hat{\alpha}] = \mathbb{P}[\mathcal{A} = \hat{\alpha}].$$

To prove Proposition 5.3, we will give an explicit construction of $\mathbb{A}$ and $\mathbb{V}$ in Lemma 5.4. This construction indicates that the frontier of $\mathbb{A}$ and of $\mathbb{V}$ forms a planar link pattern in LP_ς. Then, we prove Lemma 5.5 which indicates that for any subsequence $\delta_n \to 0$ as $n \to \infty$, there exists a coupling such that the frontier of $\tilde{\mathbb{A}}^{\delta_n}$ and of $\tilde{\mathbb{V}}^{\delta_n}$ converges to the frontier of $\mathbb{A}$ and of $\mathbb{V}$ almost surely in Hausdorff metric. This indicates the proposition.

Lemma 5.4. *The frontier of $\mathbb{A}$ is the union of level lines of the continuum GFF $\Gamma + u$ starting from y_{2j-1} with height λ for $1 \le j \le N$, the frontier of $\mathbb{V}$ is the union of level lines of $\Gamma + u$ starting from y_{2j-1} with height $-\lambda$ for $1 \le j \le N$.*

Proof. We will prove the conclusion for $\mathbb{A}$, and the proof for $\mathbb{V}$ is similar. We will argue that the first passage set $\mathbb{A}$ can be constructed as follows. Let η_j be the level line of $\Gamma + u$ starting from y_{2j-1} with height λ for $1 \le j \le N$. Suppose $S_1, \dots, S_r$ are the different connected components of $\Omega \setminus (\cup_{1\le j\le N}\eta_j)$ which have (y_{2j-1}, y_{2j}) on their boundary for some $1 \le j \le N$. Note that $(\Gamma + u)|_{S_i}$ has boundary data 2λ along ∂S_i. Conditional on $\cup_{1\le j\le N}\eta_j$, we sample the first passage set above zero of $(\Gamma + u)|_{S_i}$ in each S_i, and we denote it by $\mathbb{A}^{2\lambda}_{0,i}$ for $1 \le i \le r$. We will show that the union $A := (\cup_{1\le j\le N}\eta_j) \cup (\cup_{1\le i\le r}\mathbb{A}^{2\lambda}_{0,i}) \cup \partial\Omega$ has the same law as $\mathbb{A}$.

First, we prove that A is a local set. By construction,

$$
\begin{aligned}
\Gamma =& \Gamma_{\cup_{1\le j\le N}\eta_j} + \Gamma^{\cup_{1\le j\le N}\eta_j} \\
=& \Gamma_{\cup_{1\le j\le N}\eta_j} + \sum_{i=1}^{r} \Gamma^{\cup_{1\le j\le N}\eta_j}|_{S_i} \\
=& \Gamma_{\cup_{1\le j\le N}\eta_j} + \sum_{i=1}^{r} (\Gamma^{\cup_{1\le j\le N}\eta_j}|_{S_i})_{\mathbb{A}^{2\lambda}_{0;i}} + \sum_{i=1}^{r} (\Gamma^{\cup_{1\le j\le N}\eta_j}|_{S_i})^{\mathbb{A}^{2\lambda}_{0;i}}.
\end{aligned}
$$

Note that $\Gamma_A := \Gamma_{\cup_{1\le j\le N}\eta_j} + \sum_{i=1}^{r} (\Gamma^{\cup_{1\le j\le N}\eta_j}|_{S_i})_{\mathbb{A}^{2\lambda}_{0;i}}$ is harmonic in $\Omega \setminus A$. Conditional on (A, Γ_A), the function $\Gamma^A = \sum_{i=1}^{r} (\Gamma^{\cup_{1\le j\le N}\eta_j}|_{S_i})^{\mathbb{A}^{2\lambda}_{0;i}}$ is the continuum GFF with zero-boundary condition in $\Omega \setminus A$. This implies that A is a local set.

Next, we check the two properties in Theorem 5.1. The first one is obvious by construction. For the second one, suppose f is a positive smooth function with compact support in Ω, it suffices to prove

$$(\Gamma_{\cup_{1\le j\le N}\eta_j}, f) = (h_{\cup_{1\le j\le N}\eta_j}, f) \tag{5.1}$$

and

$$\left((\Gamma^{\cup_{1\le j\le N}\eta_j}|_{S_i})_{\mathbb{A}^{2\lambda}_{0;i}} + 2\lambda 1_{S_i}, f\right) \ge 0. \tag{5.2}$$

Eq. (5.1) is a consequence of properties of level lines of GFF. For (5.2), consider the metric graph $\tilde{S}_i^\delta = S_i \cap \delta\tilde{\mathbb{Z}}^2$, we denote by $\tilde{\Gamma}^\delta|_{\tilde{S}_i}$ the metric graph GFF with zero-boundary condition on $\tilde{S}_i^\delta$. Then by Proposition 5.2, we can couple $\left((\hat{\Gamma}^\delta|_{\tilde{S}_i})_{\tilde{\mathbb{A}}^{2\lambda}_0}\right)_{\delta>0}$ and $(\Gamma^{\cup_{1\le j\le N}\eta_j}|_{S_i})_{\mathbb{A}^{2\lambda}_{0;i}}$ together such that

$$
\begin{aligned}
\left((\Gamma^{\cup_{1\le j\le N}\eta_j}|_{S_i})_{\mathbb{A}^{2\lambda}_{0;i}} + 2\lambda 1_{S_i}, f\right) &= \left((\Gamma^{\cup_{1\le j\le N}\eta_j}|_{S_i})_{\mathbb{A}^{2\lambda}_{0;i}} + 2\lambda 1_{S_i}, f 1_{S_i}\right) \\
&= \lim_{\delta \to 0} \left((\hat{\Gamma}^\delta|_{\tilde{S}_i})_{\tilde{\mathbb{A}}^{2\lambda}_0} + 2\lambda 1_{S_i}, f 1_{S_i}\right) \ge 0.
\end{aligned}
$$

This gives (5.2). Combining with (5.1) and Theorem 5.1, we see that A has the same law as $\mathbb{A}$, and this completes the proof. □

Lemma 5.5. *Suppose $\delta_n \to 0$ as $n \to \infty$.*

- *Suppose $(\mathbb{A}_1, \dots, \mathbb{A}_r)$ are different connected components of $\overline{\mathbb{A} \cap \Omega}$ and $(y_{i,1}, \dots, y_{i,k_i})$ are the marked points on the boundary of $\mathbb{A}_i$ for each $1 \le i \le r$; and suppose $(\mathbb{V}_1, \dots, \mathbb{V}_s)$ are different connected components of $\overline{\mathbb{V} \cap \Omega}$ and $(y_{j,1}, \dots, y_{j,l_j})$ are the marked points on the boundary of $\mathbb{V}_j$ for each $1 \le j \le s$.*

- Suppose $\tilde{A}_1^{n}; \ldots; \tilde{A}_{r_n}^{n}$ are the connected components of

$$\tilde{A}^{n} \cup \left(\cup_{1 \le l \le N} \left(y_{2l-1}^{n}; y_{2l}^{n} \right) \right);$$

and suppose $\tilde{\forall}_1^{n}; \ldots; \tilde{\forall}_{s_n}^{n}$ are the connected components of

$$\tilde{\forall}^{n} \cup \left(\cup_{1 \le l \le N} \left(y_{2l}^{n}; y_{2l+1}^{n} \right) \right):$$

Then there exists a coupling of $\{(\tilde{A}^{n}; \tilde{\forall}^{n})\}_{n \ge 1}$ and $(A; \forall)$ such that the following holds almost surely.

- For n large enough, we have $r_n = r$ and $s_n = s$.
- Moreover, we can reorder $\tilde{A}_1^{n}; \ldots; \tilde{A}_r^{n}$ and we still denote them by $\tilde{A}_1^{n}; \ldots; \tilde{A}_r^{n}$, such that $y_{i;1}^{n}; \ldots; y_{i;k_i}^{n}$ are the marked points on the boundary of $\tilde{A}_i^{n}$ for $1 \le i \le r$. Similarly, we can reorder $\tilde{\forall}_1^{n}; \ldots; \tilde{\forall}_s^{n}$ and we still denote them by $\tilde{\forall}_1^{n}; \ldots; \tilde{\forall}_s^{n}$ such that $y_{j;1}^{n}; \ldots; y_{j;l_j}^{n}$ are the marked points on the boundary of $\tilde{\forall}_j^{n}$ for $1 \le j \le s$.
- Furthermore, we have that $\tilde{A}_i^{n}$ converges to A_i for each $1 \le i \le r$ and $\tilde{\forall}_j^{n}$ converges to $\forall_j$ for each $1 \le j \le s$ in Hausdorff metric.

Figure 7: Suppose $N = 2$. The frontier of the first passage sets A and $\forall$ is a collection of 4 curves connecting the four points $y_1; y_2; y_3; y_4$. There are three possibilities for the connectivity patterns as indicated in Figure 1. In this figure, in the left panel, we have the first possibility. The two red curves are frontiers of A_1 and A_2 and the two blue curves are frontiers of $\forall_1$ and $\forall_2$. In the right panel, we are in metric graph and the two red curves are frontiers of A_1^{n} and A_2^{n} and the two blue curves are frontiers of $\forall_1^{n}$ and $\forall_2^{n}$. Lemma 5.5 guarantees that there exists a coupling such that, when $_n \to 0$, the red curves in the right panel converges to the red curves in the left panel and the blue curves in the right panel converges to the blue curves in the left panel respectively.

Proof. We denote by $\tilde{R}_i^{n}$ the connected component of $\tilde{A}^{n} \cup \left(\cup_{1 \le l \le N} \left(y_{2l-1}^{n}; y_{2l}^{n} \right) \right)$ which contains $y_{2i-1}^{n}; y_{2i}^{n}$ on its boundary for $1 \le i \le N$; and we denote by $\tilde{S}_i^{n}$ the connected component of $\tilde{\forall}^{n} \cup \left(\cup_{1 \le l \le N} \left(y_{2l}^{n}; y_{2l+1}^{n} \right) \right)$ which contains $(y_{2i}^{n}; y_{2i+1}^{n})$ on its boundary for $1 \le i \le N$. Note that the sequence

$$\left\{ \tilde{R}_1^{n}; \ldots; \tilde{R}_N^{n}; \tilde{A}^{n}; \tilde{S}_1^{n}; \ldots; \tilde{S}_N^{n}; \tilde{\forall}^{n} \right\}_{n \ge 1}$$

is tight. Thus, it suffices to prove that Lemma 5.5 holds for any convergent subsequence. Given any convergent subsequence, we still denote it by

$$\left\{ \tilde R_1^n; \dots; \tilde R_N^n; \tilde A^n; \tilde S_1^n; \dots; \tilde S_N^n; \forall^n \right\}_{n \geq 1}.$$

By Skorokhod representation theorem, we can couple them on the same probability space such that there is almost sure convergence. We denote the probability measure of this coupling by P, and we denote its limit by $(R_1; \dots; R_N; R; S_1; \dots; S_N; S)$.

By Proposition 5.2, we know that $R = \overline{A \cap}$ and $S = \overline{\forall \cap}$. We will prove that R_i is the connected component of A which contains $(y_{2i-1}; y_{2i})$ on its boundary and S_i is the connected component of $\forall$ which contains $(y_{2i}; y_{2i+1})$ on its boundary. Moreover, we will show that Lemma 5.5 holds in this coupling. We only give proof for the positive first passage set, as the proof for the negative first passage set is similar. The proof is divided into two steps. First, suppose $(y_{i_1}; \dots; y_{i_k})$ are the marked points on the boundary of R_i. We will prove that $y_{i_1}^n; \dots; y_{i_k}^n$ are the marked points on the boundary of $\tilde R_i^n$ for n large enough. Then we will prove that R_i is the connected component of R which contains $(y_{2i-1}; y_{2i})$ on its boundary for each $1 \leq i \leq N$. See Figure 7.

For the first step, it is clear that if $y_j \notin R_i$ for some $1 \leq j \leq 2N$, we have $y_j^n \notin \tilde R_i^n$ for n large enough by the almost sure convergence. For $1 \leq j \leq 2N$, we define the event

$$F_j := \left\{ y_j \in R_i; \text{ but } y_j^n \notin \tilde R_i^n \text{ except for finitely many } n \right\}.$$

It suffices to prove $P[F_j] = 0$. By Lemma 5.4, we have $S \cap (y_{2i-1}; y_{2i}) = \emptyset$ for $1 \leq i \leq N$. We denote by D_i the connected component of $\setminus S$ which contains $(y_{2i-1}; y_{2i})$ and we denote by D_i^n the connected component of ${}^n \setminus \forall^n$ which contains $\left(y_{2i-1}^n; y_{2i}^n\right)$. By Carathéodory kernel theorem, the domain D_i^n converges to D_i in Carathéodory topology as $n \to \infty$. Note that $\tilde A^n \cap \tilde D_i^n$ is the first passage set $\tilde A_0^{v^n}$ of the metric graph GFF on $\tilde D_i^n$ with boundary data given by v^n which is defined as follows: v^n equals 0 on $\forall^n \cap \partial \tilde D_i^n$ and v^n equals 2 on $\partial^n \cap \partial \tilde D_i^n$. We may assume j is odd. On the event F_j, we have $(y_j; y_{j+1}) \subset \partial D_i$. Thus for n large enough, we have $(y_j^n; y_{j+1}^n) \subset \partial D_i^n$. In such case, we define the harmonic function v_1^n on $\tilde D_i^n$ as follows: it equals 0 on $\left(y_j^n; y_{j+1}^n\right)$ and it equals v^n on $\partial \tilde D_i^n \setminus \left(y_j^n; y_{j+1}^n\right)$. Then, in the construction of $\tilde A_0^{v^n}$ by loops and excursions, we can divide the excursions into two independent parts: the excursions connecting to $\left(y_j^n; y_{j+1}^n\right)$ and the excursions which do not intersect $\left(y_j^n; y_{j+1}^n\right)$. Note that the excursions connecting to $\left(y_j^n; y_{j+1}^n\right)$ correspond to the Poisson point process with intensity measure $\mu_{exc}^{\tilde D_i^n; v^n} - \mu_{exc}^{\tilde D_i^n; v_1^n}$ and that the excursions which do not intersect $\left(y_j^n; y_{j+1}^n\right)$ correspond to the Poisson point process with intensity measure $\mu_{exc}^{\tilde D_i^n; v_1^n}$. Thus, we have $\tilde R_i^n \subset \tilde A\left(\tilde L_{1=2}^{\tilde D_i^n}; \mu_{v_1^n}^{\tilde D_i^n}\right)$ if $y_j^n \notin \tilde R_i^n$. Note that $\tilde A_0^{v_1^n}$ has the same law as $\tilde A\left(\tilde L_{1=2}^{\tilde D_i^n}; \mu_{v_1^n}^{\tilde D_i^n}\right)$. According to [ALS20, Corollary 4.12], the limit of $\overline{\tilde A_0^{v_1^n} \cap D_i}$ does not intersect $(y_j; y_{j+1})$ almost surely. This implies $P[F_j] = 0$.

For the second step, we define the event $F_{i;k} := \{R_i \neq R_k; \text{ but } R_i \cap R_k \neq \emptyset\}$ for $1 \leq i < k \leq N$. It suffices to prove that $P[F_{i;k}] = 0$. Note that on the event $F_{i;k}$, we have $(y_{2k-1}; y_{2k}) \subset D_{i;k}$. This implies $(y_{2k-1}^n; y_{2k}^n) \subset D_{i;k}^n$ for n large enough. Moreover, we have $\tilde R_i^n \cap \tilde R_k^n = \emptyset$ for n large enough. This implies $\tilde R_i^n \cap (y_{2k-1}^n; y_{2k}^n) = \emptyset$. We denote by $D_{i;k}$ the connected component of $D_i \setminus R_i$ with $(y_{2k-1}; y_{2k})$ on its boundary and we denote by $D_{i;k}^n$ the connected component of $D_i^n \setminus \tilde R_i^n$ with $\left(y_{2k-1}^n; y_{2k}^n\right)$ on its boundary. Then by Carathéodory kernel theorem, the domain $D^n_{i;k}$ converges to $D_{i;k}$ as $n \to \infty$ in Carathéodory topology. Note that $\tilde{A}^n \cap \tilde{D}^n_{i;k}$ is the first passage set of the metric graph GFF with boundary data w^n on $\tilde{D}^n_{i;k}$, where w^n is defined as follows: w^n equals 0 on $\tilde{\gamma}^n \cup R_i^n \cap \partial\tilde{D}^n_{i;k}$ and w^n equals 2λ on $\partial\tilde{\Omega}^n \cap \partial\tilde{D}^n_{i;k}$. According to [ALS20, Corollary 4.12], the limit of $\overline{\tilde{A}^{w^n}_0 \cap D_{i;k}}$ does not intersect $(y_{2i-2}, y_{2i-1}) \cup \partial R \cup (y_{2i}, y_{2i+1})$. This implies $\mathbb{P}[F_{i;k}] = 0$. It completes the proof. □

5.4 Asymptotics of partition functions and proof of Proposition 1.2

In this subsection, we will give the following asymptotics of pure partition functions: Proposition 5.6. The purpose of this proposition will be clear in the proof of Theorem 1.1.

Proposition 5.6. Fix $\kappa = 4$. Fix $N \geq 1$ and the index valences $\varsigma = (2, \ldots, 2)$ of length $2N$. For each $\wedge \in \mathrm{LP}_\varsigma$, let $\alpha := \alpha(\wedge) \in \mathrm{PP}_{2N}$ be the associated planar pair partition as defined in Section 2.2, and $\mathcal{Z}_\alpha$ be the pure partition function associated to α. Then, the following limit exists: for $y_1 < \cdots < y_{2N}$ and $x_1 < \cdots < x_{4N}$,

$$\mathcal{Z}_\wedge(y_1, \ldots, y_{2N}) := \lim_{\substack{x_{2j-1}, x_{2j} \to y_j, \\ \forall 1 \leq j \leq 2N}} \frac{\mathcal{Z}_\alpha(x_1, \ldots, x_{4N})}{\prod_{j=1}^{2N} (x_{2j} - x_{2j-1})^{1/2}}. \tag{5.3}$$

We will show Proposition 5.6 by the explicit expression for $\mathcal{Z}_\alpha$ from (3.12):

$$\mathcal{Z}_\alpha(x_1, \ldots, x_{4N}) = \sum_{\beta \in \mathrm{DP}_{2N}} \mathcal{M}^{-1}_{\alpha,\beta}\, \mathcal{U}_\beta(x_1, \ldots, x_{4N}).$$

For $\beta \in \mathrm{DP}_{2N}$ such that $\beta_{2j-1} \in 2\mathbb{Z}$ for all $1 \leq j \leq 2N$, it is easy to see that $\mathcal{U}_\beta$ admits a limit when normalized by $\prod_j (x_{2j} - x_{2j-1})^{1/2}$, see Lemma 5.7. However, for other β, the conformal block $\mathcal{U}_\beta$ explodes when normalized by $\prod_j (x_{2j} - x_{2j-1})^{1/2}$. In order to derive the existence of the limit, we need to group distinct β's properly so that the explosion cancels. The proof of Proposition 5.6 involves heavy notation which we find unavoidable. As the proof is lengthy and not instructive to include in the main text, we put it in Appendix B. We suggest readers to first read the proof of Corollary 5.10 where we give the proof for Proposition 5.6 when $N = 2$.

Lemma 5.7. Fix $\kappa = 4$. Fix $N \geq 1$. Given a Dyck path $\alpha \in \mathrm{DP}_{2N}$ of length $4N$ such that $\alpha_{2j-1} \in 2\mathbb{Z}$ for all $1 \leq j \leq 2N$, define $(\alpha)_2 \in \mathrm{DP}_N$ by

$$(\alpha)_2(k) = \frac{1}{2}\alpha(2k), \quad 1 \leq k \leq 2N.$$

One may check that this is a well-defined Dyck path of length $2N$. Then, for $y_1 < \cdots < y_{2N}$ and $x_1 < \cdots < x_{4N}$, we have

$$\lim_{\substack{x_{2j-1}, x_{2j} \to y_j, \\ \forall 1 \leq j \leq 2N}} \frac{\mathcal{U}_\alpha(x_1, \ldots, x_{4N})}{\prod_{j=1}^{2N} (x_{2j} - x_{2j-1})^{1/2}} = \mathcal{U}^4_{(\alpha)_2}(y_1, \ldots, y_{2N}). \tag{5.4}$$

Proof. By the definition (3.10), we have

$$\begin{aligned} &\frac{\mathcal{U}_\alpha(x_1, \ldots, x_{4N})}{\prod_{j=1}^{2N} (x_{2j} - x_{2j-1})^{1/2}} \\ &= \prod_{1 \leq s < t \leq 2N} \left((x_{2t} - x_{2s})(x_{2t} - x_{2s-1})(x_{2t-1} - x_{2s})(x_{2t-1} - x_{2s-1})\right)^{\frac{1}{2}\vartheta_\alpha(2s;2t)}. \end{aligned}$$

Thus

$$\lim_{\substack{x_{2j-1}, x_{2j} \to y_j, \\ \forall 1 \leq j \leq 2N}} \frac{\mathcal{U}_\alpha(x_1, \ldots, x_{4N})}{\prod_{j=1}^{2N} (x_{2j} - x_{2j-1})^{1/2}} = \prod_{1 \leq s < t \leq 2N} (y_t - y_s)^{2\vartheta_\alpha(2t;2s)} = \mathcal{U}^4_{(\alpha)_2}(y_1, \ldots, y_{2N}).$$

□

Assuming Proposition 5.6 holds, we will extend definition of $\mathcal{Z}_{\hat\alpha}$ via conformal covariance.

Corollary 5.8. Assume the same notations as in Proposition 5.6. The function $\mathcal{Z}_{\hat\alpha}$ satisfies the following conformal covariance: for all Möbius maps φ of $\mathbb{H}$ such that $\varphi(y_1) < \cdots < \varphi(y_{2N})$, we have

$$\mathcal{Z}_{\hat\alpha}(y_1, \ldots, y_{2N}) = \prod_{i=1}^{2N} \varphi'(y_i) \, \mathcal{Z}_{\hat\alpha}(\varphi(y_1), \ldots, \varphi(y_{2N})). \tag{5.5}$$

For general polygon $(\Omega; y_1, \ldots, y_{2N})$, we define

$$\mathcal{Z}_{\hat\alpha}(\Omega; y_1, \ldots, y_{2N}) := \prod_{i=1}^{2N} \varphi'(y_i) \, \mathcal{Z}_{\hat\alpha}(\varphi(y_1), \ldots, \varphi(y_{2N})), \tag{5.6}$$

where φ is any conformal map from Ω onto $\mathbb{H}$ such that $\varphi(y_1) < \cdots < \varphi(y_{2N})$.

Proof. The conformal covariance (5.5) is a consequence of (3.3) and the existence of the limit (5.3). From (5.5), we see that (5.6) is well-defined: suppose φ_1 and φ_2 are conformal maps on Ω with $\varphi_n(y_1) < \cdots < \varphi_n(y_{2N})$ for $n = 1, 2$. From (5.5), we have

$$\prod_{i=1}^{2N} \varphi_1'(y_i) \, \mathcal{Z}_{\hat\alpha}(\varphi_1(y_1), \ldots, \varphi_1(y_{2N})) = \prod_{i=1}^{2N} \varphi_2'(y_i) \, \mathcal{Z}_{\hat\alpha}(\varphi_2(y_1), \ldots, \varphi_2(y_{2N})). \qquad \square$$

Assuming Proposition 5.6 holds, we are able to complete the proof of (1.6) and (1.8).

Proof of (1.6). This is immediate from Corollary 5.8. $\square$

Proof of Proposition 1.2. We will show (1.8) with $j = 1$, and the other cases can be proved similarly. For $y_1 < \cdots < y_{2N}$, we denote $y_{kl} = y_k - y_l$ for $k \neq l$. For $x_1 < \cdots < x_{4N}$, we denote $x_{kl} = x_k - x_l$ for $k \neq l$.

Fix $N \geq 1$ and the index valences $\varsigma = (2, \ldots, 2)$ of length $2N$. Fix $\hat\alpha \in \mathrm{LP}_\varsigma$ and let $\alpha = \pi(\hat\alpha) \in \mathrm{PP}_{2N}$ be the associated planar pair partition as defined in Section 2.2. We set $F_0(x_1, \ldots, x_{4N}) = \mathcal{Z}_\alpha(x_1, \ldots, x_{4N})$ for $x_1 < \cdots < x_{4N}$. We define F_j by induction on j. Fix $j \in \{1, 2, \ldots, 2N\}$ and suppose F_{j-1} is defined. For $y_1 < \cdots < y_j < x_{2j+1} < \cdots < x_{4N}$ and $y_{j-1} < x_{2j-1} < x_{2j} < x_{2j+1}$, we define

$$F_j(y_1, \ldots, y_j, x_{2j+1}, \ldots, x_{4N}) := \lim_{x_{2j-1}, x_{2j} \to y_j} \frac{F_{j-1}(y_1, \ldots, y_{j-1}, x_{2j-1}, x_{2j}, \ldots, x_{4N})}{(x_{2j} - x_{2j-1})^{1/2}}.$$

From Proposition 5.6, we see that $F_1, \ldots, F_{2N}$ are well-defined and $F_{2N} = \mathcal{Z}_{\hat\alpha}$. We will show the following PDE by induction on $j \in \{1, 2, \ldots, 2N\}$:

$$\mathcal{D}_j F_j(y_1, \ldots, y_j, x_{2j+1}, \ldots, x_{4N}) = 0, \tag{5.7}$$

where

$$\begin{aligned}\mathcal{D}_j = \frac{\partial^3}{\partial y_1^3} &- 4\left(\sum_{2 \leq i \leq j} \left(\frac{1}{y_{i1}^2} - \frac{1}{y_{i1}} \frac{\partial}{\partial y_i} \right) + \sum_{2j+1 \leq i \leq 4N} \left(\frac{1/4}{(x_i - y_1)^2} - \frac{1}{(x_i - y_1)} \frac{\partial}{\partial x_i} \right) \right) \frac{\partial}{\partial y_1} \\ &+ 2\left(\sum_{2 \leq i \leq j} \left(\frac{2}{y_{i1}^3} - \frac{1}{y_{i1}^2} \frac{\partial}{\partial y_i} \right) + \sum_{2j+1 \leq i \leq 4N} \left(\frac{1/2}{(x_i - y_1)^3} - \frac{1}{(x_i - y_1)^2} \frac{\partial}{\partial x_i} \right) \right).\end{aligned}$$

When $j=1$, PDE (5.7) holds due to (3.8) with $j=1$. For $j\geq 2$, suppose (5.7) holds for $j-1$, and we will show it for j. Comparing the two operators $\mathcal{D}_{j-1}$ and $\mathcal{D}_j$, we denote their overlap by

$$\mathcal{O}_{j-1}=\frac{\partial^3}{\partial y_1^3}-4\left(\sum_{2\leq i\leq j-1}\left(\frac{1}{y_{i1}^2}-\frac{1}{y_{i1}}\frac{\partial}{\partial y_i}\right)+\sum_{2j+1\leq i\leq 4N}\left(\frac{1/4}{(x_i-y_1)^2}-\frac{1}{(x_i-y_1)}\frac{\partial}{\partial x_i}\right)\right)\frac{\partial}{\partial y_1}$$
$$+2\left(\sum_{2\leq i\leq j-1}\left(\frac{2}{y_{i1}^3}-\frac{1}{y_{i1}^2}\frac{\partial}{\partial y_i}\right)+\sum_{2j+1\leq i\leq 4N}\left(\frac{1/2}{(x_i-y_1)^3}-\frac{1}{(x_i-y_1)^2}\frac{\partial}{\partial x_i}\right)\right).$$

Then, we have

$$\begin{aligned}\mathcal{D}_{j-1}=\mathcal{O}_{j-1}&+\left(-\frac{1}{(x_{2j-1}-y_1)^2}+\frac{4}{(x_{2j-1}-y_1)}\frac{\partial}{\partial x_{2j-1}}\right)\frac{\partial}{\partial y_1}\\&+\left(-\frac{1}{(x_{2j}-y_1)^2}+\frac{4}{(x_{2j}-y_1)}\frac{\partial}{\partial x_{2j}}\right)\frac{\partial}{\partial y_1}\\&+\left(-\frac{1}{(x_{2j-1}-y_1)^3}+\frac{2}{(x_{2j-1}-y_1)^2}\frac{\partial}{\partial x_{2j-1}}\right)\\&+\left(-\frac{1}{(x_{2j}-y_1)^3}+\frac{2}{(x_{2j}-y_1)^2}\frac{\partial}{\partial x_{2j}}\right)\\ \mathcal{D}_j=\mathcal{O}_{j-1}&+\left(-\frac{4}{y_{j1}^2}+\frac{4}{y_{j1}}\frac{\partial}{\partial y_j}\right)\frac{\partial}{\partial y_1}+\left(-\frac{4}{y_{j1}^3}+\frac{2}{y_{j1}^2}\frac{\partial}{\partial y_j}\right).\end{aligned}$$

We set $G_{j-1}=(x_{2j}-x_{2j-1})^{-1/2}F_{j-1}$. From $\mathcal{D}_{j-1}F_{j-1}=0$, we have

$$\begin{aligned}0=\mathcal{O}_{j-1}G_{j-1}&+\left(-\frac{1}{(x_{2j-1}-y_1)^2}-\frac{1}{(x_{2j}-y_1)^2}+\frac{2}{(x_{2j-1}-y_1)(x_{2j}-y_1)}\right)\frac{\partial}{\partial y_1}G_{j-1}\\&+\left(-\frac{1}{(x_{2j-1}-y_1)^3}-\frac{1}{(x_{2j}-y_1)^3}+\frac{(x_{2j}-x_{2j-1})+2(x_{2j-1}-y_1)}{(x_{2j-1}-y_1)^2(x_{2j}-y_1)^2}\right)G_{j-1}\\&+\left(\frac{4}{(x_{2j-1}-y_1)}\frac{\partial}{\partial y_1}+\frac{2}{(x_{2j-1}-y_1)^2}\right)\frac{\partial}{\partial x_{2j-1}}G_{j-1}\\&+\left(\frac{4}{(x_{2j}-y_1)}\frac{\partial}{\partial y_1}+\frac{2}{(x_{2j}-y_1)^2}\right)\frac{\partial}{\partial x_{2j}}G_{j-1}.\end{aligned}\tag{5.8}$$

We will argue that

$$KG_{j-1}(y_1,\ldots,y_{j-1};x_{2j-1},\ldots,x_{4N})\to K F_j(y_1,\ldots,y_j;x_{2j+1},\ldots,x_{4N}),\quad \text{as } x_{2j-1},x_{2j}\to y_j,\tag{5.9}$$

where

$$K\in\left\{1,\frac{\partial}{\partial y_1},\frac{\partial^3}{\partial y_1^3}\right\}\cup\left\{\frac{\partial}{\partial y_i},\frac{\partial^2}{\partial y_i\partial y_1}:2\leq i\leq j-1\right\}\cup\left\{\frac{\partial}{\partial x_n},\frac{\partial^2}{\partial x_n\partial y_1}:2j+1\leq n\leq 4N\right\},$$

and that

$$\begin{aligned}&\left(\frac{1}{(x_{2j-1}-y_1)^2}\frac{\partial}{\partial x_{2j-1}}+\frac{1}{(x_{2j}-y_1)^2}\frac{\partial}{\partial x_{2j}}\right)G_{j-1}(y_1,\ldots,y_{j-1};x_{2j-1},\ldots,x_{4N})\\&\to\frac{1}{y_{j1}^2}\frac{\partial}{\partial y_j}F_j(y_1,\ldots,y_j;x_{2j+1},\ldots,x_{4N}),\quad\text{as } x_{2j-1},x_{2j}\to y_j,\end{aligned}\tag{5.10}$$

$$\begin{aligned}&\left(\frac{1}{(x_{2j-1}-y_1)}\frac{\partial}{\partial x_{2j-1}}+\frac{1}{(x_{2j}-y_1)}\frac{\partial}{\partial x_{2j}}\right)\frac{\partial}{\partial y_1}G_{j-1}(y_1,\ldots,y_{j-1};x_{2j-1},\ldots,x_{4N})\\&\to\frac{1}{y_{j1}}\frac{\partial^2}{\partial y_j\partial y_1}F_j(y_1,\ldots,y_j;x_{2j+1},\ldots,x_{4N}),\quad\text{as } x_{2j-1},x_{2j}\to y_j.\end{aligned}\tag{5.11}$$

From the proof of Proposition 5.6, Eq. (5.9) holds for $K=\infty$. Furthermore, as G_{j-1} is a finite linear combination of terms of the form

$$\prod (x_a-x_b)^{\pm 1/2}\prod (y_k-y_l)^{-2}\prod (x_n-y_m)^{-1};$$

we view G_{j-1} as a function on distinct complex variables $(y_1,\ldots,y_{j-1};x_{2j-1},x_{2j},\ldots,x_{4N})$. We fix arbitrarily distinct complex points $(y_1,\ldots,y_{j-1},x_{2j+1},\ldots,x_{4N})$ and denote $\boldsymbol{y}=(y_1,\ldots,y_{j-1})$ and $\boldsymbol{x}=(x_{2j+1},\ldots,x_{4N})$. Then G_{j-1} is a meromorphic function of $\epsilon = x_{2j}-x_{2j-1}$ and its Laurent series can be written as:

$$\begin{aligned} H_{j-1}(\boldsymbol{y};x_{2j-1};\epsilon;\boldsymbol{x}) &:= G_{j-1}(\boldsymbol{y};x_{2j-1},x_{2j-1}+\epsilon;\boldsymbol{x}) \\ &= F_j(\boldsymbol{y};x_{2j-1};\boldsymbol{x})+\sum_{n\geq 1}K_n(\boldsymbol{y};x_{2j-1};\boldsymbol{x})\epsilon^n; \end{aligned}$$

where K_n is a finite linear combination of terms of the form

$$\prod (x_a-x_b)^{p/2}\prod (y_k-y_l)^{-2}\prod (x_n-y_m)^{q}$$

with $p,q\in\mathbb{Z}$.

Now, we prove (5.10). We have

$$\begin{aligned} &\left(\frac{1}{(x_{2j-1}-y_1)^2}\frac{\partial}{\partial x_{2j-1}}+\frac{1}{(x_{2j}-y_1)^2}\frac{\partial}{\partial x_{2j}}\right)G_{j-1}(\boldsymbol{y};x_{2j-1},x_{2j};\boldsymbol{x}) \\ =&\left(\frac{1}{(x_{2j-1}-y_1)^2}\frac{\partial}{\partial x_{2j-1}}+\left(\frac{1}{(x_{2j}-y_1)^2}-\frac{1}{(x_{2j-1}-y_1)^2}\right)\frac{\partial}{\partial \epsilon}\right)H_{j-1}(\boldsymbol{y};x_{2j-1};\epsilon;\boldsymbol{x}). \end{aligned}$$

Thus, it suffices to prove, as $x_{2j-1}\to y_j$, $\epsilon\to 0$,

$$\frac{\partial}{\partial x_{2j-1}}H_{j-1}(\boldsymbol{y};x_{2j-1};\epsilon;\boldsymbol{x})\to\frac{\partial}{\partial y_j}F_j(\boldsymbol{y};y_j;\boldsymbol{x});\quad \frac{\partial}{\partial \epsilon}H_{j-1}(\boldsymbol{y};x_{2j-1};\epsilon;\boldsymbol{x})\to K_1(\boldsymbol{y};y_j;\boldsymbol{x}). \tag{5.12}$$

We define $\delta_0=\min\{\frac{x_{2j+1}-y_j}{4},\frac{y_j-y_{j-1}}{4}\}$ and $H_{j-1}(\boldsymbol{y};x_{2j-1};0;\boldsymbol{x}) := F_j(\boldsymbol{y};x_{2j-1};\boldsymbol{x})$. Then, the function $(x_{2j-1},\epsilon)\mapsto H_{j-1}(\boldsymbol{y};x_{2j-1};\epsilon;\boldsymbol{x})$ is continuous on $[y_j-\delta_0,y_j+\delta_0]\times B(0,\delta_0)$ where $B(0,\delta_0):=\{z\in\mathbb{C}: d(z,0)<\delta_0\}$. Moreover, for every $x_{2j-1}\in[y_j-\delta_0,y_j+\delta_0]$, the function $\epsilon\mapsto H_{j-1}(\boldsymbol{y};x_{2j-1};\epsilon;\boldsymbol{x})$ is holomorphic in $B(0,\delta_0)\setminus\{0\}$. Thus, the function $\epsilon\mapsto H_{j-1}(\boldsymbol{y};x_{2j-1};\epsilon;\boldsymbol{x})$ is holomorphic in $B(0,\delta_0)$. Then, we have

$$K_n(\boldsymbol{y};x_{2j-1};\boldsymbol{x})=\frac{1}{2\pi i}\oint_{\partial B(0,\delta_0)}\frac{H_{j-1}(\boldsymbol{y};x_{2j-1};z;\boldsymbol{x})}{z^{n+1}}dz.$$

Note that, there exists $M=M(\boldsymbol{y};\boldsymbol{x};y_j)>0$ such that, for all $x_{2j-1}\in[y_j-\delta_0,y_j+\delta_0]$ and $z\in B(0,2\delta_0)\setminus B(0,\frac{\delta_0}{2})$,

$$|H_{j-1}(\boldsymbol{y};x_{2j-1};z;\boldsymbol{x})|\leq M;\quad\text{and}\quad \left|\frac{\partial}{\partial x_{2j-1}}H_{j-1}(\boldsymbol{y};x_{2j-1};z;\boldsymbol{x})\right|\leq M.$$

Thus, we have

$$|K_n(\boldsymbol{y};x_{2j-1};\boldsymbol{x})|\leq\frac{M}{\delta_0^n};$$

and

$$\frac{\partial}{\partial x_{2j-1}}K_n(\boldsymbol{y};x_{2j-1};\boldsymbol{x}) = \frac{1}{2\pi i}\oint_{\partial B(0,\delta_0)}\frac{\frac{\partial}{\partial x_{2j-1}}H_{j-1}(\boldsymbol{y};x_{2j-1};z;\boldsymbol{x})}{z^{n+1}}dz \leq \frac{M}{\delta_0^n}.$$

These imply that, for every $x_{2j-1} \in [y_j - \delta_0; y_j + \delta_0]$ and $\in B(0; \frac{\delta_0}{2})$,

$$\frac{\partial}{\partial x_{2j-1}} H_{j-1}(y; x_{2j-1}; ; x) = \frac{\partial}{\partial x_{2j-1}} F_j(y; x_{2j-1}; x) + \sum_{n \geq 1} \frac{\partial}{\partial x_{2j-1}} K_n(y; x_{2j-1}; x)^{n};$$

and

$$\frac{\partial}{\partial} H_{j-1}(y; x_{2j-1}; ; x_{2j+1}, \ldots; x_{4N}) = \sum_{n \geq 1} n K_n(y; x_{2j-1}; x)^{n-1}:$$

These give (5.12), and complete the proof of (5.10). Eq. (5.9) and (5.11) can be proved in a similar way.

Plugging (5.9)-(5.11) into (5.8), and letting $x_{2j-1}; x_{2j} \to y_j$, we obtain $D_j F_j = 0$. This completes the proof of (5.7). Taking $j = 2N$ in (5.7), we obtain the third order PDE (1.8) as desired. This completes the proof. □

5.5 Proof of Theorem 1.1

In this section, we complete the proof of Theorem 1.1. Before that, we first address the coefficient $M_{\omega;(\hat{})}$ in the theorem.

Lemma 5.9. Fix $N \geq 1$ and the index valences $\varsigma = (2; \ldots; 2)$ of length $2N$. Define $\omega \in DP_{2N}$ to be: $\forall j \in \{0; 1; \ldots; N-1\}$,

$$\omega(4j) = 0; \quad \omega(4j+1) = 1; \quad \omega(4j+2) = 2; \quad \omega(4j+3) = 1; \quad \omega(4j+4) = 0:$$

For any $\hat{} \in LP_\varsigma$, let $(\hat{}) \in PP_{2N}$ be the associated planar pair partition as defined in Section 2.2. Recall the definition of the incidence matrix M from (3.11). Then

$$M_{\omega;} = 1 \quad \text{implies} \quad = (\hat{}) \text{ for some } \hat{} \in LP_\varsigma: \tag{5.13}$$

However, the converse does not hold in general.

Proof. Recall from Section 2.2 that introduces a bijection between LP_ς and the collection $\{ \in PP_{2N} : \hat{}_{2j-1} \notin ; \text{ for all } 1 \leq j \leq 2N\}$. Thus, it suffices to prove that $\hat{}_{2j-1} \notin$ for every $1 \leq j \leq 2N$. By definition,

$$\omega = \{\{4j+1; 4j+4\}; \{4j+2; 4j+3\} : 1 \leq j \leq N-1\}:$$

Note that $M_{\omega;} = 1$ implies there exists a which is a permutation of $\{4j+3; 4j+4 : 0 \leq j \leq N-1\}$ such that

$$= \{\{4j+1; (4j+4)\}; \{4j+2; (4j+3)\} : 0 \leq j \leq N-1\}:$$

This implies $\hat{}_{2j-1} \notin$ for every $1 \leq j \leq 2N$. □

Proof of Theorem 1.1 and (1.7). We use the same notations as in Section 5.3. By conformal invariance, we may assume $= \mathbb{H}$ and $y_1 < \cdots < y_{2N}$. Suppose is zero-boundary GFF on $\mathbb{H}$ and let u be the harmonic function with the boundary data (1.5). From Proposition 5.3, we have

$$\lim_{\to 0} P[A = \hat{}] = P[A = \hat{}]:$$

Let $_{2j-1}$ be the level line of the continuum GFF $+ u$ starting from y_{2j-1} with height for $1 \leq j \leq N$; and let $_{2j}$ be the level line of $(+ u)$ starting from y_{2j} with height for $1 \leq j \leq N$. Note that the collection $\{_2; _4; \ldots; _{2N}\}$ coincides with the collection of level lines $+ u$ starting from y_{2j-1} with height for $1 \leq j \leq N$. See Figure 8. From

Figure 8: Consider continuum GFF $+ u$ in rectangle with alternating boundary data. In the left panel, we have four level lines: Let $_1$ (resp. $_3$) be the level line of $+ u$ starting from y_1 (resp. from y_3) with height . These two curves are in black.Let $_2$ (resp. $_4$) be the level line of $(+ u)$ starting from y_2 (resp. from y_4) with height . These two curves are in red. The middle panel indicates the domain H which is obtained by removing from H the four pieces $_i[0; T_i]$ with $i = 1; 2; 3; 4$. In the right panel, we see that the boundary data of $+ u$ in H is piecewise constant: $0; 2; 0; 2; 0; 2; 0; 2$.

Lemma 5.4, the frontier of A and of $\forall$ has the same law as $[_{1 \ j \ 2N} \ _j$. It suf ces to prove

$$P[f\ _1; \ldots; \ _{2N}\ g \text{ forms the planar link pattern } \ \hat{}\ ] = M_{!; \ (\hat{})} \frac{Z_{\hat{}}(y_1; \ldots; y_{2N})}{Z^{(N)}_{mGFF}(y_1; \ldots; y_{2N})};$$

where $!$ and $M_{!; \ (\hat{})}$ are de ned in Lemma 5.9.

For $1 \ j \ 2N$ and > 0 small, we denote $T_j = \inf f\, t > 0 : d(\ _j(t); y_j) = \ g$. We take to be the conformal map from

$$H := H \, n \, [_{1 \ j \ 2N} \ _j[0; T_j]$$

onto H normalized at 1. Then, we see that, given H, the event

$$ff\ _1; \ldots; \ _{2N}\ g \text{ forms the planar link pattern } \ \hat{}\ g$$

is the same as

$$ff\ (\ _1); \ldots; \ (\ _{2N})g \text{ forms the planar link pattern } \ (\hat{})g$$

where $(\hat{})$ is de ned in Section 2.2.

Now, let us consider the collection $f\ (\ _1); \ldots; \ (\ _{2N})g$. The conditional law of $+ u$ given H is a GFF in H with the following boundary data: for $1 \ j \ 2N$,

$$\begin{aligned} &2 \ \text{ on } (y^+_{2j-1}; y_{2j}); \quad 0 \text{ along the left side of } \ _{2j}[0; T_{2j}]; \\ &2 \ \text{ along the right side of } \ _{2j}[0; T_{2j}]; \\ &2 \ \text{ on } (y^+_{2j}; y_{2j+1}); \quad 0 \text{ along the left side of } \ _{2j+1}[0; T_{2j+1}]; \\ &2 \ \text{ along the right side of } \ _{2j+1}[0; T_{2j+1}]: \end{aligned}$$

See Figure 8. Then, we have

$$\begin{aligned} &P[f\ _1; \ldots; \ _{2N}\ g \text{ forms the planar link pattern } \ \hat{}\ ] \\ &= E\ P\ f\ _1; \ldots; \ _{2N}\ g \text{ forms the planar link pattern } \ \hat{} \ H \\ &= E\ P\ f\ (\ _1); \ldots; \ (\ _{2N})g \text{ forms the planar link pattern } \ (\hat{})\ H \\ &= E\ M_{!; \ (\hat{})} \frac{Z_{(\hat{})}((y_1); (\ _1(T_1)); (y_2); (\ _2(T_2)); \ldots; (y^+_{2N}); (\ _{2N}(T_{2N})))}{U_!((y_1); (\ _1(T_1)); (y_2); (\ _2(T_2)); \ldots; (y^+_{2N}); (\ _{2N}(T_{2N})))} \\ &= M_{!; \ (\hat{})} \frac{Z_{\hat{}}(y_1; \ldots; y_{2N})}{U^4_{(!)_2}(y_1; \ldots; y_{2N})}; \end{aligned}$$

where $(!\ )_2$ is defined as in Lemma 5.7. In the second last equal sign, we use Theorem 4.1: consider the GFF in H, the collection $f\ (\ _2);\ (\ _4); : : : ;\ (\ _{2N})g$ coincides with the collection of level lines starting from $y_{2j\ 1}$ with height . Therefore, the connection probability is given by $M_{!;\ (^)} Z_{(^)} = U_!$. In the last equal sign, we let $!\ 0$. Combining Proposition 5.6, Lemma 5.7, and dominated convergence theorem, we obtain the conclusion.

Finally, from the above analysis, we have

$$\lim_{!\ 0} P[A\ = ^] = P[A = ^] = M_{!;\ (^)} \frac{Z_{^}(y_1; : : : ; y_{2N})}{U^4_{(!\ )_2}(y_1; : : : ; y_{2N})}; \quad \text{for all } ^ 2 LP_{\&}:$$

Furthermore, from (3.12) and (5.13), we have

$$\sum_{^ 2 LP_{\&}} M_{!;\ (^)} \frac{Z_{^}(y_1; : : : ; y_{2N})}{U^4_{(!\ )_2}(y_1; : : : ; y_{2N})} = 1:$$

Thus

$$Z^{(N)}_{mGFF}(y_1; : : : ; y_{2N}) := \sum_{^ 2 LP_{\&}} M_{!;\ (^)} Z_{^}(y_1; : : : ; y_{2N}) = U^4_{(!\ )_2}(y_1; : : : ; y_{2N}):$$

This completes the proof of (1.7). □

Figure 9: There are six Dyck paths in this figure: in the first row, from left to right, we denote them by $_1; _2; _3$ respectively; in the second row, from left to right, we denote them by $_4; _5; _6$ respectively. We see that $_1\ \ _2\ \ _3; _4\ \ _5\ \ _6$.

Corollary 5.10. The conclusion in (1.3) holds.

Proof. We define $_1; _2; _3$ as in Figure 5 and we define $_1; : : : ; _6$ as in Figure 9. From (3.12), we have

$$\begin{aligned} Z_{\ _1} &= U_{\ _2}\ \ U_{\ _3}\ \ U_{\ _4} + U_{\ _5}\ \ 2U_{\ _6}; \\ Z_{\ _2} &= U_{\ _6}; \\ Z_{\ _3} &= U_{\ _1}\ \ U_{\ _2} + U_{\ _3} + U_{\ _4}\ \ U_{\ _5} + U_{\ _6}: \end{aligned} \tag{5.14}$$

Suppose $y_1 < y_2 < y_3 < y_4$ and we need to derive the limits as

$$x_1; x_2 ! \ y_1; \quad x_3; x_4 ! \ y_2; \quad x_5; x_6 ! \ y_3; \quad x_7; x_8 ! \ y_4:$$

We denote $x_{ji} = x_j\ \ x_i$ for $1\ \ i < j\ \ 8$ and $y_{ji} = y_j\ \ y_i$ for $1\ \ i < j\ \ 4$. Furthermore, we denote the cross-ratio by $q = (y_{21}y_{43})=(y_{31}y_{42})$.

First, for $_1$ and $_6$, we have

$$\lim_{\substack{x_1,x_2\to y_1,\ x_3,x_4\to y_2,\\ x_5,x_6\to y_3,\ x_7,x_8\to y_4}} \frac{U_1(x_1;\dots;x_8)}{\sqrt{x_{21}x_{43}x_{65}x_{87}}} = \left(\frac{y_{31}y_{42}}{y_{21}y_{41}y_{32}y_{43}}\right)^2 = \frac{1}{q^2(1-q)^2y_{31}^2y_{42}^2}; \tag{5.15}$$

$$\lim_{\substack{x_1,x_2\to y_1,\ x_3,x_4\to y_2,\\ x_5,x_6\to y_3,\ x_7,x_8\to y_4}} \frac{U_6(x_1;\dots;x_8)}{\sqrt{x_{21}x_{43}x_{65}x_{87}}} = \left(\frac{y_{21}y_{43}}{y_{31}y_{41}y_{32}y_{42}}\right)^2 = \frac{q^2}{(1-q)^2y_{31}^2y_{42}^2}. \tag{5.16}$$

Second, for $n=2;3;4;5$, we have

$$\lim_{\substack{x_1,x_2\to y_1,\\ x_7,x_8\to y_4}} \frac{U_n(x_1;\dots;x_8)}{\sqrt{x_{21}x_{87}}} = y_{41}^{2}\prod_{3\le i\le 6}(x_i-y_1)^{\#_n(i;1)}(y_4-x_i)^{\#_n(i;7)}\prod_{3\le i<j\le 6}x_{ji}^{\frac{1}{2}\#_n(i;j)}.$$

Taking the difference between U_2 and U_3 and the difference between U_4 and U_5, we have

$$\begin{aligned}
&\lim_{\substack{x_1,x_2\to y_1,\ x_3,x_4\to y_2,\\ x_7,x_8\to y_4}} \frac{(U_2-U_3)(x_1;\dots;x_8)}{\sqrt{x_{21}x_{43}x_{87}}}\\
&= y_{41}^{2}\left(\frac{(x_6-y_1)(y_4-x_5)}{(x_5-y_1)(y_4-x_6)}\frac{\sqrt{x_{65}}}{(x_5-y_2)(x_6-y_2)}+\frac{2y_{41}}{y_{21}y_{42}\sqrt{x_{65}}}\right);\\
&\lim_{\substack{x_1,x_2\to y_1,\ x_3,x_4\to y_2,\\ x_7,x_8\to y_4}} \frac{(U_4-U_5)(x_1;\dots;x_8)}{\sqrt{x_{21}x_{43}x_{87}}}\\
&= y_{41}^{2}\left(\frac{(x_5-y_1)(y_4-x_6)}{(x_6-y_1)(y_4-x_5)}\frac{\sqrt{x_{65}}}{(x_5-y_2)(x_6-y_2)}+\frac{2y_{41}}{y_{21}y_{42}\sqrt{x_{65}}}\right).
\end{aligned}$$

Taking the difference between these two, we have

$$\begin{aligned}
&\lim_{\substack{x_1,x_2\to y_1,\ x_3,x_4\to y_2,\\ x_5,x_6\to y_3,\ x_7,x_8\to y_4}} \frac{(U_2-U_3-U_4+U_5)(x_1;\dots;x_8)}{\sqrt{x_{21}x_{43}x_{65}x_{87}}}\\
&= \frac{2}{y_{41}^2y_{32}^2}+\frac{4}{y_{21}y_{31}y_{42}y_{43}} = \left(\frac{2}{(1-q)^2}+\frac{4}{q}\right)\frac{1}{y_{31}^2y_{42}^2}.
\end{aligned} \tag{5.17}$$

Plugging (5.15), (5.16) and (5.17) into (5.14), we have

$$\begin{aligned}
&\lim_{\substack{x_1,x_2\to y_1,\ x_3,x_4\to y_2,\\ x_5,x_6\to y_3,\ x_7,x_8\to y_4}} \frac{Z_1(x_1;\dots;x_8)}{U_1(x_1;\dots;x_8)} = 2q(1-q)(2-q+q^2);\\
&\lim_{\substack{x_1,x_2\to y_1,\ x_3,x_4\to y_2,\\ x_5,x_6\to y_3,\ x_7,x_8\to y_4}} \frac{Z_2(x_1;\dots;x_8)}{U_1(x_1;\dots;x_8)} = q^4;\\
&\lim_{\substack{x_1,x_2\to y_1,\ x_3,x_4\to y_2,\\ x_5,x_6\to y_3,\ x_7,x_8\to y_4}} \frac{Z_3(x_1;\dots;x_8)}{U_1(x_1;\dots;x_8)} = (1-q)^4.
\end{aligned}$$

The scaling limit of the crossing probability in (1.3) corresponds to the limit of Z_2/U_1, see Figure 1 and Figure 5. This completes the proof. □

A Technical lemmas

The following three lemmas are technical. Lemmas A.1 and A.2 are needed in the proof of Lemma A.3 which is essential in the proof of Theorem 4.1.

Lemma A.1. Let $x_1<x_2<x_3<x_4$. Suppose is a continuous simple curve in $\mathbb{H}$ starting from x_1 and terminating at x_4 at time T. Assume hits $\mathbb{R}$ only at its two end points. Let $(W_t;0\le t\le T)$ be its driving function and $(g_t;0\le t\le T)$ be the corresponding family of conformal maps. Then

$$\lim_{t\to T}\frac{(g_t(x_3)-g_t(x_2))(g_t(x_4)-W_t)}{(g_t(x_3)-W_t)(g_t(x_4)-g_t(x_2))}=0.$$

Proof. See [PW19, Lemma B.2]. □

Lemma A.2. Let $x_0 < x_1 < x_2 < x_3 < x_4$. Suppose η is a continuous simple curve in $\mathbb{H}$ starting from x_0 and terminating at x_4 at time T. Assume η hits $\mathbb{R}$ only at its two end points. Let $(W_t; 0 \le t \le T)$ be its driving function and $(g_t; 0 \le t \le T)$ be the corresponding family of conformal maps. Then there exist $C_1, C_2 > 0$, which depend on $\eta[0,T]$, such that for all $t \in [0,T]$,

$$C_1 \le \frac{(g_t(x_2) - g_t(x_1))(g_t(x_3) - W_t)}{(g_t(x_2) - W_t)(g_t(x_3) - g_t(x_1))} \le C_2.$$

Proof. To prove the conclusion, we will show the following two estimates: First, we will show that there exist $C_1, C_2 > 0$, which only depend on $\eta[0,T]$, such that for all $t \in [0,T]$,

$$C_1 \le \frac{g_t(x_2) - g_t(x_1)}{g_t(x_3) - g_t(x_1)} \le C_2. \tag{A.1}$$

Second, we will show

$$\lim_{t \to T} \frac{g_t(x_3) - W_t}{g_t(x_2) - W_t} = 1 + \lim_{t \to T} \frac{g_t(x_3) - g_t(x_2)}{g_t(x_2) - W_t} = 1. \tag{A.2}$$

In this proof, we use $\asymp$ to simplify notations: for two functions f and g, the notation $f \asymp g$ means that there exists a constant $C > 0$ which only depends on $\eta[0,T]$ such that $C^{-1} \le f/g \le C$.

We first show (A.1). Note that for an interval $[a,b]$, we have

$$b - a = \lim_{y \to \infty} \pi y P^{iy}[\text{BM hits } \partial\mathbb{H} \text{ in } [a,b]],$$

where BM is the Brownian motion starts from iy. By conformal invariance of the Brownian motion, we have

$$b - a = \lim_{y \to \infty} \pi y P^{g_t^{-1}(iy)}\left[\text{BM hits } \partial(\mathbb{H} \setminus \eta[0,t]) \text{ in } g_t^{-1}([a,b])\right].$$

We choose ϵ_0 small enough, such that the ϵ_0-neighborhood of the interval $[x_1, x_3]$ does not intersect $\eta[0,T]$. We denote the boundary of this neighborhood in $\mathbb{H}$ by γ, this is a simple curve. For the Brownian motion starting from $g_t^{-1}(iy)$, let τ be the first time the Brownian motion hits γ. Consider the connected component V of $\mathbb{H} \setminus \eta[0,T]$ which contains x_1 on its boundary and choose a point $z \in V$. Suppose U is the unit disk, and $\phi_t : \mathbb{H} \setminus \eta[0,t] \to U$ is the conformal map with $\phi_t(z) = 0$, $\phi_t'(z) > 0$. Suppose $\phi_T : V \to U$ is the conformal map with the same normalization. Then, for any compact set $K \subset \overline{V}$ which does not intersect $\eta[0,T]$, the conformal map ϕ_t converges to ϕ_T uniformly on K.

Note that

$$\begin{aligned} & P^{g_t^{-1}(iy)}[\text{BM hits } \partial(\mathbb{H} \setminus \eta[0,t]) \text{ in } [x_1, x_2]] \\ &= P^{g_t^{-1}(iy)}\left[1_{\{\tau < \infty\}} P^{B_\tau}[\text{BM hits } \partial(\mathbb{H} \setminus \eta[0,t]) \text{ in } [x_1, x_2]]\right]. \end{aligned}$$

We will compare

$$P^{B_\tau}[\text{BM hits } \partial(\mathbb{H} \setminus \eta[0,t]) \text{ in } [x_1, x_2]] \quad \text{and} \quad P^{B_\tau}[\text{BM hits } \partial(\mathbb{H} \setminus \eta[0,t]) \text{ in } [x_1, x_3]].$$

In fact we can replace B_τ by a deterministic point on γ. For every $w \in \gamma$, we have

$$P^w[\text{BM hits } \partial(\mathbb{H} \setminus \eta[0,t]) \text{ in } [x_1, x_2]] = P^{\phi_t(w)}[\text{BM hits } \partial U \text{ in } [\phi_t(x_1), \phi_t(x_2)]];$$

where $[\varphi_t(x_1), \varphi_t(x_2)]$ is the conformal image of $[x_1, x_2]$. By direct computation, the right hand-side equals

$$\frac{1}{2\pi}\left(\arg\frac{\varphi_t(x_2)-\varphi_t(w)}{1-\overline{\varphi_t(w)}\varphi_t(x_2)}-\arg\frac{\varphi_t(x_1)-\varphi_t(w)}{1-\overline{\varphi_t(w)}\varphi_t(x_1)}\right);$$

where $\arg$ is the argument principal which takes value in $[0, 2\pi)$. Note that there exists $\varepsilon_0 > 0$ such that

$$\frac{1}{2\pi}\left(\arg\frac{\varphi_t(x_2)-\varphi_t(w)}{1-\overline{\varphi_t(w)}\varphi_t(x_2)}-\arg\frac{\varphi_t(x_1)-\varphi_t(w)}{1-\overline{\varphi_t(w)}\varphi_t(x_1)}\right)\le 1-\varepsilon_0;$$

because γ is bounded away from $[x_1, x_3]$. Thus,

$$\begin{aligned}\mathbb{P}^{\varphi_t(w)}[\text{BM hits } \partial\mathbb{U} \text{ in } [\varphi_t(x_1), \varphi_t(x_2)]] &\asymp \left|\frac{\varphi_t(x_2)-\varphi_t(w)}{1-\overline{\varphi_t(w)}\varphi_t(x_2)}-\frac{\varphi_t(x_1)-\varphi_t(w)}{1-\overline{\varphi_t(w)}\varphi_t(x_1)}\right| \\ &= \frac{(1-|\varphi_t(w)|^2)|\varphi_t(x_2)-\varphi_t(x_1)|}{|1-\overline{\varphi_t(w)}\varphi_t(x_2)||1-\overline{\varphi_t(w)}\varphi_t(x_1)|}.\end{aligned}$$

Similarly, we have

$$\mathbb{P}^{\varphi_t(w)}[\text{BM hits } \partial\mathbb{U} \text{ in } [\varphi_t(x_1), \varphi_t(x_3)]] \asymp \frac{(1-|\varphi_t(w)|^2)|\varphi_t(x_3)-\varphi_t(x_1)|}{|1-\overline{\varphi_t(w)}\varphi_t(x_3)||1-\overline{\varphi_t(w)}\varphi_t(x_1)|}.$$

Therefore,

$$\frac{\mathbb{P}^{\varphi_t(w)}[\text{BM hits } \partial\mathbb{U} \text{ in } [\varphi_t(x_1), \varphi_t(x_2)]]}{\mathbb{P}^{\varphi_t(w)}[\text{BM hits } \partial\mathbb{U} \text{ in } [\varphi_t(x_1), \varphi_t(x_3)]]} \asymp \frac{|\varphi_t(x_2)-\varphi_t(x_1)|}{|\varphi_t(x_3)-\varphi_t(x_1)|}\frac{|\varphi_t(x_3)-\varphi_t(w)|}{|\varphi_t(x_2)-\varphi_t(w)|} \asymp 1.$$

The last $\asymp$ is because of the uniform convergence of φ_t. Thus, we have

$$\mathbb{P}^{B_\tau}[\text{BM hits } \partial(\mathbb{H}\setminus\gamma[0,t]) \text{ in } [x_1, x_2]] \asymp \mathbb{P}^{B_\tau}[\text{BM hits } \partial(\mathbb{H}\setminus\gamma[0,t]) \text{ in } [x_1, x_3]].$$

This implies (A.1).

Next, we show (A.2). Consider the Brownian motion starting from $g_t^{-1}(iy)$. We define $C(x_4, \varepsilon) := \{z \in \mathbb{H} : d(z, x_4) = \varepsilon\}$. Let τ be the first time that it hits the connected component of half circle $C(x_4, \varepsilon)\cap\mathbb{H}\setminus\gamma[0,t]$ which contains x_4 on its boundary and we denote this connected component by C_ε. Then we have

$$\begin{aligned}&\mathbb{P}^{g_t^{-1}(iy)}[\text{BM hits } \partial(\mathbb{H}\setminus\gamma[0,t]) \text{ in the right side of } \gamma[0,t]\cup[x_0, x_2]] \\ &\ge \mathbb{P}^{g_t^{-1}(iy)}\left(1_{\{\tau<1\}}\mathbb{P}^{B_\tau}[\text{BM hits } \partial(\mathbb{H}\setminus\gamma[0,t]) \text{ in the right side of } \gamma[0,t]\cup[x_0, x_2]]\right);\end{aligned}$$

and

$$\begin{aligned}&\mathbb{P}^{g_t^{-1}(iy)}[\text{BM hits } \partial(\mathbb{H}\setminus\gamma[0,t]) \text{ in } [x_2, x_3]] \\ &= \mathbb{P}^{g_t^{-1}(iy)}\left(1_{\{\tau<1\}}\mathbb{P}^{B_\tau}[\text{BM hits } \partial(\mathbb{H}\setminus\gamma[0,t]) \text{ in } [x_2, x_3]]\right).\end{aligned}$$

By conformal invariance of the Brownian motion, we have

$$\begin{aligned}&\mathbb{P}^{B_\tau}[\text{BM hits } \partial(\mathbb{H}\setminus\gamma[0,t]) \text{ in the right side of } \gamma[0,t]\cup[x_0, x_2]] \\ &= \mathbb{P}^{\varphi_t(B_\tau)}[\text{BM hits } \partial\mathbb{U} \text{ in } [\varphi_t(\gamma(t)), \varphi_t(x_2)]];\end{aligned}$$

where $[\phi_t(\gamma(t)), \phi_t(x_2)]$ is the conformal image of the right side of $\gamma[0,t] \cup [x_0, x_2]$. Moreover

$$\begin{aligned} & P^{B}[\text{BM hits } \partial(\mathbb{H} \setminus \gamma[0,t]) \text{ in } [x_2, x_3]] \\ = & P^{\phi_t(B)}[\text{BM hits } \partial\mathbb{U} \text{ in } [\phi_t(x_2), \phi_t(x_3)]], \end{aligned}$$

where $[\phi_t(x_2), \phi_t(x_3)]$ is the conformal image of $[x_2, x_3]$. We replace B by a deterministic point on C for the same reason as in the proof of (A.1) . For every $w \in C$, by Beurling estimate and conformal invariance, there exists $C > 0$ such that

$$P^{\phi_t(w)}[\text{BM hits } \partial\mathbb{U} \text{ in } [\phi_t(x_2), \phi_t(x_3)]] \geq C\left(\frac{\epsilon}{x_4 - x_3}\right)^{\frac{1}{2}}.$$

This implies that there exists $\epsilon_0 > 0$ such that

$$P^{\phi_t(w)}[\text{BM hits } \partial\mathbb{U} \text{ in } [\phi_t(x_2), \phi_t(x_3)]] \leq 1 - \epsilon_0.$$

Thus, by the same method as in the proof of (A.1), we have

$$P^{\phi_t(w)}[\text{BM hits } \partial\mathbb{U} \text{ in } [\phi_t(x_2), \phi_t(x_3)]] \gtrsim \frac{(1 - |\phi_t(w)|^2)|\phi_t(x_3) - \phi_t(x_2)|}{|1 - \overline{\phi_t(w)}\phi_t(x_3)||1 - \overline{\phi_t(w)}\phi_t(x_2)|}.$$

Moreover,

$$\begin{aligned} & P^{\phi_t(w)}[\text{BM hits } \partial\mathbb{U} \text{ in } [\phi_t(\gamma(t)), \phi_t(x_2)]] \\ = & \frac{1}{2\pi}\left(\arg\frac{\phi_t(x_2) - \phi_t(w)}{1 - \overline{\phi_t(w)}\phi_t(x_2)} - \arg\frac{\phi_t(\gamma(t)) - \phi_t(w)}{1 - \overline{\phi_t(w)}\phi_t(\gamma(t))}\right) \\ \leq & \frac{1}{2}\left|\frac{\phi_t(x_2) - \phi_t(w)}{1 - \overline{\phi_t(w)}\phi_t(x_2)} - \frac{\phi_t(\gamma(t)) - \phi_t(w)}{1 - \overline{\phi_t(w)}\phi_t(\gamma(t))}\right| \\ = & \frac{1}{2}\frac{(1 - |\phi_t(w)|^2)|\phi_t(\gamma(t)) - \phi_t(x_2)|}{|1 - \overline{\phi_t(w)}\phi_t(\gamma(t))||1 - \overline{\phi_t(w)}\phi_t(x_2)|}. \end{aligned}$$

Combining these two together, there exists $C > 0$ such that

$$\frac{P^{\phi_t(w)}[\text{BM hits } \partial\mathbb{U} \text{ in } [\phi_t(\gamma(t)), \phi_t(x_2)]]}{P^{\phi_t(w)}[\text{BM hits } \partial\mathbb{U} \text{ in } [\phi_t(x_2), \phi_t(x_3)]]} \leq C\frac{|\phi_t(\gamma(t)) - \phi_t(x_2)|}{|\phi_t(x_3) - \phi_t(x_2)|}\frac{|\phi_t(x_3) - \phi_t(w)|}{|\phi_t(\gamma(t)) - \phi_t(w)|}.$$

We denote the connected component of $\mathbb{H} \setminus (\gamma[0,t] \cup C)$ which contains ∞ by A. By the relation between diameter and harmonic measure, there exists $C_1 > 0$, such that

$$\begin{aligned} \frac{1}{C_1}\operatorname{diam}(\phi_t(A)) \geq P^{0}[\text{BM hits } \phi_t(C) \text{ before } \partial\mathbb{U}] & = P^{z}[\text{BM hits } C \text{ before } \partial(\mathbb{H} \setminus \gamma[0,t])] \\ & \geq P^{z}[\text{BM hits } C \text{ before } \partial\mathbb{H}] \\ & \geq C_1. \end{aligned}$$

Thus, we have

$$|\phi_t(\gamma(t)) - \phi_t(w)| \geq C_1^2.$$

For $|\phi_t(\gamma(t)) - \phi_t(x_2)|$ and $|\phi_t(x_3) - \phi_t(w)|$, we have

$$\begin{aligned} |\phi_t(\gamma(t)) - \phi_t(x_2)| & \leq \min\left\{\left|\phi_t\left(\frac{x_4 + x_3}{2}\right) - \phi_t(x_2)\right|, \left|\phi_t\left(\frac{x_1 + x_0}{2}\right) - \phi_t(x_2)\right|\right\}; \\ |\phi_t(w) - \phi_t(x_3)| & \geq |\phi_t(\gamma(t)) - \phi_t(x_3)| - |\phi_t(\gamma(t)) - \phi_t(w)| \\ & \geq \min\left\{\left|\phi_t\left(\frac{x_4 + x_3}{2}\right) - \phi_t(x_3)\right|, \left|\phi_t\left(\frac{x_1 + x_0}{2}\right) - \phi_t(x_3)\right|\right\} - C_1^2. \end{aligned}$$

Thus, by the uniform convergence, there exists $C_2>0$ such that

$$\frac{|\phi_t(\eta(t))-\phi_t(x_2)||\phi_t(x_3)-\phi_t(w)|}{|\phi_t(x_3)-\phi_t(x_2)|}\le C_2.$$

This implies that there exists $C>0$ such that

$$\frac{\mathbb{P}^{\phi_t(w)}[\text{BM hits } \partial U \text{ in } [\phi_t(\eta(t)),\phi_t(x_2)]]}{\mathbb{P}^{\phi_t(w)}[\text{BM hits } \partial U \text{ in } [\phi_t(x_2),\phi_t(x_3)]]}\ge C^{-1}.$$

Therefore, we have (A.2). Combining (A.1) and (A.2), we obtain the conclusion. □

We set $B_{\emptyset}=1$, and for $\alpha,\beta\in\mathrm{PP}_N$ and $x_1<\cdots<x_{2N}$, we define

$$B_\alpha(x_1,\dots,x_{2N}):=\prod_{\{a,b\}\in\alpha}|x_a-x_b|^{-1};\quad F_{\alpha,\beta}(x_1,\dots,x_{2N}):=\frac{B_\alpha(x_1,\dots,x_{2N})}{U_\beta(x_1,\dots,x_{2N})^2}.$$

Lemma A.3. Fix $\alpha,\beta\in\mathrm{PP}_N$ such that $\alpha\overset{()}{\leftarrow}\beta$. Fix $j\in\{1,2,\dots,2N-1\}$, we assume that $\wedge_j\in\alpha$ and $\wedge_j\in\beta$. Fix $n\in\{1,\dots,j-1,j+2,\dots,2N\}$ such that $\beta(n-1)=\alpha(j)$ and $\beta(n)=\alpha(j-1)$. Fix $x_1<\cdots<x_{2N}$. Suppose η is a continuous simple curve in $\mathbb{H}$ starting from x_j and terminating at x_n at time T. Assume η hits $\mathbb{R}$ only at its two end points. Let $(W_t;0\le t\le T)$ be its driving function and $(g_t;0\le t\le T)$ be the corresponding family of conformal maps. Then

$$\lim_{t\to T}F_{\alpha,\beta}(g_t(x_1),\dots,g_t(x_{j-1}),W_t,g_t(x_{j+1}),\dots,g_t(x_{2N}))=0.$$

Proof. We may assume $j+1<n$. The other case can be proved similarly. By definition, we have

$$F_{\alpha,\beta}(x_1,\dots,x_{2N})=\prod_{\substack{1\le i\le 2N\\ i\ne j,j+1}}\left(\frac{x_i-x_{j+1}}{x_i-x_j}\right)^{\vartheta^{\#}(i;j)}F_{\alpha/\wedge_j,\beta/\wedge_j}(x_1,\dots,x_{j-1},x_{j+2},\dots,x_{2N}).$$

To get the conclusion, we will prove the following two estimates:

$$\lim_{t\to T}\prod_{\substack{1\le i\le 2N\\ i\ne j,j+1}}\left(\frac{g_t(x_i)-g_t(x_{j+1})}{g_t(x_i)-W_t}\right)^{\vartheta^{\#}(i;j)}=0;\tag{A.3}$$

and

$$\sup_{0\le t\le T}F_{\alpha/\wedge_j,\beta/\wedge_j}(g_t(x_1),\dots,g_t(x_{j-1}),g_t(x_{j+2}),\dots,g_t(x_{2N}))<\infty.\tag{A.4}$$

Suppose $\beta=\{\{a_1,b_1\},\dots,\{a_N,b_N\}\}$ is ordered as in (2.2). The number of elements in two sets of indexes $A=\{i: j+1<i\le n, i\in\{a_1,\dots,a_N\}\}$ and $B=\{i: j+1<i\le n, i\in\{b_1,\dots,b_N\}\}$ are equal. Note that $n\in B$. We choose the increasing bijection $\sigma:A\to B$ and suppose $\sigma(i_0)=n$.

We first show (A.3). We write

$$\begin{aligned}
&\prod_{\substack{1\le i\le 2N\\ i\ne j,j+1}}\left(\frac{g_t(x_i)-g_t(x_{j+1})}{g_t(x_i)-W_t}\right)^{\vartheta^{\#}(i;j)}\\
=&\prod_{i<j\text{ or }i>n}\left(\frac{g_t(x_i)-g_t(x_{j+1})}{g_t(x_i)-W_t}\right)^{\vartheta^{\#}(i;j)}\prod_{\substack{i\in A\\ i\ne i_0}}\frac{g_t(x_i)-g_t(x_{j+1})}{g_t(x_i)-W_t}\,\frac{g_t(x_{\sigma(i)})-W_t}{g_t(x_{\sigma(i)})-g_t(x_{j+1})}\\
&\frac{g_t(x_{i_0})-g_t(x_{j+1})}{g_t(x_{i_0})-W_t}\,\frac{g_t(x_n)-W_t}{g_t(x_n)-g_t(x_{j+1})}
\end{aligned}$$

By Lemma A.1, we have

$$\lim_{t\to T} \frac{g_t(x_{i_0}) - g_t(x_{j+1})}{g_t(x_{i_0}) - W_t}\,\frac{g_t(x_n) - W_t}{g_t(x_n) - g_t(x_{j+1})} = 0. \tag{A.5}$$

By Lemma A.2, there exist $C_1, C_2 > 0$, which only depend on $[0,T]$, such that for any $i \in A$ with $\sigma(i) \neq n$, we have for all $t \in [0,T]$,

$$C_1 \le \frac{g_t(x_i) - g_t(x_{j+1})}{g_t(x_i) - W_t}\,\frac{g_t(x_{\sigma(i)}) - W_t}{g_t(x_{\sigma(i)}) - g_t(x_{j+1})} \le C_2. \tag{A.6}$$

For $i \notin A \cup B$,

$$\lim_{t\to T} \frac{g_t(x_i) - g_t(x_{j+1})}{g_t(x_i) - W_t} = \frac{g_T(x_i) - g_T(x_{j+1})}{g_T(x_i) - W_T}.$$

Combining with (A.6) and (A.5), we obtain (A.3).

Next, we prove (A.4). We write

$$\begin{aligned}
&F_{\beta=\wedge_j;\,\alpha=\wedge_j}(g_t(x_1),\ldots,g_t(x_{j-1}),g_t(x_{j+2}),\ldots,g_t(x_{2N}))\\
&= \frac{\prod_{\substack{\{a_i,b_i\}\in\wedge_j\\ a_i\notin A \text{ or } b_i\notin B}} (g_t(x_{b_i}) - g_t(x_{a_i}))^{-1}}{\prod_{\substack{i\notin A\cup B\\ \text{or } k\notin A\cup B}} |g_t(x_k) - g_t(x_i)|^{\#(i,k)}} \times S_t,
\end{aligned}$$

where

$$S_t = \frac{\prod_{\substack{\{a_i,b_i\}\in\wedge_j\\ a_i\in A \text{ and } b_i\in B}} (g_t(x_{b_i}) - g_t(x_{a_i}))^{-1}}{\prod_{\substack{i\in A\cup B\setminus\{n\}\\ \text{and } k\in A\cup B\setminus\{n\}}} |g_t(x_k) - g_t(x_i)|^{\#(i,k)} \prod_{i\in A\cup B\setminus\{n\}} |g_t(x_n) - g_t(x_i)|^{\#(i,n)}}.$$

In this decomposition, we have

$$\frac{\prod_{\substack{\{a_i,b_i\}\in\wedge_j\\ a_i\notin A \text{ or } b_i\notin B}} (g_t(x_{b_i}) - g_t(x_{a_i}))^{-1}}{\prod_{\substack{i\notin A\cup B\\ \text{or } k\notin A\cup B}} |g_t(x_k) - g_t(x_i)|^{\#(i,k)}} \asymp 1,$$

because both the numerator and the denominator converge to a bounded and nonzero quantity as $t \to T$. Here the notation is defined in the same way as in the proof of Lemma A.2. By (A.1), for distinct $i,k \in A \cup B \setminus \{n\}$, we have

$$|g_t(x_k) - g_t(x_i)| \ge g_t(x_{j+2}) - g_t(x_{j+1}).$$

By the same method as in the proof of (A.2), for $i \in A \cup B \setminus \{n\}$, we have

$$\lim_{t\to T} \frac{g_t(x_n) - g_t(x_i)}{g_t(x_n) - g_t(x_{n-1})} = 1.$$

Thus we have

$$S_t \lesssim \prod_{\substack{\{a_i,b_i\}\in\wedge_j\\ a_i\in A \text{ and } b_i\in B}} (g_t(x_{b_i}) - g_t(x_{a_i}))^{-1} (g_t(x_{j+2}) - g_t(x_{j+1}))^{\#A-1} (g_t(x_n) - g_t(x_{n-1})).$$

When there is $a \in A$ such that $\{a,n\} \in \wedge_j$, and $\#A - 1$ pairs $\{a_i,b_i\} \in \wedge_j$ such that $a_i \in A$ and $b_i \in B$, we have $S_t \asymp 1$. Otherwise, we have $\lim_{t\to T} S_t = 0$. This gives (A.4) and completes the proof. □

B Proof of Proposition 5.6

Proof of Proposition 5.6. From (3.12), we have

$$\mathcal{Z}_\alpha(x_1,\dots,x_{4N}) = \sum_{\beta\in\mathrm{DP}_{2N}} \mathcal{M}^{-1}_{\alpha,\beta}\,\mathcal{U}_\beta(x_1,\dots,x_{4N}).$$

For $\beta\in\mathrm{DP}_{2N}$, there exists $J\subset\{1,2,\dots,2N\}$ such that

$$\wedge_{2j-1}\in\beta \quad\text{for all } j\in J;\ \text{and}\quad \vee_{2j-1}\in\beta\quad \text{for all } j\in\{1,\dots,2N\}\setminus J.$$

Then, from the definition (3.10), the following limit exists:

$$\lim_{\substack{x_{2j-1},x_{2j}\to y_j,\\ \forall j\notin J}} \frac{\mathcal{U}_\beta(x_1,\dots,x_{4N})}{\prod_{j\notin J}(x_{2j}-x_{2j-1})^{1/2}}.$$

To obtain the desired limit, we need to group distinct β's according to the location of their local extremes.

Let J be any subset of $\{1,2,\dots,2N\}$, and define

$$\mathrm{P}_J=\{\beta\in\mathrm{DP}_{2N}:\ ;\ \wedge_{2j-1}\in\beta\ \text{ for all } j\in J;\ \vee_{2j-1}\in\beta\ \text{ for all } j\in\{1,2,\dots,2N\}\setminus J\}.$$

It suffices to show that the following limit exists for all possible α and J:

$$\lim_{\substack{x_{2j-1},x_{2j}\to y_j,\\ \forall j\in J}} \frac{\sum_{\beta\in\mathrm{P}_J}\mathcal{M}^{-1}_{\alpha,\beta}\,\mathcal{U}_\beta(x_1,\dots,x_{4N})}{\prod_{j\in J}(x_{2j}-x_{2j-1})^{1/2}}.$$

Suppose $n=\#J\ge 1$. For some $\beta_0\in\mathrm{P}_J$ such that $\underline{\vee}_{2j-1}\in\beta_0$ for all $j\in J$, we define

$$\mathrm{P}_J^{\beta_0}=\{\beta\in\mathrm{DP}_{2N}:\exists\{i_1,\dots,i_k\}\subset J \text{ such that } \beta=\beta_0\uparrow\Diamond_{2i_1-1}\cdots\uparrow\Diamond_{2i_k-1}\}.$$

It is clear that $\#\mathrm{P}_J^{\beta_0}=2^n$. Furthermore, for distinct $\beta_0,\beta_0'\in\mathrm{P}_J$ such that $\underline{\vee}_{2j-1}\in\beta_0$ and $\underline{\vee}_{2j-1}\in\beta_0'$ for all $j\in J$, we see that $\mathrm{P}_J^{\beta_0}\cap\mathrm{P}_J^{\beta_0'}=\emptyset$. Thus $\{\mathrm{P}_J^{\beta_0}:\beta_0\in\mathrm{P}_J \text{ with } \underline{\vee}_{2j-1}\in\beta_0\ \forall j\in J\}$ gives a disjoint partition of P_J. Therefore, it suffices to show that the following limit exists for each such β_0:

$$\lim_{\substack{x_{2j-1},x_{2j}\to y_j,\\ \forall j\in J}} \frac{\sum_{\beta\in\mathrm{P}_J^{\beta_0}}\mathcal{M}^{-1}_{\alpha,\beta}\,\mathcal{U}_\beta(x_1,\dots,x_{4N})}{\prod_{j\in J}(x_{2j}-x_{2j-1})^{1/2}}. \tag{B.1}$$

To derive (B.1), we will show a more general conclusion. Suppose $K\subset J$ and suppose $\beta_0\in\mathrm{P}_J$ such that $\underline{\vee}_{2j-1}\in\beta_0$ for all $j\in K$. We define

$$\mathrm{P}_{J;K}^{\beta_0}=\{\beta\in\mathrm{DP}_{2N}:\exists\{i_1,\dots,i_k\}\subset K \text{ such that } \beta=\beta_0\uparrow\Diamond_{2i_1-1}\cdots\uparrow\Diamond_{2i_k-1}\}.$$

We denote $R_K:=\{2j-1,2j: j\in K\}$, and we denote

$$Z_{K;j}:=\sum_{l\notin R_K}\frac{\vartheta_\alpha(l;2j-1)}{x_l-y_j}.$$

We denote by $\mathfrak{S}_n$ the set of permutations of $\{1,2,\dots,n\}$. Suppose $K=\{j_1,\dots,j_n\}$. For any $\beta_0\in\mathrm{P}_J$ such that $\underline{\vee}_{2j-1}\in\beta_0$ for all $j\in K$, we claim that

$$\lim_{\substack{x_{2j-1},x_{2j}\to y_j,\\ \forall j\in K}} \frac{\sum_{\beta\in\mathrm{P}_{J;K}^{\beta_0}}\mathcal{M}^{-1}_{\alpha,\beta}\,\mathcal{U}_\beta(x_1,\dots,x_{4N})}{\prod_{j\in K}(x_{2j}-x_{2j-1})^{1/2}} \tag{B.2}$$

$$=\mathcal{M}^{-1}_{\alpha,\beta_0}\prod_{\substack{1\le t<s\le 4N\\ t,s\notin R_K}}(x_s-x_t)^{\frac12\vartheta_{\beta_0}(t,s)}\left(\sum_{m=0}^{\lfloor n/2\rfloor}\sum_{\sigma\in\mathfrak{S}_n^m}\frac{2^m Z^{\beta_0}_{K;j_{\sigma(2m+1)}}\cdots Z^{\beta_0}_{K;j_{\sigma(n)}}}{(y_{j_{\sigma(1)}}-y_{j_{\sigma(2)}})^2\cdots(y_{j_{\sigma(2m-1)}}-y_{j_{\sigma(2m)}})^2}\right);$$

where J_n^m is a subset of S_n:

$$J_n^m = \{\sigma \in S_n : \sigma_1 < \sigma_3 < \cdots < \sigma_{2m-1};\ \text{and}\ \sigma_{2j-1} < \sigma_{2j}\ \text{for}\ j \le m;\ \text{and}\ \sigma_{2m+1} < \cdots < \sigma_n\}.$$

Fix β and J, we will show (B.2) by induction on $n = \# K$. It is true for $K = \emptyset$ as it is the same as the definition of U_{ϖ_0}. Suppose (B.2) holds for $\# K = i$. We need to show it for $\# K = i+1$. Suppose $K = \{j_1, \ldots, j_{i+1}\}$. We will take the limit in the left hand side of (B.2) in a particular order: we first let $x_{2j-1}, x_{2j} \to y_j$ with $j \in K \setminus \{j_{i+1}\}$ and then let $x_{2j_{i+1}-1}, x_{2j_{i+1}} \to y_{j_{i+1}}$. It will be clear from the calculation that the limit in (B.2) for $\# K = i+1$ does not depend on the order of taking limits.

For any $\varpi_0 \in P_J$ such that $\_{2j_1-1}, \ldots, \_{2j_{i+1}-1} \in \varpi_0$, we have the decomposition

$$P_{J;K}^{\beta;\varpi_0} = P_{J;K\setminus\{j_{i+1}\}}^{\beta;\varpi_0} \bigsqcup P_{J;K\setminus\{j_{i+1}\}}^{\beta;\varpi_0 \setminus \{2j_{i+1}-1\}}.$$

Denote by $\varpi_1 = \varpi_0 \setminus \{2j_{i+1}-1\}$ and $K_1 = \{j_1, \ldots, j_i\} = K \setminus \{j_{i+1}\}$. Then we have

$$\begin{aligned}
&\lim_{x_{2j-1},x_{2j}\to y_j;\,\forall j\in K} \frac{\sum_{\beta\in P_{J;K}^{\beta;\varpi_0}} M_{\beta}^{-1} U_\beta(x_1,\ldots,x_{4N})}{\prod_{j\in K}(x_{2j}-x_{2j-1})^{1/2}} \\
&= \lim_{x_{2j_{i+1}-1},x_{2j_{i+1}}\to y_{j_{i+1}}} \frac{1}{(x_{2j_{i+1}}-x_{2j_{i+1}-1})^{1/2}} \\
&\quad \left(\lim_{x_{2j-1},x_{2j}\to y_j;\,\forall j\in K_1} \frac{\sum_{\beta\in P_{J;K_1}^{\beta;\varpi_0}} M_{\beta}^{-1} U_\beta(x_1,\ldots,x_{4N})}{\prod_{j\in K_1}(x_{2j}-x_{2j-1})^{1/2}} + \lim_{x_{2j-1},x_{2j}\to y_j;\,\forall j\in K_1} \frac{\sum_{\beta\in P_{J;K_1}^{\beta;\varpi_1}} M_{\beta}^{-1} U_\beta(x_1,\ldots,x_{4N})}{\prod_{j\in K_1}(x_{2j}-x_{2j-1})^{1/2}} \right).
\end{aligned} \tag{B.3}$$

By the induction hypothesis, we have

$$\lim_{x_{2j-1},x_{2j}\to y_j;\,\forall j\in K_1} \frac{\sum_{\beta\in P_{J;K_1}^{\beta;\varpi_0}} M_{\beta}^{-1} U_\beta(x_1,\ldots,x_{4N})}{\prod_{j\in K_1}(x_{2j}-x_{2j-1})^{1/2}} = M_{\beta;\varpi_0}^{-1} \prod_{\substack{1\le t<s\le 4N \\ t,s\notin R_{K_1}}} (x_s - x_t)^{\frac{1}{2}\theta_{\varpi_0}(t,s)} S_0;$$

$$\lim_{x_{2j-1},x_{2j}\to y_j;\,\forall j\in K_1} \frac{\sum_{\beta\in P_{J;K_1}^{\beta;\varpi_1}} M_{\beta}^{-1} U_\beta(x_1,\ldots,x_{4N})}{\prod_{j\in K_1}(x_{2j}-x_{2j-1})^{1/2}} = M_{\beta;\varpi_1}^{-1} \prod_{\substack{1\le t<s\le 4N \\ t,s\notin R_{K_1}}} (x_s - x_t)^{\frac{1}{2}\theta_{\varpi_1}(t,s)} S_1;$$

where

$$S_u = \sum_{m=0}^{\lfloor \frac{i}{2}\rfloor} \sum_{\sigma\in J_i^m} \frac{2^m Z^u_{K_1;j_{\sigma_{2m+1}}} \cdots Z^u_{K_1;j_{\sigma_i}}}{(y_{j_{\sigma_1}} - y_{j_{\sigma_2}})^2 \cdots (y_{j_{\sigma_{2m-1}}} - y_{j_{\sigma_{2m}}})^2}; \quad \text{for } u = 0;1.$$

Comparing the two expressions in the right hand side, we have $M_{\beta;\varpi_1}^{-1} = M_{\beta;\varpi_0}^{-1}$, and

$$\begin{aligned}
&\prod_{\substack{1\le t<s\le 4N \\ t,s\notin R_{K_1}}} (x_s - x_t)^{\frac{1}{2}\theta_{\varpi_0}(t,s)} \\
&= \prod_{\substack{1\le t<s\le 4N \\ t,s\notin R_{K}}} (x_s - x_t)^{\frac{1}{2}\theta_{\varpi_0}(t,s)} \prod_{\substack{1\le n\le 4N \\ n\notin R_K}} \left(\frac{x_n - x_{2j_{i+1}-1}}{x_n - x_{2j_{i+1}}}\right)^{\frac{1}{2}\theta_{\varpi_0}(n;2j_{i+1}-1)} (x_{2j_{i+1}} - x_{2j_{i+1}-1})^{\frac{1}{2}};
\end{aligned}$$

$$\begin{aligned}
&\prod_{\substack{1\le t<s\le 4N \\ t,s\notin R_{K_1}}} (x_s - x_t)^{\frac{1}{2}\theta_{\varpi_1}(t,s)} \\
&= \prod_{\substack{1\le t<s\le 4N \\ t,s\notin R_{K}}} (x_s - x_t)^{\frac{1}{2}\theta_{\varpi_0}(t,s)} \prod_{\substack{1\le n\le 4N \\ n\notin R_K}} \left(\frac{x_n - x_{2j_{i+1}-1}}{x_n - x_{2j_{i+1}}}\right)^{\frac{1}{2}\theta_{\varpi_0}(n;2j_{i+1}-1)} (x_{2j_{i+1}} - x_{2j_{i+1}-1})^{\frac{1}{2}}.
\end{aligned}$$

Plugging into (B.3) and denoting $\varepsilon = x_{2j_{i+1}} - x_{2j_{i+1}-1}$, we have

$$\lim_{\substack{x_{2j-1},x_{2j}\to y_j;\\ \forall j\in K}} \frac{\sum_{\beta\in 2\mathrm{P}_{;K}} \mathcal{M}^{-1}_{;0}\, \mathcal{U}(x_1;\dots;x_{4N})}{\prod_{j\in K}(x_{2j}-x_{2j-1})^{1/2}} \tag{B.4}$$

$$= \mathcal{M}^{-1}_{;0} \prod_{\substack{1\le t<s\le 4N\\ t,s\notin R_K}} (x_s - x_t)^{\frac12 \vartheta_0(t,s)}$$

$$\lim_{\substack{\varepsilon\to 0\\ x_{2j_{i+1}-1}\to y_{j_{i+1}}}} \left(\frac{1}{\varepsilon}\left(\left(\prod_{\substack{1\le n\le 4N\\ n\notin R_K}}\left(\frac{x_n - x_{2j_{i+1}-1}}{x_n - x_{2j_{i+1}}}\right)^{\vartheta_0(n;2j_{i+1}-1)} - 1\right) S_0 + S_0 - S_1\right)\right).$$

Note that

$$Z^{0}_{K_1;j_{2m+1}} Z^{0}_{K_1;j_i}$$
$$= \left(Z^{0}_{K;j_{2m+1}} + \frac{\varepsilon}{\left(x_{2j_{i+1}-1} - y_{j_{2m+1}}\right)\left(x_{2j_{i+1}} - y_{j_{2m+1}}\right)}\right)$$
$$\left(Z^{0}_{K;j_i} + \frac{\varepsilon}{\left(x_{2j_{i+1}-1} - y_{j_i}\right)\left(x_{2j_{i+1}} - y_{j_i}\right)}\right);$$

$$Z^{1}_{K_1;j_{2m+1}} Z^{1}_{K_1;j_i}$$
$$= \left(Z^{0}_{K;j_{2m+1}} - \frac{\varepsilon}{\left(x_{2j_{i+1}-1} - y_{j_{2m+1}}\right)\left(x_{2j_{i+1}} - y_{j_{2m+1}}\right)}\right)$$
$$\left(Z^{0}_{K;j_i} - \frac{\varepsilon}{\left(x_{2j_{i+1}-1} - y_{j_i}\right)\left(x_{2j_{i+1}} - y_{j_i}\right)}\right).$$

Plugging into S_0 and S_1, we have

$$\lim_{\substack{\varepsilon\to 0\\ x_{2j_{i+1}-1}\to y_{j_{i+1}}}} \left(\frac{1}{\varepsilon}\left(\prod_{\substack{1\le n\le 4N\\ n\notin R_K}}\left(\frac{x_n - x_{2j_{i+1}-1}}{x_n - x_{2j_{i+1}}}\right)^{\vartheta_0(n;2j_{i+1}-1)} - 1\right) S_0\right)$$

$$= Z^{0}_{K;j_{i+1}} \left(\sum_{m=0}^{\lfloor \frac{i}{2}\rfloor} \sum_{\in J^m_i} \frac{2^m Z^{0}_{K;j_{2m+1}} \cdots Z^{0}_{K;j_i}}{(y_{j_1} - y_{j_2})^2 \cdots (y_{j_{2m-1}} - y_{j_{2m}})^2}\right);$$

$$\lim_{\substack{\varepsilon\to 0\\ x_{2j_{i+1}-1}\to y_{j_{i+1}}}} \frac{1}{\varepsilon}(S_0 - S_1)$$

$$= \sum_{m=0}^{\lfloor \frac{i}{2}\rfloor} \sum_{\in J^m_i} \sum_{r=2m+1}^{i} \frac{2^{m+1} Z^{0}_{K;j_{2m+1}} \cdots Z^{0}_{K;j_{r-1}} Z^{0}_{K;j_{r+1}} \cdots Z^{0}_{K;j_i}}{(y_{j_1} - y_{j_2})^2 \cdots (y_{j_{2m-1}} - y_{j_{2m}})^2 (y_{j_{i+1}} - y_{j_r})^2}.$$

Plugging into (B.4), we see that it remains to show

$$
\begin{aligned}
&\sum_{m=0}^{\lfloor\frac{i+1}{2}\rfloor}\sum_{\sigma\in J^m_{i+1}}\frac{2^m Z'_{K;j_{\sigma_{2m+1}}}\cdots Z'_{K;j_{\sigma_{i+1}}}}{(y_{j_{\sigma_1}}-y_{j_{\sigma_2}})^2\cdots(y_{j_{\sigma_{2m-1}}}-y_{j_{\sigma_{2m}}})^2} \qquad \text{(B.5)}\\
&=\sum_{m=0}^{\lfloor\frac{i}{2}\rfloor}\sum_{\sigma\in J^m_{i}}\sum_{r=2m+1}^{i}\frac{2^{m+1} Z'_{K;j_{\sigma_{2m+1}}}\cdots Z'_{K;j_{\sigma_{r-1}}}Z'_{K;j_{\sigma_{r+1}}}\cdots Z'_{K;j_{\sigma_{i}}}}{(y_{j_{\sigma_1}}-y_{j_{\sigma_2}})^2\cdots(y_{j_{\sigma_{2m-1}}}-y_{j_{\sigma_{2m}}})^2(y_{j_{i+1}}-y_{j_{\sigma_r}})^2}\\
&\quad+\sum_{m=0}^{\lfloor\frac{i}{2}\rfloor}\sum_{\sigma\in J^m_{i}}\frac{2^m Z'_{K;j_{\sigma_{2m+1}}}\cdots Z'_{K;j_{\sigma_{i}}}Z'_{K;j_{i+1}}}{(y_{j_{\sigma_1}}-y_{j_{\sigma_2}})^2\cdots(y_{j_{\sigma_{2m-1}}}-y_{j_{\sigma_{2m}}})^2}.
\end{aligned}
$$

For $\sigma\in J^m_{i+1}$, let us consider the location of $i+1$ in σ. If $\sigma_{i+1}=i+1$, we define $\sigma'_j=\sigma_j$ for $1\le j\le i$, then $\sigma'\in J^m_i$. Thus, for $0\le m\le\lfloor\frac{i}{2}\rfloor$, we have

$$
\begin{aligned}
&\sum_{\substack{\sigma\in J^m_{i+1}:\\ \sigma_{i+1}=i+1}}\frac{2^m Z'_{K;j_{\sigma_{2m+1}}}\cdots Z'_{K;j_{\sigma_{i+1}}}}{(y_{j_{\sigma_1}}-y_{j_{\sigma_2}})^2\cdots(y_{j_{\sigma_{2m-1}}}-y_{j_{\sigma_{2m}}})^2} \qquad \text{(B.6)}\\
&=\sum_{\sigma\in J^m_{i}}\frac{2^m Z'_{K;j_{\sigma_{2m+1}}}\cdots Z'_{K;j_{\sigma_{i}}}Z'_{K;j_{i+1}}}{(y_{j_{\sigma_1}}-y_{j_{\sigma_2}})^2\cdots(y_{j_{\sigma_{2m-1}}}-y_{j_{\sigma_{2m}}})^2}.
\end{aligned}
$$

If $\sigma_{i+1}<i+1$, we define a mapping for each $m\in\{1,2,\ldots,\lfloor\frac{i}{2}\rfloor\}$:

$$
T_m:\{\sigma\in J^m_{i+1}:\sigma_{i+1}<i+1\}\to J^{m-1}_i\times\{2m-1,\ldots,i\},\quad \sigma\mapsto(\sigma',r)
$$

in the following way. For $\sigma\in J^m_{i+1}$ and $\sigma_{i+1}<i+1$, we must have $\sigma_{2k}=i+1$ for some $1\le k\le m$. We set $\sigma'_j=\sigma_j$, for $j\le 2k-2$; we set $\sigma'_j=\sigma_{j+2}$, for $2k-1\le j\le 2m-2$; and we set $\{\sigma'_{2m-1},\ldots,\sigma'_i\}=\{\sigma_{2k-1},\sigma_{2m+1},\ldots,\sigma_{i+1}\}$ such that $\sigma'_{2m-1}<\cdots<\sigma'_i$. Suppose $\sigma'_r=\sigma_{2k-1}$ for some $r\in\{2m-1,\ldots,i\}$. This defines the map $T_m(\sigma)=(\sigma',r)$. We argue that T_m is a bijection. For any $(\sigma',r)\in J^{m-1}_i\times\{2m-1,\ldots,i\}$, we can define σ as follows: $\{\{\sigma_1,\sigma_2\},\ldots,\{\sigma_{2m-1},\sigma_{2m}\}\}=\{\{\sigma'_1,\sigma'_2\},\ldots,\{\sigma'_{2m-3},\sigma'_{2m-2}\},\{\sigma'_r,i+1\}\}$ and $\{\sigma_{2m+1},\ldots,\sigma_{i+1}\}=\{\sigma'_{2m-1},\ldots,\sigma'_{r-1},\sigma'_{r+1},\ldots,\sigma'_i\}$, such that $\sigma_1<\sigma_3<\cdots<\sigma_{2m-1}$, $\sigma_{2j-1}<\sigma_{2j}$ for $j\le m$ and $\sigma_{2m+1}<\cdots<\sigma_{i+1}$. Then we have $\sigma\in J^m_{i+1}$ and $\sigma_{i+1}<i+1$. This implies T_m is a bijection. Thus, we have

$$
\begin{aligned}
&\sum_{m=1}^{\lfloor\frac{i}{2}\rfloor}\sum_{\substack{\sigma\in J^m_{i+1}:\\ \sigma_{i+1}<i+1}}\frac{2^m Z'_{K;j_{\sigma_{2m+1}}}\cdots Z'_{K;j_{\sigma_{i+1}}}}{(y_{j_{\sigma_1}}-y_{j_{\sigma_2}})^2\cdots(y_{j_{\sigma_{2m-1}}}-y_{j_{\sigma_{2m}}})^2} \qquad \text{(B.7)}\\
&=\sum_{m=1}^{\lfloor\frac{i}{2}\rfloor}\sum_{\sigma\in J^{m-1}_{i}}\sum_{r=2m-1}^{i}\frac{2^{m} Z'_{K;j_{\sigma_{2m-1}}}\cdots Z'_{K;j_{\sigma_{r-1}}}Z'_{K;j_{\sigma_{r+1}}}\cdots Z'_{K;j_{\sigma_{i}}}}{(y_{j_{\sigma_1}}-y_{j_{\sigma_2}})^2\cdots(y_{j_{\sigma_{2m-3}}}-y_{j_{\sigma_{2m-2}}})^2(y_{j_{i+1}}-y_{j_{\sigma_r}})^2}\\
&=\sum_{m=0}^{\lfloor\frac{i}{2}\rfloor-1}\sum_{\sigma\in J^m_{i}}\sum_{r=2m+1}^{i}\frac{2^{m+1} Z'_{K;j_{\sigma_{2m+1}}}\cdots Z'_{K;j_{\sigma_{r-1}}}Z'_{K;j_{\sigma_{r+1}}}\cdots Z'_{K;j_{\sigma_{i}}}}{(y_{j_{\sigma_1}}-y_{j_{\sigma_2}})^2\cdots(y_{j_{\sigma_{2m-1}}}-y_{j_{\sigma_{2m}}})^2(y_{j_{i+1}}-y_{j_{\sigma_r}})^2}.
\end{aligned}
$$

Combining (B.6) and (B.7), we obtain (B.5) for even i. Next, suppose i is odd and denote

$\ell = \frac{i+1}{2}$. By (B.6) and (B.7), we have

$$\sum_{m=0}^{\ell} \sum_{\substack{\sigma \in J^m_{i+1}: \\ \sigma_{i+1} = i+1}} \frac{2^m Z^0_{K;j_{\sigma_{2m+1}}} \cdots Z^0_{K;j_{\sigma_{i+1}}}}{(y_{j_{\sigma_1}} - y_{j_{\sigma_2}})^2 \cdots (y_{j_{\sigma_{2m-1}}} - y_{j_{\sigma_{2m}}})^2}$$

$$= \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} \sum_{\sigma \in J^m_i} \frac{2^m Z^0_{K;j_{\sigma_{2m+1}}} \cdots Z^0_{K;j_{\sigma_i}} Z^0_{K;j_{i+1}}}{(y_{j_{\sigma_1}} - y_{j_{\sigma_2}})^2 \cdots (y_{j_{\sigma_{2m-1}}} - y_{j_{\sigma_{2m}}})^2} + \sum_{\substack{\sigma \in J^\ell_{i+1}: \\ \sigma_{i+1} = i+1}} \frac{2^\ell}{(y_{j_{\sigma_1}} - y_{j_{\sigma_2}})^2 \cdots (y_{j_{\sigma_i}} - y_{j_{\sigma_{i+1}}})^2};$$

$$\sum_{m=0}^{\ell} \sum_{\substack{\sigma \in J^m_{i+1}: \\ \sigma_{i+1} < i+1}} \frac{2^m Z^0_{K;j_{\sigma_{2m+1}}} \cdots Z^0_{K;j_{\sigma_{i+1}}}}{(y_{j_{\sigma_1}} - y_{j_{\sigma_2}})^2 \cdots (y_{j_{\sigma_{2m-1}}} - y_{j_{\sigma_{2m}}})^2}$$

$$= \sum_{m=1}^{\lfloor \frac{i}{2} \rfloor} \sum_{\substack{\sigma \in J^m_{i+1}: \\ \sigma_{i+1} < i+1}} \frac{2^m Z^0_{K;j_{\sigma_{2m+1}}} \cdots Z^0_{K;j_{\sigma_{i+1}}}}{(y_{j_{\sigma_1}} - y_{j_{\sigma_2}})^2 \cdots (y_{j_{\sigma_{2m-1}}} - y_{j_{\sigma_{2m}}})^2} + \sum_{\substack{\sigma \in J^\ell_{i+1}: \\ \sigma_{i+1} < i+1}} \frac{2^\ell}{(y_{j_{\sigma_1}} - y_{j_{\sigma_2}})^2 \cdots (y_{j_{\sigma_i}} - y_{j_{\sigma_{i+1}}})^2}$$

$$= \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor - 1} \sum_{\sigma \in J^m_i} \sum_{r=2m+1}^{i} \frac{2^{m+1} Z^0_{K;j_{\sigma_{2m+1}}} \cdots Z^0_{K;j_{\sigma_{r-1}}} Z^0_{K;j_{\sigma_{r+1}}} \cdots Z^0_{K;j_{\sigma_i}}}{(y_{j_{\sigma_1}} - y_{j_{\sigma_2}})^2 \cdots (y_{j_{\sigma_{2m-1}}} - y_{j_{\sigma_{2m}}})^2 (y_{j_{i+1}} - y_{j_{\sigma_r}})^2} + \sum_{\substack{\sigma \in J^\ell_{i+1}: \\ \sigma_{i+1} < i+1}} \frac{2^\ell}{(y_{j_{\sigma_1}} - y_{j_{\sigma_2}})^2 \cdots (y_{j_{\sigma_i}} - y_{j_{\sigma_{i+1}}})^2}.$$

Combining these two, in order to get (B.5), it remains to show

$$\sum_{\sigma \in J^\ell_{i+1}} \frac{2^\ell}{(y_{j_{\sigma_1}} - y_{j_{\sigma_2}})^2 \cdots (y_{j_{\sigma_i}} - y_{j_{\sigma_{i+1}}})^2} = \sum_{\tau \in J^{\ell-1}_i} \frac{2^\ell}{(y_{j_{\tau_1}} - y_{j_{\tau_2}})^2 \cdots (y_{j_{\tau_{i-2}}} - y_{j_{\tau_{i-1}}})^2 (y_{j_{i+1}} - y_{j_{\tau_i}})^2}. \tag{B.8}$$

To derive (B.8), we define $T : J^\ell_{i+1} \to J^{\ell-1}_i$ in the following way. For $\sigma \in J^\ell_{i+1}$, we must have $\sigma_{2k} = i+1$ for some $1 \le k \le \ell$. We set $\tau_j = \sigma_j$, for $j \le 2k-2$; we set $\tau_j = \sigma_{j+2}$, for $2k-1 \le j \le 2\ell-2$; and we set $\tau_i = \sigma_{2k-1}$. This defines the map $T(\sigma) = \tau$. We argue that T is a bijection. For any $\tau \in J^{\ell-1}_i$, we can define σ as follows: $\{\{\sigma_1, \sigma_2\}, \ldots, \{\sigma_i, \sigma_{i+1}\}\} = \{\{\tau_1, \tau_2\}, \ldots, \{\tau_{i-2}, \tau_{i-1}\}, \{\tau_i, i+1\}\}$ such that $\sigma_1 < \sigma_3 < \cdots < \sigma_i$, $\sigma_{2j-1} < \sigma_{2j}$ for $j \le \ell$. Then we have $\sigma \in J^\ell_{i+1}$. This implies T is a bijection, and gives (B.8). Hence, we complete the proof of (B.5) for odd i, and complete the proof of (B.5).

Eq. (B.5) gives (B.2) for $\# K = i+1$ and completes the induction. Hence, Eq. (B.2) holds for all $K \subseteq J$. Taking $K = J$ in (B.2), we obtain (B.1). This completes the proof. □

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