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Ergodicity for stochastic equations of Navier–Stokes type*

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Abstract

In this note we consider a simple example of a finite dimensional system of stochastic differential equations driven by a one dimensional Wiener process with a drift, that displays some similarity with the stochastic Navier-Stokes Equations (NSEs), and investigate its ergodic properties depending on the strength of the drift. If the latter is sufficiently small and lies below a critical threshold, then the system admits a unique invariant probability measure which is Gaussian. If, on the other hand, the strength of the noise drift is larger than the threshold, then in addition to a Gaussian invariant probability measure, there exist another one. In particular, the generator of the system is not hypoelliptic.

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1 Introduction

Study of ergodic properties of dynamical (inclusive random) systems is of profound importance from both applied and theoretical standpoints. Two examples of such properties are the existence and uniqueness (or possibly non-uniqueness) of invariant probability measures. These are often linked to the not-yet fully explained aspects of turbulence such as e.g. the rigorous proof of the form of the Kolmogorov spectrum.

In the case of stochastic hydrodynamics, i.e. for the stochastic Navier–Stokes equations of the following form

$$\partial_t u + (u \cdot \nabla) u = (\mu \Delta u - \nabla p + f) + \xi, \qquad \nabla \cdot u = 0, \tag{1.1}$$

where u is the velocity field, p is the pressure scalar (both unknown), f is the external force acting on the fluid and ξ is a noise, the first results in those directions are due to

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Flandoli [14], who showed the existence of an invariant probability measure for the 2D Navier–Stokes equations (NSE) driven by an additive Gaussian noise. The question of the uniqueness of an invariant probability measure for the same system has been later addressed by Flandoli and Maslowski [16], Ferario [13] and E, Mattingly and Sinai [12]. The first two papers assumed that noise ξ has been sufficiently non-degenerate (which had to be counterbalanced by a requirement that it is sufficiently spatially regular to ensure the solvability of the system (1.1)). The direction of research initiated in first two of these papers has been followed by Da Prato and Debussche in [8] who proved the unique solvability of (1.1) (in Besov spaces of negative order) and the existence of an invariant probability measure for the stochastic Stokes equation (on a 2D torus) when ξ is the space-time noise; see also the paper by Albeverio and Ferrario [1].

The paper [12] by E, Mattingly and Sinai looked at the question of uniqueness of an invariant probability measure in the case of a degenerate noise, which happens to be mathematically more challenging than the non-degenerate noise. In this case the corresponding Markov process is only a Feller rather than strong Feller. This case was also studied by Mattingly in [27, 29] in the case the external force f is equal to zero and in [18, 25, 7] for nonzero force f. The culminating work on this topic is due to Hairer and Mattingly [21] who, using a new concept of an asymptotically strong Feller semigroup, proved that the Markov process generated by the stochastic NSEs on a 2D torus has a unique invariant probability measure provided the Gaussian perturbation is of mean 0 and acts on at least two modes that are of different length and whose integer linear combinations generate the two dimensional integer lattice. Such a system can be called a hypoelliptic. Later on Friedlander et al. [17] and Andreis et al. [2], proved that the hypoellipticity still holds for certain stochastic inviscid dyadic models and hence such models have a unique invariant probability measure even if the centered noise acts only on a single mode.

It is still an open question whether similar properties hold in the presence of a large deterministic force, i.e. when the noise in not centered and its mean is large. For instance the method from [21] still works when the force is small so that the corresponding deterministic system has a unique stationary solution which is exponentially stable. Another open question is whether whether similar properties hold when the noise is more degenerate than the noise considered in the paper [21].

The modest aim of this note is to prove that for a certain finite dimensional system modelling the true SNSE, introduced by Minea in [30], such a result is not true. To be precise, in Theorem 2.3 we show that if $\kappa > \lambda_1 \min\{\lambda_2, \lambda_3\}$, then the stochastic system (2.1), i.e. (2.2), has at least two invariant probability measures, and, since any convex combination of these invariant probability measures is also an invariant probability measure, the stochastic system (2.1), i.e. (2.2), has infinitely many invariant probability measures.

One of the measures, denoted by $\nu_{\sigma,\kappa}$, is Gaussian. This measure is also the unique invariant probability measure for the corresponding stochastic "Stokes system" (2.14). Let us finish this paragraph by recalling that the set of stationary solutions for the corresponding deterministic system (2.13) has quite a complicated structure. Thus the present note shows that this also could be the case for its stochastic perturbation.

2 Main results

Let us consider the following Stochastic Differential Equations (SDEs) in $\mathbb{R}^3$:

$$\begin{aligned} \mathrm{d}u_1 &= \left[-\lambda_1 u_1 - \left(u_2^2 + u_3^2\right)\right] \mathrm{d}t + \kappa \mathrm{d}t + \sigma \mathrm{d}W(t), \\ \mathrm{d}u_2 &= \left[-\lambda_2 u_2 + u_1 u_2\right] \mathrm{d}t, \\ \mathrm{d}u_3 &= \left[-\lambda_3 u_3 + u_1 u_3\right] \mathrm{d}t, \end{aligned} \tag{2.1}$$

where $\sigma > 0$, $\kappa \in \mathbb{R}$, and W is a standard real-valued Wiener process.

Clearly we can write the SDEs (2.1) in the form

$$\mathrm{d}u = [Au + B(u,u) + \kappa f_1]\,\mathrm{d}t + \sigma f_1 \mathrm{d}W(t), \tag{2.2}$$

where the maps A and B are defined by

$$A \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = -\begin{pmatrix} \lambda_1 u_1 \\ \lambda_2 u_2 \\ \lambda_3 u_3 \end{pmatrix}, \quad B\left[\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right] = \begin{pmatrix} -u_2 v_2 - u_3 v_3 \\ u_2 v_1 \\ u_3 v_1 \end{pmatrix} \tag{2.3}$$

and $(f_i)_{i=1}^3$ is the canonical orthonormal basis of $\mathbb{R}^3$.

Note that the mapping $B\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is bilinear and

$$(B(u,v),w) = b(u,v,w),$$

where b is a trilinear form on $\mathbb{R}^3$ defined by

$$b(u,v,w) = -(u_2 v_2 + u_3 v_3) w_1 + u_2 v_1 w_2 + u_3 v_1 w_3.$$

We have

$$b(u,v,w) = -b(u,w,v), \qquad u,v,w \in \mathbb{R}^3, \tag{2.4}$$

$$(B(u,v),v) = 0, \qquad u,v \in \mathbb{R}^3. \tag{2.5}$$

Remark 2.1. The 2D stochastic Navier-Stokes Equations (1.1) with periodic boundary conditions can be written in the form

$$\mathrm{d}u = [Au + B(u,u) + f]\,\mathrm{d}t + \mathrm{d}W(t), \tag{2.6}$$

where $-A$ is a self-adjoint, positive and of compact reselvent, linear operator on the Hilbert space H of all divergence free and mean zero square integrable vector fields defined on the 2-d torus $\mathbb{T}^2$ and B is a bilinear bounded map from $D(A) \times D(A)$ to H having the following properties

$$(B(u,v),w) = -(B(u,w),v), \qquad u,v,w \in D(A), \tag{2.7}$$

$$(B(u,v),v) = 0, \qquad u,v \in D(A), \tag{2.8}$$

$$B(e,e) = 0 \qquad \text{for any eigenvector } e \text{ of } A, \tag{2.9}$$

$$(B(v,v),Av) = 0, \qquad v \in D(A). \tag{2.10}$$

It is worth noting that (2.9) and (2.10) hold only in the periodic 2-dimensional case, whereas (2.7) and (2.8) hold both in dimensions 2 and 3 and also in the case of the Dirichlet boundary conditions. The proof of property (2.9) can be found in [15].

The finite dimensional model we consider satisfies (2.7), (2.8), but contrary to the 2D Navier–Stokes equations with periodic boundary conditions, condition (2.9) is not satisfied, see (2.12). Instead we only have

$$B(f_1, f_1) = 0 \tag{2.11}$$

and

$$B(f_j, f_j) = -f_1 \neq 0 \qquad \text{for } j = 2,3. \tag{2.12}$$

Remark 2.2. Let us emphasize that the condition (2.11) above corresponds to the assumption (2.9) which, as we have pointed out earlier, is satisfied for the 2D Navier–Stokes equations with periodic boundary conditions, see [15]. Thus our equation (2.2) can be seen as a simple finite dimensional model of such a problem. Let us point out here that a more general, but still finite dimensional, model has been recently investigated by Hairer and Coti-Zelati in [20]. They proved the ergodicity of the non-unique invariant probability measures.

One should also mention a less recent paper [3], by Baňas et al, who studied the uniqueness and non-uniqueness of invariant probability measures for second order stochastic differential equations on a sphere.

Given $v \in \mathbb{R}^3$ we denote by $u(\cdot; v)$ the solution of (2.1) starting at time 0 from v. Note that

$$u_1 \equiv \kappa/\lambda_1, \quad u_2 \equiv 0 \equiv u_3,$$

is a stationary solution to the deterministic problem

$$\begin{aligned} \mathrm{d}u_1 &= \left[-\lambda_1 u_1 - \left(u_2^2 + u_3^2\right) + \kappa\right] \mathrm{d}t, \\ \mathrm{d}u_2 &= \left[-\lambda_2 u_2 + u_1 u_2\right] \mathrm{d}t, \\ \mathrm{d}u_3 &= \left[-\lambda_3 u_3 + u_1 u_3\right] \mathrm{d}t. \end{aligned} \tag{2.13}$$

Note that if $\kappa \le \lambda_1 \min\{\lambda_2, \lambda_3\}$, then there is unique stationary solution to the system, whereas if $\kappa > \lambda_1 \min\{\lambda_2, \lambda_3\}$, then there exists more than one such a solution. The set of solutions different from the described above can be characterized as follows:

$$\begin{cases} \text{If } \lambda_2 = \lambda_3, \text{ then } u_1 = \lambda_2, \quad u_2^2 + u_3^2 = \kappa - \lambda_1\lambda_2. \\ \text{If } \lambda_2 > \lambda_3,\ \lambda_2\lambda_3 \ge \kappa,\ \lambda_3\lambda_1 < \kappa \text{ then } u_1 = \lambda_3,\ u_2 = 0 \text{ and } u_3^2 = \kappa - \lambda_1\lambda_3. \\ \text{If } \lambda_3 > \lambda_2,\ \lambda_3\lambda_2 \ge \kappa,\ \lambda_2\lambda_1 < \kappa \text{ then } u_1 = \lambda_2,\ u_3 = 0 \text{ and } u_2^2 = \kappa - \lambda_1\lambda_2. \\ \text{If } \lambda_1 \max\{\lambda_2, \lambda_3\} < \kappa,\ \lambda_2 \ne \lambda_3, \text{ then:} \\ \qquad \text{(i) } u_1 = \lambda_2,\ u_3 = 0 \text{ and } u_2^2 = \kappa - \lambda_1\lambda_2, \\ \qquad\quad \text{or} \\ \qquad \text{(ii) } u_1 = \lambda_3,\ u_2 = 0 \text{ and } u_3^2 = \kappa - \lambda_1\lambda_3. \end{cases}$$

A natural question arises whether the stochastic differential equation (2.1) exhibits a similar phenomena as its deterministic counterpart (2.13). We have the following result.

Theorem 2.3. *In the framework described above the following holds.*

(i) *For arbitrary parameters, there exists an invariant probability measure to* (2.1)*. In fact for any initial value $v \in \mathbb{R}^3$, there exists a sequence $t_n \nearrow +\infty$ such that the following sequence of Borel probability measures on $\mathbb{R}^3$*

$$\mathcal{L}\left(\frac{1}{t_n}\int_0^{t_n} u(s; v)\mathrm{d}s\right)$$

converges weakly to a Borel probability measure on $\mathbb{R}^3$. Consequently, by the Krylov–Bogolyubov theorem, the simplified stochastic NSE (2.1) *has at least one invariant probability measure.*

(ii) *For arbitrary $\lambda_1 > 0$ and $\kappa, \sigma \in \mathbb{R}$, the law $\nu_{\sigma,\kappa}$ of the following random variable*

$$\frac{\kappa}{\lambda_1} f_1 + \sigma \int_0^{+\infty} \mathrm{e}^{-\lambda_1 t} \mathrm{d}W(t) f_1$$

in $(\mathbb{R}^3, \mathcal{B}(\mathbb{R}^3))$ *is Gaussian and invariant both for* (2.1) *and for the stochastic linear "Stokes" equation*

$$dz = Az\mathrm{d}t + (\kappa f_1 \mathrm{d}t + \sigma f_1 \mathrm{d}W). \tag{2.14}$$

(iii) *If*

$$\kappa < \lambda_1 \min\{\lambda_2, \lambda_3\}, \tag{2.15}$$

then for any $\sigma \geq 0$*, the measure* $\nu_{\sigma,\kappa}$ *is the unique invariant probability measure of the simplified stochastic NSEs* (2.1)*. Moreover,* $\nu_{\sigma,\kappa}$ *is stochastically stable, i.e. for any initial data* $v \in \mathbb{R}^3$*, the laws* $\mathcal{L}(u(t;v))$ *converge weakly as* $t \to +\infty$ *to* $\nu_{\sigma,\kappa}$*.*

(iv) *If*

$$\kappa > \lambda_1 \min\{\lambda_2, \lambda_3\}, \tag{2.16}$$

then there exists an invariant probability measure to the simplified stochastic NSE (2.1) *which is different from the Gaussian measure* $\nu_{\sigma,\kappa}$*.*

Remark 2.4. We have recently learnt from a talk given by Francesco Morandin about two papers [17] and [2], in which infinite dimensional models of NSE-s are studied with the noise acting only on the first mode. Contrary to our case, that model is hypoelliptic and admits a unique invariant probability measure.

Remark 2.5. The uniqueness of the invariant measure for the stochastic Navier-Stokes equations (on a 2D torus) when external force $f = \kappa e$ is equal to 0, the noise is one-dimensional and viscosity is large with respect to the diffusion coefficient, is known, see e.g. the paper [12, Theorem 1] by E, Mattingly and Sinai. See also later works [18, 25, 7].

Remark 2.6. The fact that an invariant probability measure for the stochastic Stokes equations (on a 2D torus) driven by a canonical cylindrical Wiener process on H is also an invariant probability measure for the corresponding stochastic Navier–Stokes equations (1.1) is known, see e.g. the paper [8] by Da Prato and Debussche, where this statement is made rigorous, and also the paper by Albeverio and Ferrario [1].

3 Proof of Theorem 2.3

Without loss of generality we can assume in the proof that all processes considered here are continuous.

Proof of (i). By the Itô formula and (2.5) we have

$$\begin{aligned}\mathbb{E}\,|u(t;v)|^2 &= \mathbb{E}\,|v|^2 + 2\mathbb{E}\int_0^t \left[\langle u(s;v), Au(s;v)\rangle + \langle u(s;v), \kappa f_1\rangle + \frac{\sigma^2}{2}\right]\mathrm{d}s \\ &\leq \mathbb{E}\,|v|^2 - \rho\int_0^t \mathbb{E}\,|u(s;v)|^2 + ct, \quad t \geq 0,\end{aligned}$$

where $\rho = \min\{\lambda_1, \lambda_2, \lambda_3\}$ and $c = c(\rho, \sigma, \kappa)$ is independent of t. Here $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^3$. Thus

$$\sup_{t>0} \mathbb{E}\left[\frac{1}{t}\int_0^t |u(s;v)|^2\,\mathrm{d}s\right] < +\infty.$$

Consequently, the laws of the following family of random variables

$$\frac{1}{t}\int_0^t u(s;v)\mathrm{d}s, \quad t > 0,$$

are tight in $\mathbb{R}^3$, and hence relatively weakly compact. Therefore, the existence of an invariant probability measure follows from the Krylov–Bogoliubov theorem. □

Proof of (ii). This part follows follows immediately from the fact that $B(f_1, f_1) = 0$. □

Proof of (iii). Note that, for $t \geq 0$,

$$u_i(t;v) = \exp\left\{\int_0^t u_1(s;v)\mathrm{d}s - \lambda_i t\right\} v_i, \qquad i = 2, 3,$$

and

$$u_1(t;v) = \mathrm{e}^{-\lambda_1 t} v_1 - \int_0^t \mathrm{e}^{-\lambda_1(t-s)} X(s;v)\mathrm{d}s + Z(t), \tag{3.1}$$

where

$$Z(t) := \int_0^t \mathrm{e}^{-\lambda_1(t-s)} \left(\kappa \mathrm{d}s + \sigma \mathrm{d}W(s)\right) \tag{3.2}$$

and

$$X(t;v) := u_2^2(t;v) + u_3^2(t;v) \geq 0.$$

Thus, by (2.1), for $t \geq 0$,

$$u_1(t;v) \leq \mathrm{e}^{-\lambda_1 t} v_1 + Z(t),$$

and consequently,

$$\begin{aligned} X(t;v) &= \mathrm{e}^{2\int_0^t u_1(s;v)\mathrm{d}s} \left(\mathrm{e}^{-2\lambda_2 t} v_2^2 + \mathrm{e}^{-2\lambda_3 t} v_3^2\right) \\ &\leq \mathrm{e}^{2\int_0^t Z(s)\mathrm{d}s} \left(\mathrm{e}^{-2\lambda_2 t} v_2^2 + \mathrm{e}^{-2\lambda_3 t} v_3^2\right) \exp\left\{\frac{2|v_1|}{\lambda_1}\right\}. \end{aligned}$$

Clearly, for $t \geq 0$,

$$\mathrm{e}^{-2\lambda_2 t} v_2^2 + \mathrm{e}^{-2\lambda_3 t} v_3^2 \leq \mathrm{e}^{-2\lambda t} \left(v_2^2 + v_3^2\right),$$

where $\lambda = \min\{\lambda_2, \lambda_3\} > 0$, and therefore

$$X(t;v) \leq \mathrm{e}^{2\int_0^t Z(s)\mathrm{d}s - 2\lambda t} \left(v_2^2 + v_3^2\right) \exp\left\{\frac{2|v_1|}{\lambda_1}\right\}.$$

By the law of large numbers we deduce that

$$\frac{1}{t}\int_0^t Z(s)\mathrm{d}s \to \frac{\kappa}{\lambda_1}, \qquad \mathbb{P}\text{-a.s. as } t \to +\infty.$$

Thus, as $\kappa < \lambda_1 \lambda$ we have

$$\lim_{t\to+\infty} X(t;v) = 0, \qquad \mathbb{P}\text{-a.s.}$$

From the first equation of (2.1) we conclude that for $t \geq 0$,

$$\begin{aligned} u_1(t;v) &= \mathrm{e}^{-\lambda_1(t-T)} u_1(T;v) - \int_T^t \mathrm{e}^{-\lambda_1(t-s)} X(s;v)\mathrm{d}s + \int_T^t \mathrm{e}^{-\lambda_1(t-s)}(\kappa \mathrm{d}s + \sigma \mathrm{d}W(s)) \\ &= R(t,T) + Z(t), \end{aligned}$$

where

$$R(t,T;v) := \mathrm{e}^{-\lambda_1(t-T)} u_1(T;v) - \int_T^t \mathrm{e}^{-\lambda_1(t-s)} X(s;v)\mathrm{d}s - \int_0^T \mathrm{e}^{-\lambda_1(t-s)}(\kappa \mathrm{d}s + \sigma \mathrm{d}W(s)).$$

Since $R(t,T;v) \to 0$, $\mathbb{P}$ a.s., as $t \gg T$ and $t, T \to +\infty$ and $Z(t)$ converges in law to

$$\tilde{\nu}_{\sigma,\kappa} := \mathcal{N}\left(\frac{\kappa}{\lambda_1}, \frac{\sigma^2}{2\lambda_1}\right) \tag{3.3}$$

it follows that $u_1(t;v)$ converges in law to $\tilde{\nu}_{\sigma,\kappa}$, and the desired conclusion follows with

$$\nu_{\sigma,\kappa} := \tilde{\nu}_{\sigma,\kappa} \otimes \delta_0 \otimes \delta_0. \tag{3.4}$$

This concludes the proof of (iii). □

Proof of (iv). Assume that $\lambda_2 = \min\{\lambda_2, \lambda_3\}$. Let u be the solution to (2.1) with the initial data $u_1(0) = u_3(0) = 0$ and $u_2(0) = 1$. Then, for $t \geq 0$,

$$u_1(t) = -\int_0^t \mathrm{e}^{-\lambda_1(t-s)} X(s)\mathrm{d}s + Z(t),$$

where the process Z is defined in (3.2) and

$$X(t) = \exp\left\{2\int_0^t (u_1(s) - \lambda_2)\mathrm{d}s\right\}.$$

Note that under the prescribed initial condition we have, for $t \geq 0$,

$$X(t) = u_2^2(t). \tag{3.5}$$

Since $X \geq 0$ and $\lambda_1 > 0$, we have, for $t \geq 0$,

$$\begin{aligned}
\int_0^t u_1(s)\mathrm{d}s &= -\int_0^t \int_0^s \mathrm{e}^{-\lambda_1(s-r)} X(r)\mathrm{d}r\mathrm{d}s + \int_0^t Z(s)\mathrm{d}s \\
&= -\int_0^t \int_r^t \mathrm{e}^{-\lambda_1(s-r)}\mathrm{d}s X(r)\mathrm{d}r + \int_0^t Z(s)\mathrm{d}s \\
&= \int_0^t \left[-\frac{1}{\lambda_1}\left(1 - \mathrm{e}^{-\lambda_1(t-s)}\right) X(s) + Z(s)\right]\mathrm{d}s \\
&\geq \int_0^t \left[-\frac{1}{\lambda_1} X(s) + Z(s)\right]\mathrm{d}s.
\end{aligned}$$

Therefore, we infer that

$$X(t) \geq \exp\left\{2\int_0^t \left(-\frac{1}{\lambda_1}X(s) + Z(s) - \lambda_2\right)\mathrm{d}s\right\}, \quad t \geq 0.$$

Next, let us observe that by the law of large numbers for any ρ such that

$$0 < \rho < \frac{\kappa}{\lambda_1} - \lambda_2,$$

there exists a random variable ξ such that $\mathbb{P}(\xi > 0) = 1$ and $\mathbb{P}$-a.s

$$X(t) \geq \xi \exp\left\{2\int_0^t \left(-\frac{1}{\lambda_1}X(s) + \rho\right)\mathrm{d}s\right\} \quad \text{for all } t > 0.$$

Thus, for $t \geq 0$,

$$X(t)\exp\left\{\frac{2}{\lambda_1}\int_0^t X(s)\mathrm{d}s\right\} \geq \xi \mathrm{e}^{2\rho t}.$$

Equivalently, for $t \geq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp\left\{\frac{2}{\lambda_1}\int_0^t X(s)\mathrm{d}s\right\} \geq \frac{2}{\lambda_1}\xi \mathrm{e}^{2\rho t},$$

and hence

$$\exp\left\{\frac{2}{\lambda_1}\int_0^t X(s)\mathrm{d}s\right\} \geq \frac{\xi}{\rho\lambda_1}\left(\mathrm{e}^{2\rho t} - 1\right) + 1.$$

Finally, for t large enough we have

$$\frac{1}{t}\frac{2}{\lambda_1}\int_0^t X(s)\mathrm{d}s \geq \frac{1}{t}\log\left\{\frac{\xi}{\rho\lambda_1}\left(\mathrm{e}^{2\rho t} - 1\right) + 1\right\}.$$

Since

$$\lim_{t\to+\infty}\frac{1}{t}\log\left\{\frac{\xi}{\rho\lambda_1}\left(\mathrm{e}^{2\rho t}-1\right)+1\right\}=2\rho$$

we can see that

$$\liminf_{t\to+\infty}\frac{1}{t}\frac{2}{\lambda_1}\int_0^t X(s)\mathrm{d}s\geq 2\rho,\qquad \mathbb{P}\text{-a.s. }. \tag{3.6}$$

This implies that there exists an invariant probability measure different from $\nu_{\sigma,\kappa}$ defined in (3.4). Indeed, consider the Markov process $(u_1,u_2,u_3,X=u_2^2)$, see (3.5), for initial value $(0,1,0,1)$. From the first part of the theorem, the sequence of laws

$$\mathcal{L}\left(\frac{1}{t}\int_0^t(u_1(s),u_2(s),u_3(s),X(s))\mathrm{d}s\right)$$

is tight and hence there is a sequence $t_n\nearrow+\infty$ and a probability measure ν on $\mathbb{R}^3\times[0,+\infty)$ such that

$$\mathcal{L}\left(\frac{1}{t_n}\int_0^{t_n}(u_1(s),u_2(s),u_3(s),X(s))\mathrm{d}s\right)$$

converge to ν. The probability measure

$$\tilde\nu(\Gamma)=\nu(\Gamma\times[0,+\infty)),\quad \Gamma\in\mathcal{B}(\mathbb{R}^3)$$

is invariant for the process $(u_1(t),u_2(t),u_3(t))$, $t\geq 0$. Since, thanks to (3.6), its marginal with respect to the second variable is not δ_0, it is different from $\nu_{\sigma,\kappa}$. □

We finish this paper with the following strengthening of our main result.

Proposition 3.1. *Let $\tilde\nu_{\sigma,\kappa}$ and $\nu_{\sigma,\kappa}$, be given by* (3.3) *and respectively by* (3.4). *If*

$$2\int_{\mathbb{R}}|z|\tilde\nu_{\sigma,\kappa}(\mathrm{d}z)<\min\{\lambda_1,\lambda_2,\lambda_3\},\tag{3.7}$$

then $\nu_{\sigma,\kappa}$ is the unique invariant probability measure for the nonlinear equation (2.1).

Remark 3.2. Condition (3.7) is stronger than the condition (2.15) from Theorem 2.3(iii). In particular, for a fixed $\kappa\geq 0$, (3.7) is not satisfied for large σ.

Proof of Proposition 3.1. To see that (3.7) is a sufficient condition for the ergodicity we denote by z the solution of the linear equation

$$\mathrm{d}z=Az\mathrm{d}t+(\kappa f_1\mathrm{d}t+\sigma f_1\mathrm{d}W(t)),\qquad z(0)=0.$$

Let $v\in\mathbb{R}^3$. Then $y=u(\cdot;v)-z$ satisfies

$$\mathrm{d}y=\left[Ay+B(y+z,y+z)\right]\mathrm{d}t,\qquad y(0)=v.$$

Hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|y(t)|^2=\langle Ay,y\rangle+b(y(t),z(t),y(t)).$$

Clearly

$$\langle Ay,y\rangle\leq-\overline{\lambda}\,|y(t)|^2\,,$$

where

$$\overline{\lambda}:=\min\{\lambda_1,\lambda_2,\lambda_3\}.$$

Next, it is easy to see that

$$|b(y(t), z(t), y(t))| \leq 2\,|z(t)|\,|y(t)|^2 \,.$$

Consequently we have the estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,|y(t)|^2 \leq \left(-\overline{\lambda} + 2\,|z(t)|\right)|y(t)|^2 \,,$$

and hence

$$|y(t)|^2 \leq |v|^2 \exp\left\{2\int_0^t \left(-\overline{\lambda} + 2\,|z(s)|\right)\mathrm{d}s\right\}.$$

Since, by the ergodicity of $\tilde{\nu}_{\sigma,\kappa}$ for z, we have

$$\frac{1}{t}\int_0^t |z(s)|\ \mathrm{d}s \to \int_{\mathbb{R}} |z|\tilde{\nu}_{\sigma,\kappa}(\mathrm{d}z), \qquad \mathbb{P}\text{-a.s.},$$

the desired conclusion follows. □

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